MAT 342: Homework 1

INSTRUCTIONS: Please read carefully the following instructions for this assignment.

1. You have time to submit this assignment until July 15, you need to do by uploading it in **Gradescope**. Unless you can do it by writing it down in a tablet, use an app to scan it (like CamScanner, not just pictures) and send it in .pdf format.
2. This an assignment, so you can discuss these problems, but in case you do that, please make sure that the solutions that you send me are in your own words. This is our chance to correct any mathematical mistake that you do when writing.
3. You will get a grade from 0 to 5 depending on how many problems you attempt.

Problem 1.

(a) Find the real and imaginary parts of each of the following:

(i) $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

(ii) $\frac{3+5i}{7i+1}$.

(iii) $\frac{z-a}{z+a}$, *for any* $a \in \mathbb{R}$.

(iv) i^n, *for any* $n \in \mathbb{Z}$.

(b) Write in rectangular form (that is $z = x + iy$*):*

(i) $34e^{i\frac{3\pi}{4}}$.

(ii) $-e^{i250\pi}$.

(iii) $\sqrt{2}e^{i\frac{3\pi}{4}}$.

Problem 2. *Define*

$$f(z) = \frac{x^2 y}{x^4 + y^2} \quad \text{where} \quad z = x + iy \neq 0.$$

(i) Parametrize all straight lines in $\mathbb{C} = \mathbb{R}^2$ *that go through the origin.*

(ii) Show that the limits of f *at* 0 *along all the previous lines through the origin exist and are equal.*

(iii) Show that the limit of f *at* 0 *along the parabola* $y = x^2$ *is different from the value found in (ii).*

(iv) Can we extend this function as a continuous function at $z = 0$*?*

Problem 3. *Remember the notation:*

$$\frac{\partial f}{\partial z} = \partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{z}} = \bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Remember also the following useful fact:

If f is differentiable at $z_0 \in \mathbb{C}$, then the set

$$\left\{ w \in \mathbb{C} \colon \exists (z_n)_{n\geq 1} \to z_0 \text{ with } \lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = w \right\}$$

is a circle of center $\partial f(z_0)$ and radius $|\bar{\partial} f(z_0)|$.

Use it to give an explanation for the following fact:

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is $\mathbb{R}^2-$differentiable is holomorphic if and only if $\bar{\partial} f \equiv 0$ (that means that the function $\bar{\partial} f$ is identically 0).

We want to prove now that the $\mathbb{R}^2$-differentiable function $f(z) = \bar{z}^3$ is not $\mathbb{C}$-differentiable in three different (but equivalent) ways.

(a) *Find two sequences $z_n \to 1$ and $w_n \to 1$ so that*

$$\lim_{n\to\infty} \frac{f(z_n) - f(1)}{z_n - 1} \neq \lim_{n\to\infty} \frac{f(w_n) - f(1)}{w_n - 1},$$

where you also show that both limits exist.

(b) *Using the Cauchy-Riemann equations.*

(c) *Computing the $\bar{\partial}$-derivative and using the previous useful fact to show all the limits that we could have found in (a).*

Problem 4. (a) *Prove the triangle inequality, that is, if $z_1, z_2 \in \mathbb{C}$, then*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

When do we have equality? Represent this set in the complex plane $\mathbb{C}$.

(b) *Prove that $|z_1 - z_2| \geq ||z_1| - |z_2||$ and that*

$$||z_1| - |z_2|| = \max\{|z_1| - |z_2|, |z_2| - |z_1|\}.$$

(c) *Use the previous to show that for $z \in \partial D(0,2) = \{z \in \mathbb{C} \colon |z| = 2\}$, we have:*

$$\left|\frac{1}{z^2 - 1}\right| \leq \frac{1}{3}.$$

(d) *Let* G *be the set of points* $z \in \mathbb{C}$ *such that* $|z|^2 + |z| < 1$.

   (i) *Is this set bounded?*

   (ii) *Is this set open? Is this set closed?*

   (iii) *Represent graphically this set.*

Problem 5. *Show that,*

(a) z *is a real number if and only if* $z = \bar{z}$.

(b) $|z| = 1$ *if and only if* $\bar{z} = 1/z$.

(c) z *is either real or purely imaginary if and only if* $(\bar{z})^2 = z^2$.