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REPRESENTATIONS OF $GL_2(\boldsymbol{R})$ AND $GL_2(\boldsymbol{C})$

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1. $SL_2(\boldsymbol{R})$. We shall give lists of the irreducible finite-dimensional representations, the irreducible unitary representations, and the nonunitary principal series. Then we discuss reducibility questions, asymptotic expansions, and the Langlands classification. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a typical element of $G = SL_2(\boldsymbol{R})$.

Irreducible finite-dimensional representations. $\mathscr{F}_n$, $n \geqq 0$, an integer.

$$\text{Space} = \{f \text{ polynomial on } \boldsymbol{R} \text{ of degree } n\},$$
$$\mathscr{F}_n(g)f(x) = (bx+d)^n f((ax+c)/(bx+d)).$$

Finite-dimensional representations of G are fully reducible.

Unitary representations. The irreducible unitary representations were classified by Bargmann [**1**]. We give realizations in function spaces on the line or upper half-plane. Realizations on the circle or disc are possible also.

(1) Discrete series $\mathscr{D}_n^+$ and $\mathscr{D}_n^-$, $n \geqq 2$.

$$\text{Space for } \mathscr{D}_n^+ = \left\{ f \text{ analytic for } \operatorname{Im} z > 0 \,\middle|\, \|f\|^2 = \iint_{\operatorname{Im} z > 0} |f(z)|^2 y^{n-2}\, dx\, dy < \infty \right\},$$
$$\mathscr{D}_n^+(g)f(z) = (bz+d)^{-n} f\left(\frac{az+c}{bz+d}\right).$$

The space for $\mathscr{D}_n^+$ is not 0 because $(z+i)^{-n}$ is in it. The representation $\mathscr{D}_n^-$ is obtained by using complex conjugates. All these representations are irreducible, unitary, and square-integrable. The square-integrability (of a matrix coefficient) will be shown below.

(2) Principal series $\mathscr{P}^{+,iv}$ and $\mathscr{P}^{-,iv}$, $v \in \boldsymbol{R}$.

Space for $\mathscr{P}^{\pm,iv} = L^2(\boldsymbol{R})$,

$$\begin{aligned} \mathscr{P}^{\pm,iv}(g)f(x) &= |bx+d|^{-1-iv} f((ax+c)/(bx+d)) && \text{if } +, \\ &= \operatorname{sgn}(bx+d)|bx+d|^{-1-iv} f((ax+c)/(bx+d)) && \text{if } -. \end{aligned}$$

These representations are all unitary, and all but $\mathscr{P}^{-,0}$ are irreducible. Equivalences $\mathscr{P}^{+,iv} \cong \mathscr{P}^{+,-iv}$ and $\mathscr{P}^{-,iv} \cong \mathscr{P}^{-,-iv}$ are implemented by analytic continuations of intertwining operators that we give below. $\mathscr{P}^{\pm,iv}$ is really the induced representation $\operatorname{Ind}_{MAN}^G(\sigma \otimes e^{iv\cdot} \otimes 1)$ with G acting by right translation and with the functions restricted to $\bar{N} = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Here MAN is the upper triangular group, σ is trivial or signum on $M = \{\pm I\}$, and the character of A is

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \to e^{ivt}.$$

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(3) Complementary series $\mathscr{C}^s$, $0 < s < 1$.

$$\text{Space for } \mathscr{C}^s = \left\{ f\colon \boldsymbol{R} \to \boldsymbol{C} \,\middle|\, \|f\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}\,dx\,dy}{|x-y|^{1-s}} < \infty \right\},$$

$$\mathscr{C}^s(g)f(x) = |bx+d|^{-1-s} f\left(\frac{ax+c}{bx+d}\right).$$

These are irreducible unitary. They arise from certain nonunitary principal series (see below) by redefining the inner product.

(4) Others. There is the trivial representation, and there are two "limits of discrete series," $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$. The group action with $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ is like that in discrete series, but the norm is different. We have the relation $\mathscr{P}^{-,0} \cong \mathscr{D}_1^+ \oplus \mathscr{D}_1^-$.

Nonunitary principal series. $\mathscr{P}^{\pm,\zeta}$, $\zeta \in \boldsymbol{C}$.

Space $= L^2(\boldsymbol{R}, (1+x^2)^{\operatorname{Re}\zeta} dx)$

$$\begin{aligned} \mathscr{P}^{\pm,\zeta}(g)f(x) &= |bx+d|^{-1-\zeta} f((ax+c)/(bx+d)) && \text{if } +, \\ &= \operatorname{sgn}(bx+d)|bx+d|^{-1-\zeta} f((ax+c)/(bx+d)) && \text{if } -. \end{aligned}$$

Reducibility. We can see some reducibility in $\mathscr{P}^{\pm,\zeta}$ on a formal level by specializing the parameter ζ and by passing from z in the upper half-plane to x on the real axis. We obtain the following continuous inclusions:

$$\begin{aligned} \mathscr{F}_n &\subseteq \mathscr{P}^{+,-(n+1)} && \text{if } n \text{ even}, \\ &\subseteq \mathscr{P}^{-,-(n+1)} && \text{if } n \text{ odd}, n \geqq 0; \\ \mathscr{D}_n^+ \oplus \mathscr{D}_n^- &\subseteq \mathscr{P}^{+,n-1} && \text{if } n \text{ even}, \\ &\subseteq \mathscr{P}^{-,n-1} && \text{if } n \text{ odd}, n \geqq 1. \end{aligned}$$

There is no other reducibility. The quotient by an $\mathscr{F}$ is the sum of two $\mathscr{D}$'s, and vice versa.

Asymptotics. Let k_θ be the rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The maximal compact subgroup $K = \{k_\theta\}$ is abelian, and its irreducible representations are one-dimensional, $k_\theta \to e^{im\theta}$ with m an integer. We have

$$G = KA^+K \quad \text{with} \quad A^+ = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \geqq 0 \right\}$$

and Haar measure is of the form $dg = c \sinh 2t\, dk_\theta\, dk_{\theta'}\, dt$ if $g = k_\theta a_t k_{\theta'}$. Let $U(g)$ be an admissible representation of G, and let φ_1 and φ_2 transform under K according to $k_\theta \to e^{im_1\theta}$ and $k_\theta \to e^{im_2\theta}$. Then

$$(U(g)\varphi_1, \varphi_2) = (U(k_\theta a_t k_{\theta'})\varphi_1, \varphi_2) = \exp(i(m_1\theta' + m_2\theta))\,(U(a_t)\varphi_1, \varphi_2).$$

Thus to test whether a matrix coefficient is in some L^p class on G, it is enough to test $(U(a_t)\varphi_1, \varphi_2)$ and use the measure $\sinh 2t\, dt$, $t \geqq 0$.

EXAMPLE. $\mathscr{D}_n^+(k_\theta)(z+i)^{-n} = e^{in\theta}(z+i)^{-n}$. Then

$$(\mathscr{D}_n^+(a_t)(z+i)^{-n}, (z+i)^{-n})$$
$$= \iint_{\operatorname{Im} z>0} e^{nt}[x + i(y+1)]^{-n}[e^{2t}x - i(e^{2t}y+1)]^{-n} y^{n-2}\, dx\, dy.$$

By residues the right side is

$$= c_n \int_0^\infty e^{-nt}(y + 1 + e^{-2t})^{1-2n} y^{n-2}\, dy,$$

and this in turn, after the change of variables $y = y'(1+e^{-2t})$, is $= c_n'(\cosh t)^{-n}$. Then

$$\int_G |\ldots|^2\, dg = cc_n' \int_0^\infty (\cosh t)^{-2n} \sinh 2t\, dt,$$

which is finite for $n > 1$. Thus this matrix coefficient is square-integrable on G. A theorem in functional analysis due to Godement [3] implies that all matrix coefficients are square-integrable on G.

In the example, we could see the matrix coefficient was square-integrable by computation. There is a general technique, due to Harish-Chandra, for getting at the behavior of matrix coefficients by means of differential equations. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a basis for the Lie algebra of G. The *Casimir operator* $\Omega = \frac{1}{2}h^2 + ef + fe$ is a member of the universal enveloping algebra. For $SL_2(\boldsymbol{R})$, Ω generates the center of the universal enveloping algebra. (For larger groups, it must be replaced in this discussion by the whole center of the universal enveloping algebra.) It acts as a scalar on each representation in our lists, hence on each matrix coefficient. Take a matrix coefficient whose two K-dependences are according to known characters of K, and regard the matrix coefficient as an unknown function. Then the equation Ω(coefficient) $= c$(coefficient) leads to a second order ordinary differential equation on A^+, with t as independent variable. The classical substitution is $s = \cosh t$, and the resulting differential equation has three singularities, all regular; we are interested in the behavior at $s = \infty$. (If the "known characters" of K are trivial, this is Legendre's equation.) This substitution does not generalize well, and Harish-Chandra's treatment of this equation amounts to making the substitution $z = e^{-2t}$ instead. The resulting differential equation has four singularities, all regular, and we expand about $z = 0$, using standard regular-singular-point theory. The result is that

$$\text{coefficient}(a_t) = e^{-(1+\zeta)t} \sum_{n=0}^{\infty} c_n e^{-2nt} + e^{-(1-\zeta)t} \sum_{n=0}^{\infty} d_n e^{-2nt}$$

except when ζ is an integer, in which case there may be factors of t that arise from factors $\log z$ in the solution. If one of the leading terms vanishes, the whole corresponding infinite sum vanishes.

The eigenvalue of Ω determines ζ, and in particular the matrix coefficients of $\mathscr{P}^{\pm,\zeta}$ lead to the expansion with ζ present. From this expansion, we can read off L^p-integrability conditions, since we are to integrate for $t \geqq 0$ the pth power against $\sinh 2t\, dt$, which is comparable with $e^{2t}dt$. We see that $\mathscr{P}^{\pm,iv}$ has coefficients in

$L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, but not in $L^2(G)$. *Discrete series* $\mathscr{D}_n^\pm$ with $n \geqq 2$ have one sum absent, in order to have coefficients in $L^2(G)$. Representations with coefficients in $L^{2+\varepsilon}$ for every $\varepsilon > 0$ are said to be *tempered.* The tempered representations are $\mathscr{P}^{\pm,iv}$ and $\mathscr{D}_n^\pm$ with $n \geqq 1$. Notice how in general the two leading terms give some information about where imbeddings occur as subrepresentations in the nounitary principal series; Wallach dealt with this point in his lectures.

Langlands classification. For general G, Langlands parametrizes the irreducible admissible representations by triples (P, π, ν), where $P = MAN$ is a standard parabolic, π is (the class of) an irreducible tempered representation of M, and ν is a complex-valued linear functional on the Lie algebra of A with real part in the open positive Weyl chamber. The Langlands representation $J_P(\pi, \nu)$ is the unique irreducible quotient of $\mathrm{Ind}_P^G(\pi \otimes e^\nu \otimes 1)$. In our case, $P = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ is minimal parabolic, or $P = G$.

Case P minimal. There are two (one-dimensional) representations of $M = \{\pm I\}$, and the functional ν enters as the complex number ζ with $\operatorname{Re} \zeta > 0$; the character of A is $a_t = \exp(\nu \log a_t) = \exp(\zeta t)$. The Langlands list then includes the unique irreducible quotient of $\mathscr{P}^{\pm,\zeta}$ for each ζ with $\operatorname{Re} \zeta > 0$.

Case P = G. Here ν is irrelevant, and $M = G$. We simply get the irreducible tempered representations of G. The Langlands classification itself does not address the question of what these are, though one of the theorems implies for our G that they are subrepresentations of discrete series or unitary principal series.

Intertwining operators. The Langlands classification theorem describes the unique irreducible quotient more precisely than we have done. Kunze and Stein [4] showed in 1960 that the operator

$$f \to \int_{-\infty}^{\infty} \frac{f(y)\, dy}{|x-y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{+,\zeta},$$

$$\to \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y) f(y)\, dy}{|x-y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{-,\zeta},$$

intertwines $\mathscr{P}^\zeta$ with $\mathscr{P}^{-\zeta}$. Note that the integral is convergent only if $\operatorname{Re} \zeta > 0$. Later [5] they found a formula in the induced picture, namely $f \to \int_{\bar{N}} f(\bar{n} w^{-1} g)\, d\bar{n}$, where $\bar{N} = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This is the composition of two operators, $f \to \int_{\bar{N}} f(\bar{n} g)\, d\bar{n}$ and a relatively trivial translation operator by w^{-1}. Define $A(\bar{P} : P : \pi : \nu) f(x) = \int_{\bar{N}} f(\bar{n} x)\, d\bar{n}$. Under the Langlands conditions on ν, this integral is convergent if f is K-finite. The theorem is that

$$J_P(\pi, \nu) = \mathrm{Ind}_P^G(\pi \otimes e^\nu \otimes 1)/\ker A(\bar{P} : P : \pi : \nu) \cong \operatorname{Image} A(\bar{P} : P : \pi : \nu).$$

2. Other groups.

$\mathrm{GL}_2(\boldsymbol{R})$. To pass from $\mathrm{SL}_2(\boldsymbol{R})$ to the group $\mathrm{SL}_2^\pm(\boldsymbol{R})$ of matrices of determinant ± 1, we first induce the representations of $\mathrm{SL}_2(\boldsymbol{R})$. The $\mathscr{P}$'s and $\mathscr{F}$'s split into two equivalent pieces, and the $\mathscr{D}$'s yield irreducibles on $\mathrm{SL}_2^\pm(\boldsymbol{R})$ that restrict back to $\mathscr{D}^+ \oplus \mathscr{D}^-$ on $\mathrm{SL}_2(\boldsymbol{R})$. This construction gives us the representations of $\mathrm{SL}_2^\pm(\boldsymbol{R})$. Then to pass to $\mathrm{GL}_2(\boldsymbol{R})$, we paste on a character of the group $\boldsymbol{R}^+\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$\mathrm{SL}_2(\boldsymbol{C})$. This group has finite-dimensional representations given by two integer parameters; the representations can be realized in spaces of polynomials in z and $\bar{z}$ on $\boldsymbol{C}$. The group has no discrete series. We have

$$M = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right\} \quad \text{and} \quad A = \left\{\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\right\},$$

so that the nonunitary principal series is parametrized by an integer n (for M) and a complex number ζ (for A); by restriction of functions to $\bar{N} = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$, we can realize these representations in spaces of functions on $\boldsymbol{C}$. See [2] for more detail. The unitary principal series is all irreducible and provides the only tempered irreducibles, and parameters (n, iv) and $(-n, -iv)$ lead to equivalent representations. The Langlands classification points to the Langlands quotients of the nonunitary principal series with Re $\zeta > 0$ and to the irreducible tempered representations.

$GL_2(\boldsymbol{C})$. To an irreducible representation of $SL_2(\boldsymbol{C})$, we paste on a character of $\boldsymbol{C}^{\times}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ that agrees with the representation on $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In this way we obtain all irreducible representations of $GL_2(\boldsymbol{C})$.

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