Notes of a Course

Functional Analysis

Given by William Feller

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(Some lectures near the end were given by Christopher Anagnostakis.)

Personal Notes of A. W. Knapp

Adjoint transformations

Banach spaces: $T : X \to Y$, T bounded linear, domain all of X

Kernel $N_X = \{x \mid Tx = 0\}$.

Quotient space X/N_x is mapped into R_Y = closure of range. Can look at map either way.

Adjoint transformation: $X^* \xleftarrow{T^*} Y$. To every x and y^*

(y^*, Tx) is defined.

So exists unique x^* such that $(y^*, Tx) = (x^*, x)$. Define $x^* = T^* y^*$

Can use matrix notation $y^* T x$

T induces two multiplications: Tx and y^*T.

Three ways of looking at adjoint

1) T^*

2) $N_X^\perp = (X/N_X)^* \longleftarrow R_Y^*$

3) $Y^*/N_{Y^*} \longrightarrow R_{X^*}$

(2) and (3) should be the same

$R_{X^*} = N_{X^\perp}$: closure of range of adjoint = annihilator of kernel

$R_Y^\perp = N_{Y^*}$: $N_{Y^*} = \{y \mid y^* T x = 0,\ y^* \perp Tx,\ y^* \perp \text{range}$, and conversely$\}$

Example:

Cont fcns on $[0,1]$. Define $y(t) = \int_0^t x(s)\,ds$. $y = Tx$, $y(0) = 0$.

Adjoint consists of measures on $[0,1]$

$$\int_{0^-}^{1^+} x(s)\,\xi(ds) \underset{\text{somehow}}{=} \int_{0^-}^{1^+} y(s)\,\eta(ds)$$. y has a derivative
Integrate by parts

$$= y(s)\,\eta(s)\Big|_0^1 - \int_0^1 \eta(s)\,x(s)\,ds$$ where $\eta(t) = \eta[-\infty, t)$
$\eta(0) = 0$.

$= y(1)\eta(1) - \eta(0)y(0) - \int_0^1 \eta(s)\,x(s)\,ds$

So $\xi(t) = \eta(1)t - \int_t^1 \eta(s)\,ds$. Check by integrating x wrt this.

Adjoint mapping maps measures into absolutely cont. measures.
So look at adjoint as mapping abs. cont measures into themselves or
as mapping functions into functions.

Example: Harmonic functions in disc

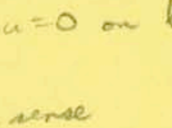
$\Delta u = f$. Given f, find u. Domain $f = C$, cont fcns in closed disk, Banach space with sup norm. Put boundary condition, say $\bar{u} = 0$, $u = 0$ on boundary in some sense. We can put any topology on U; but we choose one (like sup) which makes u cont. Formally write

$$\int v(p)\, f(p)\, dp =$$

Harmonic function example:

Want $\Delta u = f$, $\bar{u} = 0$, ($u = 0$ on boundary)

$\uparrow$ in some sense



Look at as transformation $f \to u$ of bnded cont fcns $\to$ bnded cont. fcns.

If $f > 0$, look where u assumes maximum. Then $\Delta u \leq 0$. So u does not assume + maximum. So solution is negative everywhere. So write $\Delta u = -f$. Positive transformation, so monotone. If $\|f\| \leq 1$, worst case is $f = 1$. Hence transformation is bounded.

Adjoint transformation $\begin{matrix} f \to u \\ v \leftarrow g \end{matrix}$

Backwards we map measures $g \to$ measures v. Assume absolutely cont. wrt Lebesgue measure. Want

$$\int f v \overset{?}{=} \int u g. \quad \leftarrow \text{kernel is Green's function}$$

$$\shortparallel$$

$$\int (\Delta u) v$$

But $\int (v \Delta u - u \Delta v) = \oint \left(\bar{v} \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial \bar{v}}{\partial n} \right) ds$ $\uparrow$ arc length

So above $= \int u \Delta v + \int \bar{v} \frac{\partial \bar{u}}{\partial n} \, ds$

Want $\Delta v = g$, $\bar{v} = 0$. This works for a dense set of measures; say ones with smooth densities. Then use approximation.

Get $u(p) = -\int G(p, q) \, f(q) \, dq$, Lebesgue integration. If there is a solution, we can write it in this form. Kernel is non-negative. Adjoint transformation is formally given in same way

$$v(q) = -\int g(p) \, G(p, q) \, dp$$

For cont fcns they are the same. So we get

$$G(p, q) = G(q, p)$$

Adjoint is again bounded. This situation shows how we do not consider the whole adjoint space.

Dirichlet problem

$\Delta u = 0$, $\bar{u} = f$

Map $f \to u$.

But map is entirely different; f is on boundary, u in interior.

Existence will be assumed.

Bounded transformation: norm 1.

Function does not assume max and min in interior. Hence sup & inf are at f, norm is 1.

Adjoint

$$\begin{array}{c} f \to u \\ V \leftarrow g \end{array}$$

$\leftarrow$ functions in interior

$\uparrow$ functions on boundary

Want $\int ug = \oint fV$

Let $\Delta V = g$, $\bar{V} = 0$. V is unique. We have this from 1st problem

$$\int u \Delta V = \underset{\substack{\uparrow \\ \Delta u = 0}}{0} + \oint f \frac{\partial \bar{V}}{\partial n}$$

Hence V gives required: $g \to \frac{\partial \bar{V}}{\partial n}$ = function on boundary

Poisson integral

$$V(p) = \int G(p,q)\, g(q)\, dq$$

$$\frac{\partial}{\partial n} V(p) = \int \frac{\partial}{\partial n} \text{ - - - }, \quad p \text{ on boundary}$$

$$\text{So } u(q) = \int \frac{\partial}{\partial n} G(p,q) \quad f(p)\, dp$$

Norm of adjoints

$\|T^*\| = \|T\|$ claimed.

$$\|T\| = \underset{\substack{\uparrow \\ \text{understood} \\ \text{with norm 1}}}{\sup} (x^*, Tx) = \sup (T^* x^*, x) = \|T^*\|$$

Let $X \xrightarrow{T} Y$. Assume $Tx \neq 0$ if $x \neq 0$; no restriction because (3)
we can consider quotient spaces. Can take closure of range as Banach
space; hence range T is dense in Y. Adjoint has same
properties from last time.

Look at T^{-1}; for every element in range, T^{-1} is defined. Inverse does
not have to be bounded.

T^{-1} bounded $\Leftrightarrow$ range $= Y$

$(T^*)^{-1}$ is bounded and equals $(T^{-1})^*$.

If T^* has bounded inverse, then adjoint to adjoint has bounded inverse. But
second adjoint is extension of given result. Second adj of inv.
maps range T into X and hence all of Y into X by boundedness.

Bounded linear transformations on Hilbert space

$T: X \to X$ bounded, X a Hilbert space

We will look for transformations associated with it. We will get polynomials
and approximations in some topology to e^T, etc.

Assume $T = T^*$, T self-adjoint. Then
$$(y, Tx) = (Ty, x).$$

Let $\xi \in X$ and look at $T\xi, T^2\xi, T^3\xi$; take closure of span. This is
normally everything (in finite case, if two eigenvalues are same. Space is
separable. Take $\|T\| = 1$. Map
$$\xi \to 1, \; T^k\xi \to t^k$$
This maps dense set of space into polynomials. We shall construct a
real isomorphism into square integrable fcns. Transformation will be
multiplication by t. It is an entirely different thing to go to the operator
algebra $\Sigma a_k T^k \xi \to \Sigma a_k T^k$.

$\mathcal{H}$ a Hilbert space

T a bounded transformation, $\|T\| \le 1$

$(y, Tx) = (Ty, x)$, symmetric

(x, Tx) is real: In complex case $(x, Tx) = \overline{(Tx, x)}$.

Exercise to show (x, Tx) real $\Rightarrow T$ symmetric

$$\|T\| = \sup_{\|x\| \le 1} |(x, Tx)|$$

Proof: $N = \sup |(x, Tx)|$

$\|T\| = \sup (y, Tx)$, definition

$N \le \|T\|$, trivially

$$4(y, Tx) = (x+y, T(x+y)) - (x-y, T(x-y))$$

$$4\|T\| \le N\left[\|x+y\|^2 + \|x-y\|^2\right] \quad \text{in real case}$$

$$= 2N\left(\|x\|^2 + \|y\|^2\right) \le 4N$$

$$N \ge \|T\|.$$

Symmetric = Hermitian. All operators are like this

T is positive if $(x, Tx) \ge 0$. Gives partial ordering.

$T + \lambda I$ and T have same kinds of properties

Look at $1 - T$.

Norm: $(x, X - Tx) = 1 - (x, Tx)$

So $\left\| \frac{1-T}{2} \right\| \le 1.$

$\frac{1-T}{2} \ge 0$

So we may assume that bounded operators are positive and have norm 1.

Situation: $0 \le (x, Tx) \le 1$ if $\|x\| \le 1$ without loss

Lemma: For every j and k (if T is positive),

$$(x, T^j(1-T)^k x) \ge 0.$$

Proof:

Induction $(x, Tx) \ge 0$

$(x, (1-T)x) \ge 0$

Claim $(x, T(1-T)x) \ge 0$

$(x, (\frac{1}{2}-T)^2 x) \le \frac{1}{4}$ since $((\frac{1}{2}-T)x, (\frac{1}{2}-T)x) \le \frac{1}{4}$. since $\|\frac{1}{2}-T\| \le \frac{1}{2}$ and $|(x, (\frac{1}{2}-T)x)| \le \frac{1}{2}$

$0 \le Tx \le 1$

So $(x, \frac{1}{4}x) - (x, T(1-T)x) \le 1/4$ and claim holds.

If j and k are even

$j = 2n$, $k = 2m$

$(T^n(1-T)^m x, T^n(1-T)^m x) \ge 0$ shove back

If j odd, k even, we use $(y, Ty) \ge 0$

If j even, k odd, $(y, (1-T)y) \ge 0$

If both odd $(y, T(1-T)y) \ge 0$. Q.E.D.

Single out ξ, look at closure of span of $\xi, T\xi, T^2\xi, \ldots$ This is a Hilbert space $\mathcal{H}_\xi$. $\|\xi\| = 1$

Associate $a_0\xi + a_1 T\xi + \ldots + a_n T^n \xi \to a_0 + a_1 t + \ldots + a_n t^n$.

Dense set is mapped into polynomials.

Call polynomial $p(T)\xi \to p(t)$, $0 \le t \le 1$ since $0 \le T \le 1$.

$(\xi, p(T)\xi)$ is a linear functional $E(p)$ on set of polynomials

Claim functional is positive and E is expectation w.r.t. prob. measure or $$E(p) = \int_{0-}^{1+} p(t)\,\mu(dt) \quad \text{with } \mu\{[0,1]\} \le 1.$$

Measures are bounded.

Does there exist a measure μ such that

$$\int_0^1 t^k \mu(dt) = c_k \ , \ k=0,1,\ldots$$

This gives value for every polynomial.

Problem reduced to moment problem (Hausdorff moment problem).

Thm: A necessary and sufficient condition that μ exist is that

$$E\left(t^j (1-t)^k\right) \ge 0.$$

Necessity: If μ exists, polynomial is ≥ 0 and $\int$ is ≥ 0.

Sufficiency:

Bernstein polynomials p_n corresponding to f continuous

$$p_n = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right)$$

As $n \to \infty$ $p_n \to f$ uniformly. Assume this for a moment.

Let $h>0$, define $\underset{h}{\Delta} f(x) = \dfrac{f(x+h)-f(x)}{h}$, function of x, defined on $[0,1-h]$

$$\underset{h}{\Delta}^2 f(x) = \frac{\underset{h}{\Delta} f(x+h) - \underset{h}{\Delta} f(x)}{h}$$

$$\underset{h}{\Delta}^n f(x) = \frac{1}{h^n} \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu} f(x+\nu h) \quad \text{by induction}$$

Bernstein $$p_n = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k \sum_{r=0}^{n-k} \binom{n-k}{r} (-t)^r$$

Collect terms where $k+r=\lambda$

$$= \sum_{\lambda=0}^{n} \binom{n}{\lambda} t^\lambda \sum_{k=0}^{\lambda} \binom{\lambda}{k} (-1)^{r=\lambda-k} f\left(\frac{k}{n}\right)$$

since $\binom{n}{k}\binom{n-k}{r} = \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{r!\,\underbrace{(n-k-r)}_{n-\lambda}!} = \binom{n}{\lambda}\binom{\lambda}{k}$

$$\overset{\text{with } h=\frac{1}{n}}{=} \sum_{\lambda=0}^{n} \binom{n}{\lambda} t^{\lambda} h^{\lambda} \Delta_h^{\lambda} f(0).$$

Let f be a non-negative polynomial of degree m.

$E(B_n) \geq 0$. Everything of degree $\geq m$ disappears.

Effective degree of B_n is m, constant.

So every coefficient converges to coefficient of f, individually.

(Note $B_n \neq f$).

We get $a_0^{(n)} c_0 + \dots + a_m^{(n)} c_m$

$$\downarrow$$

$a_0 c_0 + \dots + a_m c_m$ uniformly

So $E(B_n) \to E(f)$. If $E(B_n) \geq 0$, then $E(f) \geq 0$.

So every positive polynomial has ≥ 0 expectation.

If $0 \leq p(t) \leq 1$, then $0 \leq E(p) \leq E(1)$

since $1 - p \geq 0$.

So $|E(p)| \leq \max |p(t)|$.

Hence with uniformly convergent sequence of positive polynomials, functional converges, and we get the result on all continuous functions.

Some analysis

μ a mass function on Borel sets on line, finite. $\mu(R)=1$.

Center of gravity $\int x\,\mu(dx) = E(X)$, X = coordinate function

This is defined if integral exists.

Translate: Look at $\mu\{R+t\}$.

$$E(X-a) = \int (x-a)\,\mu(dx) = E(X) - a$$

So the center of gravity shifts linearly.

Let $a = E(X)$ and we may assume 0 is center of gravity. Set $m = E(X)$

Moment of inertia $E(X^2) = \int x^2 \mu(dx)$

$E((X-a)^2) = E(X^2) - 2am + a^2$

Minimum occurs when $a = m$. So if center of gravity is at 0

$\mathrm{Var}(X) = E(X^2) - m^2$. This equals second moment if first moment is at 0.

Chebychev's inequality

Want estimate for $\mu\{|X| > t\}$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 \mu(dx) \geq \int_{|X|>t} x^2 \mu(dx) \geq t^2 \mu\{|X|>t\}$$

$$\boxed{\mu\{|X|>t\} \leq \frac{1}{t^2} E(X^2)}$$

$\Rightarrow$ Let $\mu_n\{\theta, \cdot\}$ be a one-parameter family of probability measures for each n

Suppose for every θ $\quad m_n(\theta) = \theta$, mean.

$\sigma_n^2(\theta) = o(1), n\to\infty \quad \sigma^2$ = variance

Let u be a bounded uniformly continuous function on $[-\infty, +\infty]$. Then

$$\int u(x)\,\mu_n(\theta, dx) \to u(\theta)$$

Proof: $\int = \int_{|x-\theta|\leq\epsilon} + \int_{|x-\theta|>\epsilon}$

For second integral

$$\mu\{|X-m|>t\} \leq \frac{1}{t^2}\sigma^2$$

$$\mu_n\{|X-m|>\epsilon\} \leq \frac{1}{\epsilon^2}\sigma_n^2(\theta)$$

$\downarrow$ 0 for every fixed ϵ.

For first one mass concentrates to 1.

More is true; convergence is uniform if variances go uniformly to 0.

Example:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad \text{with} \quad v(0,x) = f(x)$$

$$v(t,\theta) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x)\, e^{-\frac{(x-\theta)^2}{4t}}$$

$\uparrow$ like $\frac{1}{n}$

$$m_n(\theta) = \theta$$

variance is like $\frac{1}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t}}$

This goes uniformly to f if f is continuous

Poisson integral: t becomes radius

Weierstrass theorem

Consider sequence (n) of probability distributions on points $\frac{k}{n} \in [0,1]$

$$\binom{n}{k}\theta^k(1-\theta)^{n-k} \quad \text{= fraction of successes in } n \text{ tries}$$

$m_n(\theta) = \theta$ (see interpretation)

$\sigma_n^2(\theta) = \frac{1}{n}\theta(1-\theta)$, $\theta \in [0,1]$. This goes to 0 uniformly with n.

$$m_n(\theta) = \sum \frac{k}{n}\binom{n}{k}\theta^k(1-\theta)^{n-k} = \theta \sum_{k-1=0}^{n-1} \binom{n-1}{k-1}\theta^{k-1}(1-\theta)^{n-1-(k-1)}$$

$$= \theta.$$

Second moment comes from doing this twice

Theorem says for any continuous u

$$\sum_{k=0}^{n} u\left(\frac{k}{n}\right) \binom{n}{k} \theta^k (1-\theta)^{n-k} \xrightarrow{\text{uniformly}} u(\theta)$$

This is the Bernstein formulation of the Weierstrass theorem

Another method: By the Poisson distribution

Let $h>0$. Distribution on $\{\nu h\}$

Mass at k-th point is $e^{-\frac{1}{h}\theta}\left(\frac{1}{h}\theta\right)^k \frac{1}{k!}$

Mean $=\theta$, variance $= h\theta$

Hence as $h\to 0$ with u continuous on $[0,\infty]$,

$$e^{\frac{1}{h}\theta} \sum_0^\infty u(kh)\frac{1}{k!}\left(\frac{1}{h}\theta\right)^k \to u(\theta).$$

Apply to $u(t+\theta)$:

$$\boxed{u(t+\theta) = \lim_{h\to 0} \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \Delta_h^n u(t)},$$ Taylor's formula with differences

Interchange of limit and $\sum$ is okay for analytic functions.

This holds for arbitrary continuous u.

More about Bernstein polynomials, $h = 1/n$

Left side is

$$\sum_{j=0}^{n} \binom{n}{j} (\theta h)^j \Delta_h^j u(0)$$

<u>Corollary</u>: Let $u \in C([0,1])$ and suppose $\Delta_h^j u(0) \geq 0$, $h = 1/n$, $j=0,\dots,n$. Then $u(\theta)$ is of the form

$$u(\theta) = \sum_0^\infty u_k \theta^k, \quad 0 \le \theta \le 1, \text{ with } u_k \geq 0.$$

Proof: Assume $u(1)=1$. Bernstein polynomial has all positive coefficients. For every n, Bernstein polynomial is like

$$\sum_{k=0}^{n} u_k^{(n)}\theta^k$$

and $\sum u_k^{(n)}=1$ from other form of Bernstein polynomial. We thus have prob. distribution on integers. By positivity extract convergent subsequence. For that subsequence we get the representation $u(\theta)=\sum_0^\infty u_k\theta^k$.

A function is absolutely monotone in $[0,1)$ if continuous and if $u^{(n)}\geq 0$, derivatives. A function is absolutely monotone if and only if it is of the form $\sum_0^\infty u_k\theta^k$.

Proof:

If $u(1)<\infty$, then u satisfies differencing condition

If $u(1)$ unbounded, look at $u(a\theta)$, $a<1$. Use uniqueness in $[0,a]$.

A function is completely monotone on $(0,\infty)$ if continuous and

$$(-1)^n v^{(n)}(x)\geq 0$$

It is necessary and sufficient that v be of the form

$$v(x)=\int_0^\infty e^{-xt}\mu(dt)$$

Then $u(e^{-x})$ is absolutely monotone on $[0,1)$

Let ϕ be completely monotone on $(0,\infty)$, where $(-1)^n \phi^{(n)} \geq 0$. ①

Assume $\phi(\infty) = 0$.

For fixed x look at $\phi(x - x\theta)$ in $0 < \theta < 1$

This function is absolutely monotone

Hence for fixed x, $\phi(x - x\theta) = \sum_{k=0}^{\infty} \frac{(-x\theta)^k}{k!} \phi^{(k)}(x)$

Set $\theta = e^{-\lambda}$. Then $0 < \lambda < \infty$.

Therefore $\qquad \phi(x - xe^{-\lambda}) = \int_0^\infty e^{-\lambda t}\, U_x(dt)$

U_x is a discrete measure attaching weight $\frac{(-x)^k \phi^{k}(x)}{k!}$

to the point k, $0 \leq k < \infty$.

Change λ into λ/x and get

$$\phi(x - xe^{-\lambda/x}) = \int_0^\infty e^{-\frac{\lambda}{x}t}\, U_x(dt)$$

Change variable $\frac{t}{x} \to s$

$\phi(x - xe^{-\lambda/x})$ is the Laplace Transform of a measure V_x

attaching weight $\frac{(-x)^k \phi^{(k)}(x)}{k!}$ to k/x

$$V_x(s) = \sum_{k=0}^{[xs]} \frac{(-x)^k \phi^{(k)}(x)}{k!}$$

Let $x \to \infty$. ϕ is continuous

$$\phi(x - xe^{-\lambda/x}) \to \phi(\lambda)$$

$$\phi_n(\lambda) = \int_{0^-}^\infty e^{-\lambda t}\, \mu_n(dt)$$

If $\phi_n(\lambda) \to \phi(\lambda)$, then $\mu_n \to \mu$, and conversely (continuity of Laplace Transform

If We have $V_x(s) \to V(s) = \lim_{x\to\infty} \sum_{k=0}^{[xs]} \frac{(-x)^k \phi^{(k)}(x)}{k!}$ at points of continuity ②

Claim $\quad \phi(\lambda) = \int_{0^-}^{\infty} e^{-\lambda s} V(ds)$

($\phi(\lambda)$ is limit of Laplace transform const. function measures)

Function is completely monotone if and only if

$$\phi(\lambda) = \int_{0^-}^{\infty} e^{-\lambda s} V(ds), \qquad \leftarrow \text{Serge-Bernstein.}$$

and $V(s) = \lim_{x\to\infty} \sum_{k=0}^{[xs]} \frac{(-x)^k \phi^{(k)}(x)}{k!}$

↑
distribution
function

Suppose $\mu_n \to \mu$, μ such that L.T. exists for one λ.

For λ_0 $\quad \int_0^\infty e^{-\lambda_0 t}\mu(dt)$ exists

Define measure by $\int_0^s e^{-\lambda_0 t}\mu(dt)$. Bounded measure

For bigger λ

$$\int_0^\infty e^{-(\lambda-\lambda_0)t} \underline{e^{-\lambda_0 t}\mu_n(dt)} \quad \text{exists}$$

With bounded measures $\nu_n \to \nu$, can extract conv. subsequence

$$\int_0^\infty u(x)\,\nu_n(dx) \to \int_0^\infty u(x)\,\nu(dx) \quad \text{if } u \to 0 \text{ at } \infty.$$

So extract subsequence of $\mu_n \to \mu$. Either measures go to ∞ or we get convergence for λ_0 and all bigger λ.
(Can forget about μ_n and work with $e^{-\lambda_0 t}\mu_n(dt)$)

Uniqueness of Laplace transforms for measures V.

Corresponding uniqueness of $\int_0^1 t^k \mu(dt) = c_k$, $k=0,1,2,\ldots$

Is the measure uniquely determined?

$$\phi(\lambda) = \int_0^\infty e^{-\lambda s} V(ds)$$ is same with $e^{-\lambda} \to \theta$ if $\phi(\lambda)$ is known at integer points.

In moment problem μ is necessarily bounded.

If $p(t) = \sum_0^m a_k t^k$, define

$$E(p) = \sum_0^m a_k c_k$$

For $E\left(t^j(1-t)^k\right)$, $E \geq 0$. This is a necessary condition on c's for existence. If $E(t^j(1-t)^k) \geq 0$, then moment problem has a solution and the solution is unique.

If $p \geq 0$ is of degree n, the Bernstein polynomial B_n of degree n is actually of order n, and $B_n \Rightarrow p$. So every coefficient must converge. Then $E(B_n) \to E(p)$. Hence $E(p) \geq 0$ whenever $p \geq 0$.

$E(1) = c_0$. If polynomial p has norm $\|p\|$, then $\|p\| - p \geq 0$ and $E(\|p\| - p) \geq 0$, so that

$$E(p) \leq c_0 \|p\|$$

Moreover with $-p$, $|E(p)| \leq c_0 \|p\|$

Polynomials are dense in continuous fcns. If

$$\|p_n - p_m\| < \epsilon \quad (p_n \Rightarrow f),$$

$$E(p_n) \longrightarrow E(f).$$

Is consistent and is still linear.

We have a positive linear functional on cont. fcns. By Riesz Theorem a measure μ exists with

$$E(f) = \int_{0^-}^{1^+} f(t)\,\mu(dt)$$

Riesz Theorem also gives uniqueness.

Condition that $E(t^j(1-t)^k) \geq 0$ is as follows

For $k=0$, $c_j \geq 0$

For $k=1$, $-t^j(1-t) = t^{j+1} - t^j$

So $c_{j+1} - c_j \leq 0$

or $\Delta c_j = c_{j+1} - c_j \leq 0$

or $(-1)\Delta c_j \geq 0$

For $k=2$, $t^{j+1}(1-t) - t^j(1-t) = -t^j(1-t)^2$

Hence condition is identical with

$$(-1)^k \Delta^k c_j \geq 0 \quad \text{for every } j \text{ and } k.$$

(completely monotone sequence)

If moments ≥ 17 are known, call $t^{17}\mu(dt) = \nu(dt)$, problem is

$$\int_0^1 s^j \nu(ds) = c_{17+j}$$

Measure ν exists. May or may not get a finite μ. No uniqueness, because can attach weight to 0.

If can extrapolate (cannot necessarily), then it is unique except at 0.

Could look at $\int_0^1 t^\theta \mu(dt)$ for all $\theta \geq 0$. Call it $\phi(\theta)$. It is completely monotone. Could write as Laplace transform when $t = e^{-x}$. $\phi(\theta) = \int_0^\infty e^{-\theta x}\nu(dx)$. Hausdorff problem gives completely monotone function, which can written as L.T. It is enough to know ϕ at integer points in reverse direction; can interpolate uniquely.

Müntz theorem: generalization to arbitrary points $x_1, \ldots, x_n, \ldots$

Situation:

$[0,1]$ closed. Given $c_k = E(t^k)$ and hence $E(p)$, a linear functional. There exists a measure μ such that $E(p) = \int_{0^-}^{1^+} p(t)\,\mu(dt)$ if and only if $E\left(t^j(1-t)^k\right) \geq 0$. (This condition is that $(-1)^k \Delta^k c_{m+j} \geq 0$.) In particular no two measures can have the same moments.

If $\int_0^1 u(t)\, t^u \,\mu(dt) = \gamma_k$, is u determined? Then $\int_0^1 u(t)\, t^u \mu(dt) = 0$ or $\int_0^1 u^+(t)\, t^u \mu(dt) = \int_0^1 u^-(t)\, t^u \mu(dt)$. Define $\mu^+(dt) = u^+(t)\,\mu(dt)$ and similarly with u^-. Then $\mu^+ = \mu^-$. Hence u is uniquely defined.

H is a real Hilbert space and S an operator such that $\|S\| \leq 1$ and S is symmetric: $(x, Sy) = (Sx, y)$

<u>Lemma</u>: $\|S\| = \sup\limits_{\|x\| \leq 1} |(x, Sx)|$

Proof: Call right side $N(s)$.

$|(x, Sx)| \leq \|S\|\,\|x\|^2$. Hence $N(s) \leq \|S\|$

$$\begin{aligned}
\underbrace{4(y, Sx)}_{\text{do not need absolute value}} &= (x+y, S(x+y)) - (x-y, S(x-y)) \\
&\leq N(S)\left[\|x+y\|^2 + \|x-y\|^2\right] \quad \leftarrow \text{symmetry here} \\
&= N(S)\left[2\|x\|^2 + 2\|y\|^2\right] \\
&\leq 4N(S) \quad \text{if } \|x\| \leq 1,\ \|y\| \leq 1
\end{aligned}$$

Find x with $\|Sx\| \geq (1-\epsilon)\|x\|\,\|S\|$

$$(y, Sx) \geq (1-\epsilon)\|y\|\,\|Sx\|$$

So $|(y, Sx)| \geq (1-\epsilon)^2 \|S\|$ or something

Write $T = \frac{1+S}{2}$, a new bounded operator. Symmetric and non-negative:

$$0 \le (x, Tx) \le 1 \quad \text{for every } x \text{ with } \|x\| \le 1$$

For $2(x,Tx) = (x, x+Sx) = \|x\|^2 + (x, Sx)$

$$= 1 + (x, Sx)$$

Lemma: If T is symmetric, is positive, and has norm ≤ 1, then

$$(x, T^j(1-T)^k x) \geq 0.$$

Proof: We have

$(y,y) \geq 0$, $(y, Ty) \geq 0$, and $(y, (1-T)y) \geq 0$.

Let $y = T^m(1-T)^n x$. Therefore

$$(x, T^{2m}(1-T)^{2n} x) \geq 0$$

$$(x, T^{2m+1}(1-T)^{2n} x) \geq 0$$

$$(x, T^{2m}(1-T)^{2n+1} x) \geq 0.$$

Need j and k odd. Comes from

$\frac{1}{4} \geq (x, (\frac{1}{2}-T)^2 x)$. Need $\|\frac{1}{2} - T\| \le \frac{1}{2}$

So difference is ≥ 0.

$$\|\tfrac{1}{2} - T\| = \sup (x, (\tfrac{1}{2}-T)x) = \sup\left|\tfrac{1}{2} - (x, Tx)\right| = \tfrac{1}{2}.$$

$$\frac{1}{4} - \frac{1}{4}(x,x) + (x, T(1-T)x) \geq 0$$

Let $\xi \in \mathcal{H}$ be arbitrary and fixed. Define $\mathcal{H}_\xi =$ closure of subspace spanned by $(\xi, T\xi, T^2\xi, \dots)$. Hence $\mathcal{H} = \mathcal{H}_\xi \oplus \mathcal{H}_\xi^\perp$. $\mathcal{H}_\xi$ is invariant subspace for T: $p_n(T)\xi \to x$ implies

$T p_n(T)\xi \to Tx$ by continuity of T.

If $y \in \mathcal{H}_\xi^\perp$, then also $Ty \in \mathcal{H}_\xi^\perp$. For

$(y, T^k \xi) = 0$ for $k = 0, \ldots$

Or $(y, T^{k+j}\xi) = (T^j y, T^k \xi) = 0$.

Thus $T^j y \in \mathcal{H}_\xi^\perp$ and $\mathcal{H}_\xi^\perp$ is invariant.

We restrict attention to $\mathcal{H}_\xi$. For we may repeat procedure with $\mathcal{H}_\xi^\perp$ later.

Introduce $c_k = (\xi, T^k \xi)$. By lemma the c_k are a moment sequence.

If $p(t) = \sum_{k=0}^{n} a_k t^k$, define $E(p) = \sum_0^n a_k c_k = (\xi, p(T)\xi)$.

This is a linear functional on polynomials. There exist a μ such that $E(p) = \int_0^1 p(t)\mu(dt)$. O transformation goes into δ-measure. If $c_0 = 1$ or if $\|\xi\| = 1$, then μ is a probability measure.

For two real-valued functions (L^2) on $[0,1]$ define $(u, v) = \int_0^1 u(t) v(t) \mu(dt)$.

Call this $\mathcal{L}_2(\mu)$. Map

$$T^k \xi \longleftrightarrow t^k, \quad k = 0, 1, 2, \ldots$$

linearly: $\quad p(T)\xi \longleftrightarrow p(t)$ $\quad$ (is one-one)

This map is an isometry:

$$(p(T)\xi, q(T)\xi) = (\xi, p(T) q(T) \xi) = E(pq)$$
$$= \int_0^1 p q \mu = (p, q)$$

It is one-one.

Closure of $p(T)\xi$ is mapped onto closure of $p(t)$ space.

Hence $x \in \mathcal{H}_\xi \xrightarrow{\text{biunique}} u(t) \in \mathcal{L}_2$

$$\|x\| \longrightarrow \int_0^1 u^2(t)\mu(dt)$$

If $x \leftrightarrow u$, the $Tx \leftrightarrow t\,u(t)$ and so on.

$$p(t)\, x \leftrightarrow p(t)\, u(t)$$

$$\| p(T)\|^2 = \sup_{\|u\|=1} \int p^2(t)\, u^2(t)\, \mu\, dt$$

Spectrum of $p(t)$ = intersection of all closed sets of measure 1.

Then $\| p(T) \| = \max_{t \in \text{spectrum}} |p(t)|$

$p_n(T) \to \tau(T)$ if sequence is Cauchy on $[0,1]$: Need uniform convergence on spectrum, which is a closed set.

Let Ψ be bounded on the spectrum
If $x \to Sx$, the induced is $u \to \int_0^1 u(t)\,\Psi(t)\ \mu(dt)$.

Choose $\eta \in \mathcal{H}_\xi$ (or $\phi \in \mathcal{L}_2$). Representation in terms of η gives
$T^k \eta \to t^k \phi(t)$. First question is whether $p(t)\,\phi(t)$ is dense.
Dense if $\phi(t)$ is bounded and does not vanish anywhere.

And $$E_\xi(t^k) = \int t^k \underbrace{\phi(t)\,\mu(dt)}_{\text{gives } \nu(dt)} \text{ if } \phi \geq 0.$$

$(T\xi, T^2\xi, T^3\xi, \ldots)$ spans $\mathcal{H}_\xi$ if origin has zero measure, and conversely.

Spectrum of μ is uniquely defined. Can only be decreased by introducing ϕ.
Norm of $p(t) = \max_{t \in \text{spectrum}} |(p(t))|$ is unique, independent of ξ.

Lemma: If A is closed $\varepsilon [0,1]$, then $\chi_A \varepsilon \mathcal{L}_2$ (image of $\mathcal{H}_\xi$).

Proof:

Every continuous function is in $\mathcal{L}_2$.

$$\phi_n = \max\{0, n \text{ dist to } A\}$$

$$\|\phi_n - \phi_m\|^2 = \int(\phi_n(t)-\phi_m(t))^2 \leq \mu(A_n - A), \quad A_n \supset A.$$

$A_n \downarrow A$; so $\mu(A_n - A) \downarrow 0$.

Hence indicators of open and closed sets are in $\mathcal{L}_2$.

A partition will be a covering by non-overlapping intervals.

$I = [0,1] = \{I_\nu\}$

Take subspace of $\mathcal{L}_2$ consisting of all functions carried by A. This subspace is closed and is a Hilbert space $\mathfrak{M}_A^*$.

Projection of ψ on $\mathcal{L}_2$ is $\chi_A \psi$.

If $A \cap B = 0$, then $\mathfrak{M}_A^* \perp \mathfrak{M}_B^*$. χ_A and $\chi_{\tilde{A}}$ give projection operators into orthogonal complements.

$$\phi = \chi_A \phi + \chi_{A'} \phi$$

Resolution of identity: Write $I = \cup I_\nu$, $I_j \cap I_k = 0$, finite number.

Then $\phi = \sum \chi_{I_\nu} \phi$.

$t\phi(t) = \sum t \chi_{I_\nu} \phi(t)$. Let diam $I_\nu < \epsilon$. t is practically constant in I_ν. Let $\lambda_\nu \varepsilon I_\nu$, and consider

$$\sum \lambda_\nu \chi_{I_\nu}(t)\phi(t) - t\phi(t)$$

$$= \sum(\lambda_\nu - t)\chi_{I_\nu}(t)\phi(t).$$

Norm: $\| \|^2 < \epsilon^2 \|\phi\|^2$ since χ's are orthogonal.

Meaning in original space:

<u>Theorem</u> (Spectral theorem):

To any interval I_j there corresponds a subspace $\mathfrak{M}_{I_j}$ of $\mathcal{H}_\xi$, and if $I = \cup I_j$ disjointly, then $\mathcal{H}_\xi = \mathfrak{M}_{I_1} \perp \mathfrak{M}_{I_2} \perp \cdots$.

If $x = \sum_\Delta x_i$, where x_i is the projection on $\mathfrak{M}_{I_i}$, then

$\sum_\Delta \lambda_i x_i \to Tx$ in norm, where λ_i = midpoint of interval.

Resolution of identity (relative to part of λ-axis): abstract definition

To every interval I of the real axis there corresponds a subspace $\mathfrak{M}_I \subset \mathcal{H}$. If $I_1 \cap I_2 = 0$, $\mathfrak{M}_{I_j} \perp \mathfrak{M}_{I_k}$. $\mathfrak{M}_{I_1 \cup I_2} = \mathfrak{M}_{I_1} \oplus \mathfrak{M}_{I_2}$.

Furthermore $\mathfrak{M}_{(-\infty,\infty)} = \mathcal{H}$.

We have a resolution for $\mathcal{H}_\xi$; it is carried by $[0,1]$.

Fix a resolution:

Denote $E(\lambda)$ = projection on $\mathfrak{M}_{(-\infty,\lambda]}$

$E(\lambda_2) - E(\lambda_1)$ = projection to corresponding subspace.

Write $x = \int_{-\infty}^{\infty} x\, E(d\lambda \quad) = \sum_\Delta x_\nu$

Or $1 = \sum [E(\lambda_{n+1}) - E(\lambda_n)]$, $\lambda_0 = 0$.

$1 = \int E(d\lambda)$.

In case for $\mathcal{H}_\xi$.

$$x = \sum (E(\lambda_n) - E(\lambda_{n+1}))x, \quad \sum \lambda_n [E(\lambda_n) - E(\lambda_{n-1})]x \to Tx$$

So $T = \int \lambda E(d\lambda)$; get Tx in limit

For general $\mathcal{H}$, split off $\mathcal{H}_{\xi_1}$, apply procedure to complement. Continue. ③

In separable space, we need only countably many steps. For each $\mathcal{H}_{\xi_j}$ we get a resolution on $[0,1]$. Then we get a resolution of identity in whole space $\mathfrak{M}_I = \mathfrak{M}_I^1 \oplus \mathfrak{M}_I^2 \oplus \dots$ by superposition.

$$x = \sum [E(\lambda_{n+1}) - E(\lambda_n)]\, x$$

$$Tx = \lim \sum \lambda_n [E(\lambda_{n+1}) - E(\lambda_n)]\, x$$

Example: compact operators, T as before

T (symmetric for us) is compact if whenever $\|x_n\| \le 1$, then there is a convergent sequence in $\{Tx_n\}$.

Example

$\int K(x,y) f(y)\, dy$, K nice (say two derivatives). Then Tf_n has two derivatives, is of bounded variation

Compact transformation always has eigenvalue and eigenelement.

We shall prove measure in image L_2

Let $\xi \neq 0 \in \mathcal{H}$. Form $\mathcal{H}_\xi$. If $\exists$ acc pt., then 0 is one

is discrete on countably many pts.

of them.

Suppose to point τ, every interval $(\tau - h_n, \tau + h_n)$

has measure > 0. Let ϕ_n be carried on $(\tau - h_n, \tau + h_n)$

e.g. — take indicators $\rightarrow$ with $\|\phi_n\| = 1$, and suppose $h_n \downarrow 0$. (τ is in spectrum.)

Out of $t\,\phi_n(t)$, we can extract sequence $\rightarrow \psi(t)$ in norm.

Then $\tau\,\phi_n(t)$ subsequence converges

$$\| t\,\phi_n(t) - \tau\,\phi_n(t) \| \le h_n \|\phi_n\| = h_n .$$

Then it follows that ψ is carried by τ itself. Must have $\psi = c\chi_\tau$, $c \neq 0$ (if $\tau \neq 0$) Then measure μ must be > 0 at τ. $\|\chi_\tau\|$ = mass of τ. So either $\tau\phi_n \to 0$ or we get an eigenvalue. Norm $\tau\phi_n \not\to 0$ if $\tau \neq 0$ though.

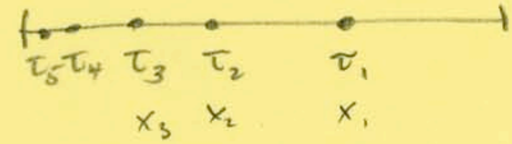
For every τ there exists a nbd with 0 mass or point is an eigenvalue. Only denumerably many points by Lindelöf, only denumerably many eigenvalues.

Repeat argument for $(\tau, \tau + h_n)$, $\tau \neq 0$. Conclude

↑ open

all points except 0 are isolated. Only pt. of accumulation is therefore 0.

Picture: ξ



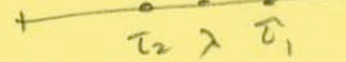
$T x_n = \tau_n x_n$

$(x_i, x_j) = \delta_{ij}$. $\tau_n \to 0$ if infinitely many

0 may or may not be eigenvalue.

Eigen element in H_ξ is eigenelement in H. Repeat for other split factors of H. Get all eigen elements.

For another η if



λ is eigenvalue for η,

we could have chosen $\xi + \eta$

$T^n(\xi + \eta) = T^n\xi + T^n\eta$, get all eigenvalues for η and ξ

If $\lambda = \tau_1$, double eigenvalue, we could not have done better.

Note $\xi = \sum x_k$

$$T^n \xi = \sum \tau_k^n x_k \,, \quad \frac{1}{\tau_1^n} T^n \xi \to x_1$$

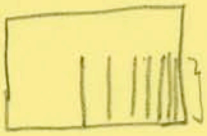
Note measure does not enter into position of τ's. τ's do not depend on ξ.

Two kinds of unbounded solutions

1) Ones and zeroes in boundary give some unbounded solution in their span

2) $(q/p)^i$ is walk on line — is outside cone

Irrelevant parts of boundary



Green's function $\sum \pi^n(i,j) = G(i,j)$

Assume finite

Martin's technique

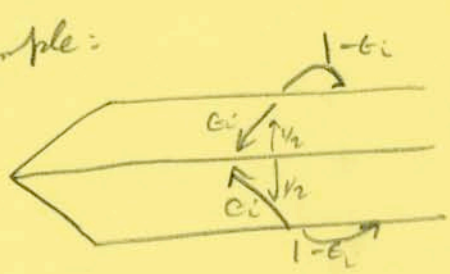
$$\frac{G(i,j)}{G(0,j)}$$

Introduce metric

Distance between m and n is $\dfrac{G(i,m)}{G(0,m)} - \dfrac{G(i,n)}{G(0,n)}$, denominator is weighting factor; want to sum over i's, but need to bound by factor c_i and divide by 2^i for convergence.

Complete and get a compact space.

Example:



$$\sum \epsilon_i < \infty$$

Go far out on top line.

If i, m, n are on top line $G(i,m) \simeq G(i,n) \simeq 1$.

$G(0,m) \simeq G(0,n)$.

So any two top points are close.

m and n on different lines: distance is not small (consider fixed bad i)

Distance goes to 0 on middle line

Gives arithmetic mean of adjacent lines

Method is then to throw out points not give minimal solutions; hard to prove this is a Borel set.

Example: $\overset{q}{\frown}\ \overset{p}{\frown}$

Number of passages $\underset{\sim(q/p)^m}{-m}\quad \underset{0}{\overset{q\ p}{\frown}}\quad \sim 1$

For all i's not between m and n, distance of 0.

Get ordering nbds of $+\infty$

In other direction distance is small, for i's between m and n give + part.

Entrance boundary

Nice if invariant measure a exists and is a probability measure

So to reverse chain $\dfrac{a_i\,\pi(i,j)}{a(j)} = Pr_j$ [to have come from i] $= \pi^*(j,i)$

Works with $a \geq a\pi$

(How about increasing masses with $a \leq a\pi$)

Use exit boundary for reverse chain to get entrance boundary

If we use another invariant measure α. Ratio $\dfrac{\alpha(i)}{a(i)}$ gives solution of $\pi^* x^* = x^*$

So result is invariant under choice of a if we take the whole boundary to begin with

Hunt approach is to get entrance boundary more generally.

No connection between topologies of entrance and exit boundaries.

Time-dependent processes (Kolmogorov)

Stay at state for awhile and then leave with certain probabilities

Markov property: exponential requirement

$\pi(i,j) \qquad e^{-p_i t} = P\{\text{sojourn time}_i > t\}$

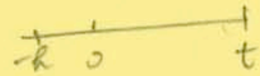
$u_i(t) =$ mass from i

$$u_k'(t) = -\underbrace{p_k u_k(t)}_{\text{outward flow}} + \sum_j \underbrace{p_j u_j(t)\ \pi(j,k)}_{\text{inward}}$$

Convergence question for series, although intuition is clear.

Reverse direction $v_i(t) =$ to be at i at time t

$P_{ik}(t)$



$P_{ik}(t+h) = (1-p_i h)\, P_{ik}(t)$

see picture ↗

$$+\, p_i h \sum_j \pi(i,j)\, P_{jk}(t) + o(h)$$

This can be axiomitized

$P_{ik}' = -p_i P_{ik} + p_i \sum_j \pi_{ij} P_{jk}(t)$ function of first coordinate

Convergence is okay

Let $p = \begin{pmatrix} p_1 & & 0 \\ & p_2 & \\ 0 & & \ddots \end{pmatrix}$. $\qquad P' = -pP + p\pi P$

Other one is $\quad P' = -Pp + Pp\pi$

In case of p_i of moving to right, they are speeds, as above.

If $\sum \frac{1}{p_i} < \infty$, positive prob. of reaching boundary in finite time.

Uniform mortality

$e^{-\lambda t}$ of staying alive

Let $\hat{P} = \int_0^\infty e^{-\lambda t} P(t)$

$$(\lambda + p) P = p \pi \hat{P} + I$$

$\lambda \hat{P}$ is substochastic

Boundary (exit) does not depend on λ.

p is afad matrix : tells whether boundary is actually reached.

Part approached is active boundary, other is passive boundary.

Theorem : $\lambda \hat{P}$'s exit boundary is active boundary, independently of λ.

With compact operator

$\lambda_n \to 0$

$\{\phi_n\}$ orthonormal

Any $x = \sum a_n \phi_n$

$Tx = \sum \lambda_n a_n \phi_n$, $\sum a_n^2 < \infty$

Measure depends on representation; carrier of measure is invariant

In general
Consider L_2 wrt μ. Let $m(dt) = a(t)\mu(dt)$. $m \ll \mu$, density a

Require a integrable wrt μ. Let $a > 0$.

$(\phi, \psi)_\mu = \int \phi(t)\psi(t)\,\mu(dt)$

$(f, g)_m = \int f(t) g(t) a(t)\,\mu(dt)$ Claimed isomorphic

Map $f\sqrt{a} \leftrightarrow \phi$. This is an isomorphism, one-one if $a > 0$ or $a = 0$ on set of μ-measure 0.

$(\xi, T\xi, \dots)$ space: $\to (1, t, t^2, \dots)$

$\sqrt{a}$ is in space and corresponds to $\eta \in H_\xi$

$t\eta \leftrightarrow t\sqrt{a}$

$t^2\eta \leftrightarrow t^2\sqrt{a}$. So depending on which element is chosen, we get different measures

If a vanishes somewhere, we get only $H_\eta \subsetneq H_\xi$. Functions carried by interval where $a=0$ are left out.

Measures are equivalent for this if they are abs. cont. wrt each other.

Comparison

m gives $\mathcal{L}_m$

μ gives $\mathcal{L}_\mu$.

Case 1: m and μ carried by disjoint sets.

Then $\mathcal{L}_{m+\mu} = \mathcal{L}_m \oplus \mathcal{L}_\mu$ (from function representation)

Case 2: If $m \ll \mu$ and $\mu \ll m$, isomorphic spaces

Case 3: μ carried by subset for m. Write $m = \mu + (m - \mu)$, apply 1.

ξ can be replaced by an element $\eta \Leftrightarrow \chi_I$, instead of 1, to give representation on a subset of $[0,1]$.

Consideration of all of $\mathcal{H}$

$\mathcal{H} = \mathcal{H}_\xi \oplus \mathcal{H}_\xi^\perp$, measure μ for $\mathcal{H}_\xi$

$\eta \in \mathcal{H}_\xi^\perp$ gives $\mathcal{H}_\eta$ and measure ν.

If μ and ν are disjoint (μ singular wrt ν and vice versa),

then take $\mu+\nu$. Space comes from $\varsigma = \xi + \eta$

$$T^n \varsigma = T^n \xi + T^n \eta$$

$$\mathcal{H}_\varsigma = \mathcal{H}_\xi \oplus \mathcal{H}_\eta$$

So if we can find η with disjoint spectrum, we could have taken $\xi + \eta$.

For arbitrary η, if ν not sing wrt μ, split $\nu = \underset{\substack{\uparrow \\ \text{abs. cont.}}}{\nu_a} + \underset{\substack{\uparrow \\ \text{singular}}}{\nu_\sigma}$

Take the element giving ν_σ (from remark above); add it to ξ.

Either every element gives smaller spectrum or exists disjoint spectrum.

In separable space, it is possible (with comps) to choose ξ maximal (uniquely) — gives a maximal spectrum.

Then in $\mathcal{H} = \mathcal{H}_\xi \oplus \mathcal{H}_\xi^\perp$, spectrum of latter is contained in first ③

Apply to compact operator:

Spectrum discrete; things left after first step are points corresponding to multiple eigenvalues. Etc.

Operator algebra: Let $\mathcal{H} = \mathcal{H}_\xi$

We have T and we got $p(T)$ at beginning

In L_2 $\quad p(T): \ \phi \to p(t)\phi(t)$

$$\| p(T) \| = \sup_{\phi} \frac{\| p(t)\phi(t) \|}{\| \phi(t) \|} \qquad p(t) \text{ assume maximum on spectrum}$$

$$\leq \sup_{\text{spectrum}} |p(t)|.$$ Also $\geq |p(t)| - \epsilon$. If bigger than a, it is bigger than a in nbd. Take function carried in small nbd.

$$\| p(T) \| = \sup_{\text{spectrum}} |p(t)|$$

If p_n converge uniformly on spectrum to $f(t)$, then f gives a new bounded operator $f(T): \phi \to f(t)\phi(t)$.

$$\| f(T) \| = \sup_{t \in \text{spectrum}} |f(t)|.$$

All these operators commute. We get a big algebra of commuting algebra. Are these all the symmetric operators commuting with T? Heller would like to see modified form of this argument give this.

Unbounded operators

T^{-1}: must be $1/t$, bounded iff ~~essentially~~ bounded on spectrum, spectrum at finite distance from origin.

If spectrum goes to 0, for every $\mathcal{M}_{[\epsilon,1]}$, T^{-1} is bounded.

Let $\lambda_n \to 0$ be eigenvalues

$$x = \sum a_n \phi_n$$

$$T^{-1}x = \sum \lambda_n^{-1} a_n \phi_n$$

This is an element if $\sum \lambda_n^{-2} a_n^2 < \infty$. This is a linear set, dense, but not a subspace. Inverse exists on this set.

Spectrum of T^2 is square of spectrum, pointwise

T^{-1} has reciprocals, pointwise as spectrum

Under map t^{-1}, disjoint intervals $\to$ disjoint intervals; resolution of identity is essentially the same.

Spectral theorem for unitary operators

Situation: We shall be considering circle as the spectrum: $e^{i\theta}$, $0\le\theta<2\pi$, periodically. Whole Hilbert space is complex.

Given $\{a_n\}$ complex, $n=0,\pm1,\pm2,\ldots$. When is this a moment sequence?

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,\mu(dx) \quad , \ -\pi \text{ and } \pi \text{ identified.}$$

a_n is the nth Fourier coefficient of the measure μ. μ is a (real non-negative) measure.

NASC: $\{a_n\}$ must be bounded and

$$f_r(\theta) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \quad , \ 0<r<1,$$

which is well-defined, must be ≥ 0 for each r.

Proof of necessity:

$$\sum_{-\infty}^{\infty} r^{|n|} e^{in\alpha} = \frac{1}{1-re^{i\alpha}} + \frac{1}{1-re^{-i\alpha}} - 1$$

$$= \frac{1-r^2}{1-2r\cos\alpha+r^2}$$

$$f_r(\theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \int r^{|n|} e^{in(\theta-x)}\,\mu(dx)$$

By uniform convergence for interchange

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-x)+r^2}\,\mu(dx) \geq 0.$$

Note f is harmonic in r and θ; it is ≥ 0. Integral is Poisson integral. If μ has continuous density, $\theta - x$ is fixed, $\cos(\theta-x)\neq 1$, then integrand $\Rightarrow 0$ uniformly.

For every r, integral over circle is 1. Mass concentrate at $(1,0)$.

If density is $f(x)$, $f_r(\theta)\to f(\theta)$ as $r\to 1$.

Proof of sufficiency:

For every fixed r interpret $f_r(\theta)$ as a density on the unit circle. Total mass is $\int_{-\pi}^{\pi} f_r(\theta)\,d\theta = 2\pi a_0$, integration term by term. Measures are bounded. Take convergent subsequence going to limit measure.

We have $$a_n = \frac{1}{|r|^n}\frac{1}{2\pi}\int e^{-in\theta} f_r(\theta)\,d\theta$$

$$\downarrow$$

$$\frac{1}{2\pi}\int e^{-in\theta} \underbrace{\mu(d\theta)}_{\text{limit measure.}}$$

Definition of positive definite sequence

$a(n)$, $n=0,\pm1,\pm2,\dots$ is positive definite

if $\sum\sum_{\text{finite}} a(n-m)\rho_n\overline{\rho_m} \geq 0$ for every finite sum.

Necessary conditions:

1) If $\rho_0 \neq 0$, we find $a(0) \geq 0$

2) Let $\rho_0 = 1$, $\rho_1 = 1$ or i

Then $a(k) = \overline{a(-k)}$

3) Let $\rho_0 = 1$, $\rho_k = e^{i\gamma}\lambda$, where we define γ so that $a(k) = |a(k)|e^{-i\gamma}$.

Then $a(0) + 2\lambda|a(k)| + a(0)\lambda^2 \geq 0$

So $a(0) \geq |a(k)|$

Hence every positive definite sequence is positive definite and bounded.

In order that $a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\mu(dx)$,

it is NAS that sequence be positive definite.

Proof:

Necessity:

$$\sum\sum a(n-m)\rho_n\overline{\rho_m} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum\sum(\rho_n e^{inx})(\overline{\rho_m}e^{-imx})\,\mu(dx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum \rho_n e^{inx}\right|\mu(dx) \geq 0$$

Sufficiency:

Let $\rho_n = 0$ for $n<0$

$\rho_n = r^n e^{inx}$, $n \geq 0$ for fixed r in $(0,1)$

Then $\sum\sum a(n-m)\rho_n\bar{\rho}_m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a(n-m)\, r^{n+m} e^{i(n-m)\alpha}$ ②

$$= \sum_{k=-\infty}^{\infty} a(k)\, e^{ik\alpha} \sum_{\substack{n\geq 0 \\ n\geq k}} r^{2n-k} \qquad \text{with } k=n-m$$

$$= \sum_{k=-\infty}^{-1} a(k) e^{ik\alpha} \frac{r^{|k|}}{1-r^2} + \underset{\substack{\uparrow \\ n-k\geq 0}}{\sum_{k=0}^{\infty}} \text{ same}$$

$$= \sum_{k=-\infty}^{\infty} a(k) e^{ik\alpha} \frac{r^{|k|}}{1-r^2} = f_r(\alpha) \geq 0. \quad \text{Q.E.D.}$$

Need limiting argument since we assumed things only for finite sums.

$\mathcal{H}$ a Hilbert space, complex

$T^{-1} = T^*$ is a unitary transformation

$(x, Ty) = (T^{-1}x, y)$

T is an isometry

$$(x,y) = (T^{-1}Tx, y) = (Tx, Ty).$$

Let ξ be arbitrary. We consider the subspace spanned by

$\xi, T\xi, T^2\xi, \dots, T^{-1}\xi, T^{-2}\xi, \dots$

$\mathcal{H}_\xi = \{T^k \xi\}$, $k = 0, \pm 1, \dots$

$\mathcal{H}_\xi$ and orthogonal complements are invariant under T.

Let $a(n) = (\xi, T^n \xi)$. Claim it is positive definite.

$$\sum\sum a(n-m)\rho_n\bar{\rho}_m = \sum\sum\left(\rho_n\xi, \rho_m T^{n-m}\xi\right)$$
$$= \sum\sum\left(\rho_n T^{-n}\xi, \rho_m T^{-m}\xi\right)$$
$$= \|\sum \rho_n T^{-n}\xi\|^2$$
$$\geq 0.$$

Hence $a(n)$ is positive definite.

In L_μ inner product is $\int_{-\pi}^{\pi} \phi(t)\overline{\psi(t)}\,\mu(dt)$ (periodic)

Map $T^k\xi \longleftrightarrow e^{ikt}$

Now $(T^m\xi, T^n\xi) = \int_{-\pi}^{\pi} e^{-(m-n)t}\,\mu(dt)$; so we have an isometry.

Map is between whole spaces, and things are as in symmetric case.

If $\eta \longleftrightarrow \phi(t)$, $T\eta \Leftrightarrow e^{it}\phi(t)$

Everything else with maximal spectrum, etc, goes as before

Stone - Von Neumann Theorem (for unitary operators)

Any measure μ can occur, for we can start with H as L_μ.

Prediction theory example

$\{X_k\}$ a sequence of stationary random variables

$E(X_k)=0$, $E(X_k^2)=1$.

$E(X_j X_k) = (X_j, X_k)$. Hilbert space

Inner product is covariance.

We get $(X_j, X_k) = a(k-j)$. This is called wide sense stationarity.

Introduce T, a shift operator

Map $(Tx)_n = x_{n-1}$, right shift of sequences

T is an isometry.

Choose $\xi = X_0$. We get a model for these sequences.

Suppose all X_k for $k \leq 0$ are all known. Want to predict X_1, for example. Knowing past gives subspace of Hilbert space. X_1 may not be in space (interesting case). Find its component (in principle) in subspace; this is known. Orthogonal component is completely unknown. Prediction is that component is component of X_1 determined by past.

Note independent random variables Y_n give Lebesgue measure for μ.

Model for random influences: example - $X_k = \frac{1}{2}(X_{k-1} + X_{k-2}) + \underbrace{Y_k}_{\text{random influence}}$

(independent random variables)

Best approximation = projection on past.

Example: $X_k = \sum_{-\infty}^{k} \frac{1}{2^{k-n}} Y_n$

Particular example

$\{\xi_n\}$ as before

$\xi_n \leftrightarrow e^{int}$. We shall consider what μ is.

Suppose $\xi_n = \Sigma c_k \phi_{k+n}$, where $\Sigma |c_k|^2 < \infty$ and $\{\phi_k\}$ orthonormal.

We claim μ has density $\frac{1}{2\pi}|c(t)|^2$, where ← special class of stochastic processes

$$c(t) = \sum_{-\infty}^{\infty} c_k e^{ikt}$$

Conversely if spectral density of this form then process is of above form.

Proof:

$a_n = (\xi_{n+k}, \xi_k)$ independent of k by stationarity

$= (\xi_n, \xi_0)$

$= \Sigma\Sigma\, c_k \bar{c}_j (\phi_{k+n}, \phi_j)$ (terms are 0 except when $j = k+n$)

$= \Sigma\Sigma\, c_k \overline{c_{j+n}}$

To prove $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} c(t)\overline{c(t)}\,dt$ modulo factor of 2π

$$c(t)\overline{c(t)} = \sum_k \sum_j c_k \bar{c}_j e^{i(k-j)t}$$

Multiply by e^{int} and integrate and we get the results in one direction.

Conversely suppose density is given in above form.

We suppose (for simplicity) that $|c(t)| > \epsilon > 0$

System will be $\frac{e^{ikt}}{\overline{c(t)}} = \phi_k(t)$, square integrable because bounded.

Note ϕ's are not unique because c's are not.

Orthonormality $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_k(t)\overline{\phi_n(t)} \cdot c(t)\overline{c(t)}\,dt =$

②

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt} e^{-int} dt = \delta_{kn}$$

By definition
$c_k = \int c(t) e^{ikt} dt$ modulo factor of 2π

Want $\sum c_k \varphi_{k+n}(t) = \xi_n$ (in original) $= e^{int}$ in model

And this follows by pulling out a factor.

Another example:
U and V two points in a Hilbert space
$\xi_n = U \cos n + V \sin n$
Two observations determine everything

Generally
Define
$\mathfrak{M}_n = \{\xi_n, \xi_{n-1}, \ldots\}$

$\mathfrak{M}_{-1} \subset \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \ldots$

Let $\mathfrak{M}_{-\infty} = \cap \mathfrak{M}_n$, called remote past in prediction business.

The most general stationary stochastic process is

$$\xi_n = \sum_{k=0}^{\infty} c_k \eta_{n-k} + \nu_n$$

where η_k are orthonormal and $\nu_n \perp \eta_k$ for all n and k,
$\nu_n \perp \mathfrak{M}_{n-1}$. ξ is decomposed into two orthogonal processes, each stationary

Proof:
ξ_0 is in $\mathfrak{M}_{-1}$ and a part orthogonal to it
$c_0\eta_0$ = component of $\xi_0 \perp \mathfrak{M}_{-1}$, where $c_0\eta_0$ is chosen to make $(\eta_0, \eta_0) = 1$

$$\xi_0 = \underbrace{c_0 \eta_0}_{\perp \mathfrak{M}_{-1}} + \text{something in } \mathfrak{M}_{-1}$$

Inductively $\xi_0 = c_0 \eta_0 + \underbrace{c_1 \eta_{-1}}_{\perp \mathfrak{M}_{-2}} + \text{something in } \mathfrak{M}_{-2}$

$$\xi_0 = \sum_{k=0}^{n} c_k \eta_{-k} + \text{remainder}$$

Series converges ($\Sigma |c_k|^2$ cannot exceed $\|\xi_0\|^2$), remainder goes to $\mathfrak{M}_{-\infty}$

Etc., and we get the decomposition.

Interpretation

η's are noise effects, tending to zero with time and depending only on past.

Remote past is completely deterministic.

For prediction, if process has been observed for awhile, everything is known except c_0. Best prediction is to predict remainder.

Example:

Space L_2 on real line $(-\infty, \infty)$, Lebesgue measure, $(f,g) = \int_{-\infty}^{\infty} f(s)\bar{g}(s)\,ds$

$$T : f \to Tf,\ T f(x) = f(x+h)$$

Operator is unitary. What is measure?

$-\pi$ |—— ( ) ——| π Want measure.

When does $f(s+h) = \lambda f(s)$ for arbitrary function?

$e^{ia(s+h)} = e^{iah}\, e^{ias}$ But these are not in

No point can carry a measure; for there are no individual eigenfunctions.

If e^{ias} were in space, it would have e^{iah} as eigenvalue.

Let ϕ be a function in L_2. Consider function $\int_{\lambda-\epsilon}^{\lambda+\epsilon} e^{ias}\phi(a)\,da$.

These span a subspace. If $\phi = 1$, value is

$$e^{i\lambda s} \frac{\sin \epsilon s}{s}$$, which is in Hilbert space.

Approximate, etc.

Fix λ and ϵ, let ϕ flop around. We get resolution of identity in Hilbert space. It should be shown that inner products on two different intervals are zero. For ϵ small transformation has approximate eigenvalue $e^{i\lambda h}$.

Operator T_t, translation by t

T_{nt} is a discrete group of operators

Interpolation of all real powers can be done.

Example:

If $f \in L_2$, Fourier integral $\frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(s)\, e^{ist}\, ds$ exists and converges as $T \to \infty$ in L_2 norm to an L_2 function $\phi(t)$. This is an isometry of L_2 onto L_2:

$$(f_1, f_2) = (\varphi_1, \varphi_2)$$

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) \cdot e^{-ist}\, dt.$$

Fourth power of this transformation is always identity. Spectrum has 4 points — $1, -1, i, -i$.

Let $\mathfrak{M}_I$ = subspace with support in I. Can talk of $\mathfrak{M}(-\infty, \lambda] = \mathfrak{M}(-\infty, \lambda)$

Let $\mathfrak{N}_I$ = those functions whose Fourier transform is carried by I. Subspace because of isometry. This gives a different resolution of identity. Starting from $\mathfrak{H}$, we get $\mathfrak{M}$ space and $\mathfrak{N}$ space. If $\mathfrak{N}$ is taken as replica, space of Fourier transforms is space of conjugates of elements in $\mathfrak{M}$.

If $f \leftrightarrow \phi$, then $T_a f \leftrightarrow e^{-iat}\phi(t)$.

T_s gives group with s

Question of continuity:

$$T_{s+h} \xrightarrow{?} T_s \quad \text{in norm}$$

or more weakly

$$T_{t+h} f \to T_s f \quad \text{for each } f$$ ← called norm topology; other should be called that.

For translations first is false but second holds because

$$\| f(s+t+h) - f(s+t) \| = \| f(t+h) - f(t) \| \to 0.$$

Does derivative exist?

Does $\frac{T_{\xi+h} - T_\xi}{h} f$ converge, say to Ωf

If $f \in C_\infty$ and of compact support, yes.

This exists for nice functions, but not all. Ω is differentiation operator when possible. Ω is defined on dense set, the differentiable functions.

Derivative in $\mathfrak{N}$ space

$$\left[\frac{e^{-i(\xi+h)t} - e^{i\xi t}}{h}\right]\phi(t) \to -e^{i\xi t}\, t\,\phi(t),$$ unbounded operator

NASC for derivative above to exist is that $t\phi(t) \in \mathcal{L}_2$ by isometry.

Let μ be finite on $(-\infty,\infty)$ and let $\eta =$ 

Map $\eta \longrightarrow 1$

$T_t \eta \to e^{its}$

Set $\mu(ds) = \underbrace{\frac{1}{\pi}}_{\text{approximately}} \frac{1-\cos s}{s^2}\, ds$

If $f \to \Phi$, then $T_t f \to e^{its}\phi(s)$

Given $\mathcal{H}$ and a group of unitary operators $\mathcal{U}_t$ (isometry for each t, u_{-t} is inverse of u_t). Assume $u_t x \to x$ as $t \to 0$ (Von Neumann showed measurability is enough.) Single out η.

Set $\phi(t) = (\eta, u_t \eta)$ (covariance function)

We had this for t taking integer values before. Now we have t arbitrary real

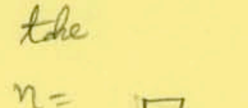
We shall show $\exists$ measure ~~such that~~

$$\phi(t) = \int_{-\infty}^{\infty} e^{it\xi} F(ds)$$, F a bounded measure

F = prob measure if $\|\eta\|^2 = 1$.

Get isometry of subspace of $\mathcal{H}$ generated by $\mathcal{H}$ and $\mathcal{H}(F)$.

With translations take

$\eta =$  . Subspace is all of $\mathcal{H}$.

$$(\eta, T_t \eta) = \begin{cases} 1-|t| & \text{if } |t| \le 1 \\ 0 & \text{if } |t| \ge 1 \end{cases}$$

ϕ looks like 

Inversion formula $\frac{1}{2\pi}\int_{-1}^{1} e^{-its}(1-|t|)\,dt$ gives density of F

Integrate by parts and get $\frac{1-\cos s}{s^2}\,ds$ formula.

Get a decomposition with this measure. Measure depends on η. A different spanning η would give another measure abs cont and vice versa. In every case we get an isometry. In $\mathcal{N}$ space, mapping with η and S is a multiplication mapping and preserves supports.

$$T_t \eta \leftrightarrow e^{its} \text{ in F-space}$$

Given M and N finite, we want a Hilbert space.

Define $N^-(I) = N(-I)$. For fixed F define ϕ by

$$\phi(t) = \int_{-\infty}^{\infty} e^{it\xi} F(ds).$$

Consider $\int \phi(\alpha-\beta)\, M(d\alpha)\, N^-(d\beta)$. In space of characteristic fcns, flipping of measure corresponds to conjugation. Now

$$\int \phi(\alpha-\beta)\, M(d\alpha)\, N^-(d\beta) = \int_{-\infty}^{\infty} M(d\alpha) \int e^{i(\alpha-\beta)\xi} F(ds)$$

$$= \int_{-\infty}^{\infty} \mu(\xi)\, e^{-i\beta\xi} F(ds) \quad \text{BLAH}$$

This gives Hilbert space of measures. Close it. $= \int \mu(\xi)\overline{\nu(\xi)}\, F(ds)$, μ is function from M, ν ... N.

Necessary condition on ϕ is that for any M, with $N = M$, $\int \phi(\alpha-\beta) M(d\alpha) N^{-}(d\beta) \geq 0$.
Bochner's theorem is that this is NAS for every M carried by finitely many points.

Let ϕ be continuous with $\phi(0)=1$.

When is $\phi(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} F(dx)$, $F(-\infty,\infty)=1$? Bochner Thm.

Suppose first that $\phi \in \mathcal{L}$.

Then a NASC is that $\int_{-\infty}^{\infty} \phi(\alpha) e^{-i\alpha x} d\alpha \geq 0$ for every x

(This integral exists for $\phi \in \mathcal{L}$.)

We need a pair g, γ with $g(x) \geq 0$, $\int_{-\infty}^{\infty} g(x)dx = 1$

$$\gamma(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} dx \quad , \gamma \geq 0 \leftarrow \text{So } g \text{ is even}$$

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \gamma(\alpha) d\alpha. \quad \text{Also } g(0) \geq g(x)$$

Normal density is example: $g = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\gamma = e^{-\alpha^2/2}$

Or $g = \frac{1}{2} e^{-|x|}$, $\frac{1}{\pi} \frac{1}{1+\alpha^2}$

Sufficiency: $\int_{-\infty}^{\infty} \phi(\alpha) e^{-i\alpha x} d\alpha = f(x) \geq 0$

f is continuous. We shall prove that f is density.

Form $g\left(\frac{x}{n}\right)\frac{1}{n}$. It corresponds to $\gamma(\alpha n)$

$$g\left(\frac{x-y}{n}\right)\frac{1}{n} \leftrightarrow \gamma(\alpha n) e^{i\alpha y}$$

$$\int_{-\infty}^{\infty} f(x)\, g\left(\frac{x-y}{n}\right) \frac{dx}{n} = \int_{-\infty}^{\infty} \phi(\alpha)\gamma(\alpha n) e^{i\alpha y} d\alpha.$$

Take $y=0$. $\int_{-\infty}^{\infty} f(x) g\left(\frac{x}{n}\right) d\alpha = \int \phi(\alpha)\gamma(\alpha n)\, n\, d\alpha$

Let $n \to \infty$, $g\left(\frac{x}{n}\right) \to g(0)$. Left side $\to \int_{-\infty} f(x)$

Right side converges, to $\leq \int_{-\infty}^{\infty} \gamma(\alpha) d\alpha$ So $f \in \mathcal{L}$.

Convolution $f(x)\, g\left(\frac{x-y}{n}\right)$ is a density. We calculate characteristic function of left side $e^{ity} dy$

We get $\quad \gamma(t\eta) \int_{-\infty}^{\infty} f(x)\, e^{ix\alpha}\, dx$ ②

As $\eta \to 0$, left side as a measure goes to measure with density f

(Want to prove $f(x)\, g\left(\frac{x-t}{\eta}\right)$ has char. funct. $\phi(\alpha)\gamma(\alpha\eta)$. To be repaired.)

Assume we have proved f is a density

$$\phi(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{ix\alpha} dx$$

Hypothesis of $\phi \in \mathcal{L}$.

ϕ is characteristic fcn if $\phi(\alpha)\gamma(\epsilon\alpha)$ is for every ϵ (uniform limit is char. fcn)

γ is integrable and by special case, we have that

$$\int_{-\infty}^{\infty} \phi(\alpha)\gamma(\epsilon\alpha)\, e^{-i\alpha x}\, d\alpha \geq 0 \text{ is sufficient}$$

Let $u(x)$ be a density with function $\omega(\alpha)$.

$$u(-x) \leftrightarrow \overline{\omega(\alpha)}$$

Call $u(-x)$ u^-; $\quad u * u^-$ is symmetric and $\leftrightarrow |\omega(\alpha)|^2$

For measures $\quad \mu * \mu^- \leftrightarrow |\mu|^2$.

Take g to be $u * u^-$ and γ to be $|\omega(\alpha)|^2$ and we get Bochner's criterion.

Want to interpret

$$\iint_{-\infty}^{+\infty} \phi(\alpha - \beta)\, u(\alpha)\, u(\beta)\, d\alpha\, d\beta$$
$$= E(\phi(X - Y))$$
$$= E(\phi(X + Y^-))$$

X and Y independent r.v. gives product measures

$$\int_{-\infty}^{\infty} \phi(\alpha)\, |\omega(\alpha)|^2 e^{-i\alpha x}\, d\alpha = E(\phi(Z)),\ Z \text{ has density } \gamma.$$

First integral is same as $\int \phi(t)\,(u * u^-)(t)\, dt$ by change of variable

Bochner criterion:

Take finitely many points α_j α_k, arbitrary z_j

$$\sum \phi(\alpha_j - \alpha_k)\, z_j \bar{z}_k \geq 0.$$

Want to say this same as integral condition.

Sum gives by Riemann sums

$$\iint \phi(\alpha - \beta)\, \omega(\alpha)\, \overline{\omega(\beta)} \geq 0$$

for every choice of ω.

If $\omega(\alpha) = e^{i\alpha s}\, \gamma(\alpha)$, then from $\gamma \geq 0$

$\overline{\omega(\beta)} = e^{-i\beta s}\, \gamma(\beta)$. Then put in to get criterion

$$\int \phi(\alpha)\, |\omega(\alpha)|^2\, e^{-i\alpha x}\, d\alpha \geq 0.$$

We had Hilbert space with inner product

$\iint \phi(\alpha - \beta)\, M(d\alpha)\, N(d\beta)$. If M and N are delta measures, we get

$$\phi(\alpha_0 - \alpha_1)$$

z_j may be considered as giving linear combination of delta measures.

Bochner's criterion says do this for complex discrete measures and we must get a pre-Hilbert space.

Groups of transformations

$\mathcal{H}$ a Hilbert space

Fix η, generate space by $U_t\,\eta$, $-\infty<t<\infty$. Assume this generates all of $\mathcal{H}$.

$\phi(\alpha)=(\eta, U_\alpha \eta)$

Criterion:

$$\sum_{j,k}\phi(\alpha_j-\alpha_k)\,z_j\bar{z}_k=\sum_{j,k}\left(z_j\eta, z_k U_{\alpha_j-\alpha_k}\eta\right)$$
$$=\sum\left(z_j U_{-\alpha_j}\eta\,,\ z_k U_{-\alpha_k}\eta\right)$$
$$=\left\|\sum z_j U_{\alpha_j}\eta\right\|^2\geq 0.$$

We assume that ϕ is continuous in α. Then

$$\phi(\alpha)=\int_{\infty}^{\infty}e^{i\alpha x}F(dx)\ ;$$ this gives spectral measure.

$\phi(0)=\|\eta\|^2$, assume $=1$. F is probability measure

$\mathcal{H}\leftrightarrow \mathcal{L}^2_F$.

Map $\eta\leftrightarrow 1$

$U_\alpha\eta\leftrightarrow e^{i\alpha x}$

is an isometry.

To every interval $\mathfrak{M}_I$ there corresponds a subspace, functions vanishing outside I. Subspaces for different intervals are orthogonal. Write

$$X=\int_{-\infty}^{\infty}E(d\lambda, x)$$

$$U_\alpha X=\int_{\infty}^{\infty}e^{i\alpha\lambda}E(d\lambda,x)$$, group structure

F as usual is not canonical. If $\xi\to z(\lambda)$,

$$F(d\lambda)\longrightarrow \underbrace{\frac{1}{z(\lambda)}}_{\text{or } z(\lambda)}F(d\lambda)$$

$z(\lambda)\neq 0$ a.e. if ξ generates.

Subspaces are fixed

Unbounded operators

Does $\lim_{\alpha \to 0} \frac{u_\alpha x - x}{\alpha}$ exist?

If $x \to g(\lambda)$, we get $g(\lambda) \left(\frac{e^{i\alpha\lambda} - 1}{\alpha} \right)$

Formally this goes to $i\lambda g(\lambda)$. If g has compact support convergence does take place in $\mathcal{L}^2$ norm. Limit corresponds to unbounded operator

Define A to be an operator defined when $\lambda g(\lambda) \in \mathcal{L}_2$ and

$$A x \to \lambda g(\lambda).$$

Let $g(x)$ be a probability density, $\int g(x) = 1$

Fourier integral formula says $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma(t)\, e^{-itx}\, dt$

We do not use this. Take any such pair g, γ for which this holds and $\gamma \geq 0$ and $\gamma \in \mathcal{L}$.

Sufficient conditions for ϕ bounded cont., $\phi(0) = 1$ to be a characteristic function

$$\phi(\xi) = \int e^{i\xi x}\; F(dx)$$

1) $\phi \in \mathcal{L}$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\alpha)\, e^{-i\alpha t}\, d\alpha \geq 0 \quad \text{for every } t.$$

Proof:

Call left side $f(t)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\alpha)\, e^{it(x-\alpha)}\, d\alpha = f(t)\, e^{itx}$$

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\, \gamma(\varepsilon t) \int_{-\infty}^{\infty} \phi(\alpha)\, e^{it(x-\alpha)} d\alpha = \int_{-\infty}^{\infty} \gamma(\varepsilon t)\, f(t)\, e^{itx}$$

Integral exists because γ is integrable

$$\int_{-\infty}^{\infty} \phi(\alpha)\, g\left(\frac{x-\alpha}{\epsilon}\right) \frac{1}{\epsilon}\; d\alpha = \int_{-\infty}^{\infty} f(t)\, \gamma(\varepsilon t)\, e^{itx}\, dt.$$

Put $x = 0$.

$$\int_{-\infty}^{\infty} f(t)\, \gamma(\varepsilon t)\, dt \leq \max |\phi|$$

since ϕ bounded $\int g = 1$ or $\int g\left(\frac{x}{\epsilon}\right) \frac{dx}{\epsilon} = 1$

Let $\epsilon \to 0$. $\gamma(\varepsilon t) \to \gamma(0) = 1$. Hence $f \in \mathcal{L}$.

Then we can apply Dominated Convergence as $\epsilon \to 0$ above.

g concentrates mass at x $\int_{-\infty}^{\infty} f(t)\, e^{itx} dt = \phi(x)$. Since $\phi(0) = 1$,

f has integral 1.

2) $\int_{-\infty}^{\infty} \phi(\alpha)\, g\left(\frac{\alpha}{\epsilon}\right) \frac{1}{\epsilon}\, e^{-i\alpha t}\, d\alpha \geq 0$ for every t, every ϵ, where

g is one such density $(g(\alpha) = g(-\alpha))$.

Drop $\frac{1}{\epsilon}$.

By (1) $\phi(\alpha)\, g\left(\frac{\alpha}{\epsilon}\right)$ is a characteristic function for any ϵ by (1).

Let $\epsilon \to \infty$. $\phi(\alpha)\, g\left(\frac{\alpha}{\epsilon}\right) \to \phi(\alpha)$. Limit of characteristic functions is a characteristic function. Hence ϕ is a characteristic function.

3) Let m be a probability density.

Define $m^-(x) = m(-x)$. Let

$$g(x) = \int_{-\infty}^{\infty} m(x-y)\, m(-y)\, dy.$$

Let m have characteristic function. Then $\gamma = |\mu|^2$. Hence it is sufficient that $\gamma \geq 0$. If μ is square integrable, then it is sufficient that

$$\int \phi(\alpha)\, g\left(\frac{\alpha}{\epsilon}\right) e^{-i\alpha t}\, d\alpha \geq 0$$

for $g = m * m^-$. More generally it is sufficient that

$$\boxed{\int \phi(\alpha)\, e^{-i\alpha t}\, \tilde{M}(d\alpha) \geq 0} \quad \text{where } \tilde{M} = M * M^-$$

for every t and every prob. measure M.

Remarks.

If X has distribution M and Y has distribution $-M$, then $X - Y$ has dist. $M * M^-$. Condition is then that

$$E\left(\phi(X-Y)\, e^{-i(X-Y)t}\right) \geq 0$$

Take M to be concentrated at finitely many points, value α_j

$$\sum \phi(\alpha_j - \alpha_k)\, e^{-i(\alpha_j - \alpha_k)t}\, p_j p_k \geq 0$$

or $\quad \sum \phi(\alpha_j - \alpha_k)\, z_j \bar{z}_k \geq 0$, where $z_j = p_j e^{-i\alpha_j t}$.

Bochner says sufficient if use arbitrary z's.

Next, put $\alpha = a-b$. Condition gives

$$\iint \phi(a-b)\, e^{-i(a-b)t}\, M(da)M(db) \geq 0.$$

Absorb e's into M's and talk of M's complex. Make it worse and assume above holds for all complex measures

$$\iint \phi(\alpha-\beta)\, M(d\alpha)\, \overline{M(d\beta)} \geq 0.$$

Moreover, if $\phi(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} F(dx)$, then

$$\iint \phi(\alpha-\beta) M(d\alpha)\overline{M(d\beta)} = \int \mu(x)\overline{\mu(x)}\, F(dx) \geq 0$$

gives a pre-Hilbert space since characteristic fcns give pre-Hilbert space

Criterion is therefore that we can get a pre-Hilbert space.

Unbounded operators.

Representation of group of unitary operators wrt measure.

Let $\{\phi_k\}$ be a complete orthonormal system. Assign ϕ_k to a point λ_k of axis. Get resolution of identity.

If $x = \sum a_k \phi_k$, let $Tx = \sum \lambda_k a_k \phi_k$.

Projection operator E: $\int_{-\infty}^{17} E(d\lambda, x) = \sum_{\lambda_k \leq 17} a_k \phi_k$.

$u(T)\,x = \sum u(\lambda_k)\, a_k \phi_k$

Before $U_t x = \int e^{i\lambda} E(d\lambda, x)$ or $U_t x = \sum e^{i\lambda_n t} a_n \phi_n$.

For fixed t, then λ_n can be changed by $2\pi t$ and we can get spectrum in $[-2\pi, 2\pi]$. We can form powers of U_t but not extract roots.

Take a resolution of identity. Call an interval resolvent if the zero space is assigned to it. There is a maximal open resolvent set. (If ϕ_k's are assigned to rationals, set is empty.) Remainder is called spectrum.

Let T be bounded symmetric, with $\|T\|=1$. Then spectrum is such that $|\lambda| \le 1$. positive $(Tx,x)\ge 0$

Differential operator can be made negative definite

$$\int_0^1 f''(s) f(s) ds = \underbrace{f'f\Big|_0^1}_{0 \text{ on a dense set.}} - \int_0^1 f'^2(s)\, ds$$

Thus let T be negative definite $= (Tx, x) \le 0$, norm of $T = a$

Spectrum of $T: (-a, 0)$, $-T: (0, a)$, $\gamma - T: (\gamma, a+\gamma)$, $\gamma > 0$.

$(\gamma-T)^{-1}$ exists: $\left(\frac{1}{a+\gamma}, \frac{1}{\gamma}\right)$

$(\gamma-T)^{-1}$ is bounded; so $\gamma(\gamma-T)^{-1}$ has spectrum between $\left(\frac{\gamma}{a+\gamma}, 1\right)$.

Operator is positive for $\gamma > 0$.

Let A be a negative definite operator with spectrum in $(-\infty, 0)$

$$Ax = \int_{-\infty}^{0} \lambda\, E(d\lambda, x).$$

Domain of A is those x's for which this converges.

γR_γ = resolvent operator $= \gamma(\gamma - A)^{-1}$ is a bounded symmetric operator with spectrum in $(0,1)$. In spectral notation

$$\gamma R_\gamma x = \int_{-\infty}^{0} \frac{\gamma}{\gamma - \lambda} E(d\lambda, x) \qquad (\lambda \le 0)$$

Let $(\gamma - A)^{-1} x = y$, $(\gamma-A)^{-1}$ bounded, defined for all x.

Then $(\gamma - A) y = x$. $\quad y \in R_\gamma$.

For $\gamma > 0$, $(\gamma - A) y = x$ possesses a unique solution corresponding to the domain of A. Hence range of R_γ does not depend on γ.

We wanted to every x a y such that $(\gamma-A)^{-1}x = y$.

Range R is independent of γ.

We have $(\gamma-A)y = x$

$(\gamma-A)y = 0$ is impossible-

For fixed x and γ varying let

$$\gamma y_\gamma - Ay_\gamma = x$$

$$\nu y_\nu - Ay_\nu = x$$

$$\gamma(y_\gamma - y_\nu) - A(y_\gamma - y_\nu) = (\nu-\gamma)y_\nu$$

$y_\gamma - y_\nu$ is in range of trans. corresponding to γ. Since y_γ is in, so is y_ν. Hence range is invariant.

Conversely if R is fixed as γ varies, then A is uniquely defined with domain R and range H. Just substitute

Example:

$\mathcal{L}^2$ with Lebesgue measure on $[0,1]$ Let $A = \frac{d^2}{ds^2}$. Claim nothing like above exists, but that operator can be repaired to give inverse. We can impose conditions by saying, for example, for $\gamma = 1$ make function vanish at 0 and 1. Take $\gamma = 0$ and fix a boundary condition that will make solution to $\gamma y - y'' = x$ unique for $\gamma = 0$. Then let γ vary and take invariant part. Operator is restriction $d^2/ds^2 \,\big|\, R$ for some set R

Take for R the twice diff (in $\mathcal{L}^2$) functions which vanish at 0 and 1.

For every γ there is exactly one solution to $\gamma y - y'' = x$.

More generally solution is unique for $p_0 y'(0) + q_0 y(0) = 0$

$$p_1 y'(1) + q_1 y(1) = 0$$

and operator is symmetric

$$y(s) = u(s)\int_0^s x(t)v(t)\,dt + v(s)\int_s^1 x(t)u(t)\,dt$$

satisfies $\gamma y - y'' = x$ up to num. constant. u and v satisfy $\gamma y - y'' = 0$ and one bndry condition

Let $u'v - uv' = 1$.

Condition: $u(0)=0$, $v(1)=0$.

Call solution $\bar{y}$ to homogeneous.

$$\bar{y} = \int_0^1 G(s,t)\, x(t)\,dt \qquad G(s,t) = \begin{cases} u(t)v(s) \\ u(s)v(t) \end{cases}$$

$$\bar{G} = G(s,t) + \underbrace{P_1(t)\,u_1(s) + P_2(t)\,u_2(s)}_{\sum_{j,k=1}^{2} P_{jk}\, u_j(s)\, u_k(t)}$$ must be symmetric.

, P depends on γ.

Call $(\gamma - A)^{-1} = R_\gamma$ the resolvent if $(\gamma-A)^{-1}$ is bounded linear from $\mathcal{H} \to \mathcal{R}$.

Above $y_\gamma = R_\gamma x$

$$R_\gamma(y_\nu - y_\gamma) = R_\gamma(\nu - \gamma)y$$

$$\boxed{(R_\gamma - R_\nu)\,x = (\nu - \gamma)\, R_\gamma R_\nu\, x}$$ for every x

$$\boxed{(R_\gamma - R_\nu) = -(\gamma - \nu)\, R_\gamma R_\nu}$$

Called the resolvent operator.

If this equation holds, then range is fixed.

For all of them commute and range $R_\nu \subset R_\gamma$ since range of right side $\subset$ range R_γ. Hence range R_ν = range R_γ.

Define A by $(\gamma - A)R_\gamma(x) = x$ for fixed γ. Claim this is satisfied for all other γ. Ranges are same. Result comes from resolvent equation.

Semigroups of operators (domain Banach space)

Space $\mathcal{B}$. For initial data x we want to know what happens at time t.

Assume transformation is linear: $T_t x$. We suppose time homogeneity (intervals of same length play same role). Postulate

$$T_s(T_t x) = T_{t+s}(x)$$

(This is physicists' notion of no external forces)

$$T_{t+s} = T_s T_t$$

Domain of T_t is all of $\mathcal{B}$. T_t is closed. So $\underline{T_t \text{ is bounded.}}$

This condition corresponds to equation $u(t+s)=u(t)u(s)$. We need a condition. u is continuous is sufficient.

Assumption: For $t>0$, semigroup is strongly continuous

$$T_{t+h}\, x \to T_t\, x \quad \text{in norm topology as } h \downarrow 0$$

(strong continuity)

Set $T_0 = I$ and assume also for $t=0$.

Example: $\mathcal{B} = C(R)$, vanishing at ∞.

$T_t f(x) = f(x+t)$

Strong continuity: $\|T_t f - f\| \to 0$ means uniform continuity, which is satisfied

If functions are just bounded, use $f =$ 

and strong continuity is not satisfied

Solution should be $e^{t\frac{d}{dx}} f(x) = \sum \frac{t^n}{n!} f^{(n)}(x)$

$= f(x+t)$ by Taylor formula

This does not make sense in general, but $e^{t\frac{\Delta}{h}}$ does make sense

$$\lim_{h\to 0} \sum \frac{t^n}{n!} \frac{\Delta}{h}^n f(x) \to f(x+t)$$

Example:

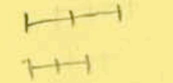
Bounded $\to 0$ at ∞ continuous. Norm $\|f\| = \max\{\max|f|, 17|f(1)|\}$.

$T_t f(\alpha) = f(\alpha + t)$. Norm is 17. Still strongly continuous.

Cut off space at 0 and 17. Then norm is 17 for $t < 1$, norm is 1 for $1 < t < 17$, norm is 0 beyond 17.

Example:

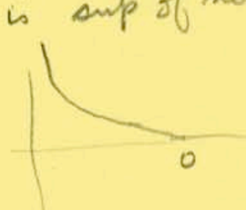
Denumerably many spaces as above



Space is direct sum

Norm is sup of norms in individual spaces.

Norm can be made like



Still strong continuity for $t > 0$.

Introduction:

$\mathfrak{X}$ a Banach space, $\mathcal{L}\mathfrak{X}$ the space of bounded linear transformations of $\mathfrak{X}$ into $\mathfrak{X}$

$\mathcal{L}\mathfrak{X}$ is a Banach algebra

<u>Definition 1</u>:

A <u>strongly continuous semigroup</u> of bounded transformations on $\mathfrak{X}$ is a mapping $[0,\infty) \ni t \to P^t \in \mathcal{L}\mathfrak{X}$ such that

(1) $P^0 = 1$

(2) $P^s P^t = P^{s+t}$

(3) $\lim\limits_{t\downarrow 0} P^t x = x$ for $x \in X$

<u>Theorem 2</u>:

Let P^t be a semigroup on X. Then there exists an $M \geq 1$ and an $\omega \geq 0$ such that $\|P^t\| \leq Me^{\omega t}$ for $t \geq 0$. The function $t \to P^t x$ from $[0,\infty) \to \mathfrak{X}$ is continuous in the norm topology for all x.

Proof:

Let $\quad \mathfrak{X}_n = \left\{ x \;\middle|\; \sup\limits_{0\le t\le \frac{1}{n}} \|P^t x\| \le n \right\}$

$$= \bigcap_{0\le t\le\frac{1}{n}} (P^t)^{-1} \left\{ u \;\middle|\; \|u\| \le n \right\}$$, which is closed since P^t is continuous

Also $\bigcup\limits_{n=1}^{\infty} \mathfrak{X}_n = \mathfrak{X}$ since $P^t x \to x$ for $t\downarrow 0$ and $\|P^t x\|$ must be bounded. By Baire Category Theorem, for some $n \geq 1$, $X_n^0 \neq \phi$. Then there exists an x_0 and an $1 > \epsilon > 0$ such that $\|u - x_0\| \le \epsilon$ implies $u \in \mathfrak{X}_n$. If $\|w\| \le \epsilon$, look at

$$\begin{aligned}\|P^t w\| &= \|P^t(x_0 + w) - P^t x_0\| \\ &\le \|P^t(x_0+w)\| + \|P^t x_0\| \le 2n \quad \text{for } 0 \le t \le \tfrac{1}{n}\end{aligned}$$

since each is in $\mathfrak{X}_n$.

By linearity $\|P^t\| \le \frac{2m}{\epsilon}$ for $0 \le t \le \frac{1}{m}$

Now $\|AB\| \le \|A\|\|B\|$. For $0 \le t \le 1$

$$\|P^t\| \le \left(\frac{2m}{\epsilon}\right)^{m+1} \text{ since } P^t = P^{\frac{1}{m}} \cdot P^{\frac{1}{m}} \cdots P^{t - n/m}$$

$$\le \left(\frac{2m}{\epsilon}\right)^m = M$$

$$\|P^t\| = \|P^{[t]} P^{t-[t]}\|$$

$$\le \|P^{[t]}\| \|P^{t-[t]}\|$$

$$\le \|P^1\|^{[t]} \cdot M \le M^{[t]+1} \le M^{t+1} = Me^{\omega t} \text{ with } \omega = \log M.$$

For the second part let $0 \le t_1 < t_2 \le T$. Then

$$\|P^{t_1}x - P^{t_2}x\| = \|P^{t_1}(x - P^{t_2 - t_1}x)\|$$

$$\le \|P^{t_1}\| \|x - P^{t_2-t_1}x\|$$

$$\le Me^{\omega T} \|P^{t_2-t_1}x - x\| \to 0 \text{ as } t_2 - t_1 \to 0$$

Hence the continuity.

3. Linear operators

Linear transformation $L : \mathcal{D} \xrightarrow{\text{into}} \mathfrak{X}$. $\operatorname{Im} L = \{Lx \mid x \in \mathcal{D}_L\}$. $\mathcal{D}$ is assumed a subspace

Graph $= \{(x, Lx) \mid x \in \mathcal{D}_L\} \subset \mathfrak{X} \times \mathfrak{X}$.

L is closed if its graph is closed in $\mathfrak{X} \times \mathfrak{X}$, i.e. if whenever $x_\nu \to x$, and $Lx_\nu \to y$, then $x \in \mathcal{D}_L$ and $Lx = y$

L is densely defined if $\overline{\mathcal{D}_L} = \mathfrak{X}$.

If $x \ne y$ and $x, y \in \mathcal{D}_L \Rightarrow Lx \ne Ly$, we can define L^{-1} with domain equal to $\operatorname{Im} L$ and range $\mathcal{D}_L$.

4. Differentiation of Banach-valued real functions

$(a,b) \xrightarrow{f} \mathfrak{X}$ with $a<b$.

If $\dfrac{f(x+h_\nu)-f(x)}{h_\nu}$ converges for every sequence h_ν, then limit exists

and is defined to be $f'(x)$

If $[a,b] \xrightarrow{f} \mathfrak{X}$ is continuous, let

$$a = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_n = b \text{ with } \alpha_{\nu-1} \le \xi_\nu \le \alpha_\nu .$$

As in real case,

$$\sum_{1 \le \nu \le n} (\alpha_\nu - \alpha_{\nu-1}) f\, \xi_\nu \quad \text{converges when mesh} \to 0$$

Call limit $\int_a^b f$ or $\int_a^b f(t)\,dt$.

Note $\int_a^b f \in \mathfrak{X}$.

Results:

1) $\| \int_a^b f \| \le \int_a^b \|f(t)\|\,dt$

2) Let $f : [t, t+\delta] \to \mathfrak{X}$ be continuous. Then

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} f = f(t) .$$

Proof:

$$\left\| \frac{1}{h} \int_t^{t+h} f - f(t) \right\| = \frac{1}{h} \left\| \int_t^{t+h} (f - f(t)) \right\|$$

$$\le \frac{1}{h} \int_t^{t+h} \| f - f(t) \|$$

$$\to \| f(t) - f(t) \| = 0.$$

3) $T\left(\int_a^b f\right) = \int_a^b T \circ f$ because T commutes with sums and T is continuous (so can take limit).

5. The infinitesimal generator is the linear operator A defined by ④

$$Ax = \lim_{h \downarrow 0} \frac{P^h x - x}{h} \text{ with domain } \{x \mid \text{limit exists}\}.$$

Remark: Ax is the derivative of $P^t x$ at 0.

Theorem 5-2: A is densely defined.

Proof: $\int_\alpha^\beta P^\cdot x = \int_\alpha^\beta P^t x \, dt$

Now
$$\frac{P^h\left(\int_0^t P^\cdot x\right) - \int_0^t P^\cdot x}{h} = \frac{1}{h}\int_0^t P^{h+\cdot} x - \frac{1}{h}\int_0^t P^\cdot x$$
$$= \frac{1}{h}\int_h^{h+t} P^\cdot x - \frac{1}{h}\int_0^t P^\cdot x$$
$$= \frac{1}{h}\int_t^{t+h} P^\cdot x - \frac{1}{h}\int_0^h P^\cdot x$$
$$\longrightarrow P^t x - P^0 x = P^t x - x.$$

Hence $\int_0^t P^\cdot x \in \mathcal{D}A$.

Therefore $\frac{1}{t}\int_0^t P^\cdot x \in \mathcal{D}A$

And as $t \downarrow 0$, this goes to x. Therefore $\mathcal{D}$ is dense.

Proposition 5-3:

For $x \in \mathcal{D}A$ $P^t x \in \mathcal{D}A$ and $AP^t x = P^t A x$

Proof:
$$\frac{P^h P^t x - P^t x}{h} = P^t\left(\frac{P^h - 1}{h} x\right) \longrightarrow P^t A x \text{ by continuity}$$

Hence $P^t x \in \mathcal{D}A$ and since left side is $AP^t x$, we have the second part.

Proposition 5-4: $(P^{\cdot}x)' = P^{\cdot}Ax$ for $x \in \mathcal{D}A$.

Proof:

For $h > 0$ $\quad \dfrac{P^{t+h}x - P^t x}{h} \underset{h \downarrow 0}{\longrightarrow} P^t A x$

For $t > 0$ and h small look at $\left\| \dfrac{P^{t-h}x - P^t x}{-h} - P^t A x \right\|$

$$= \left\| P^{t-h}\left(\frac{1-P^h}{-h} x - P^h A x \right) \right\|$$

$$\le \| P^{t-h} \| \left\| \frac{P^h - 1}{h} x - P^h A x \right\|$$

$$\le M e^{\omega t} \| \quad \| \to 0$$

since $\dfrac{P^h - 1}{h} x \to Ax$

and $P^h A x \to Ax$.

Corollary 5-5: $P^b x - P^a x = \int_a^b P^{\cdot} A x$.

Proposition 5-6: A is closed.

Proof:

Suppose $x_\nu \in \mathcal{D}A$ and $x_\nu \to x$ and $Ax_\nu \to y$. Form

$$\| P^t A x_\nu - P^t y \| \le \| P^t \| \| A x_\nu - y \|$$

Hence $P^t A x_\nu \to P^t y$ uniformly in t for $t \in$ bounded interval.

For fixed h $\int_0^h P^{\cdot} A x_\nu \longrightarrow \int_0^h P^{\cdot} y$ by uniform convergence.

Left side by 5.5 is

$$P^h x_\nu - x_\nu \longrightarrow P^h x - x.$$

Then $\frac{1}{h}(P^h x - x) = \frac{1}{h}\int_0^h P^\cdot y \to y$

Therefore $x \in \mathcal{D}A$ and $Ax = y$ Q.E.D.

Example 6:

Let $A \in \mathcal{L}\mathfrak{X}$. $P^t = e^{tA}$ is a semigroup on $\mathfrak{X}$.

Notes:

1) If $B \in \mathcal{L}\mathfrak{X}$, define e^B by

$$e^B = \sum \frac{B^\nu}{\nu!}$$

We have $\left\| \sum_n^{n+p} \frac{B^\nu}{\nu!} \right\| \le \sum_n^{n+p} \frac{\|B\|^\nu}{\nu!} \to 0$.

Hence series is Cauchy in the Banach algebra and e^B exists.

2) $BC = CB$ implies $e^B e^C = e^{B+C} = e^{C+B} = e^C e^B$

3) $e^{0A} = e^0 = 1$

4) $P^t P^s = P^{t+s}$ since tA and sA commute

5) $\|P^t - 1\| \to 0$ for $t \downarrow 0$ (uniform topology)

$$\left\| \sum_{0\le\nu} \frac{(tA)^\nu}{\nu!} - 1 \right\| = \left\| \sum_{1\le\nu} \frac{(tA)^\nu}{\nu!} \right\| \le t\|A\| \left\| \sum_{1\le\nu} \frac{(tA)^{\nu-1}}{\nu!} \right\|$$

$$\le t\|A\| \sum_{1\le\nu} \frac{(t\|A\|)^{\nu-1}}{(\nu-1)!} \to 0$$

This is uniform continuity rather than strong continuity (If uniform continuity holds, then A is a bounded operator — to be shown later.)

6) $$\left\| \frac{P^h x - x}{h} - Ax \right\| = \left\| \frac{\sum_{0\le\nu} \frac{h^\nu A^\nu x}{\nu} - x}{h} - Ax \right\|$$

$$= \left\| \frac{1}{h}\left(\sum_{1\le\nu} \frac{h^\nu A^\nu x}{\nu!}\right) - Ax \right\| = \left\| \frac{1}{h}\left(\sum_{2\le\nu} \frac{h^\nu A^\nu x}{\nu!}\right) \right\|$$

$$= h \left\| \sum_{2\le\nu} \frac{h^{\nu-2} A^\nu x}{\nu!} \right\| \to 0.$$

Proposition 7:

The resolvent or Laplace transform of the semigroup: If $\|P^t\| \leq Me^{\omega t}$ and $\operatorname{Re}\lambda > \omega$, then

$$R_\lambda x = \int_0^\infty e^{-\lambda t} P^t x \, dt \quad \text{exists for } x \in \mathfrak{X}$$

$R_\lambda \in \mathcal{L}\mathfrak{X}$ and $\|R_\lambda\| \leq \dfrac{M}{\operatorname{Re}\lambda - \omega}$.

Proof:

Let $0 \leq a_\nu < \infty$ and look at $\int_0^{a_\nu}$. $\quad a_\nu \uparrow \infty$

$$\left\| \left(\int_0^{a_{n+p}} - \int_0^{a_n} \right) e^{-\lambda t} P^t x \, dt \right\|$$

$$= \left\| \int_{a_n}^{a_{n+p}} e^{-\lambda t} P^t x \, dt \right\|$$

$$\leq \int_{a_n}^{a_{n+p}} |e^{-\lambda t}| \, M e^{\omega t} \|x\| \, dt$$

$$= \|x\| \cdot M \int_{a_n}^{a_{n+p}} e^{(\omega - \operatorname{Re}\lambda)t} \, dt$$

$$= \|x\| \cdot M \, \frac{e^{(\omega - \operatorname{Re}\lambda) a_{n+p}} - e^{(\omega - \operatorname{Re}\lambda) a_n}}{\omega - \operatorname{Re}\lambda}$$

$$\leq \|x\| \cdot M \, \frac{e^{(\omega - \operatorname{Re}\lambda) a_n}}{\operatorname{Re}\lambda - \omega} \longrightarrow 0 \quad \text{as } a_n \to \infty.$$

Theorem: P^t is a semigroup with $\|P^t\| \le Me^{\omega t}$ for $t \ge 0$. For $\mathrm{Re}\lambda > \omega$, (1)

$R_\lambda x = \int_0^\infty e^{-\lambda t} P^t x \, dt$ exists for all x, and $R_\lambda \in \mathcal{L}\mathfrak{X}$ and

$\|R_\lambda\| \le \dfrac{M}{\mathrm{Re}\lambda - \omega}$

Proof:

We have seen for $a_\nu \uparrow \infty$, $\lim_{\nu\to\infty} \int_0^{a_\nu} e^{-\lambda t} P^t x \, dt \in \mathfrak{X}$ exists.

Define this to be $\int_0^\infty e^{-\lambda t} P^t x \, dt$. R_λ is linear

$$
\begin{aligned}
\|R_\lambda x\| &= \left\|\int_0^\infty e^{-\lambda t} P^t x \, dt\right\| \\
&= \left\| \lim_{n\to\infty} \int_0^n \quad \right\| \\
&\le \lim_{n\to\infty} \int_0^n \|e^{-\lambda t} P^t x \, dt\| \\
&\le \int_0^\infty e^{-\lambda t} M e^{\omega t} \|x\| dt \\
&= \|x\| \frac{M}{\mathrm{Re}\lambda - \omega} .
\end{aligned}
$$

Assert $R_\lambda = (\lambda - A)^{-1}$ for $\mathrm{Re}\lambda > \omega$. Im $(\lambda - A) = \mathfrak{X}$. $(\lambda - A)$ is one-one.

Proof:

Form

$$
\begin{aligned}
\frac{P^h - 1}{h} R_\lambda x &= \int_0^\infty e^{-\lambda t} \frac{P^{h+t} - P^t}{h} x \, dt \\
&= \frac{1}{h} \int_h^\infty e^{-\lambda(t-h)} P^t x \, dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} P^t x \, dt \\
&= \frac{1}{h}\int_0^\infty \left(e^{-\lambda(t-h)} - e^{-\lambda t}\right) P^t x \, dt - \frac{1}{h}\int_0^h e^{-\lambda(t-h)} P^t x \, dt \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} P^t x \, dt - \frac{1}{h}\int_0^h e^{-\lambda(t-h)} P^t x \, dt
\end{aligned}
$$

$\frac{e^{\lambda h}}{h}(\quad)$

$\rightarrow \lambda R_\lambda x - x$

So $R_\lambda x \in \mathcal{D}A$ and $AR_\lambda x = \lambda R_\lambda x - x$. So $R_\lambda \subset \mathcal{D}A$ and

$(\lambda - A) R_\lambda = I$

If $x \in \mathcal{D}A$, then $x \in \mathcal{D}(\lambda - A)$ and

$$R_\lambda A x = \int_0^\infty e^{-\lambda t} P^t A x \, dt$$

$$= \int_0^\infty e^{-\lambda \cdot} (P^{\cdot} x)'$$

$$= e^{-\lambda \cdot} P^{\cdot} x \Big|_0^\infty + \lambda \int_0^\infty e^{-\lambda \cdot} P^{\cdot} x$$

$$= -x + \lambda R_\lambda x .$$

$$R_\lambda (\lambda - A) x = x .$$

We can write $R_\lambda(\lambda - A) \subset I$ (restriction of I).

First part shows $\lambda - A$ is onto $\mathcal{X}$

Second part shows $\lambda - A$ is one-one.

So $(\lambda - A)^{-1}$ exists with domain $\mathcal{X}$

Also $(\lambda - A)(R_\lambda - (\lambda - A)^{-1}) = 0$

So $R_\lambda = (\lambda - A)^{-1}$. Q.E.D.

Theorem: (Yosida and Hille)
Let A be a densely defined operator on a Banach space $\mathcal{X}$, and suppose there exists a sequence $\lambda_\nu \uparrow \infty$ such that $(\lambda_\nu - A)^{-1} \in \mathcal{L}\mathcal{X}$ and $\|(\lambda_\nu - A)^{-1}\| \le \frac{1}{\lambda_\nu}$ for $1 \le \lambda_\nu$. Then A is the infinitesimal generator of a contraction ($\|P^t\| \le 1$) semigroup

Proof:
$\lambda A (\lambda - A)^{-1}$ is everywhere defined (Note for reals $\frac{\lambda A}{\lambda - A} = \frac{A}{1 - \frac{A}{\lambda}} \to A$ as $\lambda \to \infty$)

Let λ, μ be some λ_ν's. $\lambda \to \infty$ means $\lambda_\nu \to \infty$.

$$A = \lambda - (\lambda - A)$$

$\operatorname{Im}(\lambda - A)^{-1} = \mathcal{D}(\lambda - A) = \mathcal{D}A$. Then

$$A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I$$

Let $A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda \in \mathcal{L}\mathcal{H}$.

Set $P_\lambda^t = e^{tA_\lambda}$

$$\|P_\lambda^t\| = \|e^{tA_\lambda}\| = \|e^{t\lambda^2(\lambda-A)^{-1}} e^{-\lambda t}\| \quad \text{since operators commute}$$

$$\leq e^{-\lambda t} e^{\|t\lambda^2(\lambda-A)^{-1}\|}$$

$$\leq e^{-\lambda t} e^{\lambda t \cdot 1} \quad \text{since } \|\lambda(\lambda-A)^{-1}\| \leq 1 \text{ by hypothesis}$$

$$= 1.$$

Now $(\lambda-A)^{-1}(\lambda-\mu)(\mu-A)^{-1} = (\lambda-A)^{-1}[(\lambda-A) - (\mu-A)](\mu-A)^{-1}$

$$= (\mu-A)^{-1} - (\lambda-A)^{-1}$$

So $(\mu-A)^{-1}$ commutes with $(\lambda-A)^{-1}$ and A_λ commutes with A_μ. P_λ^t commutes with P_μ^s. And P_λ^t commutes with A_μ, which is a power series in P_μ^s.

$$P_\lambda^t x - P_\mu^t x = \int_0^t (P_\mu^{t-\cdot} P_\lambda^{\cdot} x)'$$

$$= \int_0^t \{-P_\mu^{t-\cdot} A_\mu P_\lambda^{\cdot} x + P_\mu^{t-\cdot} P_\lambda^{\cdot} A_\lambda x\}$$

commutativity used in product rule

$$= \int_0^t P_\mu^{t-\cdot} P_\lambda^{\cdot} (A_\lambda x - A_\mu x)$$

Then $\|P_\lambda^t x - P_\mu^t x\| \leq t \|A_\lambda x - A_\mu x\|$

Suppose $x \in \mathcal{D}A$. Then

$$\lambda(\lambda-A)^{-1}x - x = A(\lambda-A)^{-1}x \underset{\text{for } x \in \mathcal{D}A}{=} (\lambda-A)^{-1}Ax$$

$\|\lambda(\lambda-A)^{-1}x - x\| \le \|(\lambda-A)^{-1}\| \, \|Ax\| \xrightarrow{\lambda\to\infty} 0$

So $\quad x = \lim_{\lambda\to\infty} \lambda(\lambda-A)^{-1}x \quad$ for $x \in \mathcal{D}A$

Since A is densely defined, this holds for all x.

(We have a sequence $\|B^t_\lambda\| \le M$. $B^t_\lambda x \to 0$ for $x \in \mathcal{D}$ with $\bar{\mathcal{D}} = \mathfrak{X}$
uniformly for $t \in T$

Then $B^t_\lambda x \underset{\lambda\to\infty}{\Rightarrow} 0$ for all x.

Proof:
$y \in \mathfrak{X}$. Let $\mathcal{D} \ni x_\nu \to y$.

$$\|B^t_\lambda y\| \le \|B^t_\lambda x_\nu - B^t_\lambda y\| + \|B^t_\lambda x_\nu\|$$
$$\le M\|x_\nu - y\| + \|B^t_\lambda x_\nu\|$$
$$< \epsilon \text{ uniformly in } t \text{ for } \lambda \text{ and } \nu \text{ large})$$

Thus $(\lambda(\lambda-A)^{-1} - I)x \to 0$ for $x \in \mathfrak{X}$.

For $x \in \mathcal{D}A$, $\quad \underset{\displaystyle A_\lambda x}{\underset{\|}{\lambda(\lambda-A)^{-1}Ax}} \to Ax$ as $\lambda\to\infty$

Thus $\{P^t_\lambda x\}$ is Cauchy for each x and t.

In fact $(P^t_\lambda - P^t_\mu)x \overset{\lambda,\mu\to\infty}{\Longrightarrow} 0$ for $x \in \mathcal{D}A$, hence for all x.
uniformly for $t \in$ bounded interval

Uniform Cauchy implies uniform convergence

$P^t_\lambda x \to P^t x$, uniform for $t \in [0,T]$, holding for each x.

By pointwise convergence, $\|P^t\| \le 1$, $P^0 = 1$.
$$P^{t+s} = P^t P^s.$$

Now $P^{\cdot}_\lambda x$ is continuous. This goes uniformly to $P^{\cdot}x$ on bounded intervals. So $P^{\cdot}x$ is continuous, especially $P^t x \to x$ for $t \to 0$.

Thus P^t is a semigroup.

We also have

$$\| P_\lambda^t x - P^t x \| \le t \, \| A_\lambda x - A x \| \quad \text{for } x \in \mathcal{D}A.$$

For $x \in \mathcal{D}A$

$$\| P_\lambda^t A_\lambda x - P^t A x \| \le \| P_\lambda^t A_\lambda x - P_\lambda^t A x \| + \| P_\lambda^t A x - P^t A x \|$$

$$\le \| A_\lambda x - A x \| + \| P_\lambda^t A x - P^t A x \|$$

Hence uniform convergence since $P_\lambda^t y \Rightarrow P^t y$

So $\quad P_\lambda^t A_\lambda x \to P^t A x$ uniformly in t on any $[0,T]$

$$\int_0^T P_\lambda^t A_\lambda x \underset{\text{by } 5.5}{=} P_\lambda^T x - x$$

by uniform conv. on finite intervals $\left\{ \downarrow \lambda \to \infty \right.$ $\qquad \downarrow \lambda \to \infty$

$$\int_0^T P^t A x = P^T x - x$$

Or $\quad \displaystyle\int_0^T P^t A x = P^T x - x.$

$$\frac{P^t x - x}{t} = \frac{1}{t} \int_0^t P^{\cdot} A x$$

$$\underset{\text{by continuity of } P^t}{\longrightarrow} A x$$

Thus $x \in \mathcal{D}A \Rightarrow x \in \mathcal{D}\tilde{A} = \mathcal{D}$ (inf generator). And $\tilde{A} x = A x$. Thus $A \subset \tilde{A}$.

$$(\lambda_2 - \tilde{A})^{-1} \supset (\lambda_2 - A)^{-1} \in \mathcal{L}\mathfrak{X}$$

$\| P^t \| \le 1 \cdot e^{\omega t}$ with $\omega = 0$. Resolvent exists for $\lambda > 0$.

So $(\lambda_2 - \tilde{A})^{-1} = (\lambda_2 - A)^{-1}$ and $\lambda_2 - \tilde{A} = \lambda_2 - A$, $\tilde{A} = A$. $\qquad$ Q.E.D.

Extension of theorem:

Assume $\|(\lambda_\nu - A)^{-n}\| \le \dfrac{M}{(\lambda_\nu - \omega)^n}$ for each ν and all n

Then we conclude $\|P^t\| \le M e^{\omega t}$.

Uniqueness:

If P^t and Q^t are semigroups with the same generator A, then $P^t = Q^t$.

Proof: $\int_0^\infty e^{-\lambda t} P^t x \, dt = (\lambda - A)^{-1} x$ for $\operatorname{Re} \lambda > \omega$ and $\|P^t\|, \|Q^t\| \le M e^{\omega t}$

$$= \int_0^\infty e^{-\lambda t} Q^t x \, dt$$

Take $y^* \in X^*$. Apply to integrals and take inside

$$\int_0^\infty e^{-\lambda t} (P^t x, y^*) \, dt = \int_0^\infty e^{-\lambda t} (Q^t x, y^*) \, dt$$

These are Laplace transforms. By their uniqueness theorem

$$(P^t x, y^*) = (Q^t x, y^*)$$

By Hahn-Banach, $P^t x = Q^t x$. $P^t = Q^t$. Q.E.D.

10. <u>Theorem</u>: P^t a semigroup with

$$\limsup_{t\downarrow 0} \|P^t - 1\| \le 1 \quad , \quad A \text{ the infinitesimal generator.}$$

Then $A \in \mathcal{L}\mathfrak{X}$ and $P^t = e^{At}$ and $\|P^t - 1\| \to 0$.

Proof:

Suppose $1 > \epsilon > \limsup \|P^t - 1\|$. Choose $\delta > 0$ such that for $0 < t \le \delta$ we have $\|P^t - 1\| < \epsilon$. Take $x \in \mathcal{D}A$.

$$\frac{P^t x - x}{t} = \frac{1}{t}\int_0^t P^{\cdot} A x = \frac{1}{t}\int_0^t \{Ax + (P^{\cdot}Ax - Ax)\}$$

$$= Ax + \frac{1}{t}\int_0^t (P^{\cdot} - 1)(Ax)$$

For $t = \delta$ $\qquad Ax = \frac{P^\delta - 1}{\delta} x - \frac{1}{\delta}\int_0^\delta (P^{\cdot} - 1) Ax$

$$\|Ax\| \le \frac{1}{\delta}\|P^\delta - 1\|\|x\| + \frac{1}{\delta}\delta\,\epsilon\,\|Ax\|$$

because $\|(P^{\cdot} - 1)(Ax\| \le \epsilon \|Ax\|$

$$\|Ax\| \le \left\{\frac{1}{1-\epsilon}\,\frac{1}{\delta}\,\|P^\delta - 1\|\right\}\|x\|$$

So A is bounded on $\mathcal{D}A$. But A is ai densely defined and closed. Therefore $A \in \mathcal{L}\mathfrak{X}$

(Note: Let x be given. Take $x_\nu \to x$.

$$\|Ax_\nu - Ax_\mu\| \le \text{const}\,\|x_\nu - x_\mu\|$$

So Ax_ν converges. Since A is closed, $x \in \mathcal{D}A$ and $Ax_\nu \to Ax$.)

Then $P^t = e^{At}$ by uniqueness theorem. For such a semigroup we know $\|P^t - 1\| \to 0$.

11. Theorem: Let A be densely defined on $\mathfrak{X}$. Suppose $\exists\, \lambda_\nu \uparrow \infty$, M, ω such that $(\lambda_\nu - A)^{-1} \in \mathcal{L}\mathfrak{X}$ and

$$\|(\lambda_\nu - A)^{-n}\| \le \frac{M}{(\lambda_\nu - \omega)^n} \text{ for } 1 \le n, 1 \le \nu.$$

Then A is the infinitesimal generator of a semigroup P^t such that $\|P^t\| \le M e^{\omega t}$ for $t \ge 0$

Convergence theorems for semigroups.

12. Let P^t be a semigroup with infinitesimal generator A. Let $R_\lambda = (\lambda - A)^{-1}$ for $\lambda > \omega$. Then

$$e^{-\lambda t} e^{t\lambda^2 R_\lambda} x \longrightarrow P^t x \quad \text{as } \lambda \to \infty \text{ for } t \ge 0, x \in \mathfrak{X}$$

Also $\lambda R_\lambda x \to x$ as $\lambda \uparrow \infty$ for $x \in \mathfrak{X}$

(For contraction case we have proved this:

$$\lambda(\lambda - A)^{-1} x \to x \quad \text{and} \quad e^{t\lambda^2(\lambda - A)^{-1}} e^{-\lambda t} = P^t_\lambda \to P^t$$

strongly

Hence by uniqueness we have the result.)

In addition $$\lim_{\lambda \to \infty} e^{-\lambda t} \sum_{0 \le \nu} \frac{(t\lambda^2)^\nu}{\nu!} R_\lambda^\nu x = P^t x.$$

From such formulas we get information about P^t from R_λ^ν

For the reverse $R_\lambda = \int_0^\infty P^t \, dt$.

Example: Positivity preserving for either implies for other.

$$\lim_{h\downarrow 0} e^{t\frac{P^h-1}{h}} x = P^t x \quad \text{if } \|P^t\| \le 1.$$

Proof: $$\|e^{t\frac{P^h-1}{h}}\| = \|e^{-\frac{t}{h}} e^{tP^h/h}\|$$
$$\le e^{-t/h} \sum \frac{t^\nu \|P^h\|^\nu}{h^\nu} \frac{1}{\nu!}$$
$$\le e^{-t/h} \sum \frac{t^\nu}{h^\nu} \frac{1}{\nu!} = 1.$$

Look at $e^{(s-t)\frac{P^h-1}{h}} P^t x$ Differentiable because these commute

This has derivative wrt t.

$$e^{(s-t)\frac{P^h-1}{h}}\left(-\frac{P^h-1}{h} P^t x + P^t A x\right).$$

$$\int_0^s \left(e^{(s-\cdot)\frac{P^h-1}{h}} P^\cdot x\right)' = P^s x - e^{s\frac{P^h-1}{h}} x$$

$$\|$$

$$\int_0^s e^{(s-\cdot)\frac{P^h-1}{h}} P^\cdot\left(Ax - \frac{P^h-1}{h}x\right)$$

$$\|P^s x - e^{s\frac{P^h-1}{h}} x\| \le \int_0^s \|e^{(s-\cdot)\frac{P^h-1}{h}}\| \, \|P^\cdot\| \, \|Ax - \frac{P^h-1}{h}x\|$$

$$\le s\|Ax - \frac{P^h-1}{h}x\|$$

$\longrightarrow 0$ uniformly for s in compact interval.

Hence the result for $x \in \mathcal{D}A$

$$\left(P^s - e^{s\frac{P^h-1}{h}}\right)x \to 0 \text{ as } h \downarrow 0.$$

By lemma from last time (norm here bnded by 2), this holds for all x.

Q.E.D.

14. <u>Theorem</u>: P^t a semigroup with $\|P^t\| \le Me^{\omega t}$, A the infinitesimal generator. Let $B \in \mathcal{L}\mathfrak{X}$. Then $A+B$ is the infinitesimal generator of a semigroup Q^t with $\|Q^t\| \le Me^{\omega_1 t}$, where $\omega_1 = \omega + M\|B\|$

Proof:

We use 11.

1) $A+B$ is defined on $\mathcal{D}A$ and is densely defined.

2) Suppose $\lambda > \omega_1$. Then

$$\|B(\lambda-A)^{-1}\| \le \|B\| \frac{M}{\lambda-\omega} < 1.$$

Let $C = (\lambda-A)^{-1} \sum_{0\le\nu} \left(B(\lambda-A)^{-1}\right)^\nu \in \mathcal{L}\mathfrak{X}$. (geometric series.

Denote $(\lambda-A)^{-1} = R_\lambda$

Let $x \in \mathcal{D}A$. Form

$$\begin{aligned} C(\lambda-A-B)x &= R_\lambda \sum_{0\le\nu} (BR_\lambda)^\nu (\lambda-A-B)x \\ &= R_\lambda(\lambda-A-B)x + R_\lambda \sum_{1\le\nu} (BR_\lambda)^{\nu-1} BR_\lambda(\lambda-A)x \\ &\quad - R_\lambda \sum_{1\le\nu} (BR_\lambda)^\nu Bx \\ &= x - R_\lambda Bx + R_\lambda \sum_{1\le\nu} (BR_\lambda)^{\nu-1} Bx \\ &\quad - R_\lambda \sum_{1\le\nu} (BR_\lambda)^\nu Bx \quad = x \\ &= x. \end{aligned}$$

So $C(\lambda-A-B) \subset 1$.

For $x \in \mathfrak{X}$, $Cx \in \mathcal{D}A$ since im $(\lambda - A)^{-1} \subset \mathcal{D}A$. Hence

$$(\lambda - A - B)\, Cx = (\lambda - A) R_\lambda \sum_{0 \le \nu} (BR_\lambda)^\nu x - BR_\lambda \sum_{0 \le \nu} (BR_\lambda)^\nu x$$

$$= x + \sum_{1 \le \nu} (BR_\lambda)^\nu x - \sum_{1 \le \nu} (BR_\lambda)^\nu x = x$$

$$(\lambda - A - B)\, C = 1.$$

Therefore $(\lambda - A - B)^{-1} \in \mathcal{L}\mathfrak{X}$ for $\lambda > \omega_1$

3) $$\Big\{ \sum_{0 \le \nu} R_\lambda (BR_\lambda)^\nu \Big\}^n = \sum_{0 \le \nu_1, \dots, \nu_n} R_\lambda (BR_\lambda)^{\nu_1} R_\lambda (BR_\lambda)^{\nu_2} \dots R_\lambda (BR_\lambda)^{\nu_n}$$

Look at terms with $k = \nu_1 + \dots + \nu_n$ B's. Here have $n+k$ R_λ's.
The R_λ's are in $k+1$ groups separated by B's.

We know that
$$\| R_\lambda^k \| \le \frac{M}{(\lambda - \omega)^k}$$, true for any semigroup.

The norm of such a term is
$$\le \|B\|^k \frac{M^{k+1}}{(\lambda - \omega)^{n+k}}$$ ← since $k+1$ groups only

$*\ \ *\ \ *\ \ \cdots\ \ *$ B's

Divide into at most n sets. Draw in $n-1$ lines. Equivalent to choosing from $n+k-1$ objects $n-1$ of them. Order is irrelevant. Get
$\binom{n+k-1}{n-1}$ terms with k B's.

$$(1-x)^{-n} = \sum_{0 \le k} \binom{-n}{k} (-1)^k x^k$$

$$\frac{(-n)(-n-1)\dots(-n-k+1)}{k!} = (-1)^k \frac{(n+k-1)\dots(n)}{k!} = (-1)^k \binom{n+k-1}{k}$$

$$(1-x)^{-n} = \sum_{0 \le k} \binom{n+k-1}{n-1} x^k$$

$$\left\| \left(\sum_{0\le \nu} R_\lambda (BR_\lambda)^\nu \right)^n \right\| \le \sum_{0\le k} \binom{n+k-1}{n-1} \|B^k\| M^{k+1} \frac{1}{(\lambda-\omega)^{n+k}}$$

$$= \frac{M}{(\lambda-\omega)^n} \left(1 - \frac{\|B\| M}{\lambda - \omega}\right)^{-n}$$

$$= M(\lambda - \omega - \|B\|M)^{-n}$$

$$= \frac{M}{(\lambda-\omega_1)^n} \qquad \text{Q.E.D.}$$

15. Lemma: P^t a contraction semigroup on a Hilbert space with infinitesimal generator A. Then $\operatorname{Re}(Ax, x) \le 0$ for $x \in \mathcal{D}A$.

Proof: For $x \in \mathcal{D}A$

$$0 \leftarrow \frac{P^h - 1}{h} x - Ax = P^h \frac{x - x - hAx}{h}$$

$$P^h x = x + hAx + o(h)$$

$$\|P^h x\|^2 = \|x\|^2 + h(Ax, x) + h(x, Ax) + o(h)$$

$$= \|x\|^2 + 2h \operatorname{Re}(Ax, x) + o(h)$$

$$\frac{\|P^h x\|^2 - \|x\|^2}{h} = 2\operatorname{Re}(Ax, x) + \frac{o(h)}{h}$$

But $\|P^h x\|^2 \le \|P^h\|^2 \|x\|^2 \le \|x\|^2$. So left side is ≤ 0.
Let $h \to 0$. Then

$$2\operatorname{Re}(Ax, x) \le 0.$$

16. Definition: A an operator on Hilbert space. A is called dissipative if and only if $\operatorname{Re}(Ax, x) \le 0$ for $x \in \mathcal{D}A$.

17. <u>Theorem</u>:

A densely defined on a Hilbert space $\mathcal{H}$. Then A is the infinitesimal generator of a contraction semigroup if and only if

1) A is dissipative
2) Image $(\lambda_0 - A) = \mathcal{H}$ for some $\lambda_0 > 0$.

Proof:

We have seen one direction. Now for converse. Let $\lambda > 0$ and $x \in \mathcal{D}A$. Then

$$\|(\lambda - A)x\|^2 = \lambda^2 \|x\|^2 - 2\lambda \operatorname{Re}(Ax, x) + \|Ax\|^2$$
$$\geq \lambda^2 \|x\|^2.$$

Therefore $(\lambda - A)$ is one-one. So $(\lambda - A)^{-1}$ is defined on $\mathcal{I}m(\lambda - A)$. Now

$\frac{1}{\lambda}\|(\lambda - A)x\| \geq \|x\|$ so that since $x = (\lambda - A)^{-1}y$

$$\|(\lambda - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\| \text{ for } y \in \mathcal{D}(\lambda - A)^{-1}$$

We know $\mathcal{D}(\lambda_0 - A)^{-1} = \mathcal{H}$.

So $(\lambda_0 - A)^{-1}$ is bounded and hence closed.

Therefore $\lambda_0 - A$ closed, A is closed, $\lambda - A$ is closed, $(\lambda - A)^{-1}$ is closed for $\lambda > 0$.

Let $\lambda = \mu + \lambda_0$, $\mu > 0$. Suppose $z \perp \mathcal{I}m(\lambda - A)$. Then

$0 = (z, (\lambda - A)y)$ for $y \in \mathcal{D}A$. Let $z = (\lambda_0 - A)x$ with $x \in \mathcal{D}A$. Put $y = x$

$$0 = ((\lambda_0 - A)x, (\lambda - A)x) = \|(\lambda_0 - A)x\|^2 + \mu((\lambda_0 - A)x, x)$$
$$= \|(\lambda_0 - A)x\|^2 + \mu$$

X a Banach space, T_t a semigroup with $\|T_t\| \le 1$, strongly continuous

We write "inner product" as (x^*, x) or $x^* x$

Define for each x a value (x^*, Tx). This gives a unique $x^* T$ with

$$(x^*, Tx) = (x^* T, x)$$

This is the adjoint transformation. Gives a semigroup, strongly continuous fails.

Then $x^* T_t x = (x^* T_t, x) = (x^*, T_t x)$

This $\uparrow$ is real-valued function in t.

Example:

Continuous functions on a circle, sup norm.

$T_t\, x(s) = x(s+t)$ modulo 1. Is strongly continuous.

Adjoint space is measures on circle.

Pushing measure in opposite direction is same as measure on shifted function

$$\mu T_t (I) = \mu(I - t)$$

Take Dirac measure μ. $\|\mu T_t - \mu\| = 2$.

Example:

Continuous functions bounded on open line.

General situation is that there is a subspace on which semigroup is strongly cont.

In first example if μ has a density m

$$\int |m(s-t) - m(s)|\, ds \longrightarrow 0 \quad \text{by Lebesgue's theorem.}$$

Example: Heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2}$

Abs. cont measure; set

$$m_t(\cdot) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} m(s)\, e^{-\frac{(\cdot - s)^2}{4t}}\, ds.$$

$u(t,\cdot) = m_t(\cdot)$ satisfies above equation

This is density of heat distribution at time t starting with δ distribution

Look at T_t as mapping $X^* \times X$ into R.

Semigroup maps all of $X^* \times X$ into real-valued functions of t.

We want these functions to behave. If T_t is strongly cont., these functions are continuous. Reason: T_t maps $x^* T_\alpha$ and $T_\beta x$ into $x^* T_{\alpha+t+\beta} x$, which is the translation of the given function.

We can put a topology on functions, e.g.

$$\sup_t |x^* T_t x|$$

This gives a Banach space $\mathcal{B}$. T_t maps $X^* \times X \to \mathcal{B}$.

Hille's definition: T_t is weakly continuous if $x^* T_t x$ is always bnded measurable.

We assume this condition holds.

Suppose for each x^* and x that $x^* T_t x$ is bounded measurable

$\int_\alpha^\beta (x^* T_t x)\, dt$ exists. Fix x^*. As x runs through X, get linear functional: $x^*_{\alpha,\beta}(x) \overset{def.}{=} \int_\alpha^\beta (x^* T_t x)\, dt$

Notation: $x^*_{\alpha,\beta} = \int_\alpha^\beta x^* T_t\, dt$. Weak integral.

Now we use semigroup property.

$$x^*_{\alpha,\beta} T_\tau(x) = \int_\alpha^\beta (x^* T_\tau) T_t \, x \, dt$$
$$= \int_\alpha^\beta x^* T_{\tau+t} \, x \, dt$$
$$= x^*_{\alpha+\tau, \beta+\tau}$$

If $\|T_t\| \le 1$, then
$\|x^*_{\alpha,\beta}\| \le \|x^*\|(\beta-\alpha)$

Assert strong continuity in a form.

$$x^*_{\alpha\beta} T_h - x^*_{\alpha,\beta} = x^*_{\beta,\beta+h} - x^*_{\alpha,\alpha+h}$$

As $h \to 0$, this goes to zero in norm.
The elements $x^*_{\alpha,\beta}$ form a linear set as x^*, α, β vary. Close it. Get a subspace.
Semigroup is strongly cont on dense set and hence everywhere.
Call subspace $Y \subset X^*$. Y is invariant $Y T_t \subset Y$ because dense set is invariant. If $y \in Y$, then $y T_h \to y$.
Hence exists subspace on which semigroup is strongly continuous.

Size of Y: Y is big enough to distinguish members of X. (to be shown next time). Symmetrically X is big enough to distinguish members of Y.

Also Y is big enough so that

$$\|x\| = \sup_{\|y\|=1} |yx| .$$

Y is called representative of X^* if all of this holds.

Assumptions

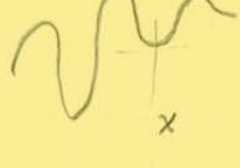
1) axis homogeneous, commutativity with translations

2) A of local character

If $f=0$ in a nbd of x, then $\frac{T_t f - f}{t}(x) \to 0$

Or $T_t f(x) = f(x) + o(t)$

A has a local minimum property:

If f looks like  , then $Af(x) \geq 0$ if $f \in \mathcal{D}A$.

We have $A1 = 0$, so we may shift coordinates. By local character, we can assume f looks like .

Now $T_t f \geq 0$. $\frac{T_t f - f}{t}(x) = \frac{T_t f}{t} \geq 0$.

Limit must be ≥ 0.

$Af(x) \geq 0$ is the heat equation.

Example. If $a>0$, then $af'' + bf'$ has the local minimum property.

If A is of local character and has the local minimum property, then there is a reparametrization such that every function is once differentiable and A is of the form $\frac{d}{dm}\frac{df}{dx}$.

If we have axis homogeneity, we must have Lebesgue measure, and operator becomes $\frac{d^2 f}{dx^2} + a\frac{df}{dx}$