MINIMAL K-TYPE FORMULA

A. W. Knapp*

In an effort to attach new invariants to group representations, D. A. Vogan introduced in [9] a notion of minimal or "lowest" K-types for representations of semisimple Lie groups and used it as a starting point for several deep investigations in representation theory. What we shall do here is to announce a simple formula for all the minimal K-types of the standard representations induced from parabolic subgroups MAN when the inducing data include a discrete series or nondegenerate limit of discrete series representation of M and when the total group is linear. If we anticipate that certain results of Vogan's extend to all of our representations, then it follows from Theorem 5 of [5] that we obtain a minimal K-type formula for all irreducible admissible representations of linear semisimple groups in terms of their Langlands parameters [7].

Some applications of our formula appear in the joint paper [4] with B. Speh.

Let G be a linear connected semisimple Lie group, let K be a maximal compact subgroup, and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. Fix a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{k}$, and let

$$\Delta_K = \{\text{roots of } (\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})\} \subseteq (i\mathfrak{b})'$$

$$\Delta_K^+ = \text{some positive root system for } \Delta_K$$

$$\rho_K = \text{half the sum of the members of } \Delta_K^+.$$

* Research supported by grants from the National Science Foundation through the University of Chicago and Cornell University.

To each dominant integral form Λ on $\mathfrak{b}^{\mathbb{C}}$, we associate the irreducible representation τ_Λ of K with highest weight Λ. We introduce an inner product $\langle\cdot,\cdot\rangle$ and a norm $|\cdot|$ on $(i\mathfrak{b})'$ in the usual way.

If π is an irreducible admissible representation of G, we say that τ_Λ is a <u>minimal K-type</u> of π if, among all irreducible representations $\tau_{\Lambda'}$ occurring in $\pi|_K$, $|\Lambda' + 2\rho_K|^2$ is minimized for $\Lambda' = \Lambda$. Existence of minimal K-types for any π is clear; there may be several. It is important to note that this notion is independent of the choice of the positive system Δ_K^+.

Let $P = MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup of G, let σ be a discrete series or nondegenerate limit of discrete series[1] representation of M, let ν be a complex-valued linear functional on the Lie algebra of A, and form the induced representation[2]

$$U(P,\sigma,\nu) = \mathrm{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1) . \qquad (0.1)$$

The minimal K-types of $U(P,\sigma,\nu)$ are independent of ν, and we shall give a formula for them. For the precise formula we need to define suitably compatible orderings for various root systems that occur. But if we ignore this difficulty for the moment, we can give the formula approximately. Disregarding the possible disconnectedness of M, let λ be the Blattner parameter of σ; this is the highest weight of the minimal $K \cap M$ type of σ. Then the minimal K-types τ_Λ of $U(P,\sigma,\nu)$ are given by

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[1] See §§1 and 12 of [6] for the definition and elementary properties of limits of discrete series and nondegeneracy.

[2] The notation refers to unitary induction with G operating on the left.

$$\Lambda = \lambda - E(2\rho_K) + 2\rho_{K_r} + \mu . \qquad (0.2)$$

Here E is the orthogonal projection to the subspace orthogonal to the Lie algebra of M. The term $2\rho_{K_r}$ refers to the $2\rho_K$ for a certain split subgroup G_r of G determined by G and M, and μ refers to any of several fine[3] K_r-types for G_r. In practice the group G_r is often locally just a product of copies of $SL(2,\mathbb{R})$, and μ is easy to understand; in principle G can be split and P can be a minimal parabolic subgroup, in which case $G_r = G$, $\Lambda = \mu$, and the formula gives no information.

The notation needed to make sense out of (0.2) and to define the compatible orderings is assembled in §1. The reader is asked to think first in terms of the case that rank G = rank K, where $\Delta_K \subseteq \Delta$ and where the other notation simplifies greatly. The precisely stated minimal K-type formula appears as Theorems 1 and 2 in §2, and Theorem 4 of §2 gives additional information about μ when G_r is locally a product of copies of $SL(2,\mathbb{R})$.

1. Notation

We continue with G, $\mathfrak{g}$, K, $\mathfrak{k}$, $\mathfrak{b}$, and Δ_K as in the introduction, but we postpone defining the positive system Δ_K^+. Let

θ = Cartan involution of $\mathfrak{g}$ determined by $\mathfrak{k}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: corresponding Cartan decomposition

$\mathfrak{t}$ = centralizer of $\mathfrak{b}$ in $\mathfrak{g}$.

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[3] See §1 below for a definition of "fine." The notion was introduced by Bernstein, Gelfand, and Gelfand [1] and developed further by Vogan [10]. We use some of Vogan's results.

Here $\mathfrak{t}$ is a maximally compact θ-stable Cartan subalgebra of $\mathfrak{g}$ and is of the form $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$, where $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. (This $\mathfrak{a}$ will usually not coincide with the Lie algebra of the group A in the introduction.) Let

$$B = \exp \mathfrak{b}$$

$$\Delta = \{\text{roots of } (\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})\}$$

$$\Delta_B = \{\text{roots of } (\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) \text{ vanishing on } \mathfrak{a}\}.$$

The root vectors for the members of Δ_B lie either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$, and we call the corresponding roots <u>compact</u> or <u>noncompact</u>, accordingly. Let

$$\Delta_{B,c} = \{\text{compact roots in } \Delta_B\}$$

$$\Delta_{B,n} = \{\text{noncompact roots in } \Delta_B\}.$$

One can show that restriction from $\mathfrak{t}^{\mathbb{C}}$ to $\mathfrak{b}^{\mathbb{C}}$ carries $\Delta - \Delta_{B,n}$ onto Δ_K; consequently we can regard $\Delta_{B,c}$ as a subset of Δ_K.

To characterize the M of our parabolic subgroup up to conjugacy, it is enough (by Harish-Chandra's construction in [2]) to specify a conjugacy class of θ-stable Cartan subalgebras in $\mathfrak{g}$, and this conjugacy class in turn is determined by specifying a sequence $\alpha_1, \ldots, \alpha_\ell$ of strongly orthogonal members of $\Delta_{B,n}$. (See §2 of Schmid [8] for an exposition.) Thinking of the effect of a Cayley transform, we say that

$$\text{a root in } \Delta \text{ is } \begin{cases} \underline{\text{real}} & \text{if in } \sum \mathbb{R}\alpha_j \oplus \mathfrak{a}' \\ \underline{\text{imaginary}} & \text{if orthogonal to } \sum \mathbb{R}\alpha_j \oplus \mathfrak{a}' \\ \underline{\text{complex}} & \text{otherwise.} \end{cases}$$

Let

$$\Delta_r = \{\text{real roots in } \Delta\}$$

$$\mathfrak{b}_r = \sum \mathbb{R} i H_{\alpha_j}\ , \quad \text{where } H_{\alpha_j} \text{ is dual in } \mathfrak{t}^{\mathbb{C}} \text{ to } \alpha_j$$

$$\mathfrak{b}_- = \text{orthocomplement in } \mathfrak{b} \text{ to } \mathfrak{b}_r\ , \quad \text{so that } \mathfrak{b} = \mathfrak{b}_- \oplus \mathfrak{b}_r$$

$$\mathfrak{t}_r = \mathfrak{b}_r \oplus \mathfrak{a}$$

$$E = \text{orthogonal projection of } (\mathfrak{t}^{\mathbb{C}})' \text{ onto } (\mathfrak{t}_r^{\mathbb{C}})'\ .$$

The subalgebra

$$\mathfrak{g}_r = \mathfrak{g} \cap (\mathfrak{t}_r^{\mathbb{C}} \oplus \sum_{\beta \in \Delta_r} \mathbb{C} X_\beta)$$

is a θ-stable reductive subalgebra of $\mathfrak{g}$ that is split over $\mathbb{R}$.

Let G_r be the analytic subgroup of G with Lie algebra $\mathfrak{g}_r$. The group $K_r = K \cap G_r$ is a maximal compact subgroup of G_r, and its Lie algebra is $\mathfrak{k}_r = \mathfrak{k} \cap \mathfrak{g}_r$. Moreover, $\mathfrak{b}_r$ is a maximal abelian subspace of $\mathfrak{k}_r$, $\mathfrak{t}_r$ is a maximally compact θ-stable Cartan subalgebra of $\mathfrak{g}_r$, and Δ_r is the root system of $(\mathfrak{g}_r^{\mathbb{C}}, \mathfrak{t}_r^{\mathbb{C}})$.

To obtain M, we build a Cayley transform $\underset{\sim}{c}$ out of the roots $\alpha_1, \ldots, \alpha_\ell$ and construct a new θ-stable Cartan subalgebra $\mathfrak{g} \cap \underset{\sim}{c}(\mathfrak{t}^{\mathbb{C}})$, as in [8]. Then we construct M and its Lie algebra $\mathfrak{m}$ in the standard way [2]. With

$$\Delta_- = \{\beta \in \Delta \mid \beta|_{\mathfrak{t}_r} = 0\}\ ,$$

$\mathfrak{m}$ is equal to the intersection of $\mathfrak{g}$ with

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{b}_-^{\mathbb{C}} \oplus \sum_{\beta \in \Delta_-} \mathbb{C}\, \underset{\sim}{c}(X_\beta)\ .$$

Each root vector $\underset{\sim}{c}(X_\beta)$ for $\mathfrak{m}^{\mathbb{C}}$ is either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$, and we call β <u>M-compact</u> or <u>M-noncompact</u> accordingly. Let

$$\Delta_{-,c} = \{\text{M-compact roots in } \Delta_-\}$$

$$\Delta_{-,n} = \{\text{M-noncompact roots in } \Delta_-\}\ .$$

Since $\mathfrak{t}_r \supseteq \mathfrak{a}$, we have $\Delta_- \subseteq \Delta_B$. However, $\Delta_{-,c}$ need not be contained in $\Delta_{B,c}$, since $\underset{\sim}{c}$ may move X_β from $\mathfrak{p}^{\mathbb{C}}$ to $\mathfrak{t}^{\mathbb{C}}$.

Every discrete series or limit of discrete series representation of M is known to be induced[4] from the subgroup

$$M^{\#} = M_e Z_M ,$$

where M_e is the identity component of M and Z_M is the center of M. The algebra $\mathfrak{b}_-$ is a compact Cartan subalgebra of $\mathfrak{m}$; let $B_- = \exp \mathfrak{b}_-$. By Lemma 2.1c of [6], we have

$$M^{\#} = M_e M_r , \tag{1.1}$$

where M_r is defined as the finite abelian group

$$M_r = F(B_-) = \operatorname{span}\{\gamma_{\underset{\sim}{c}(\beta)} \mid \beta \in \Delta \text{ and } \beta|_{\mathfrak{b}_-} = 0\}$$

$$= \operatorname{span}\{\gamma_{\underset{\sim}{c}(\beta)} \mid \beta \in \Delta_r\} .$$

The element $\gamma_{\underset{\sim}{c}(\beta)}$ is the element of G corresponding to the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in the $SL(2,\mathbb{R})$ subgroup built from the root $\underset{\sim}{c}(\beta)$. The group M_r is the M of a minimal parabolic subgroup of the split group G_r.

Let $\sigma^{\#}$ be a discrete series or nondegenerate limit of discrete series representation of $M^{\#}$. Because of (1.1), it follows from §1 of [6] that $\sigma^{\#}$ is determined by a triple (λ_0, C, χ), where

- λ_0 is a Harish-Chandra parameter of $\sigma^{\#}$ relative to $(\mathfrak{m}, \mathfrak{b}_-)$
- C is a Weyl chamber with respect to which λ_0 is dominant
- χ is the scalar $\sigma|_{M_r}$.

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[4] See §1 of [6] for an exposition in the discrete-series case and a proof in the limits-of-discrete-series case.

This triple will allow us to define compatible positive systems for the various root systems we have introduced.

Define $(\Delta_-)^+$ so that C is the dominant chamber, and define $\rho_{-,c}$ and $\rho_{-,n}$ as the corresponding half sums of positive members of $\Delta_{-,c}$ and $\Delta_{-,n}$. The Blattner parameter of $\sigma^\#$, given by

$$\lambda = \lambda_0 + \rho_{-,n} - \rho_{-,c} ,$$

has the property that the unique minimal $K \cap M^\#$ type of $\sigma^\#$ is

$$\sigma_\lambda = \begin{cases} \text{irreducible representation with} & \\ \quad \text{highest weight } \lambda & \text{on } K \cap M_e \\ \chi & \text{on } M_r . \end{cases}$$

(This follows from Theorem 1.3 of Hecht and Schmid [3].)

To define Δ^+, let

$$\Delta_1^+ = \{\beta \in \Delta \mid \langle \lambda_0, \beta \rangle > 0\}$$

$$\Delta_0 = \{\beta \in \Delta \mid \langle \lambda_0, \beta \rangle = 0\}$$

$$\mathfrak{t}_0 = \Big(\sum_{\beta \in \Delta_0} \mathbb{C} H_\beta \Big) \cap \mathfrak{g}$$

$$\Delta_{-,0} = \Delta_- \cap \Delta_0 .$$

Then $\Delta_{-,0}$ is generated by $(\Delta_-)^+$ simple roots $\varepsilon_1, \ldots, \varepsilon_k$. Since $\mathfrak{t}_0 \supseteq \mathfrak{b}_r \oplus \mathfrak{a}$, we have

$$\mathfrak{t}_0 = (\mathfrak{t}_0 \cap \mathfrak{b}_-) \oplus \mathfrak{b}_r \oplus \mathfrak{a} .$$

Therefore the following list provides an ordered basis of real elements in $(\mathfrak{t}_0^{\mathbb{C}})'$:

$$\varepsilon_1, \ldots, \varepsilon_k, \text{ orthogonal basis of remainder of } i(\mathfrak{t}_0 \cap \mathfrak{b}_-)',$$

$$\alpha_1, \ldots, \alpha_\ell, \text{ basis of } \mathfrak{a} . \tag{1.2}$$

We use this ordered basis to define a lexicographic ordering. This ordering defines Δ_0^+, and we take

$$\Delta^+ = \Delta_1^+ \cup \Delta_0^+ .$$

Then one can check that Δ^+ is a positive system with $(\Delta_-)^+ \subseteq \Delta^+$ and that

$$\Delta_K^+ = \{\delta \in \Delta_K \mid \delta = \text{restriction to } \mathfrak{b}^{\mathbb{C}} \text{ of a member of } \Delta^+ - \Delta_{B,n}\}$$

is a positive system for Δ_K.

Finally the inclusions $\Delta_r \subseteq \Delta$ and $\Delta_{K_r} \subseteq \Delta_K$ define Δ_r^+ and $\Delta_{K_r}^+$ for us, and these definitions are compatible within G_r with the above construction for passing from G_r to K_r.

If $\sigma = \mathrm{ind}_{M^\#}^{M}(\sigma^\#)$, then the representation (0.1) satisfies

$$U(P,\sigma,\nu)|_K = \mathrm{ind}_{K\cap M^\#}^{K}(\sigma^\#) ,$$

and we shall work with it in this form. Correspondingly the restriction to K_r of the nonunitary principal series of G_r induced from data including a character ω of M_r is

$$\mathrm{ind}_{M_r}^{K}(\omega) .$$

A minimal K_r-type τ_μ in this case is called a <u>fine</u> K_r-type; τ_μ contains no other characters of M_r besides ω and its conjugates by the Weyl group.

2. Results

Now we come to the theorems. Let $\sigma^\# \leftrightarrow (\lambda_0, C, \chi)$ be a discrete series or nondegenerate limit of discrete series representation of $M^\#$, and let the notation and orderings be as in §1.

Theorem 1. Every minimal K-type τ_Λ of $\mathrm{ind}_{K\cap M^\#}^{K}(\sigma^\#)$ has Λ of the form

$$\boxed{\Lambda = \lambda - E(2\rho_K) + 2\rho_{K_r} + \mu}\ , \tag{2.1}$$

where τ_μ is a fine K_r-type whose restriction to M_r contains the character

$$\omega = \chi \cdot \exp(E(2\rho_K) - 2\rho_{K_r})|_{M_r}\ ; \tag{2.2}$$

here $\exp(E(2\rho_K) - 2\rho_{K_r})$ is a well-defined one-dimensional representation of $K_r \supseteq M_r$. Conversely every fine K_r-type τ_μ with $\tau_\mu|_{M_r} \supseteq \omega$ is such that Λ in (2.1) is integral; if Λ is also Δ_K^+ dominant, then τ_Λ is a minimal K-type of $\mathrm{ind}_{K\cap M^\#}^{K}(\sigma^\#)$.

Generically Δ_0 is equal to Δ_r, and then Theorem 2 below says that every Λ defined by (2.1) is automatically Δ_K^+ dominant; in this case the minimal τ_Λ's and the fine τ_μ's are in one-one correspondence. In the exceptional cases when $\Delta_0 \supsetneq \Delta_r$, the fine μ's that lead to minimal Λ's are exactly those that satisfy certain conditions relative to the members of $\Delta_0 - \Delta_r$. The theorem uses the following notation: t_i denotes +1 or -1, α_i is a member of our strongly orthogonal set in $\Delta_{B,n}$, and ε is a member of $(\mathfrak{t}^{\mathbb{C}})'$ orthogonal to $(\mathfrak{t}_r^{\mathbb{C}})'$.

Theorem 2. If τ_μ is a fine K_r-type with $\tau_\mu|_{M_r}$ containing the character ω in (2.2), then the integral form Λ defined by (2.1) is Δ_K^+ dominant if and only if μ satisfies all of the following conditions:

(i) $2\langle\mu,\beta\rangle/|\beta|^2 > -1/2$ for each Δ_K^+ simple root β in $\Delta_0 - \Delta_r$ of the form $\beta = \varepsilon - \frac{1}{2}t\alpha$ such that $|\beta| = |\alpha|$ and also $\varepsilon - \frac{1}{2}\alpha$ and α are simple for Δ^+.

(ii) $2\langle\mu,\beta\rangle/|\beta|^2 > -1$ for each Δ_K^+ simple root β in $\Delta_0 - \Delta_r$ of the form $\beta = \varepsilon - \frac{1}{2}t_i\alpha_i - \frac{1}{2}t_j\alpha_j$ such that $|\beta| = |\alpha_i| = |\alpha_j|$, $\frac{1}{2}(\alpha_i + \alpha_j)$ is not in Δ, index i precedes index j for the ordering, $\varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$ and α_j are simple for Δ^+, and either $t_i = 1$ or α_i is simple for Δ^+.

(iii) $2\langle\mu,\beta\rangle/|\beta|^2 > -1$ for each Δ_K^+ simple root β in $\Delta_0 - \Delta_r$ of the form $\beta = \varepsilon - \frac{1}{2}t_j\alpha_j$ such that $2|\beta|^2 = |\alpha_j|^2$, $\varepsilon - \frac{1}{2}\alpha_1$ is simple for Δ^+ when α_1 is the first α such that $\frac{1}{2}(\alpha + \alpha_j)$ is in Δ, and either $t_j = 1$ or α_j is simple for Δ^+.

Prototypes for the situations described in (i), (ii), and (iii) occur with the minimal parabolic subgroup of SU(2,1) in the case of (i), the minimal parabolic subgroup of SU(2,2) in the case of (ii), and the maximal parabolic subgroup of $Sp(2,\mathbb{R})$ with nonabelian N in the case of (iii). Case (iii) may be dropped from the theorem if $\sigma^{\#}$ is a genuine discrete series representation.

The proofs of the two theorems are straightforward but rather long. One proves the integrality first, and then the long step is Theorem 2. Next one constructs some μ satisfying the conditions in Theorem 2, and the rest is comparatively easy. We isolate from the proof one key lemma, which we shall use elsewhere.

<u>Lemma 3</u>. $2(\rho_K - \rho_{-,c}) = \rho - \rho_- - \rho_r + E(2\rho_K)$.

We conclude with some information about μ. It is always true that μ is a linear combination of the α_j's with coefficients 0, $\frac{1}{2}$, or $-\frac{1}{2}$. When G_r is locally a product of copies of $SL(2,\mathbb{R})$, i.e., when $\mathfrak{a} = 0$ and Δ_r is a product of root systems A_1, we can be more precise. This condition on G_r is satisfied, for example, whenever the restricted roots of G form a system of type $(BC)_n$.

For each α_j let ρ_{α_j} be half the sum of the roots in Δ whose inner product with α_j is > 0 and whose inner product with all other α_k is $= 0$.

Theorem 4. Suppose G_r is locally a product of copies of $SL(2,\mathbb{R})$. If τ_μ is a fine K_r-type with $\tau_\mu|_{M_r}$ containing the character ω in (2.2), then μ is of the form

$$\mu = \Sigma\, s_j \alpha_j, \quad s_j = \pm \tfrac{1}{2}, \tag{2.3}$$

with the sum extended over exactly those j for which

$$\chi(\gamma_{\alpha_j}) = (-1)^{2\langle \rho_{\alpha_j}, \alpha_j \rangle / |\alpha_j|^2}. \tag{2.4}$$

Moreover, every choice of signs in (2.3) leads to another such μ.

There is a mnemonic for this result. To each α_j, §7 of [6] associates[5] a "Plancherel factor" μ_{σ,α_j}. When (2.4) holds, μ_{σ,α_j} is the product of a polynomial and a cotangent; when (2.4) fails, μ_{σ,α_j} is the product of a polynomial and a tangent. Consequently Theorem 4 says that each cotangent-type α_j contributes to the fine K_r-type μ in a pair of ways, via coefficients $s_j = \pm \frac{1}{2}$, while the tangent-type α_j's contribute uniquely via coefficient $s_j = 0$.

It is known from Theorem 12.6 of [6] that reducibility of $U(P,\sigma,0)$ arises when these Plancherel factors fail to vanish at the origin. Theorems 2 and 4 say that this same phenomenon accounts for multiple minimal K-types of $U(P,\sigma,0)$. When σ is a discrete series representation, Theorem 1.4 of Vogan [9] explains this correspondence.

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5 See also §10 and Corollary 12.5 of [6].

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Department of Mathematics
Cornell University
Ithaca, New York 14853, U.S.A.