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Irreducible Unitary Representations of $SU(2, 2)$

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We give an explicit classification of the irreducible unitary representations of the simple Lie group $SU(2, 2)$.

1. INTRODUCTION

We shall give a classification of the irreducible unitary representations of the simple Lie group $SU(2, 2)$. By definition this group is the subgroup of $SL(4, \mathbb{C})$ preserving a Hermitian quadratic from with two plus signs and two minus signs. The group is locally isomorphic to the conformal group of space-time, also to the group $SO_e(4, 2)$, and also to the group of holomorphic automorphisms of the tube domain over the forward light cone in $\mathbb{R}^4$.

Although the classification of irreducible unitary representations of general semisimple Lie groups is far from understood, it is unlikely that a group-by-group approach will be the ultimate answer. Despite this fact, it has seemed advisable to us to do the classification for $SU(2, 2)$ separately. There are two reasons: this group appears to be of special interest in mathematical physics (see the bibliographies of [26, 30, 39]), and it provides a prototype for a number of Lie-theoretic phenomena that one does not see in easier examples.

Qualitatively our classification involves no surprises. In terms of our parameters, no representations are isolated except for discrete series, limits of discrete series, and the trivial representation. No wholly unexpected continuous series arise. The ladder representations [5, 11, 26] occur at the ends of the longest complementary series for a maximal parabolic subgroup, and some representations used by Strichartz [34] for the analysis of $SO(4, 2)/SO(3, 2)$ occur at the ends of complementary degenerate series.

Langlands [25] has given a classification of irreducible admissible

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representations. Since irreducible unitary implies irreducible admissible, what we have to do is decide which Langlands parameters correspond to unitary representations. Actually we shall not use the original formulation of the Langlands classification but rather the reformulation in [22]; the test that we apply to decide what representations are unitary is the one in [17]. But in any event, the Langlands classification is vital to our work.

Decisions whether particular Langlands parameters give unitary representations are made by means of a number of techniques. None of the ones we use is new, and the list of the main ones is as follows:

(1) Continuity of Hermitian forms ([18] or Sect. 16 of [20]). If a continuous family of nonsingular Hermitian matrices is defined on a connected set and is positive definite at one point, then it is positive definite everywhere.

(2) Isolation of the trivial representation in the unitary dual. See Kazhdan [13] and Wang [38].

(3) Vanishing at infinity of the K-finite matrix coefficients of any nontrivial irreducible unitary representation. See Howe and Moore [9].

(4) Detailed analysis of intertwining operators by means of formulas for real-rank-one operators. See Duflo [3] and Kostant [24].

(5) Techniques for proving irreducibility of degenerate series [31, 33].

(6) Minimal K-type argument for showing a representation is not unitary. See Baldoni Silva [1].

Both the notation and the classification appear in Section 2. In Section 3 we reduce the proof of the classification to eight lemmas, which are then stated and proved in Section 4. It appears to us that a classification of irreducible unitary representations is much more useful when accompanied by some additional information. Better understanding comes from knowledge of a number of standard realizations of unitary representations, including the relationships among them and the values of the corresponding parameters in the classification. Some information in this direction is assembled in Section 5 without proof.

2. Langlands Parameters of Irreducible Unitary Representations

In this section we shall specialize to $SU(2,2)$ some general facts concerning the Langlands classification of irreducible admissible representations in order to identify what has to be done in classifying irreducible unitary representations. Then we shall state our classification for $SU(2, 2)$ of the irreducible unitary representations.

Our notation is as follows. We let G be $SU(2,2)$, the group of 4-by-4 complex matrices g of determinant one such that $g^*jg = j$, where j is the diagonal matrix $j = \mathrm{diag}(1, 1, -1, -1)$. Let

$$\mathfrak{g} = \text{Lie algebra of } G,$$

$$\theta = \text{conjugate transpose inverse},$$

$$K = \{g \in G; \theta g = g\} = \left\{ \overset{\begin{matrix} 2 & 2 \end{matrix}}{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}} \begin{matrix} \}2 \\ \}2 \end{matrix} \right\},$$

$$\mathfrak{a}_{\min} = \left\{ \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}; s \in \mathbb{R},\ t \in \mathbb{R} \right\},$$

and define linear functions f_1 and f_2 on $\mathfrak{a}_{\min}$ by

$$f_1 \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix} = s \quad \text{and} \quad f_2 \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix} = t.$$

The nonzero simultaneous eigenvalues for the action of $\mathrm{ad}(\mathfrak{a}_{\min})$ on the Lie algebra $\mathfrak{g}$ are the *restricted roots*, and the corresponding eigenspaces are the *restricted-root spaces*. If α is a restricted root, we let $\mathfrak{n}^{(\alpha)}$ be the corresponding restricted-root space.

The restricted roots are $\pm f_1 \pm f_2$, $\pm 2f_1$, and $\pm 2f_2$, and they form a root system of type C_2. Here, $\pm f_1 \pm f_2$ have multiplicity two, and the others have multiplicity one. We define $f_1 \pm f_2$, $2f_1$, and $2f_2$ to be *positive*, and we call

$$\{af_1 + bf_2 : 0 \leqslant b \leqslant a\}$$

the *closed positive Weyl chamber* in the dual space $\mathfrak{a}'_{\min}$. Let

$$N_{\min} = \exp\left(\sum_{\alpha > 0} \mathfrak{n}^{(\alpha)}\right),$$

$$A_{\min} = \exp \mathfrak{a}_{\min},$$

$$T = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta})\},$$

$$M_{\min} = \{i^n t; 0 \leqslant n \leqslant 3 \text{ and } t \in T\},$$

$$P_{\min} = M_{\min} A_{\min} N_{\min}.$$

Matters have been arranged so that $G = KA_{\min}N_{\min}$ is an Iwasawa decomposition of G and $P_{\min}$ is a minimal parabolic subgroup of G. To each restricted root α, we associate an element γ_α of order 2 in G by

$$\gamma_{\pm f_1 \pm f_2} = -I, \qquad \gamma_{\pm 2f_1} = \operatorname{diag}(-1, 1, -1, 1), \qquad \gamma_{\pm 2f_2} = -\gamma_{2f_1}.$$

A *standard parabolic subgroup* P of G is any closed subgroup of G containing $P_{\min}$. There are four such subgroups, two of them being $P_{\min}$ and G, and all four can be written as $P = MAN$. (For $P = G$, we use $A = N = \{1\}$.) The other two are denoted

$$P_{f_1-f_2} = M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2} \qquad \text{and} \qquad P_{2f_2} = M_{2f_2}A_{2f_2}N_{2f_2},$$

where

$$\begin{aligned}
\mathfrak{a}_{f_1-f_2} &= \{H \in \mathfrak{a}_{\min};\, (f_1 - f_2)(H) = 0\},\\
\mathfrak{a}_{2f_2} &= \{H \in \mathfrak{a}_{\min};\, 2f_2(H) = 0\},\\
A_{f_1-f_2} &= \exp \mathfrak{a}_{f_1-f_2},\\
A_{2f_2} &= \exp \mathfrak{a}_{2f_2},\\
(M_{f_1-f_2})_0 &= \text{analytic subgroup of } G \text{ with Lie algebra}\\
&\quad \left\{ \begin{pmatrix} i\theta & w & x & z \\ -\bar{w} & -i\theta & \bar{z} & -x \\ x & z & i\theta & w \\ \bar{z} & -x & -\bar{w} & -i\theta \end{pmatrix};\ x, \theta \in \mathbb{R} \text{ and } w, z \in \mathbb{C} \right\},\\
M_{f_1-f_2} &= (M_{f_1-f_2})_0 \rtimes \{1, \gamma_{2f_2}\},\\
M_{2f_2} &= T \oplus \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \bar{\beta} & 0 & \bar{\alpha} \end{pmatrix};\ \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 - |\beta|^2 = 1 \right\}\\
N_{f_1-f_2} &= \exp(\mathfrak{n}^{(2f_1)} \oplus \mathfrak{n}^{(f_1+f_2)} \oplus \mathfrak{n}^{(2f_2)}),\\
N_{2f_2} &= \exp(\mathfrak{n}^{(2f_1)} \oplus \mathfrak{n}^{(f_1+f_2)} \oplus \mathfrak{n}^{(f_1-f_2)}).
\end{aligned}$$

The group T is a circle group, and we observe that

$$\begin{aligned}
M_{\min} &= T \oplus \{1, \gamma_{2f_2}\},\\
M_{f_1-f_2} &\cong SL(2, \mathbb{C}) \rtimes \{1, \gamma_{2f_2}\},\\
M_{2f_2} &\cong T \oplus SL(2, \mathbb{R}).
\end{aligned}$$

For each $P = MAN$, we define the *roots* of $(\mathfrak{g}, \mathfrak{a})$ to be the nonzero restrictions to $\mathfrak{a}$ of the restricted roots: $\{\pm(f_1 + f_2)\}$ in the case of $P_{f_1-f_2}$ and $\{\pm f_1, \pm 2f_1\}$ in the case of P_{2f_2}. We continue to call $f_1 + f_2$ and $2f_1$ *positive*. In the respective spaces $\mathfrak{a}'$, the nonnegative multiples of $f_1 + f_2$ and $2f_1$ comprise the closed positive Weyl chamber of $\mathfrak{a}'$. Half the sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$, with multiplicities counted, is denoted ρ_A; thus

$$\rho_{A_{\min}} = 3f_1 + f_2, \qquad \rho_{A_{f_1-f_2}} = 2(f_1 + f_2), \qquad \rho_{A_{2f_2}} = 3f_1.$$

For each parabolic the Weyl group $W(A : G)$ is the quotient of the normalizer of A in K by the centralizer of A in K. If s_α denotes reflection in α, then

$$W(A_{f_1-f_2} : G) = \{1, s_{f_1+f_2}\},$$
$$W(A_{2f_2} : G) = \{1, s_{2f_1}\},$$
$$W(A_{\min} : G) = \{1, s_{f_1-f_2}, s_{f_1+f_2}, s_{2f_2}, s_{2f_1}, s_{2f_2}s_{f_1-f_2}, s_{f_1-f_2}s_{2f_2}, s_{2f_1}s_{2f_2}\}.$$

An *admissible representation* of G is a representation of the universal enveloping algebra of $\mathfrak{g}$ on a complex vector space such that

(i) the action of the Lie algebra of K exponentiates to K;

(ii) every vector lies in a finite-dimensional K-stable subspace (is "K-finite");

(iii) every irreducible representation of K occurs with finite multiplicity.

From [6] we know that the space of K-finite vectors in any irreducible unitary representation is stable under $\mathfrak{g}$ and gives an irreducible admissible representation. Also an irreducible admissible representation comes from the space of K-finite vectors of a unitary representation if and only if it is *infinitesimally unitary* in the sense of admitting a $\mathfrak{g}$-invariant inner product, and in this case the unitary representation is unique (up to unitary equivalence) and irreducible.

The irreducible admissible representations have been classified by Langlands [25], and thus the problem of classifying the irreducible unitary representations of G amounts to deciding which Langlands parameters correspond to infinitesimally unitary representations. We shall describe a variant [22] of this classification after first defining the constructs that go into it.

Let $P = MAN$ be a standard parabolic subgroup. An irreducible unitary representation of M is in the *discrete series* of M if its matrix coefficients are in $L^2(M)$. The group P is said to be *cuspidal* if the discrete series of M is nonempty. Since it is known [8] that M has a nonempty discrete series if and only if rank $M = \text{rank}(K \cap M)$, it follows that P is cuspidal when $P = G$ or $P = P_{2f_2}$ or $P = P_{\min}$, but not when $P = P_{f_1 - f_2}$.

We shall recall a parametrization for the discrete series of M in each case. For $P_{\min}$, the group

$$M_{\min} = T \oplus \{1, \gamma_{2f_2}\}$$

is compact abelian, all irreducible unitary representations are one-dimensional and are in the discrete series, and a parametrization is by pairs

$$(n, \pm), \qquad n \in \mathbb{Z}, \tag{2.1}$$

with n indexing a character of T and with the sign indexing a character of $\{1, \gamma_{2f_2}\}$. For $P_{f_1 - f_2}$, we have

$$\begin{aligned} M_{f_1 - f_2} &\cong SL(2, \mathbb{C}) \rtimes \{1, \gamma_{2f_2}\} \\ &= SL(2, \mathbb{C}) \qquad \text{with scalars } \{i^k I\} \text{ adjoined;} \end{aligned}$$

since $SL(2, \mathbb{C})$ has no discrete series, the same thing is true of $M_{f_1 - f_2}$. For P_{2f_2}, we have

$$M_{2f_2} \cong T \oplus SL(2, \mathbb{R}),$$

and the discrete series of $SL(2, \mathbb{R})$ was obtained by Bargmann [2]. Thus a parametrization of the discrete series of M_{2f_2} is by pairs

$$(n, D_k^{\pm}), \qquad n \in \mathbb{Z},\ k \in \mathbb{Z},\ k \geqslant 2, \tag{2.2}$$

where $D_k^{\pm}$ is a discrete series representation of $SL(2, \mathbb{R})$. Finally for $P = M = G$, the discrete series parametrization was obtained by Harish-Chandra [8] in terms of character formulas. Let $\mathfrak{b}$ be the diagonal subalgebra of $\mathfrak{g}$, and let e_j be evaluation of the jth diagonal entry. A linear functional on $\mathfrak{b}^{\mathbb{C}}$ is of the form

$$\sum_{j=1}^{4} c_j e_j \qquad \operatorname{mod} \mathbb{C}(e_1 + e_2 + e_3 + e_4).$$

It is integral if each $c_i - c_j$ is an integer, and it is nonsingular if $c_i \neq c_j$ for $i \neq j$. The Harish-Chandra parameters are

$$\text{nonsingular integral linear functionals on } \mathfrak{b}^{\mathbb{C}}, \tag{2.3}$$

and interchange of indices 1 and 2, or indices 3 and 4, or both, leads to the only equivalences for the corresponding representations.

In the cases of M_{2f_2} and G, there are also irreducible unitary representations called *nondegenerate limits of discrete series*. These are obtained in

[40] from discrete series by a tensor product construction (see also [23]). For M_{2f_2} they are parametrized by pairs $(n, D_1^{\pm})$ with n in $\mathbb{Z}$. For G, let us first observe for discrete series that a parameter (2.3) uniquely determines an enumeration of $\{1, 2, 3, 4\}$ so that $c_i - c_j$ is positive if i precedes j; thus the parameter could as well be a pair consisting of the linear functional (2.3) and the consistent enumeration. For nondegenerate limits of discrete series, a parameter is a pair

$$\begin{pmatrix} \text{singular integral linear functional} & \text{consistent} \\ \text{on } \mathfrak{b}^{\mathbb{C}} \text{ with } c_1 \neq c_2 \text{ and } c_3 \neq c_4, & \text{enumeration} \end{pmatrix}, \tag{2.4}$$

and equivalences are given as in the case of discrete series.

Whenever $P = MAN$ is a standard parabolic subgroup and π is an irreducible unitary representation of M and ν is in $(\mathfrak{a}')^{\mathbb{C}}$, we can form the induced representation

$$U(P, \pi, \nu) = \operatorname{ind}_P^G(\pi \otimes e^\nu \otimes 1),$$

and this will be unitary if ν is imaginary. For general ν, the paper [20] defines a *standard intertwining operator* $A_P(w, \pi, \nu)$ for each w in the normalizer of $\mathfrak{a}$ in K by analytic continuation from a convergent integral; this operator varies meromorphically in ν and has the property that

$$U(P, w\pi, w\nu)\, A_P(w, \pi, \nu) = A_P(w, \pi, \nu)\, U(P, \pi, \nu) \tag{2.5}$$

as an identity of meromorphic functions.[1] Various normalizations of these operators are possible and will be denoted by $\mathcal{A}_P$ instead of A_P.

Now let $P = MAN$ be a cuspidal standard parabolic subgroup of G, let σ be a discrete series or nondegenerate limit of discrete series representation of M, and suppose ν in $(\mathfrak{a}')^{\mathbb{C}}$ has $\operatorname{Re} \nu$ in the closed positive Weyl chamber of $\mathfrak{a}'$. We shall associate a larger standard parabolic $P^* = M^*A^*N^*$ and a unitary representation[2] π of M^* to this situation. If $P = G$, we use $P^* = G$ and $\pi = \sigma$. If $P = P_{2f_2}$, there are two cases. For $\operatorname{Re} \nu = 0$, we use $P^* = G$ and $\pi = U(P, \sigma, \nu)$, and for $\operatorname{Re} \nu \neq 0$, we use $P^* = P_{2f_2}$ and $\pi = \sigma$.

If $P = P_{\min}$, there are four cases depending on $\operatorname{Re} \nu$. Let

$$S = \{\alpha = \text{restricted root}; \langle \operatorname{Re} \nu, \alpha \rangle = 0\}.$$

If S is empty, let $P^* = P_{\min}$. If $S = \{\pm(f_1 - f_2)\}$, let $P^* = P_{f_1 - f_2}$. If $S =$

[1] Technically such an identity is proved on a restricted domain, and the precise formulation is in Proposition 7.8 of [20].

[2] In the usual Langlands classification [25], π becomes the tempered representation parameter.

$\{\pm(2f_2)\}$, let $P^* = P_{2f_2}$. If S is the set of all restricted roots, let $P^* = G$. Then we take

$$\pi = \operatorname{ind}_{M(A \cap M^*)(N \cap M^*)}^{M^*}(\sigma \otimes \exp(\text{restriction of } \nu) \otimes 1) \tag{2.6}$$

as the associated unitary representation of M^*. It will be important to know when π is irreducible.

REFORMULATED COMPLETENESS OF LANGLANDS CLASSIFICATION [22]. *Let $P = MAN$ be a cuspidal standard parabolic subgroup, let σ be a discrete series or nondegenerate limit of discrete series representation of M, and let ν be a member of $(\mathfrak{a}')^{\mathbb{C}}$ with* Re ν *in the closed positive Weyl chamber. Let P^* and π be constructed from σ and ν as in* (2.6), *and suppose that π is irreducible. Then $U(P, \sigma, \nu)$ has a unique irreducible quotient $J'(P, \sigma, \nu)$, and every irreducible admissible representation of G is of the form $J'(P, \sigma, \nu)$ for some such triple (P, σ, ν).*

If we take into account the correspondence [22] between the above formulation and the original version of the Langlands classification, then we can translate into the present language the criterion of [22] for a Langlands quotient to be infinitesimally unitary (see the Criterion below). We can also apply this correspondence to sort out equivalences; the result of this step is incorporated as part of the Main Theorem.

CRITERION [17]. *$J'(P, \sigma, \nu)$ is infinitesimally unitary if and only if*

(i) *there exists w in $W(A : G)$ such that $w^2 = 1$, $w\sigma \cong \sigma$, and $w\nu = -\bar{\nu}$, and*

(ii) *the standard intertwining operator for w, when normalized to be pole-free and not identically zero as*

$$\sigma(w)\, \mathcal{A}_P(w, \sigma, \nu), \tag{2.7}$$

is positive or negative semidefinite.

If $J'(P, \sigma, \nu)$ is infinitesimally unitary, then every w satisfying (i) *is such that the operator* (2.7) *is semidefinite.*

MAIN THEOREM. *Let*

$P = MAN =$ a cuspidal standard parabolic subgroup of $SU(2, 2)$,

$\sigma =$ discrete series or nondegenerate limit of discrete series on M,

$\nu =$ member of $(\mathfrak{a}')^{\mathbb{C}}$ with Re ν *in the closed positive Weyl chamber of $\mathfrak{a}'$.*

Then the representation $J'(P, \sigma, \nu)$ of the reformulated Langlands *classification is defined as an irreducible admissible representation just in the following circumstances:*

(i) $P = G$. *In all cases.*[3]

(ii) $P = P_{2f_2}$. *In all cases except when* $\nu = 0$ *and* $\sigma \leftrightarrow (n, D_k^{\pm})$ *with* $n \not\equiv k \bmod 2$ *and* $|n| \neq k - 1$.

(iii) $P = P_{\min}$. *In all cases except the following:*

(a) $\nu = 0$ *and* $\sigma \leftrightarrow (n, \pm)$ *with n odd or the sign negative*

(b) $\nu = zf_1$ *with* $z \neq 0$, $\operatorname{Re} z \geqslant 0$, *and* $\sigma \leftrightarrow (n, -)$ *for any integer n*

(c) $\nu = iyf_2$ *with y nonzero real and either* $\sigma \leftrightarrow (n, -)$ *with n even or* $\sigma \leftrightarrow (n, +)$ *with n odd.*

In these circumstances, $J'(P, \sigma, \nu)$ *is infinitesimally unitary exactly in the following situations:*

(i) $P = G$. *In all cases.*[3]

(ii) $P = P_{2f_2}$, *with* $\nu = 3zf_1$ *and with*

(a) *z imaginary, or*

(b) *z positive and satisfying*

$$0 < z \leqslant \tfrac{1}{3} \qquad \text{when} \quad n \equiv k \bmod 2$$

or

$$0 < z \leqslant \tfrac{2}{3} \qquad \text{when} \quad |n| = k - 1.$$

(iii) $P = P_{\min}$, *with*

(a) ν *imaginary and* σ *arbitrary, or*

(b) $\nu = iy(f_1 - f_2) + x(f_1 + f_2)$ *when* $0 < x \leqslant 1$ *if* $\sigma \leftrightarrow (0, \pm)$, *or*

(c) $\nu = iyf_2 + xf_1$ *when* $0 < x \leqslant 1$ *and either* $\sigma \leftrightarrow (n, +)$ *for n even or* $\sigma \leftrightarrow (n, -)$ *for n odd, or*

(d) $\nu = af_1 + bf_2$ *when* $a \geqslant b > 0$ *and*

$a \leqslant 1$	*when* $\sigma \leftrightarrow (n, +)$, *n nonzero even*
$a + b \leqslant 2$	*when* $\sigma \leftrightarrow (0, -)$
$a \leqslant 1$ *or* $(a, b) = (3, 1)$	*when* $\sigma \leftrightarrow (0, +)$.

The following is a complete list of equivalences for the above irreducible infinitesimally unitary representations $J'(P, \sigma, \nu)$:

(i) $P = G$. *Equivalences occur in* (2.3) *and* (2.4) *under interchange of indices* 1 *and* 2, *or* 3 *and* 4, *or both.*

[3] A parametrization of these representations is given in (2.3) and (2.4).

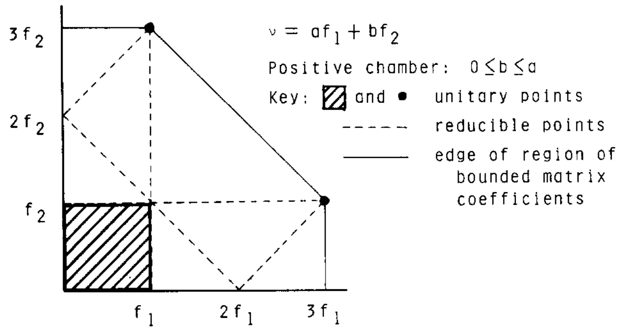

FIG. 1. Unitary $J'(P_{\min}, \sigma, \nu)$ with ν real and $\sigma \leftrightarrow (0, +)$.

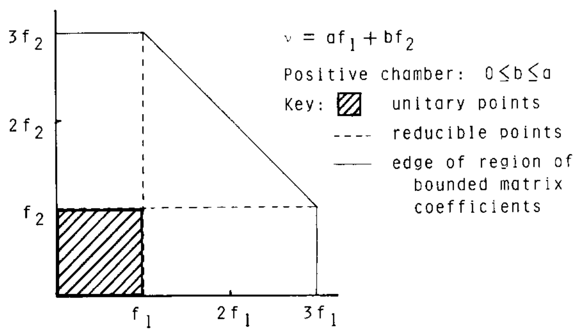

FIG. 2. Unitary $J'(P_{\min}, \sigma, \nu)$ with ν real and $\sigma \leftrightarrow (n, +)$ with n even and nonzero.

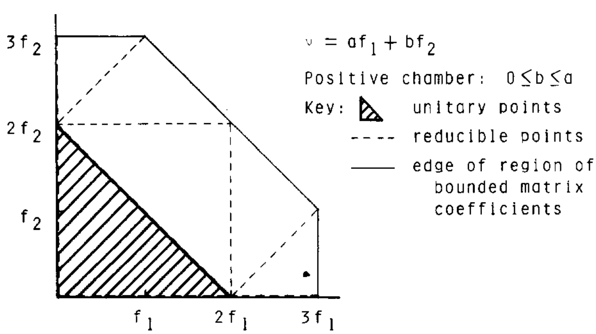

FIG. 3. Unitary $J'(P_{\min}, \sigma, \nu)$ with ν real and $\sigma \leftrightarrow (0, -)$.

(ii) $P = P_{2f_2}$. *Equivalences occur when* z *is imaginary and is replaced by* $-z$.

(iii) $P = P_{\min}$. *Equivalences occur in* (a) *when* (σ, ν) *is replaced by* $(w\sigma, w\nu)$ *for* w *in* $W(A_{\min} : G)$, *they occur in* (b) *and* (c) *when* $\operatorname{Im} \nu$ *is replaced by* $-\operatorname{Im} \nu$, *and they occur in* (d) *for the case* $a = b$ *when* n *is replaced by* $-n$.

The three accompanying figures show what parameters yield unitary representations in situation (iii,d) above. Parameters for which the induced representation is reducible (and J' is a proper quotient) are indicated in each figure.

3. Separation of Steps Requiring Proof

The bulk of the proof of the Main Theorem is the verification that the various representations in the theorem are unitary or nonunitary as claimed. For each of the three cuspidal standard parabolics MAN, we must deal with representations obtained from discrete series or nondegenerate limits of discrete series on M and from suitable ν on $\mathfrak{a}$. Most of the argument is quite easy, and the purpose of this section is to sort out what statements really need proving.

The final step is to unravel the equivalences by passing to the original formulation of the Langlands classification by means of the correspondence given in [22]. This final step is a routine computation using Theorem 14.2 of [23], and its proof is omitted.

We shall make extensive use of the standard intertwining operators of [20]. We mentioned the operators $A_P(w, \sigma, \nu)$ and their intertwining property (2.5) earlier (see Proposition 7.8 of [20]). When $w\sigma \cong \sigma$, we can extend σ so as to define $\sigma(w)$, and the operator $\sigma(w) A_P(w, \sigma, \nu)$ satisfies[4]

$$U(P, \sigma, w\nu)[\sigma(w) A_P(w, \sigma, \nu)] = [\sigma(w) A_P(w, \sigma, \nu)]\ U(P, \sigma, \nu)$$

as an identity of meromorphic functions. These operators can be normalized so as to eliminate certain poles; the normalized operators, denoted $\mathcal{A}$ instead of A, have additional properties if the system of normalizing factors satisfies certain axioms (see Sect. 8 of [20]). In the presence of these axioms, a normalized operator $\sigma(w) \mathcal{A}_P(w, \sigma, \nu)$ will be well defined for ν imaginary and will be a unitary self-intertwining operator for $U(P, \sigma, \nu)$ if also $w\nu = \nu$. When σ is a discrete series or nondegenerate limit of discrete series representation (and ν is imaginary), these self-intertwining operators span the commuting algebra of $U(P, \sigma, \nu)$, and a basis for their span is given by a

[4] See footnote 1.

computable subgroup $R_{\sigma,\nu}$ of $W(A:G)$, according to Theorem 13.4 of [20] and Theorem 12.6 of [23]. The group $R_{\sigma,\nu}$ will appear in several places in the steps below.

1. *Series for* $M = G$

The representations in question will be the members of the discrete series and those limits of discrete series having singularities only with respect to noncompact roots. For the parametrization of the discrete series, see Theorem 16 of [8]. For the appropriate facts about limits of discrete series, see Theorem 1.1 and Proposition 12.4 of [23].

2. *Series for* $M = M_{2f_2}$

We study the induced representation $U(P_{2f_2}, \sigma, \nu)$ and quotients of it. As in Section 2, σ is parametrized by a pair $(n, D_k^{\pm})$, where k and n are integers and k is $\geqslant 1$, and we write $\nu = z\rho_A$ with z complex.

a. *Computation of R group for* $\nu = 0$. There is a subgroup $R_{\sigma,0}$ of the Weyl group $W(A_{2f_2}:G) = \{1, s_{2f_1}\}$ that measures reducibility of $U(P_{2f_2}, \sigma, 0)$. We compute it using results of [15] and [23]. From Proposition 7.1 of [23], the Plancherel factor $\mu_{\sigma,2f_1}$ is given by

$$\begin{aligned}\mu_{\sigma,2f_1}(z\rho_A) &= cz(9z^2 - (n+k-1)^2)(9z^2 - (n-k+1)^2)\tan(3\pi z/2) \\ &\qquad\qquad \text{if} \quad \sigma(\gamma_{2f_1}) = +I \\ &= cz(9z^2 - (n+k-1)^2)(9z^2 - (n-k+1)^2)\cot(3\pi z/2) \\ &\qquad\qquad \text{if} \quad \sigma(\gamma_{2f_1}) = -I.\end{aligned}$$

Using Lemma 3.8b and Proposition 4.9b of [15], Corollary 12.5 of [23], and the formulas of Section 10 of [23], we consequently obtain

$$\begin{aligned}R_{\sigma,0} &= \{1\} && \text{if } n \equiv k \bmod 2 \text{ or } |n| = k-1 \\ &= \{1, s_{2f_1}\} && \text{otherwise.}\end{aligned}$$

b. *Analysis of z parameter.* If z is imaginary, $U(P_{2f_2}, \sigma, \nu)$ is induced from a unitary representation and so is unitary. In this case if $z \neq 0$, then $U(P_{2f_2}, \sigma, \nu)$ is irreducible by Corollary 9.2 of [23]. For $z = 0$, $U(P_{2f_2}, \sigma, \nu)$ is irreducible exactly when $R_{\sigma,0} = \{1\}$, by Theorem 12.6 of [23]. For the remaining values of z, the formal symmetry conditions (i) in the Criterion of Section 2 show that we may confine our attention to z real and positive.

c. *Case that z is positive and* $R_{\sigma,0} \neq \{1\}$. From Vogan's theory of minimal K-types [35], there are two minimal K-types, and the standard intertwining operator $A_P(s_{2f_1}, \sigma, z\rho_A)$ for $z > 0$ is nonvanishing on each of them. At $\nu = 0$, the normalized standard intertwining operator

$\sigma(s_{2f_1})\,\mathcal{O}\!\ell_P(s_{2f_1}, \sigma, 0)$ has opposite signs on these two K-types, in order to exhibit the reducibility. Therefore the operator $\sigma(s_{2f_1})\,\mathcal{O}\!\ell_P(s_{2f_1}, \sigma, z\rho_A)$ continues to have opposite signs on these two K-types for $z > 0$, and the operator cannot be semidefinite. (This style of argument was communicated to us by Baldoni Silva.)

d. *Case that z is positive and* $R_{\sigma,0} = \{1\}$. For a unitary representation the K-finite matrix coefficients must be bounded, and thus we need check only $0 < z \leqslant 1$. Let

$$\begin{aligned} z_0 &= \tfrac{1}{3} \qquad \text{if} \quad \sigma(\gamma_{2f_1}) = +I \\ &= \tfrac{2}{3} \qquad \text{if} \quad \sigma(\gamma_{2f_1}) = -I. \end{aligned}$$

In Lemma 1 we shall see that $U(P_{2f_2}, \sigma, z\rho_A)$ is reducible for $0 < z < 1$ only when $z = z_0$. Since $R_{\sigma,0} = \{1\}$, the intertwining operator $\sigma(s_{2f_1})\,\mathcal{O}\!\ell_P(s_{2f_1}, \sigma, z\rho_A)$ can be taken as the identity at $z = 0$, and the standard continuity argument [20, Sect. 16] shows that it must remain semidefinite until the first reducibility point z_0. If it is semidefinite somewhere between $z = z_0$ and $z = 1$, the same continuity argument shows it is semidefinite for $z = 1$. On the other hand, the possibility of a unitary representation at $z = 1$ is excluded by a theorem of Howe and Moore [9] that K-finite matrix coefficients tend to 0 (at infinity) for every nontrivial irreducible unitary representation. Thus our representation is infinitesimally unitary for $0 < z \leqslant z_0$ and not for $z_0 < z < \infty$.

3. *Series for* $M_{\min}$ *when A Parameter Is Real and Nonzero*

We study the induced representation $U(P_{\min}, \sigma, \nu)$ and quotients of it. As in Section 2, σ is parametrized by a pair $(n, \pm)$, where n is an integer and the sign $\pm$ is the sign of $\sigma(\gamma_{2f_2})$. We write $\nu = af_1 + bf_2$, and we need consider only the closed chamber

$$0 \leqslant \operatorname{Re} b \leqslant \operatorname{Re} a. \tag{3.1}$$

Here $\rho_A = 3f_1 + f_2$, and there will be unbounded matrix coefficients outside the convex hull of the orbit of ρ_A under the Weyl group $W(A_{\min} : G)$. Hence, in addition to (3.1), we may assume

$$\operatorname{Re} a \leqslant 3 \qquad \text{and} \qquad \operatorname{Re}(a+b) \leqslant 4. \tag{3.2}$$

a. *Computation of R group for* $\nu = 0$. The group $R_{\sigma,0}$ is a subgroup of $W(A_{\min} : G)$, and we compute it by means of the techniques of Sections 4 and 6 of [15]. The result is

$$\begin{aligned} R_{\sigma,0} &= \{1\} && \text{for } (n, +) \text{ when } n \text{ is even} \\ &= \{1, s_{2f_2}\} && \text{for } (0, -) \\ &= \{1, s_{2f_1}, s_{2f_2}, s_{2f_1}s_{2f_2}\} && \text{for } (n, -) \text{ when } n \neq 0 \text{ is even} \\ &= \{1, s_{2f_1}\} && \text{for } (n, +) \text{ when } n \text{ is odd} \\ &= \{1, s_{2f_2}\} && \text{for } (n, -) \text{ when } n \text{ is odd.} \end{aligned}$$

b. *Case that n is odd.* In Lemma 2 below, we use essentially Duflo's technique [3] and find two K-types on which the standard intertwining operator for $s_{2f_1}s_{2f_2}$ is nonzero scalar and has opposite sign. Since $s_{2f_1}s_{2f_2}$ satisfies (i) in the Criterion of Section 2, no unitary representations arise. (By way of motivation, we might expect from the formula for $R_{\sigma,0}$ that the argument of Step 2c above with two minimal K-types should apply here. Vogan's results [35] are not directly applicable here on the walls of the Weyl chamber, but Lemma 2 effectively shows that similar conclusions are valid here anyway.)

c. *Case of $(n, +)$ with n even and nonzero.* (See Fig. 2.) By Lemma 3 below, $U(P_{\min}, \sigma, \nu)$ is reducible within the set

$$a, b \text{ real}; \qquad 0 \leqslant b \leqslant a < 3, \qquad a + b < 4$$

just when $a = 1$ or $b = 1$. (See the dotted lines in the figure.) According to the formula for $R_{\sigma,0}$, the normalized standard intertwining operator for $s_{2f_1}s_{2f_2}$ is the identity at $\nu = 0$, and $s_{2f_1}s_{2f_2}$ satisfies the formal symmetry conditions. Thus standard arguments [20, Sect. 16] show that we get a semidefinite operator for $a \leqslant 1$ and that a further point with $b \neq 1$ where the operator is semidefinite would lead to a semidefinite operator somewhere on the boundary (where $a = 3$ or $a + b = 4$), in contradiction to the Howe–Moore theorem [9]. A special argument is needed to eliminate points with $b = 1$ (and $a > 1$). For such a point, Lemma 3 says that the Langlands quotient J' is a full degenerate series representation

$$\operatorname{ind}_{M_{2f_2}A_{2f_2}N_{2f_2}}^{G}((n, \text{trivial}) \otimes e^{af_1} \otimes 1),$$

which is irreducible for $1 < a < 3$. We can form a standard intertwining operator for this series, as in [20], and the operator must be nonsingular on each K-type. The Langlands quotient will be infinitesimally unitary if and only if this operator is semidefinite. Thus we can use the standard continuity argument [20, Sect. 16] to show that if the Langlands quotient is infinitesimally unitary for some a with $1 < a < 3$, then it is infinitesimally unitary for $a = 3$. Since $(a, b) = (3, 1)$ gives the coordinates of ρ_A, we have a contradiction to the Howe–Moore theorem [9]. Thus we get unitary representations for $a \leqslant 1$ and not otherwise.

d. *Case of* $(0, +)$. (See Fig. 1.) By Lemma 4 below, $U(P_{\min}, \sigma, \nu)$ is reducible within the set

$$a, b \text{ real}; \qquad 0 \leqslant b \leqslant a < 3, \qquad a + b < 4$$

just when

$$a = 1$$

or

$$b = 1$$

or

$$a + b = 2$$

or

$$a - b = 2.$$

(See the dotted lines in Fig. 1.) The argument is similar in spirit to that in the previous case. As in that case we get unitary representations for $a \leqslant 1$. Also the trivial representation, which arises from $(a, b) = (3, 1)$, is unitary. By the Howe–Moore theorem [9], no other points with $a = 3$ or $a + b = 4$ can yield unitary representations. The standard continuity argument [20, Sect. 16] then shows that it is enough to eliminate the points

$$b = 1 \qquad (\text{and } 1 < a < 3), \tag{3.3a}$$

$$a + b = 2 \qquad (\text{and } 1 < a \leqslant 2), \tag{3.3b}$$

$$a - b = 2 \qquad (\text{and } 2 \leqslant a < 3). \tag{3.3c}$$

Lemma 4 says that the Langlands quotient for (3.3a) is a full degenerate series representation

$$\operatorname{ind}_{M_{2f_2} A_{2f_2} N_{2f_2}}^{G} (1 \otimes e^{af_1} \otimes 1),$$

which is irreducible for $1 < a < 3$. The standard intertwining operators of [20] for this series $(1 < a < 3)$ are nonsingular Hermitian on each K-type, and it follows that all of them are semidefinite or else none of them is. Thus the whole series is infinitesimally unitary, or else none of it is. Since the trivial representation (occurring at $a = 3$) is known to be isolated in the unitary dual (by a theorem of Kazhdan [13]), we can conclude that none of the series is infinitesimally unitary.

We can argue similarly with (3.3b) and (3.3c), since by Lemma 4 the relevant quotients form a single degenerate series

$$\mathrm{ind}_{M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2}}^G(1 \otimes e^{(a-1)(f_1+f_2)} \otimes 1),$$

which is irreducible for $1 < a < 3$. The same argument, including the use of Kazhdan's theorem [13], shows that none of the series is infinitesimally unitary.

e. *Case of* $(n, -)$ *with n even and nonzero.* The formula for $R_{\sigma,0}$ shows that there is reducibility of the intermediate tempered representation when $b = 0$. Consequently we can ignore $b = 0$. By Lemma 5 below, $U(P_{\min}, \sigma, \nu)$ is reducible within the set

$$a, b \text{ real}; \qquad 0 < b \leqslant a < 3, \qquad a + b < 4 \tag{3.4}$$

just when $a = 2$, and also $a = 2$ coresponds to an irreducible degenerate series. A unitary point with $a = 3$ or $a + b = 4$ would immediately contradict the Howe–Moore theorem [9]. Any other unitary point would, by the arguments in Step 3d above, force some point of the degenerate series $a = 2$, $0 < b < 2$, to be unitary, say, for $b = b_0$. We shall show this is impossible.

In fact, since $SL(2, \mathbb{R})$ is isomorphic with $SU(1, 1)$, the inducing representation F is unitary with respect to an indefinite Hermitian form, and also it is fixed by s_{2f_1}. Let the lifted form for the induced representation be $(\cdot, \cdot)_F$, and let the standard intertwining operator for s_{2f_1} for the degenerate series be B. Then it follows that

$$\langle f, g \rangle = (Bf, g)_F \tag{3.5}$$

is a nonzero invariant Hermitian form for the induced representation. By irreducibility of the degenerate series, (3.5) must be the form (up to a constant) that exhibits the representation at b_0 as unitary. Then the standard continuity argument [20, Sect. 16] gives us unitary representations for $0 < b \leqslant 2$, and this conclusion at $b = 2$ contradicts the Howe–Moore theorem [9]. We conclude there are no unitary points.

f. *Case of* $(0, -)$. (See Fig. 3.) As in Step 3e, we can ignore $b = 0$. The computation of the formula for $R_{\sigma,0}$ shows that the normalized standard intertwining operator for $s_{2f_1}s_{2f_2}$ at $\nu = 0$ is scalar. Since Lemma 6 says that $U(P_{\min}, \sigma, \nu)$ is reducible within the set (3.4) just when $a = 2$ or $a + b = 2$ or $a - b = 2$, the standard continuity argument [20, Sect. 16] shows that we have unitary representations for $a + b \leqslant 2$. Since Lemma 6 says that the points with $a - b = 2$ and $2 < a < 3$ have Langlands quotients equivalent with degenerate series

$$\mathrm{ind}_{M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2}}^G(\mathrm{signum} \otimes e^{(a-1)(f_1+f_2)} \otimes 1),$$

the same argument as that in Step 3d shows that none of these representations is infinitesimally unitary or else all are. If all are unitary, then so is the one for $a = 3$, in contradiction to the Howe–Moore theorem [9]. So the points $a - 2 = 2$ with $2 < a < 3$ do not give unitary representations. The remaining points are eliminated by the same argument as in Step 3e, and thus $a + b \leqslant 2$ gives the only unitary points.

4. *Series for $M_{\min}$ when A Parameter Is Not Real (or Is Zero)*

We continue with the same notation as in Step 3, working within the set given by (3.1) and (3.2).

a. *Analysis of A parameter.* The first case is that $\operatorname{Re} \nu = 0$. Then $U(P_{\min}, \sigma, \nu)$ is unitary (and tempered), and only the irreducibility is at issue. By Theorem 13.4 of [20], $U(P_{\min}, \sigma, \nu)$ is irreducible if and only if $R_{\sigma,\nu} = \{1\}$. From Step 3a, $U(P_{\min}, \sigma, 0)$ is irreducible only for σ parametrized by $(n, +)$ with n even. Computing the remaining R groups, we find that $U(P_{\min}, \sigma, \nu)$ is irreducible for imaginary $\nu = af_1 + bf_2$ unless $(a, b) = (iy, 0)$ or $(0, iy)$. If $(a, b) = (iy, 0)$ with $y \neq 0$, there is irreducibility for $(n, +)$ for all n; if $(a, b) = (0, iy)$ with $y \neq 0$, there is irreducibility for $(n, +)$ for even n and for $(n, -)$ for odd n.

Now suppose that $\operatorname{Re} \nu \neq 0$. The case $\operatorname{Im} \nu = 0$ was treated in Step 3. Otherwise the constraints (3.1) and (3.2) and the formal symmetry conditions force

$$\nu = iy(f_1 - f_2) + x(f_1 + f_2) \qquad \text{with } x, y \in \mathbb{R} \text{ and } x > 0,\ y \neq 0$$

or

$$\nu = iyf_2 + xf_1 \qquad \text{with } x, y \in \mathbb{R} \text{ and } x > 0,\ y \neq 0.$$

b. *Case that $\nu = iy(f_1 - f_2) + x(f_1 + f_2)$.* Here x and y are real, x is positive, and y is nonzero. The formal symmetry conditions allow us to work only with the standard intertwining operator for the Weyl group element $s_{f_1+f_2}$ and to assume that σ corresponds to $(0, \pm)$. Fix y and regard x as varying. The standard intertwining operator for $x = 0$ is scalar by Step 4a, and Lemma 7 below shows that $U(P_{\min}, \sigma, \nu)$ is reducible for $0 < x < 2$ only when $x = 1$. Hence the standard continuity argument [20, Sect. 16] and the Howe–Moore theorem [9] show that we have unitary representations for $0 < x \leqslant 1$ but not for $x > 1$.

c. *Case that $\nu = iyf_2 + xf_1$.* Here x and y are real, x is positive, and y is nonzero. The formal symmetry conditions lead us to work with the standard intertwining operator for the Weyl group element s_{2f_1}. According to Lemma 8, this operator is semidefinite if and only if a certain standard intertwining operator for $SL(2, \mathbb{R})$ is semidefinite. But then we can read off the desired results from the facts for $SL(2, \mathbb{R})$ that unitary representations occur for the series obtained from the trivial representation on the minimal M with a positive A parameter out to rho and that unitary representations do not occur for M parameter nontrivial and A parameter positive.

4. Proofs of Lemmas

In this section we shall state and prove the eight lemmas referred to in Section 3. The first lemma uses the notation of Step 2d.

LEMMA 1. *The representation* $U(P_{2f_2}, \sigma, z\rho_A)$ *is reducible for* $0 < z < 1$ *only when* $z = z_0$, *where*

$$\begin{aligned} z_0 &= \tfrac{1}{3} \qquad \text{if} \quad \sigma(\gamma_{2f_1}) = +I \\ &= \tfrac{2}{3} \qquad \text{if} \quad \sigma(\gamma_{2f_1}) = -I. \end{aligned} \tag{4.1}$$

Proof. The question is where the Langlands intertwining operator has a nonzero kernel. This operator has the same kernel as the operator denoted[5]

$$A_{P_{2f_2}}(s_{2f_1}, (n, D_k^{\pm}), z\rho_A) \tag{4.2}$$

in Section 7 of [20]. Following the techniques of [20], we imbed this operator as the restriction to a subspace of a standard intertwining operator for the series obtained from $P_{\min}$. To do so, we observe that $2f_2$ plays the role of the positive root in $SL(2, \mathbb{R})$ and that the representation $D_k^{\pm}$ of $SL(2, \mathbb{R})$ imbeds as a subrepresentation in the nonunitary principal series of $SL(2, \mathbb{R})$ with parameters $((\gamma_{2f_2} \to (-1)^k), (k-1)f_2)$. Since $\rho_A = 3f_1$ in (4.2), the operator (4.2) is

$$\subseteq A_{P_{\min}}(s_{2f_1}, (n, \gamma_{2f_2} \to (-1)^k), (k-1)f_2 + 3zf_1). \tag{4.3}$$

Writing s_{2f_1} as a minimal product of simple reflections $s_{f_1-f_2}s_{2f_2}s_{f_1-f_2}$, we see from Theorem 7 of [19] that (4.3) is

$$\begin{aligned} &= A_{P_{\min}}(s_{f_1-f_2}, (-n, \gamma_{2f_2} \to (-1)^{k+n}), (k-1)f_1 - 3zf_2) \\ &\quad \times A_{P_{\min}}(s_{2f_2}, (-n, \gamma_{2f_2} \to (-1)^{k+n}), (k-1)f_1 + 3zf_2) \\ &\quad \times A_{P_{\min}}(s_{f_1-f_2}, (n, \gamma_{2f_2} \to (-1)^k), (k-1)f_2 + 3zf_1). \end{aligned}$$

[5] Technically we should choose a representative for s_{2f_1}, but the analysis we do will be independent of the representative.

By Lemma 56 of [19], these operators behave like standard intertwining operators in $SL(2, \mathbb{C})$, $SL(2, \mathbb{R})$, and $SL(2, \mathbb{C})$, respectively, namely, like

$$\begin{aligned} &A^{\mathbb{C}}(-n, \tfrac{1}{2}(k-1+3z) \times \text{root}) \\ &\quad \times A^{\mathbb{R}}((-1)^{k+n}, 3z \times \text{rho}) \\ &\quad \times A^{\mathbb{C}}(n, \tfrac{1}{2}(3z-k+1) \times \text{root}). \end{aligned}$$

The locations of the poles of these operators are discussed in Section 16 of [19]. We see immediately that a pole can occur for $-1 < z < 0$ only when $z = -\frac{2}{3}$ or $-\frac{1}{3}$. Now $\sigma(\gamma_{2f_1}) = (-1)^{k+n} I$, and we see from (4.1) that neither $A^{\mathbb{C}}$ factor has a pole for $z = z_0 - 1$. Also any pole from $A^{\mathbb{R}}$ is at most simple. Thus the only possible poles of (4.2) for $-1 < z < 0$ are at $z = -\frac{2}{3}$ and $z = -\frac{1}{3}$, and a pole at $z = z_0 - 1$ is necessarily simple. The operator (4.2) is known from [25] to have no poles for $z > 0$, and the composition of (4.2) at z with (4.2) at $-z$ is a multiple of the reciprocal of the Plancherel factor $\mu_{\sigma, 2f_1}(z\rho_A)$, by Proposition 7.3 of [20] and formula (7.3) of [23]. The formula for $\mu_{\sigma, 2f_1}$ in Step 2a of Section 3 thus shows that this composition is nonzero for $0 < z < 1$ and $z \neq \frac{1}{3}$ or $\frac{2}{3}$, and it has a pole if $z = 1 - z_0$. Comparing this fact with the worst possible poles of the operator at $-z$, we conclude that the operator is nonsingular for $0 < z < 1$ and $z \neq z_0$. The lemma follows.

For the next lemma we observe, in the spirit of [3], that a standard intertwining operator for the series for $P_{\min}$, when decomposed into the product of operators corresponding to simple reflections, is effectively exhibited as the composition of operators for $SL(2, \mathbb{R})$ and operators for $SL(2, \mathbb{C})$. For a K-type of multiplicity one, all these operators are scalar, and the given operator is a product of scalar factors corresponding to $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$. A necessary condition for the intertwining operator to be semidefinite is that such product scalars be all $\geqslant 0$ or all $\leqslant 0$.

We shall use some integral formulas for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$. (See Sect. 5 of [17] and also [14] and [3].) In each of these groups let KAN be the usual Iwasawa decomposition, with an element written as $g = \kappa(g)(\exp H(g))n$, let $V = \theta N$, and let ρ be the rho for A.

For the K of $SL(2, \mathbb{R})$, define

$$\tau_N \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{iN\theta}$$

and

$$a_N(\zeta) = \int_V e^{-(1+\zeta)\rho H(v)} \tau_N(\kappa(v))^{-1}\, dv.$$

Then an easy recursive computation gives

$$a_N(\zeta) = \pi^{1/2} \left[\prod_{j=1}^{|N|} \frac{\zeta - j}{\zeta + |N| + 1 - 2j} \right] \frac{\Gamma(\frac{1}{2}(\zeta - |N|))}{\Gamma(\frac{1}{2}(\zeta + 1 - |N|))}. \tag{4.4}$$

For the case of $SL(2, \mathbb{C})$, K is $SU(2)$, and we let τ_N be the irreducible unitary representation of dimension $N+1$, realized on homogeneous polynomials of degree N in two variables and written in the basis of monomials. Define

$$a_{\tau_N}(\zeta) = \int_V e^{-(1+\zeta)\rho H(v)} \tau_N(\kappa(v))^{-1} \, dv.$$

Then $a_{\tau_N}(\zeta)$ can be seen to be diagonal, and we let $a^N_{k,l}(\zeta)$ be its scalar value on the monomial $z_1^k z_2^l$, $k + l = N$. Another computations yields

$$a^N_{k,l}(\zeta) = \frac{\pi \{\prod_{j=1}^k (\zeta + \frac{1}{2}(k - l - 2j))\} \{\prod_{j=1}^l (\zeta + \frac{1}{2}(l - k - 2j))\}}{\prod_{j=1}^{k+l+1} (\zeta + \frac{1}{2}(k + l + 2 - 2j))}. \tag{4.5}$$

We shall use only a special case, namely,

$$a^N_{N,0}(\zeta) = a^N_{0,N}(\zeta) = \frac{\pi}{\zeta + \frac{1}{2}N}. \tag{4.6}$$

LEMMA 2. *Fix σ on $M_{\min}$ as given by $(n, \pm)$ with n odd. Then there exist two K-types τ and τ' occurring in the induced representation $U(P_{\min}, \sigma, af_1 + bf_2)$ with multiplicity one such that the restriction of the standard intertwining operator for $s_{2f_1} s_{2f_2}$ to the τ space (respectively, the τ' space) is a positive (respectively, negative) scalar for all real a and b with $0 \leqslant b \leqslant a$.*

Proof. We shall compute

$$\sigma(s_{2f_1} s_{2f_2}) A_{P_{\min}}(s_{2f_1} s_{2f_2}, \sigma, \nu) \tag{4.7}$$

by means of the formulas in Section 5 of [17]. We expand

$$s_{2f_1} s_{2f_2} = s_{f_1 - f_2} s_{2f_2} s_{f_1 - f_2} s_{2f_2}$$

and use the same notation for Weyl group elements and their representatives. Let τ be any irreducible representation of K (acting in a space V^τ) such that $\tau|_M$ contains σ, and let σ act on V_σ. From Proposition 5.3 of [17] we know that the restriction of (4.7) to the τ space can be regarded as the tensor product of an identity operator and the composition of the following five endomorphisms of $\operatorname{Hom}_M(V^\tau, V_\sigma)$:

$$E \to \sigma(s_{2f_1}s_{2f_2})\, E\tau(s_{2f_1}s_{2f_2})^{-1} \tag{4.8a}$$

$$\text{right by} \quad \tau(s_{2f_1}s_{2f_2})\, a_{\tau,2f_2}(\nu)\, \tau(s_{2f_1}s_{2f_2})^{-1} \tag{4.8b}$$

$$\text{right by} \quad \tau(s_{2f_1})\, a_{\tau,f_1-f_2}(s_{2f_2}\nu)\, \tau(s_{2f_1})^{-1} \tag{4.8c}$$

$$\text{right by} \quad \tau(s_{f_1-f_2}s_{2f_2})\, a_{\tau,2f_2}(s_{f_1-f_2}s_{2f_2}\nu)\, \tau(s_{f_1-f_2}s_{2f_2})^{-1} \tag{4.8d}$$

$$\text{right by} \quad \tau(s_{f_1-f_2})\, a_{\tau,f_1-f_2}(s_{f_1+f_2}\nu)\, \tau(s_{f_1-f_2})^{-1}, \tag{4.8e}$$

where $a_{\tau,\alpha}$ is defined for the simple restricted root α by

$$a_{\tau,\alpha}(\lambda) = \int_{V_\alpha} e^{-(\rho_\alpha+\lambda)H(v)}\, \tau(\kappa(v))^{-1}\, dv$$

in terms of the Iwasawa decomposition $v = \kappa(v)(\exp H(v))\, n$.

If $\tau|_M$ contains σ just once, then $\mathrm{Hom}_M(V^\tau, V_\sigma)$ is one-dimensional, and each of the five operators (4.8) is scalar. We are to compute the product of the five scalars in such cases.

In the usual realization of $SU(2, 2)$ the diagonal matrices form a compact Cartan subalgebra, and we let e_j be evaluation of the jth entry. The roots are $\{e_i - e_j,\ i \neq j\}$, and we choose our ordering to make $e_1 - e_2$ and $e_3 - e_4$ the positive compact roots. We can arrange matters so that our $\mathfrak{a}_{\min}$ roots $2f_2$ and $2f_1$ are the Cayley transforms of $e_2 - e_3$ and $e_1 - e_4$, respectively, and so that the fundamental character on the circle T in $M_{\min}$ is the exponential of $\frac{1}{4}(e_1 - e_2 - e_3 + e_4)$. An irreducible representation τ of K has highest weight

$$\frac{r}{2}(e_1 - e_2) + \frac{s}{2}(e_3 - e_4) + \frac{t}{2}(e_1 + e_2 - e_3 - e_4)$$

with r and $s \geqslant 0$; the integrality conditions are

$$r \in \mathbb{Z}, \qquad s \in \mathbb{Z}, \qquad t \in \mathbb{Z} + \tfrac{1}{2}(r+s).$$

Here $K = S(U(2) \times U(2))$, and r and s determine the restriction of τ to $SU(2) \times SU(2)$. Since the representations of $SU(2)$ are well understood, we can read off the weights of τ:

$$\frac{r-2m}{2}(e_1 - e_2) + \frac{s-2n}{2}(e_3 - e_4) + \frac{t}{2}(e_1 + e_2 - e_3 - e_4) \tag{4.9}$$

with $0 \leqslant m \leqslant r$ and $0 \leqslant n \leqslant s$. We can regroup (4.9) as

$$\frac{(2t-r-s)+2m+2n}{4}(e_2 - e_3) + \frac{r-s-2m+2n}{4}(e_1 - e_2 - e_3 + e_4)$$

$$+ \frac{2t+r+s-2m-2n}{4}(e_1 - e_4). \tag{4.10}$$

To see whether $\tau|_M$ contains σ, we can read off $\tau(\gamma_{2f_2})$ on a weight vector from the coefficient of $e_2 - e_3$, and we can look for a match with σ on the circle T in $M_{\min}$ from the coefficient of $e_1 - e_2 - e_3 + e_4$.

Suppose σ is given by $(2N+1, +)$ with $N \geqslant 0$. (The other cases are handled in similar fashion, and their proofs will be omitted.) Let τ be given by (r, s, t). The conditions on τ for $\tau|_M$ to contain σ are that there exist integers m and n such that the 5-tuple (r, s, t, m, n) satisfies

(i) $0 \leqslant m \leqslant r$ and $0 \leqslant n \leqslant s$,

(ii) $r - s - 2m + 2n = 2N + 1$,

(iii) $2t - r - s + 2m + 2n$ is in $4\mathbb{Z}$.

If the pair (m, n) is unique, then $\tau|_M$ contains σ just once.

For τ, we use $(r, s, t) = (N+1, N, \frac{1}{2})$, and then $(m, n) = (0, N)$ is the unique pair satisfying (i), (ii), (iii). For τ', we use $(r, s, t) = (N, N+1, -\frac{1}{2})$, and then $(m, n) = (0, N+1)$ is the unique pair satisfying (i), (ii), (iii).

Consider the restriction of τ and τ' to the maximal compact subgroup $SU(2)$ of $(M_{f_1 - f_2})_o \cong SL(2, \mathbb{C})$. A Cartan subgroup of $SU(2)$ is the circle T in $M_{\min}$, and again the fundamental character is the exponential of $\frac{1}{4}(e_1 - e_2 - e_3 + e_4)$. Referring to (i) and (ii) above, we see that $2N+1$ is the largest value that the left side of (ii) attains when m and n satisfy (i); this is true for both τ and τ'. Hence the restrictions of τ and τ' to $SU(2)$ both contain the representation with highest weight $(2N+1) \times$ (fundamental) once and otherwise contain only lower-dimensional representations. Hence the scalars for (4.8c) are what we have called

$$a^{2N+1}_{2N+1,0}(\langle s_{2f_2}\nu, f_1 - f_2\rangle / |f_1 - f_2|^2)$$

or

$$a^{2N+1}_{0,2N+1}(\langle s_{2f_2}\nu, f_1 - f_2\rangle / |f_1 - f_2|^2).$$

By (4.6), both of these are equal to

$$2\pi/(a + b + 2N + 1). \tag{4.11}$$

Similarly the scalars for (4.8e) are

$$a^{2N+1}_{2N+1,0}(\langle s_{f_1+f_2}\nu, f_1 - f_2\rangle / |f_1 - f_2|^2)$$

or

$$a^{2N+1}_{0,2N+1}(\langle s_{f_1+f_2}\nu, f_1 - f_2\rangle / |f_1 - f_2|^2).$$

By (4.6), both of these are equal to

$$2\pi/(a - b + 2N + 1). \tag{4.12}$$

Next consider the restriction of τ and τ' to the $SL(2, \mathbb{R})$ circle within the maximal compact subgroup

$$(SL(2, \mathbb{C}) \text{ circle}) \times (SL(2, \mathbb{R}) \text{ circle}) \subseteq (SL(2, \mathbb{C}) \text{ circle}) \times SL(2, \mathbb{R}) = M_{2f_2}.$$

This restriction corresponds to using the coefficient of $e_2 - e_3$ in (4.10). The scalars for (4.8b) are

$$a_j(\langle \nu, f_2 \rangle / |f_2|^2),$$

where $j = \frac{1}{2}(2t - r - s + 2m + 2n)$. Here $j = 0$ for both τ and τ'. Thus by (4.4), both τ and τ' contribute the factor

$$\pi^{1/2} \Gamma(\tfrac{1}{2}b) / \Gamma(\tfrac{1}{2}(b + 1)). \tag{4.13}$$

To determine the scalars for (4.8d), we need to know the effect of $\tau(s_{f_1 - f_2} s_{2f_2})$. Since s_{2f_2} centralizes $M_{\min}$, the $\tau(s_{2f_2})$ has no effect. However, to handle $\tau(s_{f_1 - f_2})$, we are led to look at the effect of $a_{\tau, 2f_2}$ on $\operatorname{Hom}_M(V^\tau, V_{s_{f_1 - f_2}\sigma})$, i.e., of using a pair (m', n') for which (ii) is replaced by

$$r - s - 2m' + 2n' = -(2N + 1).$$

For τ we take $(m', n') = (N + 1, 0)$, and for τ' we take $(m', n') = (N, 0)$. The scalars for (4.8d) are

$$a_j(\langle s_{f_1 - f_2} s_{2f_2} \nu, f_2 \rangle / |f_2|^2),$$

where $j = \frac{1}{2}(2t - r - s + 2m' + 2n')$. Then $j = 1$ for τ and $j = -1$ for τ'. Thus by (4.4), both τ and τ' contribute the factor

$$\pi^{1/2} \Gamma(\tfrac{1}{2}(a + 1)) / \Gamma(\tfrac{1}{2}(a + 2)). \tag{4.14}$$

Putting (4.11), (4.12), (4.13), and (4.14) together, we see that the scalars from (4.8b) to (4.8e) are the same for τ as they are for τ', and they are nonzero for both τ and τ'. Thus the lemma will be proved, in view of (4.8a), if we show that the scalar operators $\tau(s_{2f_1} s_{2f_2})$ and $\tau'(s_{2f_1} s_{2f_2})$ are negatives of each other.

Since rank G = rank K, $s_{2f_1} s_{2f_2}$ has a representative w such that $\operatorname{Ad}(w)$ equals the Cartan involution θ. This means we can choose w to be in the center of K and given by

$$\begin{aligned} w &= \exp(\pi i H_{e_2 - e_3} / |e_2 - e_3|^2) \exp(\pi i H_{e_1 - e_4} / |e_1 - e_4|^2) \\ &= \exp(\pi i H_{e_1 + e_2 - e_3 - e_4} / |e_1 - e_4|^2). \end{aligned}$$

Then $\tau(w)$ and $\tau'(w)$ are given by the effect on the respective highest weight spaces as

$$\exp\left[\frac{\pi it}{2}(e_1+e_2-e_3-e_4)(H_{e_1+e_2-e_3-e_4})/|e_1-e_4|^2\right]=e^{\pi it}.$$

For τ, this is i, and for τ', this is $-i$. The proof of Lemma 2 is complete.

LEMMA 3. *If σ is given by $(n,+)$ with n even and nonzero, then the representation $U(P_{\min},\sigma, af_1+bf_2)$ is reducible within the set*

$$a, b \text{ real};\qquad 0\leqslant b\leqslant a<3,\qquad a+b<4 \tag{4.15}$$

just when $a=1$ or $b=1$. At the points of reducibility where $b=1$ and $1<a<3$, the Langlands quotient J' is equivalent with an irreducible degenerate series representation

$$\operatorname{ind}_{M_{2f_2}A_{2f_2}N_{2f_2}}^{G}((n, \text{trivial})\otimes e^{af_1}\otimes 1), \tag{4.16}$$

where $(n, \text{trivial})$ denotes the representation of M_{2f_2} that is trivial on $SL(2,\mathbb{R})$ and is the nth character of the circle T.

Proof. If the representation is reducible, then the Langlands operator of [25] will have a nonzero kernel. At points in the interior of the set (4.15), the Langlands operator is essentially just the standard intertwining operator for $s_{2f_1}s_{2f_2}$, and at the remaining points the normalized standard intertwining operator for $s_{2f_1}s_{2f_2}$ is the composition of the Langlands operator and something else. So the first statement will follow if we show that the normalized operator[6]

$$\mathcal{A}_{P_{\min}}(s_{2f_1}s_{2f_2}, (n,+), af_1+bf_2) \tag{4.17}$$

is nonsingular within the set (4.15) except when $a=1$ or $b=1$. We expand the operator in terms of operators for simple reflections as

$$\begin{aligned}&=\mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (-n,+), -af_2-bf_1)\,\mathcal{A}_{P_{\min}}(s_{2f_2}, (-n,+), af_2-bf_1)\\ &\quad\times\mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (n,+), af_1-bf_2)\,\mathcal{A}_{P_{\min}}(s_{2f_2}, (n,+), af_1+bf_2).\end{aligned} \tag{4.18}$$

For purposes of computing kernels, this product behaves like a composition of standard operators $\mathcal{A}^{\mathbb{R}}$ and $\mathcal{A}^{\mathbb{C}}$ for $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$, namely,

$$\begin{aligned}&\mathcal{A}^{\mathbb{C}}(-n, \tfrac{1}{2}(a-b)\times\text{root})\,\mathcal{A}^{\mathbb{R}}(+, a\times\text{rho})\\ &\quad\times\mathcal{A}^{\mathbb{C}}(n, \tfrac{1}{2}(a+b)\times\text{root})\,\mathcal{A}^{\mathbb{R}}(+, b\times\text{rho}).\end{aligned} \tag{4.19}$$

[6] For definitions and properties, see Section 8 of [20].

Since $|n| \geqslant 2$, the $\mathcal{A}^{\mathbb{C}}$ factors are nonsingular for $|\frac{1}{2}(a+b)| < 2$ and $|\frac{1}{2}(a-b)| < 2$. Thus only the $\mathcal{A}^{\mathbb{R}}$ factors can contribute to the kernel, and they do so only when a or b is an odd integer. This proves the first statement of the lemma.

For $b = 1$ and $1 < a < 3$, the only contribution to the kernel of (4.19) is from the right-hand factor, and thus the Langlands quotient, which is necessarily irreducible, is equivalent with the image of

$$\mathcal{A}_{P_{\min}}(s_{2f_2}, (n, +), af_1 + f_2),$$

which is equivalent with

$$\operatorname{ind}_{M_{2f_2} A_{2f_2} N_{2f_2}}^{G}((n, \text{image}\, \mathcal{A}^{\mathbb{R}}(+, \text{rho})) \otimes e^{af_1} \otimes 1),$$

which is (4.16).

Lemma 4. *If σ is given by $(0, +)$, then the representation $U(P_{\min}, \sigma, af_1 + bf_2)$ is reducible within the set (4.15) just when $b = 1$ or $a + b = 2$ or $a - b = 2$. At the points of reducibility where $b = 1$ and $1 < a < 3$, the Langlands quotient J' is equivalent with an irreducible degenerate series representation*

$$\operatorname{ind}_{M_{2f_2} A_{2f_2} N_{2f_2}}^{G}(1 \otimes e^{af_1} \otimes 1).$$

At the points of reducibility where $a + b = 2$ (and $1 < a \leqslant 2$) or $a - b = 2$ (and $2 \leqslant a < 3$), the Langlands quotient J' is equivalent with an irreducible degenerate series representation

$$\operatorname{ind}_{M_{f_1 - f_2} A_{f_1 - f_2} N_{f_1 - f_2}}^{G}(1 \otimes e^{(a-1)(f_1 + f_2)} \otimes 1). \tag{4.20}$$

Proof. The argument for Lemma 3 applies through (4.19). This time we have $n = 0$, and the $\mathcal{A}^{\mathbb{C}}$ operators can contribute to the kernel within the parameter set (4.15), namely, when $\frac{1}{2}(a-b) = 1$ or $\frac{1}{2}(a+b) = 1$. The first statement of the lemma follows. The Langlands quotient for $b = 1$ and $1 < a < 3$ can then be identified in the same manner as in Lemma 3.

To handle the set where $a - b = 2$ and $2 \leqslant a < 3$, we rewrite (4.17) as

$$\begin{aligned} &= \mathcal{A}_{P_{\min}}(s_{2f_2}, (0, +), -af_1 + bf_2)\, \mathcal{A}_{P_{\min}}(s_{f_1 - f_2}, (0, +), -af_2 + bf_1) \\ &\quad \times \mathcal{A}_{P_{\min}}(s_{2f_2}, (0, +), af_2 + bf_1)\, \mathcal{A}_{P_{\min}}(s_{f_1 - f_2}, (0, +), af_1 + bf_2), \end{aligned} \tag{4.21}$$

and this behaves like

$$\begin{aligned} &\mathcal{A}^{\mathbb{R}}(+, b \times \text{rho})\, \mathcal{A}^{\mathbb{C}}(0, \tfrac{1}{2}(a+b) \times \text{root}) \\ &\qquad \times \mathcal{A}^{\mathbb{R}}(+, a \times \text{rho})\, \mathcal{A}^{\mathbb{C}}(0, \tfrac{1}{2}(a-b) \times \text{root}). \end{aligned} \tag{4.22}$$

For $a-b=2$ and $2<a<3$, only the right-hand factor contributes to the kernel. Thus the Langlands quotient is equivalent with the image of

$$\mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (0,+), af_1+bf_2),$$

which is equivalent with

$$\operatorname{ind}_{M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2}}^{G}((\text{image } \mathcal{A}^{\mathbb{C}}(0, \text{root}), +) \otimes e^{(1/2)(a+b)(f_1+f_2)} \otimes 1),$$

which is (4.20) because $\frac{1}{2}(a+b)=a-1$ when $a-b=2$.

A special supplement is needed when $(a,b)=(2,0)$. The Langlands operator is then substantially the one for s_{2f_1}, which is the product of the last three factors of (4.21). However, the first factor of (4.21) is the identity, in view of (4.22). Thus we can as well use the full operator (4.21). Now both the second and last factors of (4.22) might contribute to the kernel, and the argument in the previous paragraph shows that the Langlands quotient is an irreducible quotient of (4.20). But (4.20) is irreducible by Theorem 2 of [32], and hence the Langlands quotient is equivalent with (4.20).

To handle the set where $a+b=2$ and $1<a<2$, we note from (4.19) that the only contribution to the kernel of (4.18) is from the third factor. We can multiply (4.18) on the right by the inverse of the fourth factor (which is nonsingular) and see that the Langlands quotient is equivalent with the image of

$$\mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (0,+), af_1-bf_2),$$

which is equivalent with

$$\operatorname{ind}_{M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2}}^{G}((\text{image } \mathcal{A}^{\mathbb{C}}(0, \text{root}), +) \otimes e^{(1/2)(a-b)(f_1+f_2)} \otimes 1),$$

which is (4.20) because $\frac{1}{2}(a-b)=a-1$ when $a+b=2$. This proves the lemma.

LEMMA 5. *If σ is given by $(n,-)$ with n even and nonzero, then the representation $U(P_{\min}, \sigma, af_1+bf_2)$ is reducible within the set*

$$a, b \text{ real}; \qquad 0<b\leqslant a<3, \qquad a+b<4 \tag{4.23}$$

just when $a=2$. At the points of reducibility where $a=2$ and $0<b<2$, the Langlands quotient J' is equivalent with an irreducible degenerate series representation

$$\operatorname{ind}_{M_{2f_2}A_{2f_2}N_{2f_2}}^{G}((-n,F)\otimes e^{-bf_1}\otimes 1), \tag{4.24}$$

where $(-n,F)$ denotes the representation of M_{2f_2} that is the standard representation F on $SL(2,\mathbb{R})$ and is the $(-n)$th character of the circle T.

Proof. We argue as in Lemma 3, replacing the pair $(n, +)$ by $(n, -)$. We are led to study

$$\mathcal{O}\!\ell_{P_{\min}}(s_{f_1-f_2}, (-n, -), -af_2 - bf_1)\, \mathcal{O}\!\ell_{P_{\min}}(s_{2f_2}, (-n, -), af_2 - bf_1)$$
$$\times\, \mathcal{O}\!\ell_{P_{\min}}(s_{f_1-f_2}, (n, -), af_1 - bf_2)\, \mathcal{O}\!\ell_{P_{\min}}(s_{2f_2}, (n, -), af_1 + bf_2), \qquad (4.25)$$

which behaves like

$$\mathcal{O}\!\ell^{\mathrm{C}}(-n, \tfrac{1}{2}(a-b) \times \text{root})\, \mathcal{O}\!\ell^{\mathrm{R}}(-, a \times \text{rho})$$
$$\times\, \mathcal{O}\!\ell^{\mathrm{C}}(n, \tfrac{1}{2}(a+b) \times \text{root})\, \mathcal{O}\!\ell^{\mathrm{R}}(-, b \times \text{rho}). \qquad (4.26)$$

Since $|n| \geqslant 2$, only the $\mathcal{O}\!\ell^{\mathrm{R}}$ factors can contribute to the kernel, and they do so only when a or b is an even integer. This proves the first statement.

When $a = 2$ and $0 < b < 2$, only the second factor of (4.25) has a kernel, in view of (4.26), and thus the Langlands quotient is equivalent with the image of

$$\mathcal{O}\!\ell_{P_{\min}}(s_{2f_2}, (-n, -), 2f_2 - bf_1), \qquad (4.27)$$

which is equivalent with

$$\operatorname{ind}^G_{M_{2f_2}A_{2f_2}N_{2f_2}}((-n, \text{image}\, \mathcal{O}\!\ell^{\mathrm{R}}(-, 2 \times \text{rho})) \otimes e^{-bf_1} \otimes 1),$$

which is (4.24).

LEMMA 6. *If σ is given by $(0, -)$, then the representation $U(P_{\min}, \sigma, af_1 + bf_2)$ is reducible within the set (4.23) just when $a = 2$ or $a + b = 2$ or $a - b = 2$. At the points of reducibility where $a = 2$ and $0 < b < 2$, the Langlands quotient J' is equivalent with the irreducible degenerate series representation (4.24) for $n = 0$. At the points of reducibility where $a - b = 2$ and $2 < a < 3$, the Langlands quotient is equivalent with the irreducible degenerate series representation*

$$\operatorname{ind}^G_{M_{f_1-f_2}A_{f_1-f_2}N_{f_1-f_2}}(\text{signum} \otimes e^{(a-1)(f_1+f_2)} \otimes 1).$$

Proof. The argument for the first two statements is just as in Lemma 5, the extra reducibility for $n = 0$ coming from the $\mathcal{O}\!\ell^{\mathrm{C}}$ factors in (4.26). The final statement is proved in the same way as the analogous statement in Lemma 4.

LEMMA 7. *Let x and y be nonzero and real. If σ is given by $(0, \pm)$, then the representation*

$$U(P_{\min}, \sigma, iy(f_1 - f_2) + x(f_1 + f_2))$$

is reducible for $0 < x < 2$ only when $x = 1$.

Proof. The Langlands operator has the same kernel as the standard intertwining operator for $s_{f_1+f_2}$, which decomposes as a product

$$\begin{aligned}&\mathcal{A}_{P_{\min}}(s_{2f_2}, (0, \pm), iy(f_1+f_2) - x(f_1-f_2))\\&\quad\times \mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (0, \pm), iy(f_1+f_2) + x(f_1-f_2))\\&\quad\times \mathcal{A}_{P_{\min}}(s_{2f_2}, (0, \pm), iy(f_1-f_2) + x(f_1+f_2)),\end{aligned}$$

which in turn behaves like

$$\mathcal{A}^{\mathbb{R}}(\pm, (x+iy)\times \mathrm{rho})\, \mathcal{A}^{\mathbb{C}}(0, x\times \mathrm{root})\, \mathcal{A}^{\mathbb{R}}(\pm, (x-iy)\times \mathrm{rho}).$$

The end operators are invertible, and the lemma then follows from the known properties of $\mathcal{A}^{\mathbb{C}}$.

LEMMA 8. *Let x and y be nonzero and real. If σ is given by $(n, \pm)$, then*

$$\sigma(s_{2f_1})\, \mathcal{A}_{P_{\min}}(s_{2f_1}, (n, \pm), iyf_2 + xf_1)$$

is semidefinite if and only if the $SL(2, \mathbb{R})$ operator

$$\mathcal{A}^{\mathbb{R}}(\pm(-1)^n, x \times rho)$$

is semidefinite.

Proof. We decompose the given operator as a product

$$\begin{aligned}&\sigma(s_{2f_1})\, \mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (-n, \pm(-1)^n), iyf_1 - xf_2)\\&\quad\times \mathcal{A}_{P_{\min}}(s_{2f_2}, (-n, \pm(-1)^n), iyf_1 + xf_2)\\&\quad\times \mathcal{A}_{P_{\min}}(s_{f_1-f_2}, (n, \pm), iyf_2 + xf_1).\end{aligned}$$

The first intertwining operator in this product is the K-space-by-K-space adjoint of the third, by Proposition 8.6(iii) of [20], and these operators are invertible since $y \neq 0$. Moreover $\sigma(s_{2f_1})$ is scalar if the representative is chosen properly. Thus the question is whether a nonzero multiple of the middle intertwining operator is semidefinite, and the lemma follows.

5. OTHER REALIZATIONS OF UNITARY REPRESENTATIONS

In this section we shall discuss without proof how the irreducible unitary representations of $SU(2, 2)$ with certain Langlands parameters can be realized in some other ways.

1. *Unitarily Induced Representations with Nonreal Infinitesimal Character*

In the Main Theorem, the irreducible unitary representations with nonreal infinitesimal character are those of types (ii, a), (iii, a), (iii, b), and (iii, c), with the imaginary parameter assumed nonzero. In cases (ii, a) and (iii, a), we have $J'(P, \sigma, \nu) = U(P, \sigma, \nu)$ with ν imaginary, and thus J' is exhibited as a unitarily induced representation. In case (iii, b), $J'(P, \sigma, \nu)$ can be viewed as unitarily induced from $P_{f_1-f_2}$ with a complementary series representation of $SL(2, \mathbb{C})$ on $(M_{f_1-f_2})_o$, with the sign part of σ attached to iI, and with $\exp iy(f_1+f_2)$ on $A_{f_1-f_2}$; if $x = 1$, the complementary series on $SL(2, \mathbb{C})$ is replaced by the trivial representation. In case (iii, c), $J'(P, \sigma, \nu)$ can be viewed as unitarily induced from P_{2f_2} with the nth character used on $T \subseteq M_{2f_2}$, with a complementary series used on $SL(2, \mathbb{R}) \subseteq M_{2f_2}$, and with $\exp iyf_1$ on A_{2f_2}; if $x = 1$, the complementary series on $SL(2, \mathbb{R})$ is replaced by the trivial representation. In short, all irreducible unitary representations of $SU(2, 2)$ with nonreal infinitesimal character can be realized as unitarily induced from a proper parabolic subgroup MAN with a unitary character on A and a unitary representation with real infinitesimal character on M; this is a special case of an unpublished theorem of Vogen (see Item 4 in Sect. 4 of [17]).

2. *Complementary Degenerate Series*

The representations

$$\operatorname{ind}_{P_{f_1-f_2}}^G(\text{signum} \otimes \exp x(f_1+f_2) \otimes 1)$$

are irreducible and infinitesimally unitary for $0 \leqslant x < 1$ and occur in the classification in case (iii, d) with $\sigma \leftrightarrow (0, -)$ and

$$\nu = (1+x)f_1 + (1-x)f_2.$$

The representations

$$\operatorname{ind}_{P_{2f_2}}^G((n, \text{trivial}) \otimes \exp xf_1 \otimes 1)$$

are irreducible and infinitesimally unitary for n even and $0 \leqslant x < 1$. For $x = 0$, they occur in case (iii, c) with $\sigma \leftrightarrow (n, +)$ and $\nu = f_1$; for $0 < x < 1$, they occur in case (iii, d) with $\sigma \leftrightarrow (n, +)$ and $\nu = f_1 + xf_2$.

3. *Strichartz Series*

In Theorem 2 of [34], Strichartz decomposes the (nonempty) discrete spectrum of $O(4, 2)/O(3, 2)$, or equivalently of $SO_e(4, 2)/SO_e(3, 2)$, as $\mathscr{H}(0, \infty) = \sum_{d > -2} E_d$, and he describes E_d explicitly. Since $SO_e(4, 2)$ is a quotient of $SU(2, 2)$, we may regard E_d as a representation space for $SU(2, 2)$. The representation in question is the one in case (iii, d) of the Main Theorem with $\sigma \leftrightarrow (2(2+d), +)$ and $\nu = f_1 + f_2$. (This is the corner of the square in Fig. 2.) See also [28] for earlier discussion of the analysis of $SO_e(4, 2)/SO_e(3, 2)$.

4. *Effects of Outer Automorphisms*

For the remaining discussion, it will be useful for us to know the effects of outer automorphisms of $SU(2, 2)$ on the classification in the Main Theorem. Complex conjugation of matrices and group conjugation by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ give rise to two distinct nontrivial outer automorphisms φ_1 and φ_2 of $SU(2, 2)$.

For case (i) in the Main Theorem, the representations in question are discrete series or limits of discrete series. The effects of φ_1 and φ_2 on the parameters in (2.3) and (2.4), adjusted so as to the K-dominant, are

$$\varphi_1 \left(\sum c_j e_j\right) = -c_2 e_1 - c_1 e_2 - c_4 e_3 - c_3 e_4,$$

$$\varphi_2 \left(\sum c_j e_j\right) = c_3 e_1 + c_4 e_2 + c_1 e_3 + c_2 e_4.$$

For case (ii) in the Main Theorem, the representations are those arising from P_{2f_2}, and we write their parameters as $(n, D_k^\pm, z)$. Then

$$\varphi_1(n, D_k^\pm, z) = (-n, D_k^\mp, z),$$
$$\varphi_2(n, D_k^\pm, z) = (+n, D_k^\mp, z).$$

Finally for case (iii) in the Main Theorem, the representations are those arising from $P_{\min}$, and we write their parameters as $(n, \pm, \nu)$. Then

$$\varphi_1(n, \pm, \nu) = (-n, \pm, \nu),$$
$$\varphi_2(n, \pm, \nu) = (+n, \pm, \nu).$$

5. *Ladder Representations*

Ladder representations have been studied by many authors (e.g., [5, 11, 26]). They occur in the classification in case (ii, b) with $z = \frac{2}{3}$ and $|n| = k - 1$. For a particular series of ladder representations, one fixes the sign of n and the sign of $D_k^\pm$, and then $k \geqslant 1$ is the parameter. The representations with $k = 1, 2,$ and 3 have associated with them the terms "wave equation," "Dirac equation," and "Maxwell's equations," respectively, in the mathematical physics literature. The above discussion of automorphisms shows that the other choices for the sign of n and the sign of $D_k^\pm$ arise by applying outer automorphisms to a particular series of ladder representations. In some of the mathematical physics literature, outer automorphisms allow for the passage from particles to antiparticles.

6. *Holomorphic Discrete Series and Analytic Continuations*

As we recalled in (2.3), we can choose parameters for the discrete series of $SU(2,2)$ as

$$c_1e_1 + c_2e_2 + c_3e_3$$

with c_1, c_2, c_3 distinct nonzero integers such that $c_1 > c_2$ and $c_3 > 0$. The special case $c_1 < 0$ leads to the "holomorphic discrete series" studied earlier by Harish-Chandra [7]; these representations have a highest weight vector. If we replace c_1 by an integer variable c, we obtain a one-parameter family of representations π_c with Harish-Chandra parameter

$$ce_1 + [c-(c_1-c_2)]e_2 + c_3e_3,\ c_1-c_2>0 \text{ and } c_3>0,$$

studied by Wallach [36, 37], Rossi and Vergne [29], Gross and Holman [4], Jakobsen [10], and others.

For $c < 0$, π_c is in the holomorphic discrete series, necessarily unitary, but further values of c lead to unitary representations. For $c = 0$, π_c is a (unitary) limit of holomorphic discrete series, studied in [16]. To describe what happens for $c > 0$, we have to be more precise about the notation $\sigma \leftrightarrow (n, D_k^{\pm})$ used for representations of M_{2f_2}: let us agree that $e_2 - e_3$ is the positive root of M_{2f_2} (relative to the obvious Cartan subalgebra) and that

$$\mp\tfrac{1}{2}k(e_2-e_3) + \tfrac{1}{2}n(e_2+e_3)$$

is a minimal $(K \cap M_{2f_2})$-type of the σ corresponding to $(n, D_k^{\pm})$.

For $c = 1$, π_c occurs as the irreducible quotient of

$$U(P_{2f_2}, (n, D_k^{+}), f_1),$$

where

$$n = c_3 - (c_1 - c_2),$$

$$k = c_3 + (c_1 - c_2).$$

This is an instance of case (ii, b) of the Main Theorem with $n \equiv k \bmod 2$ and with $z = \frac{1}{3}$. Thus the representation π_1 is infinitesimally unitary.

For $c = 2$, π_c occurs as the irreducible quotient of

$$U(P_{2f_2}, (n, D_k^{+}), 2f_1),$$

where

$$n = c_3 - (c_1 - c_2),$$

$$k = c_3 + (c_1 - c_2) - 1.$$

When $|n| = k - 1$, this is an instance of case (ii, b) of the Main Theorem with $z = \frac{2}{3}$. Thus π_c is unitary if $c_1 - c_2 = 1$ or if $c_3 = 1$.

The special case in which $c_1 - c_2 = 1$ and $c_3 = 1$ leads to representations that can be realized in a space of scalar-valued holomorphic functions on G/K with a power of determinant as multiplier. When $c = 1$, the representation is the Hardy space representation of $SU(2, 2)$; the imbedding of this representation in

$$\mathrm{ind}_{P_{f_1 - f_2}}^{G}(1 \otimes 1 \otimes 1)$$

has been studied in [12] and [21]. When $c = 2$, the representation is the first ladder representation.

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