REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 1, Pages 1–24 (November 4, 1996)
S 1088-4165(97)00001-0

EXCEPTIONAL UNITARY REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

A. W. KNAPP

ABSTRACT. Let G be a noncompact simple Lie group with finite center, let K be a maximal compact subgroup, and suppose that rank $G =$ rank K. If G/K is not Hermitian symmetric, then a theorem of Borel and de Siebenthal gives the existence of a system of positive roots relative to a compact Cartan subalgebra so that there is just one noncompact simple root and it occurs exactly twice in the largest root. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ stable parabolic obtained by building $\mathfrak{l}$ from the roots generated by the compact simple roots and by building $\mathfrak{u}$ from the other positive roots, and let $L \subseteq K$ be the normalizer of $\mathfrak{q}$ in G. Cohomological induction of an irreducible representation of L produces a discrete series representation of G under a dominance condition. This paper studies the results of this cohomological induction when the dominance condition fails. When the inducing representation is one-dimensional, a great deal is known about when the cohomologically induced representation is infinitesimally unitary. This paper addresses the question of finding Langlands parameters for the natural irreducible constituent of these representations, and also it finds some cases when the inducing representation is higher-dimensional and the cohomologically induced representation is infinitesimally unitary.

Let G be a simple Lie group with finite center, let K be a maximal compact subgroup, and suppose that rank $G =$ rank K. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification. In this paper we investigate some representations of G first studied in [EPWW] that are closely related to a fundamental kind of discrete series representations of G. We are especially interested in the Langlands parameters associated to these representations. We begin with some background.

In an effort to find unusual irreducible unitary representations of G in the case that G/K is Hermitian symmetric, Wallach [W1] studied "analytic continuations of holomorphic discrete series." When G/K is Hermitian symmetric, $\mathfrak{p}$ splits as the direct sum of two abelian subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$, and $\mathfrak{k} \oplus \mathfrak{p}^+$ is a parabolic subalgebra of $\mathfrak{g}$. An irreducible representation τ_Λ of K leads via this parabolic subalgebra to a generalized Verma module that is a $(\mathfrak{g}, K)$ module. If the highest weight Λ of τ_Λ satisfies suitable inequalities, this $(\mathfrak{g}, K)$ module arises from a holomorphic discrete series representation [HC1]. Wallach [W1] studied the "scalar case," in which τ_Λ is one-dimensional. By adjusting Λ on the center of $\mathfrak{k}$, he was able to move Λ in a one-parameter family. For values of Λ outside the range that yields holomorphic discrete series, he determined necessary and sufficient conditions for the unique irreducible quotient of the generalized Verma module to be infinitesimally unitary.

Received by the editors June 19, 1996 and, in revised form, August 5, 1996.
1991 *Mathematics Subject Classification.* Primary 22E46, 22E47.
Presented to the Society August 7, 1995 at the AMS Summer Meeting in Burlington, Vermont.

Later Enright-Howe-Wallach [EHW] and Jakobsen [Ja] independently generalized this study to the "vector case," in which τ_Λ is allowed to be higher-dimensional, and they obtained a similar classification.

With hindsight one could adjust this construction a little, relating it to cohomological induction, and then related constructions become apparent. In the adjusted construction, one forms a generalized Verma module from $\mathfrak{k} \oplus \mathfrak{p}^-$ and a translate of the parameter Λ. The result is the 0^{th} cohomological induction functor

$$\mathcal{L}_0(F) = \operatorname{ind}_{\mathfrak{k}\oplus\mathfrak{p}^-,K}^{\mathfrak{g},K}(F \otimes \textstyle\bigwedge^{\text{top}}\mathfrak{p}^+),$$

where F is an irreducible representation of K whose highest weight is a translate of Λ. (See [Kn-Vo, p. 328].) A special feature of this situation is that, relative to a compact Cartan subalgebra of $\mathfrak{g}_0$, there is just one noncompact simple root, and it occurs just once in the largest root.

Enright-Parthasarathy-Wallach-Wolf [EPWW] undertook a parallel study of the situation in which there is just one noncompact simple root β_0 and it occurs exactly twice in the largest root. This situation is rich with examples: According to a theorem of Borel and de Siebenthal [Bo-deS], any G with rank G = rank K and G/K not Hermitian has a positive system of roots with this property. Following the line of [EPWW] but not the notation, we define L to be the subgroup of K built from the simple roots that are compact, and we let $\mathfrak{u}$ be the sum of the root spaces in $\mathfrak{g}$ for the positive roots requiring β_0 in their expansions. Then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ stable parabolic subalgebra in the sense of [Kn-Vo, §IV.6], and the representations to study are

$$\mathcal{L}_S(F) = (\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S(\operatorname{ind}_{\overline{\mathfrak{q}},L}^{\mathfrak{g},L}(F \otimes \textstyle\bigwedge^{\text{top}}\mathfrak{u})),$$

where F is an irreducible representation of L with highest weight λ, $\overline{\mathfrak{q}}$ is the opposite parabolic of $\mathfrak{q}$, the $\mathfrak{l}$ module $F \otimes \bigwedge^{\text{top}}\mathfrak{u}$ is extended to a $\overline{\mathfrak{q}}$ module by having the nilpotent radical act by 0, S is $\dim(\mathfrak{u} \cap \mathfrak{k})$, and $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S$ is the S^{th} derived functor of the Bernstein functor [Kn-Vo, p. 106].

The work of [EPWW] was a forerunner of the Unitarizability Theorem, which was proved by Vogan [Vo2] and reproved in the spirit of [EPWW] by Wallach [W2]. [EPWW] proved that $\mathcal{L}_S(F)$ is infinitesimally unitary if $\operatorname{ind}_{\overline{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes (\bigwedge^{\text{top}}\mathfrak{u})^{t+1})$ is irreducible for all $t \geq 0$. In particular, [EPWW] found that this condition is satisfied in the "scalar case" (i.e., F one-dimensional) if the infinitesimal character $\lambda + \delta$ of $\mathcal{L}_S(F)$ is dominant, where δ is half the sum of the positive roots. (The range where the infinitesimal character of $\mathcal{L}_S(F)$ is dominant is called the "weakly good" range in [Kn-Vo].) The general Unitarizability Theorem of [Vo2] and [W2], which came later despite the dates on the papers, proved that unitarity is always preserved by cohomological induction in the weakly good range.

For the scalar case, [EPWW] determined exactly when $\operatorname{ind}_{\overline{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes (\bigwedge^{\text{top}}\mathfrak{u})^{t+1})$ is irreducible for all $t \geq 0$. They stated no result for F higher-dimensional (the "vector case"), but they had the tools to observe that weakly good implies infinitesimally unitary for this situation.

A part of $\mathcal{L}_S(F)$ can be infinitesimally unitary even outside the range when $\operatorname{ind}_{\overline{\mathfrak{q}}}^{\mathfrak{g}}(F \otimes (\bigwedge^{\text{top}}\mathfrak{u})^{t+1})$ is irreducible for all $t \geq 0$. In their study, [EPWW] assumed that $\Lambda = \lambda + 2\delta_n$ is dominant for K, where $2\delta_n$ is the sum of the positive noncompact roots. Then the K type Λ occurs with multiplicity 1 in $\mathcal{L}_S(F)$, and it makes sense to consider the unique irreducible subquotient of $\mathcal{L}_S(F)$ containing the K type Λ. Various authors (see [Ba], [Bi-Z], [Br-Ko], [G-W], [Ka-S], [McG1], [McG2]) have found additional scalar cases beyond the range treated by [EPWW] where this irreducible subquotient is infinitesimally unitary.

In this paper we undertake a further investigation of the representations studied by [EPWW], maintaining the assumption that $\Lambda = \lambda + 2\delta_n$ is K dominant. Our main interest is in the Langlands parameters of the irreducible subquotient of $\mathcal{L}_S(F)$ containing the K type Λ, so that this representation is located in the classification of all irreducible admissible representations of G. After preliminaries in §§1–2, we prove in §3 a strong vanishing theorem for $\mathcal{L}_j(F)$ when $j \neq S$. One consequence is a formula for the multiplicities of the K types in $\mathcal{L}_S(F)$ that involves no cancellation. Another consequence is that Λ is the unique minimal K type of $\mathcal{L}_S(F)$ in the scalar case; in the vector case it need not be. In §4 we establish some unitarity of the vector case of $\mathcal{L}_S(F)$ outside the weakly good range.

Finally in §5 we use combinatorial methods to address Langlands parameters. We show by example that these methods lead to ambiguous results in the vector case. Thus we concentrate on the scalar case, where we conjecture a natural algorithm for computing these parameters. The algorithm has the property that one can see through to the answer without computation when G is classical; this algorithm is different from the one given by Vogan in [Vo1, Proposition 4.1]. We show that the algorithm gives the correct Langlands parameters when G is classical. The line of proof works at least sometimes when G is exceptional, but we have not carried it through in general.

1. Preliminary Identities with $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ Cohomology

Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup. We denote corresponding Lie algebras by the corresponding Gothic letters with subscripts 0, and we denote complexifications by dropping the subscripts. Let θ be the Cartan involution of $\mathfrak{g}_0$ corresponding to K, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the associated Cartan decomposition. Let $(\mathfrak{g}, K)$ be a reductive pair built from G and K as in [Kn-Vo, §IV.3], and let $\mathcal{C}(\mathfrak{g}, K)$ be the category of all $(\mathfrak{g}, K)$ modules.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ stable parabolic subalgebra of $\mathfrak{g}$ in the sense of [Kn-Vo, §IV.6]; here $\mathfrak{l}$ is the theta stable Levi factor, and $\mathfrak{u}$ is the nilpotent radical. The normalizer L of $\mathfrak{q}$ in G is connected and has Lie algebra $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$.

We let bar denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then $\overline{\mathfrak{q}} = \theta\overline{\mathfrak{q}}$, $\overline{\mathfrak{u}} = \theta\overline{\mathfrak{u}}$, and $\overline{\mathfrak{q}}$ and $\mathfrak{q}$ are opposite parabolic subalgebras. We use bar also to stand for the passage of a module to its conjugate. If V is a $(\mathfrak{q} \cap \mathfrak{k}, L \cap K)$ module, then §VI.2 of [Kn-Vo] shows that the conjugate module $\overline{V}$ is naturally a $(\overline{\mathfrak{q}} \cap \mathfrak{k}, L \cap K)$ module.

We shall make use of the invariants functor from $\mathcal{C}(\mathfrak{q} \cap \mathfrak{k}, L \cap K)$ to $\mathcal{C}(\mathfrak{l} \cap \mathfrak{k}, L \cap K)$ that is usually written as $V \mapsto V^{\mathfrak{u} \cap \mathfrak{k}}$. Proposition 3.12 of [Kn-Vo] shows that the derived functors are the functors from $\mathcal{C}(\mathfrak{q} \cap \mathfrak{k}, L \cap K)$ to $\mathcal{C}(\mathfrak{l} \cap \mathfrak{k}, L \cap K)$ that are usually written $V \mapsto H^j(\mathfrak{u} \cap \mathfrak{k}, V)$.

Lemma 1. *The two functors F_1 and F_2 from $\mathcal{C}(\mathfrak{q} \cap \mathfrak{k}, L \cap K)$ to $\mathcal{C}(\mathfrak{l} \cap \mathfrak{k}, L \cap K)$ given by*

$$F_1(V) = \overline{V^{\mathfrak{u} \cap \mathfrak{k}}} \qquad \text{and} \qquad F_2(V) = (\overline{V})^{\overline{\mathfrak{u}} \cap \mathfrak{k}}$$

are naturally isomorphic. Consequently

$$\overline{H^j(\mathfrak{u} \cap \mathfrak{k}, V)} \cong H^j(\overline{\mathfrak{u}} \cap \mathfrak{k}, \overline{V})$$

as $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ *modules whenever* V *is a* $(\mathfrak{q}\cap\mathfrak{k}, L\cap K)$ *module.*

Proof. The first statement is immediate from the definition of the action on $\overline{V}$ given in [Kn-Vo]. We have noted that the functor from $\mathcal{C}(\mathfrak{q}\cap\mathfrak{k}, L\cap K)$ to $\mathcal{C}(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ given by $H^j(\mathfrak{u}\cap\mathfrak{k},\,\cdot\,)$ is the j^{th} derived functor of $(\,\cdot\,)^{\mathfrak{u}\cap\mathfrak{k}}$, and similar remarks apply to $\overline{\mathfrak{u}}\cap\mathfrak{k}$. Let us compute the derived functors of F_1 and F_2, which are each compositions. Since bar is exact, (C.27a) of [Kn-Vo] shows that the derived functors of F_1 are $\text{bar}\circ H^j(\mathfrak{u}\cap\mathfrak{k},\,\cdot\,)$. Since bar is exact and sends injectives to injectives, (C.28a2) of [Kn-Vo] shows that the derived functors of F_2 are $H^j(\overline{\mathfrak{u}}\cap\mathfrak{k},\,\cdot\,)\circ\text{bar}$. The natural isomorphism of F_1 with F_2 yields natural isomorphisms of the respective derived functors, and the lemma follows.

Proposition 2. *If* V *is a finite-dimensional* $(\mathfrak{k}, K)$ *module, then*

$$H_j(\mathfrak{u}\cap\mathfrak{k}, V)\cong H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, V)$$

as $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ *modules for every* $j\geq 0$.

Remark. This result is given in [Kn-Vo, Lemma 4.82] for the case $j=0$, but the general case does not appear in [Kn-Vo].

Proof. Let $(\,\cdot\,)^c$ denote contragredient, and let $(\,\cdot\,)^h$ denote Hermitian dual, which is defined as the composition of bar and contragredient. Every finite-dimensional $(\mathfrak{k}, K)$ module is infinitesimally unitary and hence is isomorphic with its Hermitian dual, and a similar remark applies to $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ modules. Then we have isomorphisms on the level of $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ modules, given by

$$\begin{aligned}
H_j(\mathfrak{u}\cap\mathfrak{k}, V) &\cong H_j(\mathfrak{u}\cap\mathfrak{k}, V)^h && \\
&\cong \overline{H_j(\mathfrak{u}\cap\mathfrak{k}, V)^c} && \text{by definition of } (\,\cdot\,)^h\\
&\cong \overline{H^j(\mathfrak{u}\cap\mathfrak{k}, V^c)} && \text{by [Kn-Vo, Theorem 3.1]}\\
&\cong H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, \overline{V^c}) && \text{by Lemma 1}\\
&\cong H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, V) && \text{since } V\cong V^h.
\end{aligned}$$

Corollary 3. *Let* $S=\dim(\mathfrak{u}\cap\mathfrak{k})$. *If* V *is a* $(\mathfrak{k}, K)$ *module, then*

$$H^{S-j}(\mathfrak{u}\cap\mathfrak{k}, V)\cong H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, V)\otimes{\textstyle\bigwedge}^S(\overline{\mathfrak{u}}\cap\mathfrak{k})$$

as $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ *modules for every* $j\geq 0$.

Proof. This follows by combining Proposition 2 and Hard Duality [Kn-Vo, Corollary 3.13].

2. Setting

We shall now specialize from the generality of §1 to the setting of this paper. We assume throughout that G is simple and that $\operatorname{rank} G=\operatorname{rank} K$. Let $T\subseteq K$ be a Cartan subgroup, and let $\Delta=\Delta(\mathfrak{g},\mathfrak{t})$ be the set of roots. We introduce in the usual way an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and a norm squared $|\,\cdot\,|^2$ on the real linear span of the roots. Each root vector lies in $\mathfrak{k}$ or in $\mathfrak{p}$, and roots are called compact or noncompact accordingly. Let $\Delta_K=\Delta(\mathfrak{k},\mathfrak{t})$ be the set of compact roots.

Fix a positive system $\Delta^+=\Delta^+(\mathfrak{g},\mathfrak{t})$. The **key assumption** is that there is exactly one noncompact simple root β_0 and that β_0 has multiplicity 2 in the largest root. The largest root and the simple roots that are compact then span the dual $\mathfrak{t}^*$. (In particular, K is semisimple and G/K is not Hermitian.)

Let Δ_L be the subset of roots not requiring β_0 for their expansions in terms of simple roots, and let $\Delta_L^+ = \Delta_L \cap \Delta^+$. Then we can define a θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ by requiring that $\mathfrak{l}$ is spanned by $\mathfrak{t}$ and the root vectors for the members of Δ_L while $\mathfrak{u}$ is spanned by the root vectors for the positive roots not in Δ_L^+. As in §2 we let L be the normalizer of $\mathfrak{q}$ in G. We have $\mathfrak{l} \subseteq \mathfrak{k}$ and $L \subseteq K$.

Define $\Delta(\mathfrak{u})$ to be the roots contributing to $\mathfrak{u}$, and let $\Delta(\mathfrak{u}\cap\mathfrak{k})$ and $\Delta(\mathfrak{u}\cap\mathfrak{p})$ be the subsets of compact and noncompact roots in $\Delta(\mathfrak{u})$. Let δ, δ_K, δ_L, $\delta(\mathfrak{u})$, $\delta(\mathfrak{u}\cap\mathfrak{k})$, and $\delta(\mathfrak{u}\cap\mathfrak{p})$ be the half sums of the members of Δ^+, Δ_K^+, Δ_L^+, $\Delta(\mathfrak{u})$, $\Delta(\mathfrak{u}\cap\mathfrak{k})$, and $\Delta(\mathfrak{u}\cap\mathfrak{p})$, respectively.

When a positive root γ is expanded in terms of simple roots, the coefficient of β_0 is 0, 1, or 2 because of the key assumption above. The coefficient is therefore 1 if and only if γ is noncompact. The key assumption will play a role in the results of the next section through the following lemma.

Lemma 4. *If ε is in $\Delta(\mathfrak{u}\cap\mathfrak{p})$ and α is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$, then $\langle\varepsilon,\alpha\rangle \geq 0$.*

Remark. [EPWW] calls a general θ stable parabolic subalgebra $\mathfrak{q}$ "quasi abelian" when the property in this lemma holds. The setting of this section is the subject of [EPWW, §13].

Proof. If not, then $\langle\varepsilon,\alpha\rangle < 0$, and it follows that $\varepsilon+\alpha$ is a root. When $\varepsilon+\alpha$ is expanded in terms of simple roots, the coefficient of β_0 has to be 3, since ε yields coefficient 1 and α yields coefficient 2. Coefficient 3 is not allowed by our assumptions on Δ^+ and β_0, and we have a contradiction.

Let F be an $(\mathfrak{l}, L)$ module. We shall be interested in the cohomological induction functors $\mathcal{L}_j : \mathcal{C}(\mathfrak{l}, L) \to \mathcal{C}(\mathfrak{g}, K)$ defined by

$$\mathcal{L}_j(F) = \Pi_j(\operatorname{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L}(F \otimes \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u})),$$

where Π_j is the j^{th} derived functor of the Bernstein functor $\Pi = \Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$ defined in [Kn-Vo, p. 106]. The interesting degree for cohomology is $j = S$, where $S = \dim(\mathfrak{u}\cap\mathfrak{k})$ as in Corollary 3. The number S is the complex dimension of the complex manifold K/L.

Suppose that the $(\mathfrak{l}, L)$ module F is irreducible with highest weight λ, hence with infinitesimal character $\lambda+\delta_L$. Corollary 5.25 of [Kn-Vo] shows that the $(\mathfrak{g}, K)$ module $\mathcal{L}_j(F)$ has infinitesimal character $(\lambda+\delta_L)+\delta(\mathfrak{u}) = \lambda+\delta$. If $\langle\lambda+\delta,\gamma\rangle > 0$ for every $\gamma \in \Delta(\mathfrak{u})$, then the parameter λ is said to be in the **good range**. In this case it is well known (and it is proved in [Kn-Vo]) that $\mathcal{L}_j(F) = 0$ for $j \neq S$ and that $\mathcal{L}_S(F)$ is a discrete series $(\mathfrak{g}, K)$ module with Harish-Chandra parameter $\lambda+\delta$. The unique minimal K type parameter of $\mathcal{L}_S(F)$ (called "lowest" in [Vo1]) is

$$\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}) = (\lambda+\delta) + \delta - 2\delta_K.$$

In the special case that F is one-dimensional with unique weight λ, we write $F = \mathbb{C}_\lambda$. The module $\mathcal{L}_S(\mathbb{C}_\lambda)$ is commonly denoted $A_{\mathfrak{q}}(\lambda)$ in the literature and is known as a Zuckerman module.

When λ is in the **weakly good range** (i.e., $\langle\lambda+\delta,\gamma\rangle \geq 0$ for every $\gamma \in \Delta(\mathfrak{u})$), $\mathcal{L}_j(F) = 0$ for $j \neq S$ and $\mathcal{L}_S(F)$ is a discrete series or limit of discrete series. Our interest in this paper is in the modules $\mathcal{L}_j(F)$ when the parameter λ goes outside the weakly good range. We shall always assume that the parameter $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is Δ_K^+ dominant. Under this assumption, Theorem 5.80a of [Kn-Vo] shows that the K type Λ occurs in $\mathcal{L}_S(F)$ with multiplicity 1.

The modules $A_{\mathfrak{q}}(\lambda)$ with λ outside the weakly good range have been studied by Enright-Parthasarathy-Wallach-Wolf [EPWW]. Using continuity arguments with generalized Verma modules, they established that a number of these modules are infinitesimally unitary.

A later theorem of Vogan (given in [Vo2] originally and appearing in [Kn-Vo] as Corollary 9.70) implies that $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary if λ is in the **weakly fair range**, i.e., λ is orthogonal to Δ_L and has $\langle \lambda + \delta(\mathfrak{u}), \gamma\rangle \geq 0$ for all $\gamma \in \Delta(\mathfrak{u})$.

In this paper we shall study the vanishing of $\mathcal{L}_j(F)$ for $j \neq S$, the K spectrum of $\mathcal{L}_S(F)$, some unitarity of $\mathcal{L}_S(F)$ that can be obtained from continuity arguments, and the Langlands parameters of the irreducible subquotient of $\mathcal{L}_S(F)$ containing the K type Λ.

3. K Spectrum

We work in the setting of §2. Let $S(\mathfrak{u}\cap\mathfrak{p})$ be the symmetric algebra of $\mathfrak{u}\cap\mathfrak{p}$, and let $S^n(\mathfrak{u}\cap\mathfrak{p})$ be the subspace of elements homogeneous of degree n.

Lemma 5. *If γ is any L highest weight in $S^n(\mathfrak{u}\cap\mathfrak{p})$, then γ is Δ_K^+ dominant.*

Proof. Let α be a Δ_K^+ simple root. If α is in Δ_L, then $\langle\gamma,\alpha\rangle \geq 0$ by the Δ_L^+ dominance of γ. If α is not in Δ_L, then α is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$. Suppose that $\langle\gamma,\alpha\rangle < 0$. Since γ is a weight of $S^n(\mathfrak{u}\cap\mathfrak{p})$, γ is the sum of members of $\Delta(\mathfrak{u}\cap\mathfrak{p})$, and there must exist $\varepsilon \in \Delta(\mathfrak{u}\cap\mathfrak{p})$ with $\langle\varepsilon,\alpha\rangle < 0$. We obtain a contradiction to Lemma 4, and we conclude that γ is Δ_K^+ dominant.

Theorem 6. *Let V be an irreducible representation of K, and let F be an irreducible representation of L whose highest weight is Δ_K^+ dominant. Then*

$$\mathrm{Hom}_L(H^j(\mathfrak{u}\cap\mathfrak{k}, V), S^n(\mathfrak{u}\cap\mathfrak{p})\otimes F) = 0$$

for all $j > 0$ and all $n \geq 0$.

Proof. Let Λ' be the highest weight of V, and let W_K^1 be the set of all w in the Weyl group of Δ_K such that the conditions $\alpha \in \Delta_K^+$ and $w^{-1}\alpha < 0$ can happen only if α is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$. Arguing by contradiction, suppose that the Hom in the statement of the theorem is nonzero. Then Kostant's Theorem [Kn-Vo, Theorem 4.139] implies that there is some $w \in W_K^1$ of length j such that $w(\Lambda'+\delta_K)-\delta_K$ is a Δ_L^+ highest weight of $S^n(\mathfrak{u}\cap\mathfrak{p})\otimes F$. Any highest weight of $S^n(\mathfrak{u}\cap\mathfrak{p})\otimes F$ is, by [Kn2, Problem 16 on p. 285 and p. 554] the sum of a weight γ of $S^n(\mathfrak{u}\cap\mathfrak{p})$ and the highest weight Λ of F. Thus we must have

$$w(\Lambda'+\delta_K)-\delta_K = \gamma+\Lambda.$$

Since $j > 0$ and w has length j, w is not 1. Then there exists a root $\alpha \in \Delta_K^+$ such that $w^{-1}\alpha < 0$. Since w is in W_K^1, α is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$. Taking the inner product of α with the equation

$$w(\Lambda'+\delta_K) = \gamma + (\Lambda+\delta_K),$$

we obtain

$$\langle\Lambda'+\delta_K, w^{-1}\alpha\rangle = \langle\gamma,\alpha\rangle + \langle\Lambda+\delta_K,\alpha\rangle.$$

The left side is < 0 since $\Lambda' + \delta_K$ is Δ_K^+ dominant nonsingular and since $w^{-1}\alpha$ is < 0. The first term on the right is ≥ 0 by Lemma 4 since γ is a sum of members of $\Delta(\mathfrak{u} \cap \mathfrak{p})$, and the second term on the right is > 0 since $\Lambda + \delta_K$ is Δ_K^+ dominant nonsingular. Thus we have a contradiction, and the proof is complete.

Corollary 7. *Let F be an irreducible representation of L whose highest weight λ has the property that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is Δ_K^+ dominant.*

(a) *If $j \neq S$, then $\mathcal{L}_j(F) = 0$.*

(b) *If Λ' is any Δ_K^+ dominant integral form, then the multiplicity of the K type Λ' in $\mathcal{L}_S(F)$ equals the multiplicity of the L type Λ' in the $(\mathfrak{l}, L)$ module $S(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \bigwedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p})$.*

Proof. Let $F' = F \otimes \bigwedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p})$ as an $(\mathfrak{l}, L)$ module. By Theorem 6,

$$\mathrm{Hom}_L(S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes F', H^{S-j}(\mathfrak{u} \cap \mathfrak{k}, V)) = 0$$

for $0 \leq j < S$ and for $n \geq 0$. Substituting from Corollary 3, we have

$$\mathrm{Hom}_L(S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes F', H^j(\overline{\mathfrak{u}} \cap \mathfrak{k}, V) \otimes \textstyle\bigwedge^{\mathrm{top}}(\overline{\mathfrak{u}} \cap \mathfrak{k})) = 0.$$

Therefore

$$\mathrm{Hom}_L(S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u}, H^j(\overline{\mathfrak{u}} \cap \mathfrak{k}, V)) = 0.$$

By Theorem 5.35a of [Kn-Vo], $\mathcal{L}_j(F) = 0$ for $j \neq S$. This proves (a).

Since $\mathcal{L}_j(F) = 0$ for $j \neq S$, Theorem 5.64 of [Kn-Vo] gives

$$\begin{aligned}(-1)^S &\dim \mathrm{Hom}_K(\mathcal{L}_S(F), V) \\ &= \sum_{j=0}^{S}(-1)^j \sum_{n=0}^{\infty} \dim \mathrm{Hom}_L(S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u}, H^j(\overline{\mathfrak{u}} \cap \mathfrak{k}, V)).\end{aligned}$$

We have just seen that the terms on the right side are 0 for $j < S$, and therefore

$$\dim \mathrm{Hom}_K(\mathcal{L}_S(F), V) = \sum_{n=0}^{\infty} \dim \mathrm{Hom}_L(S^n(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u}, H^S(\overline{\mathfrak{u}} \cap \mathfrak{k}, V)).$$

Corollary 3 shows that the right side is

$$\begin{aligned}&= \dim \mathrm{Hom}_L(S(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u}, H^0(\mathfrak{u} \cap \mathfrak{k}, V) \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{k})) \\ &= \dim \mathrm{Hom}_L(S(\mathfrak{u} \cap \mathfrak{p}) \otimes F \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p}), V^{\mathfrak{u} \cap \mathfrak{k}}).\end{aligned}$$

Since $V^{\mathfrak{u} \cap \mathfrak{k}}$ is an irreducible $(\mathfrak{l}, L)$ module of type Λ', conclusion (b) follows.

Corollary 8. *Let $\mathbb{C}_\lambda$ be a one-dimensional representation of L whose unique weight λ has the property that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is Δ_K^+ dominant.*

(a) *The K type Λ is the unique minimal K type of $A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$.*

(b) *If Λ' is any Δ_K^+ dominant integral form, then the multiplicity of the K type Λ' in $A_{\mathfrak{q}}(\lambda)$ equals the multiplicity of the L type $\Lambda' - \Lambda$ in $S(\mathfrak{u} \cap \mathfrak{p})$.*

Proof. Let V be an irreducible $(\mathfrak{k}, K)$ module of type Λ'. By Corollary 7b the multiplicity of the K type Λ' in $A_{\mathfrak{q}}(\lambda)$ is

$$\begin{aligned}&= \dim \mathrm{Hom}_L(S(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})}, V^{\mathfrak{u} \cap \mathfrak{k}}) \\ &= \dim \mathrm{Hom}_L(S(\mathfrak{u} \cap \mathfrak{p}), V^{\mathfrak{u} \cap \mathfrak{k}} \otimes \mathbb{C}_{-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p})}).\end{aligned}$$

This proves (b). If this multiplicity is positive, then $\Lambda' - \Lambda$ is an L highest weight γ in $S(\mathfrak{u}\cap\mathfrak{p})$. Hence

$$\Lambda' + 2\delta_K = (\Lambda + 2\delta_K) + \gamma$$

and

$$|\Lambda' + 2\delta_K|^2 = |\Lambda + 2\delta_K|^2 + 2\langle \Lambda + 2\delta_K, \gamma\rangle + |\gamma|^2.$$

Since γ is an L highest weight in some $S^n(\mathfrak{u}\cap\mathfrak{p})$, Lemma 5 shows that γ is Δ_K^+ dominant. Thus $\Lambda + 2\delta_K$ and γ are both Δ_K^+ dominant. Remembering that K is semisimple, we see from [Kn1, Lemma 8.57] that $\langle \Lambda + 2\delta_K, \gamma\rangle \geq 0$. Therefore

$$|\Lambda' + 2\delta_K|^2 \geq |\Lambda + 2\delta_K|^2 + |\gamma|^2 \geq |\Lambda + 2\delta_K|^2$$

with equality at the right only if $\gamma = 0$.

The conclusion of minimality of the K type Λ in Corollary 8a is special to the case that the representation of L is one-dimensional. Here is an example of what can go wrong in a higher-dimensional case.

Example. Let $\mathfrak{g}_0 = \mathfrak{so}(4,5)$ with the usual $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, and positive system. The simple roots are $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$, and e_4, and $e_2 - e_3$ is the unique noncompact simple root. Let

$$\lambda = (a, a, 1, 0) = ae_1 + ae_2 + e_3,$$

so that

$$\lambda + \delta = (a + \tfrac{7}{2}, a + \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{1}{2}).$$

The parameter λ is outside the weakly good range when $a < 0$. We readily calculate that $\delta(\mathfrak{u}\cap\mathfrak{p}) = (\frac{5}{2}, \frac{5}{2}, 0, 0)$. Therefore

$$\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}) = (a+5, a+5, 1, 0)$$

is Δ_K^+ dominant for $a \geq -5$.

First take $a = -5$, so that F has highest weight $(-5, -5, 1, 0)$ and Λ equals $(0,0,1,0)$. Since $2\delta_K = (2,0,3,1)$, we find that $|\Lambda + 2\delta_K|^2 = |(2,0,4,1)|^2 = 21$. Put $\Lambda' = (1,0,0,0)$. Then $|\Lambda' + 2\delta_K|^2 = |(3,0,3,1)|^2 = 19$. To show that the K type Λ' occurs in $\mathcal{L}_S(F)$, it is enough by Corollary 7b to show that the L type $\Lambda' = (1,0,0,0)$ occurs in $S^1(\mathfrak{u}\cap\mathfrak{p}) \otimes F \otimes \bigwedge^{\mathrm{top}}(\mathfrak{u}\cap\mathfrak{p})$, i.e., the tensor product of the L types $(1,0,1,0)$ and $(0,0,1,0)$. Now $\mathfrak{l}$ is a direct sum $\mathfrak{l}_1 \oplus \mathfrak{l}_2$ corresponding to the first two coordinates plus the last two coordinates. In the first two coordinates the $\mathfrak{l}_1$ type $(1,0)$ actually equals $(1,0)\otimes(0,0)$. In the last two coordinates, the $\mathfrak{l}_2$ type $(0,0)$ occurs in $(1,0)\otimes(1,0)$ since $(1,0)$ is its own contragredient. Hence the K type Λ' occurs in $\mathcal{L}_S(F)$, and it has $|\Lambda' + 2\delta_K|^2 < |\Lambda + 2\delta_K|^2$.

Next take $a = -4$, so that F has highest weight $(-4,-4,1,0)$ and Λ equals $(1,1,1,0)$. Then $|\Lambda + 2\delta_K|^2 = |(3,1,4,1)|^2 = 27$. Put $\Lambda' = (2,1,0,0)$. Then $|\Lambda' + 2\delta_K|^2 = |(4,1,3,1)|^2 = 27$. To show that the K type Λ' occurs in $\mathcal{L}_S(F)$, it is enough by Corollary 7b to show that the L type $\Lambda' = (2,1,0,0)$ occurs in $S^1(\mathfrak{u}\cap\mathfrak{p})\otimes F \otimes \bigwedge^{\mathrm{top}}(\mathfrak{u}\cap\mathfrak{p})$, i.e., in $(1,0,1,0)\otimes(1,1,1,0)$. Arguing as when $a=-5$, we find that this is the case. Hence the K type Λ' occurs in $\mathcal{L}_S(F)$, and it has $|\Lambda' + 2\delta_K|^2 = |\Lambda + 2\delta_K|^2$.

4. Unitarity

We continue with the setting of §2. When λ is orthogonal to Δ_L and λ is in the weakly fair range, $\mathcal{L}_S(\mathbb{C}_\lambda) = A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary as a consequence of a theorem of Vogan mentioned near the end of §2.

When the irreducible $(\mathfrak{l}, L)$ module F has highest weight λ and λ is not orthogonal to Δ_L, the theorem of Vogan does not apply directly, and we get no information outside the weakly good range. But Vogan's theorem can be combined with double cohomological induction to get a positive result.

Let λ be dominant integral for Δ_L^+. Define a parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ of $\mathfrak{g}$ by

$$\begin{aligned} \Delta_{L'} &= \{\alpha \in \Delta_L \mid \langle \lambda, \alpha\rangle = 0\} \\ \Delta(\mathfrak{u}') &= \Delta(\mathfrak{u}) \cup \{\alpha \in \Delta_L^+ \mid \alpha \notin \Delta_{L'}\}. \end{aligned}$$

Then $\mathfrak{l}' \subseteq \mathfrak{l}$ and $\mathfrak{q}' \subseteq \mathfrak{q}$. The group L' is compact. We say that λ is in the **weakly fair range** if $\langle \lambda + \delta(\mathfrak{u}'), \gamma\rangle \geq 0$ for all $\gamma \in \Delta(\mathfrak{u}')$.

Theorem 9. *Let λ be dominant integral for Δ_L^+, and let F be an irreducible $(\mathfrak{l}, L)$ module with highest weight λ. If λ is in the weakly fair range, then $\mathcal{L}_S(F)$ is infinitesimally unitary.*

Proof. Let us write $(\mathcal{L}_{\mathfrak{q},L}^{\mathfrak{g},K})_j$ to refer to the usual cohomological induction functor $\mathcal{L}_j$. We introduce also the cohomological induction functors $(\mathcal{L}_{\mathfrak{q}'\cap\mathfrak{l},L'}^{\mathfrak{l},L})_i$ and $(\mathcal{L}_{\mathfrak{q}',L'}^{\mathfrak{g},K})_k$. Let $S' = \dim(\mathfrak{u}' \cap \mathfrak{l})$. By [Kn-Vo, Proposition 4.173], we have

$$(\mathcal{L}_{\mathfrak{q}'\cap\mathfrak{l},L'}^{\mathfrak{l},L})_i(\mathbb{C}_\lambda) = \begin{cases} F & \text{if } i = S' \\ 0 & \text{if } i \neq S'. \end{cases}$$

Since $(\mathcal{L}_{\mathfrak{q}'\cap\mathfrak{l},L'}^{\mathfrak{l},L})_i(\mathbb{C}_\lambda)$ is nonvanishing in only one degree, the double induction result in [Kn-Vo, Corollary 11.86a] is applicable. When combined with a supplementary argument to take $\bigwedge^{\mathrm{top}}\mathfrak{u}$ into account (cf. [Kn-Vo, §XI.7]), it gives

$$(\mathcal{L}_{\mathfrak{q}',L'}^{\mathfrak{g},K})_{S+S'}(\mathbb{C}_\lambda) \cong (\mathcal{L}_{\mathfrak{q},L}^{\mathfrak{g},K})_S(\mathcal{L}_{\mathfrak{q}'\cap\mathfrak{l},L'}^{\mathfrak{l},L})_{S'}(\mathbb{C}_\lambda) \cong (\mathcal{L}_{\mathfrak{q},L}^{\mathfrak{g},K})_S(F).$$

Since λ is weakly fair in the sense of the definition preceding the theorem, Vogan's theorem implies that the left side is infinitesimally unitary. Hence the right side is infinitesimally unitary.

Example. As in the example in §3, let $\mathfrak{g}_0 = \mathfrak{so}(4,5)$ with the usual $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, and positive system. Let the irreducible $(\mathfrak{l}, L)$ module F have highest weight

$$\lambda = (a+b, a, 0, 0) = (a+b)e_1 + ae_2.$$

For G simply connected with Lie algebra $\mathfrak{g}_0$, the conditions for λ to be dominant integral for Δ_L^+ are that $a \in \frac{1}{2}\mathbb{Z}$, $b \in \mathbb{Z}$, and $b \geq 0$. Here

$$\lambda + \delta = (a+b+\tfrac{7}{2}, a+\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}),$$

and λ is outside the weakly good range if $a < -1$. The parameter Λ is

$$\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}) = (a+b+5, a+5, 0, 0),$$

and it is Δ_K^+ dominant for $a \geq -5$.

First suppose $b = 0$. Then F is one-dimensional, and Vogan's theorem is directly applicable. Since $\delta(\mathfrak{u}) = (3, 3, 0, 0)$, we have

$$\lambda + \delta(\mathfrak{u}) = (a + 3, a + 3, 0, 0).$$

Thus the weakly fair range is $a \geq -3$. Vogan's theorem says that $\mathcal{L}_S(F)$ is infinitesimally unitary for $a \geq -3$.

Next suppose $b > 0$. Then F is no longer one-dimensional. The simple roots defining the parabolic subalgebra $\mathfrak{q}'$ are $e_3 - e_4$ and e_4. Since $\delta(\mathfrak{u}') = (\frac{7}{2}, \frac{5}{2}, 0, 0)$, we have

$$\lambda + \delta(\mathfrak{u}') = (a + b + \tfrac{7}{2}, a + \tfrac{5}{2}, 0, 0).$$

Thus the weakly fair range is $a \geq -\frac{5}{2}$. Theorem 9 says that $\mathcal{L}_S(F)$ is infinitesimally unitary for $a \geq -\frac{5}{2}$.

5. Langlands Parameters

By **Langlands parameters** for an irreducible $(\mathfrak{g}, K)$ module V, we mean a triple (MAN, σ, ν) with the following properties:

(i) MAN is a cuspidal parabolic subgroup of G
(ii) σ is a discrete series or limit of discrete series on M
(iii) ν is a complex-valued linear functional on the Lie algebra $\mathfrak{a}_0$ of A with Re ν in the closed positive Weyl chamber
(iv) the induced representation $\operatorname{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$, given by normalized induction, has a unique irreducible quotient, called the **Langlands quotient** and denoted $J(MAN, \sigma, \nu)$
(v) V is equivalent with the underlying Harish-Chandra module of the Langlands quotient $J(MAN, \sigma, \nu)$.

Property (iv) is automatic if Re ν is in the open positive Weyl chamber (cf. [Kn1, Theorem 7.24]). Since every irreducible $(\mathfrak{g}, K)$ module globalizes to an irreducible admissible representation of G, the Langlands classification of irreducible admissible representations of G (cf. [Kn1, Theorem 14.91]) implies that every irreducible $(\mathfrak{g}, K)$ module has Langlands parameters. Such parameters are not necessarily unique, but for given MAN the value of Re ν is uniquely determined (by the asymptotics of the K finite matrix coefficients of the representation).

We continue with the setting of §2. In this section we are interested in Langlands parameters for the irreducible subquotient V of $A_\mathfrak{q}(\lambda)$ containing the K type Λ.

For $A_\mathfrak{q}(\lambda)$, Corollary 8a tells us that this V has minimal K type Λ, and we know that V has infinitesimal character $\lambda + \delta$. Minimal K type and infinitesimal character together almost completely determine an irreducible $(\mathfrak{g}, K)$ module up to equivalence. In fact, the work of Vogan [Vo1] shows how to determine a pair (M, σ) from the minimal K type. (There is a question about whether (iv) above will be satisfied when the full triple (MAN, σ, ν) is in place, but this issue is not important for our current purposes, and we set it aside in this introductory discussion.) If λ_σ is the infinitesimal character of σ, then the sum of λ_σ and ν has to match the infinitesimal character of V. The infinitesimal character is determined only up to a member of the complex Weyl group. Thus the question is whether the ambiguity from the Weyl group allows for more than one ν in the positive Weyl chamber.

If we drop the assumption that we are working with $A_{\mathfrak{q}}(\lambda)$, the answer is that more than one ν can sometimes yield the correct infinitesimal character. Here is an example.

Example. As in the example in §3, let $\mathfrak{g}_0 = \mathfrak{so}(4,5)$ with the usual $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, and positive system. Starting from the Cartan subalgebra $\mathfrak{t}_0$, we form a new Cartan subalgebra by Cayley transform from the roots $\alpha_1 = e_2 - e_3$ and $\alpha_2 = e_1 - e_4$. Then we take $\mathfrak{a}_0$ to be the $\mathfrak{p}_0$ part of this Cartan subalgebra, and we put $A = \exp \mathfrak{a}_0$. Once A has been fixed, MAN is determined completely by the choice of a positive Weyl chamber in the dual of $\mathfrak{a}_0$. The Weyl group relative to A is transitive on the Weyl chambers in this example, and the choice of positive Weyl chamber will therefore not be important. To establish the notation, let us use the names α_1 and α_2 also for the Cayley transforms of $e_2 - e_3$ and $e_1 - e_4$, and let us fix the positive Weyl chamber of the dual of $\mathfrak{a}_0$ as all $\nu = c_1\alpha_1 + c_2\alpha_2$ with $c_1 \geq c_2 \geq 0$.

The roots defining M are $\pm(e_2 + e_3)$ and $\pm(e_1 + e_4)$. Define $\lambda_\sigma = (4,1,1,4)$. There exist discrete series representations σ of M with λ_σ as infinitesimal character, and [Kn-Vo, §XI.11] shows how to obtain a minimal K type that leads to this σ. Put $\nu_1 = 9\alpha_1 + 2\alpha_2$ and $\nu_2 = 7\alpha_1 + 6\alpha_2$. These are both in the open positive Weyl chamber, and hence (iv) above is satisfied. The parameters $\lambda_\sigma + \nu_1 = (6,10,-8,2)$ and $\lambda_\sigma + \nu_2 = (10,8,-6,-2)$ are conjugate by the complex Weyl group (permutations and sign changes) and thus represent the same infinitesimal character. Since the ν parameters are distinct, we have two inequivalent Langlands quotients $J(MAN,\sigma,\nu_1)$ and $J(MAN,\sigma,\nu_2)$ with the same minimal K type and the same infinitesimal character.

This kind of ambiguity does not appear to occur for $A_{\mathfrak{q}}(\lambda)$ as in §2. When λ is in the weakly good range, the Langlands parameters are simply $(G, A_{\mathfrak{q}}(\lambda), 0)$. When λ is outside the weakly good range, we propose the Conjectural Method below for determining Langlands parameters almost completely.

The Conjectural Method is intended to produce candidates for MAN, ν, and the infinitesimal character of σ. When M is disconnected, these data need not determine the full Langlands parameters since the Harish-Chandra parameter of σ does not necessarily determine σ.

Before stating the Conjectural Method, we carry it out in an example. The example will also suggest a line of proof that the method is successful in a particular case:

(1) show that there is no obstruction to carrying the method through to completion,
(2) use the minimal K type formula of [Kn1, (15.9)] and [Kn-Vo, §XI.11] to show that the candidate for the infinitesimal character of σ leads back to Λ as minimal K type,
(3) show that ν is the unique member of the dual of $\mathfrak{a}_0$ whose sum with the infinitesimal character of σ is Weyl-group equivalent with the infinitesimal character $\lambda + \delta$ of $A_{\mathfrak{q}}(\lambda)$.

We should emphasize that the Conjectural Method is different from the well known algorithm of Vogan [Vo1, Proposition 4.1]. That algorithm starts from an ordering in which $\Lambda + 2\delta_K$ is dominant for Δ^+ and shows how to obtain the infinitesimal character of σ. For the example below, there is a unique positive system Δ^+ in which $\Lambda + 2\delta_K$ is dominant, and Δ^+ does not arise in the discussion of the example.

Example. As in the example in §3, let $\mathfrak{g}_0 = \mathfrak{so}(4,5)$ with the usual $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, and positive system. Again $e_2 - e_3$ is the unique noncompact simple root. Let $\lambda = (-\frac{7}{2}, -\frac{7}{2}, 0, 0)$, so that

$$\lambda + \delta = (0, -1, \tfrac{3}{2}, \tfrac{1}{2}),$$

and

$$\Lambda = (\tfrac{3}{2}, \tfrac{3}{2}, 0, 0).$$

The parameter $\lambda + \delta$ has inner product > 0 with all simple roots that are compact, and it fails to have inner product ≥ 0 with one noncompact simple root, namely $\alpha_1 = e_2 - e_3$. We decompose $\lambda + \delta$ into its components perpendicular and parallel to α_1 as

$$\lambda + \delta = (0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}) - \tfrac{5}{4}\alpha_1.$$

Put $\lambda_1 + \delta_1 = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. A one-dimensional group A_1 is obtained by Cayley transform with $e_2 - e_3$ (cf. [Kn2, §VI.7]), and the corresponding M_1 is a group with simple roots $e_2 + e_3$, $e_1 - e_4$, and e_4. Of these simple roots, $e_2 + e_3$ and $e_1 - e_4$ are noncompact for M_1. The parameter $\lambda_1 + \delta_1$ has inner product > 0 with all M_1 simple roots that are compact, and it fails to have inner product ≥ 0 with one M_1 noncompact simple root for M_1, namely $\alpha_2 = e_1 - e_4$. Then we write

$$\lambda_1 + \delta_1 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) - \tfrac{1}{4}\alpha_2,$$

so that

$$\lambda + \delta = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) - \tfrac{5}{4}\alpha_1 - \tfrac{1}{4}\alpha_2.$$

Put $\lambda_2 + \delta_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. A two-dimensional group A_2 is obtained by Cayley transform with $e_1 - e_4$, and the corresponding M_2 is a group with simple roots $e_2 + e_3$ and $e_1 + e_4$. Both these roots are noncompact for M_2. The parameter $\lambda_2 + \delta_2$ is dominant for M_2. At this stage, because of the M_2 dominance of $\lambda_2 + \delta_2$, the Conjectural Method announces that the Langlands parameters of $A_{\mathfrak{q}}(\lambda)$ should be (MAN, σ, ν), where $M = M_2$, $A = A_2$, N is determined by the Weyl chamber $\{c_1\alpha_1 + c_2\alpha_2 \mid c_1 \geq c_2 \geq 0\}$ in the dual of $\mathfrak{a}_0$, σ has Harish-Chandra parameter $\lambda_2 + \delta_2$, and $\nu = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2$.

This completes step (1) for the example. Step (2) is to apply the minimal K type formula. Put $\lambda_\sigma = \lambda_2 + \delta_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and form the induced series for a corresponding σ. To use the formula, we introduce a new ordering in which λ_σ is still dominant and α_1 and α_2 are simple. Write δ and δ_K for the half sums of positive roots and positive compact roots in this ordering. Also write δ_r and δ_{K_r} for the corresponding sums in the split group built from α_1 and α_2. The formula says that all minimal K types of such series are given by all dominant integral expressions

$$\Lambda_{\text{new}} = \lambda_\sigma + (\delta - \delta_r) - 2\delta_K + (\mu + 2\delta_{K_r}),$$

where μ is a fine K_r type in the sense of Vogan [Vo1] and where μ is related to σ in a particular way. Each fine K_r type is related to some σ. The validity of this formula has a proviso, namely that some such expression is Δ_K^+ dominant. But since we are trying to achieve $\Lambda_{\text{new}} = \Lambda$, the proviso will not be an issue. We choose the ordering in which $e_2 \geq e_3 \geq e_1 \geq e_4 \geq 0$, so that

$$\begin{aligned} \delta &= (\tfrac{3}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{1}{2}) \\ -2\delta_K &= (0, -2, -3, -1) \\ \delta_r &= (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}) \\ 2\delta_{K_r} &= 0. \end{aligned}$$

Then we find that

$$\Lambda_{\text{new}} = (\tfrac{5}{4}, \tfrac{5}{4}, \tfrac{1}{4}, \tfrac{1}{4}) + \mu.$$

If $\mu = (\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$, then the K_r type μ is fine and we obtain

$$\Lambda_{\text{new}} = (\tfrac{3}{2}, \tfrac{3}{2}, 0, 0) = \Lambda,$$

as required.

This completes step (2) for the example. Step (3) is to show that $\nu = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2$ is the only possibility for obtaining the correct infinitesimal character. The target infinitesimal character is $\lambda + \delta = (0, -1, \frac{3}{2}, \frac{1}{2})$, while the infinitesimal character of an induced representation with the linear functional $c_1\alpha_1 + c_2\alpha_2$ on $\mathfrak{a}_0$ is

$$\begin{aligned} \lambda_\sigma + c_1\alpha_1 + c_2\alpha_2 &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) + (c_2, c_1, -c_1, -c_2) \\ &= (\tfrac{1}{4} + c_2, \tfrac{1}{4} + c_1, \tfrac{1}{4} - c_1, \tfrac{1}{4} - c_2). \end{aligned}$$

Since $c_1 \geq c_2 \geq 0$, the largest entry in absolute value is $\frac{1}{4} + c_1$, and this must match the largest entry in absolute value for $(0, -1, \frac{3}{2}, \frac{1}{2})$, namely $\frac{3}{2}$. Therefore $c_1 = \frac{5}{4}$. Then $\frac{1}{4} - c_1 = -1$. We have used the two entries $\frac{3}{2}$ and -1 of $(0, -1, \frac{3}{2}, \frac{1}{2})$. The larger of the absolute values of the remaining two entries 0 and $\frac{1}{2}$ must match $\frac{1}{4} + c_2$, and thus $c_2 = \frac{1}{4}$. Thus $\nu = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2$, as required.

Conjectural Method for Obtaining Langlands Parameters. Let $A_{\mathfrak{q}}(\lambda)$ be given in the setting of §2. The Langlands parameters are to be obtained recursively. For the initial stage, let $M_0 = G$, $A_0 = \{1\}$, $\lambda_0 = \lambda$, $\delta_0 = \delta$, $\nu_0 = \nu$. Here $\dim A_0 = 0$ and $\lambda_0 + \delta_0$ is dominant nonsingular relative to all simple roots of M_0 that are M_0 compact. Suppose that M_j, A_j, λ_j, δ_j, and ν_j are given with $\dim A_j = j$ and with $\lambda_j + \delta_j$ dominant nonsingular relative to all simple roots of M_j that are M_j compact. There are now two cases:

(a) If $\langle \lambda_j + \delta_j, \alpha \rangle \geq 0$ for all simple roots of M_j that are M_j noncompact, the recursive construction ends. Define $M = M_j$, $A = A_j$, $\lambda_\sigma = \lambda_j + \delta_j$, and $\nu = \nu_j$. Define N so that ν is dominant relative to N. Then MAN, λ_σ, and ν are the cuspidal parabolic subgroup, the infinitesimal character of the M representation, and the parameter on $\mathfrak{a}_0$ of a set of Langlands parameters for the irreducible subquotient of $A_{\mathfrak{q}}(\lambda)$ containing the K type $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$.

(b) Otherwise let α_{j+1} be a simple root of M_j that is noncompact and has $\langle \lambda_j + \delta_j, \alpha \rangle < 0$. Build A_{j+1} with $\dim A_{j+1} = j + 1$ from A_j by Cayley transform relative to α_{j+1} (cf. [Kn2, §VI.7]). Let $M_{j+1} = {}^0Z_G(A_{j+1})$ in the notation of [HC2] and [Kn2, p. 391]. The roots of M_{j+1} may be identified with the roots of M_j orthogonal to α_{j+1}, and we let δ_{j+1} be half the sum of the positive roots. Define $\lambda_{j+1} + \delta_{j+1}$ to be the projection of $\lambda_j + \delta_j$ orthogonal to α_{j+1}, so that

$$\lambda_j + \delta_j = (\lambda_{j+1} + \delta_{j+1}) + \frac{\langle \lambda_j + \delta_j, \alpha_{j+1} \rangle}{|\alpha_{j+1}|^2}\, \alpha_{j+1}.$$

Identifying α_{j+1} with its Cayley transform, put

$$\nu_{j+1} = \nu_j - \frac{\langle \lambda_j + \delta_j, \alpha_{j+1} \rangle}{|\alpha_{j+1}|^2} \alpha_{j+1},$$

so that

$$\lambda + \delta = (\lambda_{j+1} + \delta_{j+1}) - \nu_{j+1}.$$

Then $\lambda_{j+1} + \delta_{j+1}$ is dominant nonsingular relative to all simple roots of M_{j+1} that are M_{j+1} compact, and the recursive construction continues.

We cannot prove the Conjectural Method completely. In the results that follow, we shall carry out step (1) above in general (showing that there is no obstruction to carrying out the method), and we shall carry out steps (2) and (3) in two cases: when the induction terminates with $j = 1$ and with $\lambda_1 + \delta_1$ dominant for Δ^+ and also when $\mathfrak{g}_0 = \mathfrak{so}(2m, 2n)$. From the proof of steps (2) and (3) for $\mathfrak{g}_0 = \mathfrak{so}(2m, 2n)$, it will be apparent that the argument works for all classical $\mathfrak{g}_0$.

Proposition 10. *Let $\mathbb{C}_\lambda$ be a one-dimensional representation of L whose unique weight λ has the property that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is Δ_K^+ dominant. Then the Conjectural Method runs into no obstruction in finding parameters MAN, λ_σ, and ν.*

Remark. The proof will give more information than is in the statement. This information is of some use in handling steps (2) and (3) for $A_{\mathfrak{q}}(\lambda)$ in the setting of §2, and it may be of some use in finding Langlands parameters for more general $\mathcal{L}_S(F)$'s.

Proof. In essence the argument will proceed by induction on j in the statement of the Conjectural Method. However, in order to induct successfully, we shall enlarge the set of situations under consideration. For the inductive step, suppose that G is a reductive Lie group in the Harish-Chandra class with trivial split component, and let $\mathfrak{k}_0$, $\mathfrak{p}_0$, and K be as usual. We assume rank $G =$ rank K, and we let $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$. Fix a positive system Δ^+ for $\Delta(\mathfrak{g}, \mathfrak{t})$, and define Δ_K^+, δ, and δ_K as usual. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra with Δ_L built from the simple roots that are compact and with $\Delta(\mathfrak{u}) \subseteq \Delta^+$, let λ be an analytically integral form on $\mathfrak{t}$ that is dominant for $\Delta_L^+ = \Delta^+ \cap \Delta_L$, and suppose that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is dominant for Δ_K^+. We make no assumption on the number of noncompact simple roots.

If there is a noncompact simple root α with $\langle \lambda + \delta, \alpha \rangle < 0$, fix such a root. Use α to form a Cayley transform, writing $\mathfrak{t}^- \oplus \mathfrak{a}^-$ for the transformed version of $\mathfrak{t}$. Here $\dim \mathfrak{a}^- = 1$. Form $\mathfrak{m}$ from $\mathfrak{a}^-$ as usual. We may identify $\Delta(\mathfrak{m}, \mathfrak{t}^-)$ with the subset of $\Delta(\mathfrak{g}, \mathfrak{t}^- \oplus \mathfrak{a}^-)$ orthogonal to α. In turn we may identify $\Delta(\mathfrak{g}, \mathfrak{t}^- \oplus \mathfrak{a}^-)$ with $\Delta(\mathfrak{g}, \mathfrak{t})$ via the Cayley transform. With these identifications in place, we define $\Delta_M^+ = \Delta^+ \cap \Delta(\mathfrak{m}, \mathfrak{t}^-)$. Define $\Delta_{M,K}^+$, δ_M, and $\delta_{M,K}$ correspondingly. Form a parabolic subalgebra $\mathfrak{q}_M = \mathfrak{l}_M \oplus \mathfrak{u}_M$ of $\mathfrak{m}$ in the same way that $\mathfrak{q}$ is formed in $\mathfrak{g}$. Thus $\mathfrak{l}_M$ is formed from the subset $\Delta_{M,L}$ of $\Delta(\mathfrak{m}, \mathfrak{t}^-)$ generated by the simple roots of Δ_M^+ whose root vectors are in $\mathfrak{k}$, and $\mathfrak{u}_M$ is formed from the subset $\Delta(\mathfrak{u}_M)$ of all the remaining members of Δ_M^+. Define a form λ_M on $\mathfrak{t}^-$ by requiring that $\lambda_M + \delta_M$ is the orthogonal projection of $\lambda + \delta$ on the dual of $\mathfrak{t}^-$. We shall prove that

(i) λ_M is analytically integral,

(ii) $\Lambda_M = \lambda_M + 2\delta_M(\mathfrak{u}_M \cap \mathfrak{p})$ is dominant for $\Delta^+_{M,K}$, where $2\delta_M(\mathfrak{u}_M \cap \mathfrak{p})$ is the sum of the members of $\Delta^+_M(\mathfrak{u})$ with root vectors in $\mathfrak{p}$,

(iii) λ_M is dominant for $\Delta^+_{M,L} = \Delta^+_M \cap \Delta_{M,L}$.

This will prove the proposition.

Throughout the proofs of (i) through (iii), it is important to keep in mind what happens to compactness and noncompactness of roots in passing from G to M. The property of being compact or noncompact is retained by a root β in $\Delta(\mathfrak{g}, \mathfrak{t})$ that is strongly orthogonal to α when β is viewed in $\Delta(\mathfrak{m}, \mathfrak{t}^-)$. But if β is orthogonal to α but not strongly orthogonal, then compactness and noncompactness are interchanged in passing from $\Delta(\mathfrak{g}, \mathfrak{t})$ to $\Delta(\mathfrak{m}, \mathfrak{t}^-)$. See [Kn2, Proposition 6.72].

Let us prove (i). By definition $\lambda_M + \delta_M$ is the restriction to $\mathfrak{t}^-$ of $\lambda + \delta$. Now 2δ is the sum of $2\delta_M$, α, and the members of the two-element sets $\{\beta, s_\alpha\beta\}$ of positive roots other than α whose inner product with α is not 0. The sum for a pair $\{\beta, s_\alpha\beta\}$ has the same restriction to $\mathfrak{t}^-$ as 2β. Thus $\delta - \delta_M$ has the same restriction to $\mathfrak{t}^-$ as the sum of a certain collection of roots and is therefore integral on $\mathfrak{t}^-$. Since λ is integral on $\mathfrak{t}$, it follows that λ_M is integral on $\mathfrak{t}^-$. This proves (i).

Next let us prove (ii). Since Δ_L contains only compact roots, $\delta(\mathfrak{u} \cap \mathfrak{p})$ equals δ_n, the half sum of the positive noncompact roots. Similarly $\delta_M(\mathfrak{u}_M \cap \mathfrak{p})$ equals $\delta_{M,n}$, the half sum of the positive noncompact roots in Δ^+_M. Then we have

$$\begin{aligned}\Lambda_M &= \lambda_M + 2\delta_M(\mathfrak{u}_M \cap \mathfrak{p})\\ &= (\lambda_M + \delta_M) + (2\delta_{M,n} - \delta_M)\\ &= (\lambda_M + \delta_M) + (\delta_M - 2\delta_{M,K})\end{aligned}$$

and similarly

$$\Lambda = (\lambda + \delta) + (\delta - 2\delta_K).$$

Taking the inner product of both equations with $\beta \in \Delta^+_M$ and subtracting, we obtain

$$\langle \Lambda_M, \beta \rangle = \langle \Lambda, \beta \rangle + \langle 2(\delta_K - \delta_{M,K}) - (\delta - \delta_M), \beta \rangle.$$

The right side, by [Kn-Vo, Lemma 11.231], is

$$= \langle \Lambda, \beta \rangle + \langle E(2\delta_K) - \tfrac{1}{2}\alpha, \beta \rangle,$$

where E is the orthogonal projection on $\mathbb{R}\alpha$. The second inner product on the right side is 0, and hence

$$\langle \Lambda_M, \beta \rangle = \langle \Lambda, \beta \rangle.$$

When we specialize to $\beta \in \Delta^+_{M,K}$, there are two cases. If β is strongly orthogonal to α, then β is in Δ^+_K and $\langle \Lambda, \beta \rangle \geq 0$. Hence $\langle \Lambda_M, \beta \rangle \geq 0$. If β is orthogonal to α but not strongly orthogonal, then $\beta \pm \alpha$ are in Δ^+_K and $\langle \Lambda, \beta \pm \alpha \rangle \geq 0$. Thus

$$\langle \Lambda_M, \beta \rangle = \langle \Lambda, \beta \rangle = \tfrac{1}{2}(\langle \Lambda, \beta + \alpha \rangle + \langle \Lambda, \beta - \alpha \rangle) \geq 0.$$

This proves (ii).

Finally let us prove (iii). If γ is in $\Delta^+_{M,L}$, then

$$\langle \lambda_M, \gamma \rangle = \langle \Lambda_M - 2\delta_M(\mathfrak{u}_M \cap \mathfrak{p}), \gamma \rangle = \langle \Lambda_M, \gamma \rangle,$$

and this is ≥ 0 by (ii) since $\Delta^+_{M,L} \subseteq \Delta^+_{M,K}$. This completes the proof.

Lemma 11. *Let $\mathbb{C}_\lambda$ be a one-dimensional representation of L whose unique weight λ has the property that $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is Δ_K^+ dominant. Then the Conjectural Method stops with $j=1$ and with $\lambda_1+\delta_1$ dominant for Δ^+ if and only if the unique noncompact simple root β_0 has the property that*

$$-\frac{2}{n} \le \frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2} < 0,$$

where n is the maximum value of $2|\langle\beta_0,\gamma\rangle|/|\gamma|^2$ over all simple roots $\gamma \ne \beta_0$.

Proof. The condition that $\lambda_1+\delta_1$ is dominant for Δ^+ is a condition on the simple roots adjacent to β_0 in the Dynkin diagram since $\langle\lambda_1+\delta_1,\beta_0\rangle = 0$ and since $\langle\lambda_1+\delta_1,\gamma\rangle > 0$ for γ orthogonal to β_0. Thus let γ be a simple root adjacent to β_0. Since $\langle\lambda,\gamma\rangle = 0$, we have

$$\begin{aligned}
2|\gamma|^{-2}\langle\lambda_1+\delta_1,\gamma\rangle &= 2|\gamma|^{-2}\langle\lambda_1+\delta_1,\gamma - \tfrac{\langle\gamma,\beta_0\rangle}{|\beta_0|^2}\beta_0\rangle \\
&= 2|\gamma|^{-2}\langle\lambda+\delta,\gamma - \tfrac{\langle\gamma,\beta_0\rangle}{|\beta_0|^2}\beta_0\rangle \\
&= \frac{2\langle\delta,\gamma\rangle}{|\gamma|^2} - \left(\frac{\langle\gamma,\beta_0\rangle}{|\gamma|^2}\right)\left(\frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\right) \\
&= 1 - \frac{1}{2}\left(\frac{2\langle\gamma,\beta_0\rangle}{|\gamma|^2}\right)\left(\frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\right).
\end{aligned}$$

The right side is

$$\ge 1 - \frac{n}{2}\left|\frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\right|$$

for all γ, and equality holds for some γ. This proves the lemma.

Proposition 12. *Let $\mathbb{C}_\lambda$ be a one-dimensional representation of L whose unique weight λ has the property that $\Lambda = \lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$ is Δ_K^+ dominant. If the Conjectural Method stops with $j=1$ and with $\lambda_1+\delta_1$ dominant for Δ^+, then the Conjectural Method produces a triple (MAN,λ_σ,ν) with the property that (MAN,σ,ν) is a set of Langlands parameters for $A_{\mathfrak{q}}(\lambda)$ for some σ with infinitesimal character λ_σ.*

Remarks.

1) The proof will effectively show what σ is, as well.

2) In most situations as in §2, the unique noncompact simple root β_0 has exactly two neighbors γ_1 and γ_2 in the Dynkin diagram, both connected to β_0 by single lines. In this case, $\gamma_1+\beta_0+\gamma_2$ is a noncompact simple root of M_1 with

$$\frac{2\langle\lambda_1+\delta_1,\gamma_1+\beta_0+\gamma_2\rangle}{|\gamma_1+\beta_0+\gamma_2|^2} = \frac{2\langle\lambda+\delta,\gamma_1+\beta_0+\gamma_2\rangle}{|\beta_0|^2} = 2 + \frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}.$$

If the Conjectural Method stops with $j=1$, then this quantity is ≥ 0, and hence $-2 \le \frac{2\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}$. By Lemma 11, $\lambda_1+\delta_1$ is dominant for Δ^+. In other words, the hypothesis "and with $\lambda_1+\delta_1$ dominant for Δ^+" may be dropped in most situations of the kind in §2.

Proof. Since the Conjectural Method stops when $j=1$, we have $\dim A = 1$, $\lambda_\sigma = \lambda_1+\delta_1$, and $\nu = -\frac{\langle\lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\beta_0$, where β_0 is the unique noncompact simple root. Let σ be a discrete series or limit of discrete series representation of M with infinitesimal character λ_σ. The assumed dominance of λ_σ means that we can use the minimal K type formula (see [Kn1, (15.9)] and [Kn-Vo, §XI.11]) with the given Δ^+ to compute the minimal K types of the series of representations induced from σ. Such a minimal K type Λ_{new} has to be of the form

$$\Lambda_{\text{new}} = \lambda_\sigma + (\delta - \tfrac{1}{2}\beta_0) - 2\delta_K + \mu,$$

where μ is a fine K_r type for the group G_r corresponding to the $\mathfrak{sl}(2,\mathbb{R})$ built from β_0. If δ_n denotes half the sum of the positive noncompact roots, then the above expression is

$$\begin{aligned} &= \lambda + \delta - \tfrac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0 + \delta - \tfrac{1}{2}\beta_0 - 2\delta_K + \mu \\ &= \lambda + 2\delta_n - \tfrac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0 - \tfrac{1}{2}\beta_0 + \mu \\ &= \Lambda - \tfrac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0 - \tfrac{1}{2}\beta_0 + \mu. \end{aligned}$$

According to Lemma 11, $\frac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0$ is equal to $-c\beta_0$ with $0 < c \le 1$. Therefore

$$\mu = \tfrac{1}{2}\beta_0 + \tfrac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0$$

is a fine K_r type, and the corresponding Λ_{new} equals Λ. Since the result is integral, there exists a σ leading to this μ. Then it follows from [Kn1, Proposition 15.5] that any irreducible admissible representation with minimal K type Λ is an irreducible quotient of some $\text{ind}_{MAN}^{G}(\sigma\otimes e^{\nu'}\otimes 1)$ with Re ν' in the closed positive Weyl chamber. Since $\lambda+\delta$ is real, ν' must be real. The equality

$$|\lambda_\sigma|^2 + |\nu'|^2 = |\lambda+\delta|^2 = |\lambda_1+\delta_1|^2 + \left|\tfrac{\langle \lambda+\delta,\beta_0\rangle}{|\beta_0|^2}\,\beta_0\right|^2$$

and the one-dimensionality of A imply that $\nu' = \nu$, as required. Since $\nu \ne 0$, condition (iv) is satisfied in the definition of Langlands parameters. The other conditions have already been verified, and thus (MAN, σ, ν) is a set of Langlands parameters for the irreducible subquotient of $A_{\mathfrak{q}}(\lambda)$ containing the K type Λ.

Proposition 13. *Let $\mathbb{C}_\lambda$ be a one-dimensional representation of L whose unique weight λ has the property that $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is Δ_K^+ dominant. If $\mathfrak{g}_0 = \mathfrak{so}(2m,2n)$ with $m>1$ and $n>1$, then the Conjectural Method produces a triple $(MAN, \lambda_\sigma, \nu)$ with the property that (MAN,σ,ν) is a set of Langlands parameters for $A_{\mathfrak{q}}(\lambda)$ for some σ with infinitesimal character λ_σ.*

Remark. The proof will effectively show what σ is, as well.

Proof. For $\mathfrak{g}_0 = \mathfrak{so}(2m,2n)$ within the setting of §2, the root system has to be of type D_{m+n}, and the unique noncompact simple root has to be $\beta_0 = e_m - e_{m+1}$ or $\beta_0 = e_n - e_{n+1}$. There is no loss of generality in assuming that $\beta_0 = e_m - e_{m+1}$. The compact roots are the ones involving only indices $1,\ldots,m$ or only indices $m+1,\ldots,m+n$. If we take

$$\lambda = (a,\ldots,a,0,\ldots,0) \qquad \text{with } a \in \tfrac{1}{2}\mathbb{Z},$$

then we have

$$\begin{aligned} \delta &= (m+n-1,\ldots,n,n-1,\ldots,0) \\ \lambda+\delta &= (a+m+n-1,\ldots,a+n,n-1,\ldots,0) \\ -2\delta_K &= (-2(m-1),\ldots,-2,0,-2(n-1),\ldots,-2,0) \\ 2\delta(\mathfrak{u}\cap\mathfrak{p}) &= (2n,\ldots,2n,0,\ldots,0) \\ \Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}) &= (a+2n,\ldots,a+2n,0,\ldots,0). \end{aligned}$$

The condition that λ be outside the weakly good range is that $a < -1$. The condition that Λ be Δ_K^+ dominant is that $a \geq -2n$.

In the Conjectural Method, the first roots whose inner products are tested with $\lambda+\delta$ can be $e_m - e_{m+1}$, $e_{m-1} - e_{m+2}$, and so on. Afterward the roots $e_m + e_{m+1}$, $e_{m-1} + e_{m+2}$, and so on can have their inner products tested. For the latter kind, the inner products are all the same, namely $a + 2n - 1$, and the proof divides into cases according to the sign of this quantity.

Case I. $a + 2n - 1 \geq 0$. For $j \geq 0$, we have

$$\langle \lambda+\delta, e_{m-j} - e_{m+1+j} \rangle = (a+n+j) - (n-1-j) = a + 2j + 1.$$

Let p be the largest integer with $a + 2p + 1 < 0$. By the assumption of Case I, $p \leq n-1$. Put $p' = \min\{m-1, p\}$. The successive roots that arise in the Conjectural Method are

$$e_{m-j} - e_{m+j+1} \qquad \text{for } 0 \leq j \leq p'.$$

We are going to apply the minimal K type formula (see [Kn1, (15.9)] and [Kn-Vo, §XI.11]) for the induced series from σ, where σ has infinitesimal character λ_σ. To do so, we need a new positive system in which λ_σ is dominant and the roots $e_{m-j} - e_{m+j+1}$ with $0 \leq j \leq p'$ are simple. We shall produce σ and show that the minimal K type Λ_{new} of the induced series from σ coincides with Λ, and then we shall be able to complete Case I. The verification of the equality $\Lambda_{\text{new}} = \Lambda$ breaks into two subcases, $p' = p$ and $p' = m-1$.

Subcase Ia. $p \leq m-1$, so that $p' = p$. Since p is as large as possible, we have $a + 2p + 3 \geq 0$. The indices $\{1, \ldots, m+n\}$ break into three sets, namely $\{1, \ldots, m-p-1\}$, $\{m-p, \ldots, m+p+1\}$, and $\{m+p+2, \ldots, m+n\}$. The first or the third set or both may be empty. For these three sets of indices, the corresponding entries of λ_σ are

$$\begin{array}{ccc} a+m+n-1 & \cdots & a+n+p+1 \\ \frac{1}{2}(a+2n-1) & \cdots & \frac{1}{2}(a+2n-1) \\ n-p-2 & \cdots & 0. \end{array}$$

The desired ordering is the one that takes the first set of indices in its given ordering, followed by the second set in the ordering

$$m, m+1, m-1, m+2, m-2, m+3, \ldots, m-j, m+j+1, \ldots, m-p, m+p+1,$$

followed by the third set in its given ordering. Let us write δ and δ_K for the usual quantities in this ordering, and let us write δ_r and δ_{K_r} for the corresponding quantities for the roots spanned by all $e_{m-j} - e_{m+j+1}$ with $0 \leq j \leq p$. The minimal K type formula says that the induced series from σ (with σ having infinitesimal character λ_σ) has minimal K type

$$\Lambda_{\text{new}} = \lambda_\sigma + (\delta - \delta_r) - 2\delta_K + (\mu + 2\delta_{K_r})$$

for a fine K_r type μ. For the first and third sets of indices, we have

$$\lambda_\sigma = \begin{cases} a+m+n-1 & \cdots & a+n+p+1 \\ n-p-2 & \cdots & 0 \end{cases}$$

$$\delta - \delta_r = \begin{cases} m+n-1 & \cdots & n+p+1 \\ n-p-2 & \cdots & 0 \end{cases}$$

$$-2\delta_K = \begin{cases} -2(m-1) & \cdots & -2(p+1) \\ -2(n-p-2) & \cdots & 0 \end{cases}$$

$$\mu + 2\delta_{K_r} = \begin{cases} 0 & \cdots & 0 \\ 0 & \cdots & 0. \end{cases}$$

Addition gives

$$\Lambda_{\text{new}} = \begin{cases} a+2n & \cdots & a+2n \\ 0 & \cdots & 0, \end{cases}$$

and therefore Λ_{new} matches Λ in the first and third sets of indices.

In the second set of indices, we define

$$\mu = \begin{cases} \frac{1}{2}a+p+1 & \text{in position } m-j \\ -\frac{1}{2}a-p-1 & \text{in position } m+j+1. \end{cases}$$

Since

$$a+2p+1<0 \qquad \text{and} \qquad a+2p+3 \geq 0,$$

the contribution to μ from positions $m-j$ and $m+j+1$ is $c(e_{m-j}-e_{m+j+1})$ with $-\frac{1}{2} \leq c < \frac{1}{2}$. Therefore μ is a fine K_r type. Then we have

$$\lambda_\sigma = \begin{cases} \frac{1}{2}(a+2n-1) & \text{in position } m-j \\ \frac{1}{2}(a+2n-1) & \text{in position } m+j+1 \end{cases}$$

$$\delta - \delta_r = \begin{cases} n+p-2j-\frac{1}{2} & \text{in position } m-j \\ n+p-1-2j+\frac{1}{2} & \text{in position } m+j+1 \end{cases}$$

$$-2\delta_K = \begin{cases} -2(p-j) & \text{in position } m-j \\ -2(n-j-1) & \text{in position } m+j+1 \end{cases}$$

$$\mu + 2\delta_{K_r} = \begin{cases} \frac{1}{2}a+p+1 & \text{in position } m-j \\ -\frac{1}{2}a-p-1 & \text{in position } m+j+1. \end{cases}$$

Addition gives

$$\Lambda_{\text{new}} = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1, \end{cases}$$

and therefore Λ_{new} matches Λ in the second set of indices. Since Λ_{new} is integral, μ gives rise to a well defined σ with infinitesimal character λ_σ and the induced series from σ has Λ as minimal K type.

Subcase Ib. $p > m-1$, so that $p' = m-1$. Since $p \leq n-1$, we must have $m < n$. The indices break into two nonempty sets $\{1, \ldots, 2m\}$ and $\{2m+1, \cdots, m+n\}$. For these two sets of indices, the corresponding entries of λ_σ are

$$\begin{matrix} \frac{1}{2}(a+2n-1) & \cdots & \frac{1}{2}(a+2n-1) \\ n-m-1 & \cdots & 0. \end{matrix}$$

Here

$$n-p-2 \le \tfrac{1}{2}(a+2n-1) < n-p-1.$$

Since $p \ge m$, λ_σ is not dominant. The desired ordering is the one that takes the first $p-m+1$ indices from the block $\{2m+1,\dots,m+n\}$, followed by the set $\{1,\dots,2m\}$ in the ordering

$$m, m+1, m-1, m+2, m-2, m+3, \dots, m-j, m+j+1, \dots, 1, 2m,$$

followed by the remaining indices from the block $\{2m+1,\dots,m+n\}$. Let us write δ and δ_K for the usual quantities in this ordering, and let us write δ_r and δ_{K_r} for the corresponding quantities for the roots spanned by all $e_{m-j}-e_{m+j+1}$ with $0 \le j \le m-1$. We define μ in the same way as for Subcase Ia, and μ is again a fine K_r type. For the new first and third sets of indices, we have

$$\begin{aligned}
\lambda_\sigma &= \begin{cases} n-m-1 & \cdots \quad n-p-1 \\ n-p-2 & \cdots \qquad 0 \end{cases} \\
\delta-\delta_r &= \begin{cases} n+m-1 & \cdots \quad n+2m-p-1 \\ n-p-2 & \cdots \qquad 0 \end{cases} \\
-2\delta_K &= \begin{cases} -2(n-1) & \cdots \quad -2(n+m-p-1) \\ -2(n-p-2) & \cdots \qquad 0 \end{cases} \\
\mu+2\delta_{K_r} &= \begin{cases} 0 & \cdots \quad 0 \\ 0 & \cdots \quad 0. \end{cases}
\end{aligned}$$

Addition gives

$$\Lambda_{\text{new}} = \begin{cases} 0 & \cdots \quad 0 \\ 0 & \cdots \quad 0, \end{cases}$$

and therefore Λ_{new} matches Λ in the new first and third sets of indices.

Now we check the contribution from the indices $\{1,\dots,2m\}$. We have

$$\begin{aligned}
\lambda_\sigma &= \begin{cases} \tfrac{1}{2}(a+2n-1) & \text{in position } m-j \\ \tfrac{1}{2}(a+2n-1) & \text{in position } m+j+1 \end{cases} \\
\delta-\delta_r &= \begin{cases} n+2m-p-2-2j-\tfrac{1}{2} & \text{in position } m-j \\ n+2m-p-3-2j+\tfrac{1}{2} & \text{in position } m+j+1 \end{cases} \\
-2\delta_K &= \begin{cases} -2(m-1-j) & \text{in position } m-j \\ -2(n+m-p-2-j) & \text{in position } m+j+1 \end{cases} \\
\mu+2\delta_{K_r} &= \begin{cases} \tfrac{1}{2}a+p+1 & \text{in position } m-j \\ -\tfrac{1}{2}a-p-1 & \text{in position } m+j+1. \end{cases}
\end{aligned}$$

Addition gives

$$\Lambda_{\text{new}} = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1, \end{cases}$$

and therefore Λ_{new} matches Λ in the set of indices $\{1,\dots,2m\}$. Once again, since Λ_{new} is integral, μ gives rise to a well defined σ with infinitesimal character λ_σ and the induced series from σ has Λ as minimal K type.

This completes the construction of σ in the two subcases of Case I, as well as the proof that the induced series from σ has Λ as minimal K type. To complete

Case I, we need to verify that ν is the Langlands A parameter. Let ν' be the actual Langlands parameter, so that

$$\nu' = \begin{cases} c_j & \text{in position } m-j \text{ for } 0 \le j \le p' \\ -c_j & \text{in position } m+j+1 \text{ for } 0 \le j \le p' \\ 0 & \text{in all other positions.} \end{cases}$$

The Weyl group of A is transitive on the Weyl chambers for this situation, and we may thus assume that $c_0 \geq c_1 \geq \cdots \geq c_{p'}$. Then $\lambda + \delta$ and $\lambda_\sigma + \nu'$ must agree, up to a member of the complex Weyl group. For indices outside the range $m - p' \leq j \leq m+1+p'$, the entries of $\lambda + \delta$ and $\lambda_\sigma + \nu'$ match exactly. Thus we have only to consider indices in the range $m - p' \leq j \leq m+1+p'$. For this range of indices, the entries of λ_σ are constant and positive. Thus we can determine c_0 from the largest entry of $\lambda + \delta$ in absolute value, c_1 from the next largest, and so on. This argument shows that there is only one candidate for ν'. Since ν has the property that $\lambda + \delta = \lambda_\sigma + \nu$, we must have $\nu' = \nu$. Since ν is in the open positive Weyl chamber, condition (iv) in the definition of Langlands parameters is satisfied.

Case II. $a + 2n - 1 < 0$, so that $a = -2n + \frac{1}{2}$ or $a = -2n$. Define $p' = \min\{m-1, n-1\}$. The successive roots that arise in the Conjectural Method are

$$e_{m-j} \pm e_{m+j+1} \qquad \text{for } 0 \le j \le p',$$

and λ_σ is 0 in entries $m - p', \ldots, m + p' + 1$. To apply the minimal K type formula for the induced series from σ, we need a positive system in which λ_σ is dominant and the Lie algebra generated by all simple roots (and their negatives) contributing to all $e_{m-j} \pm e_{m+j+1}$ is split. We shall produce σ and show that the minimal K type Λ_{new} of the induced series from σ coincides with Λ, and then the argument is completed as in Case I. The verification of the equality $\Lambda_{\text{new}} = \Lambda$ breaks into two subcases, $n \leq m$ and $n > m$.

Subcase IIa. $n \leq m$, so that the successive roots are

$$e_{m-n+1} \pm e_{m-n+2}, e_{m-n+3} \pm e_{m-n+4}, \ldots, e_{m+n-1} \pm e_{m+n}.$$

The indices $\{1, \ldots, m+n\}$ break into two sets, namely $\{1, \ldots, m-n\}$ and $\{m-n+1, \ldots, m+n\}$. The first set is empty if and only if $n = m$. For these two sets of indices, the corresponding entries of λ_σ are

$$\begin{matrix} a+m+n-1 & \cdots & a+2m \\ 0 & \cdots & 0. \end{matrix}$$

Here $a+2m \geq a+2n \geq 0$, and the desired ordering for the minimal K type formula is the standard ordering. For the first set of indices, we have

$$\begin{aligned} \lambda_\sigma &= a+m+n-1 \quad \cdots \quad a+m \\ \delta - \delta_r &= m+n-1 \quad \cdots \quad m \\ -2\delta_K &= -2(m-1) \quad \cdots \quad -2(m-n) \\ \mu + 2\delta_{K_r} &= 0 \quad \cdots \quad 0. \end{aligned}$$

Addition gives

$$\Lambda_{\text{new}} = a+2n \quad \cdots \quad a+2n,$$

and therefore Λ_{new} matches Λ in the first set of indices.

In the second set of indices, we define

$$\mu = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

for $0 \le j \le n-1$. This K_r type is trivial or is a spin representation on one factor of K_r. In either case it is fine for $\mathfrak{so}(n,n)$. For $0 \le j \le n-1$, we then have

$$\lambda_\sigma = \begin{cases} 0 & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

$$\delta - \delta_r = \begin{cases} 0 & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

$$-2\delta_K = \begin{cases} -2j & \text{in position } m-j \\ -2(n-j-1) & \text{in position } m+j+1 \end{cases}$$

$$\mu = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1. \end{cases}$$

$$2\delta_{K_r} = \begin{cases} 2j & \text{in position } m-j \\ 2(n-j-1) & \text{in position } m+j+1. \end{cases}$$

Addition shows for $0 \le j \le n-1$ that

$$\Lambda_{\text{new}} = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1. \end{cases}$$

Thus $\Lambda_{\text{new}} = \Lambda$. Since Λ_{new} is integral, μ gives rise to a well defined σ with infinitesimal character λ_σ and the induced series from σ has Λ as minimal K type.

Subcase IIb. $n > m$, so that the successive roots are

$$e_1 \pm e_2, e_3 \pm e_4, \ldots, e_{2m-1} \pm e_{2m}.$$

The indices $\{1, \ldots, m+n\}$ break into two nonempty sets, namely $\{1, \ldots, 2m\}$ and $\{2m+1, \ldots, m+n\}$. For these two sets of indices, the corresponding entries of λ_σ are

$$\begin{matrix} 0 & \cdots & 0 \\ n-m-1 & \cdots & 0. \end{matrix}$$

The desired ordering for the minimal K type formula takes the second set of indices followed by the first. For the second set of indices, we have

$$\begin{aligned} \lambda_\sigma &= n-m-1 \quad \cdots \quad 0 \\ \delta - \delta_r &= m+n-1 \quad \cdots \quad 2m \\ -2\delta_K &= -2(n-1) \quad \cdots \quad -2m \\ \mu + 2\delta_{K_r} &= 0 \quad \cdots \quad 0. \end{aligned}$$

Addition gives

$$\Lambda_{\text{new}} = 0 \quad \cdots \quad 0,$$

and therefore Λ_{new} matches Λ in this set of indices.

In the other set of indices, we define

$$\mu = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

for $0 \le j \le m-1$. Here μ is a fine K_r type for $\mathfrak{so}(m,m)$. For $0 \le j \le m-1$, we then have

$$\lambda_\sigma = \begin{cases} 0 & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

$$\delta - \delta_r = \begin{cases} 0 & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

$$-2\delta_K = \begin{cases} -2j & \text{in position } m-j \\ -2(m-j-1) & \text{in position } m+j+1 \end{cases}$$

$$\mu = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1. \end{cases}$$

$$2\delta_{K_r} = \begin{cases} 2j & \text{in position } m-j \\ 2(m-j-1) & \text{in position } m+j+1. \end{cases}$$

Addition gives

$$\Lambda_{\text{new}} = \begin{cases} a+2n & \text{in position } m-j \\ 0 & \text{in position } m+j+1 \end{cases}$$

for $0 \le j \le m-1$. Thus $\Lambda_{\text{new}} = \Lambda$, and the proof is complete.

References

[Ba] D. Barbasch, *Representations with maximal primitive ideal*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (A. Connes, M. Duflo, A. Joseph, and R. Rentschler, eds.), Birkhäuser, Boston, 1990, pp. 317–331. MR **92j:**17006

[Bi-Z] B. Binegar and R. Zierau, *Unitarization of a singular representation of $SO(p,q)$*, Commun. Math. Phys. **138** (1991), 245–258. MR **92h:**22027

[Bo-deS] A. Borel and J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helvetici **23** (1949), 200–221. MR **11:**326d

[Br-Ko] R. Brylinski and B. Kostant, *Minimal representations, geometric quantization, and unitarity*, Proc. Nat. Acad. Sci. USA **91** (1994), 6026–6029. MR **95d:**58059

[EHW] T. Enright, R. Howe, and N. Wallach, *A classification of unitary highest weight modules*, Representation Theory of Reductive Groups (P. C. Trombi, ed.), Birkhäuser, Boston, 1983, pp. 97–143. MR **86c:**22028

[EPWW] T. J. Enright, R. Parthasarathy, N. R. Wallach, and J. A. Wolf, *Unitary derived functor modules with small spectrum*, Acta Math. **154** (1985), 105–136. MR **86j:**22026

[G-W] B. H. Gross and N. R. Wallach, *A distinguished family of unitary representations for the exceptional groups of real rank = 4*, Lie Theory and Geometry: in Honor of Bertram Kostant (J.-L. Brylinski, R. Brylinski, V. Guillemin, and V. Kac, eds.), Birkhäuser, Boston, 1994, pp. 289–304. MR **96i:**22034

[HC1] Harish-Chandra, *Representations of semisimple Lie groups IV*, Amer. J. Math. **77** (1955), 743–777; V, **78** (1956), 1–41; VI, **78** (1956), 564–628.

[HC2] Harish-Chandra, *Harmonic analysis on real reductive groups I*, J. Func. Anal. **19** (1975), 104–204. MR **53:**3201

[Ja] H. P. Jakobsen, *Hermitian symmetric spaces and their unitary highest weight modules*, J. Func. Anal. **52** (1983), 385–412. MR **85a:**17004

[Ka-S] D. Kazhdan and G. Savin, *The smallest representations of simply laced groups*, Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion on His Sixtieth Birthday, Part I (S. Gelbart, R. Howe, and P. Sarnak, eds.), Israel Math. Conf. Proc. 2, Weizmann Science Press of Israel, Jerusalem, 1990, pp. 209–223. MR **93f:**22019

[Kn1] A. W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, N.J., 1986. MR **87j:**22022

[Kn2] A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[Kn-Vo] A. W. Knapp and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University Press, Princeton, N.J., 1995. MR **96c:**22023

[McG1] W. M. McGovern, *Dixmier algebras and the orbit method*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (A. Connes, M. Duflo, A. Joseph, and R. Rentschler, eds.), Birkhäuser, Boston, 1990, pp. 397–416. MR **92f:**17010

[McG2] W. M. McGovern, *Rings of regular functions on nilpotent orbits II*, Commun. in Algebra **22** (1994), 765–772. MR **95b:**22035

[Vo1] D. A. Vogan, *The algebraic structure of the representation of semisimple Lie groups I*, Ann. of Math. **109** (1979), 1–60. MR **81j:**22020

[Vo2] D. A. Vogan, *Unitarizability of certain series of representations*, Ann. of Math. **120** (1984), 141–187. MR **86h:**22028

[W1] N. Wallach, *The analytic continuation of the discrete series II*, Trans. Amer. Math. Soc. **251** (1979), 19–37. MR **81a:**22009

[W2] N. Wallach, *On the unitarizability of derived functor modules*, Invent. Math. **78** (1984), 131–141. MR **86f:**22018

Department of Mathematics, State University of New York, Stony Brook, New York 11794

E-mail address: `aknapp@ccmail.sunysb.edu`