First Steps with the Langlands Program*

A. W. Knapp

Abstract

This article describes ways in which to get an overview of some of the goals and methods of Langlands program, and it points to treatments of some examples that are worth keeping in mind.

Keywords and Phrases: Langlands program, Hecke operator, L-function, class field theory, automorphic form, local Langlands correspondence, trace formula, base change.

Mathematics Subject Classification: Primary, 11F70, 11R39. Secondary, 11S37, 22E55.

The overall nature of the Langlands program is captured well by some quotations from introductions to Langlands's own papers that are cited here. Langlands writes,[1] "There are at least three different problems with which one is confronted in the study of L-functions: the analytic continuation and functional equation; the location of the zeros; and in some cases, the determination of the values at special points. The first may be the easiest. It is certainly the only one with which I have been closely involved.

"There are two kinds of L-functions, and they will be described below: motivic L-functions which generalize the Artin L-functions and are defined purely arithmetically, and automorphic L-functions, defined by data which are largely transcendental. Within the automorphic L-functions a special class can be singled out, the class of standard L-functions, which generalize Hecke L-functions and for which the analytic continuation and functional equation can be proved directly.

"For the other L-functions the analytic continuation is not so easily effected. However all evidence indicates that there are fewer L-functions than the definitions suggest, and that every L-function, motivic or automorphic, is equal to a standard L-function. Such equalities are often deep, and are called reciprocity laws, for historical reasons. Once a reciprocity law can be proved for an L-function, analytic continuation follows, and so, for those who believe in the validity of the reciprocity

[1] In his Helsinki paper, "L-functions and automorphic representations," 1980.

laws, they and not analytic continuation are the focus of attention, but very few such laws have been established.

"The automorphic L-functions are defined representation-theoretically, and it should be no surprise that harmonic analysis can be applied to some effect in the study of reciprocity laws. ..."

In an earlier paper[2] Langlands elaborates on this thought. He defines in the usual way classical modular forms f of weight k on the upper half plane relative to $SL(2, Z)$ and then converts f to an associated function ϕ_f on the group $G = SL(2, R)$ by setting

$$\phi_f(g) = (ci + d)^{-k} f\big((ai + b)(ci + d)^{-1}\big)$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. He continues, "If Γ is the group of integral matrices in G, then ϕ_f is a function on $\Gamma\backslash G$. The functions ϕ_f associated to those f which have a finite norm in the Petersson metric[3] can be characterized in terms of the representations of G. Associating to each h in G the operator $\lambda(h)$ on $L^2(\Gamma\backslash G)$ defined by

$$(\lambda(h)\phi)(g) = \phi(gh),$$

we obtain a representation of G on $L^2(\Gamma\backslash G)$. There is a representation π_k of G on a Hilbert space H_k and a distinguished subspace H_k^0 of H_k so that ϕ is a ϕ_f if and only if there is a G-invariant map of H_k to $L^2(\Gamma\backslash G)$ which takes H_k^0 to the space generated by ϕ.

"This is the first hint that it might not be entirely unprofitable to study automorphic forms in the context of group representations. The Hecke operators, which play a major role in the study of modular forms, provide a second. ..."

Langlands goes on to say what the Hecke operators look like in this setting. For this purpose he passes to an adele group defined by G, extends λ so that $\lambda(f)$ is meaningful for certain compactly supported functions on the adele group, and discusses the situation briefly. Then he says, "The theory of modular forms and the operators $\lambda(f)$ is far from complete. Indeed very little attempt has been made, so far as I can see, to understand what the goals of the theory should be. Once it is put in the above form it is clear that the concepts at least admit of extension to any reductive algebraic group defined over a number field. It may be possible to give some coherency to the subject by introducing the simple principle, implicit in the work of Harish-Chandra, that what can be done for one reductive group should be done for all."

As Langlands says, it was unclear at the time what the goals of the theory should be. But a year later, he was completely clear what the theory should be in the simplest nontrivial case. This is not the case of $GL(2)$ but of a general abelian reductive group. The special case of the abelian group $GL(1)$ is the case of class field theory. Langlands writes,[4] "There is reason to believe that there is a close

[2] In the written version of his Whittemore lectures "Euler products" at Yale, given in 1967.
[3] This condition forces f to be a cusp form.
[4] In his paper "Representations of abelian algebraic groups," first drafted in 1968.

relation between the irreducible representations, in the sense of harmonic analysis, of the group of rational points on a reductive algebraic group over a local field and the representations of the Weil group of the local field in a certain complex group. There should also be a relation, although it will not be so close, between the representations of the global Weil group, in the associated complex group and the representations of the adèle group that occur in the space of automorphic forms. The nature of these relations will be explained elsewhere. For now all I want to do is explain and prove the relations when the group is abelian. I should point out that this case is not typical. For example, in general there will be representations of the algebraic group not associated to representations of the Weil group."

The first steps in the Langlands program, as envisioned in the present notes, are to understand some of the detail of what Langlands has just said. As the discussion proceeds, there will sometimes be references to papers in one of the following three conference proceedings:

Corvallis proceedings: Borel, A., and W. Casselman (editors), *Automorphic Forms, Representations, and L-Functions, Corvallis, 1977*, Proceedings of Symposia in Pure Mathematics, vol. 33, Parts 1 and 2, American Mathematical Society, 1979.

Motives proceedings: Jannsen, U., S. Kleiman, and J.-P. Serre (editors), *Motives, Seattle, 1991*, Proceedings of Symposia in Pure Mathematics, vol. 55, American Mathematical Society, 1994.

Edinburgh proceedings: Bailey, T. N., and A. W. Knapp (editors), *Representation Theory and Automorphic Forms, Instructional Conference, Edinburgh, 1996*, Proceedings of Symposia in Pure Mathematics, vol. 61, American Mathematical Society, 1997, download table of contents from
`http://www.math.sunysb.edu/~aknapp/books/edinburgh/edinburgh-contents.pdf`.

Expository overviews

This section contains a list of five expositions of the Langlands program and includes some comments about each:

Langlands, R. P., Representation theory: its rise and its role in number theory, *Proceedings of the Gibbs Symposium, Yale, 1989*, American Mathematical Society, 1990, pp. 181–210, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/gibbs-ps.pdf`.

Langlands, R. P., L-functions and automorphic representations, *Proceedings International Congress of Mathematicians, Helsinki, 1978*, 1980, pp. 165–175, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/lfunct-ps.pdf`.

Knapp, A. W., Introduction to the Langlands program, *Edinburgh proceedings*, pp. 245–302, download from
`http://www.math.sunysb.edu/~aknapp/pdf-files/245-302.pdf`.

Gelbart, S., An elementary introduction to the Langlands program, *Bull. Amer. Math. Soc.* 10 (1984), 177–219.

Bump, D., J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, S. S. Kudla, *An Introduction to the Langlands Program, Jerusalem, 2001*, Birkhäuser, 2003.

Borel, A., Formes automorphes et séries de Dirichlet (d'après R. P. Langlands), *Séminaire Bourbaki (1974/1975: Exposés No. 453–470)*, Lecture Notes in Mathematics, Springer-Verlag, vol. 514, 1976, Exp. No. 466, pp. 183–222.

The Langlands paper for the Gibbs Symposium is the most elementary of these expositions. It works with examples of reciprocity, one quadratic and one involving a polynomial of degree 5, and then it gives a brief introduction to p-adic numbers and the representation theory of $SL(2)$ over R and over p-adic fields. It concludes with some discussion of automorphic L-functions and some philosophy.

The Langlands paper for the Helsinki congress is an overview at a higher level of sophistication. It begins by discussing Artin L-functions and Artin's approach to prove their regularity by showing that they match some of Hecke's L-functions. Langlands introduces an L-function for any nonsingular projective variety over a number field (at least at all but finitely many places) and speculates on its regularity. Next he introduces the standard L-functions, discusses the possibilities for handling Artin L-functions when the representation is 2-dimensional, sketches how to associate an automorphic L-function to each irreducible representation of $G(A)$ (with G being a connected reductive group over a global field and A being the adeles of the global field), mentions how the conjectural functoriality implies that this automorphic L-function equals a standard one, and concludes with some examples related to his base change for $GL(2)$.

In the Knapp paper, the first five sections summarize abelian class field theory largely for number fields, introduce the Weil group of a local field, and apply Artin reciprocity to prove the equality of Artin and Hecke L-functions for one-dimensional representations of the Galois group of an abelian extension. The other five sections say something about the Langlands program. They discuss linear algebraic groups, introduce the L^2 space of interest for automorphic forms, discuss how classical modular forms lead to representation-theoretic automorphic forms, state Harish-Chandra's key theorem that the space of functions automorphic relative to a certain set of data is finite-dimensional, define cusp forms, state what the theory is for $GL(n)$, give a rough description of how the theory is framed for more general reductive groups, and say something about functoriality.

Gelbart's paper in its Section II introduces four classical themes—the local-global principle, Hecke theory and the centrality of automorphic forms, Artin L-functions, and group representations in number theory. Section III briefly discusses the theory of automorphic representations of $GL(n)$. Section IV is about the Langlands program, defining all but finitely many factors of L-functions, saying some words about L-groups and functoriality, mentioning some results that are known (as of 1984), and briefly summarizing some techniques of proof.

The book by Bump et al. is a collection of expository articles. It begins with the elementary theory of L-functions and classical automorphic forms, introduces

Artin L-functions and the L-functions of elliptic curves, explains Tate's thesis, brings in spectral theory and the trace formula, discusses the theory for $GL(n)$, discusses L-groups and functoriality, and introduces the "geometric Langlands program."

Borel's Séminaire Bourbaki paper is a gentle version of Borel's article "Automorphic L-functions" in the Corvallis proceedings (see below). It goes through the theory in successively greater degrees of generality and does not strive for the degree of completeness found in the Corvallis article.

Automorphic forms on $GL(2)$

Gelfand and Fomin realized already in the 1950s that the theory of modular forms had something to do with group representations for $GL(2)$. There are three expository books that are largely about $GL(2)$ and expand on this notion:

Gelfand, I. M., M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Forms*, W. B. Saunders, Philadelphia, 1969.

Gelbart, S., *Automorphic Forms on Adele Groups*, Princeton University Press, 1975.

Bump, D., *Automorphic Forms and Representations*, Cambridge University Press, 1997.

The differences among these books are relatively small and result somewhat from the differing years of publication. They all discuss classical modular forms, embed their theory eventually into $GL(2, A)$, where A is the ring of adeles of Q or a number field, discuss aspects of the representation theory of $GL(2)$ over a local field, study the space of "cusp forms" of $GL(2, A)$, and prove that this space decomposes discretely. The Gelbart and Bump books establish the theory of Eisenstein series for $GL(2)$ and work with ingredients of the trace formula, explicitly or implicitly. Some of these results are influenced by the Jacquet–Langlands theory to be discussed next, and they have bearing on the theory of standard L-functions in the next subsection.

The representation-theoretic aspect of the above three expositions appears within the often-quoted research monograph

Jacquet, H., and R. P. Langlands, *Automorphic Forms on GL(2)*, Lecture Notes in Mathematics, vol. 114, 1970, download from
http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/a-ps.pdf

This book goes beyond the mathematics in the above three expository books, finding a sufficient condition for the "converse theorem" to hold, i.e., for an irreducible representation of the adele group to contribute to the space of cusp forms of $GL(2, A)$. The condition of contributing to the space of cusp forms needs some explanation, which is deferred to the next subsection. Readers who want to stop preparations and get right into the theory can begin with the Jacquet–Langlands book. Those who try the Jacquet–Langlands book and decide that they want to begin with a gentler treatment may consult

Godement, R., Notes on Jacquet–Langlands' theory, duplicated notes, Institute for Advanced Study, 1970.

This book may be found in many research libraries, but it was never officially published and therefore is not in the Library of Congress.

Automorphic representations of $GL(n)$: Local Langlands correspondence, standard L-functions

Work on generalizing Tate's thesis for $GL(1)$ so that it applies to $GL(n)$ dates to a paper of Tamagawa in 1963. Further progress was made by Maloletkin in 1971, and a complete theorem for "cuspidal" irreducible representations of $GL(n)$ was obtained in

Godement, R., and H. Jacquet, *Zeta Functions of Simple Algebras*, Lecture Notes in Mathematics, vol. 260, Springer-Verlag, 1972.

The word "cuspidal" requires explanation. Write $G(\,\cdot\,)$ for $GL(n,\,\cdot\,)$. As in the theory for $GL(2,\,\cdot\,)$ mentioned above, one wants to understand the space of complex-valued functions on $G(F)\backslash G(A)$, where F is a number field and A is the ring of adeles of F. Already for $n=1$, this quotient does not have finite measure, and one cannot expect L^2 of it to be very interesting. Instead, one defines $Z(\,\cdot\,)$ to be the group of scalar matrices, so that $Z(F)\backslash Z(A)$ is isomorphic to the group of ideles. If χ is any character of $Z(F)\backslash Z(A)$, define $L^2(G(F)\backslash G(A))_\chi$ to be the space of complex-valued functions on $G(A)$ such that f is left-invariant under $G(F)$, $|f|$ is left-invariant under $Z(A)$, $|f|$ is in $L^2((Z(A)G(F))\backslash G(A))$, and $f(zx)=\chi(z)f(x)$ for all $z\in Z(A)$ and $x\in G(A)$.

The cuspidal subspace of $L^2(G(F)\backslash G(A))_\chi$ is described precisely in Section 7 of the Knapp exposition mentioned above in the subsection "Expository overviews." Roughly speaking, this is the subspace of f's for which $f(ng)$, for each $g\in G(A)$, always has integral 0 taken over $N(F)\backslash N(A)$ if N is the unipotent radical of any proper parabolic subgroup of G; but this condition needs to be made precise because a general f is not well defined on sets of measure 0 like Ng. At any rate, it is a theorem of Gelfand and Piatetski–Shapiro that the cuspidal subspace $L^2(G(F)\backslash G(A))_\chi$ decomposes discretely into irreducible representations of $G(A)$ with finite multiplicities. The irreducible constituents are the cuspidal irreducible representations of $G(A)$.

It was known for general linear groups G that any irreducible "admissible" representation of $G(A)$ is a kind of infinite tensor product over all places v of irreducible admissible representations of each group $G(F_v)$, and the paper

Flath, D., Decomposition of representations into tensor products, *Corvallis proceedings*, Part 1, pp. 179–183.

eventually generalized this theorem to all reductive groups. Applying this conclusion to a cuspidal irreducible representation π in the case of the general linear group exhibits π as this kind of tensor product of irreducible representations π_v of $GL(n,F_v)$. Godement and Jacquet found a function-theoretic way of associating a local L factor $L(s,\pi_v)$ and an ε factor $\varepsilon(s,\pi_v)$ to π_v, and they defined $L(s,\pi)$ and $\varepsilon(s,\pi)$ to be the corresponding products over all places v. Their construction agrees with Hecke's L and ε factors for Grossencharacters (the case $n = 1$). They proved that the global L and ε satisfy a functional equation relating matters at s for π to matters at $1 - s$ for the contragredient of π. From the functional equation it follows that these functions are meromorphic in the whole complex plane.

In 1973 Langlands gave his classification of irreducible admissible representations over R and C, and Silberger and Wallach proved enough of a corresponding classification over nonarchimedean local fields that the condition "cuspidal" could be removed in the above construction of L and ε. Since the noncuspidal part of $L^2(G(F)\backslash G(A))_\chi$ does not decompose discretely, some condition has to remain in place of "cuspidal." Langlands called this condition "automorphic," and it is defined in

Langlands, R. P., On the notion of an automorphic representation, *Corvallis proceedings*, Part 1, pp. 203–207, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/notion-ps.pdf`.

With this advance, Jacquet was able to extend the Godement–Jacquet construction of L and ε, including the resulting functional equation, to all automorphic representation of $G(A)$ for $G = GL(n, \cdot)$. The result and an exposition, respectively, appear in

Jacquet, H., Principal L-functions of the linear group, *Corvallis proceedings*, Part 2, pp. 63–86,

Jacquet, H., Principal L-functions for $GL(n)$, *Edinburgh proceedings*, pp. 321–329.

The L-functions obtained for $GL(n)$ in this way are what Langlands calls the "standard L-functions" in his Helsinki paper.

The 1973 classification paper of Langlands of irreducible admissible representations of real algebraic groups handled matters in two stages. One stage, purely a matter of harmonic analysis on reductive groups, was to give a parametrization of the irreducible representations useful in harmonic analysis. The second stage was to recast this parametrization in terms of finite-dimensional complex representations of the Weil group of R. This latter parametrization is what is called the local Langlands correspondence over R. An exposition of the two stages for $GL(n)$ appears in

Knapp, A. W., Local Langlands correspondence: the archimedean case, *Motives proceedings*, Part 2, pp. 393–410.

Meanwhile, to each finite-dimensional representation of the Weil group of R corresponds a local L-function and a local ε factor, and one can check that under the local Langlands correspondence over R, L and ε match the factors of Godement–Jacquet and Jacquet. Consequently the local L and ε in the Jacquet papers are consistent with Langlands's framework for reciprocity theorems for $GL(n)$, at least over the field R. These remarks apply to the field C also.

What about nonarchimedean local fields? Let us leave aside general G and consider only general linear groups. Is there a nice parametrization of the irreducible admissible representations of each $GL(n)$ by n-dimensional complex representations of the Weil group of the local field in such a way that the associated local L and ε factors match the ones of Godement and Jacquet? One necessary adjustment found by Deligne is that the Weil group of the field is not quite large enough. A modification, now known as the Weil–Deligne group, is to be used instead. Even with this change, making this question precise was not a trivial matter, but Henniart managed to find in 1991 a set of conditions on local L and ε factors ensuring uniqueness of a local Langlands correspondence. An exposition of these conditions and their setting is in

Kudla, S. S., The local Langlands correspondence: the non-archimedean case, *Motives proceedings*, Part 2, pp. 365–391.

Henniart's 1991 result did not address existence, but it at least identified a precise problem. The existence was finally proved by Harris and Taylor and by Henniart in the late 1990s. A nice exposition of this result and its meaning for the Langlands program is given in

Rogawski, J., The nonabelian reciprocity law for local fields, *Notices of the American Mathematical Society*, vol. 47 (2000), no. 1, 35–41, download from `http://www.ams.org/notices/200001/comm-rogawski.pdf`.

Initial details

The initial steps in the Langlands program are taken in three papers, all of which have informative introductions and all of which are otherwise difficult to read. The list of papers follows, and then comes some commentary about each. Afterward comes information about where one can find this mathematics in other sources:

Langlands, R. P., Representations of abelian algebraic groups, duplicated notes, Yale University, 1968, published as *Pacific J. Math.* 61 (1998), 231–250, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/Abel-ianAlg-ps.pdf`.

Langlands, R. P., On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced Study, 1973, published in P. J. Sally and D. A. Vogan (editors), *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Math. Surveys and Monographs, vol. 31, American mathematical Society, 1989, pp. 101–170, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/irred-ps.pdf`.

Langlands, R. P., Problems in the theory of automorphic forms, *Lectures in Modern Analysis and Applications III*, Lecture Notes in Mathematics, Springer-Verlag, vol. 170, 1970, pp. 18–61, download from
`http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/prob-lems-ps.pdf`

The first of these papers carries out a certain duality calculation for connected abelian reductive groups, i.e., algebraic tori that are not necessarily split. In retrospect this paper is showing that the full Langlands program works for tori.

The second paper classifies the irreducible admissible representations of reductive algebraic groups G over R. Sections 3–4 carry out the relevant harmonic analysis, and their main result parametrizes the irreducible admissible representations by triples (MAN, π, ν), where MAN is a parabolic subgroup containing a fixed minimal parabolic subgroup, π is an equivalence class of irreducible tempered representations of M, and ν is a complex-valued linear functional on the Lie algebra of A whose real part is in the open positive Weyl chamber. Two other theorems in these sections show (a) that each irreducible tempered representation is an irreducible constituent of a standard representation unitarily induced from a parabolic subgroup MAN with a discrete series representation on M and a unitary character on A and (b) that any two standard unitarily induced representations are unitarily equivalent or else have no irreducible constituent in common. Section 2 translates these results into a conclusion that the irreducible admissible representations of G are parametrized by certain homomorphisms of the Weil group of R into a certain complex group called the L-group of G.

The third paper outlines the general program, including functoriality. Functoriality on the level of a local field F concerns the following situation. If two groups G_1 and G_2 are given and if a representation of G_2 is given by a homomorphism of φ of the Weil or Weil–Deligne group of F into the complex L-group of G_2, then any homomorphism of the L-group of G_2 into the L-group of G_1 yields by composition with φ a homomorphism of the Weil/Weil–Deligne group of F into the L-group of G_1. There ought to be a corresponding representation of G_1, and the local L factors should be compatible. Functoriality makes specific conjectures about this situation and about a global version of it. Most of the paper consists of conjectures; rather little is proved. An interesting example is given how the validity of functoriality when one of the groups is trivial implies the "Artin conjecture," which says that the Artin L-function for each nontrivial irreducible Galois representation is entire.

Parenthetically the expectation that functoriality holds very generally or completely generally is what lies behind the expectation that the standard L-functions from $GL(n)$ should be so ubiquitous. In fact, if G is any reductive group and ${}^LG$ is its L-group, then ${}^LG$ is a complex matrix group and hence is a subgroup of some $GL(n, C)$. Unwinding what functoriality suggests, one sees that any automorphic L-function for G ought to be an automorphic L-function for some $GL(n)$. Arthur has investigated the validity of this case of functoriality for classical groups.

A difficult but more readable account of matters in the above three papers of Langlands is the paper

Borel, A., Automorphic L-functions, *Corvallis proceedings*, Part 2, pp. 27–61.

It may be helpful to get into Borel's account gradually, and Borel's Séminaire Bourbaki account listed above under "Expository overviews" can help with the transition. Borel omits most of the details concerning the classification of irreducible admissible representations of real algebraic groups. A messy but well motivated proof of this classification appears in

Knapp, A. W., *Representation Theory of Semisimple Groups, An Overview Based on Examples*, Princeton University Press, 1986; reprinted in paperback, 2001.

A tidy but less well motivated proof of the classification appears in

Borel, A., and N. R. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Princeton University Press, 1980; second edition, American Mathematical Society, 2000.

Trace formula and base change

A key point of the Langlands program is to show that motivic L-functions are automorphic, specifically that they are standard L-functions for $GL(n)$, and one can ask what techniques are available to show this kind of thing. A technique that has yielded some success is the trace formula, and the first stunning use of the trace formula in the Langlands program to prove something classical was the proof by Langlands in

Langlands, R. P., *Base Change for* $GL(2)$, Princeton University Press, 1980, download from
http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/book-ps.pdf

that Artin L-functions corresponding to 2-dimensional "tetrahedral" representations are entire. This subsection will discuss the circle of ideas that revolves around the trace formula and base change.

A beautiful qualitative description of the trace formula and its uses has recently appeared in

Arthur, J., A (very brief) history of the trace formula, *Pacific Institute for the Mathematical Sciences*, vol. 10, no. 2, winter 2007, pp. 8–11, download from
http://www.claymath.org/cw/arthur/pdf/HistoryTraceFormula.pdf.

As this paper makes clear, the trace formula is an identity for a particular reductive group saying that two quantities can be interpreted as the same trace computed in two different ways. The variable in the identity is a certain kind of function that one can shape as desired. In applications the trace formula is typically to be applied to two different groups. For certain functions on one group, one introduces corresponding functions on the other group in such a way that most of the terms in the two trace formulas match. An equality results for each such matching pair of functions, and some conclusion is drawn.

Base change provides a context for discussing this matter in more detail. The introduction to *Base Change for* $GL(2)$ by Langlands contains a discussion, partly historical, of base change and its relation to Artin's conjecture. It is fairly readable. The full proof involves some delicate computations that may make it hard to understand what is really happening. In that spirit, an exposition of the proof may be a helpful way to begin. One such exposition, minus the use of the trace formula, appears in

Rogawski, J. D., Functoriality and the Artin conjecture, *Edinburgh proceedings*, pp. 331–353, download from http://www.math.ucla.edu/∼jonr/eprints/artin.pdf.

A detailed development of the trace formula for $GL(2)$ gives a first sense of the nature of the formula. (However, $GL(2)$ is only one group, and applications typically involve two different groups.) One such development is in Section 9 of *Automorphic Forms on Adele groups* by Gelbart. Another is in the paper

Knapp, A. W., Theoretical aspects of the trace formula for $GL(2)$, *Edinburgh proceedings*, pp. 355–405, download from
http://www.math.sunysb.edu/∼aknapp/pdf-files/355-405.pdf.

A book that treats the trace formula more generally, providing detail about Arthur's early papers and sometimes specializing to $GL(2)$ for illustration is

Gelbart, S., *Lectures on the Arthur–Selberg Trace Formula*, American Mathematical Society, 1996.

A beginning introduction to how to apply the trace formula is the article

Knapp, A. W., and J. D. Rogawski, Applications of the trace formula, *Edinburgh proceedings*, pp. 413–431, download from
http://www.math.sunysb.edu/∼aknapp/pdf-files/413-431.pdf.

This can well be read in conjunction with the short paper

Kottwitz, R., Orbital integrals and base change, *Corvallis proceedings*, Part 2, pp. 111–113,

which gives a better sense of the kind of thing that has to be verified for the Langlands proof.

Finally two other expositions of *Base Change for* $GL(2)$ are

Gelbart, S., Automorphic forms and Artin's conjecture, *Modular Functions of One Variable VI, Bonn, 1976*, Lecture Notes in Mathematics, vol. 627, Springer, 1977, pp. 241–276,

Gérardin, P., and J. P. Labesse, The solution of a base change problem for GL(2) (following Langlands, Saito, Shintani), *Corvallis proceedings*, Part 2, pp. 115–133.