POSITIVE (p,p) FORMS, WIRTINGER'S INEQUALITY, AND CURRENTS

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The purpose of this paper is to discuss positivity of (p,p) forms, to generalize Wirtinger's Inequality [2], and to indicate a relationship among the generalization, the results of [7], [4], and [5], and the Hodge Conjecture. In the first section three notions of positivity (weak, regular, and strong) for (p,p) forms over a complex vector space are examined, and a canonical form for positive (p,p) forms is described. The canonical form implies one of the two generalizations of Wirtinger's Inequality that we obtain, and it allows us to answer negatively two questions of Lelong [10] concerning weak positivity. In the second section a conjecture (I) concerning positive Plateau problems on a complex projective manifold is stated. This conjecture is equivalent to another conjecture (II) proposing a sufficient condition for a cohomology class to be determined by a rational positive analytic cycle. The conjecture (II) easily implies the Hodge Conjecture (III).

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1. POSITIVE (p,p) FORMS

Suppose E is a complex vector space of complex dimension n with i

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the operation of multiplication by $\sqrt{-1}$. See Weil [13] for a more complete discussion of the complex exterior algebra described below. Let $\Lambda^r_{\mathbb{R}}E$ denote the space of exterior r-vectors over E considered as a real vector space of real dimension 2n, and let $\Lambda^r_{\mathbb{C}}E$ denote the complexified space of exterior r-vectors over E or, equivalently, the space of exterior r-vectors over $E_{\mathbb{C}}$, the complexification of E. The notion of conjugation on $E_{\mathbb{C}}$ extends to $\Lambda^r_{\mathbb{C}}E$ and will be denoted by bar. The operation i extends to $\Lambda^1_{\mathbb{C}}E$ with eigenvalues $\pm i$. Let $\Lambda^{1,0}E$ denote the eigenspace corresponding to i and $\Lambda^{0,1}E$ the eigenspace corresponding to $-i$. Then $\Lambda^1_{\mathbb{C}}E = \Lambda^{1,0}E \oplus \Lambda^{0,1}E$. This decomposition induces a natural direct sum decomposition $\Lambda^r E = \Sigma\, \Lambda^{p,q}E$, where $\Lambda^{p,q}$ denotes the space of r-vectors of type (p,q) and the summation extends over all (p,q) where $p + q = r$. The map of E into $E_{\mathbb{C}}$ followed by projection on $\Lambda^{1,0}E$ is a natural complex isomorphism.

Let E' denote the space of real-valued linear functionals on E considered as a real vector space, and let E* denote the space of complex linear functionals on the complex space E. The operation on E' dual to the operation i on E will also be denoted i. Note that $E^* = \Lambda^{1,0}E'$. It is customary to identify E' with $\Lambda^{1,0}E'$ by taking the projection on $\Lambda^{1,0}E'$ of *twice* the natural injection of E' into its complexification $E'_{\mathbb{C}}$, since this identification agrees with the usual isomorphism $E' \cong E^*$ defined by sending $F(x)$ into $F(x) - iF(ix)$. The natural isomorphism $\Lambda^{p,p}_{\mathbb{R}}E' = (\Lambda^{p,p}_{\mathbb{R}}E)'$ will be used below.

Let I be a multi-index, say $I = (i_1,\ldots,i_p)$. If $\{e_1,\ldots\}$ is a set in $E^* = \Lambda^{1,0}E'$, let $e^I = e_{i_1} \wedge \ldots \wedge e_{i_p}$. If $\{e_1,\ldots,e_n\}$ is a basis for $E^* = \Lambda^{1,0}E'$, then

$$\{e^I \wedge \bar{e}^J : \text{I and J strictly increasing, } |I| = p,\ |J| = q\}$$

is a basis for $\Lambda^{p,q}E'$. Therefore each A in $\Lambda^{p,q}E'$ may be uniquely expressed as $\Sigma'\, a_{IJ}e^I \wedge \bar{e}^J$ with the coefficients in $\mathbb{C}$, with Σ' denoting summation over strictly increasing multi-indices, and with $|I| = p$ and $|J| = q$. In such a basis, $\bar{A} = \Sigma'\, \bar{a}_{IJ}\bar{e}^I \wedge e^J$.

Let $\sigma_k = 2^{-k}$ if k is even and $\sigma_k = i2^{-k}$ if k is odd. For For $e_1,\ldots,e_k$ in $E^* = \Lambda^{1,0}E'$ and $I = (1,\ldots,k)$, we have

$$\frac{i}{2}\, e_1 \wedge \bar{e}_1 \wedge \frac{i}{2}\, e_2 \wedge \bar{e}_2 \wedge \ldots \wedge \frac{i}{2}\, e_k \wedge \bar{e}_k = \sigma_k e^I \wedge \bar{e}^I .$$

If ξ and η are in $\Lambda^{k,0}E$, then the conjugate of $\sigma_k \xi \wedge \bar{\eta}$ is

$\sigma_k \eta \wedge \bar{\xi}$.

A form A in $\Lambda^r_{\mathbb{C}} E'$ is said to be *real* if $A = \bar{A}$ or equivalently if A is in $\Lambda^r_{\mathbb{R}} E'$. Let Herm E be the real vector space of complex-valued functions $H(u,v)$ on $E \times E$ that are i-linear in u and satisfy $\overline{H(u,v)} = H(v,u)$. Passage from H to minus its imaginary part gives us a real-linear isomorphism

$$\text{Herm } E \to \Lambda^{1,1}_{\mathbb{R}} E', \tag{1.1}$$

expressed in terms of a basis $\{e_1,\ldots,e_n\}$ for $E^* = \Lambda^{1,0}E'$ as $\Sigma\, a_{ij} e_i \otimes \bar{e}_j \to \frac{i}{2} \Sigma\, a_{ij} e_i \wedge \bar{e}_j$ with the matrix (a_{ij}) Hermitian. In addition, we have a real-linear isomorphism

$$\Lambda^{p,p}_{\mathbb{R}} E' \to \text{Herm}(\Lambda^{p,0}E) \tag{1.2}$$

given by sending A into the Hermitian form H with $H(\xi,\eta) = A(\sigma_p^{-1} \xi \wedge \bar{\eta})$. This map is given in terms of a basis $\{e_1,\ldots,e_n\}$ for $E^* = \Lambda^{1,0}E'$ and its dual basis $\{g_1,\ldots,g_n\}$ for $\Lambda^{1,0}E$ by sending an element $A = \Sigma'\, a_{IJ} \sigma_p e^I \wedge \bar{e}^J$ of $\Lambda^{p,p}E'$ into $H \in \text{Herm}(\Lambda^{p,0}E)$ defined by $H(\Sigma'\, \zeta_I g^I, \Sigma'\, \eta_J g^J) = \Sigma'\, a_{IJ} \zeta_I \bar{\eta}_J$. Combining (1.2) and (1.1), we obtain a real-linear isomorphism

$$\Lambda^{p,p}_{\mathbb{R}} E' \to \Lambda^{1,1}_{\mathbb{R}} (\Lambda^{p,0}E)'. \tag{1.3}$$

An exterior $(p,0)$ form that can be expressed as e^I with $e_{i_1},\ldots,e_{i_p}$ in $E^* = \Lambda^{1,0}E'$ is called *decomposable*. An exterior (p,p) form will be called *elementary* if it can be expressed as $c\zeta \wedge \bar{\zeta}$ with c in $\mathbb{C}$ and ζ in $\Lambda^{p,0}E'$, and it will be called *decomposable* if in addition ζ can be chosen decomposable.

A form $A \in \Lambda^{2n}E' = \Lambda^{n,n}E'$ is said to be a *positive volume form* if $A = c\, ie_1 \wedge \bar{e}_1 \wedge \ldots \wedge ie_n \wedge \bar{e}_n$, with $c \geq 0$. This notion of positive volume form is of course independent of the choice of basis $\{e_1,\ldots,e_n\}$.

Definition 1.1: A form $A \in \Lambda^{p,p}E'$ is *weakly positive* if

$A \wedge \sigma_k e^I \wedge \bar{e}^I$ is a positive volume form for all
$e_1,\ldots,e_k \in \Lambda^{1,0}E' = E^*$ with $I = (1,\ldots,k)$ and (1.4)
$p + k = n$.

A form $A \in \Lambda^{p,p}E'$ is *positive* if

$$A \wedge \sigma_k \zeta \wedge \overline{\zeta} \text{ is a positive volume form for all } \zeta \in \Lambda^{k,0}E' \text{ with } p + k = n. \tag{1.5}$$

A form $A \in \Lambda^{p,p}E'$ is *strongly positive* if

$$A \text{ can be expressed as } \Sigma\, \sigma_p \zeta_j \wedge \overline{\zeta}_j \text{ with each } \zeta_j \in \Lambda^{p,0}E' \text{ decomposable.} \tag{1.6}$$

Let WP^p, P^p and SP^p denote the cones in $\Lambda^{p,p}_{\mathbb{R}}E'$ defined by (1.4), (1.5), and (1.6) respectively. If ζ is any element of $\Lambda^{p,0}E'$, then $\sigma_p \zeta \wedge \overline{\zeta}$ is in P^p, since $\sigma_p \zeta \wedge \overline{\zeta} \wedge \sigma_k \eta \wedge \overline{\eta}$ can be checked to be equal to the positive volume form $\sigma_n \zeta \wedge \eta \wedge \overline{\zeta} \wedge \overline{\eta}$ for an arbitrary η in $\Lambda^{k,0}E'$. Therefore $SP^p \subset P^p \subset WP^p$. If $p = 1$ or $n - 1$, then $P^p = WP^p$ since every form in $\Lambda^{1,0}E'$ or $\Lambda^{n-1,0}E'$ is decomposable. An immediate consequence of Theorem 1.2 below (for $p = 1$ and $p = n-1$) is that $SP^p = P^p = WP^p$ for $p = 1$ or $n - 1$.

The condition (1.4) may be equivalently stated as follows: If $L: E_1 \to E_2$ is a complex linear map let $L^*: \Lambda^{p,q}E_2' \to \Lambda^{p,q}E_1'$ denote the usual pull-back of (p,q) forms from E_2 to E_1.

$$L^*A \text{ is a positive volume form for all complex linear maps } L: F \to E \text{ with } \dim_{\mathbb{C}} F = p. \tag{1.4'}$$

To check that (1.4) and (1.4)' are equivalent consider the case where F is a subspace of E and L is inclusion. Let $\{e_1,\dots,e_n\}$ be a basis for $\Lambda^{1,0}E' = E^*$ with $\{e_1,\dots,e_k\}$ vanishing on F. Then $L^*A \wedge \sigma_k e^I \wedge \overline{e}^I = A \wedge \sigma_k e^I \wedge \overline{e}^I$ where $I = (1,\dots,k)$; and L^*A is a positive volume form if and only if $L^*A \wedge \sigma_k e^I \wedge \overline{e}^I$ is a positive volume form.

A weakly positive (p,p) form A is said to be *non-degenerate* or *definite* if $A \wedge \sigma_k e^I \wedge \overline{e}^I$ in (1.4) is greater than zero for all $\{e_1,\dots,e_k\}$ linearly independent. Similarly a positive (p,p) form A is said to be *non-degenerate* or *definite* if $A \wedge \sigma_k \zeta \wedge \overline{\zeta}$ in (1.5) is greater than zero for all $\zeta \in \Lambda^{k,0}E'$, $\zeta \neq 0$. Note that the set of (weakly) positive (p,p) forms that are non-degenerate is the interior of (WP^p) P^p.

Suppose E is furnished with a Hermitian inner product. In terms of the extension of this inner product to $\Lambda^r_{\mathbb{C}}E'$, we obtain a canonical form for real (p,p) forms and for positive (p,p) forms. This theorem is well known for $p = 1$ but we include the proof for the sake of completeness. One can easily show that for ζ in $\Lambda^{p,0}E'$, $|\zeta|^2 = 2^p$ if and only if $|\sigma_p \zeta \wedge \bar{\zeta}| = 1$.

Theorem 1.2: *Let* $N = \binom{n}{p}$. *If* A *is in* $\Lambda^{p,p}_{\mathbb{R}}E'$, *there exist real numbers* $\{\lambda_1,\ldots,\lambda_N\}$ *and a set* $\{\zeta_1,\ldots,\zeta_N\}$ *of mutually orthogonal vectors with* $|\zeta_j|^2 = 2^p$ *in* $\Lambda^{p,0}E'$ *such that* $A = \sum_{j=1}^{N} \lambda_j \sigma_p \zeta_j \wedge \bar{\zeta}_j$. *The numbers* λ_j *and their multiplicities are unique, and the subspaces spanned by* $\{\zeta_j : \lambda_j = \lambda\}$ *are unique. In addition,* A *is positive if and only if each* λ_j *is* ≥ 0.

Proof (for $p = 1$): Choose a basis $\{e_1,\ldots,e_n\}$ of $E^* = \Lambda^{1,0}E'$ with $\langle e_i, e_j\rangle = 0$ for $i \neq j$ and $|e_i|^2 = 2$. Then $|\frac{i}{2} e_i \wedge \bar{e}_j| = 1$. Write $A = \frac{i}{2} \Sigma a_{ij} e_i \wedge \bar{e}_j$ with (a_{ij}) Hermitian. Diagonalize (a_{ij}) by a unitary matrix and change the e's accordingly. The existence of the decomposition follows. Any different choice of orthogonal basis $\{e_i\}$ with $|e_i|^2 = 2$ (in particular, a second choice for the ζ_j) leads to a matrix (b_{ij}) that is conjugate to (a_{ij}) by a unitary matrix. Consequently the uniqueness follows from standard linear algebra. From (1.4') A is weakly positive if and only if (a_{ij}) is positive semidefinite and hence if and only if all λ_j are ≥ 0. Q.E.D.

Remark: Since $\frac{i}{2} \zeta \wedge \bar{\zeta}$ with $\zeta \in \Lambda^{1,0}E'$ is positive, this argument proves that weakly positive implies positive for $p = 1$ (and for $p = n - 1$ by duality).

Proof (for general p): Let $L: \Lambda^{p,p}_{\mathbb{R}}E' \to \Lambda^{1,1}_{\mathbb{R}}(\Lambda^{p,0}E')$ be the isomorphism of (1.3). If A is in $\Lambda^{p,p}_{\mathbb{R}}E'$, A can be expanded in terms of a basis as $\sigma_p \Sigma' a_{IJ} e^I \wedge \bar{e}^J$ with (a_{IJ}) Hermitian. Then $L(A)$ is positive (weakly positive = positive for $L(A)$) if and only if (a_{IJ}) is positive semidefinite, and (a_{IJ}) is positive semidefinite if and only if A is positive. Hence the result for general p follows from the case $p = 1$. Q.E.D.

Remark: The notion of positivity does not depend upon the Hermitian inner product on E, and hence the theorem shows that all the λ_j are ≥ 0

for one inner product if and only if they are ≥ 0 for any other inner product. More generally the numbers of λ_j that are respectively > 0, < 0, and $= 0$ are independent of the inner product.

The Hermitian inner product on E induces an operator on $\Lambda_{\mathbb{C}} E'$ usually denoted $*$. If K is a cone in $\Lambda_{\mathbb{R}}^{p,p} E'$, the dual cone K^0 is by definition the set of all $B \in \Lambda_{\mathbb{R}}^{k,k} E'$ $(p + k = n)$ such that $A \wedge B$ is a positive volume form for all $A \in K$.

Corollary 1.3:

(a) *If* A *and* B *are positive then* $A \wedge B$ *is positive.*

(b) *If* A *is positive then* $*A$ *is positive.*

(c) *If* A *is positive and* $L: F \to E$ *then* L^*A *is positive.*

(d) *The dual cone* $(P^p)^0$ *equals* P^k *with* $p + k = n$.

Proof: (a) By Theorem 1.2 it suffices to check the result for $A = \sigma_p \zeta \wedge \bar{\zeta}$ and $B = \sigma_q \eta \wedge \bar{\eta}$. Then $A \wedge B = \sigma_p \zeta \wedge \bar{\zeta} \wedge \sigma_q \eta \wedge \bar{\eta} = \sigma_{p+q} \zeta \wedge \eta \wedge \bar{\zeta} \wedge \bar{\eta}$, which is a positive volume form. Similarly (b) and (c) need only be checked for $A = \sigma_p \zeta \wedge \bar{\zeta}$ with $\zeta \in \Lambda^{p,0} E'$ because of Theorem 1.2; hence (b) and (c) follow. Part (d) is an immediate consequence of Theorem 1.2. Q.E.D.

Corollary 1.4: *The set of extreme rays in the cone of positive* (p,p) *forms is the set of rays determined by positive elementary* (p,p) *forms* (i.e., $\{\sigma_p \zeta \wedge \bar{\zeta} : \zeta \in \Lambda^{p,0} E'\}$).

Proof: First consider the case $p = 1$. Suppose $A = \sum_{j=1}^{n} \lambda_j \frac{i}{2} e_j \wedge \bar{e}_j$ determines an extreme ray in $P^1 \subset \Lambda_{\mathbb{R}}^{1,1} E'$. Then obviously all λ_j must vanish but one. Hence each extreme ray is determined by a decomposable vector $\frac{i}{2} e \wedge \bar{e}$ for some e in $E^* = \Lambda^{1,0} E'$. Conversely, each decomposable vector $\frac{i}{2} e \wedge \bar{e}$ determines an extreme ray in P^1. Suppose $\frac{i}{2} e \wedge \bar{e}$ is expressed as the sum of two positive elements $\alpha, \beta \in P^1$. Then α and β must be dependent since otherwise the rank of $\frac{i}{2} e \wedge \bar{e} = \alpha + \beta$ would be at least 2. This proves Corollary 1.4 for $p = 1$. The case $p \geq 1$ easily follows from the case $p = 1$ by using the isomorphism (1.3). Q.E.D.

In fact the cone P of positive (p,p) forms is nicely stratified with strata S_j equal to the set of A in P with exactly j positive eigenvalues. The lowest stratum S_1, which is the set of extreme rays, is $G(k,n,\mathbb{C}) \times \mathbb{R}^+$, where $G(k,n,\mathbb{C})$ denotes the Grassmannian of complex k-planes in complex n-space.

Our notion of weak positivity coincides with the notion of positivity occurring in Lelong [10]. Lelong ([10], p. 60) asked two questions, which translate into (A) and (B) below:

(A) *Is every weakly positive* (p,p) *form strongly positive?*

(B) *If* A *is in* $\Lambda^{p,p}_{\mathbb{R}}E'$ *and* B *is in* $\Lambda^{q,q}_{\mathbb{R}}E'$ *and both are weakly positive, is* $A \wedge B$ *weakly positive?*

Obviously (A) implies (B). In addition (B) for $q = k = n - p$ implies (A) as follows. Since SP^p is the cone on the convex hull of the compact set $G(k,n,\mathbb{C})$ in $\Lambda^{p,p}_{\mathbb{R}}E'$, it is a closed cone. By definition $(SP^p)^0 = WP^k$ and hence $SP^p = (WP^k)^0$. That is,

$$\text{the cones } SP^p \text{ and } WP^k \text{ are dual.} \tag{1.7}$$

If (B) is true for $q = k = n - p$ then $WP^p \subset (WP^k)^0$. Since $(WP^k)^0 = SP^p$ this proves that (B) implies (A). Now there is a variety of ways of showing (A) and (B) are false for $2 \le p \le n - 2$.

For example if (A) were true then certainly the notions of positive and strongly positive would have to agree. However the next result shows that if $\zeta \in \Lambda^{p,0}E'$ is not decomposable, then although $\sigma_p \zeta \wedge \bar{\zeta}$ is positive it is not strongly positive. (If $2 \le p \le n - 2$, then there exists $\zeta \in \Lambda^{p,0}E'$ not decomposable.)

Proposition 1.5: *An elementary vector* $A = \sigma_p \zeta \wedge \bar{\zeta}$ *with* $\zeta \in \Lambda^{p,0}E'$ *is strongly positive if and only if* A *is decomposable (i.e.,* ζ *is a decomposable* $(p,0)$ *form).*

Proof: If A is strongly positive, then by Theorem 1.2 $A = \Sigma \sigma_p \zeta_j \wedge \bar{\zeta}_j$ with each ζ_j a decomposable $(p,0)$ form. Since $A = \sigma_p \zeta \wedge \bar{\zeta}$ determines an extreme ray in $P^p \subset \Lambda^{p,p}_{\mathbb{R}}E$ by Corollary 1.4, it must be a positive multiple of $\sigma_p \zeta_j \wedge \bar{\zeta}_j$ for some j. This implies ζ is a multiple of ζ_j and hence decomposable. Q.E.D.

In particular, if $\{e_1,e_2,e_3,e_4\}$ is a basis for $\mathbb{C}^4$ then $A = (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (\bar{e}_1 \wedge \bar{e}_2 + \bar{e}_3 \wedge \bar{e}_4)$ is a positive $(2,2)$ form on $\mathbb{C}^4$ which cannot be expressed as the sum of positive decomposable $(2,2)$

forms, since $e_1 \wedge e_2 + e_3 \wedge e_4$ is not a decomposable (2,0) form.

Corollary 1.6: *For* $2 \le p \le n-2$ *the inclusions* $SP^p \subset P^p \subset WP^p$ *are proper.*

Proof: $SP^p \ne P^p$ from above, and $P^p \ne WP^p$ by duality since $P^k \ne SP^k$ with $p+k=n$. Q.E.D.

To explicitly construct a weakly positive (p,p) form which is not positive, suppose $\zeta_1^* \in \Lambda^{k,0}E'$ is not decomposable and $|\sigma_k \zeta_1^* \wedge \overline{\zeta}_1^*| = 1$. Extend ζ_1^* to an orthogonal basis $\{\zeta_1^*, \ldots, \zeta_N^*\}$ for $\Lambda^{k,0}E'$. Let $\{\zeta_1, \ldots, \zeta_N\}$ denote the dual basis for $\Lambda^{p,0}E'$, and set $A_1 = \sum_{j=2}^{N} \lambda_j \sigma_p \zeta_j \wedge \overline{\zeta}_j$ with each $\lambda_j > 0$. Let $A = A_1 - \varepsilon\sigma_p \zeta_1 \wedge \overline{\zeta}_1$ with $\varepsilon > 0$ to be chosen. $A_1 \wedge \sigma_k \eta \wedge \overline{\eta}$ is ≥ 0 for all $\eta \in \Lambda^{k,0}E'$ and equal to zero if and only if $\eta = c\zeta_1^*$. Since ζ_1^* is not decomposable, $A_1 \wedge \sigma_k \eta \wedge \overline{\eta}$ is > 0 for each $\eta \in \Lambda^{k,0}E'$ with η decomposable and nonzero. Choose $\varepsilon > 0$ strictly less than the minimum of $A_1 \wedge \sigma_k \eta \wedge \overline{\eta}$ over all decomposable η with $|\sigma_k \eta \wedge \overline{\eta}| = 1$. Then A is weakly positive (in fact non-degenerate) since

$$A_1 \wedge \sigma_k \eta \wedge \overline{\eta} - \varepsilon\sigma_n (\zeta_1 \wedge \eta) \wedge (\overline{\zeta}_1 \wedge \overline{\eta}) > 0$$

for all decomposable nonzero η. Since $A \wedge \sigma_k \zeta_1^* \wedge \overline{\zeta}_1^*$ equals $-\varepsilon$ times a positive volume form, A cannot be strongly positive. The above is just a matter of finding a hyperplane (linear functional) through zero in $\Lambda_{\mathbb{R}}^{k,k}E'$ separating $\sigma_k \zeta_1^* \wedge \overline{\zeta}_1^*$ and SP^k. Also note that although A is in WP^p and $B = \sigma_k \zeta_1^* \wedge \overline{\zeta}_1^*$ is in SP^k (with $k+p=n$), the product $A \wedge B$ is a negative volume form (cf. (B) and Corollary 1.3 (a); of course $A, B \in SP$ imply $A \wedge B \in SP$).

If $A = \sigma_p \zeta \wedge \overline{\zeta}$ with ζ decomposable, say $\zeta = e_1 \wedge \ldots \wedge e_p$ with $\{e_1, \ldots, e_n\}$ an orthogonal basis for $E^* = \Lambda^{1,0}E'$ and $|e_j|^2 = 2$, then $*A = \sigma_k \eta \wedge \overline{\eta}$ where $\eta = e_{p+1} \wedge \ldots \wedge e_n$. Therefore:

$*$ is an isomorphism of SP^p onto SP^k and hence by duality an isomorphism of WP^p onto WP^k (where $p+k=n$). (1.8)

Also by (1.4') and an elementary calculation

$$\text{part (c) of Corollary 1.3 remains valid with } P \text{ replaced by } WP \text{ or } SP. \tag{1.9}$$

The statements (1.7), (1.8) and (1.9) above provide the analogue of Corollary 1.3 for WP and SP.

Next we generalize Wirtinger's Inequality. We continue to assume E has a Hermitian inner product, and we consider the induced $*$ operator mapping $\Lambda^{p,q}E'$ isometrically onto $\Lambda^{n-p,n-q}E'$. If A and B are in $\Lambda^r_{\mathbb{C}}E'$, then $(A,B) = *(A \wedge \overline{*B})$ defines the Hermitian inner product on $\Lambda^r_{\mathbb{C}}E'$. Let $|\cdot|_2$ denote the norm associated with this inner product. Let $\omega \in \Lambda^{1,1}_{\mathbb{R}}E'$ denote the image of the given element of Herm(E) under the map defined by (1.1). This positive (1,1) form ω is called the standard Kähler form on the Hermitian space E. In terms of an orthogonal basis $\{e_1,\ldots,e_n\}$ for E^* with $|e_j|_2^2 = 2$, one has $\omega = \frac{i}{2}\sum_{j=1}^{n} e_j \wedge \bar{e}_j$. Let ω^k denote $\omega\wedge\ldots\wedge\omega$ taken k times.

The classical Wirtinger Inequality says that for a decomposable form $A \in \Lambda^{2p}_{\mathbb{R}}E'$ and $p + k = n$

$$*(A \wedge \tfrac{1}{k!}\,\omega^k) \le |A|_2 \text{ with equality if and only if } A \text{ is a positive decomposable } (p,p) \text{ form.} \tag{1.10}$$

See Federer [2], p. 40, for a nice proof. Before generalizing (1.10) to forms A that are not necessarily decomposable, we must discuss some other norms on $\Lambda_{\mathbb{C}}E'$ that agree with $|\cdot|_2$ on decomposable 2p-forms.

Definition 1.7: (1) Let $|\cdot|_1$ (called the *mass norm*) denote the norm on $\Lambda^{2p}_{\mathbb{R}}E'$ whose closed unit ball is the convex hull of $\mathcal{D} = \{A \in \Lambda^{2p}_{\mathbb{R}}E' \colon A \text{ is decomposable and } |A|_2 = 1\}$.

(2) Let $\|\cdot\|_1$ denote the norm on $\Lambda^{2p}_{\mathbb{R}}E'$ whose closed unit ball is the convex hull of $\mathcal{D} \cup \mathcal{E}$ where

$\mathcal{E} = \{A \in \Lambda^{p,p}_{\mathbb{R}}E' \colon A = \pm\sigma_p \zeta \wedge \bar{\zeta}$ with $\zeta \in \Lambda^{p,0}E'$ and $|A|_2 = 1\}$.

Note that $\mathcal{D} \subset \mathcal{D} \cup \mathcal{E} \subset \{A\colon |A|_2 = 1\}$ implies that $|A|_2 \le \|A\|_1 \le |A|_1$ for all $A \in \Lambda^{2p}_{\mathbb{R}}E'$.

Theorem 1.8 (Generalized Wirtinger Inequality): *Suppose* A *is in* $\Lambda^{2p}_{\mathbb{R}}E'$

(and $p + k = n$). *Then*

(a) $*(A \wedge \frac{1}{k!}\omega^k) \le |A|_1$ *with equality if and only if* $A \in SP^p$.

(b) $*(A \wedge \frac{1}{k!}\omega^k) \le \|A\|_1$ *with equality if and only if* $A \in P^p$.

Of course the inequality in (b) is stronger than the inequality in (a) since $\|A\|_1 \le |A|_1$. Before proving Theorem 1.8 we give some alternate descriptions of the norms $|\cdot|_1$ and $\|\cdot\|_1$.

If A is in $\mathcal{D}$, then A belongs to the convex hull of $\mathcal{D}$ so that $|A|_1 \le 1$. Also $1 = |A|_2 \le |A|_1$ for $A \in \mathcal{D}$. That is,

$$|A|_1 = |A|_2 \quad \text{for} \quad A \in \mathcal{D}. \tag{1.11}$$

(In fact $|A|_1 = |A|_2$ if and only if a multiple of A belongs to $\mathcal{D}$, since every point of $\{A: |A|_2 = 1\}$ is extreme.) Since for arbitrary A, $A/|A|_1$ belongs to the convex hull of $\mathcal{D}$, there exist $0 \le t_j \le 1$ with $\Sigma_1^N t_j \le 1$ and $B_j \in \mathcal{D}$ such that $A/|A|_1 = \Sigma\, t_j B_j$. Let $A_j = t_j |A|_1 B_j$, so that $A = \Sigma\, A_j$. Note that $|A_j|_1 = t_j |A|_1$ since, by (1.11), $|B_j|_1 = 1$. Therefore $\Sigma\, |A_j|_1 \le |A|_1$ and $|A|_1 = \Sigma\, |A_j|_1$. This proves

(1') $|A|_1$ equals the infimum of $\Sigma\, |A_j|_2$ taken over all collections A_j with a multiple of each A_j belonging to $\mathcal{D}$ and $A = \Sigma\, A_j$.

Similarly, one can show that a multiple of A belongs to $\mathcal{D} \cup \mathcal{E}$ if and only if $\|A\|_1 = |A|_2$ and that

(2') $\|A\|_1$ equals the infimum of $\Sigma\, |A_j|_2$ taken over all collections A_j with a multiple of each A_j belonging to $\mathcal{D} \cup \mathcal{E}$ and $A = \Sigma\, A_j$.

Let $|B|_\infty = \sup\{|A \wedge B|_2 : A \in \mathcal{D}\}$. This defines a norm (the *comass norm*) on $\Lambda_{\mathbb{R}}^{2k} E'$ $(p + k = n)$ dual to $|\cdot|_1$.

(1") $|A|_1 = \sup |A \wedge B|_2$ taken over all $B \in \Lambda_{\mathbb{R}}^{2k} E'$ with $|B|_\infty \le 1$.

Similarly $\|B\|_\infty = \sup\{|A \wedge B|_2 : A \in \mathcal{D} \cup \mathcal{E}\}$ defines a norm on $\Lambda_{\mathbb{R}}^{2k} E'$ dual to $\|\cdot\|_1$.

(2") $\|A\|_1 = \sup |A \wedge B|_2$ taken over all $B \in \Lambda_{\mathbb{R}}^{2k} E'$ with $\|B\|_\infty \le 1$.

Proof of the Generalized Wirtinger Inequality:

(a) Choose A_j with $A_j/|A_j|_1 \in \mathcal{D}$ and $A = \Sigma A_j$ so that $|A|_1 = \Sigma |A_j|_2$. By (1.10)

$$*(A \wedge \frac{1}{k!}\omega^k) = \Sigma *(A_j \wedge \frac{1}{k!}\omega^k) \le \Sigma |A_j|_1 = |A|_1.$$

This proves the inequality part of (a). Equality holds if and only if $*(A_j \wedge \frac{1}{k!}\omega^k) = |A_j|_1$ for each j. By the equality part of (1.10) if these equalities hold for all j, then each A_j is a positive decomposable (p,p) form and hence $A = \Sigma A_j$ is in SP^p. Conversely, if $A = \Sigma A_j$ is in SP with each A_j positive and decomposable, then equality in (1.10) for each A_j implies that $*(A \wedge \frac{1}{k!}\omega^k) = \Sigma |A_j|_1 \ge |A|_1$, which proves equality in (a).

(b) Suppose $A = \Sigma' a_{IJ}\sigma_p e^I \wedge \overline{e}^J$ belongs to $\Lambda^{p,p}_{\mathbb{R}}E'$ where $\{e_1,\dots,e_n\}$ is an orthogonal basis for $E^* = \Lambda^{1,0}E'$ with $|e_j|_2^2 = 2$. Since $\frac{1}{k!}\omega^k = \sum'_{|J|=k} \sigma_k e^J \wedge \overline{e}^J$, $*(A \wedge \frac{1}{k!}\omega^k)$ equals $\sum'_{|I|=p} a_{II}$, the trace of (a_{IJ}). Let $\{\lambda_1,\dots,\lambda_N\}$ denote the eigenvalues and $\{\zeta_1,\dots,\zeta_N\}$ the eigenvectors of A given by Theorem 1.2. Then, since the trace of (a_{IJ}) equals $\Sigma \lambda_j$,

$$*(A \wedge \frac{1}{k!}\omega^k) = \sum_{j=1}^N \lambda_j . \tag{1.12}$$

Just as in (1.11), for $A \in \Lambda^{p,p}_{\mathbb{R}}E'$, $\|A\|_1 = |A|_2$ if and only if a multiple of A belongs to $\mathcal{E}$. In particular, for $A_j = \lambda_j\sigma_p\zeta_j \wedge \overline{\zeta}_j$ we have $\|A_j\|_1 = |\lambda_j|$. Therefore if A is positive,

$$*(A \wedge \frac{1}{k!}\omega^k) = \Sigma \lambda_j = \Sigma \|A_j\|_1 \ge \|A\|_1.$$

In particular, if a multiple of A belongs to $\mathcal{E}$, then

$$|*(A \wedge \frac{1}{k!}\omega^k)| = \|A\|_1 \quad \text{and} \quad *(A \wedge \frac{1}{k!}\omega^k) \le \|A\|_1 \tag{1.13}$$

with equality if and only if A is positive.

The proof of the inequality in (b) proceeds exactly as in (a) except the A_j are chosen so that $A_j/\|A_j\|_1$ is in $\mathcal{D} \cup \mathcal{E}$. Because of (1.13) above and the fact that a multiple of A_j belongs to $\mathcal{E}$ or $\mathcal{D}$, $*(A_j \wedge \frac{1}{k!}\omega^k) \le \|A_j\|_1$ with equality if and only if A_j is positive. Therefore

$$*(A \wedge \frac{1}{k!}\omega^k) = \Sigma\, *(A_j \wedge \frac{1}{k!}\omega^k) \le \Sigma\, \|A_j\|_1 = \|A\|_1 ,$$

with equality implying that each A_j is of the form $\sigma_p \zeta \wedge \bar{\zeta}$ for some $\zeta \in \Lambda^{p,0}E'$. Therefore equality implies that A is in P^p. Q.E.D.

As noted above each positive (p,p) form is not necessarily a positive combination of positive decomposable (p,p) forms. We finish this section by proving that each real (p,p) form is the sum of real decomposable (p,p) forms.

Proposition 1.9: *There exists a basis for* $\Lambda^{p,p}_{\mathbb{R}}E'$ *consisting of positive decomposable* (p,p) *forms.*

Proof: Suppose $\{e_1,\ldots,e_n\}$ is a basis for $E^* = \Lambda^{1,0}E'$. The proof is by induction on p. It suffices to show that $\Lambda^{p,p}E'$ has a basis consisting of positive decomposable (p,p) forms. For $p = 1$ it is true by Theorem 1.2 or because $2\,\mathrm{Re}(\frac{i}{2} e_i \wedge \bar{e}_j) =$

$$= \frac{i}{2} e_i \wedge \bar{e}_j + \frac{i}{2} e_j \wedge \bar{e}_i$$

$$= \frac{i}{2}(e_i + e_j) \wedge (\bar{e}_i + \bar{e}_j) - \frac{i}{2} e_i \wedge \bar{e}_i - \frac{i}{2} e_j \wedge \bar{e}_j ,$$

and $2\,\mathrm{Im}(\frac{i}{2} e_i \wedge \bar{e}_j) =$

$$= \frac{1}{2} e_i \wedge \bar{e}_j - \frac{1}{2} e_j \wedge \bar{e}_i$$

$$= \frac{i}{2}(e_i + ie_j) \wedge (\bar{e}_i - i\bar{e}_j) - \frac{i}{2} e_i \wedge \bar{e}_i - \frac{i}{2} e_j \wedge \bar{e}_j .$$

Consider $e^I \wedge \bar{e}^J$. Let $I' = (i_2,\ldots,i_p)$ and $J' = (j_2,\ldots,j_p)$.

Then $$\sigma_p e^I \wedge \bar{e}^J = (\frac{i}{2} e_{i_1} \wedge \bar{e}_{j_1}) \wedge (\sigma_{p-1} e^{I'} \wedge \bar{e}^{J'}) ;$$

and by the induction hypothesis for 1 and $p - 1$, respectively, the two factors on the right can be expressed as linear combinations of positive decomposable forms. Q.E.D.

Corollary 1.10: *The* (p,p) *form* $\frac{1}{p!}\omega^p$ *belongs to the interior of the cone* SP^p *(i.e., is strongly positive non-degenerate).*

Proof: It suffices to show that $*(\frac{1}{p!}\omega^p \wedge B) > 0$ for all non-zero

$B \in WP^k$, by (1.7). Choose a basis $\{A_1,\ldots,A_N\}$ for $\Lambda^{p,p}_{\mathbb{R}}E'$ consisting of positive decomposable (p,p) forms with $|A_j|_2 = 1$. We may choose an orthogonal basis $\{e_1,\ldots,e_n\}$ with $|e_j|_2^2 = 2$ for $E^* = \Lambda^{1,0}E'$ so that $A_1 = \sigma_p e^I \wedge \overline{e}^I$ with $I = (1,\ldots,p)$.

Suppose $\frac{1}{p!}\omega^p \wedge B = 0$ for some $B \in WP^k$. Let $B = \Sigma' b_{IJ} e^I \wedge \overline{e}^J$. As noted before $*(\frac{1}{p!}\omega^p \wedge B) = \Sigma' b_{II}$. If $|I| = k$, $|J| = p$, I and J strictly increasing, and $I \cup J = \{1,\ldots,n\}$, then $b_{II} = *(\sigma_p e^J \wedge \overline{e}^J \wedge B)$; which is ≥ 0 since B is in WP^k. Therefore $*\frac{1}{p!}\omega^p \wedge B = \Sigma' b_{II} = 0$ if and only if each $b_{II} = 0$. In particular, this proves that if $\frac{1}{p!}\omega^p \wedge B = 0$, with $B \in WP^k$, then, for $I = (1,\ldots,p)$, $0 = b_{II} = *(A_1 \wedge B) = (A_1, *B)$. Similarly $(A_j, *B)$ must vanish for $1 \leq j \leq N$ so that $B = 0$. Q.E.D.

2. ANALYTIC CYCLES ON KÄHLER MANIFOLDS

Suppose X is a Kähler manifold of complex dimension n, with Kähler form ω. Suppose $T \in \mathcal{D}'^{2p}(X)$ is a real-valued current on X of degree 2p which is representable by integration (i.e., when expressed as a 2p form with distribution coefficients, the distributions are measures). Let $\|T\|$ denote the positive measure defined by letting $\|T\|(f) = \sup\{|T(\phi)|: |\phi(z)|_\infty \leq f(z)\}$ for each positive continuous function f on X and let $M(T) = \|T\|(1) = \sup\{|T(\phi)|: |\phi|_\infty \leq 1\}$ (the mass norm of T). Then there exists a $\|T\|$ measurable function $\vec{T}$ from X to $\Lambda^{2p}\mathbb{C}^n$ with $T(\phi) = \int \langle\vec{T},\phi\rangle_x d\|T\|$ (see [2]). A current T defined on an open set Ω contained in $\mathbb{C}^n$ and of degree (p,p) (with $p + k = n$) will be called *weakly positive* if for all $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, and all complex linear projections $\pi: \mathbb{C}^n \to \mathbb{C}^k$, the push forward $\pi_*(\psi T)$ is a positive measure. Such a current T will be called *positive* if for all $\zeta \in \Lambda^{k,0}\mathbb{C}^n$, $T \wedge \sigma_k \zeta \wedge \overline{\zeta}$ is a positive measure on Ω. The current T will be called *strongly positive* if for all $B \in \Lambda^{k,k}_{\mathbb{R}}\mathbb{C}^n$ which belong to WP^k, $T \wedge B$ is a positive measure on Ω. A current T of degree (p,p) on X is said to be *weakly positive, positive, or strongly positive* if the restriction of T to each coordinate chart is weakly positive, positive, or strongly positive, respectively. One can show that a current T on X is weakly positive, positive, or strongly positive if and only if T is representable by integration and $\vec{T}(z)$ is weakly positive, positive, or strongly positive, respectively, for almost all z in X (with respect to $\|T\|$).

For the proof that a weakly positive current T on $\mathbb{C}^n$ is representable by integration see Lelong [10] or use Proposition 1.9 to choose a basis $\{B_1,\ldots,B_N\}$ for $\Lambda^{k,k}_{\mathbb{R}}E'$ consisting of positive decomposable (k,k) forms and let $\{A_1,\ldots,A_n\}$ denote the dual basis for $\Lambda^{p,p}_{\mathbb{R}}E'$ where $p+k=n$. Then expressing T in terms of the basis $A_1,\ldots,A_N$, we see that the coefficients $T \wedge B_j$ are positive measures.

The Generalized Wirtinger's Inequality immediately applies to currents, since $T(\frac{1}{k!}\omega^k) = \int *(\vec{T} \wedge \frac{1}{k!}\omega^k)d\|T\|$ and $\int d\|T\| = M(T)$.

Theorem 2.1: *Suppose* T *is a real-valued current of degree* $2p$ *on* X *(with* $p+k=n$*) which is representable by integration. Then*

(a) $T(\frac{1}{k!}\omega^k) \le M(T)$ *and equality holds if and only if* T *is strongly positive.*

(b) $T(\frac{1}{k!}\omega^k) \le M'(T)$ *and equality holds if and only if* T *is positive, where* $M'(T) = \sup\{|T(\phi)| : \|\phi\|_\infty \le 1\}$.

The next three results are well known. First we briefly sketch how locally rectifiable currents can be used to compute cohomology with coefficients in $\mathbb{Z}$ on an oriented manifold. See Federer [2] for similar results concerning homology. Suppose X is a real n-dimensional oriented C^∞ manifold. Let $\mathcal{D}'^p = \mathcal{D}'_k$ $(p+k=n)$ denote the sheaf of germs of currents of degree p or dimension k on X. Let $\mathcal{R}^p = \mathcal{R}_k$ $(p+k=n)$ denote the subsheaf of germs of locally rectifiable currents of degree p or dimension k on X (see [2]). The exterior derivative d operating on smooth forms of degree p extends as a differential operator to currents of degree p, and we have the complex

$$0 \to \mathbb{C} \to \mathcal{D}'^0 \xrightarrow{d} \mathcal{D}'^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}'^n \to 0$$

which is an exact resolution of the sheaf $\mathbb{C}$ by fine sheaves $\mathcal{D}'^p$. This proves that

$$H^p(X,\mathbb{C}) \cong \{T \in \mathcal{D}'^p(X) : dT = 0\}/d\mathcal{D}'^{p-1}(X).$$

A locally rectifiable current is said to be *locally integral* if its exterior derivative is also locally rectifiable. Let I^p denote the sheaf of germs of locally integral currents of degree p. Then

$$0 \to \mathbb{Z} \to \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \dots \to \mathcal{I}^n \to 0 \qquad (2.1)$$

is an exact resolution of the sheaf $\mathbb{Z}$ (if X is not oriented the sequence is an exact resolution of the orientation sheaf). The usual cone construction provides a proof of local exactness of (2.1) (see [2]). Each sheaf $\mathcal{I}^p$ is soft. That is given a locally integral current T on a neighborhood U of a closed subset F of X, there exists a locally integral current S on X with S equal to T in a neighborhood of F. Choose $f \in C^\infty(X)$ with $f \equiv 1$ on a neighborhood of $X - U$ and $f \equiv 0$ on a neighborhood of F, and $0 \le f \le 1$. Let χ_ε denote the characteristic function of the set $\{x \in X: f(x) < \varepsilon\}$. Then it follows easily from Federer's theory of slicing [2] that $\chi_\varepsilon T$ is a locally integral current for almost all ε. Now take $S = \chi_\varepsilon T$. This proves each $\mathcal{I}^p$ is soft. The fact that (2.1) is an exact resolution of the sheaf $\mathbb{Z}$ by soft sheaves $\mathcal{I}^p$ implies the following theorem, on a smooth oriented manifold X.

Theorem 2.2: $H^p(X,\mathbb{Z}) = \{T \in \mathcal{R}^p_{loc}(X): dT = 0\}/d\mathcal{I}^{p-1}_{loc}(X)$.

Note that the natural map of $H^p(X,\mathbb{Z})$ into $H^p(X,\mathbb{C})$ is induced by the inclusion map of $\mathcal{R}^p_{loc}(X)$ into $\mathcal{D}'^p(X)$.

Now assume that X is a complex manifold of complex dimension n. Integration over a subvariety defines a current on X (see [10], [7], and [5]). Let $\text{Reg } V$ denote the regular points of a subvariety V. The second result is the following:

Proposition 2.3: *If* V *is a pure* k *dimensional subvariety of* X *then* $[V](\phi)$ *defined by integrating* $\phi \in \mathcal{D}^{2k}(X)$ *over* $\text{Reg } V$ *is a locally rectifiable current on* X *which is* d-*closed and positive of type* (p,p) $(p + k = n)$.

Remark: Let $T = [V]$. Since $\vec{T}(z)$ is decomposable for $z \in \text{Reg } V$, the three notions weakly positive, positive, and strongly positive all agree for T.

An *analytic cycle of dimension* k (or *holomorphic* k-*chain*) on X is a current T of type (p,p) $(p + k = n)$ which can be expressed as $\Sigma n_j[V_j]$ where each $n_j \in \mathbb{Z}$ and $V = \cup V_j$ is a pure k-dimensional subvariety of X with irreducible components $\{V_j\}$. An analytic cycle is positive if and only if each $n_j \ge 0$. Let $Z_k(X)$ denote the group of all analytic cycles of dimension k on X and let $Z_k^+(X)$ denote the set of positive analytic k-cycles. If T can be expressed as $\Sigma r_j[V_j]$ with each r_j a rational number then T is called a *rational analytic* k-*cycle*. The third result is a slight generalization of Proposition 2.3.

Corollary 2.4: *Each analytic* k-*cycle is a locally rectifiable,* d-*closed current of type* (p,p) $(p + k = n)$.

Using Theorem 2.1, we see that each analytic k-cycle $\Sigma n_j[V_j]$ determines a cohomology class in $H^{2p}(X,\mathbb{Z})$ where $p + k = n$. Let $\pi: H^{2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C})$ denote the natural inclusion map.

Now assume that X is a compact Kähler manifold. Then each $H^{2r}(X,\mathbb{C})$ can be expressed as the direct sum $\Sigma H^{s,t}(X)$ where $s + t = r$ and $H^{s,t}(X)$ denotes the space of harmonic forms of type (s,t). Suppose $T \in Z_k(X)$; then T determines a cohomology class in $H^{2p}(X,\mathbb{C})$ $(p+k=n)$. This cohomology class must belong to $H^{p,p}(X)$ since $\int_T \phi = 0$ for all $\phi \in H^{s,t}(X)$ with $s + t = k$, $s \neq t$. That is, π maps the subspace of $H^{2p}(X,\mathbb{Z})$ determined by $Z_k(X)$ into $H^{p,p}(X)$.

Now assume that X is a complex submanifold of some complex projective space. The original conjecture of Hodge [6] was that

III (*over* $\mathbb{Z}$): *Each class* $c \in H^{2p}(X,\mathbb{Z})$ *with* $\pi c \in H^{p,p}(X)$ *is determined by an analytic* k-*cycle* $(p + k = n)$.

While this is true for $p = 1$ (Kodaira-Spencer [9]), it is false for $p > 1$ (Atiyah-Hirzebruch [1]). Conjecture III has been reformulated for general p over the rationals $\mathbb{Q}$.

III (*over* $\mathbb{Q}$): *Each class* $c \in H^{p,p}(X) \cap \pi H^2(X,\mathbb{Z}) \subset H^{2p}(X,\mathbb{C})$ *is determined by a rational analytic* k-*cycle*, $\frac{1}{m} T$ $(T \in Z_k(X)$, $m \in \mathbb{Z}$ *and* $p + k = n)$.

A smooth form ϕ of type (p,p) is strongly positive definite if ϕ remains positive under small perturbations or equivalently if (for each point $z \in X$) each of the eigenvalues λ_j is positive for j from 1 to $\binom{n}{p}$, or equivalently if for each point $z \in X$, $\phi(z)$ belongs to the interior of $SP \subset \Lambda^{p,p}_{\mathbb{R}} T'_z(X)$.

Consider the following two "Plateau problems" on X (instead of fixing a boundary, we fix a cohomology class). Suppose a class $c \in \pi H^{2p}(X,\mathbb{Z})$

is given. Let $M(c)$ denote the infimum of $\{M(T)\}$ taken over all $T \in R^{2p}(X)$ with $T \in c$, and let $m(c)$ denote the infimum of $\{M(\psi)\}$ taken over all $\psi \in \mathcal{D}'^{2p}(X)$ with $\psi \in c$. Find a rectifiable current $T \in R^{2p}(X)$ with $T \in c$ such that

$$(1) \quad M(T) = M(c) \quad \text{or} \quad (2) \quad M(T) = m(c)\ .$$

It follows easily from Federer [2] that problem (1) always has a solution T; however, it is not always true that this solution T is also a minimum among the larger class of competitors in (2) (cf. Mumford's example at the end of this section). Let T_k denote a solution to problem (1) for the class kc, where $k \in \mathbb{Z}^+$ (i.e., $M(T_k) = M(kc)$). Then $m(c) \le \le \frac{1}{k} M(T_k) \le M(c)$, since $M(kc) \le kM(c)$. Federer has shown that $\lim_{k\to\infty} \frac{1}{k} M(T_k) = m(c)$.

Next we state a conjecture about "positive Plateau problems", which concludes that equality holds in the above limit for some finite number k.

I *Suppose* $c \in H^{p,p}(X) \cap \pi H^{2p}(X,\mathbb{Z}) \subset H^{2p}(X,\mathbb{C})$ *contains a strongly positive definite* (p,p) *form* ϕ. *Then there exists an integer* $m \in \mathbb{Z}^+$ *such that for the class* mc *a solution* $T \in R^{2p}(X)$ *to* (1) *above also satisfies* (2).

The following conjecture will be shown to be equivalent to I in the next theorem:

II *Suppose* $c \in H^{p,p}(X) \cap \pi H^{2p}(X,\mathbb{Z}) \subset H^{2p}(X,\mathbb{C})$ *contains a strongly positive definite* (p,p) *form* ϕ. *Then there exists an integer* $m \in \mathbb{Z}^+$ *and a positive analytic* k-*cycle* $T \in Z_k^+(X)$ *such that* $\frac{1}{m} T \in c$ $(p + k = n)$.

Theorem 2.5: *For a given compact Kähler manifold* X, I *is true if and only if* II *is true.*

Proof: First assume that I is true. Let $T \in R^{2p}(X)$ with $T \in mc$, and let $\phi' = m\phi$. Then by Theorem 2.1

$$M(\phi') = \int \phi' \wedge \frac{1}{k!}\,\omega^k \ \le\ M(T) \tag{2.2}$$

(the first equality uses the Generalized Wirtinger Inequality, as opposed to the usual Wirtinger Inequality). If T is a solution to (2) then equality must hold in (2.2). By Theorem 2.1 this implies that T is positive.

By the structure theorem of [7] (cf. [4] and [5]) T must be a positive analytic k-cycle. This proves II. Conversely, suppose $S \in Z_k^+(X)$ and $S \in mc$. Then by Theorem 2.1

$$M(S) = \int S \wedge \frac{1}{k!}\omega^k = \int \psi \wedge \frac{1}{k!}\omega^k \leq M(\psi)$$

for each $\psi \in c$. This proves (2) for $S \in Z_k^+(X)$. However if (1) is true for $T \in mc$ and $S \in Z_k^+(X) \cap c$ then T must belong to $Z_k^+(X)$ (see (2.2) with $\phi' = S$ to conclude T is positive of type (p,p)). Therefore II implies I. Q.E.D.

Next we note that II implies the Hodge Conjecture III. Assume X is a submanifold of $\mathbb{P}^N$ and let ω denote the Fubini-Study (1,1) form on $\mathbb{P}^N$ restricted to X. Assume II is true and let $c \in H^{p,p}(X) \cap H^{2p}(X,\mathbb{Z})$. Then, given a smooth representative $\psi \in c$ of type (p,p), we can find a positive integer q such that $\phi = \psi + q\frac{1}{p!}\omega^p$ is strongly positive definite because of Corollary 1.10. By assumption there exist $T \in Z_k^+(X)$ and $m \in \mathbb{Z}$ with $[T] = m[\phi]$. Since $\frac{1}{p!}\omega^p$ determines the same class as a k-linear section S of X in $\mathbb{P}^N$ $(S \in Z_k^+(X))$, $T - mqS \in mc$ (and $T - mqS \in Z_k(X)$).

Remark: It follows from Theorem 5.8 in Kleiman [14] that if the Hodge Conjecture III is true then Conjecture II (and hence I) is true for (using the above notation) classes c of the form $[\psi + q\frac{1}{p!}\omega^p]$, if q is chosen sufficiently large depending on ψ. Therefore, Conjecture I (a "Plateau problem") for classes c of the form $[\psi + q\frac{1}{p!}\omega^p]$ with q large is equivalent to the Hodge Conjecture III.

The statement II is of course true for $p = 1$ by the Kodaira Embedding Theorem as follows: There exists a line bundle L over X with first Chern form ϕ. By Kodaira [8] there exists an integer m such that the mapping $X \xrightarrow{f} P(H^0(X,(L^{-1})^m))$ is an embedding. Therefore $m\phi - f^*([H]) = d\psi$ where $[H]$ is a hyperplane section of $f(X)$.

One might conjecture that statement II (or equivalently I) is valid modulo torsion. That is, if $c \in H^{p,p}(X) \cap \pi H^{2p}(X,\mathbb{Z})$ contains a strongly positive definite (p,p) form ϕ then c contains a positive analytic k-cycle $T \in Z_k^+(X)$. David Mumford's example, mentioned in the introduction, shows that this is false in the simplest case $p = 1$ and dimension $X = 2$. See [11] and [12] for the results about ruled surfaces needed below. Suppose $X \xrightarrow{\pi} C$ is a $\mathbb{P}^1$ bundle over an algebraic curve of genus $g \geq 4$. Choose X homeomorphic to $C \times \mathbb{P}^1$ and generic. Then $H^2(X,\mathbb{Z}) = \mathbb{Z}\cdot e \oplus \mathbb{Z}\cdot f$ where e contains $C \times \{z\}$, $z \in \mathbb{P}^1$ and f contains $\{w\} \times \mathbb{P}^1$, $w \in C$. Briefly, one can show that, since X is generic, $e + mf$ does not contain a positive divisor for $m < [\frac{g}{2}]$; and using the Nakai-Moišezon criterion, that $\phi + mf$ is ample for $m \in \mathbb{Z}^+$ (i.e., $e + mf$ contains $\frac{1}{k} S$ where $k \in \mathbb{Z}^+$ and S is a hyperplane section for some projective embedding). Consequently $e + mf$ contains $\frac{1}{k}\omega$ where ω is the Kähler form induced on X by some projective embedding. Therefore $e + mf$, $m \in \mathbb{Z}^+$, contains a positive definite $(1,1)$ form, while for $m < [\frac{g}{2}]$ it does not contain a positive divisor.

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