Imbedding Discrete Series in $L^2(G/H)$

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This report partly discusses joint work with M. W. Baldoni-Silva and partly carries out a suggestion of D. A. Vogan.

Let G/H be a semisimple symmetric space, with G linear connected semisimple and with H the identity component of the group of fixed points of an involution. The problem is: For a generic discrete series representation of $L^2(G/H)$, with given Langlands parameters (in the classification of irreducible representations of G), find an explicit nonzero intertwining operator $\mathcal{E}$ carrying the Langlands quotient into the realization in $L^2(G/H)$.

HEURISTIC PRINCIPLE. A natural intertwining operator $\mathcal{T}$ from $\operatorname{ind}_{H_1}^G \pi_1$ to $\operatorname{ind}_{H_2}^G \pi_2$ is given by some interpretation of

$$\mathcal{T} f(x) = \int_{H_2/(H_1 \cap H_2)} \pi_2(h) f(xh)\, dh,$$

provided that π_1 and π_2 act on the same Hilbert space and are equal on $H_1 \cap H_2$ except for the natural combination of change-of-measure factors. This formula is to be valid, with a suitable interpretation, even when $H_2/(H_1 \cap H_2)$ has no invariant measure.

This formula, together with an analytic continuation, accounts for the standard Kunze-Stein operators.

It accounts also for a simplification in the search for $\mathcal{E}$ in the present situation. Namely the Langlands realization is as a quotient of some $\pi = \operatorname{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$ for a parabolic subgroup MAN. Thus $H_1 = MAN$, $\pi_1 = \sigma \otimes e^\nu \otimes 1$, $H_2 = H$, and $\pi_2 = 1$. For the compatibility property to apply directly, one needs σ and 1 to act on the same space. This situation is quite special. When it occurs, we can interpret $H/(H \cap MAN)$ as $H \cap K$, where K is maximal compact in G, and $\mathcal{T}$ becomes $\mathcal{T} f(x) =$

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$\int_{H\cap K} f(xk)\,dk$. This T we can take as the desired $\mathcal{E}$. In general, with σ and 1 acting on different spaces, $\mathcal{E}$ should involve as an additional ingredient a passage from σ to 1. The Reduction Theorem below will make precise the fact that the passage from σ to 1 is the only obstacle to finding $\mathcal{E}$.

If the desired $\mathcal{E}$ can be found, then the linear functional e on the analytic vectors of $\pi = \operatorname{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ given by $e = (\text{evaluation at } 1) \circ \mathcal{E}$ satisfies

(1) e is H-invariant ($e\pi(h) = e$ for $h \in H$) and continuous
(2) $\{e(\pi(x)f) \mid x \in G\}$ is bounded in $\mathbb{C}$ for each analytic vector f
(3) $e(\mathcal{E}^{-1}(\psi_\lambda)) \neq 0$, where ψ_λ is Flensted-Jensen's generating element of the given discrete series of $L^2(G/H)$.

Conversely any linear functional e on the analytic vectors of the correct π satisfying (1), (2), and

(3′) $e \neq 0$

determines a nonzero intertwining operator into $L^2(G/H)$.

For a generic Flensted-Jensen parameter λ, Schlichtkrull found the Langlands parameters of the discrete series representation of $L^2(G/H)$ with Flensted-Jensen parameter λ. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of the Lie algebra of G according to the involution, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a compatible Cartan decomposition. Flensted-Jensen's parameter is a linear functional on a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k} \cap \mathfrak{q}$ (and $\mathfrak{t}$ is assumed maximal abelian in $\mathfrak{q}$). Let L be the centralizer of $\mathfrak{t}$ in G. Let A be the abelian factor of an Iwasawa decomposition of L, and let MAN be a corresponding parabolic subgroup with N chosen suitably. Also let K be the compact subgroup corresponding to $\mathfrak{k}$, and let μ_λ be the highest weight of the K type constructed from λ by Flensted-Jensen.

According to Schlichtkrull, the Langlands parameters are $\nu = \rho_L$ (half the sum of the positive restricted roots of L, with multiplicities) and the discrete series σ of M constructed by the Vogan algorithm from the K type μ_λ. The quotient $M/(H \cap M)$ is a semisimple symmetric space, and the theorem below reduces the construction of the linear functional e for G to the construction of a corresponding linear functional l for M.

REDUCTION THEOREM. Let the discrete series σ act on V^σ. Suppose l is a linear functional on the analytic vectors of V^σ such that

(1_M) l is $(H \cap M)$-invariant ($l\sigma(h) = l$ for $h \in H \cap M$) and continuous

(2_M) $\{l(\sigma(m)v) \mid m \in M\}$ is bounded in $\mathbb{C}$ for each analytic vector v

(3_M) $l(v_{H\cap M\cap K}) \neq 0$ if $v_{H\cap M\cap K}$ is a nonzero $H \cap M \cap K$ fixed vector in the minimal $M \cap K$ type of σ.

Define e on the analytic vectors f of $\mathrm{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ by

$$e(f) = \int_{H\cap K} l(f(k))\, dk.$$

Then e satisfies (1), (2), (3) and defines the required nonzero intertwining operator into $L^2(G/H)$.

One expects l to be defined by integration:

$$l(v) = \int_{H\cap M} \langle \sigma(h)v,\, v_{H\cap M\cap K} \rangle\, dh. \qquad (*)$$

The following partial results support this expectation:

(a) The integral (*) always converges for σ generic and v analytic, and (1_M) holds

(b) For l as in (*), (2_M) and (3_M) hold

when M is compact,

when $M/(H \cap M)$ is a group case, and

when $M/(H \cap M) = SO(2,1)/SO(1,1)$.

(c) For l as in (*), (3_M) holds if σ is a holomorphic discrete series representation of M.