MAT 342: Homework 2

INSTRUCTIONS: Please read carefully the following instructions for this assignment.

1. You have time to submit this assignment until July 22, you need to do by uploading it in **Gradescope**. Unless you can do it by writing it down in a tablet, use an app to scan it (like CamScanner, not just pictures) and send it in .pdf format.
2. This an assignment, so you can discuss these problems, but in case you do that, please make sure that the solutions that you send me are in your own words. This is our chance to correct any mathematical mistake that you do when writing.
3. You will get a grade from 0 to 5 depending on how many problems you attempt.

Problem 1. *The polar coordinates in* $\mathbb{R}^2 \setminus \{0\}$ *are given by*

$$\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases} \quad \text{for } r \in \mathbb{R}^+ = \{r \in \mathbb{R} \colon r \geq 0\} \text{ and } \varphi \in \mathbb{R}.$$

We know that the Cauchy-Riemann equations for a function $f(x,y) = u(x,y) + iv(x,y)$*, where both* $u(x,y)$ *and* $v(x,y)$ *are differentiable, are*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

We want to write their analogue in the case where f *can be expressed as*

$$f(x,y) = r(x,y)e^{i\varphi(x,y)},$$

where $r(x,y)$ *and* $\varphi(x,y)$ *are both differentiable functions.*

To do so, note that we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial r}{\partial x}\cos(\varphi) - r\frac{\partial \varphi}{\partial x}\sin(\varphi) \\ \frac{\partial u}{\partial y} &= \frac{\partial r}{\partial y}\cos(\varphi) - r\frac{\partial \varphi}{\partial y}\sin(\varphi) \\ \frac{\partial v}{\partial x} &= \frac{\partial r}{\partial x}\sin(\varphi) + r\frac{\partial \varphi}{\partial x}\cos(\varphi) \\ \frac{\partial v}{\partial y} &= \frac{\partial r}{\partial y}\sin(\varphi) + r\frac{\partial \varphi}{\partial y}\cos(\varphi). \end{aligned}$$

So,

$$\begin{aligned} \frac{\partial r}{\partial x}\cos(\varphi) - r\frac{\partial \varphi}{\partial x}\sin(\varphi) &= \frac{\partial r}{\partial y}\sin(\varphi) + r\frac{\partial \varphi}{\partial y}\cos(\varphi) \\ \frac{\partial r}{\partial y}\cos(\varphi) - r\frac{\partial \varphi}{\partial y}\sin(\varphi) &= -\frac{\partial r}{\partial x}\sin(\varphi) - r\frac{\partial \varphi}{\partial x}\cos(\varphi). \end{aligned}$$

And then,

$$\frac{\partial r}{\partial x} = r\frac{\partial \varphi}{\partial y} \quad \text{and} \quad \frac{\partial r}{\partial y} = -r\frac{\partial \varphi}{\partial x}.$$

(a) *The solution before is rather short and the explanation could be improved. Fix the gaps in that procedure to obtain a complete answer for the previous problem.*

(b) *Prove that e^z is holomorphic using the equations in (a).*

(c) *Is the function*

$$g(x,y) = \frac{1}{x^2+y^2+1}\left(\cos(y) + i\sin(y)\right)$$

holomorphic according to the equations found in (a)?

(d) *Prove that if we have a holomorphic function*

$$\begin{aligned} f\colon \Omega \subset \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto f(z) \end{aligned}$$

where Ω is open and connected, such that $f(\Omega) \subset \mathbb{R}$, then f is a constant function by using both the conventional Cauchy-Riemann equations and the ones found in (a).

(e) *Prove that if a function $f\colon z \in \Omega \mapsto f(z) \in \mathbb{C}$ is such that $|f(z)| = c$ is constant, then f is constant by using the Cauchy-Riemann equations in polar coordinates.*

Problem 2. (a) *Sketch the sets $A = \{z \in \mathbb{C}\colon |z+3| < 2\}$, $B = \{z \in \mathbb{C}\colon |Im(z)| < 2\}$. Are they open, closed, or none? What about their union? What about their intersection?*

(b) *Parametrize the set $\{z \in \mathbb{C}\colon Re(z) = 3\}$.*

(c) *Find a parametrization for the boundary of the set A in (a). Parametrize the segment from -3 to $-3+2i$, which we denote by $[-3, -3+2i]$.*

(d) *Prove the set*

$$A \setminus [-3, -3+2i]$$

is open and write an oriented parametrization of its boundary.

Problem 3. *Assume $f(z) = u(z) + iv(z)\colon \Omega \to \mathbb{C}$ is such that its real part u and its imaginary part v are twice differentiable. Use the Cauchy-Riemann equations to show that both u and v are harmonic functions, that is,*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \Delta v = 0.$$

Is there a holomorphic function $f(x,y) = u(x,y) + iv(x,y)$ so that $u(x,y) = x^3$?

Problem 4. *For each of the following functions $u(x, y)$, find a function $v(x, y)$ such that*

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

is holomorphic.

(a) $u(x, y) = x + y$.

(b) $u(x, y) = x^2 - y^2$.

(c) $u(x, y) = x/(x^2 + y^2)$.

(d) *For each one of the previous holomorphic functions, what is the largest region in which they are holomorphic?*

Problem 5. (a) *What are the determinations of L the logarithm in $\mathbb{C}\backslash(-\infty, 0]$ that satisfy*

$$L(z_1 \cdot z_2) = L(z_1) + L(z_2)$$

for every $z_j \in \mathbb{C}$ with $\mathrm{Re}(z_j) > 0$, $j = 1, 2$.

(b) *If $Log(z)$ denotes the principal determination of the logarithm, i.e. $Arg(z) \in (-\pi, \pi)$, for which z do we have $Log(z^2) = 2Log(z)$?*

(c) *Let $f(z) = (1 + z)/(1 - z)$. Prove that $f(z)$ is holomorphic in the unit disk $\mathbb{D}$ and that there exists a determination of $\log(f(z))$ for $z \in \mathbb{D}$.*

(d) *Show that for all $z = x + iy$ we have $|\sin z|^2 = \sin^2(x) + \sinh^2(y)$.*

(e) *Show that $\cos(z_1) = \cos(z_2)$ if and only if $z_1 \pm z_2 \in 2\pi\mathbb{Z}$.*