Formulas for minimal K-types, rank G = rank K

Assumptions: G linear connected semisimple, rank G = rank K

G not involving split G_2

Δ = roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, where $\mathfrak{t}$ = compact Cartan subalgebra $\subseteq \mathfrak{k}$

$\alpha_1, \dots, \alpha_m$ a fixed sequence of strongly orthogonal noncompact roots

Roots are real if in $\sum \mathbb{R}\alpha_j$. Form G^r connected, K^r, Δ_r etc. relative to these.

imaginary if $\perp \sum \mathbb{R}\alpha_j$

complex otherwise. Decompose $\mathfrak{t} = \mathfrak{t}^- \oplus \mathfrak{t}^r$

P = orthogonal projection on $\sum \mathbb{R}\alpha_j$.

M and A constructed from $\alpha_1, \dots, \alpha_m$ in standard way; $\underset{\sim}{c}$ = Cayley transform

We work only with $M^\# = M_e Z_M$ since discrete series of M are induced from $M^\#$.

Let σ = discrete series of $M^\#$

λ_0 = a Harish-Chandra parameter of σ relative to $(\mathfrak{m}, \mathfrak{t}^-)$

$\lambda = \lambda_0 + \rho_n^- - \rho_c^-$ = unique minimal $K \cap M^\#$ type of σ

Positive system Δ^+:

$\Delta_0 = \{\beta \in \Delta \mid \langle \lambda_0, \beta \rangle = 0\}$, defines $\mathfrak{t}_0$

$\Delta_1^+ = \{\beta \in \Delta \mid \langle \lambda_0, \beta \rangle > 0\}$.

For Δ_0 adjoin elements of $i\mathfrak{t}_0'$ at the left to make $\dots, \alpha_1, \dots, \alpha_m$ an orthogonal basis of $i\mathfrak{t}_0'$, and order lexicographically

$\Delta^+ = \Delta_1^+ \cup \Delta_0^+$. Define Δ_K, Δ_K^+, and various ρ's in the obvious way.

For each real root β we have a corresponding odd restricted root $\underset{\sim}{c}(\beta)$ for (G^r, A). Let γ_β be the usual element corresponding to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. $F = \operatorname{span}\{\gamma_\beta\}$.

<u>Main Theorem.</u> Every minimal K-type of $\mathrm{ind}_{K\cap M^\#}^{K} \sigma$ is of the form

$$\boxed{\Lambda = \lambda - P(2\rho_c) + 2\rho_{red,c} + \mu,} \qquad (*)$$

where μ is a fine K^Ω type whose restriction to F contains the character

$$\omega = \sigma \cdot \exp(P(2\rho_c) - 2\rho_{red,c})\big|_F$$

where $\exp(P(2\rho_c) - 2\rho_{red,c})$ is a well-defined one-dimensional representation of $K_\Omega \supseteq F$. Conversely every fine K^Ω type μ with $\tau_\mu|_F \supseteq \omega$ is such that Λ is integral, and Λ is a minimal K-type of $\mathrm{ind}_{K\cap M^\#}^{K}$ if and only if it is K-dominant, which happens if and only if μ satisfies the following conditions:

(i) for any Δ_K^+ simple root $\beta = \varepsilon \pm \frac{1}{2}\alpha_j$ in Δ_0 but not Δ_Ω,

$$2\langle \mu, \beta\rangle / |\beta|^2 > -1/2$$

(ii) for any Δ_K^+ simple root $\beta = \varepsilon \pm \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$ in Δ_0 but not Δ_Ω, such that $\frac{1}{2}\alpha_i + \frac{1}{2}\alpha_j$ is not a root and $\varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$, α_i, and α_j are all simple for Δ^+, $2\langle \mu, \beta\rangle/|\beta|^2 > -1$.

<u>Lemma 1.</u> Let α be a real root of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a}+\mathfrak{b})^{\mathbb{C}})$ and let $M^*A^*N^* \supseteq MAN$ be constructed from MAN and α. Suppose that ξ_λ is a character of $B\,F(B)$ with differential λ on $\mathfrak{b}$. Then the formulas

$$\Lambda = \begin{cases} \lambda & \text{on } \mathfrak{b}^* \text{ if } \xi_\lambda(\gamma_\alpha) = +1 \\ \lambda + \tfrac{1}{2}\alpha & \text{on } \mathfrak{b}^* \text{ if } \xi_\lambda(\gamma_\alpha) = -1 \end{cases}$$

$$\xi_\Lambda(f) = \xi_\lambda(f) \quad \text{if } f \in F(B^*) \subseteq F(B)$$

unambiguously define a character ξ_Λ of $B^*F(B^*)$ such that

a) ξ_Λ has differential Λ on $\mathfrak{b}^*$

b) $\xi_\Lambda(\gamma_\alpha) = \xi_\lambda(\gamma_\alpha)$

c) $\xi_\Lambda(f) = \xi_\lambda(f)$ for f in $F(B^*)$

Remarks. See §2 of KZ for properties of $F(B)$. See the proof of Theorem 6.1 of KZ for a model for the first part of this proof.

Proof.

1) We define a character ξ_1 of $B^* = BB_\alpha$. For this purpose we regard $\lambda + c\alpha$ as a candidate to be exponentiated. Here λ exponentiates to B, and so $\lambda + c\alpha$ exponentiates to $c\alpha$ if and only if both of the following are satisfied:

(i) $c\alpha$ exponentiates to B_α, i.e., c is in $\frac{1}{2}\mathbb{Z}$ and $\gamma_\alpha \neq 1$ or c is in $\mathbb{Z}$ and $\gamma_\alpha = 1$.

(ii) $\exp\lambda$ agrees with $\exp c\alpha$ on $B \cap B_\alpha$, which is contained in $\{1, \gamma_\alpha\}$.

For (ii) it is sufficient to have

(ii') $\xi_\lambda(\gamma_\alpha) = (\exp c\alpha)(\gamma_\alpha) = (-1)^{2c}$.

If $\xi_\lambda(\gamma_\alpha) = +1$, we use $c = 0$; then (i) and (ii') hold. If $\xi_\lambda(\gamma_\alpha) = -1$, we use $c = \frac{1}{2}$; then (ii') holds, and (i) holds since γ_α cannot be 1.

2) We define ξ_2 on $F(B^*)$ by $\xi_2(f) = \xi_\lambda(f)$. This definition is meaningful since $F(B^*) \subseteq F(B)$.

3) We show that ξ_1 and ξ_2 are consistently defined on $B^* \cap F(B^*)$, so that we can consistently define ξ_Λ on $B^* F(B^*)$ by $\xi_\Lambda(b^* f) = \xi_1(b^*)\xi_2(f)$.

Thus let f be in $B^* \cap F(B^*)$. Since f is in B^*, write $f = \exp(H + i\theta H_\alpha')$ for some H in $\mathfrak{b}$ and θ in $\mathbb{R}$. By definition

$$\xi_1(f) = \begin{cases} \xi_\lambda(\exp H) & \text{if } \xi_\lambda(\gamma_\alpha) = +1 \\ e^{i\theta}\xi_\lambda(\exp H) & \text{if } \xi_\lambda(\gamma_\alpha) = -1. \end{cases}$$

Since $F(B^*) \subseteq Z_{M^*}$, f must commute with the root vector X_α for α.

We compute

$$\begin{aligned} Ad(f)X_\alpha &= Ad(\exp(i\theta H_\alpha'))\, Ad(\exp H) X_\alpha \\ &= Ad(\exp(i\theta H_\alpha'))X_\alpha \quad && \text{since } \alpha \text{ vanishes on } \mathfrak{b} \\ &= e^{2i\theta}X_\alpha && \text{by a computation in } \mathfrak{sl}(2,\mathbb{C}). \end{aligned}$$

Hence $e^{2i\theta} = 1$ and $e^{i\theta} = \pm 1$. This means

$$f = \begin{cases} \exp H & \text{if } \theta \equiv 0 \bmod 2\pi \\ \gamma_\alpha \exp H & \text{if } \theta \equiv \pi \bmod 2\pi \end{cases}$$

Now ξ_λ is defined on both γ_α and $\exp H$, and we thus have

$$\xi_2(f) = \xi_\lambda(f) = \begin{cases} \xi_\lambda(\exp H) & \text{if } \theta \equiv 0 \bmod 2\pi \\ \xi_\lambda(\gamma_\alpha)\,\xi_\lambda(\exp H) & \text{if } \theta \equiv \pi \bmod 2\pi . \end{cases}$$

If $\theta \equiv 0 \bmod 2\pi$, we have

$$\xi_1(f) = \xi_\lambda(\exp H) = \xi_2(f).$$

If $\theta \equiv \pi \bmod 2\pi$, we have

$$\xi_1(f) = \left\{ \begin{array}{ll} \xi_\lambda(\exp H) & \text{if } \xi_\lambda(\gamma_\alpha) = +1 \\ -\xi_\lambda(\exp H) & \text{if } \xi_\lambda(\gamma_\alpha) = -1 \end{array} \right\} = \xi_\lambda(\gamma_\alpha)\,\xi_\lambda(\exp H) = \xi_2(f).$$

This proves the required consistency.

4) We note that ξ_Λ has the required properties. In fact, (a) and (c) are from the definitions of ξ_1 and ξ_2. For (b) we simply calculate

$$\xi_\Lambda(\gamma_\alpha) = \exp \pi i \Lambda(H'_\alpha)$$

from the definition of Λ.

<u>Lemma 2.</u> On the compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, define

$$\Lambda_1 = \lambda + \tfrac{1}{2} \sum_{\xi_\lambda(\gamma_{\alpha_j}) = -1} \alpha_j .$$

Then Λ_1 is integral on G, and the associated character satisfies

$$\xi_{\Lambda_1}(\gamma_{\alpha_j}) = \xi_\lambda(\gamma_{\alpha_j}) \quad \text{for all } j$$

$$\xi_{\Lambda_1}(b) = \xi_\lambda(b) \quad \text{for } b \in B.$$

Proof. This is immediate by induction from Lemma 1.

Lemma 3. In any ordering, each $\alpha = \alpha_j$ is such that

$$\rho_\alpha' - \frac{2\langle \rho_c, \alpha\rangle}{|\alpha|^2}\alpha + \tfrac{1}{2}\alpha$$

is a multiple of α, and its associated character is 1 on each $\gamma_{\alpha,k}$.
Here ρ_α' is half the sum of the roots whose inner product with α is positive.

Proof. Let $c\alpha \pm \varepsilon$, with $c > 0$ and $\varepsilon \neq 0$, be roots contributing to ρ_α'.

If $c = 1$, their half-sum is α, which is a multiple of α and can be ignored. Also they are both compact or both noncompact, and their contribution to $2\rho_c$ is 2α or 0 or -2α; hence their contribution to $-\frac{2\langle \rho_c, \alpha\rangle}{|\alpha|^2}\alpha$ is $-\alpha$ or 0 or α, thus is a multiple of α and can be ignored.

Suppose $c = \frac{1}{2}$. Then the half sum of $\frac{1}{2}\alpha + \varepsilon$ and $\frac{1}{2}\alpha - \varepsilon$ is $\frac{1}{2}\alpha$. Also just one of them is compact. There are now two cases:

1) The positive roots from among $\pm\frac{1}{2}\alpha \pm \varepsilon$ are $\varepsilon \pm \frac{1}{2}\alpha$. Then the contribution to $-\frac{2\langle \rho_c, \alpha\rangle}{|\alpha|^2}\alpha$ is $\mp\frac{1}{2}\alpha$, and the total contribution from $\pm\frac{1}{2}\alpha \pm \varepsilon$ is $\frac{1}{2}\alpha \mp \frac{1}{2}\alpha$, which is a multiple of α and can be ignored

2) The positive roots from among $\pm\frac{1}{2}\alpha \pm \varepsilon$ are $\frac{1}{2}\alpha \pm \varepsilon$. Then the contribution to $-\frac{2\langle \rho_c, \alpha\rangle}{|\alpha|^2}\alpha$ is $\frac{1}{2}\alpha$, and the total contribution from $\pm\frac{1}{2}\alpha \pm \varepsilon$ is $\frac{1}{2}\alpha + \frac{1}{2}\alpha$, which is a multiple of α and can be ignored.

The remaining contribution to ρ_α' is from α itself, which contributes $\frac{1}{2}\alpha$. Then α does not contribute to $-\frac{2\langle\rho_c,\alpha\rangle}{|\alpha|^2}\alpha$, but the extra $\frac{1}{2}\alpha$ term adds to give us a multiple of α that can be ignored.

<u>Lemma 4.</u> On the compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, define

$$\Lambda_2 = \lambda - P(2\rho_c) + \frac{1}{2} \sum_{\xi_\lambda(\gamma_{\alpha_j}) = +(-1)^{2\langle\rho'_{\alpha_j},\alpha_j\rangle/|\alpha_j|^2}} \alpha_j \,,$$

where ρ'_{α_j} is half the sum of the roots whose inner product with α_j is positive. Then Λ_2 is integral on G, and the associated character satisfies

$$\xi_{\Lambda_2}(\gamma_{\alpha_j}) = \xi_\lambda(\gamma_{\alpha_j}) \quad \text{for all } j$$

$$\xi_{\Lambda_2}(b) = \xi_\lambda(b) \quad \text{for } b \in B.$$

Proof. With Λ_1 as in Lemma 2, we have

$$\Lambda_1 - \Lambda_2 = \frac{1}{2}\sum_{\xi_\lambda(\gamma_{\alpha_j})=-1} \alpha_j + P(2\rho_c) - \frac{1}{2}\sum_{\xi_\lambda(\gamma_{\alpha_j}) = +(-1)^{2\langle\rho'_{\alpha_j},\alpha_j\rangle/|\alpha_j|^2}} \alpha_j$$

Adding the expression in Lemma 3 for each α_j we obtain

$$\Lambda_1 - \Lambda_2 + \sum_{\text{all } k} n_k\alpha_k = \sum_{\text{all } k} \rho'_{\alpha_k} + \frac{1}{2}\sum_{\text{all } k}\alpha_k + \frac{1}{2}\sum_{\xi_\lambda(\gamma_{\alpha_j})=-1}\alpha_j - \frac{1}{2}\sum_{\xi_\lambda(\gamma_{\alpha_j}) = +(-1)^{2\langle\rho'_{\alpha_j},\alpha_j\rangle/|\alpha_j|^2}} \alpha_j$$

To complete the proof, it is enough to show that the right side is $\sum m_i\alpha_i$. It is enough to consider the α_i terms for fixed i. To see the coefficient m_i is an integer, it is enough to see $\exp 2\pi i \langle \alpha_i \text{ term}, \alpha_i\rangle/|\alpha_i|^2 = 1$.

We have

$$\exp\left(\frac{2\pi i\langle \alpha_i \text{ terms}, \alpha_i\rangle}{|\alpha_i|^2}\right) = (-1)^{2\langle 2\rho_{\alpha_i}', \alpha_i\rangle/|\alpha_i|^2} \times (-1)$$

$$\times\, \xi_\lambda(\gamma_{\alpha_i}) \times \left[(-1)\, \xi_\lambda(\gamma_{\alpha_i})(-1)^{2\langle \rho_{\alpha_i}', \alpha_i\rangle/|\alpha_i|^2}\right]$$

$$= +1,$$

and the lemma follows.

<u>Lemma 5.</u> Apart from indexing and signs, the following expressions $\delta = \sum c_i\alpha_i$ are the only possibilities for real roots. Each such possibility has

$$c_i = \frac{\langle \delta, \alpha_i\rangle}{|\alpha_i|^2} \quad\text{and}\quad \sum \frac{4\langle \delta, \alpha_i\rangle^2}{|\delta|^2|\alpha_i|^2} = 4.$$

(1) $\delta = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_4$ with δ and $\alpha_1, \ldots, \alpha_4$ all of the same length

(2) δ long and α_1 short, in which case

$$\delta = \alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}\alpha_3$$

with α_2 and α_3 long.

(3) δ short and α_1 long, in which case either

a) $\delta = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3$ with α_2 and α_3 short, or ← Cannot exist, forces $\alpha_2+\alpha_3$ to be a root

b) $\delta = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$ with α_2 long

(4) $\delta = \alpha_1$

Proof. The first two formulas follow from Parseval's equality. If δ and all α's have the same length, either each $\frac{4\langle\delta,\alpha_i\rangle^2}{|\delta|^2|\alpha_i|^2}$ is 1, and (1) results, or we have (4). If δ is long relative to some α, either there is one such α and we have (2), or there are two such α's, and we have $\delta = \alpha_1 + \alpha_2$, in contradiction to strong orthogonality. If δ is short relative to some α, we are led to (3).

Lemma 6. Apart from indexing and signs, the following expressions $\beta = \varepsilon + \sum c_i\alpha_i$ (with $\varepsilon \perp \alpha_i$ for all i), are the only possibilities for nonreal roots with $P(\beta) \neq 0$. Each such possibility has

$$c_i = \frac{\langle\beta,\alpha_i\rangle}{|\alpha_i|^2} \quad \text{and} \quad \sum \frac{4\langle\beta,\alpha_i\rangle^2}{|\beta|^2|\alpha_i|^2} = m < 4.$$

(1) $m=1$. $\beta = \varepsilon + \frac{1}{2}\alpha_1$, $|\beta| = |\alpha_1|$

(2) $m=2$

a) β and all α's of same length. $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$

b) β long relative to α_1. $\beta = \varepsilon + \alpha_1$

c) β short relative to α_1. $\beta = \varepsilon + \frac{1}{2}\alpha_1$

(3) $m=3$

a) β and all α's of same length. $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3$

b) β long relative to α_1. $\beta = \varepsilon + \alpha_1 + \frac{1}{2}\alpha_2$ with α_2 long

c) β short relative to α_1. $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$ with α_2 short

Proof. This is proved in the same way as Lemma 5.

Lemma 7. In Δ^+, each $\alpha = \alpha_j$ is such that

$$\rho_\alpha' - \rho_\alpha + \frac{2\langle \rho_{red,c}, \alpha\rangle}{|\alpha|^2}\alpha$$

is a multiple of α, and its associated character is 1 on each γ_{α_k}.
Here ρ_α' is half the sum of the roots whose inner product with α is positive, and ρ_α is half the sum of the roots whose inner product with $\alpha = \alpha_j$ is positive and whose inner product with all other α_k is 0.

Proof. Referring to Lemmas 5 and 6 and taking into account the various sign changes (including those of ε in Lemma 6), we see that the parity of

$$\frac{2\langle \rho_\alpha' - \rho_\alpha, \alpha\rangle}{|\alpha|^2} \qquad (*)$$

is affected only by roots $\frac{1}{2}\alpha \pm \frac{1}{2}\alpha_i$ and that $(*)$ counts 1 for each such pair for each i.

In Lemma 5, the contribution to

$$\frac{2\langle 2\rho_{red,c}, \alpha\rangle}{|\alpha|^2} \qquad (**)$$

from roots of type (1) and (3a) is even, and type (4) contributes nothing. With type (2) if $\alpha = \alpha_1$, any root of type (2) makes an even contribution to $(**)$. For a compact root δ of type (2) with $\alpha = \alpha_2$, suppose WLOG

$$\delta = \alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}\alpha_3$$

is compact. Then so are $\pm(-\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3)$, and two of these four roots are positive. The positive ones make an even contribution to (**). Hence the contribution to (**) from all roots other than type (3b) is even.

On the other hand, (**) counts ± 1 for each system $\pm\frac{1}{2}\alpha \pm \frac{1}{2}\alpha_i$ of roots of type (3b). We conclude therefore that

$$\frac{2\langle \rho_\alpha' - \rho_\alpha + 2\rho_{\mathrm{red},c}\,, \alpha\rangle}{|\alpha|^2} \quad \text{is even,}$$

and Lemma 7 follows.

<u>Lemma 8.</u> On the compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, define

$$\Lambda_3 = \lambda - P(2\rho_c) + 2\rho_{\mathrm{red},c} + \tfrac{1}{2}\textstyle\sum s_j\alpha_j$$
$$\xi_\lambda(\gamma_{\alpha_j}) = +(-1)^{2\langle \rho_{\alpha_j},\alpha_j\rangle/|\alpha_j|^2},$$

with each $s_j = \pm 1$ and with ρ_{α_j} equal to half the sum of the roots whose inner product with α_j is positive and whose inner product with all other α_k is 0. Then Λ_3 is integral on G, and the associated character satisfies

$$\xi_{\Lambda_3}(\gamma_{\alpha_j}) = \xi_\lambda(\gamma_{\alpha_j}) \quad \text{for all } j$$

$$\xi_{\Lambda_3}(b) = \xi_\lambda(b) \qquad \text{for } b \in B.$$

Proof. WLOG all $s_j = +1$. With Λ_2 as in Lemma 4, we form $\Lambda_2 - \Lambda_3$ and consider the α_i term for each i. To see the coefficient of α_i is an integer, it is enough to see $\exp 2\pi i \langle \alpha_i \text{ term}, \alpha_i\rangle / |\alpha_i|^2 = 1$.

We have

$$\exp\left(\frac{2\pi i \langle \alpha_i \text{ terms}, \alpha_i \rangle}{|\alpha_i|^2}\right) = \left[(-1)\,\xi_\lambda(\gamma_{\alpha_i})(-1)^{2\langle \rho'_{\alpha_i}, \alpha_i\rangle/|\alpha_i|^2}\right]^{-1} \times \left[(-1)\,\xi_\lambda(\gamma_{\alpha_i})(-1)^{2\langle \rho_{\alpha_j}, \alpha_i\rangle/|\alpha_i|^2}\right]$$

$$\times\, (-1)^{2\langle 2\rho_{\text{real},c}, \alpha_i\rangle/|\alpha_i|^2}$$

$$= +1,$$

by Lemma 7, and the lemma follows.

<u>Lemma 9.</u> In the positive system Δ^+, $P(2\rho) = 2\rho_{\text{real}}$.

Proof. Suppose β is a positive root with $P\beta \neq \beta$. If $P\beta = 0$, we may ignore β. If $P\beta \neq 0$, we pair β with $s_{\alpha_1}\cdots s_{\alpha_m}\beta$, which is positive since the α's come last in the ordering. Then

$$P(\beta + s_{\alpha_1}\cdots s_{\alpha_m}\beta) = 0.$$

Thus the only contribution to $P(2\rho)$ is from positive β with $P\beta = \beta$, and the lemma follows.

<u>Lemma 10.</u> In the system Δ^+, $2(\rho_c - \rho_c^-) = \rho - \rho^- - \rho_{\text{real}} + P(2\rho_c)$

Proof. If we apply P to both sides, we get $P(2\rho_c)$ from the left side and $P(\rho) - \rho_{\text{real}} + P(2\rho_c)$ from the right side, and these are equal by Lemma 9.

Thus it is enough to show equality of the two sides modulo span$\{\alpha_j\}$. Any root strongly orthogonal to all α_j contributes 0 to both sides.

Next we observe that if γ is orthogonal to some α_j but not strongly orthogonal to α_j, then there can be only one such j (depending on γ). In fact, if j and i are two such indices, then

$$\gamma \pm \alpha_j \quad \text{and} \quad \gamma \pm \alpha_i$$

are roots and $\langle \gamma \pm \alpha_j, \gamma \pm \alpha_i \rangle > 0$, whence $\pm\alpha_i \pm \alpha_j$ are roots, in contradiction to the strong orthogonality of the α's.

We now consider the contribution of all roots $\gamma > 0$ such that

$$\frac{2\langle \gamma, \alpha_j \rangle}{|\alpha_j|^2} = \pm 1. \qquad (*)$$

We pair γ with $s_{\alpha_j}\gamma$, where j is the least index satisfying (*).

change notation to greatest

We have to see that $s_{\alpha_j}\gamma > 0$. If $s_{\alpha_j}\gamma' < 0$, then the minimality of j and the definition of the ordering implies

(with no $\frac{1}{2}$'s)

$$\gamma = \tfrac{1}{2}\alpha_j \quad \text{or} \quad \tfrac{1}{2}\alpha_j \pm \alpha_k \quad \text{or} \quad \tfrac{1}{2}\alpha_j \pm \alpha_k \pm \cdots, \ k < j.$$

Referring to Lemma 5, we see that none of these is a root.

Thus $s_{\alpha_j}\gamma > 0$. Since $\gamma - s_{\alpha_j}\gamma = \pm\alpha_j$, exactly one of γ and $s_{\alpha_j}\gamma$ is compact. Thus $\gamma + s_{\alpha_j}\gamma$ contributes

$$\tfrac{1}{2}(\gamma + s_{\alpha_j}\gamma) \quad \text{to} \quad (I-P)(2\rho_c)$$

and

$$\tfrac{1}{2}(\gamma + s_{\alpha_j}\gamma) \quad \text{to} \quad (I-P)(\rho).$$

and 0 to $2\rho_c^-$ and ρ^-. Hence the contributions of $\gamma + s_{\alpha_j}\gamma$ to $I-P$ of the two sides of the identity are the same.

In view of the uniqueness proved at the top of the previous page, we are left for each j with roots $\varepsilon \pm \alpha_j$ and with roots ε that are orthogonal but not strongly orthogonal to this (unique) j. [Note if $\varepsilon \pm \alpha_j \pm \alpha_k$ + other is a root, we get roots of too many lengths after subtraction of $\pm\alpha_j$ and then $\pm\alpha_k$.] We shall examine the contribution of $\varepsilon, \varepsilon+\alpha, \varepsilon-\alpha$ to $(I-P)$ of the two sides of our identity.

The first case is that $\varepsilon \pm \alpha$ are compact and ε is G-noncompact, hence M-compact. Then the contributions are

$\varepsilon+\alpha$ and $\varepsilon-\alpha$ together contribute 2ε to $(I-P)(2\rho_c)$

ε contributes $-\varepsilon$ to $-(I-P)(2\rho_c^-)$

Left side: ε

$\varepsilon+\alpha$, $\varepsilon-\alpha$, ε together contribute $\frac{3}{2}\varepsilon$ to $(I-P)(\rho)$

ε contributes $-\frac{1}{2}\varepsilon$ to $-(I-P)(\rho^-)$

Right side ε ✓

The second case is that $\varepsilon \pm \alpha$ are noncompact and ε is G-compact, hence M-noncompact. Then the contributions are

ε contributes ε to $(I-P)(2\rho_c)$

nothing contributes 0 to $-(I-P)(2\rho_c^-)$

$\overline{}$

Left side: ε

$\varepsilon+\alpha,\ \varepsilon-\alpha,\ \varepsilon$ together contribute $\frac{3}{2}\varepsilon$ to $(I-P)(\rho)$

ε contributes $-\frac{1}{2}\varepsilon$ to $-(I-P)(\rho^-)$

$\overline{}$

Right side ε ✓

The lemma follows.

<u>Lemma 11.</u> $\langle \rho_c - \rho_c^-, \gamma \rangle \geq 0$ for every M-root $\gamma > 0$.

Proof. We may assume γ is simple for M (this does not make it simple for G). By Lemma 10,

$$\frac{4\langle \rho_c - \rho_c^-, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \rho - \rho^-, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \rho, \gamma\rangle}{|\gamma|^2} - 1 \geq 0.$$

Lemma 12. If β is a Δ_K^+-simple root, then

$$\frac{2\langle 2\rho_{\mathrm{real},c} - \rho_{\mathrm{real}} + \rho, \beta\rangle}{|\beta|^2} \qquad [1]$$

is bounded below by the following values, according to the nature of β as in Lemma 6.

Nature of β	Lower bound for [1]
(0) a) β orthogonal to all α's	1
b) β real	2
(1) $m=1$	
$\beta = \varepsilon \pm \frac{1}{2}\alpha_1$, $\lvert\beta\rvert = \lvert\alpha_1\rvert$	3/2
(2) $m=2$	
a) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$, $\lvert\beta\rvert = \lvert\alpha_1\rvert = \lvert\alpha_2\rvert$	
(i) $\pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ not roots	2
(ii) $\pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ roots	3/2
b) $\beta = \varepsilon + \alpha_1$, $\lvert\beta\rvert > \lvert\alpha_1\rvert$	3/2
c) $\beta = \varepsilon + \frac{1}{2}\alpha_1$, $\lvert\beta\rvert < \lvert\alpha_1\rvert$	2
(3) $m=3$	
a) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2 \pm \frac{1}{2}\alpha_3$, $\lvert\beta\rvert = \lvert\alpha_1\rvert = \lvert\alpha_2\rvert = \lvert\alpha_3\rvert$	
(i) No $\pm\frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$ are roots	5/2
(ii) Some $\pm\frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$ are roots	2
b) $\beta = \varepsilon + \alpha_1 + \frac{1}{2}\alpha_2$, $\lvert\alpha_1\rvert < \lvert\beta\rvert = \lvert\alpha_2\rvert$	2
c) $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$, $\lvert\alpha_1\rvert > \lvert\beta\rvert = \lvert\alpha_2\rvert$	5/2.

Proof. We begin by doing most of the evaluation of

$$2\rho_{real,c} - \rho_{real}. \qquad [2]$$

We may assume G is simple.

We examine each type of root in Lemma 5. Type (4) obviously contributes $-\frac{1}{2}\sum \alpha_i$ to [2].

For type (1), let $\alpha_1, \ldots, \alpha_4$ be in order. The total contribution of

$$\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \tfrac{1}{2}\alpha_3 \pm \tfrac{1}{2}\alpha_4$$

to $2\rho_{real,c}$ is $2\alpha_1$, since any two sign changes preserve compactness and we can add in pairs. Also these roots contribute $2\alpha_1$ to ρ_{real}. Hence the total contribution of roots of type (1) to [2] is 0.

For type (3a), we consider roots of the form

$$\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \tfrac{1}{2}\alpha_3,$$

and the same argument shows the total contribution to [2] is 0.

For type (2), we distinguish the ordered cases

$$\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \tfrac{1}{2}\alpha_3 \qquad \text{and} \qquad \tfrac{1}{2}\alpha_1 \pm \alpha_2 \pm \tfrac{1}{2}\alpha_3$$

In the first case we get contributions of $2\alpha_1$ to both $2\rho_{real,c}$ and ρ_{real}

by pairing δ and $s_{\alpha_2} s_{\alpha_3} \delta$. Hence we get a contribution of 0 for [2]. In the second case, we get a contribution of $\alpha_1 \pm \alpha_3$ to $2\rho_{red,c}$ and α_1 to ρ_{red}, hence $\pm\alpha_3$ to [2].

Each case of type (2) leads also to certain roots of type (3b). The first case leads also to $\pm\frac{1}{2}\alpha_2 \pm \frac{1}{2}\alpha_3$ and the contribution of these roots to [2] is $\pm\frac{1}{2}\alpha_3$. The second case leads also to $\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_3$. Here $\frac{1}{2}\alpha_1 \mp \frac{1}{2}\alpha_3$ is compact, in the notation of the previous paragraph, and the contribution of these roots to [2] is $\mp\frac{1}{2}\alpha_3$. Hence in either case the total contribution from type (2) and type (3b), when they are related this way, is $\pm\frac{1}{2}\alpha_3$, where α_3 is the second long root α.

When we consider all such connections between (2) and (3b), we get no overlap, because existence of roots

$$\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3 \quad \text{and} \quad \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_4$$

implies existence of a root $\alpha_3 - \alpha_4$, in contradiction to strong orthogonality.

Thus the only other contributions to [2] come from roots $\pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ of type (3b) that are not paired with roots of type (2), and these contribute $\pm\frac{1}{2}\alpha_2$ to [2].

Our conclusion is that

$$2\rho_{real,c} - \rho_{real} = -\frac{1}{2}\sum_j \alpha_j + \frac{1}{2}\sum_{\substack{i<j \\ \frac{1}{2}(\alpha_i+\alpha_j)\in\Delta}} \varepsilon_{ij}\alpha_j \quad , \quad \varepsilon_{ij} = \pm 1. \qquad [3]$$

Moreover, when $\frac{1}{2}(\alpha_i+\alpha_j)$ is in Δ and no $\frac{1}{2}(\alpha_i+\alpha_j)+\alpha_k$ is in Δ, then

$$\frac{1}{2}(\alpha_i + \varepsilon_{ij}\alpha_j) \text{ is compact.} \qquad [4]$$

If we put

$$s_j = \sum_{\substack{i<j \\ \frac{1}{2}(\alpha_i+\alpha_j)\in\Delta}} \frac{1}{2}\varepsilon_{ij} \, , \qquad [5]$$

then we can rewrite [3] as

$$2\rho_{real,c} - \rho_{real} = \sum_j \left(-\frac{1}{2} + s_j\right)\alpha_j \, . \qquad [6]$$

We shall now draw some qualitative conclusions from these equations.

1) If all roots have the same length, then all $s_j = 0$.

2) If there are roots of two lengths, then $s_j = 0$ for each short α_j.

3) If all short roots are orthogonal (B_n) and if α_i and α_j are long roots with $i<j$ such that $\frac{1}{2}(\alpha_i+\alpha_j)$ is a root, then $s_i = 0$ and $s_j = \pm\frac{1}{2}$.

4) If all long roots are orthogonal (C_n) and if α_i and α_j are long roots with $i<j$, then $\frac{1}{2}(\alpha_i+\alpha_j)$ is a root and $\frac{1}{2}(\alpha_i+\varepsilon_{ij}\alpha_j)$ is compact.

5) If α_i and α_j are the first and second long α's, respectively, then $s_i = 0$ and $|s_j| \le \frac{1}{2}$.

Now let β be a given Δ_K^+ simple root. In case (0a), [1] is equal to $2\langle\rho,\beta\rangle/|\beta|^2$, which is ≥ 1. In case (0b), [1] is equal to $2\langle 2\rho_{n,\bar{c}},\beta\rangle/|\beta|^2$, which equals 2.

Before handling cases (1), (2), and (3) directly, we shall dispose of the situation where Δ is C_n and $\langle\beta,\alpha_j\rangle \ne 0$ for some long α_j. Since all long roots are orthogonal in C_n, β is short and Lemma 6 says it is of the form

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_j \qquad [7]$$

or

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_j \pm \tfrac{1}{2}\alpha_k \quad \text{with } \alpha_k \text{ short.} \qquad [8]$$

First we deal with [7]. Let us renumber the α's, using $\alpha_1, \ldots$ to denote the long α's in order. We refer to qualitative conclusion (4) above. Suppose there is an $i<j$ such that ε_{ij} agrees with the ambiguous sign ε_j in [7]. Then

$$\beta - \tfrac{1}{2}(\alpha_i + \varepsilon_j\alpha_j) = \varepsilon - \tfrac{1}{2}\alpha_i$$

is a compact root, positive because of the presence of ε, and the equality

$$\beta = (\varepsilon - \tfrac{1}{2}\alpha_i) + \tfrac{1}{2}(\alpha_i + \varepsilon_j \alpha_j)$$

exhibits β as not simple for Δ_K^+, contradiction. We conclude that $\varepsilon_{ij} = -\varepsilon_j$ for $1 \le i < j$. Then [5] gives $s_j = -\tfrac{1}{2}(j-1)\varepsilon_j$, and we have

$$\frac{2\langle 2\rho_{n,c} - \rho_n, \beta\rangle}{|\beta|^2} = \frac{2\left(-\tfrac{1}{2} - \tfrac{1}{2}(j-1)\varepsilon_j\right)\varepsilon_j \tfrac{1}{2}|\alpha_j|^2}{|\beta|^2}$$

$$= 2\left(-\tfrac{1}{2} - \tfrac{1}{2}(j-1)\varepsilon_j\right)\varepsilon_j$$

$$= -(j-1) - \varepsilon_j . \qquad [9]$$

At the same time

$$\beta = (\varepsilon - \tfrac{1}{2}\alpha_1) + \tfrac{1}{2}(\alpha_1 - \alpha_2) + \cdots + \tfrac{1}{2}(\alpha_{j-1} - \alpha_j) + \begin{cases} \alpha_j & \text{if } \varepsilon_j = +1 \\ 0 & \text{if } \varepsilon_j = -1 \end{cases}$$

implies

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \tfrac{1}{2}\alpha_1\rangle}{|\beta|^2} + \cdots + \frac{2\langle \rho, \tfrac{1}{2}(\alpha_{j-1} - \alpha_j)\rangle}{|\beta|^2} + \begin{cases} 2\langle \rho, \alpha_j\rangle/|\beta|^2 \\ 0 \end{cases}$$

$$\ge 1 + \cdots + 1 + \begin{cases} 2 \\ 0 \end{cases}$$

$$= \begin{cases} j+2 \\ j . \end{cases} \qquad [10]$$

Adding [9] and [10], we see that [1] is ≥ 2, as asserted for case (2c) of the lemma.

Next we deal with [8]. Let us write α_0 for α_k and renumber the long α's as above. As above we conclude that $\varepsilon_{ij} = -\varepsilon_j$ for $1 \le i < j$.

Hence [5] gives $s_j = -\frac{1}{2}(j-1)\varepsilon_j$. Then [9] is replaced by

$$\frac{2\langle 2\rho_{\text{nel},c} - \rho_{\text{nel}}, \beta\rangle}{|\beta|^2} = -(j-1) - \varepsilon_j - \tfrac{1}{2}\varepsilon_0, \qquad [9']$$

where ε_0 is the sign in front of α_β in [8]. If $\varepsilon_0 = +1$, then

$$\varepsilon - \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_0 = (\varepsilon - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_0) + \alpha_0$$

improves the estimate for the analog of the first term of [10] to

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 + \alpha_0\rangle}{|\beta|^2} \geq 2,$$

and thus [10] is replaced by

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \tfrac{1}{2}(\varepsilon_0 + 1) + \begin{cases} j+2 & \text{if } \varepsilon_j = +1 \\ j & \text{if } \varepsilon_j = -1. \end{cases} \qquad [10']$$

Adding [9′] and [10′], we see that [1] is $\geq 5/2$, as asserted for case (3c) of the lemma.

We may henceforth exclude from our considerations C_n when β is nonorthogonal to some long α. Temporarily we shall exclude also F_4 when $\{\alpha\}$ consists of three long roots. Thus we shall proceed for now under the assumption

$$|s_j| \leq \tfrac{1}{2} \quad \text{for all } j. \qquad [11]$$

We take the various cases in turn.

In case (1), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$ with $|\beta| = |\alpha_1|$. Then

$$\frac{2\langle 2\rho_{real,c} - \rho_{real}, \beta\rangle}{|\beta|^2} \qquad [12]$$

equals $\pm(-\frac{1}{2}+s_1)$. First suppose β is simple for Δ^+. Then s_1 has to be 0, since otherwise there is a root $\frac{1}{2}(\alpha_0 \pm \alpha_1)$ such that

$$\beta = (\varepsilon - \tfrac{1}{2}\alpha_0) + \tfrac{1}{2}(\alpha_0 \pm \alpha_1)$$

exhibits β as not simple. Moreover, the sign must be minus, and hence [12] is $1/2$. Also

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} \qquad [13]$$

is 1, and hence [1] is $\geq 3/2$ if β is simple for Δ^+. In the nonsimple case if $s_1 = 0$, then [12] is $\geq -1/2$ and [13] is ≥ 2, and so [1] is $\geq 3/2$. If $s_1 \neq 0$ and the sign is minus, then [12] is ≥ 0 and [13] is ≥ 2, and [1] is $\geq 3/2$. If $s_1 \neq 0$ and the sign is plus, then [12] is ≥ -1 and [13] is ≥ 3 because

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_0\rangle}{|\beta|^2} + \frac{2\langle \rho, \frac{1}{2}(\alpha_0 - \alpha_1)\rangle}{|\beta|^2} + \frac{2\langle \rho, \alpha_1\rangle}{|\beta|^2}$$

$$\geq 1 + \tfrac{1}{2} + 1 \;;$$

hence [1] is $\geq 3/2$.

In case (2a), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ with $|\beta| = |\alpha_1| = |\alpha_2|$ and α_1 preceding α_2. Then [12] equals

$$\pm\left(-\tfrac{1}{2}+s_1\right) \pm \left(-\tfrac{1}{2}+s_2\right). \qquad [14]$$

If β is simple for Δ^+, then both signs are minus and $s_1 = 0$, just as in the previous case, and [14] is $\geq 1/2$ in general, $= 1$ if $s_2 = 0$; since [13] is 1, [1] is as asserted in the two parts of the statement of case (2a) in the lemma.

If β is not simple for Δ^+ but $s_1 = 0$, we distinguish three situations. If $s_2 = 0$, then [14] is

$$= 1 - \#\{\text{plus signs}\},$$

and [13] is

$$\geq 1 + \#\{\text{plus signs}\}, \qquad [15]$$

so that [1] is as asserted. If $s_2 \neq 0$ and $\frac{1}{2}(\alpha_1 + \alpha_2)$ is not a root, then some $\frac{1}{2}(\alpha_0 + \alpha_2)$ is a root and we easily see that

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 \rangle}{|\beta|^2} \geq 2.$$

Then it follows that [13] is

$$\geq 2 + \#\{\text{plus signs}\} \qquad [16]$$

and [14] is

$$\geq \tfrac{1}{2} - \#\{\text{plus signs}\}, \qquad [17]$$

In case (2c), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$ with $|\beta| < |\alpha_1|$. Then [12] is

$$= \pm 2\left(-\tfrac{1}{2} + s_1\right). \qquad [18]$$

If β is simple for Δ^+, the sign must be minus and s_1 must be 0, as in earlier cases; thus [12] equals 1. Since [13] is 1, [1] equals 2. If β is not simple for Δ^+ and the sign is minus, then [18] is ≥ 0 and [13] is ≥ 2; thus [1] is ≥ 2.

Now suppose the sign in β is plus: $\beta = \varepsilon + \frac{1}{2}\alpha_1$. If $s_1 = 0$, then [18] is ≥ -1 and [13] is ≥ 3 because

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} + \frac{2\langle \rho, \alpha_1\rangle}{|\alpha_1|^2/2} \geq 1 + 2 = 3;$$

thus [1] is ≥ 2. Finally if $s_1 \neq 0$, then [18] is ≥ -2 and [13] is ≥ 4 because

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} \geq 2$$

and

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} + \frac{2\langle \rho, \alpha_1\rangle}{|\alpha_1|^2/2} \geq 2 + 2 = 4;$$

thus [1] is ≥ 2.

In case (2c), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$ with $|\beta| < |\alpha_1|$. Then [12] is

$$= \pm 2(-\tfrac{1}{2} + s_1). \qquad [18]$$

If β is simple for Δ^+, the sign must be minus and s_1 must be 0, as in earlier cases; thus [12] equals 1. Since [13] is 1, [1] equals 2. If β is not simple for Δ^+ and the sign is minus, then [18] is ≥ 0 and [13] is ≥ 2; thus [1] is ≥ 2.

Now suppose the sign in β is plus: $\beta = \varepsilon + \frac{1}{2}\alpha_1$. If $s_1 = 0$, then [18] is ≥ -1 and [13] is ≥ 3 because

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} + \frac{2\langle \rho, \alpha_1\rangle}{|\alpha_1|^2/2} \geq 1 + 2 = 3;$$

thus [1] is ≥ 2. Finally if $s_1 \neq 0$, then [18] is ≥ -2 and [13] is ≥ 4 because

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} \geq 2$$

and

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} + \frac{2\langle \rho, \alpha_1\rangle}{|\alpha_1|^2/2} \geq 2 + 2 = 4;$$

thus [1] is ≥ 2.

In case (3a), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2 \pm \frac{1}{2}\alpha_3$ with $|\beta| = |\alpha_1| = |\alpha_2| = |\alpha_3|$, the α's not ordered. Then [12] equals

$$\pm\left(-\tfrac{1}{2}+s_1\right) \pm \left(-\tfrac{1}{2}+s_2\right) \pm \left(-\tfrac{1}{2}+s_3\right). \qquad [19]$$

If all $s_j = 0$, then [19] is

$$\geq \tfrac{3}{2} - \#\{\text{plus signs}\}$$

and [13] is

$$\geq 1 + \#\{\text{plus signs}\}, \qquad [20]$$

so that [1] is $\geq 5/2$.

If some $s_j \neq 0$, we may assume $s_1 \neq 0$, and then $\frac{1}{2}(\alpha_0 + \alpha_1)$ is a root. If $\alpha_0 \neq \alpha_1$ or α_2, then

$$\frac{2\langle \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3,\ \frac{1}{2}(\alpha_0+\alpha_1)\rangle}{|\varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3|^2} = \frac{1}{2},$$

contradiction. So as soon as some s_j is $\neq 0$ we are in (ii) of case (3a). Similarly if more than one s_j is $\neq 0$, then there are nonorthogonal short roots (as well as nonorthogonal long roots) and we are in the excluded F_4 case. Thus we may assume $s_1 \neq 0$ and $s_2 = s_3 = 0$. Then [19] is

$$\geq 1 - \#\{\text{plus signs}\},$$

and [13] is $\geq$ [20], so that [1] is ≥ 2.

In case (3b), we write $\beta = \varepsilon \pm \alpha_1 \pm \frac{1}{2}\alpha_2$ with $|\beta| = |\alpha_2| > |\alpha_1|$. We have $s_1 = 0$ by qualitative conclusion (2). Thus [12] equals

$$\pm(-\tfrac{1}{2}) \pm (-\tfrac{1}{2} + s_2).$$

Let us show that $s_2 = 0$. In fact, otherwise some $\frac{1}{2}(\alpha_0 + \alpha_2)$ is in Δ. Since $\varepsilon + \frac{1}{2}\alpha_2$ is a nonorthogonal short root, nonorthogonal short roots exist. Also β and α_2 together show that nonorthogonal long roots exist. Thus Δ is F_4. Now α_1 and $\frac{1}{2}(\alpha_0+\alpha_2)$ are short orthogonal roots, and their sum in F_4 must be a root. But

$$\frac{2\langle \beta, \alpha_1 + \frac{1}{2}(\alpha_0+\alpha_2)\rangle}{|\beta|^2} = \frac{3}{2}$$

is not an integer, contradiction. Thus $s_2 = 0$ and [12] equals

$$\pm(-\tfrac{1}{2}) \pm (-\tfrac{1}{2}). \qquad [21]$$

If both signs are negative, [21] is 1 and [13] is ≥ 1, so that [1] is ≥ 2. If one sign is positive, [21] is 0 and [13] is ≥ 2, so that [1] is ≥ 2. Finally if $\beta = \varepsilon + \alpha_1 + \frac{1}{2}\alpha_2$, then [21] is -1 and [13] is ≥ 3 because

$$\beta = (\varepsilon - \alpha_1 - \tfrac{1}{2}\alpha_2) + 2\alpha_1 + \alpha_2$$

shows

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} \geq 1 + 1 + 1 = 3;$$

thus [1] is ≥ 2.

In case (3c), we write $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ with $|\beta| = |\alpha_2| < |\alpha_1|$.

Here $s_2 = 0$ and [12] equals

$$\pm 2(-\tfrac{1}{2}+s_1) \pm (-\tfrac{1}{2}). \qquad [22]$$

If $s_1 = 0$, [22] is

$$= \tfrac{3}{2} - (2 \text{ if first sign is } +) - (1 \text{ if second sign is } +),$$

and [13] is

$$\geq 1 + (2 \text{ if first sign is } +) + (1 \text{ if second sign is } +),$$

so that [1] is $\geq 5/2$. If $s_1 \neq 0$, [22] is

$$\geq \tfrac{1}{2} - (2 \text{ if first sign is } +) - (1 \text{ if second sign is } +),$$

while [13] is

$$\geq 2 + (2 \text{ if first sign is } +) + (1 \text{ if second sign is } +)$$

because

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 \rangle}{|\beta|^2} \geq 2$$

when $s_1 \neq 0$; thus [1] is $\geq 5/2$.

This completes the regular analysis of β. We turn to the one unsettled excluded case, that of F_4 with $\{\alpha\}$ consisting exactly of three long roots $\alpha_1, \alpha_2, \alpha_3$, which we take to be in order. Lemma 6 shows that either β is short and $\beta = \varepsilon \pm \frac{1}{2}\alpha_j$ or β is long and

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \tfrac{1}{2}\alpha_3 . \qquad [23]$$

However, $\beta = \varepsilon \pm \frac{1}{2}\alpha_j$ cannot be simple for Δ_K^+, because if α_i is the first missing α, then $\frac{1}{2}(\alpha_i \pm \alpha_k)$ will be compact for a suitable choice of the sign and then

$$\beta = (\varepsilon - \tfrac{1}{2}\alpha_i \pm \tfrac{1}{2}\alpha_j \mp \tfrac{1}{2}\alpha_k) + \tfrac{1}{2}(\alpha_i \pm \alpha_k)$$

will exhibit β as not simple for Δ_K^+.

Thus β is of the form [23], and we are in situation (ii) of case (3a). Since $s_1 = 0$ (by qualitative observation (5)), [12] equals

$$= \pm(-\tfrac{1}{2}) \pm (-\tfrac{1}{2} + s_2) \pm (-\tfrac{1}{2} + s_3).$$

By [4] and [5]

$$s_2 = \tfrac{1}{2}\varepsilon_{12} \quad , \text{ where } \tfrac{1}{2}(\alpha_1 + \varepsilon_{12}\alpha_2) \text{ is compact,}$$

and

$$s_3 = \tfrac{1}{2}(\varepsilon_{13} + \varepsilon_{23}), \text{ where } \tfrac{1}{2}(\alpha_1 + \varepsilon_{13}\alpha_3) \text{ and } \tfrac{1}{2}(\alpha_2 + \varepsilon_{23}\alpha_3) \text{ are compact.}$$

Since $\frac{1}{2}(\alpha_1 + \varepsilon_{12}\alpha_2)$ and $\frac{1}{2}\varepsilon_{12}(\alpha_2 + \varepsilon_{23}\alpha_3)$ are compact, so is their difference $\alpha_1 - \varepsilon_{12}\varepsilon_{23}\alpha_3$. Thus it follows that

$$\varepsilon_{13} = -\varepsilon_{12}\varepsilon_{23}$$

and

$$s_3 = \tfrac{1}{2}\varepsilon_{23}(1-\varepsilon_{12}) = \varepsilon_{23}\left(\tfrac{1}{2}-s_2\right).$$

Thus [12] equals

$$= \pm\left(-\tfrac{1}{2}\right) \pm \left(-\tfrac{1}{2}+s_2\right) \pm \left(-\tfrac{1}{2}+\varepsilon_{23}\left(\tfrac{1}{2}-s_2\right)\right)$$

$$= \begin{cases} \pm\left(-\tfrac{1}{2}\right) \pm 0 \pm \left(-\tfrac{1}{2}\right) & \text{if } s_2 = +1/2 \\ \pm\left(-\tfrac{1}{2}\right) \pm (-1) \pm \left(-\tfrac{1}{2}+\varepsilon_{23}\right) & \text{if } s_2 = -1/2 . \end{cases}$$

$$\geq \begin{cases} 1 - \#\{\text{plus signs in } [23]\} & \text{if sign for } \alpha_2 \text{ is minus} \\ -\#\{\text{plus signs in } [23]\} & \text{if sign for } \alpha_2 \text{ is plus.} \end{cases}$$

If the sign for α_2 is minus, then [13] is

$$\geq 1 + \#\{\text{plus signs in } [23]\},$$

and [1] is ≥ 2. If the sign for α_2 is plus, then [13] is

$$\geq 2 + \#\{\text{plus signs in } [23]\}$$

because

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3\rangle}{|\beta|^2} = \frac{2\langle 8\varepsilon + \frac{3}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3 \,,\, \varepsilon - \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3\rangle}{|\beta|^2} = 3.$$

Thus [1] is ≥ 2 in the situation, too. This completes the proof of Lemma 12.

Lemma 13. Fix μ_0 in $\sum \mathbb{R}\alpha_j$ to be of the form

$$\mu_0 = \tfrac{1}{2}\sum s_j\alpha_j \quad , \text{ where } s_j = 0 \text{ or } \pm 1.$$

If β is a Δ_K^+-simple root, then

$$\left| \frac{2\langle \mu_0, \beta\rangle}{|\beta|^2} \right| \qquad [1]$$

is bounded above by the following values, according to the nature of β as in Lemma 6.

Nature of β	Upper bound for [1]
(0) a) β orthogonal to all α's	0
b) β real	2
(1) $m=1$, $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$, $\lvert\beta\rvert = \lvert\alpha_1\rvert$	1/2
(2) $m=2$	
a) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$, $\lvert\beta\rvert = \lvert\alpha_1\rvert = \lvert\alpha_2\rvert$	1
b) $\beta = \varepsilon \pm \alpha_1$, $\lvert\beta\rvert > \lvert\alpha_1\rvert$	1/2
c) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$, $\lvert\beta\rvert < \lvert\alpha_1\rvert$	1
(3) $m=3$	
a) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2 \pm \frac{1}{2}\alpha_3$, $\lvert\beta\rvert = \lvert\alpha_1\rvert = \lvert\alpha_2\rvert = \lvert\alpha_3\rvert$	3/2
b) $\beta = \varepsilon \pm \alpha_1 \pm \frac{1}{2}\alpha_2$, $\lvert\alpha_1\rvert < \lvert\beta\rvert = \lvert\alpha_2\rvert$	1
c) $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$, $\lvert\alpha_1\rvert > \lvert\beta\rvert = \lvert\alpha_2\rvert$	3/2.

Proof. Case (0a) is clear. Without loss of generality in the other cases, we may suppose μ_0 involves only those α_j's that occur in β. Write

$$\beta = \varepsilon + \tfrac{1}{2}\textstyle\sum c_j \alpha_j$$

with $c_j = \pm 1, \pm 2$, and the sum extended only over nonzero contributions. Then

$$\left| \frac{2\langle \mu_0, \beta\rangle}{|\beta|^2} \right| = \left| \frac{\frac{1}{2}(\sum s_j \alpha_j, \sum c_j \alpha_j)}{|\beta|^2} \right| = \frac{\frac{1}{2}\left|\sum s_j c_j |\alpha_j|^2\right|}{|\beta|^2}$$

$$\leq \tfrac{1}{2} \sum |c_j| \left(\frac{|\alpha_j|^2}{|\beta|^2}\right) = \tfrac{1}{2} \sum \left| \frac{2\langle \beta, \alpha_j\rangle}{|\beta|^2} \right|,$$

and now we can read off the results in the various cases.

Recall that

Δ_r = roots of the form $\sum \mathbb{R}\alpha_j$

Δ_0 = roots orthogonal to $\lambda_0 \supseteq \Delta_r$

G^r, G^0 = connected semisimple subgroups of G generated by Δ_r, Δ_0

$K^r, K^0, A^r, A^0, M^r, M^0, T^r, T^0$ defined by intersection.

<u>Lemma 14.</u> G^r is split over $\mathbb{R}$ with Iwasawa A equal to A^r. Thus $M^r = F$ $= \operatorname{span}\{\gamma_\beta \mid \beta \text{ in } \Delta_r\}$.

Proof. T^r is a compact Cartan subgroup of G^r, and A^r has the same dimension. Thus the result follows.

<u>Lemma 15.</u> G^0 is quasisplit over $\mathbb{R}$. Every root of Δ_0 not in Δ_r is of the form

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_1 \qquad \text{with } |\beta| = |\alpha_1|$$

or the form

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \quad \text{with } |\beta| = |\alpha_1| = |\alpha_2|,$$

and in the latter case $\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2$ is not a root. Any such compact β

wrong satisfies

$$\langle \lambda, \beta \rangle = \langle \lambda_0, \beta \rangle .$$

Proof. In G^0 if γ is a root orthogonal to $\sum \mathbb{R}\alpha_j$, then $\langle \gamma, \lambda_0 \rangle = 0$ by definition of Δ_0, and we have a contradiction to the fact that λ_0 corresponds to discrete series of M. Thus G^0 is quasisplit.

Similarly, for the form of β, the same argument shows that ε or 2ε cannot be a root (in the notation of Lemma 6), and thus the two listed cases are the only possible ones.

Finally the definition of the ordering is such that s_β permutes the positive roots orthogonal to all α_j and the positive compact roots orthogonal to all α_j. Thus $\langle \rho^-, \beta\rangle = \langle \rho_c^-, \beta\rangle = 0$, and it follows that $\langle \lambda, \beta\rangle = \langle \lambda_0, \beta\rangle$.

wrong

Recall that

— $\underset{\sim}{c} = \underset{\sim}{c}_1 \cdots \underset{\sim}{c}_m$ denotes the Cayley transform relative to $\alpha_1, \dots, \alpha_m$;
let $\underset{\sim}{c}(\beta)\big|_{\Sigma \mathbb{R} H_{\alpha_j}} = \beta^\#$

— in G^Ω,

— in G^Ω, an irreducible representation τ_μ of K^Ω is <u>fine</u> if
each β in Δ_n is such that $K^{(\beta^\#)}$ acts under τ_μ only via the eigenvalues $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \to e^{i\theta}$, 1, and $e^{-i\theta}$.

— Vogan (Fine K-types and the principal series) has proved the following things about each character ω of F:

(i) there exists a fine irreducible representation τ_μ of K^Ω such that $\tau_\mu\big|_F$ contains ω

(ii) that any two such μ's are conjugate via an outer automorphism of G^Ω that is inner with respect to $G^{\Omega\mathbb{C}}$

(iii) any minimal K-type of $\mathrm{ind}_F^{K^\Omega} \omega$ satisfies (i).

Facts about fine K-types

1) If τ_μ is a fine K^Ω-type containing ω, then τ_μ is a minimal K-type in $\mathrm{ind}_F^K \omega$.

Proof: This is an immediate consequence of Vogan's results (ii) and (iii) on the previous page.

Lemma 16. Let τ_μ be a fine K-type in G_n. Then the following conditions are satisfied:

(a) $\left|\frac{2\langle\mu,\gamma\rangle}{|\gamma|^2}\right| \le 1$ for every real noncompact root γ that can be included in a strongly orthogonal basis for $\sum \mathbb{R}\alpha_j$ of noncompact roots.

(b) $\left|\frac{2\langle\mu,\gamma\rangle}{|\gamma|^2}\right| \le 1$ for every real noncompact long root γ.

(c) $\left|\frac{2\langle\mu,\gamma\rangle}{|\gamma|^2}\right| \le 1$ whenever $\gamma = \pm\frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$, γ is compact, and γ is strongly orthogonal to all α's other than α_i and α_j.

Proof. (a) Without loss of generality let $\gamma = \alpha_j$. When we apply $\underset{m}{c}$, we find that the $SL(2,\mathbb{R})$ for the root $\underset{m}{c}(\alpha_j)$ is the same as the $SL(2,\mathbb{R})$ for the noncompact root α_j. In particular the K part is the same, namely $\mathbb{R}iH_{\alpha_j}$. Then $2iH_{\alpha_j}/|\alpha_j|^2$ is to act in τ_μ only with eigenvalues $i, 0,$ and $-i$, and in particular this is to happen in the μ-weight space. Then $|2\langle\mu,\alpha_j\rangle/|\alpha_j|^2| \le 1$.

(b) Such a γ satisfies the hypothesis of (a).

(c) For definiteness, let $\gamma = \frac{1}{2}(\alpha_1+\alpha_2)$. Choose root vectors E_δ for each root δ such that

$$B(E_\delta, E_{-\delta}) = 2/|\delta|^2 \quad \text{and} \quad \Theta E_\delta = -E_{-\delta}. \qquad [1]$$

Then it follows that

$$[E_\delta, E_{-\delta}] = \frac{2}{|\delta|^2} H_\delta$$

$E_\delta + E_{-\delta}$, $i(E_\delta - E_{-\delta})$ are in $\mathfrak{g}$ if δ is noncompact

$E_\delta - E_{-\delta}$, $i(E_\delta + E_{-\delta})$ are in $\mathfrak{g}$ if δ is compact

$$\left.\begin{aligned} [E_{-\delta}, [E_\delta, E_\beta]] &= q(p+1) E_\beta \\ [E_\delta, [E_{-\delta}, E_\beta]] &= p(q+1) E_\beta \end{aligned}\right\} \text{ if } \beta + n\delta \text{ is a root for } -p \le n \le q. \qquad [2]$$

See K-Wallach, Szego paper.

With $E_{\frac{1}{2}(\alpha_1+\alpha_2)}$ and $E_{-\frac{1}{2}(\alpha_1+\alpha_2)}$ thus chosen, let us check

that $\quad [E_{-\alpha_2}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}]$ and $-[E_{\alpha_2}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}] \qquad [3]$

can be used for $E_{\frac{1}{2}(\alpha_1-\alpha_2)}$ and $E_{-\frac{1}{2}(\alpha_1-\alpha_2)}$ and then that

$$\begin{aligned} [E_{\alpha_2}, E_{\frac{1}{2}(\pm\alpha_1-\alpha_2)}] &= \pm E_{\frac{1}{2}(\pm\alpha_1+\alpha_2)} \\ [E_{-\alpha_2}, E_{\frac{1}{2}(\pm\alpha_1+\alpha_2)}] &= \pm E_{\frac{1}{2}(\pm\alpha_1-\alpha_2)} \end{aligned} \qquad [4]$$

Then we shall check that

$\frac{1}{2}[E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}]$ and $-\frac{1}{2}[E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}] \qquad [5]$

can be used for E_{α_1} and $E_{-\alpha_1}$ and then that

$$\begin{aligned} [E_{\alpha_1}, E_{\frac{1}{2}(-\alpha_1\pm\alpha_2)}] &= \pm E_{\frac{1}{2}(\alpha_1\pm\alpha_2)} \\ [E_{-\alpha_1}, E_{\frac{1}{2}(\alpha_1\pm\alpha_2)}] &= \pm E_{\frac{1}{2}(-\alpha_1\pm\alpha_2)}. \end{aligned} \qquad [6]$$

First we verify that the vectors [3] satisfy [1]. We have

$$B([E_{-\alpha_2}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}], -[E_{\alpha_2}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}])$$

$$= B(E_{\frac{1}{2}(\alpha_1+\alpha_2)}, [E_{-\alpha_2}, [E_{\alpha_2}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}]])$$

$$= B(E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}) \qquad \text{by [2]}$$

$$= \frac{2}{|\frac{1}{2}(\alpha_1+\alpha_2)|^2} \qquad \text{by [1]}$$

$$= \frac{2}{|\frac{1}{2}(\alpha_1-\alpha_2)|^2} \qquad \text{as required.}$$

Also

$$\theta\,\overline{[E_{-\alpha_2}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}]} = [\theta\overline{E}_{-\alpha_2}, \theta\overline{E}_{\frac{1}{2}(\alpha_1+\alpha_2)}]$$

$$= [-E_{\alpha_2}, -E_{-\frac{1}{2}(\alpha_1+\alpha_2)}]$$

$$= -\left(-[E_{\alpha_2}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}]\right) \qquad \text{as required.}$$

Thus we now may assume

$$[E_{-\alpha_2}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}] = E_{\frac{1}{2}(\alpha_1-\alpha_2)}$$

$$[E_{\alpha_2}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}] = -E_{-\frac{1}{2}(\alpha_1-\alpha_2)}$$

Then [2] gives

$$[E_{\alpha_2}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}] = E_{\frac{1}{2}(\alpha_1+\alpha_2)}$$

$$[E_{-\alpha_2}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}] = -E_{-\frac{1}{2}(\alpha_1-\alpha_2)},$$

and [4] is proved.

Now let us check that the vectors [5] satisfy [1]. We have

$$B\left(\tfrac{1}{2}[E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}], -\tfrac{1}{2}[E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}]\right)$$

$$= +\tfrac{1}{4} B\left(E_{\frac{1}{2}(\alpha_1-\alpha_2)}, [E_{\frac{1}{2}(\alpha_1+\alpha_2)}, [E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}]]\right)$$

$$= +\tfrac{1}{2} B\left(E_{\frac{1}{2}(\alpha_1-\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}\right) \qquad \text{by } [2]$$

$$= +\tfrac{1}{2} \frac{2}{|\frac{1}{2}(\alpha_1-\alpha_2)|^2} = \frac{2}{|\alpha_1|^2} \qquad \text{by } [1]$$

Also

$$\theta\left(\overline{\tfrac{1}{2}[E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}]}\right) = \tfrac{1}{2}[\overline{\theta E_{\frac{1}{2}(\alpha_1+\alpha_2)}}, \overline{\theta E_{\frac{1}{2}(\alpha_1-\alpha_2)}}]$$

$$= \tfrac{1}{2}[-E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, -E_{-\frac{1}{2}(\alpha_1-\alpha_2)}]$$

$$= -\left(-\tfrac{1}{2}[E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}]\right)$$

Thus we may now assume

$$\tfrac{1}{2}[E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}] = E_{\alpha_1}$$

$$\tfrac{1}{2}[E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}] = -E_{-\alpha_1}$$

Then we compute

$$[E_{\alpha_1}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}] = \tfrac{1}{2}[E_{-\frac{1}{2}(\alpha_1-\alpha_2)}, [E_{\frac{1}{2}(\alpha_1-\alpha_2)}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}]]$$

$$= E_{\frac{1}{2}(\alpha_1+\alpha_2)}$$

$$[E_{\alpha_1}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}] = -\tfrac{1}{2}[E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, [E_{\frac{1}{2}(\alpha_1+\alpha_2)}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}]]$$

$$= -E_{\frac{1}{2}(\alpha_1-\alpha_2)}$$

$$[E_{-\alpha_1}, E_{\frac{1}{2}(\alpha_1+\alpha_2)}] = \tfrac{1}{2}[E_{\frac{1}{2}(\alpha_1+\alpha_2)}, [E_{-\frac{1}{2}(\alpha_1+\alpha_2)}, E_{-\frac{1}{2}(\alpha_1-\alpha_2)}]]$$

$$= E_{-\frac{1}{2}(\alpha_1-\alpha_2)}$$

$$[E_{-\alpha_1}, E_{\frac{1}{2}(\alpha_1-\alpha_2)}] = -\tfrac{1}{2}[E_{\frac{1}{2}(\alpha_1-\alpha_2)}, [E_{-\frac{1}{2}(\alpha_1-\alpha_2)}, E_{-\frac{1}{2}(\alpha_1+\alpha_2)}]]$$

$$= -E_{-\frac{1}{2}(\alpha_1+\alpha_2)} \quad ,$$

and [6] is proved.

Now we compute the Cayley transform of $E_{\frac{1}{2}(\alpha_1+\alpha_2)}$ and of $E_{-\frac{1}{2}(\alpha_1+\alpha_2)}$ in order to obtain the n and θn piece of the $sl(2,\mathbb{R})$.

Our assumption of strong orthogonality makes

$$\underset{\sim}{c}(E_{\pm\frac{1}{2}(\alpha_1+\alpha_2)}) = \underset{\sim}{c_2}\,\underset{\sim}{c_1}(E_{\pm\frac{1}{2}(\alpha_1+\alpha_2)}).$$

Here $$\underset{\sim}{c_1} = Ad\left(\exp \tfrac{\pi}{4}(E_{\alpha_1} - E_{-\alpha_1})\right)$$

So we have

$$\underset{\sim}{c}_1 (E_{\frac{1}{2}(\alpha_1+\alpha_2)}) = \left(\exp \tfrac{\pi}{4} \operatorname{ad}(E_{\alpha_1} - E_{-\alpha_1})\right)(E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2})$$

$$= E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \frac{\pi}{4} E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \frac{1}{2!}\left(\frac{\pi}{4}\right)^2 E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2}$$

$$+ \frac{1}{3!}\left(\frac{\pi}{4}\right)^3 E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} + \frac{1}{4!}\left(\frac{\pi}{4}\right)^4 E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \cdots$$

$$= \left(\cos\frac{\pi}{4}\right) E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \left(\sin\frac{\pi}{4}\right) E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2}$$

Similarly

$$\underset{\sim}{c}_2 (E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2}) = \left(\cos\frac{\pi}{4}\right) E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \left(\sin\frac{\pi}{4}\right) E_{\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}$$

and

$$\underset{\sim}{c}_2 (E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2}) = \left(\cos\frac{\pi}{4}\right) E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} + \left(\sin\frac{\pi}{4}\right) E_{-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}$$

Thus

$$\underset{\sim}{c} (E_{\frac{1}{2}(\alpha_1+\alpha_2)}) = \left(\cos^2\frac{\pi}{4}\right) E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right) E_{\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}$$

$$- \left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right) E_{-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - \left(\sin^2\frac{\pi}{4}\right) E_{-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}$$

Projecting to $\mathfrak{k}^{\mathbb{C}}$ by means of Θ, we find

$$\mathfrak{k}^{\mathbb{C}} \cap sl(2,\mathbb{R})^{\underset{\sim}{c}(\frac{1}{2}(\alpha_1+\alpha_2))} \subseteq \mathbb{C}\left(E_{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2} - E_{-\frac{1}{2}(\alpha_1+\alpha_2)}\right). \quad [7]$$

Returning to τ_μ, consider the $su(2) \subseteq \mathfrak{k}$ corresponding to $\frac{1}{2}(\alpha_1+\alpha_2)$. What [7] shows is that this $su(2)$ contains

$$\mathfrak{k} \cap sl(2,\mathbb{R})^{\underset{\sim}{c}(\frac{1}{2}(\alpha_1+\alpha_2))}.$$

Decompose τ_μ into weight string spaces according to $\frac{1}{2}(\alpha_1+\alpha_2)$, and on each $E_{\frac{1}{2}(\alpha_1+\alpha_2)} - E_{-\frac{1}{2}(\alpha_1+\alpha_2)}$ acts with the same eigenvalues as $c\,i\,H_{\frac{1}{2}(\alpha_1+\alpha_2)}$.

Thus our condition is simply $2\langle\mu, \frac{1}{2}(\alpha_1+\alpha_2)\rangle / |\frac{1}{2}(\alpha_1+\alpha_2)|^2 \leq 1$,

as required.

Lemma 17. There exists a one-dimensional representation ω_0' of K_r with highest weight $P(2\rho_c) - 2\rho_{\text{red},c}$.

Proof. For every γ in Δ_r, we have

$$\langle P(2\rho_c) - 2\rho_{\text{red},c}, \gamma \rangle = \langle 2\rho_c - 2\rho_{\text{red},c}, \gamma \rangle = 0.$$

Thus only the integrality needs proof. Define $\xi(t)$ as a restriction from T by

$$\xi(t) = \xi_{2\rho_c - 2\rho_{\text{red},c}}(t) \quad \text{for } t \in T_r.$$

For $H \in \mathfrak{t}_r$, the differential is

$$d\xi(H) = (2\rho_c - 2\rho_{\text{red},c})(H) = (P(2\rho_c) - 2\rho_{\text{red},c})(H).$$

The lemma follows.

Define a character ω of $F = \text{span}\{\gamma_\beta \mid \beta \text{ red}\}$ by

$$\omega(f) = \sigma(f)\,\omega_0(f), \quad f \in F.$$

This definition makes sense because $F \subseteq Z_M \cap K_r$.

Let τ_μ be a fine K_r-type containing ω (existence by Vogan's Theorem).

Lemma 18. With μ as above, let

$$\Lambda = \lambda - P(2\rho_c) + 2\rho_{real,c} + \mu .$$

Then

(i) Λ is integral for G.

(ii) If Λ is K-dominant, then $\tau_\Lambda|_{K\cap M^\#}$ contains the lowest $(K\cap M^\#)$-type of σ, namely

$$\sigma_\lambda(m) = \begin{cases} \text{rep of } K\cap M_0 \text{ with highest weight } \lambda \\ \text{character of } F \text{ given by } \sigma, \ F \subseteq Z_M . \end{cases}$$

Proof. (i) This is a matter of applying previous integrality lemmas.

First choose a T_λ-weight μ' of τ_μ such that

$$\tau_\mu(\gamma_{\alpha_j})\varphi_{\mu'} = \omega(\gamma_{\alpha_j})\varphi_{\mu'} \quad \text{for all } j .$$

We can do this since all γ_{α_j} are in T_λ and since $\tau_\mu|_F$ contains ω.

Then

$$\mu = \mu' + \sum \text{roots},$$

and it is enough to prove that

$$\Lambda_4 = \lambda - P(2\rho_c) + 2\rho_{real,c} + \mu'$$

is integral. Here

$$\mu' = \sum_{\omega(\gamma_{\alpha_j}) = -1} (n_j + \tfrac{1}{2})\alpha_j + \sum_{\omega(\gamma_{\alpha_j}) = +1} n_j\alpha_j$$

for suitable integers n_j. By Lemma 2, we know

$$\Lambda_1 = \lambda + \sum_{\sigma(\gamma_{\alpha_j}) = -1} \tfrac{1}{2}\alpha_j$$

is integral. Thus it is enough to prove that the form

$$\Lambda_4 - \Lambda_1 = -P(2\rho_c) + 2\rho_{n\ell,c} + \sum_{\omega(\gamma_{\alpha_j}) = -1} \tfrac{1}{2}\alpha_j - \sum_{\sigma(\gamma_{\alpha_j}) = -1} \tfrac{1}{2}\alpha_j + \sum n_j\alpha_j$$

is integral. In fact, this form is in $\sum \mathbb{Z}\alpha_j$. To verify this, we compute

$$\exp\left(\frac{2\pi i \langle \alpha_j \text{ terms of } \Lambda_4 - \Lambda_1, \alpha_j\rangle}{|\alpha_j|^2}\right)$$

$$= (-1)^{2\langle -2\rho_c + 2\rho_{n\ell,c}, \alpha_j\rangle / |\alpha_j|^2}$$

$$\times\ \omega(\gamma_{\alpha_j}) \times \sigma(\gamma_{\alpha_j})^{-1}$$

$$= (-1)^{2\langle -2\rho_c + 2\rho_{n\ell,c}, \alpha_j\rangle / |\alpha_j|^2}\ \omega_0(\alpha_j) = +1.$$

(ii) Let φ_Λ be a nonzero highest weight vector of τ_Λ, and let

$$S = \operatorname{span} \tau_\Lambda(K_n)\varphi_\Lambda .$$

Since K_n is compatibly ordered, S is irreducible under $\tau_\Lambda|_{K_n}$, and its highest weight is

$$\Lambda|_{\mathfrak{t}_n} = -P(2\rho_c) + 2\rho_{n\ell,c} + \mu$$

Since $\tau_\mu|_F$ contains ω, and since ω_0^{-1} has differential $-P(2\rho_c)+2\rho_{real,c}$,

$$\tau_{-P(2\rho_c)+2\rho_{real,c}+\mu}\Big|_F \quad \text{contains} \quad \omega\omega_0^{-1}|_F = \sigma|_F.$$

Here $\sigma|_F$ acts in a one-dimensional subspace $\mathbb{C}\varphi_0$ of S.

Now every vector v of S has the property that

$$\tau_\Lambda(H^-)v = \lambda(H^-)v \qquad \text{for } H^- \in t,\ \alpha_j(H^-)=0 \text{ for all } j$$

because X_n in k_n implies

$$\tau_\Lambda(H^-)\tau_\Lambda(X_n)\varphi_\Lambda = \tau_\Lambda(X_n)\tau_\Lambda(H^-)\varphi_\Lambda = \lambda(H^-)\tau_\Lambda(X_n)\varphi_\Lambda$$

and we can iterate matters and use the Birkhoff-Witt Theorem.

We shall show that v is a highest weight vector under $\tau_\Lambda|_{K\cap M_c}$.

We thus suppose that $\widetilde{E}_\gamma = \underset{\sim}{c}(E_\beta)$ with $\beta>0$ is a root vector for for an M-compact root $\gamma>0$ (orthogonal to all α_j). (Cf, K-Wallach, Szegos, p. 177.)

We first show $\tau_\Lambda(\widetilde{E}_\gamma)\varphi_\Lambda = 0$. There are two cases. First suppose β is G-compact. Since β fails to be strongly orthogonal to at most one α_j, Lemma 5.4 of [K-Wallach, Szegos] shows

$$\underset{\sim}{c}(E_\beta) = \begin{cases} E_\beta & \text{if } \beta \text{ strongly orthogonal to all } \alpha_j \\ \tfrac{1}{2}([E_{\alpha_j},E_\beta]-[E_{-\alpha_j},E_\beta]) & \text{if } \beta \text{ not strongly orthogonal to } \alpha_j. \end{cases}$$

The right side is to be in $(k\cap m)^{\mathbb{C}} \subseteq k^{\mathbb{C}}$, and β compact thus implies β is strongly orthogonal to all α_j and $\underset{\sim}{c}(E_\beta)=E_\beta$. Thus $\widetilde{E}_\gamma = E_\beta$.

Since $\beta > 0$, we have

$$\tau_\Lambda(\tilde{E}_\gamma)\varphi_\Lambda = \tau_\Lambda(E_\beta)\varphi_\Lambda = 0.$$

Second, suppose β is θ-noncompact. Then from above we must have

$$\tau_\Lambda(\tilde{E}_\gamma)\varphi_\Lambda = \tau_\Lambda\left(\tfrac{1}{2}([E_{\alpha_j}, E_\beta] - [E_{-\alpha_j}, E_\beta])\right)\varphi_\Lambda \text{ for some } j.$$

But $\beta+\alpha_j$ and $\beta-\alpha_j$ are positive, if they are roots, by our choice of ordering. Hence the right side is 0. This completes the proof that $\tau_\Lambda(\tilde{E}_\gamma)\varphi_\Lambda = 0$.

Now we set up an iteration, first considering

$$\tau_\Lambda(\tilde{E}_\gamma)\,\tau_\Lambda(E_\delta)\varphi_\Lambda \quad , \quad \delta \text{ in } \Delta_{K,n}.$$

From above, this equals

$$\tau_\Lambda([\tilde{E}_\gamma, E_\delta])\varphi_\Lambda = \begin{cases} \tau_\Lambda[E_\beta, E_\delta]\varphi_\Lambda \\ \tau_\Lambda\left[\tfrac{1}{2}[E_{\alpha_j}, E_\beta] - \tfrac{1}{2}[E_{-\alpha_j}, E_\beta], E_\delta\right]\varphi_\Lambda \end{cases}$$

in the two cases. This expression is still 0 because $\beta+\delta$, $\beta+\alpha_j+\delta$, and $\beta-\alpha_j+\delta$, if roots, are positive. Clearly we can iterate this argument. Thus any vector v in S is a highest weight vector under $\tau_\Lambda|_{K\cap M_e}$ with highest weight λ.

Let us apply this fact to φ_0, as constructed above. Then span$\{\tau_\Lambda(K\cap M_e)\varphi_0\}$ is irreducible under $K\cap M_e$ of type λ. Since F centralizes $K\cap M_e$, $\tau_\Lambda(F)$ acts as the scalar $\sigma(F)$ on the space. This completes the proof.

Lemma 19. With Λ as in Lemma 18, Λ is K-dominant provided each Δ_K^+ simple root β in Δ_0 but not Δ_n satisfies the following condition

(i) if $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$, then $\dfrac{2\langle\mu,\beta\rangle}{|\beta|^2} > -\frac{1}{2}$

(ii) if $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ with $\pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ not roots, and with $\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$, α_1, and α_2 all simple for Δ^+, then $\dfrac{2\langle\mu,\beta\rangle}{|\beta|^2} > -1$.

Remarks.

1) These are the only kinds of roots β in Δ_0 but not Δ_n, by Lemma 15.

2) Conditions (i) and (ii) say that $\dfrac{2\langle\mu,\beta\rangle}{|\beta|^2}$ is not assuming its minimum possible value in Lemma 13.

Proof. We write

$$\begin{aligned}
\Lambda &= \lambda - P(2\rho_c) + 2\rho_{nd,c} + \mu \\
&= \lambda_0 + \rho_n^- - \rho_c^- - P(2\rho_c) + 2\rho_{nd,c} + \mu \\
&= \lambda_0 + \rho^- - 2\rho_c^- - P(2\rho_c) + 2\rho_{nd,c} + \mu \\
&= \lambda_0 + 2\rho_{nd,c} - \rho_{nd} + \rho - 2\rho_c + \mu \qquad \text{by Lemma 10}
\end{aligned}$$

Then

$$\frac{2\langle\Lambda,\beta\rangle}{|\beta|^2} = \frac{2\langle\lambda_0,\beta\rangle}{|\beta|^2} + \frac{2\langle 2\rho_{red,c}-\rho_{red}+\rho,\beta\rangle}{|\beta|^2} - \frac{2\langle 2\rho_c,\beta\rangle}{|\beta|^2} + \frac{2\langle\mu,\beta\rangle}{|\beta|^2}$$

$$= \frac{2\langle\lambda_0,\beta\rangle}{|\beta|^2} + \frac{2\langle 2\rho_{red,c}-\rho_{red}+\rho,\beta\rangle}{|\beta|^2} - 2 + \frac{2\langle\mu,\beta\rangle}{|\beta|^2}. \qquad [1]$$

We distinguish several cases for β.

First suppose β is in Δ_n. Then the first term is 0, the second term is 2, and we have

$$\frac{2\langle\Lambda,\beta\rangle}{|\beta|^2} = \frac{2\langle\mu,\beta\rangle}{|\beta|^2}.$$

This term is ≥ 0 since μ is defined so as to be K_n-dominant.

Next suppose that β is not in Δ_0. Then, depending on its type in Lemma 6, β satisfies some estimate of the following form

$$\frac{2\langle 2\rho_{red,c}-\rho_{red}+\rho,\beta\rangle}{|\beta|^2} \geq c_1, \qquad [2]$$

according to Lemma 12. Also μ is a form μ_0 to which Lemma 13 applies, according to Lemma 16a, and Lemma 13 gives us an estimate

$$\frac{2\langle \mu, \beta \rangle}{|\beta|^2} \geq -c_2 \;, \qquad [3]$$

with c_2 depending on the type of β. In every case except (2a(ii)) and (3a(ii)) we have

$$c_1 - c_2 \geq 1 . \qquad [4]$$

Except in these special cases, [1] gives us

$$\frac{2\langle \Lambda, \beta \rangle}{|\beta|^2} \geq \frac{2\langle \lambda_0, \beta \rangle}{|\beta|^2} - 1 .$$

Since $\langle \lambda_0, \beta \rangle > 0$, we have

$$\frac{2\langle \Lambda, \beta \rangle}{|\beta|^2} > -1 , \qquad [5]$$

and since Λ is integral (Lemma 18a), we conclude

$$\frac{2\langle \Lambda, \beta \rangle}{|\beta|^2} \geq 0 . \qquad [6]$$

The two special cases are

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \quad \text{with} \begin{cases} |\beta| = |\alpha_1| = |\alpha_2| \\ \pm\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \in \Delta \end{cases} \qquad [7a]$$

and

$$\beta = \varepsilon \pm \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \tfrac{1}{2}\alpha_3 \quad \text{with} \begin{cases} |\beta| = |\alpha_1| = |\alpha_2| = |\alpha_3| \\ \text{some } \pm\tfrac{1}{2}\alpha_i \pm \tfrac{1}{2}\alpha_j \in \Delta . \end{cases} \qquad [7b]$$

In these cases Lemmas 12 and 13 give us merely

$$c_1 - c_2 \geq \tfrac{1}{2}. \qquad [8]$$

Under our special assumptions on μ, we shall improve this estimate to [4], so that the above argument goes through to give [6].

However, we first dispose of the remaining case, that β is in Δ_0 but not Δ_n. Now we have

$$\frac{2\langle\lambda_0, \beta\rangle}{|\beta|^2} = 0.$$

We still have the estimate [2], but Lemma 15 says that only types (1) and (2a(i)) are possible. Our hypothesis* is that

$$\frac{2\langle\mu, \beta\rangle}{|\beta|^2} > -c_2,$$

and [4] holds since types (2a(ii)) and (3a(iii)) are excluded. Adding, we again obtain [5], and [6] follows by integrality.

Thus if we can handle the special cases [7a] and [7b], we obtain [6] for all Δ_K^+ simple roots β, and it follows that Λ is K-dominant.

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* In case (ii), we have this only for certain β. The other β's are handled separately, and the argument is given on page 72.

We consider [7a], writing

$$\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 \quad \text{with } t_1 = \pm 1,\ t_2 = \pm 1.$$

We are assuming that

$$\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2 \quad \text{is in } \Delta,$$

and we may assume that we cannot improve upon [3], so that $2\langle\beta,\mu\rangle/|\beta|^2 = -1$

and hence

$$\mu = \tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2 + \cdots.$$

We divide matters into three subcases:

(I) $\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$ strongly orthogonal to all other α_j

(II) $\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$ not strongly orthogonal to α_3, and α_1 precedes α_3

(III) $\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$ not strongly orthogonal to α_0, and α_0 precedes α_1.

(I) $\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$ strongly orthogonal to all other α_j.

In this case Lemma 16c implies $\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$ is noncompact.

Thus we can write $t_2 = -\varepsilon_{12}t_1$ in the notation of Lemma 12.

For this case we have $s_2 = \tfrac{1}{2}\varepsilon_{12}$, from the proof of Lemma 12.

From page 25, we find

$$[12] = -t_1\left(-\tfrac{1}{2}\right) - t_2\left(-\tfrac{1}{2}+s_2\right)$$

$$= -t_1\left(-\tfrac{1}{2}\right) - t_2\left(-\tfrac{1}{2}+\tfrac{1}{2}\varepsilon_{21}\right)$$

$$= -t_1\left(-\tfrac{1}{2}\right) - t_2\left(-\tfrac{1}{2}-\tfrac{1}{2}t_2t_1\right)$$

$$= t_1 + \tfrac{1}{2}t_2 .$$

In addition, we have

$$\frac{2\langle\rho,\beta\rangle}{|\beta|^2} \geq 1 + (2 \text{ if } t_1 \text{ is minus}) + (1 \text{ if } t_2 \text{ is minus}),$$

the 2 for t_1 coming because

$$\frac{2\langle\rho, \varepsilon+\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2\rangle}{|\beta|^2} = \frac{2\langle\rho, \varepsilon-\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2\rangle}{|\beta|^2} + \frac{2\langle\rho,\alpha_1\rangle}{|\beta|^2}$$

$$\geq 1+2,$$

α_1 not being simple $\left(= \frac{1}{2}(\alpha_1-\alpha_2)+\frac{1}{2}(\alpha_1+\alpha_2)\right)$. We then have the following table:

t_1	t_2	Lower bound for $\frac{2\langle 2\rho_{nc}, \varepsilon - \rho_{nc}, \beta\rangle}{\lvert\beta\rvert^2}$	Lower bound for $\frac{2\langle\rho,\beta\rangle}{\lvert\beta\rvert^2}$
+	+	3/2	1
+	−	1/2	2
−	+	−1/2	3
−	−	−3/2	4

So we can use $c_1 = 5/2$ as a lower bound for [2]. Since $c_2 = 1$ in Lemma 13, we have $c_1 - c_2 \geq 1$, as required.

(II) $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ not strongly orthogonal to α_3, and α_1 precedes α_3.

In this case we use Lemma 16a to see that $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ is compact. [In fact, if it is noncompact, then we can replace $\alpha_1, \alpha_2, \alpha_3$ in the strongly orthogonal basis by

$$\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2\,, \quad \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 + \alpha_3\,, \quad \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 - \alpha_3\,,$$

and therefore the lemma applies with $\gamma = \frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ to give a contradiction.] Thus we can write $t_2 = \varepsilon_{12}t_1$ in the notation of Lemma 12. For this case we have $s_2 = -\frac{1}{2}\varepsilon_{12}$, from the proof of Lemma 12. As in (I), we obtain

$$[12] = t_1 + \tfrac{1}{2}t_2\,,$$

and the rest of the argument is as in (I).

(III) $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ not strongly orthogonal to α_0, and α_0 precedes α_1.

In this case,

$$\beta = (\varepsilon - \alpha_0) + (\alpha_0 - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2)$$

and β simple for Δ_K^+ imply $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ is compact. Thus we can write $t_2 = \varepsilon_{12}t_1$ in the notation of Lemma 12. For this case, we have $s_2 = +\frac{1}{2}\varepsilon_{12}$, from the proof of Lemma 12. Since ε is compact,

$$\beta = \varepsilon + \left(-\left(\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2\right)\right)$$

and β simple for Δ_K^+ imply $t_1 = +1$. Thus we have (from page 25)

$$[12] = -\left(-\tfrac{1}{2}\right) - \varepsilon_{12}\left(-\tfrac{1}{2} + \tfrac{1}{2}\varepsilon_{12}\right) = \tfrac{1}{2}\varepsilon_{12}.$$

On the other hand

$$\frac{2\langle\rho,\beta\rangle}{|\beta|^2} = \frac{2\langle\rho, \varepsilon-\alpha_0\rangle}{|\beta|^2} + \frac{2\langle\rho, \alpha_0 - \frac{1}{2}\alpha_1 - \frac{1}{2}\varepsilon_{21}\alpha_2\rangle}{|\beta|^2}$$

$$\geq \begin{cases} 1 + 1 & \text{always} \\ 1 + 2 & \text{if } \varepsilon_{12} = -1. \end{cases}$$

Thus we have the following table

ε_{12}	Lower bound for $\frac{2\langle 2\rho_{nd,c} - \rho_{nd}, \beta\rangle}{|\beta|^2}$	Lower bound for $\frac{2\langle\rho,\beta\rangle}{|\beta|^2}$
+	1/2	2
−	−1/2	3.

So again we can use $c_1 = 5/2$ as a lower bound for [2], and the proof goes through. This completes consideration of [7a].

We consider [7b], writing

$$\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 - \tfrac{1}{2}t_3\alpha_3 \quad \text{with each } t_j = \pm 1.$$

We are assuming that some $\frac{1}{2}t_i\alpha_i + \frac{1}{2}t_j\alpha_j$ is in Δ. Let us agree that $\alpha_1, \alpha_2, \alpha_3$ are initially unordered and that $i=1$ and $j=2$; then suppose α_1 precedes α_2. We are now assuming

$$\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2 \quad \text{is in} \quad \Delta,$$

and we may assume that we cannot improve upon [3], so that

$2\langle\beta,\mu\rangle/|\beta|^2 = -3/2$ and hence

$$\mu = \tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2 + \tfrac{1}{2}t_3\alpha_3 + \cdots .$$

We divide matters into four subcases:

(I) $\frac{1}{2}(\alpha_1+\alpha_2)$ is strongly orthogonal to all other α_j, and $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ are not in Δ.

(II) $\frac{1}{2}(\alpha_1+\alpha_2)$ is not strongly orthogonal to α_4, and α_1 precedes α_4. (In this case $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ cannot be in Δ.)

(III) $\frac{1}{2}(\alpha_1+\alpha_2)$ is not strongly orthogonal to α_0, and α_0 precedes α_1. (In this case $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ cannot be in Δ.)

(IV) $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ are in Δ.

(I) $\frac{1}{2}(\alpha_1+\alpha_2)$ is strongly orthogonal to all other α_j, and $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ are not in Δ. In this case Lemma 16c implies $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ is noncompact. Thus $t_2 = -\varepsilon_{12}t_1$. From the proof of Lemma 12, $s_2 = \frac{1}{2}\varepsilon_{12} = -\frac{1}{2}t_1t_2$. Thus

$$[12] = -t_1(-\tfrac{1}{2}) - t_2(-\tfrac{1}{2}+s_2) - t_3(-\tfrac{1}{2})$$

$$= \tfrac{1}{2}(t_1+t_2+t_3) - s_2 t_2$$

$$= t_1 + \tfrac{1}{2}(t_2+t_3).$$

Also

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} \geq 1 + (1-t_1) + \tfrac{1}{2}(1-t_2) + \tfrac{1}{2}(1-t_3),$$

the larger contribution for t_1 coming from the fact that α_1 not simple implies

$$\frac{2\langle \rho, \varepsilon + \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3\rangle}{|\beta|^2} \geq 3.$$

Thus we may take $c_1 = 3$, and the proof goes through.

(II) $\frac{1}{2}(\alpha_1+\alpha_2)$ is not strongly orthogonal to α_4, and α_1 precedes α_4.

From Lemma 16 a, just as for [7a], $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ is compact. Thus $t_2 = \varepsilon_{12} t_1$. From the proof of Lemma 12, $s_2 = -\frac{1}{2}\varepsilon_{12}$. Thus

$$[12] = -t_1(-\tfrac{1}{2}) - t_2(-\tfrac{1}{2}+s_2) - t_3(-\tfrac{1}{2})$$

$$= \tfrac{1}{2}(t_1+t_2+t_3) - s_2 t_2$$

$$= t_1 + \tfrac{1}{2}(t_2+t_3)$$

The same remarks apply to $2\langle\rho,\beta\rangle/|\beta|^2$ as in the previous case, and we may take $c_1 = 3$ to get the proof to go through.

(III) $\frac{1}{2}(\alpha_1+\alpha_2)$ is not strongly orthogonal to α_0, and α_0 precedes α_1.

In this case

$$\beta = (\varepsilon - \alpha_0 - \tfrac{1}{2}t_3\alpha_3) + (\alpha_0 - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2)$$

and β simple for Δ_K^+ imply $\frac{1}{2}t_1\alpha_1 + \frac{1}{2}t_2\alpha_2$ is compact. Thus $t_2 = \varepsilon_{12}t_1$. From the proof of Lemma 12, $s_2 = \frac{1}{2}\varepsilon_{12} = \frac{1}{2}t_1t_2$. Since $\varepsilon - \frac{1}{2}t_3\alpha_3$ is compact,

$$\beta = (\varepsilon - \tfrac{1}{2}t_3\alpha_3) + (-(\tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2))$$

and β simple for Δ_K^+ imply $t_1 = +1$. Thus we have

$$\begin{aligned}
[12] &= -(-\tfrac{1}{2}) - t_2(-\tfrac{1}{2}+s_2) - t_3(-\tfrac{1}{2}) \\
&= \tfrac{1}{2}(1+t_2+t_3) - s_2t_2 \\
&= \tfrac{1}{2}(1+t_2+t_3) - \tfrac{1}{2} = \tfrac{1}{2}(t_2+t_3).
\end{aligned}$$

Also
$$\begin{aligned}
\frac{2\langle\rho,\beta\rangle}{|\beta|^2} &= \frac{2\langle\rho, \varepsilon_0 - \alpha_0 - \frac{1}{2}t_3\alpha_3\rangle}{|\beta|^2} + \frac{2\langle\rho, \alpha_0 - \frac{1}{2}\alpha_1 - \frac{1}{2}t_2\alpha_2\rangle}{|\beta|^2} \\
&\geq [1 + \tfrac{1}{2}(1-t_3)] + [1 + \tfrac{1}{2}(1-t_2)] \\
&= 3 - \tfrac{1}{2}(t_2+t_3).
\end{aligned}$$

Thus we may take $c_1 = 3$, and the proof goes through.

(IV) $\frac{1}{2}(\alpha_2+\alpha_3)$ and $\frac{1}{2}(\alpha_1+\alpha_3)$ are in Δ.

By Lemma 16c, the roots

$$\tfrac{1}{2}t_1\alpha_1+\tfrac{1}{2}t_2\alpha_2, \quad \tfrac{1}{2}t_1\alpha_1+\tfrac{1}{2}t_3\alpha_3, \quad \text{and } \tfrac{1}{2}t_2\alpha_2+\tfrac{1}{2}t_3\alpha_3$$

are all noncompact. Thus

$$t_2=-\varepsilon_{12}t_1, \quad t_3=-\varepsilon_{13}t_1, \quad \text{and} \quad t_3=-\varepsilon_{23}t_2.$$

From the proof of Lemma 12,

$$s_2=\tfrac{1}{2}\varepsilon_{12}=-\tfrac{1}{2}t_1t_2 \quad \text{and} \quad s_3=\tfrac{1}{2}\varepsilon_{23}+\tfrac{1}{2}\varepsilon_{13}=-\tfrac{1}{2}(t_2t_3+t_1t_3).$$

Thus

$$\begin{aligned}[12] &= -t_1(-\tfrac{1}{2})-t_2(-\tfrac{1}{2}+s_2)-t_3(-\tfrac{1}{2}+s_3)\\ &= \tfrac{1}{2}(t_1+t_2+t_3)-s_2t_2-s_3t_3\\ &= \tfrac{1}{2}(t_1+t_2+t_3)+\tfrac{1}{2}t_1+\tfrac{1}{2}(t_2+t_1)\\ &= \tfrac{3}{2}t_1+t_2+\tfrac{1}{2}t_3.\end{aligned}$$

Now we know

$$\rho = 4(2\varepsilon)+\tfrac{3}{2}\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3,$$

and compute directly that

$$\frac{2\langle\rho,\beta\rangle}{|\beta|^2} \geq 1+\tfrac{3}{2}(1-t_1)+(1-t_2)+\tfrac{1}{2}(1-t_3).$$

Thus we may take $c_1=4$, and the proof goes through.

This completes our consideration of [7b] and the proof of the lemma.

Lemma 20. Suppose that for some choice of the fine K^Ω-type μ on page 44 the G-integral form

$$\Lambda = \lambda - P(2\rho_c) + 2\rho_{real,c} + \mu$$

satisfies the conditions of Lemma 19 that ensure that Λ is K-dominant. Then τ_Λ is a minimal K-type of $\mathrm{ind}_{K\cap M^\#}^{K}\sigma$, and every minimal K-type is of the form $\tau_{\Lambda'}$ for some Λ' with

$$\Lambda' = \lambda - P(2\rho_c) + 2\rho_{real,c} + \mu',$$

where μ' is a fine K^Ω-type such that the conditions of Lemma 19 are satisfied.

Remarks. Λ is integral by Lemma 18(i). The conditions of Lemma 19 that ensure K-dominance are conditions on μ relative to Δ_K^+ simple roots that are in Δ_0 but not Δ_Ω. Generically there are no such roots and (in such cases) Λ is automatically dominant.

Proof. τ_Λ occurs in the induced representation by Lemma 18(ii) and Frobenius reciprocity. We now investigate the nature of minimal K-types in the induced representation.

Thus let τ_{Λ_0} be a minimal K-type. By Frobenius reciprocity

$\tau_{\Lambda_0}|_{K\cap M^\#}$ contains some $K\cap M^\#$ type of σ, say $\sigma_{\lambda'}$, with

$$\sigma_{\lambda'} = \begin{cases} \text{irred. rep. of } K\cap M_e \text{ with highest weight } \lambda' \text{ on } K\cap M_e \\ \sigma \quad \text{on } F \subseteq Z_M. \end{cases}$$

Let $\varphi_{\lambda'}$ be a highest weight vector for this copy of $\sigma_{\lambda'}$, so that

$$\tau_{\Lambda_0}(H^-)\varphi_{\lambda'} = \lambda'(H^-)\varphi_{\lambda'} \quad \text{for } H^- \in t^-$$

$$\tau_{\Lambda_0}(\tilde{E}_\gamma)\varphi_{\lambda'} = 0. \quad \text{for all positive } M\text{-compact roots } \gamma.$$

Let $\mathcal{U}$ be the set of all v in the space of τ_{Λ_0} satisfying

$$\tau_{\Lambda_0}(H^-)v = \lambda'(H^-)v \quad \text{for } H^- \in t^-.$$

Then $\tau_{\Lambda_0}(K^n)$ leaves $\mathcal{U}$ stable because X_n in k^n implies

$$\tau_{\Lambda_0}(H^-)\tau_{\Lambda_0}(X_n)v = \tau_{\Lambda_0}(X_n)\tau_{\Lambda_0}(H^-)v = \lambda'(H^-)\tau_{\Lambda_0}(X_n)v \quad \text{for } H^- \in t^-.$$

Since $K^n \supseteq F$ and since $\mathbb{C}\varphi_{\lambda'}$ is stable under $\sigma|_F$, $\mathcal{U}$ contains a $\tau_{\Lambda_0}(K^n)$ stable irreducible subspace $\mathcal{V}$ that contains a copy of $\sigma|_F$. Let ν on t^n be the highest weight of this representation on $\mathcal{V}$, and let $\psi_{\lambda'+\nu}$ be a highest weight vector. Then

$$\tau_{\Lambda_0}(H)\psi_{\lambda'+\nu} = \begin{cases} \lambda'(H)\psi_{\lambda'+\nu} & \text{if } H \in t^- \\ \nu(H)\psi_{\lambda'+\nu} & \text{if } H \in t^n, \end{cases}$$

and it follows that $\lambda'+\nu$ is a weight of τ_{Λ_0}. This proves that

τ_{Λ_0} has a weight $\lambda' + \nu$ such that λ' is a $(K \cap M^{\#})$-type of σ,

ν is $K^{\wedge}$-dominant, and

$$\tau_\nu|_F \text{ contains } \sigma. \qquad [1]$$

Let us write down some consequences of this fact. Since

$\lambda' + \nu$ is a weight of τ_{Λ_0}, we have

$$\lambda' + \nu = \Lambda_0 - \sum n_i \beta_i \quad , \; n_i \geq 0, \; \beta_i \in \Delta_K^+ \qquad [2]$$

$$|\lambda' + \nu|^2 \leq |\Lambda_0|^2. \qquad [3]$$

Since the Blattner weight λ of σ is minimal for σ,

$$|\lambda' + 2\rho_c^-|^2 \geq |\lambda + 2\rho_c^-|^2. \qquad [4]$$

Since τ_Λ occurs in the induced representation (Lemma 18(ii)),

$$|\Lambda_0 + 2\rho_c|^2 \leq |\Lambda + 2\rho_c|^2. \qquad [5]$$

Then we have

$$|\lambda|^2 + |\nu|^2 = |\lambda + 2\rho_c^-|^2 - 4\langle \lambda, \rho_c^- \rangle - 4|\rho_c^-|^2 + |\nu|^2$$

$$\leq |\lambda' + 2\rho_c^-|^2 - 4\langle \lambda, \rho_c^- \rangle - 4|\rho_c^-|^2 + |\nu|^2 \qquad \text{by } [4]$$

$$= |\lambda'|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle + |\nu|^2$$

$$= |\lambda' + \nu|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle$$

$$\leq |\Lambda_0|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle \qquad \text{by [3]}$$

$$= |\Lambda_0 + 2\rho_c|^2 - 4\langle \Lambda_0, \rho_c \rangle - 4|\rho_c|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle$$

$$\leq |\Lambda + 2\rho_c|^2 - 4\langle \Lambda_0, \rho_c \rangle - 4|\rho_c|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle \qquad \text{by [5]}$$

$$= |\Lambda|^2 + 4\langle \Lambda - \Lambda_0, \rho_c \rangle + 4\langle \lambda' - \lambda, \rho_c^- \rangle$$

$$= |\lambda|^2 + |2\rho_{red,c} - P(2\rho_c) + \mu|^2 + 4\langle \Lambda - \Lambda_0, \rho_c \rangle + 4\langle \lambda' - \lambda, \rho_c^- \rangle$$

$$= |\lambda|^2 + |2\rho_{red,c} - P(2\rho_c) + \mu|^2 + 4\langle \lambda' - \lambda, \rho_c^- \rangle$$
$$+ 4\langle (\lambda + 2\rho_{red,c} - P(2\rho_c) + \mu) - (\lambda' + \nu + \Sigma m_i \beta_i), \rho_c \rangle \qquad \text{by [2]}$$

$$= |\lambda|^2 + |2\rho_{red,c} - P(2\rho_c) + \mu|^2 - 4\langle \lambda' - \lambda, \rho_c - \rho_c^- \rangle$$
$$- 4\langle P(2\rho_c) - 2\rho_{red,c} - \mu + \nu, \rho_c \rangle - 4\langle \Sigma m_i \beta_i, \rho_c \rangle$$

$$\leq |\lambda|^2 + |2\rho_{red,c} - P(2\rho_c) + \mu|^2 - 4\langle \lambda' - \lambda, \rho_c - \rho_c^- \rangle$$
$$- 4\langle P(2\rho_c) - 2\rho_{red,c} - \mu + \nu, \rho_c \rangle \qquad \text{by [2].}$$

By Schmid's Theorem, $\lambda' - \lambda$ is a sum of positive M-roots, and by Lemma 11, $\rho_c - \rho_c^-$ is M-dominant. Thus this expression is

$$\leq |\lambda|^2 + |2\rho_{red,c} - P(2\rho_c) + \mu|^2 - 4\langle P(2\rho_c) - 2\rho_{red,c} - \mu + \nu, \rho_c \rangle. \qquad [6]$$

For later reference let us note that equality throughout [6] forces the following conclusions:

$\lambda = \lambda'$ (by uniqueness of minimal K-types for discrete series [7a]

Λ is a minimal K-type (since $|\Lambda_0 + 2\rho_c|^2 = |\Lambda + 2\rho_c|^2$) [7b]

$\lambda' + \nu = \Lambda_0$ (since all $n_i = 0$). [7c]

We rewrite [7c], in the presence of [7a] and [1], as

$\Lambda_0 = \lambda + \nu$, where $\tau_\nu|_F$ contains σ. [7d]

Returning to [6] and subtracting $|\lambda|^2$ from both sides, we have

$$|\nu|^2 \leq |2\rho_{red,c} - P(2\rho_c) + \mu|^2 - 4\langle P(2\rho_c) - 2\rho_{red,c} - \mu + \nu, \rho_c\rangle$$

$$= |\mu + 2\rho_{red,c}|^2 - 2\langle \underline{\underline{\mu}} + \underline{2\rho_{red,c}}, P(2\rho_c)\rangle + |P(2\rho_c)|^2$$

$$- 2\langle P(2\rho_c) - \underline{2\rho_{red,c}} - \underline{\underline{\mu}}, P(2\rho_c)\rangle - 2\langle \nu, P(2\rho_c)\rangle$$

$$= |\mu + 2\rho_{red,c}|^2 - |P(2\rho_c)|^2 - 2\langle \nu, P(2\rho_c)\rangle.$$

Hence $\quad |\nu + P(2\rho_c)|^2 \leq |\mu + 2\rho_{red,c}|^2$

or $\quad |(\nu + P(2\rho_c) - 2\rho_{red,c}) + 2\rho_{red,c}|^2 \leq |\mu + 2\rho_{red,c}|^2,$ [8]

and equality in [8] will force [7].

Now [1] says that $\tau_\nu|_F$ contains $\sigma|_F$. Taking into account Lemma 17, we see that

$$\tau_{\nu + P(2\rho_c) - 2\rho_{red,c}}|_F$$

contains $w_0\sigma|_F = \omega$. But μ is a fine $\hat{K}$-type for ω and is minimal by the Fact on page 36. Hence equality holds in [8], and, moreover, ν must be a fine $K^\wedge$-type for ω since all minimal $K^\wedge$-types are fine.

Since equality holds in [8], this fact about ν and results [7b] and [7d] prove most of the theorem. To complete the proof, we show that the K-dominance of $\Lambda_0 = \lambda + \nu$ forces conditions (i) and (ii) in Lemma 19 to hold as appropriate. Let us write

$$\nu = 2\rho_{red,c} - P(2\rho_c) + \nu_0 .$$

We are to show that any Δ_K^+-simple root β in Δ_0 but not Δ_n satisfies the appropriate one of the following two conditions:

(i) if $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$ with $|\beta| = |\alpha_1|$, then $\dfrac{2\langle \nu_0, \beta\rangle}{|\beta|^2} > -\frac{1}{2}$

(ii) if $\beta = \varepsilon \pm \frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ with $\pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_2$ not roots and with $|\beta| = |\alpha_1| = |\alpha_2|$ and with $\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$, α_1, and α_2 all simple for Δ^+, then $\dfrac{2\langle \nu_0, \beta\rangle}{|\beta|^2} > -1$.

First we apply Lemma 10 to restore the format of the proof of Lemma 12. We have

$$\Lambda_0 = \lambda + \nu = (\lambda_0 + \rho_n^- - \rho_c^-) + (2\rho_{red,c} - P(2\rho_c) + \nu_0)$$

$$= \lambda_0 + \rho^- - 2\rho_c^- + 2\rho_{red,c} - P(2\rho_c) + \nu_0$$

$$= \lambda_0 + (2\rho_{red,c} - \rho_{red} + \rho) - 2\rho_c + \nu_0$$

For the Δ_K^+ simple root β, we have

$$0 \le \frac{2\langle \Lambda_0, \beta\rangle}{|\beta|^2} = \frac{2\langle\lambda_0,\beta\rangle}{|\beta|^2} + \frac{2\langle 2\rho_{red} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2} - 2 + \frac{2\langle\nu_0,\beta\rangle}{|\beta|^2}$$

$$= \frac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2} - 2 + \frac{2\langle\nu_0,\beta\rangle}{|\beta|^2} \quad \text{since } \beta \text{ is in } \Delta_0$$

Thus

$$\frac{2\langle\nu_0,\beta\rangle}{|\beta|^2} \ge 2 - \frac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2}.$$

We shall show this inequality implies the inequality given in (i) or (ii), as appropriate. Specifically we show that

if the conditions of (i) hold, then $\dfrac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2} = \dfrac{3}{2}$

if the conditions of (ii) hold, then $\dfrac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2} = 2.$

Now suppose $\beta = \varepsilon \pm \frac{1}{2}\alpha_1$. We observe that there is no real root of the form $\pm\frac{1}{2}\alpha_1 +$ other, because

$$\frac{2\langle\beta, \pm\frac{1}{2}\alpha_1 + \text{other}\rangle}{|\beta|^2} = \frac{2\langle\pm\frac{1}{2}\alpha_1, \pm\frac{1}{2}\alpha_1\rangle}{|\alpha_1|^2} = \pm\frac{1}{2}.$$

Nor can there be a real root of the form $\alpha_1 +$ other with other $\neq 0$, since Lemmas 5 and 15 together rule out all possibilities. It follows that $s_1 = 0$, that α_1 is Δ^+ simple, and that if $\varepsilon - \frac{1}{2}\alpha_1 = \beta' + \delta$ (with δ real) exhibits $\varepsilon - \frac{1}{2}\alpha_1$ as not Δ^+ simple, then $\delta = \alpha_1$ or $\delta \perp \alpha_1$.

We shall show that $\varepsilon - \frac{1}{2}\alpha_1$ is Δ^+ simple. First suppose that

$$\varepsilon - \tfrac{1}{2}\alpha_1 = \beta' + \delta \quad , \ \delta \text{ real}.$$

Then $\delta \perp \alpha_1$ and β' must involve both ε and α_1. Also β' is in Δ_0, so that we have either

$$\beta' = \varepsilon - \tfrac{1}{2}\alpha_1 \quad \text{or} \quad \beta' = \varepsilon - \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_i.$$

In the second case,

$$|\beta|^2 = |\varepsilon - \tfrac{1}{2}\alpha_1|^2 + \tfrac{1}{4}|\alpha_i|^2 = |\alpha_1|^2 + \tfrac{1}{4}|\alpha_1|^2 = \tfrac{5}{4}|\alpha_1|^2,$$

so that we conclude $\beta' = \varepsilon - \frac{1}{2}\alpha_1$ and $\varepsilon - \frac{1}{2}\alpha_1$ is simple.

Returning to β, write $\beta = \varepsilon - \frac{1}{2}t_1\alpha_1$. From what we have just proved, Next suppose

$$\varepsilon - \tfrac{1}{2}\alpha_1 = \beta' + \beta'' \quad \text{with } \beta' > 0, \beta'' > 0, \ \beta' \text{ and } \beta'' \text{ not in } \Delta_n$$

Then β' and β'' are in Δ_0 and exactly one involves α_1. Thus we can write

$$\beta' = \varepsilon' - \tfrac{1}{2}\alpha_1 \quad \text{or} \quad \beta' = \varepsilon' - \tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_i$$

and

$$\beta'' = \varepsilon'' \qquad \text{or} \quad \beta'' = \varepsilon'' \mp \tfrac{1}{2}\alpha_i .$$

Since ε'' cannot be a root, we must be in the second case. Then one of the choices of the sign in

$$\beta = \left(\varepsilon' - \tfrac{1}{2}t_1\alpha_1 \pm \tfrac{1}{2}\alpha_i\right) + \left(\varepsilon'' \mp \tfrac{1}{2}\alpha_i\right) \qquad \left(\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1\right)$$

exhibits β as not simple for Δ_K^+, contradiction.

Again write $\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1$. From what we have just proved,

$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1\rangle}{|\beta|^2} + \frac{2\langle \rho, \frac{1}{2}(1-t_1)\alpha_1\rangle}{|\alpha_1|^2}$$

$$= 1 + \tfrac{1}{2}(1-t_1).$$

Since $s_1 = 0$, we have

$$\frac{2\langle 2\rho_{red,c} - \rho_{red}, \beta\rangle}{|\beta|^2} = -t_1(-\tfrac{1}{2}) = \tfrac{1}{2}t_1$$

Thus we conclude

$$\frac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta\rangle}{|\beta|^2} = \frac{3}{2},$$

and the condition is necessary for case (i)

Now suppose $\beta = \varepsilon - \frac{1}{2}t_1\alpha_1 - \frac{1}{2}t_2\alpha_2$ with $|\beta| = |\alpha_1| = |\alpha_2|$ and $\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$ not a root. Then also $\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_j$ for $j \neq 1, 2$ is not a root since

$$\frac{2\langle \beta, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_j\rangle}{|\beta|^2} = \frac{2\langle -\frac{1}{2}t_1\alpha_1, \frac{1}{2}\alpha_1\rangle}{|\alpha_1|^2} = -\tfrac{1}{2}t_1$$

is not an integer. Then it follows that

$$\frac{2\langle 2\rho_{red,c} - \rho_{red}, \beta\rangle}{|\beta|^2} = -t_1(-\tfrac{1}{2}) - t_2(-\tfrac{1}{2}) = \tfrac{1}{2}(t_1 + t_2).$$

Our additional assumption in (ii) implies that

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 \rangle}{|\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2|^2} = 1 = \frac{2\langle \rho, \alpha_1 \rangle}{|\alpha_1|^2} = \frac{2\langle \rho, \alpha_2 \rangle}{|\alpha_2|^2} = 1.$$

Therefore

$$\frac{2\langle \rho, \beta \rangle}{|\beta|^2} = \frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 \rangle}{|\beta|^2} + \frac{2\langle \rho, \frac{1}{2}(1-t_1)\alpha_1 \rangle}{|\alpha_1|^2} + \frac{2\langle \rho, \frac{1}{2}(1-t_2)\alpha_2 \rangle}{|\alpha_2|^2}$$

$$= 1 + \tfrac{1}{2}(1-t_1) + \tfrac{1}{2}(1-t_2)$$

$$= 2 - \tfrac{1}{2}(t_1 + t_2).$$

We conclude that

$$\frac{2\langle 2\rho_{red,c} - \rho_{red} + \rho, \beta \rangle}{|\beta|^2} = 2,$$

and the condition is necessary for case (ii). This completes the proof of Lemma 20.

Supplement concerning β in Δ_0 that is Δ_K^+ simple and is of the form

$$\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 \qquad \text{with } |\beta| = |\alpha_1| = |\alpha_2|,$$

$$\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 \text{ not a root.}$$

(cf. Lemma 19)

If we review the computations on pages 69-70, we see that this β, with no further assumptions, satisfies

$$\frac{2\langle 2\rho_{red,c} - \rho_{red,c}, \beta\rangle}{|\beta|^2} = \tfrac{1}{2}(t_1+t_2)$$

and
$$\frac{2\langle \rho, \beta\rangle}{|\beta|^2} \geq 2 - \tfrac{1}{2}(t_1+t_2),$$

with equality if and only if

$$\frac{2\langle \rho, \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\rangle}{|\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2|^2} = \frac{2\langle\rho,\alpha_1\rangle}{|\alpha_1|^2} = \frac{2\langle\rho,\alpha_2\rangle}{|\alpha_2|^2} = 1,$$

i.e., if and only if $\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$, α_1, and α_2 are all simple for Δ^+.

Thus when at least one of these roots is not simple, we have

$$\frac{2\langle 2\rho_{red,c} - \rho_{red,c} + \rho, \beta\rangle}{|\beta|^2} \geqslant \frac{5}{2}$$

In the notation of pp. 50-52, $c_1 - c_2 \geqslant 2$ and so [1] on page 50 is automatically $\geqslant 0$. This fills in the missing detail in the proof of Lemma 19.

Lemma 21. Suppose that β is a root in Δ_0 but not Δ_n that is simple for Δ_K^+.

a) If $\beta = \varepsilon - \frac{1}{2}t_1\alpha_1$ with $|\beta| = |\alpha_1|$, then α_1 is orthogonal to all roots of Δ_n except $\pm\alpha_1$.

b) If $\beta = \varepsilon - \frac{1}{2}t_1\alpha_1 - \frac{1}{2}t_2\alpha_2$ with $|\beta| = |\alpha_1| = |\alpha_2|$ and $\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$, α_1, and α_2 all simple for Δ^+, then α_1 and α_2 are orthogonal to all roots of Δ_n except $\pm\alpha_1$ and $\pm\alpha_2$.

Proof: (a) This is proved in the top paragraph of page 68.

(b) For the moment, regard α_1 and α_2 as unordered. Suppose $\delta \neq \pm\alpha_1$ is a real root with $\langle \delta, \alpha_1 \rangle \neq 0$. If δ does not involve α_2, then

$$\frac{2\langle \varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2, \delta \rangle}{|\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2|^2} = \frac{2\langle -\frac{1}{2}\alpha_1, c_1\alpha_1 \rangle}{|\alpha_1|^2} = -c_1$$

shows that the coefficient of α_1 in δ is ± 1. From Lemma 5, the remaining part of δ consists of two terms. Adding or subtracting δ from β, we obtain a root in Δ_0 not in Δ_n that is not as in Lemma 15, contradiction.

Thus δ involves α_2 as well. WLOG, write

$$\delta = \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \text{other terms}$$

with α_1 now preceding α_2. According to Lemma 5, there are just three possibilities for "other terms":

1) $\pm\frac{1}{2}\alpha_3 \pm \frac{1}{2}\alpha_4$ with $|\alpha_3| = |\alpha_4| = |\alpha_1| = |\alpha_2|$

2) $\pm\alpha_3$ with $|\alpha_3| < |\alpha_1| = |\alpha_2|$

3) 0.

Possibilities (2) and (3) force ε to be a root, which is not the case.

Thus we may assume

$$\delta = \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{2}\alpha_4 .$$

Here indices 3 and 4 must come after 1 since otherwise

$$\left[\varepsilon - \tfrac{1}{2}(\alpha_3 + \alpha_4)\right] + \left[\tfrac{1}{2}(\alpha_3 + \alpha_4 - \alpha_1 - \alpha_2)\right] = \varepsilon - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2$$

would say $\varepsilon - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$ is not simple, contradiction. But then

$$\alpha_1 = \left[\tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\right] + \left[\tfrac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\right]$$

says α_1 is not simple, contradiction. We conclude there is no such δ, as asserted.

<u>Lemma 22.</u> If α_1 is one of the roots occurring in Lemma 21 and if μ is a fine K^n type containing ω, then $s_{\alpha_1}\mu$ is a fine K^n-type containing ω.

Proof. It follows from Lemma 21 that (a representative of) s_{α_1} gives an automorphism of G^n leaving K^n stable. If we define τ on K^n by

$$\tau(k) = \tau_\mu(s_{\alpha_1}^{-1} k s_{\alpha_1}),$$

then τ is an irreducible representation of K^n, and $s_{\alpha_1}\mu$ will be its highest weight if $s_{\alpha_1}\mu$ is K-dominant. But

$$\langle s_{\alpha_1}\mu, \gamma\rangle = \langle \mu, s_{\alpha_1}\gamma\rangle = \langle \mu, \gamma\rangle \geq 0 \quad \text{for } \gamma \in \Delta^+_{K,red}.$$

Since s_{α_1} fixes $2\rho_{red,c}$, we have $|s_{\alpha_1}\mu + 2\rho_{red,c}|^2 = |\mu + 2\rho_{red,c}|^2$. Thus the proof will be complete if we show that $\tau|_F$ contains ω.

In fact, it is enough to show that s_{α_1} centralizes F. Since $F \subseteq \exp \mathfrak{a}_n^{\mathbb{C}}$, it is enough to show that s_{α_1} centralizes $\mathfrak{a}$, in particular that it centralizes each $E_{\alpha_j} + E_{-\alpha_j}$. The formula for s_{α_1} is

$$s_{\alpha_1} = \exp \frac{\pi i}{2}(E_{\alpha_1} + E_{-\alpha_1})$$

with normalizations as in Lemma 16, and it is clear that this element centralizes each $E_{\alpha_j} + E_{-\alpha_j}$

Lemma 23. There exist no chains of roots β in Lemma 21 whose associated Δ^+ simple roots are of the following forms

1) $\beta_1' = \varepsilon_1 - \frac{1}{2}\alpha_1$
$\beta_2' = \varepsilon_2 - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$
$\beta_3' = \varepsilon_3 \qquad - \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3$
$\vdots$
$\beta_m' = \varepsilon_m \qquad - \frac{1}{2}\alpha_{m-1} - \frac{1}{2}\alpha_m$
$\beta_{m+1}' = \varepsilon_{m+1} \qquad - \frac{1}{2}\alpha_m$

2) $\beta_1' = \varepsilon_1 - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2$
$\beta_2' = \varepsilon_2 \qquad - \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_3$
$\vdots$
$\beta_{m-1}' = \varepsilon_{m-1} \qquad - \frac{1}{2}\alpha_{m-1} - \frac{1}{2}\alpha_m$
$\beta_m' = \varepsilon_m - \frac{1}{2}\alpha_1 \qquad - \frac{1}{2}\alpha_m$

Proof. (1) We shall show that $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{m+1}$ is a root in Δ_0, which would be a contradiction. Notice that $\varepsilon_1 + \dots + \varepsilon_{m+1} \neq 0$ since

$$\varepsilon_1 + \dots + \varepsilon_{m+1} = \beta_1' + \dots + \beta_{m+1}' + \alpha_1 + \alpha_2 + \dots + \alpha_m .$$

For $1 \le i \le m$, we show inductively that

$$\varepsilon_1 + \dots + \varepsilon_i - \tfrac{1}{2}\alpha_i$$

is a root, the assertion being obvious for $i=1$. If the assertion holds for i, then we observe that

$$\langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_i \rangle = \langle \beta'_{i+1}, \beta'_1 + \cdots + \beta'_i \rangle$$ since β'_{i+1} has no $\alpha_1, \ldots, \alpha_{i-1}$

≤ 0 since all the β' are simple

Thus $\langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i + \tfrac{1}{2}\alpha_i \rangle = \langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_i \rangle + \langle \beta'_{i+1}, \alpha_i \rangle$

$$\leq \langle \beta'_{i+1}, \alpha_i \rangle < 0$$

and the sum

$$\varepsilon_1 + \cdots + \varepsilon_i + \tfrac{1}{2}\alpha_i + \beta'_{i+1} = \varepsilon_1 + \cdots + \varepsilon_{i+1} - \tfrac{1}{2}\alpha_{i+1}$$

is a root. This completes the induction.

Now we consider

$$\langle \beta'_{m+1}, \varepsilon_1 + \cdots + \varepsilon_m - \tfrac{1}{2}\alpha_m \rangle = \langle \beta'_{m+1}, \beta'_1 + \cdots + \beta'_m \rangle \leq 0.$$

Then $\langle \beta'_{m+1}, \varepsilon_1 + \cdots + \varepsilon_m + \tfrac{1}{2}\alpha_m \rangle = \langle \beta'_{m+1}, \varepsilon_1 + \cdots + \varepsilon_m - \tfrac{1}{2}\alpha_m \rangle + \langle \beta'_{m+1}, \alpha_m \rangle$

$$\leq \langle \beta'_{m+1}, \alpha_m \rangle < 0$$

and the sum $\varepsilon_1 + \cdots + \varepsilon_m + \tfrac{1}{2}\alpha_m + \beta'_{m+1} = \varepsilon_1 + \cdots + \varepsilon_{m+1}$

is a root. This proves (1).

(2) We argue similarly to show $\varepsilon_1 + \cdots + \varepsilon_m$ is a (positive) root, contradiction. Here we show inductively for $1 \leq i \leq m-1$

that $$\varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_{i+1}$$

is a root. For $i = 1$, this is trivial. Assume it for i. Then

$$\langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_{i+1} \rangle = \langle \beta'_{i+1}, \beta'_1 + \cdots + \beta'_i \rangle \le 0$$

and so

$$\langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_{i+1} \rangle = \langle \beta'_{i+1}, \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_{i+1} \rangle + \langle \beta'_{i+1}, \alpha_{i+1} \rangle$$

$$\le \langle \beta'_{i+1}, \alpha_{i+1} \rangle < 0 .$$

Then
$$\beta'_{i+1} + \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_{i+1} = \varepsilon_1 + \cdots + \varepsilon_i + \varepsilon_{i+1} - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_{i+2}$$

is a root, and the induction is complete. Then we consider

$$\langle \beta'_m, \varepsilon_1 + \cdots + \varepsilon_{m-1} - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_m \rangle = \langle \beta'_m, \beta'_1 + \cdots + \beta'_{m-1} \rangle \le 0$$

and see that

$$\langle \beta'_m, \varepsilon_1 + \cdots + \varepsilon_{m-1} + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_m \rangle = \langle \beta'_m, \varepsilon_1 + \cdots + \varepsilon_{m-1} - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_m \rangle$$

$$+ \langle \beta'_m, \alpha_1 \rangle + \langle \beta'_m, \alpha_m \rangle$$

$$\le \langle \beta'_m, \alpha_1 \rangle + \langle \beta'_m, \alpha_m \rangle < 0 .$$

Then
$$\beta'_m + \varepsilon_1 + \cdots + \varepsilon_{m-1} + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_m = \varepsilon_1 + \cdots + \varepsilon_m$$

is a root. This proves (2).

Lemma 24. Let $\tau_{\mu'}$ be a fine $K^{\wedge}$ type such that $\tau_{\mu'}|_F$ contains ω. Then there exists another fine $K^{\wedge}$ type τ_μ such that $\tau_\mu|_F$ contains ω and such that μ satisfies the conditions (i) and (ii) of Lemma 19.

Remarks. In combination with Lemma 20, this lemma completes the proof of the Main Theorem.

Proof. We shall take $\mu = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_N}} \mu'$, where the s_α's are reflections. Then μ is certainly a fine $K^{\wedge}$ type of the sort described in Lemma 22. Then μ is certainly a fine $K^{\wedge}$ type such that $\tau_\mu|_F$ contains ω. What we have to do is show how to define these α's.

Consider all β's that need attention in Lemma 19. These are of the form

$$\beta_i = \varepsilon_i - \tfrac{1}{2} t_1^{(i)} \alpha_1^{(i)} \quad \text{or} \quad \varepsilon_i - \tfrac{1}{2} t_1^{(i)} \alpha_1^{(i)} - \tfrac{1}{2} t_2^{(i)} \alpha_2^{(i)}$$

as usual.

Let us call β_i equivalent with β_j if there exists a chain of β's with first term β_i, last term β_j, and overlap in the α's occurring in consecutive members of the chain. This relation is an equivalence relation. (However, notice that the α's that occur in a particular β_i do not a priori determine β_i

uniquely.) For each β_i let β_i' denote the associated Δ^+ simple root

$$\beta_i' = \varepsilon_i - \tfrac{1}{2}\alpha_1^{(i)} \quad \text{or} \quad \beta_i' = \varepsilon_i - \tfrac{1}{2}\alpha_1^{(i)} - \tfrac{1}{2}\alpha_2^{(i)}.$$

Actually the relation should be regarded as one applied to the β_i'.
Call these types (i) and (ii).

By Lemma 23, each equivalence class contains at most one β_i' of type (i). In this case it is of the form

$$\begin{aligned}
\beta_1' &= \varepsilon_1 - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 \\
&\vdots \\
\beta_{j-1}' &= \varepsilon_{j-1} - \tfrac{1}{2}\alpha_{j-1} - \tfrac{1}{2}\alpha_j \\
\beta_j' &= \varepsilon_j - \tfrac{1}{2}\alpha_j \\
\beta_{j+1}' &= \varepsilon_{j+1} - \tfrac{1}{2}\alpha_j - \tfrac{1}{2}\alpha_{j+1} \\
&\vdots \\
\beta_m' &= \varepsilon_m - \tfrac{1}{2}\alpha_{m-1} - \tfrac{1}{2}\alpha_m .
\end{aligned}$$

If the class contains no β_i' of type (i), it is of the form

$$\begin{aligned}
\beta_1' &= \varepsilon_1 - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 \\
\beta_2' &= \varepsilon_2 - \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\alpha_3 \\
&\vdots \\
\beta_m' &= \varepsilon_m - \tfrac{1}{2}\alpha_m - \tfrac{1}{2}\alpha_{m+1} .
\end{aligned}$$

Since the classes are disjoint, use of s_α's for one class does not affect any other class. We may therefore concentrate on a single class.

As it pertains to a single class, μ is of the form

$$\mu = \sum_{\alpha\text{'s for a class}} s_j \alpha_j \qquad , \; s_j = \pm\tfrac{1}{2} \text{ or } 0 .$$

Let us show that if a single s_j is 0 for a class, then all s_j are 0 for the class. If the contrary were to happen we could find

$$\beta' = \varepsilon - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 \quad , \text{say},$$

with

$$\mu = \pm\tfrac{1}{2}\alpha_1 + 0\,\alpha_2 + \text{other terms.}$$

Write

$$\beta = \varepsilon - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2 .$$

Then we have

$$\frac{2\langle \mu, \beta\rangle}{|\beta|^2} = \frac{\langle \pm\alpha_1, -\frac{1}{2}t_1\alpha_1\rangle}{|\alpha_1|^2} = -(\pm\tfrac{1}{2}t_1), \text{ not in } \mathbb{Z}.$$

But we know from Lemma 18 that

$$\Lambda = \lambda + 2\rho_{\text{red},c} - P(2\rho_c) + \mu$$

is integral, and we have seen in the course of the proofs of both Lemma 19 and Lemma 20 that

$$\frac{2\langle \Lambda, \beta\rangle}{|\beta|^2} = \frac{2\langle 2\rho_{\text{red},c} - \rho_{\text{red}} + \rho, \beta\rangle}{|\beta|^2} - 2 + \frac{2\langle \mu, \beta\rangle}{|\beta|^2} .$$

On page 71, we proved the first term on the right is 2 for the kind of β under study. Thus $2\langle\mu,\beta\rangle/|\beta|^2$ must be an integer, contradiction.

Thus all the s_j are 0 for an equivalence class, or they are all $\pm\frac{1}{2}$. Clearly we can disregard any class for which they are all 0. Consider a class with all s_j equal to $\pm\frac{1}{2}$. The idea is to make $\langle\beta_i,\mu\rangle = 0$ if β_i' is of type (ii) and $\langle\beta_i,\mu\rangle > 0$ if β_i' is of type (i). If a class has some β_i' of type (i), we start our assignment of s_α's there and work toward the ends. If a class has no β_i' of type (i), we start our assignment of s_α's at one end and work toward the other, with one sign arbitrary.

The only difficulty that can occur is if there are two versions of β_i for a single β_i'. If this happens, one version will be

$$\beta_i = \varepsilon - \tfrac{1}{2}t_1\alpha_1 - \tfrac{1}{2}t_2\alpha_2$$

and the other must then be

$$\beta_i = \varepsilon + \tfrac{1}{2}t_1\alpha_1 + \tfrac{1}{2}t_2\alpha_2$$

since both must be compact. If one is orthogonal to μ, the other will be orthogonal to μ, so that this difficulty does not impede our definition of the reflection. This completes the proof.