Formula for minimal K-types, maximally compact case

Assumptions: G linear connected semisimple

$\mathfrak{t}$ = θ-stable Cartan subalgebra of $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}$, $\mathfrak{b} \subseteq \mathfrak{k}$, $\mathfrak{a} \subseteq \mathfrak{p}$.

Δ = roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, Δ^- = roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ vanishing on $\mathfrak{a}$.

→ Maximally compact assumption: No member of Δ vanishes on $\mathfrak{b}$.

$\mathfrak{m}$ and M constructed in the usual way from $\mathfrak{a}$, so that $\mathfrak{b}$ is a compact Cartan subalgebra of $\mathfrak{m}$.

We can speak of compact and noncompact roots – roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ that vanish on $\mathfrak{a}$ and whose root vectors are in $\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{p}^{\mathbb{C}}$, respectively. (They are also in $\mathfrak{m}^{\mathbb{C}}$.)

We work only with $M^\# = M_0 Z_M$ since discrete series of M are induced from $M^\#$. (Probably $M^\#$ is connected under our assumptions.)

Let σ = discrete series of $M^\#$

λ_0 = a Harish-Chandra parameter of σ relative to $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$

$\lambda = \lambda_0 + \rho_n^- - \rho_c^-$ = unique minimal $K \cap M^\#$ type of σ, with positive system $(\Delta^-)^+$ of Δ^- chosen to make λ_0 dominant for it.

Remark: $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{k}$

Proof. Otherwise extend $\mathfrak{b}$ to a maximal abelian subspace of $\mathfrak{k}$ and then to a Cartan subalgebra of $\mathfrak{g}$, and end up with a more compact Cartan subalgebra of $\mathfrak{g}$ than $\mathfrak{b}$ is.

<u>Theorem.</u> $\Lambda = \lambda$ is the unique minimal K-type of $\mathrm{ind}_{K \cap M^\#}^{K} \sigma$.

Lemma 1. Restriction from $\mathfrak{t}^{\mathbb{C}}$ to $\mathfrak{b}^{\mathbb{C}}$ carries $\Delta - \Delta_m^-$ onto the set Δ_c of roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$.

Proof. Let β be in Δ with E_β in $\mathfrak{g}^{\mathbb{C}}$, and write $\beta = \beta_{\mathfrak{b}} + \beta_{\mathfrak{a}}$. Then $E_\beta + \theta E_\beta$ is in $\mathfrak{k}^{\mathbb{C}}$. If H is in $\mathfrak{b}^{\mathbb{C}}$, then

$$[H, E_\beta + \theta E_\beta] = [H, E_\beta] + \theta[H, E_\beta] = \beta_{\mathfrak{b}}(H) E_\beta + \beta_{\mathfrak{b}}(H)\theta E_\alpha$$

$$= \beta_{\mathfrak{b}}(H)(E_\beta + \theta E_\beta)$$

Hence $\beta_{\mathfrak{b}}$ is in Δ_c or $E_\beta + \theta E_\beta = 0$. (We know that $\beta_{\mathfrak{b}} \neq 0$, since no member of Δ vanishes everywhere on $\mathfrak{b}$.)

In the latter case, $-E_\beta = \theta E_\beta$ is a root vector for $\theta\beta = \beta_{\mathfrak{b}} - \beta_{\mathfrak{a}}$, and so $\beta = \theta\beta$, $\beta_{\mathfrak{a}} = 0$. Then β is in Δ^-. Since E_β satisfies $\theta E_\beta = -E_\beta$, E_β is in $\mathfrak{p}^{\mathbb{C}}$. Hence β is in Δ_m^-. We conclude restriction carries $\Delta - \Delta_m^-$ into Δ_c.

We show the map is onto Δ_c. Thus let β_1 be in Δ_c, with $X_{\beta_1} \in \mathfrak{k}^{\mathbb{C}}$ an associated root vector $\neq 0$. Write

$$X_{\beta_1} = \sum_{\beta \in \Delta} E_\beta + H_0 \quad , H_0 \in \mathfrak{h}^{\mathbb{C}}.$$

Then H in $\mathfrak{b}^{\mathbb{C}}$ implies

$$\beta_1(H) X_{\beta_1} = \sum_{\beta \in \Delta} \beta(H) E_\beta .$$

If H is in $\ker \beta_1$, then it follows that $\beta(H) = 0$ whenever $E_\beta \neq 0$. If $H = H_{\beta_1}$, then it follows that $\frac{\beta(H_{\beta_1})}{|\beta_1|^2} = 1$ whenever $E_\beta \neq 0$ and that $\frac{H_0}{|\beta_1|^2} = 0$. Consequently $H_0 = 0$ and $\beta|_{\mathfrak{b}} = \beta_1$ for every β for which $E_\beta \neq 0$. Applying Θ and averaging, we obtain

$$X_{\beta_1} = \sum_{E_\beta \neq 0} (E_\beta + \Theta E_\beta).$$

Choose β so that $E_\beta + \Theta E_\beta \neq 0$ in this expression; this is possible since $X_{\beta_1} \neq 0$. Then β is not in Δ_m^- and $E_\beta \neq 0$, so that $\beta|_{\mathfrak{b}} = \beta_1$. This proves the map is onto.

Remark. We can regard Δ_c^- as $\subseteq \Delta_c$, via the restriction map of Lemma 1.

Positive system Δ_c^+:

Let $\Delta_{0,c} = \{\beta|_{\mathfrak{b}} \mid \beta \in \Delta \text{ and } \langle \lambda_0, \beta \rangle = 0\} \subseteq \Delta_c$

$\Delta_{1,c}^+ = \{\beta|_{\mathfrak{b}} \mid \beta \in \Delta \text{ and } \langle \lambda_0, \beta \rangle > 0\} \subseteq \Delta_c$.

These notions depend only on $\beta|_{\mathfrak{b}}$, not on all of β, since λ_0 vanishes on $\mathfrak{a}$.

Choose a positive system $\Delta_{0,c}^+$ for $\Delta_{0,c}$, and define

$$\Delta_c^+ = \Delta_{1,c}^+ \cup \Delta_{0,c}^+ .$$

Then Δ_c^+ is a positive system for Δ_c, and $(\Delta_c^-)^+ \subseteq \Delta_c^+$.

Define ρ's in the obvious way.

<u>Theorem.</u> Relative to the positive system Δ_c^+ of roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, $\Lambda = \lambda$ is the unique minimal K-type of $\operatorname{ind}_{K \cap M^\#}^{K} \sigma$.

<u>Lemma 2.</u> $\Lambda = \lambda$ is integral for K, i.e., $\exp\Lambda$ is a well-defined character of the torus B of K.

Proof. $\exp\lambda$ is a well-defined character of B as a torus in M, since λ is the Blattner weight of σ.

<u>Lemma 3.</u> Suppose β_1 is a simple root for the system Δ_c^+ and is not the restriction of a member of $(\Delta_c^-)^+$. Then

(a) $s_{\beta_1}(\Delta^-) \subseteq \Delta^-$

(b) $s_{\beta_1}(\Delta_c^-) \subseteq \Delta_c^-$

(c) $\langle \rho^-, \beta_1 \rangle \geq 0$

(d) $s_{\beta_1}(\Delta_c^-)^+ \subseteq (\Delta_c^-)^+$ and hence $\langle \rho_c^-, \beta_1 \rangle = 0$.

Proof. (a) Let $\beta = \beta_1 + \beta_{or}$ be an extension of β_1 to a member of $\Delta - \Delta_m^-$, by Lemma 1. Since β is not in $(\Delta_c^-)^+$, by assumption, β is not in Δ^-. Thus $\beta_{or} \neq 0$. Thus

$$\frac{2\langle \beta, \theta\beta \rangle}{|\beta|^2} = \frac{2\langle \beta_1 + \beta_{or}, \beta_1 - \beta_{or} \rangle}{|\beta_1 + \beta_{or}|^2} \text{ is } -1, 0, \text{ or } 1. \qquad (*)$$

It cannot be 1, since otherwise $2\beta_{or}$ would be in Δ, and there are no members of Δ that vanish on b_r.

Suppose $(*)$ is 0. Then it follows that $s_{\beta_1} s_{\beta_{or}} = s_\beta s_{\theta\beta}$. Since $s_{\beta_{or}}$ fixes Δ^-, s_{β_1} acts as $s_\beta s_{\theta\beta}$ on Δ^- and must carry Δ^- into Δ, hence into Δ^-.

Suppose (*) is -1. Then it follows that $2\beta_1$ is in Δ. Thus s_{β_1} carries Δ^- into Δ, hence into Δ^-.

(b) Suppose (*) is 0. Then β, $\theta\beta$, and their negatives generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, and it follows that $s_\beta s_{\theta\beta}$ has a representative w in K. Thus s_{β_1} acts on $\mathfrak{h}^{\mathbb{C}}$ in the same way as an element $Ad(w)$ with w in K that normalizes $\mathfrak{a}$. The element w must then normalize $\mathfrak{m}$. Hence s_{β_1} leaves Δ_c^- and Δ_n^- stable.

Suppose (*) is 1. Here β is not a root of a split G_2 factor, and β_{or} is not useful. From "Weyl group of a cuspidal parabolic" essentially, $2\beta_1$ is a root of Δ_n^- such that $\pm 2\beta_1$ are orthogonal to all other roots of Δ^-. Then it is clear that $s_{\beta_1}(\Delta_c^-) \subseteq \Delta_c^-$.

(c) In view of (a),

$$s_{\beta_1}(2\rho^-) = \sum_{\substack{\alpha\in(\Delta^-)^+ \\ s_{\beta_1}\alpha>0}} s_{\beta_1}\alpha + \sum_{\substack{\alpha\in(\Delta^-)^+ \\ s_{\beta_1}\alpha<0}} s_{\beta_1}\alpha = \sum_{\substack{\beta>0 \\ s_{\beta_1}\beta>0}} \beta - \sum_{\substack{\beta>0 \\ s_{\beta_1}\beta<0}} \beta$$

$$= 2\rho^- - 2\sum_{\substack{\beta>0 \\ s_{\beta_1}\beta<0}} \beta$$

So
$$s_{\beta_1}(\rho^-) = \rho^- - \sum_{\substack{\beta>0 \\ s_{\beta_1}\beta<0}} \beta$$

and
$$\frac{2\langle\rho^-,\beta_1\rangle}{|\beta_1|^2} = \sum_{\substack{\beta>0 \\ s_{\beta_1}\beta<0}} \frac{2\langle\beta,\beta_1\rangle}{|\beta_1|^2}$$

In the sum on the right, we have $\beta > 0$ and $s_{\beta_1}\beta < 0$. Since λ_0 is nonsingular for Δ^-,

$$\langle \lambda_0, \beta \rangle > 0 \quad \text{and} \quad \langle \lambda_0, s_{\beta_1}\beta \rangle < 0 .$$

Hence
$$\frac{2\langle \beta, \beta_1 \rangle}{|\beta|^2} \langle \lambda_0, \beta_1 \rangle > 0 .$$

Since β_1 is in Δ_c^+, $\langle \lambda_0, \beta_1 \rangle \geq 0$. Thus $\frac{2\langle \beta_1, \beta_1 \rangle}{|\beta|^2} > 0$, and (c) follows.

(d) Regard $(\Delta_c^-)^+$ as $\subseteq \Delta_c^+$. Since β_1 is simple for Δ_c^+ and is not in Δ_c^-, $s_{\beta_1}(\Delta_c^-)^+ \subseteq \Delta_c^+$. Then (b) shows that $s_{\beta_1}(\Delta_c^-)^+ \subseteq (\Delta_c^-)^+$, and it follows that $\langle \rho_c^-, \beta_1 \rangle = 0$.

Lemma 5. $\Lambda = \lambda$ is dominant for Δ_c^+.

Proof. Let β_1 be simple for Δ_c^+. We have

$$\frac{2\langle \lambda, \beta_1\rangle}{|\beta_1|^2} = \frac{2\langle \lambda_0, \beta_1\rangle}{|\beta_1|^2} + \frac{2\langle \rho_n^- - \rho_c^-, \beta_1\rangle}{|\beta_1|^2}$$

$$= \frac{2\langle \lambda_0, \beta_1\rangle}{|\beta_1|^2} + \frac{2\langle \rho^-, \beta_1\rangle}{|\beta_1|^2} - \frac{2\langle 2\rho_c^-, \beta_1\rangle}{|\beta_1|^2}. \qquad (*)$$

If β_1 is the restriction of a member of $(\Delta_c^-)^+$, then β_1 is simple for $(\Delta_c^-)^+$ since $(\Delta_c^-)^+ \subseteq \Delta_c^+$. Hence

$$\frac{2\langle \rho^-, \beta_1\rangle}{|\beta_1|^2} \geq 1 \quad \text{and} \quad \frac{2\langle 2\rho_c^-, \beta_1\rangle}{|\beta_1|^2} = 2.$$

Since λ_0 is Δ^--nonsingular, we conclude $(*)$ is > -1, β_1 being positive. But the left side of $(*)$ is an integer, by Lemma 2, and must therefore be ≥ 0.

Now suppose β_1 is not the restriction of a member of $(\Delta_c^-)^+$. Then the first term on the right of $(*)$ is ≥ 0 since $\beta_1 > 0$, the second term is ≥ 0 by Lemma 3c, and the third term is 0 by Lemma 3d. Hence the left side of $(*)$ is ≥ 0.

<u>Lemma 6.</u> For $\Lambda = \lambda$, τ_Λ occurs in $\mathrm{ind}_{K\cap M^\#}^{K}\sigma$.

Remark. We shall use that $M^\#$ is connected here.

Proof. Let φ_Λ be a highest weight vector for τ_Λ. Then we have

$$\tau_\Lambda(H)\varphi_\Lambda = \lambda(H)\varphi_\Lambda \qquad \text{for } H \in \mathfrak{b}^{\mathbb{C}}$$

$$\tau_\Lambda(E_\beta)\varphi_\Lambda = \tau_\Lambda\left(\tfrac{1}{2}(E_\beta + \theta E_\beta)\right)\varphi_\Lambda = 0 \qquad \text{for } \beta \in (\Delta_c^-)^+ \subseteq \Delta_c^+ .$$

Also $M^\#$ is connected. Thus $\mathrm{span}\{\tau_\Lambda(\mathfrak{k}\cap\mathfrak{m}^\#)\varphi_\Lambda\}$ is an irreducible $K\cap M^\#$ module of type λ. Since σ_λ occurs in σ and $\tau_\Lambda|_{K\cap M^\#}$ has been shown to contain σ_λ, we conclude τ_Λ occurs in $\mathrm{ind}_{K\cap M^\#}^{K}\sigma$ by Frobenius reciprocity.

<u>Lemma 7.</u> $\langle \rho_c - \rho_c^-, \gamma\rangle \geq 0$ for $\gamma \in (\Delta^-)^+$.

Proof. First we observe s_γ leaves $\Delta_c - \Delta_c^-$ stable. In fact if β_1 is obtained by restriction to $\mathfrak{b}$ from $\beta = \beta_1 + \beta_{\mathfrak{a}}$ with $\beta_{\mathfrak{a}} \neq 0$ (cf. Lemma 1), then $s_\gamma\beta_1$ is obtained from $s_\gamma\beta = s_\gamma\beta_1 + \beta_{\mathfrak{a}}$.

Then we write

$$s_\gamma(\rho_c - \rho_c^-) = s_\gamma\Big(\tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-}}\beta_1\Big) = \tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1>0}} s_\gamma\beta_1 + \tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1<0}} s_\gamma\beta_1$$

$$= \tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1>0}}\beta_1 - \tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1<0}}\beta_1 = \tfrac{1}{2}\sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-}}\beta_1 - \sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1<0}}\beta_1$$

$$= \rho_c - \rho_c^- - \sum_{\substack{\beta_1\in\Delta_c^+\\ \beta_1\notin\Delta_c^-\\ s_\gamma\beta_1<0}}\beta_1$$

Expanding the left side, we obtain

$$\frac{2\langle \rho_c - \rho_c^-, \gamma\rangle}{|\gamma|^2}\gamma = \sum_{\substack{\beta_1 \in \Delta_c^+ \\ \beta_1 \notin \Delta_c^- \\ s_\gamma \beta_1 < 0}} \beta_1 .$$

Taking the inner product with λ_0 and using the inequality $\langle \beta_1, \lambda_0\rangle \geq 0$, we find

$$\frac{2\langle \rho_c - \rho_c^-, \gamma\rangle}{|\gamma|^2}\langle \gamma, \lambda_0\rangle = \sum \langle \beta_1, \lambda_0\rangle \geq 0 .$$

Since $\langle \gamma, \lambda_0\rangle > 0$ for $\gamma \in (\Delta^-)^+$, the lemma follows.

Proof of theorem. Let τ_{Λ_0} be a minimal K-type of $\mathrm{ind}_{K\cap M^\#}^{K} \sigma$. By Frobenius reciprocity, $\tau_{\Lambda_0}|_{K\cap M^\#}$ contains some $K\cap M^\#$ type $\sigma_{\lambda'}$ of σ. Then λ' is a weight of τ_{Λ_0}, and we have

$$|\Lambda_0 + 2\rho_c|^2 \leq |\Lambda + 2\rho_c|^2 \qquad \text{by Lemma 6 and minimality} \qquad (1)$$

$$|\lambda + 2\rho_c^-|^2 \leq |\lambda' + 2\rho_c^-|^2 \qquad \text{by minimality} \qquad (2)$$

$$|\lambda'|^2 \leq |\Lambda_0|^2 \qquad \text{since } \lambda' \text{ is a weight of } \tau_{\Lambda_0} \qquad (3)$$

$$\lambda' = \Lambda_0 - \sum m_i \beta_i \qquad \text{since } \lambda' \text{ is a weight of } \tau_{\Lambda_0}. \qquad (4)$$

$$(\beta_i \in \Delta_c^+,\ m_i \geq 0)$$

We write

$$
\begin{aligned}
|\lambda|^2 &= |\lambda + 2\rho_c^-|^2 - 4\langle \lambda, \rho_c^- \rangle - 4|\rho_c^-|^2 \\
&\le |\lambda' + 2\rho_c^-|^2 - 4\langle \lambda, \rho_c^- \rangle - 4|\rho_c^-|^2 && \text{by (2)} \\
&= |\lambda'|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle \\
&\le |\Lambda_0|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle && \text{by (3)} \\
&= |\Lambda_0 + 2\rho_c|^2 - 4\langle \Lambda_0, \rho_c \rangle - 4|\rho_c|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle \\
&\le |\Lambda + 2\rho_c|^2 - 4\langle \Lambda_0, \rho_c \rangle - 4|\rho_c|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle && \text{by (1)} \\
&= |\Lambda|^2 + 4\langle \Lambda - \Lambda_0, \rho_c \rangle + 4\langle \lambda' - \lambda, \rho_c^- \rangle \\
&= |\lambda|^2 + 4\langle \lambda - \lambda' - \Sigma m_i \beta_i, \rho_c \rangle + 4\langle \lambda' - \lambda, \rho_c^- \rangle && \text{by (4).}
\end{aligned}
$$

Hence $\quad 4\langle \lambda' - \lambda, \rho_c - \rho_c^- \rangle \le -4 \langle \Sigma m_i \beta_i, \rho_c \rangle \le 0$.

By Schmid's Theorem, $\lambda' - \lambda$ is the sum of members of $(\Delta^-)^+$. Then $4\langle \lambda' - \lambda, \rho_c - \rho_c^- \rangle \ge 0$ by Lemma 7.

We conclude first that $\langle \Sigma m_i \beta_i, \rho_c \rangle = 0$, from which it follows that $\lambda' = \Lambda_0$, and second that $|\lambda + 2\rho_c^-|^2 = |\lambda' + 2\rho_c^-|^2$ in the chain of inequalities above, from which it follows that $\lambda' = \lambda$. Then $\Lambda_0 = \lambda' = \lambda = \Lambda$, and the theorem is proved.