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Local Langlands Correspondence: The Archimedean Case

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The theory of group representations provides a rich supply of automorphic L functions that are candidates to be the L functions of motives. A motivic L function encodes arithmetic information in an analytic function defined as an Euler product and convergent in a right half-plane. Exploiting this information requires deriving conjectural properties of this analytic function, such as its analytic continuation and functional equation. In practice these properties are obtained only as a consequence of identifying the given motivic L function with an automorphic L function.

Actually the calculus of L functions is a rather small manifestation of a rather large enterprise known as the Langlands program. The general Langlands program works with a reductive group G over a global field F and with the representations that occur (in a suitable sense) in $L^2(G(F)\backslash G(\mathbb{A}))$, where $\mathbb{A}$ denotes the adèles of F. To each such representation the program associates an L function, and it is hoped that these L functions have the same kinds of nice analytic properties as the L functions of Hecke. Some original papers of Langlands on this program are [**19, 20, 21**]. Gelbart [7] has given an exposition of the scope of the theory.

The point of the present paper is to give an account of the relevant parts of representation theory that are occurring at the Archimedean places. Largely what we shall discuss is the Langlands treatment of representation theory of GL_n over $\mathbb{R}$ and $\mathbb{C}$. For $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, the Langlands classification theorem for irreducible admissible representations can be stated in a framework that sounds plausible for all local fields. We call this framework the "local Langlands correspondence." Within this framework, the contribution

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of the Archimedean places to L functions is a rather simple topic.[1] A companion paper by Kudla **[18]** treats the non-Archimedean places, where the analogous statements are partly theorems and partly conjectures.

Discussion of the classification theorem and the local Langlands correspondence for GL_n will occupy §§2–4 of this paper. In §5 we shall mention the extent to which the Langlands theory generalizes from GL_n to connected reductive groups. The special case of symplectic groups is relevant for the theory of Shimura varieties.

It is important for understanding the Langlands program to know some features of the historical transition from classical automorphic forms to automorphic representation theory. The theory separated into two directions at one time in the early 1970s and then came together several years later. The interplay between the two successful theories accounts for the relatively advanced state of knowledge for GL_n in comparison with other groups, and it is where we begin.

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1. Historical transition

Classical automorphic L functions in the work of Hecke arise as Mellin transforms of certain automorphic forms—particularly as transforms of a kind of θ function (in the case of Hecke's theory of grossencharacters) or as transforms of modular forms of the subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Hecke's theory of grossencharacters establishes analytic continuations and functional equations for L functions that generalize Dirichlet L functions. Tate's thesis **[28]** recast this theory of Hecke's in a representation-theoretic setting that corresponds to the regular representation of $\mathrm{GL}_1(\mathbb{A})$ on $\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A})$, $\mathbb{A}$ again being the adèles of a global field F. For a fixed grossencharacter, Tate's method in effect attaches to each place a local L factor given by an integral, as well as a local ε factor that contributes to a local functional equation. Also in effect, the method constructs a global L function as the product of the local L factors, and then it proves directly a global functional equation. We shall amplify this discussion shortly.

In the 1950s Gelfand and Fomin realized that modular forms are connected with representations of GL_2. Some expositions of this connection are in parts of Gelfand, Graev, and Pyatetskiĭ-Shapiro **[9]**, Weil **[35]**, Deligne **[2]**, Gelbart **[6]**, and Piatetski-Shapiro **[23]**. In part, the connection is that one can identify cusp forms for $\Gamma_0(N)$ in two stages with functions on groups. In the

[1] In classical terminology, it is customary to include only the factors from the non-Archimedean places in L and to give another name to the product of L with various gamma factors that come from the Archimedean places. But we shall follow the convention used in representation theory of including factors from all places in the definition of L.

first stage the identification is with certain functions on $\mathrm{SL}_2(\mathbb{R})$ transforming on the left side by $\Gamma_0(N)$ and on the right side by the rotation subgroup. In the second stage it is with functions on $\mathrm{GL}_2(\mathbb{A})$ whose integrals of a certain kind vanish, where $\mathbb{A}$ is now the ring of adèles of $\mathbb{Q}$. The cusp forms are then intimately connected with the decomposition into irreducible representations of the representation of $\mathrm{GL}_2(\mathbb{A})$ on the space of functions on $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})$ that transform by a character of the center $Z_{\mathbb{A}}$ of $\mathrm{GL}_2(\mathbb{A})$ and are square integrable on $\mathrm{GL}_2(\mathbb{Q})Z_{\mathbb{A}}\backslash \mathrm{GL}_2(\mathbb{A})$.

The Hecke operators turn out to have a nice interpretation in the representation-theoretic setting. As a result, each cusp form with an Euler product expansion corresponds, under the above two-stage identification, with a function that lies in a subspace irreducible under $\mathrm{GL}_2(\mathbb{A})$. This infinite-dimensional representation is like a grossencharacter in that it is the "tensor product" of representations for the local groups, one for $\mathrm{GL}_2(\mathbb{R})$ and one for each $\mathrm{GL}_2(\mathbb{Q}_p)$, and Jacquet and Langlands **[13]** were able to develop a theory for GL_2 parallel to the Tate theory for GL_1. They made critical use of a classification of the irreducible admissible representations of GL_2 of each local field. Expositions are in Robert **[24]** and Gelbart **[6]**.

In the early 1970s, generalization to GL_n took two different paths, because of the absence of a classification for $n > 2$. Godement and Jacquet **[10]** used the method of Tate to develop a theory that led to the analytic continuation and functional equation of L functions, without an explicit identification of all the L functions. Langlands **[21]** set up a conjectural framework for classification, with explicit L functions, but his theory did not account for the analytic continuation and functional equation. The final theory for GL_n requires both parts, and it was Jacquet **[12]** who showed the two parts are compatible.

To make clear the distinction between the two parts, let us amplify the discussion of Tate's method, using notation appropriate to GL_n. Proofs of the various steps may be found in **[28, 13, 10, 12]**. We shall sketch the theory at the Archimedean places, indicate how it can be adjusted for the non-Archimedean places, and say briefly what happens globally.

First let k be $\mathbb{R}$ or $\mathbb{C}$, and let $M_n(k)$ be n-by-n matrix space over k. Let

$$K = \begin{cases} O(n) & \text{if } k = \mathbb{R}, \\ U(n) & \text{if } k = \mathbb{C}. \end{cases} \tag{1.1}$$

Let (ρ, V) be an admissible representation of $\mathrm{GL}_n(k)$, "admissible" being defined in §2, and let $(\tilde{\rho}, \widetilde{V})$ be the admissible dual. A K **finite matrix coefficient** of ρ is a function

$$c(x) = \langle \rho(x)u, \tilde{u} \rangle$$

with $u \in V$ and $\tilde{u} \in \widetilde{V}$ both transforming in finite-dimensional spaces under K. The function $\check{c}(x) = c(x^{-1})$ is another K finite matrix coefficient, because it is given by

$$\check{c}(x) = \langle u, \tilde{\rho}(x)\tilde{u}\rangle.$$

The subspace $\mathscr{S}_0$ of the Schwartz space $\mathscr{S}(M_n(k))$ is to consist of all functions of the form

$$\begin{aligned} &P(x_{ij})\exp\left(-\pi\sum x_{ij}^2\right) && \text{if } k=\mathbb{R}, \\ &P(z_{ij}\bar{z}_{ij})\exp\left(-2\pi\sum z_{ij}\bar{z}_{ij}\right) && \text{if } k=\mathbb{C}, \end{aligned}$$

where P is an arbitrary polynomial. For any K finite matrix coefficient c of ρ and any function f in the space $\mathscr{S}_0$, we define

$$\zeta(f, c, s) = \int_{M_n(k)} f(x)c(x)|\det x|_k^s \, d^\times x \tag{1.2}$$

for s complex. Here $|z|_{\mathbb{R}} = |z|$ and $|z|_{\mathbb{C}} = |z|^2$. The measure is $d^\times x = |\det x|_k^{-n}\, dx$, where dx is a fixed invariant measure for $M_n(k)$; use of $d^\times x$ in the notation may be regarded as an additive normalization of the parameter s.

Assume ρ is irreducible. Then all the integrals (1.2) converge for s in a common right half-plane and extend to be meromorphic functions for s in $\mathbb{C}$. Moreover, there exist finitely many choices of (c, f), say (c_i, f_i), such that

$$L(s, \rho) = \sum_i \zeta(f_i, c_i, s) \tag{1.3}$$

has the following property: For any (c, f),

$$\zeta(f, c, s + \tfrac{1}{2}(n-1)) = P(f, c, s)L(s, \rho) \tag{1.4}$$

for a polynomial P in s. The function $L(s, \rho)$ is uniquely determined by these properties, up to a scalar factor, and is called a **local** L **factor**. (In Tate's original work, in which $n = 1$, the matrix coefficient c is essentially unique, and the sum on the right side of (1.3) collapses to a single term.)

Let ψ be the additive character of k given by

$$\begin{aligned} \psi(x) &= \exp(2\pi i x) && \text{if } k=\mathbb{R}, \\ \psi(z) &= \exp(2\pi i(z+\bar{z})) && \text{if } k=\mathbb{C}, \end{aligned} \tag{1.5}$$

and define the Fourier transform $\widehat{f}$ of a member f of $\mathscr{S}_0$ by

$$\widehat{f}(x) = \int_{M_n(k)} f(y)\psi(\mathrm{Tr}(xy))\, dy, \tag{1.6}$$

where dy is the self-dual Haar measure on $M_n(k)$. Then $\widehat{f}$ is again in $\mathscr{S}_0$.

With ρ still irreducible, there exists a meromorphic function $\gamma(s, \rho, \psi)$ independent of f and c such that

$$\zeta(\widehat{f}, \check{c}, 1 - s + \tfrac{1}{2}(n-1)) = \gamma(s, \rho, \psi)\zeta(f, c, s + \tfrac{1}{2}(n-1)) \tag{1.7}$$

for all f in $\mathcal{S}_0$ and all K finite matrix coefficients c of ρ. In terms of

$$\varepsilon(s, \rho, \psi) = \gamma(s, \rho, \psi)L(s, \rho)/L(1-s, \tilde{\rho}), \tag{1.8}$$

the local functional equation reads

$$\frac{\zeta(\widehat{f}, \check{c}, 1-s+\frac{1}{2}(n-1))}{L(1-s, \tilde{\rho})} = \varepsilon(s, \rho, \psi)\frac{\zeta(f, c, s+\frac{1}{2}(n-1))}{L(s, \rho)}. \tag{1.9}$$

From (1.3) and (1.4), it follows that $\varepsilon(s, \rho, \psi)$ is a polynomial. Let ω_ρ be the quasicharacter of $k^\times$ such that $\rho(a \cdot 1) = \omega_\rho(a)1$. Since $(\widehat{f})^\frown(x) = f(-x)$, the change of variables $x \to -x$ in (1.1) and iteration of (1.9) gives

$$\varepsilon(s, \rho, \psi)\varepsilon(1-s, \tilde{\rho}, \psi) = \omega_\rho(-1).$$

Therefore $\varepsilon(s, \rho, \psi)$ is a constant. Apart from evaluation of $L(s, \rho)$ and $\varepsilon(s, \rho, \psi)$, this completes the local part of Tate's method at Archimedean places.

Next let k be a non-Archimedean local field. In the definition of $\zeta(f, c, s)$, we make the following adjustments: Admissibility of (ρ, V) is defined in **[18]**, mention of K finiteness of the matrix coefficient $c(x)$ may be omitted, and $\mathcal{S}_0$ is taken as the whole Schwartz-Bruhat space of locally constant functions of compact support on $M_n(k)$.

For this new k, assume ρ is irreducible. It is still true that the integrals (1.2) converge in a common right half-plane and extend to be meromorphic functions for s in $\mathbb{C}$. Moreover, the definition of $L(s, \rho)$ in (1.3) and (1.4) still applies, with the following modifications: P in (1.4) is a polynomial in q^s and q^{-s}, where q is the number of elements in the residue field, and $L(s, \rho)$ can be normalized so as to be $1/Q(q^{-s})$, where Q is a polynomial with constant term 1.

Let ψ be a nontrivial additive character of k, and define the Fourier transform by (1.6). Then there exists a meromorphic function $\gamma(s, \rho, \psi)$ independent of f and c such that (1.7) holds for all f in $\mathcal{S}_0$ and all matrix coefficients c of ρ. With $\varepsilon(s, \rho, \psi)$ as in (1.8), the local functional equation reads as in (1.9). One sees that $\varepsilon(s, \rho, \psi)$ is a nonzero multiple of a power of q^{-s}.

Finally let F be a global field, let $\mathbb{A}$ be the adèles of F, and let $\mathbb{I}$ be the idèles of $F^\times$. Some care is required in identifying what irreducible admissible representations ρ of $\mathrm{GL}_n(\mathbb{A})$ and what matrix coefficients are to be allowed; these details are in §§10–12 of **[10]**. The representation ρ is to occur in the regular representation on $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})$. Moreover, it is to reduce to a quasicharacter ω_ρ on members of $\mathbb{I}$ (scalar matrices) in the sense that $\rho(a \cdot 1_n) = \omega_\rho(a)1_V$, and ω_ρ is to be trivial on $F^\times$. In addition, ρ is to incorporate conditions of vanishing integrals of the kind satisfied by embedded cusp forms when $n = 2$ and $F = \mathbb{Q}$.

Such an irreducible representation ρ is the adelic product of irreducible representations ρ_v of each local field k_v. This result is proved for $n = 2$ in **[13]**, and a general argument may be found in Flath **[5]**. One defines $\zeta(f, c, s)$ by (1.2), with k replaced by $\mathbb{A}$, and with f suitably restricted. The main step is to prove that $\zeta(f, c, s)$ extends to be an entire function and satisfies

$$\zeta(\widehat{f}, \check{c}, n-s) = \zeta(f, c, s). \tag{1.10}$$

We define

$$\begin{aligned} L(s, \rho) &= \prod_v L(s, \rho_v), \\ \varepsilon(s, \rho, \psi) &= \prod_v \varepsilon(s, \rho_v, \psi_v), \end{aligned} \tag{1.11}$$

where ψ is a nontrivial additive character of $\mathbb{A}/F$ with local components ψ_v. The **global L function** $L(s, \rho)$ initially is convergent in a right half-plane, and $\varepsilon(s, \rho, \psi)$ has almost all factors 1 and hence is entire. At each place, $L(s, \rho_v)$ is a sum of functions $\zeta(f_v, c_v, s)$. Assembling these f_v's and c_v's into global f and c, we find that $L(s, \rho)$ extends to be entire. Taking the product of (1.9) over all places and substituting from (1.10) and (1.11), we obtain

$$L(s, \rho) = \varepsilon(s, \rho, \psi) L(1-s, \tilde{\rho}). \tag{1.12}$$

This is the functional equation for $L(s, \rho)$ that we have sought, and our discussion of Tate's method is complete.

We have not discussed the evaluation of $L(s, \rho)$ and $\varepsilon(s, \rho, \psi)$ in the local case. Godement and Jacquet **[10]** showed that it is possible to compute these expressions for most places and most ρ's with rather little information about classification of irreducible admissible representations. Later Jacquet **[12]** showed that one can compute these functions in all cases with just a little more information about classification.

In the second part of the theory of GL_n, Langlands **[21]** took the known values of L and ε, went a long way toward classification of irreducible admissible representations in the Archimedean case, and organized the information about classification and L factors into a framework that showed promise for being valid for all local fields, Archimedean or not. This framework is called the **local Langlands correspondence**. One of its features is that it makes sense for general reductive groups, not just GL_n. We shall describe the classification for GL_n at the Archimedean places and then the local Langlands correspondence for organizing this information. We work with $\mathbb{R}$ in §§2–3 and with $\mathbb{C}$ in §4. The part of Jacquet **[12]** that deals with Archimedean places effectively shows that the local Langlands correspondence for GL_n attaches the same L and ε factors for $\mathbb{R}$ and $\mathbb{C}$ as in **[10]**.

2. Langlands classification for $\mathrm{GL}_n(\mathbb{R})$

The Langlands classification for $G = \mathrm{GL}_n(\mathbb{R})$ describes all irreducible admissible representations[2] of G up to infinitesimal equivalence.

Let $K = O(n)$ be the maximal compact subgroup of G given in (1.1). A representation (ρ, V) of G on a Hilbert space V will be said to be **admissible** if, in the restriction of ρ to K, each irreducible representation of K occurs with at most finite multiplicity. It is **irreducible** if V has no nontrivial closed invariant subspace. If ρ is admissible, let V_K be the space of K **finite** vectors, those transforming in a finite-dimensional space under K. Each member of V_K is a C^∞ vector, and V_K is a representation space for both K and the Lie algebra $\mathfrak{g}$ of G. (See [**14**, Chapter III].)

Moreover, these representations of K and $\mathfrak{g}$ are compatible, and (ρ, V_K) is a $(\mathfrak{g}, K)$ module, in the sense of [**33**]. From Harish-Chandra [**11**], it is known that the closed G invariant subspaces U of V are in one-one correspondence with the arbitrary $(\mathfrak{g}, K)$ invariant subspaces U_K of V_K, the correspondence being $U = \overline{U}_K$ and $U_K = U \cap V_K$. In particular, (ρ, V) is irreducible if and only if (ρ, V_K) is algebraically irreducible. (See [**14**, Chapter VIII].)

Two admissible representations are said to be **infinitesimally equivalent** if their underlying $(\mathfrak{g}, K)$ modules are isomorphic. Infinitesimally equivalent admissible representations have the same K finite matrix coefficients.

Any irreducible admissible representation (ρ, V) has a **central character** given as a quasicharacter $\omega_\rho : \mathbb{R}^\times \to \mathbb{C}^\times$ by $\rho(a\cdot 1_n) = \omega_\rho(a)1_V$ for $a \in \mathbb{R}^\times$.

Let $\mathrm{SL}_m^\pm(\mathbb{R})$ be the subgroup of elements g of $\mathrm{GL}_m(\mathbb{R})$ with $|\det g| = 1$. The unimodular subgroup $\mathrm{SL}_m(\mathbb{R})$ has index 2 in $\mathrm{SL}_m^\pm(\mathbb{R})$. We shall specify certain irreducible representations of $\mathrm{SL}_m^\pm(\mathbb{R})$ for the cases $m = 1$ and $m = 2$. For $m = 1$, there are only two representations, and they are both one dimensional; we write 1 for the trivial one and sgn for the nontrivial one. For $m = 2$ the representations of interest are the ones in the "discrete series," denoted D_l for integers $l \geq 1$. These representations are induced from $\mathrm{SL}_2(\mathbb{R})$ as

$$D_l = \mathrm{ind}_{\mathrm{SL}_2(\mathbb{R})}^{\mathrm{SL}_2^\pm(\mathbb{R})}(D_l^+). \tag{2.1a}$$

Here D_l^+ acts in the space of analytic functions f in the upper half-plane with

$$\|f\|^2 = \iint |f(z)|^2 y^{l-1}\, dx\, dy$$

finite, the action by $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ being

$$D_l^+(g)f(z) = (bz+d)^{-(l+1)} f\left(\tfrac{az+c}{bz+d}\right). \tag{2.1b}$$

[2] By convention a representation of a group always takes place on a complex vector space, and the group action is assumed to be jointly continuous.

The representations D_l of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ are irreducible unitary, and their matrix coefficients are square integrable. (See [**14,** p. 35].)

The building blocks for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ are the representations of $\mathrm{GL}_1(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$ obtained by tensoring the above representations on $\mathrm{SL}^{\pm}$ with a quasicharacter $a \to |\det a|_{\mathbb{R}}^t$ on the positive scalar matrices of size 1 or 2. Here, as earlier, $|\cdot|_{\mathbb{R}}$ denotes ordinary absolute value, and t is in $\mathbb{C}$. Thus the building blocks will be

$$\left.\begin{array}{c} 1 \otimes |\cdot|_{\mathbb{R}}^t \\ \mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t \end{array}\right\} \quad \text{for } \mathrm{GL}_1(\mathbb{R}), \tag{2.2a}$$

$$\left. D_l \otimes |\det(\cdot)|_{\mathbb{R}}^t \right\} \quad \text{for } \mathrm{GL}_2(\mathbb{R}). \tag{2.2b}$$

To any partition of n into 1's and 2's, say $(n_1, \dots, n_r)$ with each n_j equal to 1 or 2 and with $\sum n_j = n$, we associate the block diagonal subgroup

$$D = \mathrm{GL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{R}). \tag{2.3}$$

For each j with $1 \le j \le r$, let σ_j be a representation of $\mathrm{GL}_{n_j}(\mathbb{R})$ of the form (2.2), and write t_j for t. Then $(\sigma_1, \dots, \sigma_r)$ defines by tensor product a representation of the block diagonal subgroup (2.3), and we extend this representation to the corresponding block upper triangular subgroup $Q = DU$ by making it be the identity on the block strictly upper triangular subgroup U. We set

$$I(\sigma_1, \dots, \sigma_r) = \mathrm{ind}_Q^G(\sigma_1, \dots, \sigma_r), \tag{2.4}$$

using unitary induction as in [**14,** Chapter VII]. (That is, in the transformation law under Q, $(\sigma_1, \dots, \sigma_r)$ is tensored with a one-dimensional representation so that when $\sigma_1, \dots, \sigma_r$ are unitary, $I(\sigma_1, \dots, \sigma_r)$ is automatically unitary.)

THEOREM 1. *For $G = \mathrm{GL}_n(\mathbb{R})$,*

(a) *if the parameters $n_j^{-1} t_j$ of $(\sigma_1, \dots, \sigma_r)$ satisfy*

$$n_1^{-1} \operatorname{Re} t_1 \ge n_2^{-1} \operatorname{Re} t_2 \ge \cdots \ge n_r^{-1} \operatorname{Re} t_r, \tag{2.5}$$

then $I(\sigma_1, \dots, \sigma_r)$ has a unique irreducible quotient $J(\sigma_1, \dots, \sigma_r)$,

(b) *the representations $J(\sigma_1, \dots, \sigma_r)$ exhaust the irreducible admissible representations of G, up to infinitesimal equivalence,*

(c) *two such representations $J(\sigma_1, \dots, \sigma_r)$ and $J(\sigma_1', \dots, \sigma_r')$ are infinitesimally equivalent if and only if $r' = r$ and there exists a permutation $j(i)$ of $\{1, \dots, r\}$ such that $\sigma_i' = \sigma_{j(i)}$ for $1 \le i \le r$.*

Two ways are known for picking out the constituent $J(\sigma_1, \dots, \sigma_r)$ of $I(\sigma_1, \dots, \sigma_r)$. One way, following Langlands [**21**], is as the image of a certain standard intertwining operator on $I(\sigma_1, \dots, \sigma_r)$; see pp. 198–200 of [**14**] for an exposition. If any of the inequalities in (2.5) is an equality, some normalization of the operator may be necessary in order to eliminate poles.

Techniques for this normalization are explained in Chapter XIV of **[14]**. The other way, following Vogan **[30]**, is by the theory of minimal K types. For a fixed induced representation, the Vogan theory singles out finitely many irreducible representations of K (one such in this case) that occur in the induced representation with multiplicity one, and a vector transforming by any one of these representations of K generates $J(\sigma_1, \dots, \sigma_r)$ as a subquotient. (See **[14**, Chapter XV].) Since G is disconnected, the original Vogan theory is not quite general enough to handle this case, and one must appeal to Vogan **[32]** instead.

The original paper **[21]** of Langlands reduced the classification of irreducible admissible representations of fairly general reductive groups to the classification of the irreducible tempered representations. Here "tempered" is a term referring to the asymptotic behavior of matrix coefficients and need not be explained in this exposition. (See **[14**, p. 198], for a precise definition.) Langlands proved also that the irreducible tempered representations are exactly the irreducible constituents of the representations that in this case have $t_1, \dots, t_r$ purely imaginary. (See **[14**, Chapter VIII].) The irreducible tempered representations were classified in 1976 by Knapp and Zuckerman, and the result was reformulated in **[16]**. Detailed proofs are in **[17]**; for an exposition, see **[14**, Chapter XIV].

The papers **[16]** and **[17]** treat connected semisimple groups and are not literally applicable to $\mathrm{GL}_n(\mathbb{R})$. However, the results are still valid for $\mathrm{GL}_n(\mathbb{R})$, and no new ideas are needed for their proofs. (For general reductive groups with some disconnectedness, a new idea *is* needed. This extension of the classification was carried out by Mirković **[22]**.)

With account taken of the remarks in the previous paragraph, parts (a) and (b) of Theorem 1 are a special case of Theorem 5 of **[16]**. The latter theorem has four hypotheses, and the first three are checked by inspection. For the fourth, it is enough to check that $I(\sigma_1, \dots, \sigma_r)$ is irreducible in the tempered case, i.e., when all t_j are purely imaginary. This is easily done; see §7 of **[15]**. Part (c) is not explicitly written down in these sources, but §3 of **[16]** tells what one has to do to classify the infinitesimal equivalences; Example 3 in that section helps illustrate the technique.

We mention that the irreducible unitary representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ have been classified by Vogan **[31]**. The original Langlands program makes no predictions about this kind of result.

3. Local Langlands correspondence for $\mathrm{GL}_n(\mathbb{R})$

The Weil group of $\mathbb{R}$, denoted $W_{\mathbb{R}}$, is the nonsplit extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$. Here bar denotes complex conjugation. We shall be interested in the set of equivalence classes of n-dimensional complex representations of $W_{\mathbb{R}}$ whose images consist of semisimple elements.

The one-dimensional representations of $\mathbb{C}^\times$ are of the form

$$z \to z^\mu \bar{z}^\nu \quad \text{with } \mu \text{ and } \nu \text{ in } \mathbb{C} \text{ and } \mu - \nu \text{ in } \mathbb{Z}. \tag{3.1}$$

In fact, if we write $z = re^{i\theta}$, we obtain $re^{i\theta} \to r^{\mu+\nu} e^{i(\mu-\nu)\theta}$. Hence $\mu - \nu$ is an integer and $\mu + \nu$ is unrestricted.

Let us classify the one-dimensional representations φ of $W_{\mathbb{R}}$. On $\mathbb{C}^\times$, we have $\varphi(z) = z^\mu \bar{z}^\nu$ as in (3.1). Let $\varphi(j) = w$. Then

$$\varphi(\bar{z}) = \varphi(jzj^{-1}) = w\varphi(z)w^{-1} = \varphi(z).$$

Hence $\varphi(z) = z^\nu \bar{z}^\mu$ and $\mu = \nu$. In other words, $\varphi(re^{i\theta}) = r^{2\mu}$. Now

$$1 = \varphi(-1) = \varphi(j^2) = w^2$$

says $w = \pm 1$. Thus the one-dimensional representations are parametrized by a sign and a complex parameter $t = 2\mu$ as follows:

$$\begin{aligned} &(+, t): \quad \varphi(z) = |z|_{\mathbb{R}}^t \quad \text{and} \quad \varphi(j) = +1, \\ &(-, t): \quad \varphi(z) = |z|_{\mathbb{R}}^t \quad \text{and} \quad \varphi(j) = -1. \end{aligned} \tag{3.2}$$

Next let us classify the irreducible two-dimensional semisimple representations φ of $W_{\mathbb{R}}$, up to equivalence. The set $\varphi(\mathbb{C}^\times)$ consists of commuting diagonable transformations. Let u, v be a basis in which $\varphi(\mathbb{C}^\times)$ is diagonal. Say $\varphi(z)u = z^\mu \bar{z}^\nu u$ and $\varphi(z)v = z^{\mu'} \bar{z}^{\nu'} v$. If $\mu = \mu'$ and $\nu = \nu'$, then any one-dimensional invariant subspace for $\varphi(j)$ will exhibit φ as reducible. Hence we may assume that $\mu \neq \mu'$ or $\nu \neq \nu'$. Put $u' = \varphi(j)u$. Then

$$\varphi(z)u' = \varphi(j\bar{z}j^{-1})u' = \varphi(j)\varphi(\bar{z})u = z^\nu \bar{z}^\mu \varphi(j)u = z^\nu \bar{z}^\mu u'.$$

If $\mu = \nu$, then u' is in $\mathbb{C}u$, and $\mathbb{C}u$ is an invariant subspace, contradiction. Thus u' must be in $\mathbb{C}v$ with $\nu = \mu'$ and $\mu = \nu'$. We shall replace the basis u, v by the basis u, u'. Since $\varphi(j)^{-1} = \varphi(j)\varphi(-1) = (-1)^{\mu-\nu}\varphi(j)$ on the span of u and u', we can write the result as

$$\begin{aligned} \varphi(z)u &= z^\mu \bar{z}^\nu u, & \varphi(j)u &= u', \\ \varphi(z)u' &= z^\nu \bar{z}^\mu u', & \varphi(j)u' &= (-1)^{\mu-\nu} u. \end{aligned}$$

In terms of the basis u', $(-1)^{\mu-\nu}u$, these formulas become

$$\begin{aligned} \varphi(z)u' &= z^\nu \bar{z}^\mu u', & \varphi(j)u' &= (-1)^{\mu-\nu} u, \\ \varphi(z)((-1)^{\mu-\nu}u) &= z^\mu \bar{z}^\nu ((-1)^{\mu-\nu}u), & \varphi(j)((-1)^{\mu-\nu}u) &= (-1)^{\mu-\nu} u'. \end{aligned}$$

In view of the symmetry here, we may assume that the nonzero integer $\mu - \nu = l$ is positive. We conclude that the equivalence class of φ is classified by a pair (l, t) with $l = \mu - \nu$ an integer ≥ 1 and with $2t = \mu + \nu$ in $\mathbb{C}$. For the pair (l, t) there exists a basis u, u' such that

$$\begin{aligned} (l, t): \quad &\varphi(re^{i\theta})u = r^{2t} e^{il\theta} u, & \varphi(j)u &= u', \\ &\varphi(re^{i\theta})u' = r^{2t} e^{-il\theta} u', & \varphi(j)u' &= (-1)^l u. \end{aligned} \tag{3.3}$$

LEMMA. *Every finite-dimensional semisimple representation φ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

PROOF. Let φ act on the vector space V. Since $\varphi(\mathbb{C}^\times)$ consists of commuting diagonable transformations, V is the direct sum of spaces $V_{\mu,\nu}$ where all $\varphi(z)$ act by $z^\mu \bar{z}^\nu$. As above, we have $\varphi(j)V_{\mu,\nu} = V_{\nu,\mu}$. If $\mu = \nu$, then we can choose a basis of eigenvectors for $\varphi(j)$ in $V_{\mu,\mu}$, and the span of each eigenvector is a one-dimensional invariant subspace under $\varphi(W_{\mathbb{R}})$. If $\mu \neq \nu$, choose a basis $u_1, \dots, u_r$ of $V_{\mu,\nu}$, and put $u_i' = \varphi(j)u_i$ for $1 \le i \le r$. Then $\mathbb{C}u_i \oplus \mathbb{C}u_i'$ is a two-dimensional invariant subspace under $\varphi(W_{\mathbb{R}})$, and the direct sum of these subspaces as i varies is $V_{\mu,\nu} \oplus V_{\nu,\mu}$. This proves the lemma.

Now let φ be an n-dimensional semisimple complex representation of $W_{\mathbb{R}}$. By the lemma, φ is fully reducible. If we list the dimensions of the irreducible constituents in any order, we can regard the result as a partition of n into 1's and 2's, say $(n_1, \dots, n_r)$ with each n_j equal to 1 or 2 and with $\sum n_j = n$. Fix attention on n_j, and let φ_j be the corresponding irreducible constituent of φ. To φ_j we associate a representation σ_j from (2.2) as follows:

$$
\begin{array}{llll}
 & (+, t) \text{ in } (3.2) & \longrightarrow & 1 \otimes |\cdot|_{\mathbb{R}}^t \text{ in } (2.2\text{a}), \\
(3.4) & (-, t) \text{ in } (3.2) & \longrightarrow & \operatorname{sgn} \otimes |\cdot|_{\mathbb{R}}^t \text{ in } (2.2\text{a}), \\
 & (l, t) \text{ in } (3.3) & \longrightarrow & D_l \otimes |\det(\cdot)|_{\mathbb{R}}^t \text{ in } (2.2\text{b}).
\end{array}
$$

In this way, we associate a tuple $(\sigma_1, \dots, \sigma_r)$ of representations to φ. If the complex numbers $t_1, \dots, t_r$ do not satisfy (2.5), we permute $(\sigma_1, \dots, \sigma_r)$ so that (2.5) ends up being satisfied. Using Theorem 1, we can then make the association

$$
\varphi \longrightarrow \rho_{\mathbb{R}}(\varphi) = J(\sigma_1, \dots, \sigma_r) \tag{3.5}
$$

and come to the following conclusion.

THEOREM 2 (Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{R})$). *The association* (3.5) *is a well-defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{R}}$ and the set of all equivalence classes of irreducible admissible representations of* $\mathrm{GL}_n(\mathbb{R})$.

To each finite-dimensional semisimple complex representation φ of the Weil group of a local field, Weil [**34**] has associated a local L factor with certain nice properties. The results are summarized in Tate [**29**]; see also Shahidi [**25**, especially p. 990]. Some of Tate's results are taken from Deligne

[3]. In the case of $W_{\mathbb{R}}$, when φ is irreducible, the formula is

(3.6)
$$L(s,\varphi)=\begin{cases}\pi^{-(s+t)/2}\Gamma(\frac{s+t}{2}) & \text{if } \varphi \text{ is given by } (+,t) \text{ in (3.2)},\\ \pi^{-(s+t+1)/2}\Gamma(\frac{s+t+1}{2}) & \text{if } \varphi \text{ is given by } (-,t) \text{ in (3.2)},\\ 2(2\pi)^{-(s+t+\frac{l}{2})}\Gamma(s+t+\frac{l}{2}) & \text{if } \varphi \text{ is given by } (l,t) \text{ in (3.3)}.\end{cases}$$

(Recall our convention that $l \geq 1$.) For φ reducible, $L(s,\varphi)$ is the product of the L factors of the irreducible constituents of φ.

Fix the additive character ψ of $\mathbb{R}$ in (1.5), so that ordinary Lebesgue measure is self-dual Haar measure on $\mathbb{R}$. The ε factors are given for φ irreducible by

$$(3.7)\qquad \varepsilon(s,\varphi,\psi)=\begin{cases}1 & \text{if } \varphi \text{ is given by } (+,t) \text{ in (3.2)},\\ i & \text{if } \varphi \text{ is given by } (-,t) \text{ in (3.2)},\\ i^{l+1} & \text{if } \varphi \text{ is given by } (l,t) \text{ in (3.3)}.\end{cases}$$

For φ reducible, $\varepsilon(s,\varphi,\psi)$ is the product of the ε factors of the irreducible constituents of φ. Observe that the ε factors are constant in s.

In the terminology of **[29]**, both L and ε are "additive" in their behavior with respect to short exact sequences. Also L is "inductive" with respect to change of field, and its formula over $\mathbb{R}$ should really be considered together with its formula (4.6) over $\mathbb{C}$. The existence of ε for all local fields is a theorem of Langlands. See Theorem 3.4.1 of **[29]**. The ε factors satisfy a weaker property than "inductive"; they are "inductive in degree 0" with respect to change of field. The formula (3.7) over $\mathbb{R}$ should be considered together with the formula (4.7) over $\mathbb{C}$. The rule for how ε depends on ψ is given in (3.2.3) of **[29]**.

We can now define local factors $L(s,\rho)$ and $\varepsilon(s,\rho,\psi)$ for each irreducible admissible representation of $\mathrm{GL}_n(\mathbb{R})$ by the rule

$$(3.8)\qquad \left.\begin{aligned} L(s,\rho)&=L(s,\varphi)\\ \varepsilon(s,\rho,\psi)&=\varepsilon(s,\varphi,\psi)\end{aligned}\right\} \text{ if } \rho=\rho_{\mathbb{R}}(\varphi) \text{ in (3.5) and Theorem 2.}$$

These formulas are consistent with Jacquet and Langlands **[13]**, especially pp. 177–195. Jacquet **[12]** proved the following result.

THEOREM 3. *The definition of* $L(s,\rho)$ *in* (3.8) *satisfies the defining conditions* (1.3) *and* (1.4) *for* $L(s,\rho)$ *over* $\mathbb{R}$ *in* §1, *and the two definitions of* $\varepsilon(s,\rho,\psi)$ *in* (3.8) *and* (1.8) *coincide.*

4. Classification and correspondence for $\mathrm{GL}_n(\mathbb{C})$

There is a corresponding theory for $G=\mathrm{GL}_n(\mathbb{C})$, quite a bit less complicated. **Admissible representations** are defined for this G as in §2, but with $K=U(n)$ playing the role of maximal compact subgroup. The notions of **irreducible** representation, K **finite** vector, **infinitesimally equivalent**, and **central character** are defined as in §2.

For z in $\mathbb{C}$, let $[z] = z/|z|$. Also recall that $|z|_{\mathbb{C}} = |z|^2$. The building blocks for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{C})$ are the representations of $\mathrm{GL}_1(\mathbb{C})$ given by

$$z \to [z]^l |z|_{\mathbb{C}}^t \quad \text{with } l \in \mathbb{Z} \text{ and } t \in \mathbb{C}, \tag{4.1}$$

which we write as $[\cdot]^l \otimes |\cdot|_{\mathbb{C}}^t$. For each j with $1 \leq j \leq n$, let σ_j be the representation $[\cdot]^{l_j} \otimes |\cdot|_{\mathbb{C}}^{t_j}$ of $\mathrm{GL}_1(\mathbb{C})$. Then $(\sigma_1, \ldots, \sigma_n)$ defines a one-dimensional representation of the diagonal subgroup of $\mathrm{GL}_n(\mathbb{C})$, and we extend this to a one-dimensional representation of the upper triangular subgroup B. We set

$$I(\sigma_1, \ldots, \sigma_n) = \mathrm{ind}_B^G(\sigma_1, \ldots, \sigma_n), \tag{4.2}$$

using unitary induction.

THEOREM 4. *For $G = \mathrm{GL}_n(\mathbb{C})$,*

(a) *if the parameters t_j of $(\sigma_1, \ldots, \sigma_n)$ satisfy*

$$\mathrm{Re}\, t_1 \geq \mathrm{Re}\, t_2 \geq \cdots \geq \mathrm{Re}\, t_n, \tag{4.3}$$

then $I(\sigma_1, \ldots, \sigma_n)$ has a unique irreducible quotient $J(\sigma_1, \ldots, \sigma_n)$,

(b) *the representations $J(\sigma_1, \ldots, \sigma_n)$ exhaust the irreducible admissible representations of G, up to infinitesimal equivalence,*

(c) *two such representations $J(\sigma_1, \ldots, \sigma_n)$ and $J(\sigma_1', \ldots, \sigma_n')$ are infinitesimally equivalent if and only if there exists a permutation $j(i)$ of $\{1, \ldots, n\}$ such that $\sigma_i' = \sigma_{j(i)}$ for $1 \leq i \leq n$.*

This theorem predates the Langlands classification and is due to Želobenko and Naĭmark **[36, 37]**. For an exposition, see Duflo **[4]**. The quotient $J(\sigma_1, \ldots, \sigma_n)$ may be described within $I(\sigma_1, \ldots, \sigma_n)$ in the same two ways as in the case of $\mathrm{GL}_n(\mathbb{R})$.

The Weil group of $\mathbb{C}$, denoted $W_{\mathbb{C}}$, is given by $W_{\mathbb{C}} = \mathbb{C}^\times$. As in the case of $\mathbb{R}$, we shall be interested in the set of equivalence classes of n-dimensional complex representations of $W_{\mathbb{C}}$ whose images consist of semisimple elements.

Since $\mathbb{C}^\times$ is abelian, such a representation φ is diagonable and hence is the direct sum of one-dimensional representations. A one-dimensional representation is necessarily of the form (3.1). But now that conjugation no longer plays any role, it will be more convenient to write it as

$$(l, t)\colon z \to [z]^l |z|_{\mathbb{C}}^t \quad \text{with } l \in \mathbb{Z} \text{ and } t \in \mathbb{C}. \tag{4.4}$$

Let φ be an n-dimensional semisimple complex representation of $W_{\mathbb{C}}$. We can write φ as a direct sum of one-dimensional representations φ_j with

$\varphi_j(z) = [z]^{l}|z|_{\mathbb{C}}^{t_j}$ in the notation of (4.4). To φ_j we associate the representation $\sigma_j = [\cdot]^{l_j}|\cdot|_{\mathbb{C}}^{t_j}$ of $GL_1(\mathbb{C})$. In this way, we associate a tuple $(\sigma_1, \dots, \sigma_n)$ of representations to φ. If the complex numbers $t_1, \dots, t_n$ do not satisfy (4.3), we permute $(\sigma_1, \dots, \sigma_n)$ so that (4.3) ends up being satisfied. Using Theorem 4, we can then make the association

$$\varphi \longrightarrow \rho_{\mathbb{C}}(\varphi) = J(\sigma_1, \dots, \sigma_r) \tag{4.5}$$

and come to the following conclusion.

THEOREM 5 (Local Langlands Correspondence for $GL_n(\mathbb{C})$). *The association* (4.5) *is a well-defined bijection between the set of all equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{C}}$ and the set of all equivalence classes of irreducible admissible representations of* $GL_n(\mathbb{C})$.

The local L factor corresponding to a one-dimensional representation φ of $W_{\mathbb{C}}$ is

$$L(s, \varphi) = 2(2\pi)^{-(s+t+\frac{|l|}{2})}\Gamma(s+t+\tfrac{|l|}{2}) \quad \text{if } \varphi \text{ is given by } (l, t) \text{ in (4.4)}. \tag{4.6}$$

For φ reducible, $L(s, \varphi)$ is the product of the L factors of the irreducible constituents of φ.

Fix the additive character ψ of $\mathbb{C}$ in (1.5), so that twice the ordinary Lebesgue measure is the self-dual Haar measure on $\mathbb{C}$. The ε factors are given for φ one-dimensional by

$$\varepsilon(s, \varphi, \psi) = i^{|l|} \qquad \text{if } \varphi \text{ is given by } (l, t) \text{ in (4.4)}. \tag{4.7}$$

For φ reducible, $\varepsilon(s, \varphi, \psi)$ is the product of the L factors of the irreducible constituents of φ. As was noted in §3, formula (4.7) and the third part of (3.7) are connected by the fact that ε is "inductive in degree 0."

We can now define local factors $L(s, \rho)$ and $\varepsilon(s, \rho, \psi)$ for each irreducible admissible representation of $GL_n(\mathbb{C})$ by the rule

$$\left.\begin{aligned} L(s, \rho) &= L(s, \varphi) \\ \varepsilon(s, \rho, \psi) &= \varepsilon(s, \varphi, \psi) \end{aligned}\right\} \quad \text{if } \rho = \rho_{\mathbb{C}}(\varphi) \text{ in (4.5) and Theorem 5.} \tag{4.8}$$

Jacquet **[12]** proved the following result.

THEOREM 6. *The definition of $L(s, \rho)$ in* (4.8) *satisfies the defining conditions* (1.3) *and* (1.4) *for $L(s, \rho)$ over $\mathbb{C}$ in* §1, *and the two definitions of $\varepsilon(s, \rho, \psi)$ in* (4.8) *and* (1.8) *coincide.*

5. Results for other reductive groups

Some of the constructions and results in §§2–3 extend from GL_n to arbitrary connected reductive groups G defined over $\mathbb{R}$. For such a group G, we shall work in the context of the representation theory of $G(\mathbb{R})$. Langlands **[21]** reduced the classification of irreducible admissible representations of $G(\mathbb{R})$ to classification of the subset of irreducible tempered representations, which in turn were classified in **[16]**. Langlands **[21]** showed also that the classification for $G(\mathbb{R})$ fits into a framework that is consistent with what was described for $\mathrm{GL}_n(\mathbb{R})$ in §3. In this section we shall discuss aspects of that framework. For full details one can consult the excellent exposition by Borel **[1]**.

The first step is to introduce the L group of $G/\mathbb{R}$. The identity component ${}^LG^0$ is a certain complex reductive group depending only on G (not $G/\mathbb{R}$), having the same rank as G, and having root system dual to that of G. The exact definition requires some care, but it has the following features:

(a) if $G = \mathrm{GL}_n$, then ${}^LG^0 = \mathrm{GL}_n(\mathbb{C})$;
(b) if G is simply connected, then ${}^LG^0$ is an adjoint group;
(c) if G is an adjoint group, then ${}^LG^0$ is simply connected.

Now we bring in $\mathbb{R}$, letting $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. The definition of ${}^LG^0$ is such that the action of Γ on G yields an action of Γ on ${}^LG^0$. Then ${}^LG$ is defined as the corresponding semidirect product of ${}^LG^0$ with Γ. This semidirect product is a direct product if $G(\mathbb{R})$ is a split group, as is the case for $G(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})$.

By a **representation** of ${}^LG$ is meant any holomorphic homomorphism of ${}^LG$ into some $\mathrm{GL}_m(\mathbb{C})$. An element of ${}^LG$ is **semisimple** if its image is semisimple under every representation of ${}^LG$. Certain parabolic subgroups described in **[1]** are defined to be **relevant**; all parabolic subgroups of ${}^LG$ are relevant if $G(\mathbb{R})$ is split or quasisplit.

A continuous homomorphism $\varphi : W_{\mathbb{R}} \to {}^LG$ is said to be **admissible** if

(a) $\varphi(\mathbb{C}^\times) \subseteq {}^LG^0$, and $\varphi(j)$ is contained in the nontrivial coset of ${}^LG^0$ in ${}^LG$;
(b) $\varphi(W_{\mathbb{R}})$ is contained in the set of semisimple elements of ${}^LG$;
(c) whenever $\varphi(W_{\mathbb{R}})$ is contained in the Levi subgroup of a parabolic subgroup P of ${}^LG$, then P is relevant.

The set of such φ's, modulo the equivalence relation defined by conjugacy within ${}^LG^0$, is denoted $\Phi(G)$. For $G(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})$, Theorem 2 says that $\Phi(G(\mathbb{R}))$ parametrizes the set $\Pi(G(\mathbb{R}))$ of equivalence classes of irreducible admissible representations of $G(\mathbb{R})$. This statement needs adjustment for more general $G(\mathbb{R})$. What happens is that one associates to each admissible φ a finite subset Π_φ of $\Pi(G(\mathbb{R}))$ called an L **packet**. Inequivalent φ's lead to disjoint subsets, and the union of all Π_φ is all of $\Pi(G(\mathbb{R}))$.

Representations in the same Π_φ are called L-**indistinguishable**. Two phenomena contribute to this notion: discrete series with the same infinitesimal character, and reducibility of standard induced tempered representations. Shelstad **[27]** has quantified the statement that these are the only contributing factors, and she has explored the consequences of L-indistinguishability.

Local L and ε factors are associated not to irreducible admissible representations ρ of $G(\mathbb{R})$ but to pairs (ρ, r) in which r is a representation of ${}^LG$ in some $\mathrm{GL}_m(\mathbb{C})$. The definition is as follows: Given ρ, we find φ such that the class of ρ lies in Π_φ. Then $r \circ \varphi$ is an m-dimensional semisimple representation of $W_{\mathbb{R}}$, and we can put

$$L(s, \rho, r) = L(s, \varphi),$$
$$\varepsilon(s, \rho, r, \psi) = \varepsilon(s, \varphi, \psi),$$

in the notation of (3.6) and (3.7). When $G(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})$, these definitions reduce to (3.8) if r is the standard representation of $\mathrm{GL}_n(\mathbb{C})$.

In the Langlands program, one expects that the above local correspondence for $G(\mathbb{R})$ will be valid in some form over non-Achimedean local fields as well, that there will be a global theory, and that the local and global theories will mesh in the same way that they appear to mesh for GL_n. For accounts of progress in these matters, see Gelbart and Shahidi **[8]** and Shahidi **[26]**.

The full power of the Langlands program comes into play only when the functoriality of these constructions is considered. Almost all aspects of functoriality are still conjectural. A general statement of functoriality, accompanied by an overview of its consequences, is in Gelbart **[7]**. More detail may be found in Chapter V of Borel **[1]**.

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