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Theoretical Aspects of the Trace Formula for $GL(2)$

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The Selberg-Arthur trace formula is one of the tools available for approaching the conjecture of global functoriality in the Langlands program. Global functoriality is described within this volume in [Kn2]. We start with reductive groups G and H, say over the rationals $\mathbb{Q}$ for simplicity. We assume that G is quasisplit, and we suppose that we are given an L homomorphism $\psi : {}^LH \to {}^LG$. From an automorphic representation of the adeles of H, we use ψ to construct, place-by-place from the Local Langlands Conjecture (or at almost every place without the conjecture), an irreducible representation of the adeles of G. The question of global functoriality is whether the latter representation is automorphic (or, in the case that it is defined only at almost every place, whether it can be completed to an automorphic representation). If it is automorphic, then we want to know also what conditions ensure that a cuspidal representation of the adeles of H yields a cuspidal representation of the adeles of G under this process. It is known that these questions capture various deep conjectures in classical algebraic number theory, arithmetic algebraic geometry, and representation theory and that they unify and generalize such conjectures significantly.

The trace formula for the reductive group G gives information about the multiplicity of the occurrence of an irreducible representation of the adeles of G in the cuspidal spectrum. If Z denotes the center of G, the quotient $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ is almost compact in the sense that it has finite volume.[1] If $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ is actually compact and if R denotes the right regular representation of $G(\mathbb{A})$ on $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, then the trace formula will assert the equality of two expressions for $\mathrm{Tr}(R(\varphi))$ on this L^2 space, φ being a suitably regular function of compact support on $G(\mathbb{A})$. In the notation of [Ar4], the formula in the compact case has the shape

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\varphi) = \sum_{\chi\in\mathfrak{X}} J_{\chi}(\varphi), \tag{0.1}$$

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[1] In this paper we follow the standard convention that the group of $\mathbb{Q}$ points of any subgroup refers to the diagonally embedded version of that subgroup unless the contrary is explicitly indicated.

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in which the left side, called the **geometric side**, consists of terms that are integrals of φ over conjugacy classes, suitably normalized by volume factors. The right side, called the **spectral side**, is a sum of expressions $m_\pi \operatorname{Tr} \pi(\varphi)$, m_π being the multiplicity of an irreducible representation π in R.

When the quotient $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ is noncompact, $R(\varphi)$ is not of trace class on all of $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ but is of trace class on the cuspidal part. The computation of $\operatorname{Tr}(R_{\text{cusp}}(\varphi))$ is done with a "truncation parameter" T, $0 < T < \infty$, in place, and the result has the shape

$$\operatorname{Tr}(R_{\text{cusp}}(\varphi)) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(\varphi) - \sum_{\chi \in \mathfrak{X} - \mathfrak{X}(G)} J_\chi^T(\varphi), \tag{0.2}$$

with $\operatorname{Tr}(R_{\text{cusp}}(\varphi))$ being regarded as the sum of the terms $J_\chi^T(\varphi)$ with $\chi \in \mathfrak{X}(G)$, each of which is constant in T. The ingredients in (0.2) are more complicated than in (0.1): The set $\mathcal{O}$ now involves various kinds of conjugacy classes, and the terms J_χ^T involve Eisenstein series relative to proper parabolic subgroups of G. One can pass to the limit in (0.2) as $T \to +\infty$, taking into account various cancellations, and the result can be written in the qualitative form (0.1), but the interpretation of each side as a trace is lost.

In any event the trace formula does carry in it the multiplicity of each irreducible representation of the adeles of G in the cuspidal spectrum of the L^2 space, and the formula may therefore be expected to give some information toward answering the above functoriality question. In practice it is normally a comparison of the trace formulas for G and H that gives useful information, but this point will not concern us at this time.

In this paper we shall discuss aspects of the background and derivation of the trace formula for $G = GL_2$ when the number field is $\mathbb{Q}$, including a precise statement of the result. We shall treat also the case that G is a quaternion division algebra. Another article [Kn-Ro] in this volume gives some applications of the trace formula for various groups.

Although our interest in the trace formula will ultimately be in an adelic setting, it is helpful to keep in mind a certain classical setting, because the analysis there is more transparent and suggests approaches to the analysis in the adelic setting. Historically the trace formula was introduced by Selberg in [Se1] and [Se2]. Selberg worked initially in the context of a transitive group action on a Riemannian manifold in which the space of invariant differential operators is commutative, and he considered the analysis of the space of functions transforming suitably under a discrete subgroup that acts properly discontinuously. The case of the action of $SL_2(\mathbb{R})$ on the upper half plane, with $SL_2(\mathbb{Z})$ as the discrete subgroup, was of particular interest, and we may think in terms of an analysis of

$$L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})). \tag{0.3}$$

Let $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$. It is an elementary fact, which we prove as Theorem 1.3 below, that the right regular representation R of G on $L^2(\Gamma\backslash G)$ splits as an orthogonal direct sum

$$L^2(\Gamma\backslash G) = L^2_{\text{cusp}}(\Gamma\backslash G) \oplus L^2_{\text{cont}}(\Gamma\backslash G) \oplus \mathbb{C},$$

where the members of $L^2_{\text{cusp}}(\Gamma\backslash G)$ are functions satisfying a cuspidal condition of the kind discussed in [Kn2, §7] and where the members of $L^2_{\text{cont}}(\Gamma\backslash G)$ are essentially generated by summing the left translates by Γ of nice functions on G that have integral 0. The space $\mathbb{C}$ is the space of constant functions. The space $L^2_{\text{cont}}(\Gamma\backslash G)$ is the continuous part of the decomposition, and the known complete analysis of this space will be given in Theorem 1.4 and §2 in terms of Eisenstein series. Our analysis specializes the ones of Langlands ([Lgl1], [Lgl2], and [HC]); a different proof appears in [Go1]. See also [Lan1].

The space $L^2_{\text{cusp}}(\Gamma\backslash G)$ is the "cuspidal part" of the decomposition. It splits into a discrete sum of irreducible representations with finite multiplicities, as is shown in Theorem 1.5 and §3. Our proof specializes the one in [Go2].

Although the cuspidal part of the decomposition at first appears less complicated than the continuous part, little is known about what specific irreducible representations occur and what multiplicities they have. That is where the trace formula comes in. If φ is in $C^\infty_{\text{com}}(G)$, then the operator

$$R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy$$

is of trace class on $L^2_{\text{cusp}}(\Gamma\backslash G)$. The trace formula implies the equality of two expressions for the trace of $R(\varphi)$ on $L^2_{\text{cusp}}(\Gamma\backslash G)$. If

$$L^2_{\text{cusp}}(\Gamma\backslash G) = \bigoplus m_\pi \pi$$

is the decomposition into irreducible constituents with multiplicities, then one of the expressions for the trace is simply $\sum m_\pi \operatorname{Tr}\pi(\varphi)$. The other expression comes from realizing $R(\varphi)$ on $L^2_{\text{cusp}}(\Gamma\backslash G)$ as an integral operator on $\Gamma\backslash G$ and is the integral of the kernel of this operator over the diagonal; the trace works out to be a sum of terms encoding conjugacy class information about φ and spectral information about the action of $R(\varphi)$ on the noncuspidal part of $L^2(\Gamma\backslash G)$. The equality of the two expressions therefore gives information about multiplicities of irreducible representations in $L^2_{\text{cusp}}(\Gamma\backslash G)$ in terms of geometric information about G. We shall indicate in §4 what computation has to be made for the trace formula, but we shall omit an explicit statement of the formula in the context (0.3). See [He1], [He2], and [Ef] for a statement of this kind. For our purposes the trace formula is better understood in an adelic context, and we shall give in §7 a precise statement of that kind.

The trace formula in the classical setting does not lend itself to the kind of comparison of traces from different groups useful for global functoriality, but it does have some direct applications. One such is that it gives a formula for the trace of each Hecke operator on each space of classical cusp forms; the resulting theorem is called the Eichler-Selberg trace formula and is discussed in [Lan1] and [Mi, Ch. 6]. A degenerate case of this argument yields a proof of the dimension formula for spaces of classical cusp forms without appealing to the Riemann-Roch Theorem.

Let us now be more specific about the adelic context. The reductive group under study will largely be GL_2, and we regard it as defined over the rationals $\mathbb{Q}$. The places v of $\mathbb{Q}$ are ∞ and all the primes, and $\mathbb{Q}_v$ is correspondingly the field of reals $\mathbb{R}$ if $v = \infty$ and is the field of p-adics $\mathbb{Q}_p$ if v is a prime p. If the restricted direct product $\mathbb{A} = \prod_v \mathbb{Q}_v$ denotes the adeles of $\mathbb{Q}$, the problem of global functoriality typically leads one to representations of $GL_2(\mathbb{A}) = \prod_v GL_2(\mathbb{Q}_v)$ of the form $\pi = \prod_v \pi_v$ with π_v an irreducible admissible representation of $GL_2(\mathbb{Q}_v)$ for each v. Roughly speaking, π is automorphic if π is involved in analysis of the quotient $Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$, where $Z(\mathbb{A})$ denotes the subgroup of scalar matrices. More particularly, the question is likely to be whether π occurs in the cuspidal part of the discrete spectrum of

$$L^2(Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})). \tag{0.4}$$

The question is therefore answered by knowing whether the multiplicity of π in the cuspidal spectrum is zero or is positive, and the trace formula gives subtle information about this multiplicity.

As is noted in [Kn2, §6], the space (0.3) is a prototype for (0.4). The functions in $L^2(Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ that are invariant under the right action by $\prod_p GL_2(\mathbb{Z}_p)$ may be regarded as functions in $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}))$. Thus (0.3) may be analyzed by specializing results about (0.4) to results about (0.3). On the other hand, the techniques that are used in studying (0.3) often suggest techniques for studying (0.4).

The first people to consider the decomposition of the adelic setting (0.4) were Gelfand, Graev, and Piatetski-Shapiro in 1964, and an exposition is in [Gf-Gr-P]. Later expositions are the ones by Jacquet-Langlands [Ja-Lgl], Duflo-Labesse [Du-Lab], Gelbart [Gb1], Gelbart-Jacquet [Gb-Ja], Rogawski [Ro], and Gelbart [Gb2]. The treatment [Gb1] specializes work of Arthur [Ar1], and [Gb2] specializes later work of Arthur.

In §5 we obtain the trace formula for $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ when G is the multiplicative group of a quaternion division algebra over $\mathbb{Q}$. This space splits discretely with finite multiplicities and is considerably easier to understand than (0.4).

In §6 we give aspects of the decomposition of (0.4) into a continuous part and a discrete part, as well as aspects of the analysis of the continuous part using adelic Eisenstein series. The same section shows how some of the concepts used in studying (0.3) are adapted to yield an analysis of (0.4). For background material on adeles and automorphic representations, see [Kn2].

Finally in §7 we discuss the trace formula in the adelic setting (0.4). We relate aspects of Arthur's proof using truncation operators [Ar3], and we state the final formula and an important special case. The seven sections of this paper are thus as follows.

1. Overview of Decomposition of $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}))$
2. Decomposition of the Continuous Part
3. Discrete Decomposition of the Cuspidal Part
4. Introduction to the Trace Formula
5. Digression on Quaternion Algebras
6. Adelic Eisenstein Series
7. Adelic Trace Formula

Arthur has extended the theory of the trace formula well beyond GL_2. For the theorem in "$\mathbb{Q}$ rank one," see [Ar1], and for a theorem about general reductive G, see [Ar2] and [Ar3]. Labesse [Lab] gives a status report as of 1990, and Gelbart [Gb2] gives an exposition of Arthur's work.

1. Overview of Decomposition of $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}))$

We use the following notation: $G = SL(2,\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right\}$, $\Gamma_\infty = \Gamma \cap N$, $A = \left\{\begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix}\right\}$, and $K = SO(2)$. Let $L^2(\Gamma\backslash G)$ be the space of functions on G, up to equality almost everywhere, that are left invariant under Γ and are square integrable modulo Γ. We are interested in the decomposition of the regular representation of G on $L^2(\Gamma\backslash G)$. References are [Gb1], [Go1], [Go2], [HC], [Lgl1], and [Lgl2].

If H is a closed subgroup of G, let $\mathcal{D}(H\backslash G)$ be the space of complex-valued smooth functions on G that are compactly supported modulo H.

Lemma 1.1. *If ϕ is in $\mathcal{D}(N\backslash G)$, then the function $\widehat{\phi}$ defined by $\widehat{\phi}(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\phi(\gamma g)$ is in is in $\mathcal{D}(\Gamma\backslash G)$.*

Proof. Since $\Gamma_\infty\backslash N$ is compact, the support of ϕ is contained in $\Gamma_\infty C$ for some compact set $C\subset G$. Thus $\phi(\gamma g)\neq 0$ only for $\gamma g\in\Gamma_\infty C$. If g ranges through a compact set, then the γ's such that $\phi(\gamma g)\neq 0$ are those in a set $\Gamma_\infty C'$ with C' compact, and these form a finite subset of $\Gamma_\infty\backslash\Gamma$. Hence only finitely many terms in the sum defining $\widehat{\phi}$ contribute on any compact set of g's, and therefore $\widehat{\phi}$ is smooth. Finally the support of $\phi(\gamma\,\cdot\,)$ is contained in $\gamma^{-1}\Gamma_\infty C$, and the support of $\widehat{\phi}$ is contained in $\Gamma\Gamma_\infty C = \Gamma C$. The latter set is compact modulo Γ.

If F is any locally square integrable function on G that is left invariant under Γ_∞, we define the **constant term** of F to be the function F_0 on G given by

$$F_0(g) = \int_{\Gamma_\infty\backslash N} F(ng)\,d\dot{n}, \tag{1.1}$$

where $d\dot{n}$ has total mass 1. Since F is locally square integrable on G, Fubini's Theorem shows that $F(\,\cdot\,g)$ is locally square integrable on N for almost every g. Since $\Gamma_\infty\backslash N$ is compact, it follows for these g's that $F(\,\cdot\,g)$ is in $L^2(\Gamma_\infty\backslash N)$ and hence also is in $L^1(\Gamma_\infty\backslash N)$. Thus F_0 is defined almost everywhere.

The name "constant term" comes from the classical theory of modular forms. If the analytic function f on the upper half plane is a classical modular form of weight k relative to $SL_2(\mathbb{Z})$, then f has a Fourier expansion $f(z)=\sum_{n=0}^\infty c_n e^{2\pi i n z}$, and the constant term c_0 of this series is given by

$$c_0 = \int_{-1/2}^{1/2} f(x+iy)\,dx.$$

When f is lifted as in [Kn2, §7] to an automorphic form ϕ on G relative to Γ by means of the formula

$$\phi(g) = f(g(i))j(g,i)^{-k}, \tag{1.2}$$

in which $j(g,z) = cz+d$ when $g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, we find that the constant term ϕ_0 in the sense of (1.1) is given by $\phi_0(g) = c_0 j(g,i)^{-k}$.

Lemma 1.2. *Let ϕ be a measurable function on G left invariant under N, and let F be a measurable function on G left invariant under Γ. Define $\widehat{\phi}(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\phi(\gamma g)$. If $\widehat{|\phi|}$ and F are in $L^2(\Gamma\backslash G)$, then*

$$\langle\widehat{\phi},F\rangle_{L^2(\Gamma\backslash G)} = \langle\phi,F_0\rangle_{L^2(N\backslash G)}, \tag{1.3}$$

the indicated integrals converging.

PROOF. Formally we have

$$\langle\widehat{\phi}, F\rangle_{L^2(\Gamma\backslash G)} = \int_{\Gamma\backslash G}\sum_{\Gamma_\infty\backslash\Gamma}\phi(\gamma g)\overline{F(g)}\,d\dot{g} = \int_{\Gamma_\infty\backslash G}\phi(x)\overline{F(x)}\,d\dot{x}.$$

This computation is rigorous if ϕ and F are replaced by $|\phi|$ and $|F|$, and the hypotheses say that the left side is finite in this case. Then the right side is finite, and we see that the following continuation of the above computation is justified:

$$= \int_{N\backslash G}\int_{\Gamma_\infty\backslash N}\overline{F(ng)}\phi(g)\,d\dot{n}\,d\dot{g} = \langle\phi, F_0\rangle_{L^2(N\backslash G)}.$$

This completes the proof.

Lemma 1.1 implies that $\widehat{\phi}$ is in $L^2(\Gamma\backslash G)$ if ϕ is in $\mathcal{D}(N\backslash G)$. Using Lemma 1.2, we obtain a characterization of the closure of the subspace of all such $\widehat{\phi}$.

Theorem 1.3. *The space $L^2(\Gamma\backslash G)$ is the orthogonal direct sum of G invariant subspaces*

$$L^2(\Gamma\backslash G) = L^2_{\text{cusp}}(\Gamma\backslash G) \oplus L^2_{\text{cont}}(\Gamma\backslash G) \oplus \mathbb{C},$$

where $L^2_{\text{cusp}}(\Gamma\backslash G)$ is the subspace of functions whose constant terms are 0 almost everywhere on G, $L^2_{\text{cont}}(\Gamma\backslash G)$ is the closure of the subspace of all $\widehat{\phi}$ with $\phi \in \mathcal{D}(N\backslash G)$ of integral 0, and $\mathbb{C}$ is the space of constant functions.

PROOF. If F is in $L^2(\Gamma\backslash G)$ and ϕ is in $\mathcal{D}(N\backslash G)$, we shall use the formula (1.3) of Lemma 1.2. If F is in $L^2_{\text{cusp}}(\Gamma\backslash G)$, then $F_0 = 0$ almost everywhere, and (1.3) shows that $\widehat{\phi}$ is orthogonal to F. Conversely if $\widehat{\phi}$ is orthogonal to F for all ϕ, then (1.3) shows that F_0 is orthogonal to $\mathcal{D}(N\backslash G)$ and is 0 almost everywhere. Thus $L^2_{\text{cusp}}(\Gamma\backslash G)$ is the orthogonal complement of the closure of the subspace of all $\widehat{\phi}$.

Taking $F = 1$ in (1.3), we see that $L^2_{\text{cont}}(\Gamma\backslash G)$ is a closed invariant subspace of codimension 1 in the closure of the subspace of all $\widehat{\phi}$. Since G acts unitarily, the orthogonal complement of $L^2_{\text{cont}}(\Gamma\backslash G)$ is a G invariant one-dimensional subspace, necessarily $\mathbb{C}$. The theorem follows.

We shall now describe the representation of G on $L^2_{\text{cont}}(\Gamma\backslash G)$. The group G acts on the upper half plane by linear fractional transformations, with

$$g(z) = \frac{az+b}{cz+d} \qquad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.4}$$

Moreover,

$$\operatorname{Im} g(z) = \frac{\operatorname{Im} z}{|cz+d|^2}. \tag{1.5}$$

Let $G = NAK$ be the usual Iwasawa decomposition of G. We write the K component of $g \in G$ as $\kappa(g)$. If k is in K, then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} k(i) = x + iy. \tag{1.6}$$

Thus we can read off the N and A components of g from the real and imaginary parts of $g(i)$. We write $y(g) = \operatorname{Im} g(i)$ for the imaginary part. If $y > 0$, define

$a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$. Then $a(y(g))$ is the A component of g in the Iwasawa decomposition.

We need to normalize Haar measures. We normalize dn on N to be compatible with counting measure on Γ_∞ and the measure of total mass 1 on $\Gamma_\infty\backslash N$, we normalize da on A to correspond to $\dfrac{dy}{y}$ on $(0,\infty)$ when $y = y(a)$ and $a = a(y)$, and we normalize dk on K to have total mass 1. If $g = nak$ is the Iwasawa decomposition of an element $g \in G$, we define

$$dg = dn\, dk\, y(a)^{-1}\, \frac{dy}{y}. \tag{1.7}$$

Then dg is a Haar measure on G.

A function F on G or K will be called **even** if $F\left(x\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = F(x)$, **odd** if $F\left(x\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = -F(x)$. For s complex, let $P^{+,s}$ be the spherical principal series representation of G defined as follows: $P^{+,s}$ acts initially in the space

$$\{\text{even } F \in C^\infty(G) \mid F(nag) = y(a)^{\frac{1}{2}(1+s)} F(g)\} \tag{1.8}$$

by the right regular representation with norm squared given by $\int_K |F(k)|^2\, dk$, and then it is completed to a representation in a Hilbert space. The subspace of C^∞ vectors is exactly (1.8), and the representation is unitary if $\operatorname{Re} s = 0$. If f is a smooth even function on K, then f extends to a member f_s of (1.8) by the rule

$$f_s(nak) = y(a)^{\frac{1}{2}(1+s)} f(k).$$

For each t, $P^{+,it}$ is irreducible and is unitarily equivalent with $P^{+,-it}$. Thus there exists a unique-up-to-scalar bounded linear operator intertwining $P^{+,it}$ and $P^{+,-it}$. We denote a particular normalization of this operator by $M(t)$; $M(t)$ will be defined explicitly in (2.13), and it will be unitary with $M(-it)$ as inverse.

We shall describe a certain direct integral of the unitary representations $P^{+,it}$. The underlying Hilbert space, which is denoted $\widehat{L}^2(E)$, is the set of measurable functions

$$F : i\mathbb{R} \to \{\text{even functions in } L^2(K)\}$$

(modulo null functions) such that

$$M(it)F(it) = F(-it)$$

and such that the expression

$$\|F\|^2_{\widehat{L}^2(E)} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \|F(it)\|^2_{L^2(K)}\, dt$$

is finite. We make this into a representation space for G by having $P^{+,it}$ act on $F(it)_{it}$. More concretely, if U is to be the representation, we let

$$(U(g)F)(it) = \big(P^{+,it}(g)(F(it)_{it})\big)|_K.$$

The main theorem about $L^2_{\text{cont}}(\Gamma\backslash G)$ is as follows.

Theorem 1.4. *There exists a G equivariant unitary mapping E of $L^2_{\text{cont}}(\Gamma\backslash G)$ onto $\widehat{L}^2(E)$.*

This theorem will be proved in §2 by constructing the mapping E explicitly with the aid of Eisenstein series.

We come to the representation of G on $L^2_{\text{cusp}}(\Gamma\backslash G)$. Knowledge of how this representation decomposes remains far from complete. But we can say the following.

Theorem 1.5. *$L^2_{\text{cusp}}(\Gamma\backslash G)$ is the orthogonal Hilbert-space direct sum of irreducible representations, each occurring with finite multiplicity.*

The tool for proving Theorem 1.5 is Theorem 1.6 below, which will be proved in §3. Let φ be in $\mathcal{D}(G)$, and define a bounded operator $R(\varphi)$ on $L^2(\Gamma\backslash G)$ by $R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy$. This carries any closed G invariant subspace of $L^2(\Gamma\backslash G)$ into itself.

Theorem 1.6. *For each φ in $\mathcal{D}(G)$, the operator*

$$R(\varphi) : L^2_{\text{cusp}}(\Gamma\backslash G) \to L^2_{\text{cusp}}(\Gamma\backslash G)$$

is Hilbert-Schmidt, hence compact.

Proof that Theorem 1.6 implies Theorem 1.5. In order to obtain the discrete decomposition into irreducible closed invariant subspaces, it is enough, by Zorn's Lemma, to prove that any nonzero invariant closed subspace S of $L^2_{\text{cusp}}(\Gamma\backslash G)$ contains an irreducible invariant subspace. The operator $R(\varphi)$ is self adjoint on S if $\varphi(x^{-1}) = \overline{\varphi(x)}$, and it is nonzero if φ is nonzero and φ is supported in a sufficiently small neighborhood of the identity. By Theorem 1.6 it is compact. Therefore it has a nonzero eigenvalue λ, and that eigenvalue has finite multiplicity. Let f be a nonzero eigenvector belonging to λ, and let λ have multiplicity n. Let T be the closed invariant subspace generated by f. If T is the orthogonal sum of $n+1$ closed invariant subspaces and if $P_1, \ldots, P_{n+1}$ are the orthogonal projections, then $R(\varphi)$ has eigenvalue λ on the independent vectors $P_1 f, \ldots, P_{n+1} f$, contradiction. It follows that T decomposes fully into at most n irreducible closed invariant subspaces. Any one of these subspaces is the required irreducible subspace of S.

Thus we can write $L^2_{\text{cusp}}(\Gamma\backslash G)$ as the orthogonal Hilbert-space direct sum of irreducible subspaces. Let S be such a subspace. As in the previous paragraph, we can choose φ with $\varphi(x^{-1}) = \overline{\varphi(x)}$ so that $R(\varphi)$ is nonzero on S. Since $R(\varphi)$ is compact self adjoint on S, $R(\varphi)$ has a nonzero eigenvalue λ on a nonzero subspace of S. On each irreducible summand of $L^2_{\text{cusp}}(\Gamma\backslash G)$ that is equivalent with S, $R(\varphi)$ must act with λ as an eigenvalue on the corresponding subspace. If S occurs with infinite multiplicity, then λ occurs with infinite multiplicity as an eigenvalue of $R(\varphi)$. But this contradicts the compactness of the self adjoint operator $R(\varphi)$ on $L^2_{\text{cusp}}(\Gamma\backslash G)$.

2. Decomposition of the Continuous Part

In this section we shall prove Theorem 1.4, giving an explicit decomposition of $L^2_{\text{cont}}(\Gamma\backslash G)$ when $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$. We continue with notation as in §1. We shall proceed somewhat along the lines of Appendix IV of [Lgl2] and then [Gb1]. For a different argument leading to a conclusion that is stated differently, see [Go1]. The technique of proof will involve Eisenstein series, which we now introduce.

If f is an even function in $C^\infty(K)$, we recall that $f_s : G \to \mathbb{C}$ is defined for $s \in \mathbb{C}$ by

$$f_s(nak) = y(a)^{\frac{1}{2}(1+s)} f(k) \tag{2.1}$$

when $n \in N$, $a \in A$, and $k \in K$. This satisfies the functional equation

$$f_s(nag) = y(a)^{\frac{1}{2}(1+s)} f_s(g)$$

and hence is a member of the representation space for the spherical principal series $P^{+,s}$.

Fix a finite-dimensional representation τ of K, and let $W(\tau)$ denote the space of complex-valued even functions on K with the property that $k \mapsto f(k_0 k)$, for each $k_0 \in K$, is a linear combination of matrix coefficients of the constituents of τ. If f is in $W(\tau)$, the corresponding **Eisenstein series** $E(g, f, s)$ is defined formally by

$$E(g, f, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f_s(\gamma g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma g)^{\frac{1}{2}(1+s)} f(\kappa(\gamma g)) \tag{2.2}$$

for $g \in G$ and $s \in \mathbb{C}$.

We can understand $\Gamma_\infty \backslash \Gamma$ with the help of the right action of G on row vectors. In this action the orbit under Γ of the row vector $(\,0 \quad 1\,)$ is all row vectors $(\,c \quad d\,)$ with c and d integers such that $\mathrm{GCD}(c, d) = 1$. The isotropy subgroup at $(\,0 \quad 1\,)$ is Γ_∞, and thus $\Gamma_\infty \backslash \Gamma$ may be identified with the set of relatively prime pairs (c, d). Evidently if $\Gamma_\infty \gamma$ corresponds to (c, d), then c and d form the bottom row of γ.

For an example let us take $\tau = 1$ and $f = 1$. If we put $g(i) = z = x + iy$, then (1.5) shows that (2.2) becomes

$$E(g, 1, s) = \sum_{\mathrm{GCD}(c,d)=1} \frac{y^{\frac{1}{2}(1+s)}}{|cz + d|^{1+s}}. \tag{2.3a}$$

Taking into account that every nonzero (m, n) in $\mathbb{Z}^2$ is uniquely the product of a positive integer and a relatively prime pair, we obtain

$$\zeta(1 + s) E(g, 1, s) = \sum_{(m,n) \neq (0,0)} \frac{y^{\frac{1}{2}(1+s)}}{|mz + n|^{1+s}}, \tag{2.3b}$$

where $\zeta(\,\cdot\,)$ is the Riemann ζ function.

The original Eisenstein series historically were series of the form

$$\sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k}, \tag{2.4}$$

as well as certain variants. The series is absolutely convergent if $k > 2$. In order to make sense out of the series (2.4) when $k = 2$, Hecke considered the analytic continuation in s of expressions of the form

$$\sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k |mz + n|^{2s}}. \tag{2.5}$$

In [Mi] these are called "Eisenstein series with parameter s," and (2.3b) is an instance of (2.5). If we take τ to be a nontrivial character of K and reinterpret

$E(g, f, s)$ on the upper half plane by reversing the formula (1.2) for lifting modular forms to G, we obtain the other instances of (2.5).

Lemma 2.1. *$E(g, f, s)$ is absolutely convergent for $\operatorname{Re} s > 1$, and the convergence is uniform for g and s in compact sets.*

PROOF. It is enough to estimate $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma g)^{\frac{1}{2}(1+\operatorname{Re} s)}$. This is written explicitly in (2.3a), and the larger series in (2.3b) is known to converge for $\operatorname{Re} s > 1$.

Lemma 2.2. *For any $\varepsilon > 0$, there is a constant C_ε such that*

$$|E(g, f, s)| \leq C_\varepsilon (\sup_K |f|) y(g)^{\frac{1}{2}(1+\operatorname{Re} s)}$$

whenever $y(g) \geq \frac{1}{2}$ and $1 + \varepsilon \leq \operatorname{Re} s \leq 1 + \varepsilon^{-1}$.

PROOF. Without loss of generality, we may take $f = 1$ on K. Write $z = x+iy = g(i)$ and $\sigma = \operatorname{Re} s$. Applying (2.3a), we see that we are to estimate

$$\sum_{\mathrm{GCD}(c,d)=1} \frac{y^{\frac{1}{2}(1+\sigma)}}{((cx+d)^2 + c^2y^2)^{\frac{1}{2}(1+\sigma)}}.$$

So it is enough to show that

$$\sum_{(c,d)\neq(0,0)} ((cx+d)^2 + c^2y^2)^{-\frac{1}{2}(1+\sigma)} \tag{2.6}$$

is bounded above for $y \geq \frac{1}{2}$ and $1 + \varepsilon \leq \sigma \leq 1 + \varepsilon^{-1}$.

Fix $c \neq 0$. At most two d's give $|cx + d| < 1$. The contribution to (2.6) from such pairs (c, d) is therefore $\leq \sum_{c\neq 0} 2c^{-(1+\sigma)} y^{-(1+\sigma)} \leq C_{1,\varepsilon} y^{-(1+\sigma)}$.

For the remaining terms, we can replace $cx + d$ by the nonzero integer

$$\operatorname{sgn}(cx + d)[|cx + d|].$$

Then the contribution to (2.6) from the remaining terms is

$$\leq \sum_{\substack{(c,n),\\ n\neq 0}} \frac{1}{(n^2 + c^2y^2)^{\frac{1}{2}(1+\sigma)}} \leq 2^{1+\sigma} \sum_{\substack{(c,n),\\ n\neq 0}} \frac{1}{(4n^2 + c^2)^{\frac{1}{2}(1+\sigma)}},$$

and the result follows.

An **automorphic form** on G relative to Γ is a smooth function f with the following properties:

(a) $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$
(b) f is right K finite
(c) f is $Z(\mathfrak{g})$ finite, where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra of the complexified Lie algebra of G
(d) f satisfies the slow growth condition $|f(g)| \leq Cy(g)^N$ for some C and N and all g with $y(g) \geq \frac{1}{2}$.

(See [Kn2, §7] and [Gb1, p. 28].)

Proposition 2.3. *For any $f \in W(\tau)$, $E(\,\cdot\,, f, s)$ is an automorphic form on G relative to Γ if $\operatorname{Re} s > 1$.*

PROOF. Properties (a) and (b) are clear from the definitions, and (d) follows from Lemma 2.2. For (c), we observe that the function $g \mapsto f(\kappa(g))y(g)^{\frac{1}{2}(1+s)}$ is in the space of the principal series $P^{+,s}$. The Casimir operator Ω acts in $P^{+,s}$ by a scalar $c(s)$ depending on s. Since Ω is central, it acts on every term of (2.2) by $c(s)$, and it acts on $E(\,\cdot\,, f, s)$ by $c(s)$. The element Ω generates $Z(\mathfrak{g})$, and (c) follows.

Although $E(\,\cdot\,, f, s)$ is an automorphic form, it need not be in $L^2(\Gamma\backslash G)$. In fact, let us check that $E(\,\cdot\,, f, s)$ is not in $L^2(\Gamma\backslash G)$ if $f = 1$ and s is real. In this case we can see that $E(g, 1, s)$ is bounded below, as well as above, by a multiple of $y(g)^{\frac{1}{2}(1+s)}$. The invariant measure on $\Gamma\backslash G$ amounts to $y^{-2}\,dx\,dy$ on the standard fundamental domain

$$S = \left\{ z \,\middle|\, \operatorname{Im} z > 0, |z| \geq 1, |\operatorname{Re} z| \leq \tfrac{1}{2} \right\}$$

for Γ, and the integral of $|E(g, 1, s)|^2$ is of the order of $\int_{y=1}^{\infty} \int_{x=-1/2}^{1/2} y^{s-1}\,dx\,dy$, which is infinite for $s > 1$ (not to mention $s > 0$).

Although an individual $E(\,\cdot\,, f, s)$ is not in L^2, it turns out that suitable averages in the s variable are in L^2. Here is the construction.

Let $\mathcal{D}(N\backslash G, \tau)$ be the subspace of all $\phi \in \mathcal{D}(N\backslash G)$ such that $k \mapsto \phi(gk)$ is in $W(\tau)$ for each $g \in G$. For $\phi \in \mathcal{D}(N\backslash G, \tau)$, define the **Fourier-Laplace transform** of ϕ by

$$\Phi(g, s) = \int_0^{\infty} \phi(a(y)^{-1}g)y^{\frac{1}{2}(1+s)}\,\frac{dy}{y}. \tag{2.7}$$

This function satisfies

$$\Phi(nag, s) = y(a)^{\frac{1}{2}(1+s)}\Phi(g, s). \tag{2.8}$$

If we write $s = \sigma + it$ and $y = e^{2x}$, then we have

$$\Phi(g, \sigma + it) = \int_{-\infty}^{\infty} 2\phi(a(e^{2x})^{-1}g)e^{x(1+\sigma+it)}\,dx,$$

and Fourier inversion gives

$$2\phi(a(e^{2x})^{-1}g)e^{x(1+\sigma)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(g, \sigma + it)e^{-ixt}\,dt.$$

Taking $x = 0$ thus shows that

$$\phi(g) = \frac{1}{4\pi}\int_{\operatorname{Re} s = \sigma} \Phi(g, s)\,d|s| = \frac{1}{4\pi}\int_{\operatorname{Re} s = \sigma} y(g)^{\frac{1}{2}(1+s)}\Phi(\kappa(g), s)\,d|s|. \tag{2.9}$$

As a function of s, $\Phi(g, s)$ is a Schwartz function of $\operatorname{Im} s$ uniformly in any vertical strip of s and any compact set of g. The restriction $\Phi|_{K\times\{s\}}$ is a member of $W(\tau)$ for each s, and we shall usually abbreviate $\Phi|_{K\times\{s\}}$ as $\Phi(s)$.

Recall from Lemma 1.1 that the function

$$\widehat{\phi}(g) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \phi(\gamma g)$$

is in $\mathcal{D}(\Gamma\backslash G)$. Substituting from (2.9), we obtain

$$\widehat{\phi}(g) = \frac{1}{4\pi}\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \Big(\int_{\operatorname{Re} s = \sigma} y(\gamma g)^{\frac{1}{2}(1+s)}\Phi(\kappa(\gamma g), s)\Big)\,d|s|.$$

By Lemma 2.2, $\sum_\gamma |y(\gamma g)^{\frac{1}{2}(1+s)}|$ is bounded as a function of $\operatorname{Im} s$, and $\Phi(\kappa(\gamma g), s)$ is a Schwartz function of $\operatorname{Im} s$. Therefore the expression for $\widehat{\phi}(g)$ converges with absolute values inserted, and the sum and integral may be interchanged. The result is that

$$\widehat{\phi}(g) = \frac{1}{4\pi} \int_{\operatorname{Re} s=\sigma} E(g, \Phi(s), s)\, d|s|. \tag{2.10}$$

It is in this sense that suitable averages of Eisenstein series are in $L^2(\Gamma\backslash G)$.

Now we identify the constant term of an Eisenstein series. Recall from §1 that constant terms are indicated by a subscript 0. Let w denote the matrix $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 2.4. *For* $\operatorname{Re} s > 0$ *and for even functions* $f \in C^\infty(K)$*, the integral* $\int_N f_s(wng)\, dn$ *is convergent, and the formula*

$$A(s)f(g) = \int_N f_s(wng)\, dn \qquad \text{for } g \in G \tag{2.11}$$

defines a G *intertwining operator* $A(s) : P^{+,s} \to P^{+,-s}$*. As an operator from the space of even functions in* $C^\infty(K)$ *to itself,* $A(s)$ *has the following properties:*

(a) *it varies analytically in* s
(b) *it is uniformly bounded for* $\operatorname{Re} s \geq 1 + \varepsilon$
(c) *its adjoint relative to* $L^2(K)$ *is* $A(\bar{s})$.

Reference. This result is elementary, and $A(s)$ is known as a **standard intertwining operator**. See Donley's lecture [Do], Moeglin's lecture [Mo], and also [Kn1, Ch. VII].

Since $A(s)$ is a G intertwining operator, it is in particular a K intertwining operator and therefore carries $W(\tau)$ to itself.

Lemma 2.5. *As an operator from* $W(\tau)$ *to itself, the operator* $A(s)$*, initially defined for* $\operatorname{Re} s > 0$*, continues to a meromorphic function of* $s \in \mathbb{C}$*. The continued family of operators has the following properties:*

(a) *the only possible poles are at* $s = 0, -2, -4, \ldots$ *and are simple*
(b) *for* $f \in W(\tau)$*,* $A(s)f$ *vanishes at* $s = 1$ *if* τ *does not contain the trivial representation of* K
(c) *apart from the poles,* $A(s)$ *is of at most polynomial growth in* $\operatorname{Im} s$ *in any vertical strip*
(d) *the operator* $A(-s)A(s)$ *is a meromorphic scalar depending on* s.

Reference. This result is more subtle than Lemma 2.4 but is still not difficult. See [Do], [Mo], and also [Kn1, Ch. VII].

Proposition 2.6. *If* $\operatorname{Re} s > 1$ *and if* f *is in* $W(\tau)$*, then the constant term of the Eisenstein series for* f *is given by*

$$E_0(\,\cdot\,, f, s) = 2f_s + 2(M(s)f)_{-s}, \tag{2.12}$$

where $M(s)$ *is the operator*

$$M(s) = \frac{\zeta(s)}{\zeta(1+s)}\, A(s), \tag{2.13}$$

$A(s)$ being the operator in Lemma 2.4 and ζ being the Riemann ζ function. Here

(a) *$M(s)$ is analytic for $\operatorname{Re} s \geq 0$ except at $s = 1$, where it has at most a simple pole*
(b) *$M(s)$ is analytic at $s = 1$ if τ does not contain the trivial representation of K*
(c) *the residue of $M(s)$ at $s = 1$ is $6/\pi$ if $\tau = 1$*
(d) *the adjoint of $M(s)$ relative to the $L^2(K)$ norm on $W(\tau)$ is $M(\bar{s})$*
(e) *apart from the possible pole at $s = 1$, $M(s)$ is of at most polynomial growth in $\operatorname{Im} s$ uniformly for $0 \leq \operatorname{Re} s \leq \sigma$*
(f) *$M(-s)M(s) = 1$ as an identity of meromorphic functions.*

Remark. Lemma 101 of [HC] shows in (e) that $M(s)$ is actually uniformly bounded in this strip, apart from the pole.

Proof. Let H be the diagonal subgroup of G. We have seen that the coset of γ in $\Gamma_\infty \backslash \Gamma$ is characterized by the relatively prime pair (c, d) of entries of its bottom row. If $c = 0$, we obtain the cosets of ± 1. When $c \neq 0$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ may be uniquely decomposed according to $NHwN$ as

$$\gamma = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}.$$

Then

$$\Gamma_\infty \gamma = \begin{pmatrix} 1 & \frac{a}{c} + \mathbb{Z} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix},$$

and the member $\nu = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ of Γ_∞ has

$$\Gamma_\infty \gamma \nu = \begin{pmatrix} 1 & \frac{a}{c} + \mathbb{Z} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} + x \\ 0 & 1 \end{pmatrix}.$$

Thus we see that all the cosets $\Gamma_\infty \gamma \nu$, as ν varies, are distinct and that the number of double cosets $\Gamma_\infty \gamma \Gamma_\infty$ corresponding to a given c is $\varphi(|c|)$, where φ is Euler's φ function.

We compute

$$E_0(g, f, s) = \int_{\Gamma_\infty \backslash N} E(ng, f, s)\, d\dot{n} = \sum_{\Gamma_\infty \backslash \Gamma} \int_{\Gamma_\infty \backslash N} f_s(\gamma n g)\, d\dot{n}$$

by separating the terms $\gamma = \pm 1$ from the terms with $\gamma \in NHwN$. If we write $\gamma = \gamma(c, d)$, this expression is

$$\begin{aligned} &= 2 \int_{\Gamma_\infty \backslash N} f_s(ng)\, d\dot{n} + \sum_{\substack{\gamma(c,d) \in \Gamma_\infty \backslash \Gamma, \\ c \neq 0}} \int_{\Gamma_\infty \backslash N} f_s(\gamma(c, d) n g)\, d\dot{n} \\ &= 2 f_s(g) + \mathrm{II}, \end{aligned}$$

where

$$\begin{aligned}
\mathrm{II} &= \sum_{\substack{c\neq 0,\\ \mathrm{GCD}(c,d)=1}} \int_{\Gamma_\infty\backslash N} f_s(\gamma(c,d)ng)\,d\dot{n} \\
&= \sum_{c\neq 0} \sum_{\substack{d \bmod c,\\ \mathrm{GCD}(c,d)=1}} \sum_{k=-\infty}^{\infty} \int_{\Gamma_\infty\backslash N} f_s(\gamma(c,d+ck)ng)\,d\dot{n} \\
&= \sum_{c\neq 0} \sum_{\substack{d \bmod c,\\ \mathrm{GCD}(c,d)=1}} \sum_{\nu\in\Gamma_\infty} \int_{\Gamma_\infty\backslash N} f_s(\gamma(c,d)\nu ng)\,d\dot{n} \\
&= \sum_{c\neq 0} \sum_{\substack{d \bmod c,\\ \mathrm{GCD}(c,d)=1}} \int_{N} f_s(\gamma(c,d)ng)\,dn.
\end{aligned}$$

Write $\gamma(c,d) \in NHwN$ as $\gamma(c,d) = n'(c,d)h(c)wn''(c,d)$, noting that $h(c) = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$, independently of d. Then the above expression is

$$\begin{aligned}
&= \sum_{c\neq 0} \sum_{\substack{d \bmod c,\\ \mathrm{GCD}(c,d)=1}} \int_{N} f_s(n'(c,d)h(c)wn''(c,d)ng)\,dn \\
&= \sum_{c\neq 0} \sum_{\substack{d \bmod c,\\ \mathrm{GCD}(c,d)=1}} \int_{N} f_s(h(c)wng)\,dn
\end{aligned}$$

by the change of variables $n''(c,d)n \mapsto n$. In turn this is

$$= 2\sum_{c=1}^{\infty} \varphi(c)c^{-(1+s)} \int_N f_s(wng)\,dn.$$

Easy computation using Euler products shows that $\sum_{c=1}^{\infty} \varphi(c)c^{-(1+s)} = \dfrac{\zeta(s)}{\zeta(1+s)}$. Therefore

$$\mathrm{II} = \frac{2\zeta(s)}{\zeta(1+s)} \int_N f_s(wng)\,dn = \frac{2\zeta(s)}{\zeta(1+s)}\,(A(s)f)_{-s}(g)$$

in the notation of Lemma 2.4, and we conclude that

$$\tfrac{1}{2}E_0(g,f,s) = f_s(g) + \frac{\zeta(s)}{\zeta(1+s)}\,(A(s)f)_{-s}(g).$$

This proves (2.12) with $M(s)$ as in (2.13).

Conclusions (a) and (b) are immediate from Lemma 2.5, and (d) is immediate from Lemma 2.4c, (2.13), and analytic continuation. Before proving (c), we need an identity. The operator $A(s)$ carries 1_s to a multiple of 1_{-s} since $A(s)$ carries $W(1) = \mathbb{C}$ to itself. To compute the multiple, we calculate

$$(A(s)1)_{-s}(1) = \int_N 1_s(wn)\,dn = \int_N y(wn)^{\frac{1}{2}(1+s)}\,dn.$$

The measure dn is to be normalized consistently with the measure of total mass 1 on $\Gamma_\infty \backslash N$ and the counting measure on Γ_∞. Thus if $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, then dn is Lebesgue measure dx. Since

$$wn(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} i = \frac{-1}{i+x} = \frac{-x+i}{x^2+1},$$

we have $y(wn) = \operatorname{Im} wn(i) = (x^2+1)^{-1}$. Thus

$$(A(s)1)_{-s}(1) = \int_{-\infty}^{\infty} (x^2+1)^{-\frac{1}{2}(1+s)}\, dx.$$

Consequently a trick of Euler's yields

$$\begin{aligned} \Gamma(\tfrac{1}{2}(1+s))(x^2+1)^{-\frac{1}{2}(1+s)} &= \int_0^\infty (x^2+1)^{-\frac{1}{2}(1+s)} t^{\frac{1}{2}(1+s)} e^{-t}\, \frac{dt}{t} \\ &= \int_0^\infty t^{\frac{1}{2}(1+s)} e^{-t(x^2+1)}\, \frac{dt}{t} = \int_0^\infty t^{\frac{1}{2}(1+s)} e^{-t} e^{-tx^2}\, \frac{dt}{t} \end{aligned}$$

and then

$$\begin{aligned} \Gamma(\tfrac{1}{2}(1+s)) \int_{-\infty}^{\infty} (x^2+1)^{-\frac{1}{2}(1+s)}\, dx &= \int_0^\infty t^{\frac{1}{2}(1+s)} e^{-t} \Big(\int_{-\infty}^{\infty} e^{-tx^2}\, dx \Big) \frac{dt}{t} \\ &= \int_0^\infty t^{\frac{1}{2}(1+s)} e^{-t} \Big(\int_{-\infty}^{\infty} e^{-\pi r^2} \sqrt{\frac{\pi}{t}}\, dr \Big) \frac{dt}{t} \\ &= \sqrt{\pi} \int_0^\infty t^{s/2} e^{-t}\, \frac{dt}{t} \\ &= \sqrt{\pi}\, \Gamma(\tfrac{s}{2}). \end{aligned}$$

Hence

$$(A(s)1)_{-s}(1) = \int_{-\infty}^{\infty} (x^2+1)^{-\frac{1}{2}(1+s)}\, dx = \frac{\sqrt{\pi}\, \Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2}(1+s))}. \tag{2.14}$$

To prove (c), we use (2.13) and (2.14) to write

$$(M(s)1)_{-s} = \frac{\zeta(s)}{\zeta(1+s)} (A(s)1)_{-s} = \frac{\Lambda(s)}{\Lambda(1+s)}\, 1_{-s}, \tag{2.15}$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Therefore

$$\operatorname{Res}_{s=1}\big\{(M(s)1)_{-s}\big\} = \frac{\operatorname{Res}_{s=1}\big\{\Lambda(s)\big\}}{\Lambda(2)} = \frac{\pi^{-1/2}\Gamma(\frac{1}{2})\operatorname{Res}_{s=1}\big\{\zeta(s)\big\}}{\pi/6} = \frac{6}{\pi}.$$

For (f), we combine Lemma 2.5d, (2.13), and (2.15) to obtain

$$M(-s)M(s) = \frac{\Lambda(s)}{\Lambda(1-s)}\, \frac{\Lambda(-s)}{\Lambda(1+s)},$$

and (f) follows from the functional equation $\Lambda(1-s) = \Lambda(s)$ of the ζ function.

Finally to prove (e), we use (2.13). Lemma 2.5c tells us that $A(s)$ is of at most polynomial growth in $\operatorname{Im} s$, apart from the pole at $s = 0$, for $0 \le \operatorname{Re} s \le \sigma$. Also $\zeta(s)$ is bounded in any vertical strip, apart from its pole. And $|\zeta(1+s)|^{-1}$ is known to be at most polynomial growth in $\operatorname{Im} s$ uniformly for $0 \le \operatorname{Re} s \le \sigma$; see [Ti, p. 44]. Thus (e) follows.

Corollary 2.7. *Let ϕ and ψ be members of $\mathcal{D}(N\backslash G,\tau)$, and let Φ and Ψ be the Fourier-Laplace transforms of ϕ and ψ. Then*

$$\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(\Gamma\backslash G)}=\frac{1}{2\pi}\int_{\operatorname{Re}s=\sigma}\big(\langle\Phi(s),\Psi(-\bar{s})\rangle_{L^2(K)}+\langle M(s)\Phi(s),\Psi(\bar{s})\rangle_{L^2(K)}\big)\,d|s|$$

for any $\sigma>1$.

Proof. By (2.10), we have

$$\widehat{\phi}(g)=\frac{1}{4\pi}\int_{\operatorname{Re}s=\sigma}E(g,\Phi(s),s)\,d|s|.$$

Taking the constant term of both sides and applying Proposition 2.6, we obtain

$$\begin{aligned}\widehat{\phi}_0(g)&=\frac{1}{4\pi}\int_{\operatorname{Re}s=\sigma}E_0(g,\Phi(s),s)\,d|s|\\&=\frac{1}{2\pi}\int_{\operatorname{Re}s=\sigma}\big(\Phi(s)_s(g)+(M(s)\Phi(s))_{-s}(g)\big)\,d|s|.\end{aligned}$$

If we write $g=na(y)k$, then Haar measure dg decomposes as $y^{-1}\,dn\,dk\,\frac{dy}{y}$, according to (1.7). Thus the invariant measure on $N\backslash G$ is $y^{-1}\,dk\,\frac{dy}{y}$. Lemma 1.2 therefore gives

$$\begin{aligned}&\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(\Gamma\backslash G)}\\&\quad=\langle\widehat{\phi}_0,\psi\rangle_{L^2(N\backslash G)}\\&\quad=\frac{1}{2\pi}\int_{N\backslash G}\int_{\operatorname{Re}s=\sigma}\big(\Phi(s)_s(g)+(M(s)\Phi(s))_{-s}(g)\big)\overline{\psi(g)}\,d|s|\,d\dot{g}\\&\quad=\frac{1}{2\pi}\int_{N\backslash G}\int_{\operatorname{Re}s=\sigma}\big(y(g)^{\frac{1}{2}(1+s)}\Phi(\kappa(g),s)+y(g)^{\frac{1}{2}(1-s)}(M(s)\Phi(s))(\kappa(g))\big)\\&\qquad\qquad\times\overline{\psi(g)}\,d|s|\,d\dot{g}\\&\quad=\frac{1}{2\pi}\int_{\operatorname{Re}s=\sigma}\int_K\int_0^\infty\big(y^{\frac{1}{2}(1+s)}\Phi(k,s)+y^{\frac{1}{2}(1-s)}(M(s)\Phi(s))(k)\big)\\&\qquad\qquad\times\overline{\psi(a(y)k)}y^{-1}\,\frac{dy}{y}\,dk\,d|s|\\&\quad=\frac{1}{2\pi}\int_{\operatorname{Re}s=\sigma}\big(\langle\Phi(s),\Psi(-\bar{s})\rangle_{L^2(K)}+\langle M(s)\Phi(s),\Psi(\bar{s})\rangle_{L^2(K)}\big)\,d|s|.\end{aligned}$$

This completes the proof.

Now we move the line of integration in Corollary 2.7 to $\operatorname{Re}s=0$. The integrand is meromorphic, the functions $\Phi(s)$ and $\Psi(s)$ are Schwartz functions of $\operatorname{Im}z$ uniformly in vertical strips, and the growth of $M(s)$ is controlled by Proposition 2.6e. Thus we can move the line of integration by the Cauchy Integral Formula, picking up a residue term from $s=1$. The result is

$$\begin{aligned}\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(\Gamma\backslash G)}=\frac{1}{2\pi}&\int_{-\infty}^{\infty}\big(\langle\Phi(it),\Psi(it)\rangle_{L^2(K)}+\langle M(it)\Phi(it),\Psi(-it)\rangle_{L^2(K)}\big)\,dt\\&+\operatorname{Res}_{s=1}\big\{\langle M(s)\Phi(s),\Psi(\bar{s})\rangle_{L^2(K)}\big\}.\end{aligned}\tag{2.16}$$

By Proposition 2.6, parts (b) and (e), the second term is

$$\operatorname{Res}_{s=1}\big\{\langle M(s)\Phi(s),\Psi(\bar{s})\rangle_{L^2(K)}\big\} = \begin{cases} \frac{6}{\pi}\langle \Phi(1),\Psi(1)\rangle_{L^2(K)} & \text{if } \tau = 1 \\ 0 & \text{if 1 is not in } \tau. \end{cases}$$

We can simplify the right side of the residue term since

$$\Phi(k,1) = \int_0^\infty \phi(a(y)^{-1}k)\,dy = \int_A \phi(ak)y(a)^{-1}\,da.$$

When $\tau = 1$, this expression is constant in k and yields $\int_{N\backslash G}\phi(g)\,d\dot{g}$. When 1 is not in τ, the integral of this expression over $k \in K$ is 0. We conclude that

$$\operatorname{Res}_{s=1}\big\{\langle M(s)\Phi(s),\Psi(\bar{s})\rangle_{L^2(K)}\big\} = \frac{6}{\pi}\Big(\int_{N\backslash G}\phi(g)\,d\dot{g}\Big)\overline{\Big(\int_{N\backslash G}\psi(g)\,d\dot{g}\Big)} \qquad (2.17)$$

for all τ.

Corollary 2.8. *Let ϕ and ψ be members of $\mathcal{D}(N\backslash G,\tau)$, and let Φ and Ψ be the Fourier-Laplace transforms of ϕ and ψ. Then*

$$\begin{aligned}\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(\Gamma\backslash G)} = {} & \frac{1}{4\pi}\int_{-\infty}^{\infty}\langle\Phi(it)+M(-it)\Phi(-it),\Psi(it)+M(-it)\Psi(-it)\rangle_{L^2(K)}\,dt \\ & + \frac{6}{\pi}\Big(\int_{N\backslash G}\phi(g)\,d\dot{g}\Big)\overline{\Big(\int_{N\backslash G}\psi(g)\,d\dot{g}\Big)}.\end{aligned}$$

Proof. Averaging the effect of leaving alone the first term on the right side of (2.16) and replacing t by $-t$, we obtain

$$\begin{aligned}\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(\Gamma\backslash G)} = {} & \frac{1}{4\pi}\int_{-\infty}^{\infty}\big(\langle\Phi(it),\Psi(it)\rangle_{L^2(K)} + \langle M(it)\Phi(it),\Psi(-it)\rangle_{L^2(K)} \\ & \quad + \langle\Phi(-it),\Psi(-it)\rangle_{L^2(K)} + \langle M(-it)\Phi(-it),\Psi(it)\rangle_{L^2(K)}\big)\,dt \\ & + (\text{residue term}).\end{aligned} \qquad (2.18)$$

It follows from Proposition 2.6, parts (d) and (f), that $M(it)$ is unitary with inverse $M(-it)$. Therefore

$$\langle M(it)\Phi(it),\Psi(-it)\rangle_{L^2(K)} = \langle\Phi(it),M(-it)\Psi(-it)\rangle_{L^2(K)}$$

and $$\langle\Phi(-it),\Psi(-it)\rangle_{L^2(K)} = \langle M(-it)\Phi(-it),M(-it)\Psi(-it)\rangle_{L^2(K)}.$$

Substituting in (2.18) for the second and third terms of the integrand, we obtain the t integral of the corollary. The residue term has been evaluated in (2.17).

Corollary 2.9. *Let ϕ be in $\mathcal{D}(N\backslash G,\tau)$, and let Φ be its Fourier-Laplace transform. Then $\widehat{\phi} = 0$ if and only if $\int_{N\backslash G}\phi(g)\,d\dot{g} = 0$ and $\Phi(it) = -M(-it)\Phi(-it)$ for $-\infty < t < \infty$.*

Proof. This is immediate from Corollary 2.8 with $\psi = \phi$.

From these results we obtain the analysis of $L^2_{\text{cont}}(\Gamma\backslash G)$. In fact, let $\mathcal{PW}(\tau)$ be the space of Fourier transforms of the space $C^\infty_{\text{com}}(i\mathbb{R}, W(\tau))$ of compactly supported smooth functions on $i\mathbb{R}$ with values in $W(\tau)$. The Fourier-Laplace transform $\phi \mapsto \Phi$ is a one-one map of $\mathcal{D}(N\backslash G, \tau)$ onto $\mathcal{PW}(\tau)$. For $\Phi \in \mathcal{PW}(\tau)$, define

$$\Phi_1(it) = \Phi(it) + M(-it)\Phi(-it).$$

The map $\Phi \mapsto \Phi_1$ is a linear map of $\mathcal{PW}(\tau)$ into the subspace $\widehat{L}^2(E, \tau)$ of functions h in $L^2(i\mathbb{R}, W(\tau))$ such that $M(it)h(it) = h(-it)$, and Corollary 2.9 says that the composition $\phi \mapsto \Phi \mapsto \Phi_1$ descends to a map $\widehat{\phi} \mapsto \Phi_1$. Let us call this descended map $\widetilde{E}_\tau$, writing it as

$$\widetilde{E}_\tau : \{\widehat{\phi} \mid \phi \in \mathcal{D}(N\backslash G, \tau)\} \to \widehat{L}^2(E, \tau).$$

By Lemma 1.2 with $F = 1$, $\widehat{\phi}$ has integral 0 over $\Gamma\backslash G$ if and only if ϕ has integral 0 over $N\backslash G$. Let us restrict $\widetilde{E}_\tau$ to a map

$$E_\tau : \{\widehat{\phi} \mid \phi \in \mathcal{D}(N\backslash G, \tau) \text{ and } \int_{N\backslash G} \phi(g)\, d\dot{g} = 0\} \to \widehat{L}^2(E, \tau). \tag{2.19a}$$

Corollary 2.9 shows that E_τ is one-one, and Corollary 2.8 shows that E_τ is actually isometric apart from a factor $1/4\pi$. Let $L^2_{\text{cont}}(\Gamma\backslash G, \tau)$ be the subspace of functions $h \in L^2_{\text{cont}}(\Gamma\backslash G)$ such that $k \mapsto h(\Gamma gk)$ is in $W(\tau)$ for all $g \in G$. Theorem 1.3 shows that E_τ extends to an isometric map

$$E_\tau : L^2_{\text{cont}}(\Gamma\backslash G, \tau) \to \widehat{L}^2(E, \tau). \tag{2.19b}$$

Meanwhile, consideration of Fourier transforms shows that $\mathcal{PW}(\tau)$ is dense in $L^2(i\mathbb{R}, W(\tau))$, and so is the subspace where $\Phi(1) = 0$ (corresponding to ϕ of integral 0). Hence the image under $\phi \mapsto \Phi \mapsto \Phi_1$ of functions of integral 0 is dense in $\widehat{L}^2(E, \tau)$. Thus the map (2.19a) has dense image. Since (2.19a) is isometric, (2.19b) is onto. We may summarize as follows.

Theorem 2.10. *Let $\phi \in \mathcal{D}(N\backslash G, \tau)$ have integral 0, let Φ be its Fourier-Laplace transform, and define*

$$\Phi_1(it) = \Phi(it) + M(-it)\Phi(-it).$$

The composition of the linear maps $\phi \mapsto \Phi \mapsto \Phi_1$ descends to a well defined linear map $\widehat{\phi} \to \Phi_1$, which extends to a bounded linear map E_τ of $L^2_{\text{cont}}(\Gamma\backslash G, \tau)$ onto $\widehat{L}^2(E, \tau)$ such that

$$\|\widehat{\phi}\|^2_{L^2(\Gamma\backslash G)} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \|\Phi_1(it)\|^2_{L^2(K)}\, dt.$$

The map E_τ has an equivariance property. Since $\mathcal{D}(N\backslash G, \tau)$ is not closed under translation by G, we cannot hope for G equivariance. But we can hope for as much equivariance as τ permits. Thus let R be the right regular representation of G on $\mathcal{D}(N\backslash G)$, and define

$$R(f)\phi(x) = \int_G \phi(xg) f(g)\, dg$$

for all $f \in C^\infty_{\text{com}}(G)$ such that $k \mapsto f(k^{-1}g)$ is in $W(\tau)$ for all $g \in G$. If ϕ is in $\mathcal{D}(N\backslash G, \tau)$, then a change of variables shows that $R(f)\phi$ is in $\mathcal{D}(N\backslash G, \tau)$. Let $\Phi_\phi(g, s)$ be the Fourier-Laplace transform of ϕ. Remembering from (2.8) that $\Phi_\phi(\,\cdot\,, s)$ is in the space for $P^{+,s}$, we readily check that

$$P^{+,s}(f)\Phi_\phi(x, s) = \Phi_{R(f)\phi}(x, s).$$

Passing from Φ to Φ_1 and using the intertwining property of $M(-it)$ implicit in Lemma 2.4 and analytic continuation, we obtain, in obvious notation,

$$P^{+,it}(f)(\Phi_1)_\phi(x, it) = (\Phi_1)_{R(f)\phi}(x, it).$$

Consequently E_τ is equivariant with respect to the operation of all members f of $C^\infty_{\text{com}}(G)$ such that $k \mapsto f(k^{-1}x)$ is in $W(\tau)$ for all $x \in G$.

Now we pass to the limit, in effect taking the union over all τ. Let $\widehat{L}^2(E)$ be the set of all square integrable functions h from $i\mathbb{R}$ into the even functions on K such that $M(it)h(it) = h(-it)$. The union E of the E_τ gives us an isometric map (apart from the factor $1/4\pi$) of a dense subspace of $L^2_{\text{cont}}(\Gamma\backslash G)$ onto a dense subspace of $\widehat{L}^2(E)$, and this is equivariant with respect to all members f of $C^\infty_{\text{com}}(G)$ such that $k \mapsto f(k^{-1}x)$ is in a common $W(\tau)$ for all $x \in G$. Such f's form an approximate identity, and therefore E extends to an isometry of $L^2_{\text{cont}}(\Gamma\backslash G)$ onto $\widehat{L}^2(E)$ equivariant with respect to G. This proves Theorem 1.4.

3. Discrete Decomposition of the Cuspidal Part

In this section we shall prove Theorem 1.6, giving Godement's variation [Go2] of a proof of Langlands [Lgl2]. We continue to let $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$, and we use other notation as in §1. Fix φ in $\mathcal{D}(G)$. Our objective is to show that the operator $R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy$ is Hilbert-Schmidt (hence compact) on the subspace $L^2_{\text{cusp}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$. The main step is to prove the following lemma.

Lemma 3.1. *For any integer $M \geq 0$, there exists a constant $C(\varphi, M)$ such that*

$$|R(\varphi)f(g)| \leq C(\varphi, M)y(g)^{-M}\|f\|_{L^2(\Gamma\backslash G)}$$

for all $f \in L^2_{\text{cusp}}(\Gamma\backslash G)$ and for all $g \in G$ such that $g(i)$ is in the standard fundamental domain

$$S = \left\{z \,\middle|\, \operatorname{Im} z > 0, |z| \geq 1, |\operatorname{Re} z| \leq \tfrac{1}{2}\right\}$$

for Γ.

Remark. We need this estimate only for $M = 0$, but the estimate for general M is no harder.

Proof. Writing

$$R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy = \int_G f(y)\varphi(x^{-1}y)\,dy = \int_{\Gamma_\infty\backslash G} \sum_{\gamma\in\Gamma_\infty} f(y)\varphi(x^{-1}\gamma y)\,dy$$

shows that
$$R(\varphi)f(x) = \int_{\Gamma_\infty\backslash G} K_\varphi(x, y)f(y)\,dy,$$

where
$$K_\varphi(x, y) = \sum_{\gamma\in\Gamma_\infty} \varphi(x^{-1}\gamma y).$$

Define functions $n : \mathbb{R} \to N$ and $t : N \to \mathbb{R}$ by $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = x$. The function $\varphi_{x,y}(t) = \varphi(x^{-1}n(t)y)$ is in $C^\infty_{\text{com}}(\mathbb{R})$, and the Poisson summation formula gives

$$\sum_{m=-\infty}^{\infty} \varphi_{x,y}(m) = \sum_{m=-\infty}^{\infty} \widehat{\varphi}_{x,y}(m),$$

where

$$\widehat{\varphi}_{x,y}(s) = \int_{\mathbb{R}} \varphi_{x,y}(t) e^{-2\pi its}\, dt.$$

Thus the kernel defining $R(\varphi)$ on $L^2(\Gamma\backslash G)$ is given by

$$K_\varphi(x, y) = \sum_{m=-\infty}^{\infty} \widehat{\varphi}_{x,y}(m).$$

The contribution to $R(\varphi)f$ from $m = 0$ is the main term in the sense that we shall use the hypothesis that f is in $L^2_{\text{cusp}}(\Gamma\backslash G)$ to handle it. The contribution from the other terms will be treated as an error term. The term for $m = 0$ gives

$$\begin{aligned}
\int_{\Gamma_\infty\backslash G} \widehat{\varphi}_{x,y}(0) f(y)\, dy &= \int_{\Gamma_\infty\backslash G} \int_N \varphi(x^{-1}ty) f(y)\, dt\, dy \\
&= \int_{N\backslash G} \int_{s\in\Gamma_\infty\backslash N} \int_{t\in N} \varphi(x^{-1}tsy) f(sy)\, dt\, ds\, dy \\
&= \int_{N\backslash G} \int_{t\in N} \int_{s\in\Gamma_\infty\backslash N} \varphi(x^{-1}ty) f(sy)\, ds\, dt\, dy
\end{aligned}$$

after a change of variables, and the right side is 0 since f is in $L^2_{\text{cusp}}(\Gamma\backslash G)$.

Now we consider the contribution to $R(\varphi)f$ from $m \neq 0$. Let C be the support of φ, and write the Iwasawa decomposition of $x \in G$ relative to $G = NAK$ as $x = n_x a_x k_x$. Since K and C are compact, we have $KC \subset N\Omega_A K$ for some compact subset Ω_A of A. If $\varphi(x^{-1}n(t)y) \neq 0$, then $x^{-1}n(t)y$ is in C. Hence $k_x^{-1}a_x^{-1}n_x^{-1}n(t)y$ is in C, and y is in $n(-t)n_x a_x k_x C \subset N a_x N \Omega_A K \subset N a_x \Omega_A K$. In other words, $a_y = a_x \omega_A$ for some $\omega_A \in \Omega_A$. If $\widehat{\varphi}_{x,y}(m) \neq 0$, we therefore have

$$\begin{aligned}
\widehat{\varphi}_{x,y}(m) &= \int_{\mathbb{R}} \varphi(x^{-1}n(t)y) e^{-2\pi itm}\, dt \\
&= \int_{\mathbb{R}} \varphi(k_x^{-1}a_x^{-1}n_x^{-1}n(t)n_y a_y k_y) e^{-2\pi itm}\, dt \\
&= e^{2\pi it(n_x^{-1}n_y)m} \int_{\mathbb{R}} \varphi(k_x^{-1}a_x^{-1}n(t)a_x\omega_A k_y) e^{-2\pi itm}\, dt \\
&= e^{2\pi it(n_x^{-1}n_y)m} \int_{\mathbb{R}} \varphi(k_x^{-1}n(y(x)^{-1}t)\omega_A k_y) e^{-2\pi itm}\, dt \\
&= e^{2\pi it(n_x^{-1}n_y)m} \int_{\mathbb{R}} \varphi(k_x^{-1}n(t)\omega_A k_y) e^{-2\pi iy(x)tm} y(x)\, dt.
\end{aligned}$$

As k and k' vary through K and a varies through Ω_A, the functions $t \mapsto \varphi(kn(t)ak')$ vary in a compact family in $\mathcal{D}(\mathbb{R})$ and therefore satisfy uniform estimates. Thus we obtain

$$|\widehat{\varphi}_{x,y}(m)| \leq C_{M,\varphi} y(x) |y(x)m|^{-M} = C_{M,\varphi} y(x)^{1-M} |m|^{-M}$$

for every positive integer M and all x and y in G.

Since we have seen that the $m=0$ term gives 0, we obtain

$$\begin{aligned}|R(\varphi)f(x)| &\le \int_{y\in\Gamma_\infty\backslash G,\, y\in Na_x\Omega_A K} \sum_{m\ne 0} |\widehat{\varphi}_{x,y}(m)||f(y)|\,dy \\ &\le \int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} \sum_{m\ne 0} |\widehat{\varphi}_{x,y}(m)||f(y)|\,dy \\ &\le \int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} \sum_{m\ne 0} C_{M,\varphi} y(x)^{1-M}|m|^{-M}|f(y)|\,dy \\ &\le C_\varphi y(x)^{1-M} \int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} |f(y)|\,dy,\end{aligned}$$

with the last inequality valid for $M\ge 2$. By the Schwarz inequality this is

$$\le C_\varphi y(x)^{1-M}\Big(\int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} dy\Big)^{1/2}\Big(\int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} |f(y)|^2\,dy\Big)^{1/2}.$$

Since $n[-\frac12,\frac12]$ has N measure 1 and K has total measure 1, we see that

$$\int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} dy = \int_{a\in a_x\Omega_A} y(a)^{-1}\,da = y(x)\int_{a\in\Omega_A} y(a)^{-1}\,da.$$

Also if $y(x)\ge\frac12$, then the set $n[-\frac12,\frac12]a_x\Omega_A K$ is covered by finitely many Γ translates of the fundamental domain S. If the number of such translates is q, then

$$\int_{y\in n[-\frac12,\frac12]a_x\Omega_A K} |f(y)|^2\,dy \le q\int_{y(i)\in S} |f(y)|^2\,dy = q\|f\|^2_{L^2(\Gamma\backslash G)}.$$

Putting these facts together, we find that

$$|R(\varphi)f(x)| \le C'_\varphi y(x)^{\frac32-M}\|f\|_{L^2(\Gamma\backslash G)} \tag{3.1}$$

if $y(x)\ge\frac12$ and $M\ge 2$. Here C'_φ is $C_\varphi\big(q\int_{\Omega_A} y(a)^{-1}\,da\big)^{1/2}$. If $x(i)$ is in S, then $y(x)\ge\frac12$. When $y(x)\ge\frac12$, the inequality (3.1) for all exponents $\frac32-M$ with $M\ge 2$ implies the inequality for all integer exponents and a constant depending on the exponent. This proves the lemma.

Proof of Theorem 1.6. We take $M=0$ in Lemma 3.1. The lemma says that, for each $g\in\Gamma\backslash G$, $f\mapsto R(\varphi)f(g)$ is a bounded linear functional on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Hence there exists a function K_g in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ such that

$$R(\varphi)f(g) = \int_{\Gamma\backslash G} K_g(x)f(x)\,dx$$

for all $f\in L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Moreover, $\|K_g\|_{L^2(\Gamma\backslash G)}\le C(\varphi,0)$ for all $g\in\Gamma\backslash G$. Put $K(g,x)=K_g(x)$. If $K(\,\cdot\,,\,\cdot\,)$ is jointly measurable, then

$$\int_{\Gamma\backslash G\times\Gamma\backslash G} |K(g,x)|^2\,dx\,dg \le \int_{\Gamma\backslash G} C(\varphi,0)^2\,dg < \infty$$

since $\Gamma\backslash G$ has finite volume, and $R(\varphi)$ is exhibited as the restriction to $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ of a Hilbert-Schmidt operator on $L^2(\Gamma\backslash G)$ that leaves $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ stable. Hence $R(\varphi)$ is Hilbert-Schmidt on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$.

To complete the proof, we need to address the joint measurability of the kernel. If X is a left invariant first-order derivative, then $X(R(\varphi)f) = -R(X\varphi)f$. Applying the lemma to $X\varphi$, we conclude that $\sup|X(R(\varphi)f)| \le C\|f\|_{L^2(\Gamma\backslash G)}$. If ε and Γg are given, it follows that $|R(\varphi)f(g') - R(\varphi)f(g)| \le \varepsilon\|f\|_{L^2(\Gamma\backslash G)}$ for all $f \in L^2_{\text{cusp}}(\Gamma\backslash G)$ and for all g' sufficiently close to g. Therefore $g \mapsto K_g$ is continuous as a map of $\Gamma\backslash G$ into $L^2_{\text{cusp}}(\Gamma\backslash G)$, and we saw above that it is bounded. It is a general fact that if M is in $L^2(\Gamma\backslash G \times \Gamma\backslash G)$ and $M_g(x) = M(g,x)$, then $g \mapsto M_g$ is in $L^2(\Gamma\backslash G, L^2(\Gamma\backslash G))$. Thus we can use $\{K_g\}$ to define a continuous linear functional on $L^2(\Gamma\backslash G \times \Gamma\backslash G)$ by

$$M \mapsto \int_{g\in\Gamma\backslash G} (M_g, K_g)_{L^2(\Gamma\backslash G)}\, d\dot{g}.$$

This linear functional must be given by the complex conjugate of a (jointly measurable) member K' of $L^2(\Gamma\backslash G \times \Gamma\backslash G)$. We can replace $K(\,\cdot\,,\,\cdot\,)$ by $K'(\,\cdot\,,\,\cdot\,)$ above, have the required joint measurability, and still have $R(\varphi)f = \int_{\Gamma\backslash G} K'(\,\cdot\,,x)f(x)\,dx$ almost everywhere for each $f \in L^2_{\text{cusp}}(\Gamma\backslash G)$.

4. Introduction to the Trace Formula

A first insight into what to look for in a trace formula comes from the compact quotient case. Let G be a unimodular Lie group, let Γ be a discrete subgroup such that $\Gamma\backslash G$ is compact, and let R be the right regular representation of G on $L^2(\Gamma\backslash G)$.

Let φ be in $C^\infty_{\text{com}}(G)$, and define $R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy$. The computation

$$R(\varphi)f(x) = \int_G f(xy)\varphi(y)\,dy = \int_G f(y)\varphi(x^{-1}y)\,dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(y)\varphi(x^{-1}\gamma y)\,dy$$

shows that

$$R(\varphi)f(x) = \int_{\Gamma\backslash G} K(x,y)f(y)\,dy,$$

where $K(x,y) = \sum_{\gamma\in\Gamma}\varphi(x^{-1}\gamma y)$. This sum is locally finite, and it follows that K is in $C^\infty(\Gamma\backslash G \times \Gamma\backslash G)$. Thus we can apply the following lemma.

Lemma 4.1. *Let X be a compact C^∞ manifold, and let dx be a measure on X that is a smooth function times Lebesgue measure in each coordinate neighborhood. Let K be in $C^\infty(X\times X)$, and define a bounded operator B on $L^2(X,dx)$ by $Bf(x) = \int_X K(x,y)f(y)\,dy$. Then B is of trace class, and its trace is*

$$\operatorname{Tr} B = \int_X K(x,x)\,dx.$$

Reference. [Kn1, p. 341].

By the lemma, $R(\varphi)$ is of trace class. Referring to the proof in §1 that Theorem 1.6 implies Theorem 1.5, we see that $L^2(\Gamma\backslash G)$ decomposes into the direct sum of irreducible representations of G, each occurring with finite multiplicity. Let us write

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} m_\pi \pi. \tag{4.1}$$

The lemma also gives us a formula for the trace of $R(\varphi)$, namely

$$\operatorname{Tr} R(\varphi) = \int_{\Gamma\backslash G} K(x,x)\,dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(x^{-1}\gamma x)\,dx. \tag{4.2}$$

We can refine the right side of (4.2) by lumping terms whose elements γ are conjugate in G. For a group U, let U^γ be the centralizer of γ in U. From each conjugacy class $\mathfrak{o}$ of elements in Γ, we select a representative. Say that γ is a representative of $\mathfrak{o}_\gamma$. Then $\mathfrak{o}_\gamma$ consists of all $\delta^{-1}\gamma\delta$, where δ varies through $\Gamma^\gamma\backslash\Gamma$. Thus

$$\begin{aligned}\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(x^{-1}\gamma x)\,dx &= \sum_{\mathfrak{o}_\gamma} \sum_{\delta\in\Gamma^\gamma\backslash\Gamma} \int_{\Gamma\backslash G} \varphi(x^{-1}\delta^{-1}\gamma\delta x)\,dx \\ &= \sum_{\mathfrak{o}_\gamma} \int_{\Gamma^\gamma\backslash G} \varphi(x^{-1}\gamma x)\,dx \\ &= \sum_{\mathfrak{o}_\gamma} \int_{G^\gamma\backslash G} \int_{\Gamma^\gamma\backslash G^\gamma} \varphi(x^{-1}y^{-1}\gamma yx)\,dy\,dx \\ &= \sum_{\mathfrak{o}_\gamma} \operatorname{vol}(\Gamma^\gamma\backslash G^\gamma) \int_{G^\gamma\backslash G} \varphi(x^{-1}\gamma x)\,dx. \end{aligned} \tag{4.3}$$

We arrive at the following result.

Theorem 4.2. *Let G be a unimodular Lie group, let Γ be a discrete subgroup such that $\Gamma\backslash G$ is compact, let R be the right regular representation of G on $L^2(\Gamma\backslash G)$, and let φ be in $C^\infty_{\mathrm{com}}(G)$. Then $R(\varphi)$ is of trace class, and*

$$\operatorname{Tr} R(\varphi) = \sum_{\mathfrak{o}_\gamma} \operatorname{vol}(\Gamma^\gamma\backslash G^\gamma) \int_{G^\gamma\backslash G} \varphi(x^{-1}\gamma x)\,dx. \tag{4.4a}$$

Consequently if the decomposition of $L^2(\Gamma\backslash G)$ into irreducible representations of G is as in (4.1), *then*

$$\sum_{\pi\in\widehat{G}} m_\pi \operatorname{Tr}\pi(\varphi) = \sum_{\mathfrak{o}_\gamma} \operatorname{vol}(\Gamma^\gamma\backslash G^\gamma) \int_{G^\gamma\backslash G} \varphi(x^{-1}\gamma x)\,dx. \tag{4.4b}$$

Let us consider two examples. The first example is the case that G is compact and $\Gamma = \{1\}$. If dx is normalized to have total mass 1, then (4.4b) gives

$$\varphi(1) = \sum_{\pi\in\widehat{G}} m_\pi \operatorname{Tr}\pi(\varphi), \tag{4.5}$$

which is the Fourier inversion formula for G. It is typical of the trace formula that we can get information about the multiplicities m_π by specializing φ. Indeed, if in (4.5) we take φ to be the complex conjugate of the character of π, the Schur orthogonality relations tell us that m_π equals the degree of π.

The second example with compact quotient is the case that $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Assuming that the measure on $\mathbb{Z}\backslash\mathbb{R}$ has total mass one, we find that the right side of (4.4b) is just $\sum_{n=-\infty}^{\infty} \varphi(n)$, while the left side is $\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)$ if $\widehat{\varphi}(n) = \int_{\mathbb{Z}\backslash\mathbb{R}} \varphi(x) e^{-2\pi i n x}\,dx$. Formula (4.4b) is therefore the Poisson summation formula for smooth functions φ of compact support.

The example that we have been studying in this paper has $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$. For this case, $\Gamma\backslash G$ is noncompact and (4.4b) is not directly applicable. Indeed, we saw in §3 that $L^2(\Gamma\backslash G)$ has a continuous part to its decomposition, and $R(\varphi)$ cannot always be of trace class. What we know from Theorem 1.5 is that $R(\varphi)$ is Hilbert-Schmidt on $L^2_{\text{cusp}}(\Gamma\backslash G)$ if φ is in $C^\infty_{\text{com}}(G)$. Since the composition of two Hilbert-Schmidt operators is of trace class, $R(\varphi)$ is of trace class on $L^2_{\text{cusp}}(\Gamma\backslash G)$ if φ is a finite sum of convolutions of pairs of members of $C^\infty_{\text{com}}(G)$. A theorem of Dixmier and Malliavin [Di-Ma] says that this is always the case on a Lie group, and we arrive at the following theorem.

Theorem 4.3. *For $G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$, $R(\varphi)$ is of trace class on $L^2_{\text{cusp}}(\Gamma\backslash G)$ if φ is in $C^\infty_{\text{com}}(G)$.*

Following the line of argument in the compact quotient case, we want to obtain a formula for $\operatorname{Tr} R(\varphi)$ on $L^2_{\text{cusp}}(\Gamma\backslash G)$ by integrating a kernel on its diagonal. Although the computation at the beginning of this section shows that $R(\varphi)$ is given by the kernel

$$K(x, y) = \sum_{\gamma\in\Gamma} \varphi(x^{-1}\gamma y),$$

this kernel reflects the action of $R(\varphi)$ on all of $L^2(\Gamma\backslash G)$. It is necessary to subtract terms to account for the contributions of $L^2_{\text{cont}}(\Gamma\backslash G)$ and the constant functions. On the constant functions, $R(\varphi)$ acts as the scalar $\int_G \varphi(x)\,dx$, and this scalar is the trace. Thus we need to know the kernel $K_{\text{cont}}(x, y)$ for the action of $R(\varphi)$ on $L^2_{\text{cont}}(\Gamma\backslash G)$.

The derivation of a formula for $K_{\text{cont}}(x, y)$ is a little complicated, and we shall carry out only the formal argument, omitting the justification for some interchanges of limits. Also we shall assume that φ is two-sided K finite. See [Gb-Ja] for more details. The argument requires knowing that there is a meromorphic continuation for an Eisenstein series $E(g, f, s)$ itself (with f in some $W(\tau)$, say), not just for its constant term. Moreover, the only poles for the continued Eisenstein series are simple and coincide with the poles of the constant term, and the continued Eisenstein series satisfies growth estimates in $\operatorname{Im} s$ in any strip $0 \le \operatorname{Re} s \le \sigma$. For a proof of these facts, see [Go1] or Appendix IV of [Lgl2]. These facts have an analog in the adelic setting (0.4), and the paper [Ja] in this volume discusses this analog.

Lemma 4.4. *Let ϕ be a K finite even function in $\mathcal{D}(N\backslash G)$, and let Φ be its Fourier-Laplace transform. Then the analytically continued Eisenstein series satisfies*

$$E(g, M(s)\Phi(s), -s) = E(g, \Phi(s), s). \tag{4.6}$$

Proof. The constant term of the right side is $2\Phi(s)_s + 2(M(s)\Phi(s))_{-s}$ when $\operatorname{Re} s > 1$, and it is this at all points where there is no pole, by analytic continuation. Similarly the constant term of the left side is $2(M(s)\Phi(s))_{-s} + 2(M(-s)M(s)\Phi(s))_s$ when $\operatorname{Re} s < -1$, and it is this at all points where there is no pole. Since $M(-s)M(s) = 1$ by Proposition 2.6f, the two sides of (4.6) have equal constant terms.

For fixed $s = s_0$, let $b(g)$ be the difference of the two sides of (4.6). Then $b(g)$ has constant term 0, and Lemma 1.2 shows that $b(g)$ is orthogonal to any L^2 function of the form $\widehat{\phi}$. Thus $b(g)$ is orthogonal to $E(g, \Phi(s), s)$ in the region of convergence $\operatorname{Re} s > 1$ and then, by analytic continuation, for all s where there is no pole. Similarly $b(g)$ is orthogonal to $E(g, M(s)\Phi(s), -s)$ in the region of convergence $\operatorname{Re} s < -1$ and then for all s where there is no pole. Therefore $b(g)$ is orthogonal to itself, and $b(g) = 0$.

Let H be the space of even functions in $L^2(K)$. As in §2, we introduce

$$\widehat{L}^2(E) = \{F \in L^2(i\mathbb{R}, H) \mid M(it)F(it) = F(-it)\}.$$

Recall that the construction in §2 started from an even K finite $\phi \in \mathcal{D}(N\backslash G)$ of integral 0 and gave a map $\phi \to \Phi \to \Phi_1$ with $\Phi_1 \in \widehat{L}^2(E)$, and this map descended to be a well defined linear map S carrying $\widehat{\phi}$ to Φ_1. Theorem 2.10 shows that S preserves norms, in the sense that

$$\langle \widehat{\phi}, \widehat{\psi} \rangle_{L^2(\Gamma\backslash G)} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle \Phi_1(it), \Psi_1(it) \rangle_{L^2(K)}\, dt, \tag{4.7}$$

and S has dense image in $\widehat{L}^2(E)$. Hence S completes to a unitary mapping of $L^2_{\text{cont}}(\Gamma\backslash G)$ onto $\widehat{L}^2(E)$.

Lemma 4.5. *Let $\phi \in \mathcal{D}(N\backslash G)$ be K finite of integral 0, and let h be a K finite member of the space H of even functions in $L^2(K)$. Then*

$$\langle h, S\widehat{\phi}(it) \rangle_{L^2(K)} = \frac{1}{2} \int_{\Gamma\backslash G} E(g, h, it) \overline{\widehat{\phi}(g)}\, dg.$$

Proof. Since ϕ has integral 0 over $N\backslash G$, $\Phi(1)$ has integral 0 over K. Thus $M(s)\Phi(s)$ has no pole at $s = 1$, and $E(g, \Phi(s), s)$ has no pole at $s = 1$. For $\sigma > 1$, it follows that

$$\begin{aligned}
&\langle \widehat{\phi}, \widehat{\psi} \rangle_{L^2(\Gamma\backslash G)} \\
&\quad = \frac{1}{4\pi} \int_{\Gamma\backslash G} \Big[\int_{\operatorname{Re} s = \sigma} E(g, \Phi(s), s) \overline{\widehat{\psi}(g)}\, d|s| \Big]\, dg \quad \text{by (2.10)} \\
&\quad = \frac{1}{4\pi} \int_{\operatorname{Re} s = \sigma} \Big[\int_{\Gamma\backslash G} E(g, \Phi(s), s) \overline{\widehat{\psi}(g)}\, dg \Big]\, d|s| \quad \text{by interchange} \\
&\quad = \frac{1}{4\pi} \int_{t=-\infty}^{\infty} \Big[\int_{\Gamma\backslash G} E(g, \Phi(it), it) \overline{\widehat{\psi}(g)}\, dg \Big]\, dt \quad \text{by moving the line of integration}
\end{aligned} \tag{4.8}$$

since there is no pole at $s = 1$.

In (4.8), Lemma 4.4 and the change of variables $t \to -t$ allow us to replace $\Phi(it)$ by $M(-it)\Phi(-it)$. Averaging the two results yields

$$\langle \widehat{\phi}, \widehat{\psi} \rangle_{L^2(\Gamma\backslash G)} = \frac{1}{8\pi} \int_{t=-\infty}^{\infty} \Big[\int_{\Gamma\backslash G} E(g, \Phi_1(it), it) \overline{\widehat{\psi}(g)}\, dg \Big]\, dt. \tag{4.9}$$

Comparing (4.7) and (4.9), we see that

$$\int_{-\infty}^{\infty} \langle \Phi_1(it), \Psi_1(it) \rangle_{L^2(K)}\, dt = \tfrac{1}{2} \int_{t=-\infty}^{\infty} \Big[\int_{\Gamma\backslash G} E(g, \Phi_1(it), it) \overline{\widehat{\psi}(g)}\, dg \Big]\, dt. \tag{4.10}$$

On each side of (4.10), we write the integral as a sum of integrals over $(0, \infty)$ and $(-\infty, 0)$ and in the $(-\infty, 0)$ integral replace t by $-t$ and then $\Phi_1(-it)$ by

$M(it)\Phi_1(it)$. Finally on the left side we replace $\Psi_1(-it)$ by $M(it)\Psi_1(it)$, and on the right side we substitute from Lemma 4.4. The result is that

$$\int_0^\infty \langle \Phi_1(it), \Psi_1(it)\rangle_{L^2(K)}\, dt = \tfrac{1}{2}\int_{t=0}^\infty \Big[\int_{\Gamma\backslash G} E(g, \Phi_1(it), it)\overline{\widehat{\psi}(g)}\, dg\Big]\, dt. \tag{4.11}$$

The functions $t \to \Phi_1(it)$ are dense in $L^2((0,\infty), H)$, and we can pass to the limit in the Eisenstein series if we stick to a K finite function in $L^2((0,\infty), H)$. Thus (4.11) persists if $\Phi_1(it)$ is replace by any K finite function in $L^2((0,\infty), H)$. Let us use a function of the form $c(t)h$, where h is a K finite member of H and $c(\,\cdot\,)$ is in $L^2((0,\infty), \mathbb{C})$. Then we obtain

$$\int_0^\infty c(t)\langle h, \Psi_1(it)\rangle_{L^2(K)}\, dt = \tfrac{1}{2}\int_{t=0}^\infty c(t)\Big[\int_{\Gamma\backslash G} E(g, h, it)\overline{\widehat{\psi}(g)}\, dg\Big]\, dt.$$

Since $c(t)$ is arbitrary, the integrands are equal at every point of continuity, i.e., everywhere. This proves the lemma.

Proposition 4.6. *Let $\{f_\alpha\}$ be an orthonormal basis of K finite functions in H, and let φ be two-sided K finite in $C^\infty_{\mathrm{com}}(G)$. Then $R(\varphi)$ is given on $L^2_{\mathrm{cont}}(\Gamma\backslash G)$ by the kernel*

$$K_{\mathrm{cont}}(x, y) = \frac{1}{16\pi}\sum_{\alpha,\beta}\int_{-\infty}^\infty \langle P^{+,it}(\varphi)f_\beta, f_\alpha\rangle E(x, f_\alpha, it)\overline{E(y, f_\beta, it)}\, dt,$$

Proof. Extend the linear map S to all of $L^2(\Gamma\backslash G)$ by setting S equal to 0 on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ and $\mathbb{C}$. For ϕ and ψ of integral 0, we have

$$\langle \widehat{\phi}, \widehat{\psi}\rangle_{L^2(\Gamma\backslash G)} = \langle S\widehat{\phi}, S\widehat{\psi}\rangle_{\widehat{L}^2(E)},$$

and it follows that S^*S is the orthogonal projection of $L^2(\Gamma\backslash G)$ on $L^2_{\mathrm{cont}}(\Gamma\backslash G)$. Since S is an intertwining operator, we have

$$S^*SR(\varphi)S^*S = S^*P^{+,\,\cdot}(\varphi)S,$$

where $P^{+,\,\cdot}$ is the representation on $\widehat{L}^2(E)$. Consequently

$$\begin{aligned}
&\langle S^*SR(\varphi)S^*S\widehat{\phi}, \widehat{\psi}\rangle_{L^2(\Gamma\backslash G)}\\
&\quad= \langle S^*P^{+,\,\cdot}(\varphi)S\widehat{\phi}, \widehat{\psi}\rangle_{L^2(\Gamma\backslash G)}\\
&\quad= \langle P^{+,\,\cdot}(\varphi)S\widehat{\phi}, S\widehat{\psi}\rangle_{\widehat{L}^2(E)}\\
&\quad= \frac{1}{4\pi}\int_{-\infty}^\infty \langle P^{+,it}(\varphi)S\widehat{\phi}(it), S\widehat{\psi}(it)\rangle_{L^2(K)}\, dt\\
&\quad= \frac{1}{4\pi}\int_{-\infty}^\infty \sum_\alpha \langle P^{+,it}(\varphi)S\widehat{\phi}(it), f_\alpha\rangle_{L^2(K)}\langle f_\alpha, S\widehat{\psi}(it)\rangle_{L^2(K)}\, dt\\
&\quad= \frac{1}{4\pi}\int_{-\infty}^\infty \sum_\alpha \overline{\langle P^{+,it}(\varphi)^*f_\alpha, S\widehat{\phi}(it)\rangle}_{L^2(K)}\langle f_\alpha, S\widehat{\psi}(it)\rangle_{L^2(K)}\, dt\\
&\quad= \frac{1}{16\pi}\int_{-\infty}^\infty \sum_\alpha \Big[\int_{\Gamma\backslash G}\overline{E(g, P^{+,it}(\varphi)^*f_\alpha, it)}\widehat{\phi}(g)\, dg\Big]\Big[\int_{\Gamma\backslash G} E(g', f_\alpha, it)\overline{\widehat{\psi}(g')}\, dg'\Big]\, dt
\end{aligned}$$

by Lemma 4.5

$$= \int_{\Gamma\backslash G\times\Gamma\backslash G} \Big[\frac{1}{16\pi}\int_{-\infty}^{\infty}\sum_\alpha \overline{E(g,P^{+,it}(\varphi)^* f_\alpha, it)}E(g',f_\alpha,it)\,dt\Big]\widehat{\phi}(g)\overline{\widehat{\psi}(g')}\,dg\,dg'.$$

Therefore $S^*SR(\varphi)S^*S$ is given by the kernel

$$K_{\mathrm{cont}}(g',g) = \frac{1}{16\pi}\int_{-\infty}^{\infty}\sum_\alpha \overline{E(g,P^{+,it}(\varphi)^* f_\alpha, it)}E(g',f_\alpha,it)\,dt.$$

If we expand $P^{+,it}(\varphi)^* f_\alpha = \sum_\beta \langle P^{+,it}(\varphi)^* f_\alpha, f_\beta\rangle f_\beta$, then we get the result of the proposition.

As a consequence of Proposition 4.6, the kernel of $R(\varphi)$ on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)\oplus\mathbb{C}$ along the diagonal is $K(x,x) - K_{\mathrm{cont}}(x,x)$. This difference is integrable over $\Gamma\backslash G$, but the separate terms are not. Some process of truncation needs to be used to avoid $\infty - \infty$ as integral, and we shall not pursue the details in this setting. See [He2] and [Ef] for further information about the classical trace formula. Actually the mechanism of the trace formula is more understandable in the adelic setting, where the interplay between characters and conjugacy classes is fairly clear, than in the setting of $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$, where the complicated nature of $SL_2(\mathbb{Z})$'s conjugacy classes obscures matters. In addition, significant applications require having the formula for two different algebraic groups, and it is therefore appropriate to have a derivation that can be generalized to groups other than SL_2 or GL_2. We shall therefore proceed directly to the adelic setting.

5. Digression on Quaternion Algebras

This section is the first of three sections in which we discuss the trace formula in the setting of adeles. The base number field will be $\mathbb{Q}$, and the adeles of $\mathbb{Q}$ will be denoted $\mathbb{A}$. For background on adeles and reductive algebraic groups, see the exposition [Kn2].

Before treating $G = GL_2$, we consider the case that G' is the multiplicative group of a quaternion algebra over $\mathbb{Q}$. By definition a **quaternion algebra** over a field F is a central simple algebra over F that has dimension 4 and is not equal to the full matrix algebra $M_2(F)$. Since any central simple algebra over F is a full matrix algebra over a division algebra over F, it follows that a quaternion algebra over F is a division algebra.

Let us see how to make G' into a linear algebraic group. Thus let D be a quaternion algebra over $\mathbb{Q}$. It is known that there exist integers m and n such that m, n, and mn are not squares in $\mathbb{Q}$ and such that D has a $\mathbb{Q}$ basis $\{1,u,v,w\}$ with $w = uv$ and

$$u^2 = m, \qquad v^2 = n, \qquad w^2 = -mn.$$

Furthermore

$$uv = -vu, \qquad uw = -wu, \qquad vw = -wv.$$

We may associate 2-by-2 matrices to the members of this $\mathbb{Q}$ basis by

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \qquad u \leftrightarrow \begin{pmatrix} \sqrt{m} & 0 \\ 0 & -\sqrt{m}\end{pmatrix}, \qquad v \leftrightarrow \begin{pmatrix} 0 & \sqrt{n} \\ \sqrt{n} & 0\end{pmatrix}, \qquad w \leftrightarrow \begin{pmatrix} 0 & \sqrt{mn} \\ -\sqrt{mn} & 0\end{pmatrix}.$$

These matrices may also be chosen to be defined over a quadratic extension of $\mathbb{Q}$ rather than a quartic extension, for example by taking

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \qquad u \leftrightarrow \begin{pmatrix} \sqrt{m} & 0 \\ 0 & -\sqrt{m}\end{pmatrix}, \qquad v \leftrightarrow \begin{pmatrix} 0 & 1 \\ n & 0\end{pmatrix}, \qquad w \leftrightarrow \begin{pmatrix} 0 & \sqrt{m} \\ -n\sqrt{m} & 0\end{pmatrix}.$$

In either case if we identify D with its effect under left multiplication on this basis, then G' is realized as an algebraic subgroup of GL_4 defined over $\mathbb{Q}$.

The determinant of the 2-by-2 matrix corresponding to

$$x = a1 + bu + cv + dw$$

is $a^2 - b^2m - c^2n + d^2mn$, and the determinant of the 4-by-4 matrix describing left multiplication by x is the square of this expression. For $v \in \{\infty, \text{primes}\}$, we see that $D \otimes_{\mathbb{Q}} \mathbb{Q}_v \cong M_2(\mathbb{Q}_v)$ if and only if $a^2 - b^2m - c^2n + d^2mn = 0$ is solvable nontrivially in $\mathbb{Q}_v$. Exactly in this case, $G'(\mathbb{Q}_v) \cong GL_2(\mathbb{Q}_v)$ and we say that G' is **unramified** or **split** at v. If v is an odd prime p, this always happens if $p \nmid m$ and $p \nmid n$, according to Corollaries 1 and 2 of [Bv-Sh, p. 50].

Let $\mathbb{A}$ be the adeles of $\mathbb{Q}$. The center Z' of G', namely the subgroup of scalar multiples of 1, has positive dimension, and consequently the quotient space $G'(\mathbb{Q})\backslash G'(\mathbb{A})$ has infinite volume. Thus instead of studying the right regular representation of $G'(\mathbb{A})$ on $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$, we begin by studying the right regular representation on $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}))$. The quotient space $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$ is compact as a consequence of the general theorem quoted as Theorem 6.2 in [Kn2] or a direct calculation that may be found in [Gf-Gr-P, pp. 115–119] or [We, pp. 74–75]. Despite the fact that this quotient is not a manifold, we shall see that Theorem 4.2 is still valid for it with suitable interpretations.

We study functions on $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$ by studying functions on $G'(\mathbb{A})$ that are left invariant under $Z'(\mathbb{A})$ and $G'(\mathbb{Q})$. But we can investigate more of $G'(\mathbb{Q})\backslash G'(\mathbb{A})$ if we consider further functions on $G'(\mathbb{A})$. Thus for each (unitary) character ω of $Z'(\mathbb{Q})\backslash Z'(\mathbb{A})$, we define $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}), \omega)$ to be the set of f on $G'(\mathbb{A})$ such that

$$f(z\gamma g) = \omega(z) f(g) \qquad \text{for } z \in Z'(\mathbb{A}),\ \gamma \in G'(\mathbb{Q}),\ g \in G'(\mathbb{A}) \tag{5.1}$$

and such that $|f|$ is square integrable on $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$. We denote by R_ω the right regular representation of $G'(\mathbb{A})$ on this space. We put $\overline{G}' = Z'\backslash G'$, so that we can identify $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$ with $\overline{G}'(\mathbb{Q})\backslash \overline{G}'(\mathbb{A})$.

Let us write $G'(\mathbb{A}) = G'_\infty \times G'(\mathbb{A}_f)$ for the decomposition of $G'(\mathbb{A})$ according to the infinite and finite places. Recall from §7 of [Kn2] that a complex-valued function f on $G'(\mathbb{A})$ is **smooth** if it is continuous and, when viewed as a function of two arguments $(x, y) \in G'_\infty \times G'(\mathbb{A}_f)$, it is smooth in x for each fixed y and is locally constant of compact support in y for each fixed x.

We define $C^\infty_{\text{com}}(G'(\mathbb{A}), \omega^{-1})$ to be the space of smooth functions on $G'(\mathbb{A})$ such that

$$\varphi(zg) = \omega(z)^{-1}\varphi(g) \qquad \text{for } z \in Z'(\mathbb{A}),\ g \in G'(\mathbb{A}). \tag{5.2}$$

If f is in $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}), \omega)$ and φ is in $C^\infty_{\text{com}}(G'(\mathbb{A}), \omega^{-1})$, then the function $f(xy)\varphi(y)$ on $G'(\mathbb{A}) \times G'(\mathbb{A})$ descends to a function on $G'(\mathbb{A}) \times \overline{G}'(\mathbb{A})$, and it makes sense to consider

$$R_\omega(\varphi) f(x) = \int_{\overline{G}'(\mathbb{Q})\backslash \overline{G}'(\mathbb{A})} f(xy)\varphi(y)\, dy \tag{5.3}$$

as a member of $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}), \omega)$. Since $\omega(Z'(\mathbb{Q})) = 1$, the function $\gamma \mapsto \varphi(x^{-1}\gamma y)$ on $G'(\mathbb{Q})$ descends to a well defined function on $\overline{G}'(\mathbb{Q})$. Thus we can imitate the computation at the beginning of §4 and write

$$\begin{aligned} R_\omega(\varphi)f(x) &= \int_{\overline{G}'(\mathbb{A})} f(xy)\varphi(y)\,dy \\ &= \int_{\overline{G}'(\mathbb{A})} f(y)\varphi(x^{-1}y)\,dy \\ &= \int_{\overline{G}'(\mathbb{Q})\backslash\overline{G}'(\mathbb{A})} \sum_{\gamma\in\overline{G}'(\mathbb{Q})} f(y)\varphi(x^{-1}\gamma y)\,dy. \end{aligned}$$

Therefore

$$R_\omega(\varphi)f(x) = \int_{\overline{G}'(\mathbb{Q})\backslash\overline{G}'(\mathbb{A})} K(x,y)f(y)\,dy, \tag{5.4}$$

where $K(x,y) = \sum_{\gamma\in\overline{G}'(\mathbb{Q})}\varphi(x^{-1}\gamma y)$. The function $K(x,y)$ is defined on $G'(\mathbb{A})\times G'(\mathbb{A})$, is left invariant under $G'(\mathbb{Q})$ in each variable, and satisfies

$$K(z_1x, z_2y) = \omega(z_1)\omega(z_2)^{-1}K(x,y) \qquad \text{for } z_1 \text{ and } z_2 \text{ in } Z'(\mathbb{A}).$$

Let K_1 be the open compact subgroup $\prod_p G'(\mathcal{O}_p)$ of $G'(\mathbb{A}_f)$. The function φ is left and right invariant under some open compact subgroup K_2 of K_1, and consequently the function $K(x,y)$ is right invariant under K_2 in each variable. The right invariance in y implies that $R_\omega(\varphi)f(x)$ depends only on the function $f^\#(yK_2) = \mathrm{vol}(K_2)^{-1}\int_{K_2} f(yk_2)\,dk_2$ defined on $G'(\mathbb{Q})\backslash G'(\mathbb{A})/K_2$. The right invariance in x implies that $R_\omega(\varphi)f(x) = (R_\omega(\varphi)f)^\#(xK_2)$. Thus we can regard $R_\omega(\varphi)$ as an operator from functions on $G'(\mathbb{Q})\backslash G'(\mathbb{A})/K_2$ transforming under ω to functions on the same space. By (6.3) of [Kn2], the compact space $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})/K_2$ is a (possibly disconnected) manifold. If $\omega = 1$, Lemma 4.1 is directly applicable. If $\omega \neq 1$, then Lemma 4.1 is indirectly applicable with the aid of a compactly supported function h on $G'(\mathbb{Q})\backslash G'(\mathbb{A})/K_2$ such that $\int_{Z'(\mathbb{A})} h(zx)\,dz = 1$ for all $x\in G'(\mathbb{A})$. The result for any ω is as follows.

Lemma 5.1. *If φ is in $C^\infty_{\mathrm{com}}(G'(\mathbb{A}),\omega^{-1})$, then the operator $R_\omega(\varphi)$ defined by* (5.3) *is of trace class on $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}),\omega)$, and its trace is*

$$\operatorname{Tr} R_\omega(\varphi) = \int_{\overline{G}'(\mathbb{Q})\backslash\overline{G}'(\mathbb{A})} K(x,x)\,dx,$$

where $K(x,x) = \sum_{\gamma\in\overline{G}'(\mathbb{Q})}\varphi(x^{-1}\gamma x)$.

The proof that Lemma 4.1 implies Theorem 4.2 may be adjusted to show that Lemma 5.1 implies the following result, which gives the trace formula for the multiplicative group of a quaternion algebra over $\mathbb{Q}$.

Theorem 5.2. *Let G' be the multiplicative group of a quaternion algebra over $\mathbb{Q}$, let Z' be the center, let $\overline{G}' = Z'\backslash\overline{G}'$, let R_ω be the right regular representation of $G'(\mathbb{A})$ on $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}),\omega)$, and let φ be in $C^\infty_{\mathrm{com}}(G'(\mathbb{A}),\omega^{-1})$. Then $R_\omega(\varphi)$ is of trace class, and*

$$\operatorname{Tr} R_\omega(\varphi) = \sum_{\mathfrak{o}_\gamma} \mathrm{vol}(\overline{G}'(\mathbb{Q})^\gamma\backslash\overline{G}'(\mathbb{A})^\gamma)\int_{\overline{G}'(\mathbb{A})^\gamma\backslash\overline{G}'(\mathbb{A})} \varphi(x^{-1}\gamma x)\,dx,$$

the sum being taken over conjugacy classes in $\overline{G}'(\mathbb{Q})$*. Consequently if the decomposition of* $L^2(Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A}),\omega^{-1})$ *into irreducible representations of* $G'(\mathbb{A})$ *is as in* (4.1)*, then*

$$\sum_{\pi\in\widehat{G'(\mathbb{A})}} m_\pi \operatorname{Tr}\pi(\varphi) = \sum_{\mathfrak{o}_\gamma} \operatorname{vol}(\overline{G}'(\mathbb{Q})^\gamma\backslash\overline{G}'(\mathbb{A})^\gamma)\int_{\overline{G}'(\mathbb{A})^\gamma\backslash\overline{G}'(\mathbb{A})} \varphi(x^{-1}\gamma x)\,dx.$$

6. Adelic Eisenstein Series

Now we turn our attention to the group $G = GL_2$. For this group we seek an understanding of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$, where $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. References are [Gf-Gr-P], [Ja-Lgl], [Du-La], [Ar1], [Gb1], [Gb-Ja], and [Ar4]. This quotient space does not have finite volume, and some adjustment has to be made. The same difficulty arose in §5 with the multiplicative group G' of a quaternion algebra: The quotient $G'(\mathbb{Q})\backslash G'(\mathbb{A})$ has infinite volume, and we in effect chose to study only functions that could be related to $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$, where Z' is the center. For G', we took advantage of the fact that $Z'(\mathbb{A})G'(\mathbb{Q})\backslash G'(\mathbb{A})$ is compact.

In the literature an adjustment for G is made in either of two equivalent ways. One possible adjustment, analogous to what we did for G' in §5, is to study functions that can be related to $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, where Z is the center consisting of scalar matrices. This quotient space is not compact, but it does have finite volume, as we shall see in a moment. Specifically for each character ω of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$, we define $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ to be the set of f on $G(\mathbb{A})$ such that

$$f(z\gamma g) = \omega(z)f(g) \qquad \text{for } z\in Z(\mathbb{A}),\ \gamma\in G(\mathbb{Q}),\ g\in G(\mathbb{A}) \tag{6.1a}$$

and such that $|f|$ is square integrable on $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$. We shall be interested in the right regular representation R_ω of $G(\mathbb{A})$ on this space. We put $\overline{G} = Z\backslash G$, so that we can identify $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ with $\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})$.

The other possible adjustment uses the subgroup $G^1 = G(\mathbb{A})^1$ of elements $g\in G(\mathbb{A})$ such that $|\det g|_{\mathbb{A}} = 1$. The discrete subgroup $G(\mathbb{Q})$ of $G(\mathbb{A})$ lies in G^1 by the Artin product formula (Theorem 3.3 of [Kn2]), and the quotient space $G(\mathbb{Q})\backslash G^1$ is noncompact of finite volume, by the theorem of Borel and Harish-Chandra quoted as Theorem 6.2 of [Kn2]. In this approach the objective is to understand the decomposition of the right regular representation of G^1 on $L^2(G(\mathbb{Q})\backslash G^1)$. The group G^1 has center $Z^1 = G^1\cap Z(\mathbb{A})$. If $(\mathbb{A}^\times)^1$ denotes the group of ideles of module 1, then the members of Z^1 have both diagonal entries equal to the same member of $(\mathbb{A}^\times)^1$. From Theorem 3.5 of [Kn2], we know that the abelian group $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$ is compact. Its characters are in one-one correspondence with the characters of Z^1 that are trivial on $Z^1\cap G(\mathbb{Q})$, hence with the irreducible representations of $Z^1G(\mathbb{Q})$ that are trivial on $G(\mathbb{Q})$. The formalism

$$L^2(G(\mathbb{Q})\backslash G^1) \cong \operatorname{ind}_{G(\mathbb{Q})}^{G^1} 1 \cong \operatorname{ind}_{Z^1G(\mathbb{Q})}^{G^1}\operatorname{ind}_{G(\mathbb{Q})}^{Z^1G(\mathbb{Q})} 1$$

therefore leads to the conclusion that $L^2(G(\mathbb{Q})\backslash G^1)$ decomposes as a Hilbert space orthogonal sum

$$L^2(G(\mathbb{Q})\backslash G^1) = \sum_{\omega_0\in(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\hat{}} L^2(Z^1G(\mathbb{Q})\backslash G^1,\omega_0),$$

where ω_0 is regarded as a character of $Z^1G(\mathbb{Q})$ that is trivial on $G(\mathbb{Q})$. Here $L^2(Z^1G(\mathbb{Q}))\backslash G^1, \omega_0)$ is the set of f on G^1 such that

$$f(z\gamma g) = \omega_0(z)f(g) \qquad \text{for } z \in Z^1,\ \gamma \in G(\mathbb{Q}),\ g \in G^1 \tag{6.1b}$$

and such that $|f|$ is square integrable on $Z^1G(\mathbb{Q})\backslash G^1$. Invariant integration on $Z^1G(\mathbb{Q})\backslash G^1$ can be achieved by pulling functions back to $G(\mathbb{Q})\backslash G^1$ and integrating there, and hence $Z^1G(\mathbb{Q})\backslash G^1$ has finite volume.

The inclusion of G^1 into $G(\mathbb{A})$ yields a map of G^1 into the quotient space $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, and this is onto since every member of $G(\mathbb{A})$ is the product of a member of G^1 and a positive scalar matrix at the infinite place. The map descends to a map of $Z^1G(\mathbb{Q})\backslash G^1$ onto $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, and the result is one-one since $G^1 \cap Z(\mathbb{A}) = Z^1$. Thus we may identify

$$Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}) \cong Z^1G(\mathbb{Q})\backslash G^1.$$

When a character is taken into account, matters are a little more complicated. Let ω be a character of $Z(\mathbb{A})$ trivial on $Z(\mathbb{Q})$, and let $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ be as in (6.1a). By the second isomorphism theorem, $Z(\mathbb{Q})\backslash Z(\mathbb{A})$ is isomorphic to $G(\mathbb{Q})\backslash Z(\mathbb{A})G(\mathbb{Q})$, and thus ω can be regarded as a character of $Z(\mathbb{A})G(\mathbb{Q})$ trivial on $G(\mathbb{Q})$. We can restrict ω to a character ω_0 of $Z^1G(\mathbb{Q})$ that is trivial on $G(\mathbb{Q})$, and then we obtain an identification of the function spaces

$$L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega) \cong L^2(Z^1G(\mathbb{Q})\backslash G^1, \omega_0). \tag{6.2}$$

Conversely when a character ω_0 of $Z^1G(\mathbb{Q})$ that is trivial on $G(\mathbb{Q})$ is given, we can extend ω to a unitary character ω of $Z(\mathbb{A})G(\mathbb{Q})$ that is trivial on $G(\mathbb{Q})$, and we again obtain (6.2). The complication is that the extension of ω_0 to ω is not unique.

By imposing a further condition on ω, we can get around this nonuniqueness. Let $\mathbb{Q}_\infty^+$ be the group of ideles that are trivial at all finite places and are positive at the infinite place, and let Z_∞ be the subgroup of $Z(\mathbb{A})$ whose diagonal entries are in $\mathbb{Q}_\infty^+$. Then $Z(\mathbb{A}) = Z^1 \times Z_\infty$ and $Z(\mathbb{A})G(\mathbb{Q}) = Z^1G(\mathbb{Q}) \times Z_\infty$. Hence a character ω_0 of $Z^1G(\mathbb{Q})$ trivial on $G(\mathbb{Q})$ extends uniquely to a character of $Z(\mathbb{A})G(\mathbb{Q})$ trivial on $G(\mathbb{Q})$ if we impose the condition that ω is trivial on Z_∞.

We choose to study the left side of (6.2) rather than the right side. Working with the right side would make the proof of the trace formula considerably more elegant. But as we shall see in [Kn-Ro], working with the left side makes it much easier to use the trace formula in applications. It will not simplify matters to assume that ω is trivial on Z_∞, and thus we do not assume this triviality.

Henceforth we therefore fix ω as a character of $Z(\mathbb{A})$ that is trivial on $Z(\mathbb{Q})$; by extracting the upper left entry of a scalar matrix, we may regard ω alternatively as a character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. We consider the space $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ and the right regular representation R_ω of $G(\mathbb{A})$ on this space.

Let $N = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ as algebraic subgroups of G, and put $P = MN$. If f is in $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, we define the **constant term** of f (along P) to be

$$f_P(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right)\, dx,$$

where dx has total mass one. This function is left invariant under $N(\mathbb{A})$ and $P(\mathbb{Q})$, the latter because the Artin product formula shows that conjugation by a member of $P(\mathbb{Q})$ does not change dx. Let $L^2_{\text{cusp}}(\omega)$ be the closed subspace of functions f such that $f_P(g)$ is 0 almost everywhere. This subspace is invariant under $R_\omega(G(\mathbb{A}))$.

Theorem 6.1. *If φ is in $C^\infty_{\text{com}}(G(\mathbb{A}), \omega^{-1})$, then $R_\omega(\varphi)$ is Hilbert-Schmidt, hence compact, on $L^2_{\text{cusp}}(\omega)$.*

Reference for sketch. [Gb-Ja, pp. 217–218].

Corollary 6.2. *$L^2_{\text{cusp}}(\omega)$ decomposes discretely into irreducible representations having finite multiplicity.*

Proof. The argument is the same as the proof that Theorem 1.6 implies Theorem 1.5.

Corollary 6.3. *If φ is in $C^\infty_{\text{com}}(G(\mathbb{A}), \omega^{-1})$, then $R_\omega(\varphi)$ is of trace class on $L^2_{\text{cusp}}(\omega)$.*

Proof. We can write $\varphi(x) = \int_{Z(\mathbb{A}} \psi(zx)\omega(z)\,dz$ for some smooth function ψ of compact support on $G(\mathbb{A})$. Then ψ is a finite sum of functions $\psi_\infty \times \psi_{\text{fin}}$, where ψ_∞ is smooth of compact support at the place ∞ and ψ_{fin} is locally constant of compact support at the finite places. Form φ_∞ and φ_{fin} from ψ_∞ and ψ_{fin} by integrating over the appropriate components of $Z(\mathbb{A})$, so that $\varphi_\infty \times \varphi_{\text{fin}}$ is in $C^\infty_{\text{com}}(G(\mathbb{A}), \omega^{-1})$. A theorem of Dixmier and Malliavin [Di-Ma] shows that each ψ_∞ is a sum of terms that are each the convolution of two compactly supported smooth functions. Also each ψ_{fin} is the convolution of ψ_{fin} with the characteristic function of some open compact subgroup. Consequently $\varphi_\infty \times \varphi_{\text{fin}}$ is the finite sum of convolutions of pairs of members of $C^\infty_{\text{com}}(G(\mathbb{A}), \omega^{-1})$. Then it follows from Theorem 6.1 that $R_\omega(\varphi)$ is a finite sum of products of two Hilbert-Schmidt operators and hence is of trace class.

The next step is to identify the orthogonal complement of the subspace $L^2_{\text{cusp}}(\omega)$ of $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ in a fashion analogous to Theorem 1.3. The dictionary for comparing subgroups of $SL_2(\mathbb{R})$ and $G(\mathbb{A})$ is that $\Gamma \leftrightarrow G(\mathbb{Q})$, $N \leftrightarrow N(\mathbb{A})$, and $\Gamma_\infty \leftrightarrow P(\mathbb{Q})$. The condition in §§1–4 that functions be even is analogous to the condition now that functions transform under ω. The proof of Lemma 1.1 used that $\Gamma_\infty \subset N$ and that $\Gamma_\infty\backslash N$ is compact, but it would have worked as well under the condition that $\Gamma_\infty\backslash N\Gamma_\infty$ is compact. We therefore obtain an adelic analog of that lemma: If ϕ is a continuous function on $G(\mathbb{A})$ satisfying

$$\phi(zn\gamma g) = \omega(z)\phi(g) \tag{6.3}$$

for $z \in Z(\mathbb{A})$, $n \in N(\mathbb{A})$, and $\gamma \in P(\mathbb{Q})$ and having compact support modulo $N(\mathbb{A})P(\mathbb{Q})$, then

$$\widehat{\phi}(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) \tag{6.4}$$

is a locally finite sum and defines a continuous function on $G(\mathbb{A})$ satisfying (6.1b) and having compact support modulo $Z(\mathbb{A})G(\mathbb{Q})$.

Lemma 6.4. *Let ϕ be a measurable function on $G(\mathbb{A})$ left invariant under $N(\mathbb{A})P(\mathbb{Q})$ and transforming under ω, and let F be a measurable function on $G(\mathbb{A})$ as in* (6.1b). *If $\widehat{|\phi|}$ is square integrable modulo $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$ and if F is in $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, then*

$$\langle \widehat{\phi}, F\rangle_{L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))} = \langle \phi, F_P\rangle_{L^2(Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A}))}, \tag{6.5}$$

the indicated integrals converging.

REMARKS. This is proved in the same way as Lemma 1.2. When an integral over $Z(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A})$ is written as an iterated integral over $(Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q}))\backslash G(\mathbb{A})$ and $(Z(\mathbb{A})P(\mathbb{Q}))\backslash(Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q}))$, the inner integral is rewritten over $N(\mathbb{Q})\backslash N(\mathbb{A})$ by the second isomorphism theorem. The equality (6.5) depends on normalizations of Haar measures, but we postpone this detail until after the proof of Lemma 6.7 below.

Theorem 6.5. *Within $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$, the orthogonal complement of $L^2_{\mathrm{cusp}}(\omega)$ is the closure of the space of all $\widehat{\phi}$ with ϕ continuous on $G(\mathbb{A})$, left invariant under $N(\mathbb{A})P(\mathbb{Q})$, transforming under $Z(\mathbb{A})$ by ω, and having compact support modulo $Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q})$.*

PROOF. Same as for Theorem 1.3.

Eisenstein series are used in the analysis of this orthocomplement. Let K be the maximal compact subgroup $O_2(\mathbb{R})\times\prod_p G(\mathbb{Z}_p)$ of $G(\mathbb{A})$, so that $G(\mathbb{A})=P(\mathbb{A})K$. If an element g is decomposed as $g=\left(\begin{smallmatrix} a & x\\ 0 & b\end{smallmatrix}\right)k$ with $k\in K$, we define

$$h(g)=\left|\tfrac{a}{b}\right|_{\mathbb{A}}^{1/2}. \tag{6.6}$$

This is well defined since $h(g)=1$ for any element g of $P(\mathbb{A})\cap K$. To be able to compute with the function $h(\,\cdot\,)$, we identify $\mathbb{A}^2$ with row vectors and introduce a kind of norm on $\mathbb{A}^2$. If $v_p=(\,x_p\quad y_p\,)$ is a row vector over $\mathbb{Q}_p$, we define $\|v_p\|_p=\max\{|x_p|_p,|y_p|_p\}$. A little computation shows that $\|v_pk_p\|_p=\|v_p\|_p$ for k_p in $GL_2(\mathbb{Z}_p)$. If $v_\infty=(\,x_\infty\quad y_\infty\,)$ is a row vector over $\mathbb{R}$, we define $\|v_\infty\|_\infty=\sqrt{x_\infty^2+y_\infty^2}$. Then of course, $\|v_\infty k_\infty\|_\infty=\|v_\infty\|_\infty$ for k_∞ in $O_2(\mathbb{R})$. If $v\in\mathbb{A}^2$ is decomposed as $v=v_\infty\times\prod v_p$, we let $\|v\|_{\mathbb{A}}=\|v_\infty\|_\infty\times\prod_p\|v_p\|_p$, and this norm is preserved under right multiplication by K.

Lemma 6.6. *The function $h(\,\cdot\,)$ defined in* (6.6) *is given on $G(\mathbb{A})$ by*

$$h(g)^{-1}=\frac{\|\,(\,0\quad 1\,)\,g\|_{\mathbb{A}}}{|\det g|_{\mathbb{A}}^{1/2}}.$$

PROOF. Since K preserves norms, it is sufficient to consider $g\in P(\mathbb{A})$. If $g=\left(\begin{smallmatrix} a & x\\ 0 & b\end{smallmatrix}\right)$, then

$$\frac{\|\,(\,0\quad 1\,)\,g\|_{\mathbb{A}}}{|\det g|_{\mathbb{A}}^{1/2}}=\frac{\|\,(\,0\quad b\,)\,\|_{\mathbb{A}}}{|ab|_{\mathbb{A}}^{1/2}}=\frac{|b|_{\mathbb{A}}^{1/2}}{|a|_{\mathbb{A}}^{1/2}},$$

and the result follows.

The square $h(\,\cdot\,)^2$ is an adelic analog of the function $y(\,\cdot\,)$ in §§1–4. For example, Haar measure on $G(\mathbb{A})$ may be expressed in terms of h in analogy with (1.7). If $g=pk$ is a decomposition of an element relative to $G(\mathbb{A})=P(\mathbb{A})K$, then we have

$$dg=d_lp\,dk=h(p)^{-2}\,d_rp\,dk, \tag{6.7}$$

where d_lp and d_rp are left and right Haar measures on $P(\mathbb{A})$. Normalizations of Haar measures will be discussed in more detail after the proof of Lemma 6.7 below.

The analog of summing over $\Gamma_\infty\backslash\Gamma$ will be summing over $P(\mathbb{Q})\backslash G(\mathbb{Q})$. By the Bruhat decomposition we can take as representatives γ of the cosets $P(\mathbb{Q})\gamma$ the elements 1 and $w\begin{pmatrix}1&\xi\\0&1\end{pmatrix}$ with ξ in $\mathbb{Q}$, where $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

The next lemma will reduce several estimates about $h(\cdot)$ to estimates in the setting of §§1–4.

Lemma 6.7. *Let $g=\begin{pmatrix}yu&xu\\0&u\end{pmatrix}$ vary through a compact subset X of $P(\mathbb{A})$, and let γ vary through matrices of the form $\gamma=w\begin{pmatrix}1&\xi\\0&1\end{pmatrix}$ with $\xi\in\mathbb{Q}$. Write $\xi=d/c$ with $\mathrm{GCD}(c,d)=1$, and write also $x=x_\infty\prod_p x_p$ and $y=y_\infty\prod_p y_p$. Then there exists a constant B such that*

$$h(\gamma g)\le\frac{B}{|cz_\infty+d|}$$

for all $g\in X$ and all $\xi\in\mathbb{Q}$, where $z_\infty=x_\infty+iy_\infty$ as a member of $\mathbb{C}$.

Proof. We have

$$\begin{aligned}h(\gamma g)^{-1}&=\|\,(0\quad 1)\,\gamma g\|_{\mathbb{A}}=\|\,(0\quad 1)\begin{pmatrix}0&1\\1&\xi\end{pmatrix}\begin{pmatrix}yu&xu\\0&u\end{pmatrix}\|_{\mathbb{A}}\\&=\|\,(\,yu\quad u(x+\xi)\,)\,\|_{\mathbb{A}}=|u|_{\mathbb{A}}\,\|\,(\,y\quad x+\xi\,)\,\|_{\mathbb{A}}\\&=|u|_{\mathbb{A}}\,\|\,(\,cy\quad cx+d\,)\,\|_{\mathbb{A}}=|u|_{\mathbb{A}}\,|cz_\infty+d|\prod_p\|\,(\,cy_p\quad cx_p+d\,)\,\|_p.\end{aligned}$$

Thus it is enough to bound

$$\prod_p\|\,(\,cy_p\quad cx_p+d\,)\,\|_p=\prod_p\max(|cy_p|_p,|cx_p+d|_p)$$

below. We do so by making repeated use of the inequality

$$\max(a_1b_1,a_2b_2)\ge\max(a_1,a_2)\min(b_1,b_2)$$

valid for positive reals. There are three cases. First suppose that $|d|_p<|c|_p|x_p|_p$. Then

$$\max(|cy_p|_p,|cx_p+d|_p)=|c|_p\max(|y_p|_p,|x_p|_p)\ge|d|_p\,\frac{\max(|y_p|_p,|x_p|_p)}{|x_p|_p},$$

and hence

$$\begin{aligned}\max(|cy_p|_p,|cx_p+d|_p)&\ge\max\Big(|c|_p\max(|y_p|_p,|x_p|_p),|d|_p\,\frac{\max(|y_p|_p,|x_p|_p)}{|x_p|_p}\Big)\\&\ge\max(|c|_p,|d|_p)\min\Big(\max(|y_p|_p,|x_p|_p),\frac{\max(|y_p|_p,|x_p|_p)}{|x_p|_p}\Big)\\&=\min\Big(\max(|y_p|_p,|x_p|_p),\frac{\max(|y_p|_p,|x_p|_p)}{|x_p|_p}\Big)\end{aligned}$$

since $\max(|c|_p,|d|_p)=1$. Second suppose that $|d|_p>|c|_p|x_p|_p$. Then

$$\begin{aligned}\max(|cy_p|_p,|cx_p+d|_p)&=\max(|c|_p|y_p|_p,|d|_p)\\&\ge\max(|c|_p,|d|_p)\min(|y_p|_p,1)=\min(|y_p|_p,1).\end{aligned}$$

Third suppose that $|d|_p = |c|_p |x_p|_p$. Then

$$\max(|cy_p|_p, |cx_p + d|_p) \geq |c|_p |y_p|_p = |d|_p \frac{|y_p|_p}{|x_p|_p},$$

and hence

$$\begin{aligned}\max(|cy_p|_p, |cx_p + d|_p) &\geq \max\Big(|c|_p|y_p|_p, |d|_p \frac{|y_p|_p}{|x_p|_p}\Big) \\ &\geq \max(|c|_p, |d|_p) \min\Big(|y_p|_p, \frac{|y_p|_p}{|x_p|_p}\Big) \\ &= \min\Big(|y_p|_p, \frac{|y_p|_p}{|x_p|_p}\Big).\end{aligned}$$

Combining the three cases, we see that

$$\max(|cy_p|_p, |cx_p + d|_p) \geq \min\Big(|y_p|_p, 1, \frac{|y_p|_p}{|x_p|_p}\Big). \tag{6.8}$$

We claim that the product over p of (6.8) is bounded below for $g \in X$. For a single g, this is obvious since $|x_p|_p \leq 1$ and $|y_p|_p = 1$ for all but finitely many p, so that the right side of (6.8) is 1 for all but finitely many p. If a sequence $g^{(n)} \leftrightarrow (x^{(n)}, y^{(n)})$ has the product tending to 0, we can choose a convergent subsequence, say with limit $g^{(0)} \leftrightarrow (x^{(0)}, y^{(0)})$. The convergence has to take place in one of the product spaces of which $G(\mathbb{A})$ is a union, and therefore there are only finitely many p for which we do not have $|x_p^{(n)}|_p \leq 1$ and $|y_p^{(n)}|_p = 1$ for all n. For all but finitely many p, (6.8) is therefore 1 for all n, and we have convergence for the remaining p. Thus (6.8) cannot be tending to 0, and the proof is complete.

Now let us discuss normalizations of Haar measures. Discrete groups get the counting measure, and the compact group $\mathbb{Q}\backslash\mathbb{A}$ gets the measure of total mass one. However, it will not be convenient to assume that $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$ has total mass one. Instead we proceed as follows: We fix any Haar measure on $\mathbb{A}^\times$ and give $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ the quotient measure. The group $\mathbb{Q}_\infty^+$ of ideles that are trivial at all finite places and are positive at the infinite place is isomorphic to the group $\mathbb{R}_+^\times$ of positive reals by $t \to |t|_{\mathbb{A}}$, and we transport dx/x on $\mathbb{R}_+^\times$ to a Haar measure on $\mathbb{Q}_\infty^+$. Then we can use the isomorphism $\mathbb{Q}^\times\backslash\mathbb{A}^\times \cong \mathbb{Q}_\infty^+ \times \mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$ to determine a Haar measure on $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$.

For the parabolic $P(\mathbb{A})$, we have $P(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})$ with

$$M(\mathbb{A}) = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \;\middle|\; u, v \in \mathbb{A}^\times \right\}. \tag{6.9a}$$

We identify $N(\mathbb{A})$ with $\mathbb{A}$ and define Haar measure on $N(\mathbb{A})$ accordingly. Next we identify $Z(\mathbb{A})$ with $\mathbb{A}^\times$ by $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \leftrightarrow u$, and then Haar measure is determined on $Z(\mathbb{A})$. The equality $M(\mathbb{A}) = \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right\} Z(\mathbb{A})$ follows from the decomposition $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} uv^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$, and thus we have an isomorphism $M(\mathbb{A}) \cong \mathbb{A}^\times Z(\mathbb{A})$. This isomorphism allows us to fix Haar measure on $M(\mathbb{A})$. In the notation of (6.9a), Haar measure on $M(\mathbb{A})$ is nothing more than $du\,dv$, where du and dv indicate Haar measure on $\mathbb{A}^\times$.

Next the decomposition $P(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})$ allows us to use the measures dn on $N(\mathbb{A})$ and dm on $M(\mathbb{A})$ to determine left and right Haar measures $d_l p$ and $d_r p$ on $P(\mathbb{A})$ by

$$d_r p = dn\, dm \quad \text{and} \quad d_l(p) = d_r(p^{-1}). \tag{6.9b}$$

We pick any Haar measure on K, not insisting that it have total mass one, and then we use (6.7) to determine Haar measure on $G(\mathbb{A})$. Finally we require that invariant measures on closed subgroups and quotients are to be compatible with the measure on the whole group. In particular this requirement fixes the measures on the quotients of $G(\mathbb{A})$ in (6.5). It also fixes Haar measure on $Z(\mathbb{A})G(\mathbb{Q})$ since

$$Z(\mathbb{A})\backslash Z(\mathbb{A})G(\mathbb{Q}) \cong Z(\mathbb{Q})\backslash G(\mathbb{Q}).$$

For the remainder of this section we largely follow [Gb-Ja]. For each $s \in \mathbb{C}$, we introduce a Hilbert space $H(s)$ of functions $F : G(\mathbb{A}) \to \mathbb{C}$ with

$$F\left(\begin{pmatrix} q_1 au & x \\ 0 & q_2 bv \end{pmatrix} g\right) = h\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)^{1+s} \omega(bv) F\left(\begin{pmatrix} uv^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right) \tag{6.10}$$

for q_1 and q_2 in $\mathbb{Q}^\times$, a and b in $\mathbb{Q}_\infty^+$, u and v in $(\mathbb{A}^\times)^1$, x in $\mathbb{A}$, and g in $G(\mathbb{A})$. Such functions depend on u and v only as members of $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$, and the norm squared is taken to be

$$\int_{(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K} \left| F\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} k\right)\right|^2 du\, dk. \tag{6.11}$$

If F satisfies (6.10), then F is completely determined by its values on elements $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} k$ with $u \in (\mathbb{A}^\times)^1$ and $k \in K$ since $G(\mathbb{A}) = P(\mathbb{A})K$ and since the part $\begin{pmatrix} au & x \\ 0 & bv \end{pmatrix}$ of the matrix in (6.10) is the most general member of $P(\mathbb{A})$.

Conversely let H be the Hilbert space of all f on $(\mathbb{A}^\times)^1 \times K$ such that

(i) f is left invariant under $\mathbb{Q}^\times$ in the first variable
(ii) $f(uv, k) = f\left(u, \left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right) k\right)$ whenever $\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is in $(\mathbb{A}^\times)^1 \cap K$
(iii) f is square integrable on $(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1) \times K$.

If f is in H, then we can extend f uniquely to a function $F = f_s$ in $H(s)$ by

$$f_s\left(\begin{pmatrix} au & 0 \\ 0 & bv \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) = h\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)^{1+s} \omega(bv) f(uv^{-1}, k). \tag{6.12}$$

The group $G(\mathbb{A})$ operates on $H(s)$ via the right regular representation, which we denote $P^{\omega,s}$. This representation is unitary if s is imaginary.

To postpone technical difficulties until the end, fix a finite-dimensional representation η of the compact abelian group $\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$ and a finite-dimensional representation τ of the compact group K. Both η and τ are to be thought of as large (and therefore reducible). Let $W(\eta, \tau)$ be the subspace of $f \in H$ such that $u \mapsto f(uu_0, k_0)$, for each (u_0, k_0), is a linear combination of matrix coefficients of the constituents of η and such that $k \mapsto f(u_0, k_0 k)$, for each (u_0, k_0), is a linear combination of matrix coefficients of the constituents of τ. Let $\tilde\eta = \omega\eta^c$, where η^c denotes the contragredient of η; $\tilde\eta$ will play the role of a Weyl group transform of η. Possibly by replacing η by $\eta \oplus \tilde\eta$, we may assume that $\tilde\eta = \eta$, i.e., that $\eta = \omega\eta^c$. *We make this assumption in what follows.* It will cause us no loss of generality since our interest is in what happens as η gets large.

If f is in $W(\eta,\tau)$, the **Eisenstein series** $E(g,f,s)$ corresponding to f is defined formally by

$$E(g,f,s)=\sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_s(\gamma g) \tag{6.13}$$

for $g\in G(\mathbb{A})$ and $s\in\mathbb{C}$. If a member g of $G(\mathbb{A})$ is decomposed according to $G(\mathbb{A})=P(\mathbb{A})K$ as

$$g=pk=\begin{pmatrix} au & 0\\ 0 & bv\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k$$

with a and b in $\mathbb{Q}_\infty^+$ and with u and v in $(\mathbb{A}^\times)^1$, let us write $b(g)$, $u(g)$, $v(g)$, and $\kappa(g)$ for b, u, v, and k. Then we can rewrite (6.13) as

$$E(g,f,s)=\sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} h(\gamma g)^{1+s}\omega(b(\gamma g)v(\gamma g))f(u(\gamma g)v(\gamma g)^{-1},\kappa(\gamma g)).$$

The functions ω and f are bounded. By Lemma 6.7 and the convergence of the series $\sum|cz+d|^{-(1+s)}$ for $\operatorname{Re}s>1$, the series for $E(g,f,s)$ is absolutely convergent if $\operatorname{Re}s>1$, and the convergence is uniform for g and s in compact sets. By Lemmas 6.7 and 2.2, there is a constant $C(\varepsilon,b)$ such that

$$|E(g,f,s)|\le C(\varepsilon,b)(\sup_K|f|)h(g)^{1+\operatorname{Re}s}$$

whenever $h(g)\ge b$ and $1+\varepsilon\le\operatorname{Re}s\le 1+\varepsilon^{-1}$. As a function of $g\in G(\mathbb{A})$, $E(g,f,s)$ is an automorphic form on $G(\mathbb{A})$ is the sense of the definition before Theorem 7.1 of [Kn2].

Let $\mathcal{C}_\omega((Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q}))\backslash G(\mathbb{A}),(\eta,\tau))$ be the set of continuous functions on $G(\mathbb{A})$ transforming as in (6.3), having compact support modulo $Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q})$, and satisfying the condition that $\phi\left(\left(\begin{smallmatrix} r & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} u & 0\\ 0 & 1\end{smallmatrix}\right)k\right)$ is in $W(\eta,\tau)$ for fixed $r\in\mathbb{Q}_\infty^+$ and is smooth for $r\in\mathbb{Q}_\infty^+$ when u and k are fixed, with uniform estimates on the smoothness as u and k vary. We define the **Fourier-Laplace transform** of such a function ϕ by

$$\Phi(g,s)=\int_0^\infty \phi(a(y)^{-1}g)y^{\frac12(1+s)}\,\frac{dy}{y}. \tag{6.14}$$

The function $\Phi(\,\cdot\,,s)$ on $G(\mathbb{A})$ is in $H(s)$ for each s, and the restriction to the subgroup $(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K$ is in H. We write $\Phi(s)$ for the restriction. Just as in (2.9), Fourier inversion gives

$$\begin{aligned}\phi(g)&=\frac{1}{4\pi}\int_{\operatorname{Re}s=\sigma}\Phi(g,s)\,d|s|\\ &=\frac{1}{4\pi}\int_{\operatorname{Re}s=\sigma}h(g)^{1+s}\omega(v(g))\Phi\left(\begin{pmatrix} u(g)v(g)^{-1} & 0\\ 0 & 1\end{pmatrix}\kappa(g),s\right)d|s|\end{aligned} \tag{6.15}$$

for any real σ. With $\widehat{\phi}$ defined as in (6.4), we obtain, as in (2.10),

$$\widehat{\phi}(g)=\frac{1}{4\pi}\int_{\operatorname{Re}s=\sigma}E(g,\Phi(s),s)\,d|s| \tag{6.16}$$

for $\sigma>1$.

Proposition 6.8. *If* $\operatorname{Re} s > 1$ *and if* f *is in* $W(\eta, \tau)$*, then the constant term of the Eisenstein series for* f *is given by*

$$E_P(\,\cdot\,, f, s) = f_s + (M(s)f)_{-s} \tag{6.17}$$

for an operator $M(s) : W(\eta,\tau) \to W(\eta,\tau)$ *given by*

$$(M(s)f)_{-s}(g) = \int_{N(\mathbb{A})} f_s(wng)\,dn.$$

Proof. We start from

$$E(g,f,s) = f_s(g) + \sum_{\xi\in\mathbb{Q}} f_s\left(w \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) g\right).$$

Replacing g by ng and integrating for $n \in N(\mathbb{Q})\backslash N(\mathbb{A})$ gives

$$\begin{aligned} E_P(f,g,s) &= f_s(g) + \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\xi\in\mathbb{Q}} f_s\left(w \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) ng\right)\,dn \\ &= f_s(g) + \int_{N(\mathbb{A})} f_s(wng)\,dn, \end{aligned}$$

as required. An easy change of variables shows that $M(s)$ carries $W(\eta,\tau)$ to itself because $\eta = \tilde{\eta}$, i.e., $\eta = \omega\eta^c$.

Corollary 6.9. *Let* ϕ *and* ψ *be members of* $\mathcal{C}_\omega((Z(\mathbb{A})N(\mathbb{A})P(\mathbb{Q}))\backslash G(\mathbb{A}), (\eta,\tau))$, *and let* Φ *and* Ψ *be the Fourier-Laplace transforms of* ϕ *and* ψ*. Then*

$$\begin{aligned} &\langle \widehat{\phi}, \widehat{\psi}\rangle_{L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))} \\ &\quad = \frac{1}{4\pi}\int_{\operatorname{Re} s=\sigma} \big(\langle \Phi(s), \Psi(-\bar{s})\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)} \\ &\qquad\qquad + \langle M(s)\Phi(s), \Psi(\bar{s})\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}\big)\,d|s| \end{aligned}$$

for any $\sigma > 1$.

The proof of Corollary 6.9 is almost the same as for Corollary 2.7. Two comments are in order. One is that the constant $1/2\pi$ in Corollary 2.7 has become $1/4\pi$ here because the formula for the constant term of an Eisenstein series no longer involves a factor of 2. The other comment concerns normalizations of Haar measure. Suppose that x, y, r_1, and r_2 are positive reals viewed as ideles at the infinite place such that

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

A little computation with Jacobian determinants shows that $\dfrac{dx\,dy}{xy} = \dfrac{dr_1\,dr_2}{r_1r_2}$. The right side of this identity is what was defined as Haar measure for the infinite place of $M(\mathbb{A})$, and therefore dy/y is Haar measure for the subgroup of all $a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$. Representatives of the cosets of $Z(\mathbb{A})\backslash G(\mathbb{A})$ are the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} a(y) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} k$ with $y > 0$, $u \in (\mathbb{A}^\times)^1$, $x \in \mathbb{A}$, and $k \in K$, and it follows that Haar measure on $Z(\mathbb{A})\backslash G(\mathbb{A})$ is $y^{-1}\,dx\,du\,dk\,\dfrac{dy}{y}$. The invariant measure to use on $Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})$ is therefore $y^{-1}\,du\,dk\,\dfrac{dy}{y}$.

Theorem 6.10. *The family of operators $M(s)$, initially given as an analytic family $M(s): W(\eta,\tau)\to W(\eta,\tau)$, extends to be meromorphic for $s\in\mathbb{C}$. The only possible pole for $\operatorname{Re} s\geq 0$ is at $s=1$ and is at most simple. As a function of $\operatorname{Im} s$, $M(s)$ is uniformly at most of polynomial growth, apart from the pole, in any vertical strip $0\leq \operatorname{Re} s<\sigma$. The continued operators satisfy $M(-s)M(s)=1$ as a meromorphic function of s.*

Reference. See [Gb-Ja] and also Jacquet's article [Ja] in this volume.

Now we move the line of integration in Corollary 6.9 to $\operatorname{Re} s=0$, just as in §2. The integrand is meromorphic, the functions $\Phi(s)$ and $\Psi(s)$ are Schwartz functions of $\operatorname{Im} z$ uniformly in vertical strips, and the growth of $M(z)$ is controlled by Theorem 6.10. We can move the line of integration by the Cauchy Integral Formula and an easy passage to the limit, picking up a residue term from $s=1$. The result is

$$\begin{aligned}\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))} &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\big(\langle\Phi(it),\Psi(it)\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)} \\ &\qquad + \langle M(it)\Phi(it),\Psi(-it)\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}\big)\,dt \\ &\quad + \frac{1}{2}\lim_{s\to1}\langle (s-1)M(s)\Phi(1),\Psi(1)\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)} \qquad (6.18)\end{aligned}$$

for ϕ and ψ as in Corollary 6.9.

Next we simplify this expression, using that $\tilde{\eta}=\eta$. The residue term, to which we return shortly, may be shown to be

$$c\sum_{\chi^2=\omega}\langle\Phi(1),\chi\circ\det\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}\overline{\langle\Psi(1),\chi\circ\det\rangle}_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)},$$

where c is a known positive constant. (See [Gb-Ja, p. 227].) For the integral term on the right side of (6.18), the first step is to check from the definitions that $\tilde{\eta}=\eta$ implies $M(s)^*=M(\bar{s})$ for $\operatorname{Re} s>1$. Then this relation persists for all s by analytic continuation. Since $M(-s)M(s)=1$ by Theorem 6.10, it follows that $M(it)$ is unitary with inverse $M(-it)$. Then (6.18) may be rewritten by the techniques of Corollary 2.8 as

$$\begin{aligned}&\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))} \\ &\quad=\frac{1}{8\pi}\int_{-\infty}^{\infty}\langle\Phi(it)+M(-it)\Phi(-it),\Psi(it)+M(-it)\Psi(-it)\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}\,dt \\ &\quad+c\sum_{\chi^2=\omega}\langle\Phi(1),\chi\circ\det\rangle_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}\overline{\langle\Psi(1),\chi\circ\det\rangle}_{L^2((\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1)\times K)}. \qquad (6.19)\end{aligned}$$

With this formula in place, the kind of analysis in §2, in view of Theorem 6.5, shows that $L^2_{\text{cusp}}(\omega)^\perp$ is the sum of a direct integral of the representations $H(s)$, together with a discrete contribution from the residues at $s=1$. This is the adelic analog of Theorem 1.4. For details, see [Gb-Ja, §4]. We denote the direct integral term by $L^2_{\mathrm{cont}}(\omega)$ and the term for the various residues by $L^2_{\mathrm{res}}(\omega)$. The decomposition may be summarized as

$$L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}),\omega) = L^2_{\mathrm{cusp}}(\omega) \oplus L^2_{\mathrm{cont}}(\omega) \oplus L^2_{\mathrm{res}}(\omega). \tag{6.20}$$

The residues come from one-dimensional representations of $G(\mathbb{A})$, necessarily of the form $g \mapsto \chi(\det g)$. The corresponding members of $L^2_{\mathrm{res}}(\omega)$ are the functions $f(g) = \chi(\det g)$. Since f is to be left invariant under $G(\mathbb{Q})$, χ is a character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Since f is to transform by ω under $Z(\mathbb{A})$, $\chi^2 = \omega$. Thus the decomposition of $L^2_{\mathrm{res}}(\omega)$ is a Hilbert space direct sum

$$L^2_{\mathrm{res}}(\omega) = \bigoplus_{\chi^2=\omega} \mathbb{C}_{\chi\circ\det}. \tag{6.21}$$

7. Adelic Trace Formula

We continue with notation as in §6. In the decomposition (6.20) the difficult term to understand is $L^2_{\mathrm{cusp}}(\omega)$. The operator $R_\omega(\varphi)f(x) = \int_{\overline{G}(\mathbb{A})} f(xy)\varphi(y)\,dy$, for $\varphi \in C^\infty_{\mathrm{com}}(G(\mathbb{A}),\omega^{-1})$, acts on $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ and leaves $L^2_{\mathrm{cusp}}(\omega)$ stable. It is of trace class on this subspace, by Corollary 6.3. The adelic trace formula gives an explicit expression for the trace of this operator on the subspace $L^2_{\mathrm{cusp}}(\omega)$. The final formula is stated in [Gf-Gr-P], [Ja-Lgl], [Du-La], [Ar1], [Gb1], [Gb-Ja], and [Ar4], and we shall follow [Gb-Ja].

The idea is that $R_\omega(\varphi)$ is given by manageable integral operators on the whole space and on the subspaces $L^2_{\mathrm{cont}}(\omega)$ and $L^2_{\mathrm{res}}(\omega)$. Let kernels for these integral operators be $K(x,y)$, $K_{\mathrm{cont}}(x,y)$, and $K_{\mathrm{res}}(x,y)$. Then the operator on $L^2_{\mathrm{cusp}}(\omega)$ must be given by the kernel

$$K_{\mathrm{cusp}}(x,y) = K(x,y) - K_{\mathrm{cont}}(x,y) - K_{\mathrm{res}}(x,y), \tag{7.1}$$

and the trace in question ought to be the integral of $K_{\mathrm{cusp}}(x,x)$ over the quotient $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$.

These kernels are not uniquely defined as functions on $G(\mathbb{A}) \times G(\mathbb{A})$ without some further restriction. In the case of $K(x,y)$, the same derivation as for (5.4) leads to the formula

$$K(x,y) = \sum_{\gamma\in\overline{G}(\mathbb{Q})} \varphi(x^{-1}\gamma y). \tag{7.2}$$

Then $K(x,y)$ is left invariant in each variable under $G(\mathbb{Q})$ and satisfies

$$K(z_1x, z_2y) = \omega(z_1)\omega(z_2)^{-1}K(x,y) \qquad \text{for } z_1, z_2 \in Z(\mathbb{A}). \tag{7.3}$$

It is this condition that determines $K(x,y)$ uniquely.

Similarly to determine the kernels $K_{\mathrm{cont}}(x,y)$ and $K_{\mathrm{res}}(x,y)$ uniquely, we insist that they satisfy the same invariance properties as $K(x,y)$. Then $K_{\mathrm{cont}}(x,y)$ and $K_{\mathrm{res}}(x,y)$ can be written down fairly explicitly. The techniques for $K_{\mathrm{cont}}(x,y)$ are the same as for Proposition 4.6. To get at $K_{\mathrm{cont}}(x,y)$, we need to know that the Eisenstein series themselves, not just their constant terms, admit meromorphic continuations.

Theorem 7.1. *If f is in a subspace $W(\eta,\tau)$ of H, then the function $s \mapsto E(g,f,s)$, initially given as an analytic function for* $\operatorname{Re} s > 1$*, extends to be meromorphic in $\mathbb{C}$. Its constant term is given by the analytic continuation of $E_P(g,f,s)$,* *and $E(g,f,s)$ has the same poles as $E_P(g,f,s)$. Also $E(g,f,s)$ is at most of polynomial growth in* $\operatorname{Im} s$ *in any vertical strip* $0 \le \operatorname{Re} s \le \sigma$.

Reference. For a discussion of this theorem, see [Ja] in this volume.

To obtain an expression for $K_{\text{cont}}(x,y)$, we proceed as in Proposition 4.6. We can choose an orthonormal basis $\{f_\alpha\}$ of H such that each f_α is in some $W(\eta,\tau)$. Theorem 7.1 shows that $E(x,f_\alpha,it)$ is meaningful. If $P^{\omega,it}$ is the unitary representation of $G(\mathbb{A})$ on $H(it)$, then calculations in [Gb-Ja, pp. 232–234] show that[2]

$$K_{\text{cont}}(x,y) = \frac{1}{8\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}\langle P^{\omega,it}(\varphi)f_\beta, f_\alpha\rangle E(x,f_\alpha,it)\overline{E(y,f_\beta,it)}\,dt. \tag{7.4}$$

Moreover, an easy computation with (6.21) shows that

$$K_{\text{res}}(x,y) = (\operatorname{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})))^{-1}\sum_{\chi^2=\omega}\chi(\det x)\overline{\chi(\det y)}\int_{\overline{G}(\mathbb{A})}\varphi(g)\chi(\det g)\,dg. \tag{7.5}$$

A direct attempt to integrate $K_{\text{cusp}}(x,x)$ with the aid of formulas (7.1) through (7.5) leads to $\infty-\infty$, and a more subtle approach is needed. Selberg [Se1] already saw the need for truncation in the classical setting (0.3), but his method was adapted to a fundamental domain for $SL_2(\mathbb{Z})$ in the upper half plane. We shall use the truncation operator of Arthur [Ar3], which does not require a fundamental domain for its definition. Expositions of this operator appear in [Gb-Ja] and [Mo-Wa].

Recall that $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. When w is embedded in $\mathbb{A}^\times$, we regard it as embedded diagonally.

Lemma 7.2. *For any $n \in N(\mathbb{A})$ and $g \in G(\mathbb{A})$, $h(wng) \le h(g)^{-1}$.*

Proof. Let us write $g = n'ak$ with $n' \in N(\mathbb{A})$, a diagonal, and $k \in K$. Then $wng = wnn'ak = (waw^{-1})(wn''w^{-1})k$. It follows from Lemma 6.6 that $h\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \le 1$, and therefore

$$h(wng) = h(waw^{-1})h(wn''w^{-1}) = h(g)^{-1}h(wn''w^{-1}) \le h(g)^{-1}.$$

Corollary 7.3. *If $h(\gamma_0 g) > 1$ for some $\gamma_0 \in P(\mathbb{Q})\backslash G(\mathbb{Q})$, then $h(\gamma g) < 1$ for all other $\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})$.*

Remark. Since $h(p) = 1$ for $p \in P(\mathbb{Q})$ by Artin's product formula, $h(\gamma g)$ is well defined as a function of γ in $P(\mathbb{Q})\backslash G(\mathbb{Q})$.

Proof. We may assume that $\gamma_0 = 1$. By the Bruhat decomposition, $\gamma = w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ for some $\xi \in \mathbb{Q}$. Then $\gamma = wn$ for some $n \in N(\mathbb{A})$, and Lemma 7.2 gives $h(\gamma g) = h(wng) \le h(g)^{-1} < 1$.

Fix $T \in \mathbb{R}$ with $T > 0$, and let I_T be the characteristic function of the set $[e^T, +\infty)$. For $T > 0$, the Arthur **truncation operator** Λ^T is defined on all

[2] The formula (5.20) in [Gb-Ja] for K_{cont} has a coefficient $1/4\pi$. The reason for this apparent discrepancy is that $dy = \frac{1}{2}\,dt$.

locally integrable complex-valued functions f on $G(\mathbb{A})$ that are left invariant under $G(\mathbb{Q})$ by

$$\Lambda^T f(g) = f(g) - \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_P(\gamma g) I_T(h(\gamma g)). \tag{7.6}$$

The sum[3] in (7.6) has at most one nonzero term, by Corollary 7.3, and $f_P(\gamma g)$ depends only on the coset of γ in $P(\mathbb{Q})\backslash G(\mathbb{Q})$. Thus $\Lambda^T f(g)$ is well defined. It is clearly left invariant under $G(\mathbb{Q})$. If f is **cuspidal** in the sense that $f_P = 0$, then $\Lambda^T f = f$.

Corollary 7.4. *If $T > 0$, then $(\Lambda^T f)_P(g) = 0$ unless $I_T(h(g)) = 0$.*

Proof. Assume that $I_T(h(g)) = 1$. Lemma 7.2 shows that $I_T(h(wng)) = 0$ for all $n \in N(\mathbb{A})$. Hence

$$\begin{aligned}\Lambda^T f(ng) &= f(ng) - f_P(ng) I_T(h(ng)) - \sum_{\xi \in \mathbb{Q}} f_P\left(w \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) ng\right) I_T\left(h\left(w \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) ng\right)\right) \\ &= f(ng) - f_P(g).\end{aligned}$$

Integrating over $n \in N(\mathbb{Q})\backslash N(\mathbb{A})$ therefore gives

$$(\Lambda^T f)_P(g) = f_P(g) - f_P(g) = 0.$$

Corollary 7.5. *If $T > 0$, then $\Lambda^T(\Lambda^T f) = \Lambda^T f$.*

Proof. We have

$$\Lambda^T(\Lambda^T f)(g) = (\Lambda^T f)(g) - \sum_{\gamma} (\Lambda^T f)_P(\gamma g) I_T(h(\gamma g)).$$

If $I_T(h(\gamma g)) \neq 0$, then Corollary 7.4 shows that $(\Lambda^T f)_P(\gamma g) = 0$. Hence every term in the sum is 0.

Proposition 7.6. *If $T > 0$, then Λ^T is a Hermitian operator on the space $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$.*

Reference. [Gb-Ja, p. 230] or [Ar3, pp. 91–92].

Because of Corollary 7.5 and Proposition 7.6, Λ^T is an orthogonal projection on $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, and we know that its image contains $L^2_{\text{cusp}}(\omega)$. Note, however, that the truncation operator does not commute with the action of $G(\mathbb{A})$, and its image is not $G(\mathbb{A})$ invariant.

In order to obtain more subtle properties of the truncation operator, it is helpful to understand more of the geometry of the action of $G(\mathbb{Q})$ on $G(\mathbb{A})$. Recall that products from $N(\mathbb{A}) \times M(\mathbb{A}) \times K$ cover $G(\mathbb{A})$. Let

$$\begin{aligned} M_\infty &= \{m \in M(\mathbb{A}) \mid \text{diagonal entries of } m \text{ are in } \mathbb{Q}^+_\infty\} \\ M^1 &= \{m \in M(\mathbb{A}) \mid \text{diagonal entries of } m \text{ are in } (\mathbb{A}^\times)^1\}.\end{aligned}$$

Here M_∞ is the direct product of $Z_\infty = M_\infty \cap \mathbb{Z}(\mathbb{A})$ and $A_\infty = \{a(y) \mid y \in \mathbb{Q}^+_\infty\}$. Then $M(\mathbb{A}) = M_\infty M^1$, and hence products from $N(\mathbb{A}) \times M_\infty \times M^1 \times K$ cover $G(\mathbb{A})$.

[3] Instead of using I_T, Arthur uses a function $\hat\tau_P$ and incorporates T into its argument. Arthur's notation is especially suited to higher rank groups.

A **Siegel set** $\mathcal{S}$ is a subset of $G(\mathbb{A})$ consisting of all $nrmk$ with n in a compact subset of $N(\mathbb{A})$, r in M_∞ with $h(r) \geq c > 0$, m in a compact subset of M^1, and k in K. The set $\mathcal{S}$ is the product of Z_∞ and the set $\mathcal{S}^1 = \mathcal{S} \cap G^1$. The set $\mathcal{S}^1$ may be viewed as an adelic analog of a rectangular set in the upper half plane that is unbounded above but is bounded on the other three sides.

Lemma 7.7. *Let $c' > 0$. On any compact set of elements $g \in G(\mathbb{A})$, there are only finitely many $\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})$ such that $h(\gamma g) \geq c'$ for some g in the compact set.*

PROOF. By the Bruhat decomposition we can take as coset representatives γ the elements 1 and $w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ with $\xi \in \mathbb{Q}$. Thus it is enough to consider $h(\gamma g)$ for $\gamma = w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}$. Since right translation by K does not affect $h(g)$, we may assume that g is in $P(\mathbb{A})$. Write $g = \begin{pmatrix} yu & xu \\ 0 & u \end{pmatrix}$ with $x = x_\infty \prod_p x_p$ and $y = y_\infty \prod_p y_p$, and put $\xi = d/c$ with $\mathrm{GCD}(c, d) = 1$. If $h(\gamma g) \geq c'$ for some g in the given compact set, then Lemma 6.7 gives

$$B^{-1}|cz_\infty + d| \leq h(\gamma g)^{-1} \leq c'^{-1},$$

where $z_\infty = x_\infty + iy_\infty$. This can happen for only finitely many pairs (c, d) if (x_∞, y_∞) lies in a compact subset of the upper half plane, and the lemma follows.

Proposition 7.8. *Let $\mathcal{S}$ be a Siegel set, and let $c' > 0$. Then there are only finitely many $\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})$ such that $h(\gamma g) \geq c'$ for some $g \in \mathcal{S}$.*

PROOF. Write $\mathcal{S} = Z_\infty \mathcal{S}^1$ with $\mathcal{S}^1 \subset G^1$. The subset of $g \in \mathcal{S}^1$ with $c' \leq h(g) \leq 1$ is compact and is handled by Lemma 7.7. For $z \in Z_\infty$, we have $h(\gamma z_\infty g) = h(\gamma g)$, and therefore there are only finitely many $\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})$ have $h(\gamma g) \geq c'$ for some $g \in \mathcal{S}$. To complete the proof, consider the subset of $g \in \mathcal{S}$ with $h(g) > 1$. For such g, Corollary 7.3 shows that $h(\gamma g) \leq 1$ whenever γ is nontrivial in $P(\mathbb{Q})\backslash G(\mathbb{Q})$.

Corollary 7.9. *Let $\mathcal{S}$ be a Siegel set. Then there are only finitely many $\gamma \in G(\mathbb{Q})$ such that $\gamma\mathcal{S}$ meets $\mathcal{S}$.*

PROOF. Say that $h(g) \geq c'$ for $g \in \mathcal{S}$. According to Proposition 7.8, the elements γ in $G(\mathbb{Q})$ for which $\gamma\mathcal{S}$ meets $\mathcal{S}$ lie in finitely many cosets of $P(\mathbb{Q})\backslash G(\mathbb{Q})$. If there are infinitely many such elements γ, then there is some $\gamma_0 \in G(\mathbb{Q})$ such that $\varepsilon_j \gamma_0 \mathcal{S}$ meets $\mathcal{S}$ for infinitely many ε_j in $P(\mathbb{Q})$.

Suppose that the coset of γ_0 is trivial. Then we may take $\gamma_0 = 1$, so that $\varepsilon_j \mathcal{S}$ meets $\mathcal{S}$ for infinitely many ε_j. Since ε_j is in G^1, $\varepsilon_j \mathcal{S}^1$ meets $\mathcal{S}^1$ for infinitely many ε_j. Since $h(\varepsilon_j s) = h(s)$ and since $\mathcal{S}^1$ is compact in all other directions, we obtain a contradiction to the discreteness of $P(\mathbb{Q})$.

Thus we may suppose that the coset of γ_0 in $P(\mathbb{Q})\backslash G(\mathbb{Q})$ is nontrivial. If $\varepsilon_j \gamma_0 \mathcal{S}$ meets $\mathcal{S}$, then $\varepsilon_j \gamma_0 \mathcal{S}^1$ meets $\mathcal{S}^1$. If s is in $\mathcal{S}^1$, then $h(\varepsilon_j \gamma_0 s) = h(\gamma_0 s)$. When $h(s) > 1$, Corollary 7.3 shows that $h(\gamma_0 s) \leq 1$. And the part of $\mathcal{S}^1$ where $h(s) \leq 1$ is compact. Hence h is bounded on $\varepsilon_j \gamma_0 \mathcal{S}^1$ uniformly in j. Since $\varepsilon_j \gamma_0 \mathcal{S}^1$ meets $\mathcal{S}^1$, the points of intersection lie in a compact subset of $\mathcal{S}^1$, and we may assume that these points of intersection $\varepsilon_j \gamma_0 s_j = s'_j$ converge, say to s'_0. Applying h shows that $h(\gamma_0 s_j) \to h(s'_0)$. Let $s_j = n_j a_j k_j$ with $n_j \in N(\mathbb{A})$, $a_j \in M(\mathbb{A}) \cap G^1$, and $k_j \in K$.

Since γ_0 is nontrivial, we may assume that $\gamma_0 = wn'$ with $n' \in N(\mathbb{A})$. Then

$$\begin{aligned} h(\gamma_0 s_j) = h(wn'n_j a_j k_j) &= h((wa_j w^{-1})(wa_j^{-1} n' n_j a_j w^{-1})(wk_j)) \\ &= h(wa_j w^{-1}) h(wa_j^{-1} n' n_j a_j w^{-1}) \end{aligned}$$

and hence

$$h(\gamma_0 s_j) h(s_j) = h(wa_j^{-1} n' n_j a_j w^{-1}).$$

Since $n'n_j$ is bounded within $N(\mathbb{A})$ while $h(a_j)$ is bounded below, $wa_j^{-1}n'n_j a_j w^{-1}$ lies in a compact subset of G^1. Therefore h is bounded away from 0 and $+\infty$ on it. Consequently $h(s_j)$ is bounded away from 0 and $+\infty$. We may therefore assume that s_j converges within G^1, say to s_0. Then $\lim \varepsilon_j^{-1} s_0' = \gamma_0 s_0$ exists in G^1, and ε_j converges. This is a contradiction since $P(\mathbb{Q})$ is discrete.

Proposition 7.10. *If $\mathcal{S}$ is a sufficiently large Siegel set, then $G(\mathbb{Q})\mathcal{S} = G(\mathbb{A})$.*

Remark. Corollary 7.9 and Proposition 7.10 together show that Siegel sets for many purposes are adequate substitutes for fundamental domains for the action of $G(\mathbb{Q})$ on $G(\mathbb{A})$. For a generalization to all reductive groups, see [Bo].

Proof. It is known [Lan2, p. 140] that $\mathcal{D} = [-\frac{1}{2}, \frac{1}{2}) \times \prod_p \mathbb{Z}_p$ is a fundamental domain for $\mathbb{Q}\backslash\mathbb{A}$. Then $\tilde{\mathcal{D}} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathcal{D} \right\}$ is a fundamental domain for $N(\mathbb{Q})\backslash N(\mathbb{A})$. Let $\mathcal{C}_0$ be the compact subset $\{1\} \times \prod_p \mathbb{Z}_p^\times$ of $(\mathbb{A}^\times)^1$; the set $\mathcal{C}_0$ has the property that $\mathbb{Q}^\times \mathcal{C}_0 = (\mathbb{A}^\times)^1$. Let $\mathcal{C}$ be the subset of M^1 whose diagonal entries are in $\mathcal{C}_0$, and define

$$\mathcal{S} = \tilde{\mathcal{D}} \times Z_\infty \times \{a(y) \in A_\infty \mid y \geq \tfrac{\sqrt{3}}{2}\} \times \mathcal{C} \times K.$$

Given $g \in G(\mathbb{A})$, we are to left-translate g into $\mathcal{S}$ via $G(\mathbb{Q})$. Lemma 7.7 shows that we may assume that $h(\gamma g) \leq h(g)$ for all $\gamma \in G(\mathbb{Q})$. Write $g = nak$ with $n \in N(\mathbb{A})$, $a \in M(\mathbb{A})$, and $k \in K$. Left translating by a member of $M(\mathbb{Q})$, we may assume that a is in $M_\infty \mathcal{C}$. Left translating further by $N(\mathbb{Q})$, we may assume that n is in $\tilde{\mathcal{D}}$.

We have

$$h(wnak) = h((waw^{-1})(wa^{-1}naw^{-1})wk) = h(waw^{-1})h(wa^{-1}naw^{-1}),$$

and therefore

$$h(wa^{-1}naw^{-1}) = h(wg)h(g) \leq h(g)^2. \tag{7.7}$$

We can decompose n and a according to the infinite and finite places as $n = \begin{pmatrix} 1 & x_\infty x_{\mathrm{fin}} \\ 0 & 1 \end{pmatrix}$ and $a = a(y)a_{\mathrm{fin}}$. Taking into account the form of $\tilde{\mathcal{D}}$ and $\mathcal{C}$, we see that $wa^{-1}naw^{-1}$ is $\begin{pmatrix} 1 & 0 \\ y^{-1}x_\infty & 1 \end{pmatrix}$ at the place ∞ and is $\begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}$ with $x' \in \prod \mathbb{Z}_p$ at the finite places. By Lemma 6.6

$$h(wa^{-1}naw^{-1}) = (1 + y^{-2}x_\infty^{-2})^{-1/2} = \frac{y}{\sqrt{y^2 + x_\infty^2}}. \tag{7.8}$$

Since $h(g) = h(a) = h(a(y)) = y^{1/2}$, comparison of (7.7) and (7.8) shows that $y/\sqrt{y^2 + x_\infty^2} \leq y$, i.e., $y^2 + x_\infty^2 \geq 1$. Since $|x_\infty| \leq \frac{1}{2}$, $y^2 \geq \frac{3}{4}$. Thus our particular left translate of g via $G(\mathbb{Q})$ is in $\mathcal{S}$.

Now let us return to Λ^T. Suppose that $T \geq T_0 > 0$. If Ω is a compact subset of $G(\mathbb{A})$, then only finitely many terms in the sum for $\Lambda^T f(g)$ can be nonzero for some $g \in \Omega$, by Lemma 7.7. Taking T large enough, we can make $I_T(h(\gamma g)) = 0$ for each such term. Thus we obtain the following result.

Proposition 7.11. *$\Lambda^T f$ converges to f uniformly on compact subsets of $G(\mathbb{A})$.*

Under some mild restrictions on f, $\Lambda^T f$ is small at infinity in a certain sense. To make this idea precise, we shall use Siegel sets. If $\mathcal{S}$ is a Siegel set, again write $\mathcal{S} = Z_\infty \mathcal{S}^1$ with $\mathcal{S}^1 \subset G^1$. Then the part of $\mathcal{S}^1$ where $h(g) \leq 1$ is compact. For $h(g) > 1$, Corollary 7.3 shows that

$$\Lambda^T f(g) = f(g) - f_P(g) I_T(h(g)).$$

Thus if the $\mathcal{S}^1$ component of a member g of $\mathcal{S}$ is far enough out, we obtain

$$\Lambda^T f(g) = f(g) - f_P(g).$$

To get an idea why this difference is small in favorable circumstances, suppose that $f = F * \psi$ with F bounded and left invariant under $G(\mathbb{Q})$ and with ψ continuous of compact support on $G(\mathbb{A})$. Then

$$f(g) - f_P(g) = \int_{G(\mathbb{A})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} F(x)[\psi(x^{-1}g) - \psi(x^{-1}ng)]\, dn\, dx. \tag{7.9}$$

It is easy to check that as g tends to ∞ through $\mathcal{S}$, $g^{-1}ng$ tends uniformly to 1 for n in any compact subset of $N(\mathbb{A})$. Therefore

$$\int_{G(\mathbb{A})} |\psi(x^{-1}g) - \psi(x^{-1}ng)|\, dx = \int_{G(\mathbb{A})} |\psi^{-1}(x^{-1}) - \psi(x^{-1}g^{-1}ng)|\, dx$$

tends to 0 as the $\mathcal{S}^1$ component of $g \in \mathcal{S}$ tends to ∞, and (7.9) has limit 0.

Let us state a general result. A function f on $G(\mathbb{A})$ that is left invariant under $G(\mathbb{Q})$ is said to be **slowly increasing** if, for each Siegel set $\mathcal{S}$, there are constants C and N such that

$$|f(g)| \leq Ch(g)^N \qquad \text{for all } g \in \mathcal{S}. \tag{7.10}$$

Because of Proposition 7.10, this condition controls the global growth of f at infinity for $G(\mathbb{A})$. The function f is said to be **rapidly decreasing** if, for each Siegel set $\mathcal{S}$ and integer $-N$, there is a constant C such that

$$|f(g)| \leq Ch(g)^{-N} \qquad \text{for all } g \in \mathcal{S}. \tag{7.11}$$

Let $G(\mathbb{A}_{\text{fin}})$ be the part of $G(\mathbb{A})$ corresponding to the finite places, and let K_0 be an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$. If the above function f is right invariant under K_0, then f may be viewed as a function on the space $G(\mathbb{Q})\backslash G(\mathbb{A})/K_0$, which is a smooth manifold. Let us say that f is smooth if this descended function is smooth.

Proposition 7.12. *Let K_0 be an open subgroup of $G(\mathbb{A}_{\text{fin}})$. Suppose that f is a function on $G(\mathbb{A})$ that is left invariant under $G(\mathbb{Q})$, right invariant under K_0, and smooth. If f and all its left invariant derivatives are slowly increasing, then $\Lambda^T f$ is rapidly decreasing.*

Reference. [Ar3, Lemma 1.4].

Finally we can return to the formula (7.1) for $K_{\text{cusp}}(x,y)$. We follow [Gb-Ja]. Let P_{cusp} be the orthogonal projection on $L^2_{\text{cusp}}(\Omega)$. It is not hard to see that $K_{\text{cusp}}(x,y)$ is in $L^2_{\text{cusp}}(\omega^{-1})$ as a function of the second variable. When we therefore apply the truncation operator Λ_2^T in the second variable, we obtain

$$K_{\text{cusp}}(x,y) = \Lambda_2^T K(x,y) - \Lambda_2^T K_{\text{cont}}(x,y) - \Lambda_2^T K_{\text{res}}(x,y).$$

It turns out that each term on the right side is now integrable over the diagonal and that

$$\operatorname{Tr}(P_{\text{cusp}} R_\omega(\varphi) P_{\text{cusp}}) = \int \Lambda_2^T K(x,x)\,dx - \int \Lambda_2^T K_{\text{cont}}(x,x)\,dx - \int \Lambda_2^T K_{\text{res}}(x,x)\,dx \tag{7.12}$$

with the integrals extending over $\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})$.

In place of (7.2) we have the formula

$$\begin{aligned}\Lambda^T K(x,x) \\ &= \sum_{\gamma\in\overline{G}(\mathbb{Q})} \varphi(x^{-1}\gamma x) - \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Big(\sum_{\gamma\in\overline{G}(\mathbb{Q})} \varphi(x^{-1}\gamma n\xi x) I_T(h(\xi x))\Big)\,dn.\end{aligned}$$

For T large enough, the right side may be shown to be

$$= \sum_{\gamma\in\overline{G}(\mathbb{Q})} \varphi(x^{-1}\gamma x) - \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{A})} \sum_{\mu\in\overline{M}(\mathbb{Q})} \varphi(x^{-1}\xi^{-1}\mu n\xi x) I_T(h(\xi x))\,dn. \tag{7.13}$$

We group these terms according to the type of γ or μ. We say that γ is **elliptic** if its eigenvalues are not in $\mathbb{Q}$, **hyperbolic regular** if its eigenvalues are distinct rationals, **singular** if γ is the product of a scalar matrix and a unipotent matrix. From γ elliptic we get

$$\sum_{\substack{\gamma \text{ elliptic} \\ \text{in } \overline{G}(\mathbb{Q})}} \varphi(x^{-1}\gamma x).$$

From γ and μ hyperbolic regular, we get

$$\sum_{\substack{\gamma \text{ hyperbolic} \\ \text{regular in } \overline{G}(\mathbb{Q})}} \varphi(x^{-1}\gamma x) - \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{A})} \sum_{\substack{\mu\in\overline{M}(\mathbb{Q}), \\ \mu\neq 1}} \varphi(x^{-1}\xi^{-1}\mu n\xi x) I_T(h(\xi x))\,dn.$$

From γ and μ singular we get

$$\sum_{\substack{\gamma\in\overline{G}(\mathbb{Q}), \\ \text{unipotent}}} \varphi(x^{-1}\gamma x) - \sum_{\xi\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{A})} \varphi(x^{-1}\xi^{-1} n\xi x) I_T(h(\xi x))\,dn.$$

The term with the elliptic elements is handled just as in (4.3): From each conjugacy class $\mathfrak{o}$ of elliptic elements in $\overline{G}(\mathbb{Q})$, we select a representative. Say that γ is a representative of $\mathfrak{o}_\gamma$. Then $\mathfrak{o}_\gamma$ consists of all $\delta^{-1}\gamma\delta$, where δ varies through $\overline{G}(\mathbb{Q})^\gamma\backslash\overline{G}(\mathbb{Q})$. Upon integration over $x \in \overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})$, the term with the elliptic elements gives

$$\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\gamma \text{ elliptic}} \varphi(x^{-1}\gamma x)\,dx = \sum_{\mathfrak{o}_\gamma \text{ elliptic}} \operatorname{vol}(\overline{G}(\mathbb{Q})^\gamma\backslash\overline{G}(\mathbb{A})^\gamma) \int_{\overline{G}(\mathbb{A})^\gamma\backslash\overline{G}(\mathbb{A})} \varphi(x^{-1}\gamma x)\,dx.$$

A more complicated computation shows that the contribution to the x integral from the hyperbolic regular elements is of the form

$$\sum_{\substack{\mathfrak{o}_\gamma \text{ hyperbolic} \\ \text{regular}}} \operatorname{vol}(\overline{G}(\mathbb{Q})^\gamma\backslash\overline{G}(\mathbb{A})^\gamma) \int_{\overline{G}(\mathbb{A})^\gamma\backslash\overline{G}(\mathbb{A})} \varphi(x^{-1}\gamma x)(-\log h(wx))\,dx + (T)(\text{constant}).$$

Without loss of generality, we can take $\gamma = \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $q \in \mathbb{Q}^\times$. Then $\overline{G}(\mathbb{A})^\gamma = \overline{M}(\mathbb{A})$ and $\overline{G}(\mathbb{Q})^\gamma = \overline{M}(\mathbb{Q})$, so that $\overline{M}(\mathbb{Q})\backslash\overline{M}(\mathbb{A}) \cong \mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1$. Since $\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ lie in the same conjugacy class when projected to $\overline{G}(\mathbb{A})$, indexing the γ's by q counts each class twice. Thus the part of the above expression not involving T simplifies to

$$= \tfrac{1}{2}\operatorname{vol}(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1) \sum_{\substack{q\in\mathbb{Q}^\times, \\ q\neq 1}} \int_{K\times N} \varphi\left(k^{-1}n^{-1}\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)nk\right)(-\log h(wnk))\,dn\,dk.$$

The term with $\gamma = 1$ is just $\varphi(1)$, and the integral is $\operatorname{vol}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))\varphi(1)$. There is also a contribution from the terms with γ unipotent but not the identity; this result has a constant term and a T term, but we shall not write these terms out.

This much deals with the integral of $\Lambda_2^T K(x,x)$. Next we consider the integral of $\Lambda_2^T K_{\text{cont}}(x,x)$. Referring to (7.4), we see that we should compute the inner product of a truncated Eisenstein series with an untruncated Eisenstein series—or, what comes to the same thing, of two truncated Eisenstein series.

Proposition 7.13. *For f_1 and f_2 in $W(\eta,\tau)$,*

$$\begin{aligned}\langle \Lambda^T E(\,\cdot\,, f_1, s), \Lambda^T E(\,\cdot\,, f_2, -\bar{s})\rangle &= 4\langle f_1, f_2\rangle T + 2\langle M(-s)M'(s)f_1, f_2\rangle \\ &\quad + \frac{1}{s}\{\langle f_1, M(-\bar{s})f_2\rangle e^{sT} - \langle M(s)f_1, f_2\rangle e^{-sT}\}.\end{aligned}$$

Remark. The proof will show the importance of the particular form of Arthur's truncation operator.

Sketch of proof. For $\operatorname{Re} s > 1$, we start from the identities $E(g,f,s) = \sum_\gamma f_s(\gamma g)$ and $E_P(\,\cdot\,, f, s) = f_s + (M(s)f)_{-s}$, the latter given by Proposition 6.8. Then we have

$$\begin{aligned}\Lambda^T E(g,f,s) &= E(g,f,s) - \sum_\gamma E_P(\gamma g, f, s) I_T(h(\gamma g)) \\ &= \sum_\gamma f_s(\gamma g)(1 - I_T(h(\gamma g))) - \sum_\gamma (M(s)f)_{-s}(\gamma g) I_T(h(\gamma g)).\end{aligned}$$

Let $\operatorname{Re} s_1 > \operatorname{Re} s_2 > 1$. Substituting from above and writing f and f' in place of f_1 and f_2 to simplify the notation, we obtain

$$\begin{aligned}
&\langle \Lambda^T E(\,\cdot\,, f, s_1), E(\,\cdot\,, f', \bar{s}_2)\rangle_{L^2(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))} \\
&\quad = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \big(f_{s_1}(\gamma g)(1 - I_T(h(\gamma g)))(M(s_1)f)_{-s_1}(\gamma g) I_T(h(\gamma g))\big) \\
&\qquad\qquad \times \overline{E(g, f', \bar{s}_2)}\, dg \\
&\quad = \int_{\overline{P}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \big(f_{s_1}(g)(1 - I_T(h(g))) - (M(s_1)f)_{-s_1}(g) I_T(h(g))\big)\overline{E(g, f', \bar{s}_2)}\, dg \\
&\quad = \int_{\overline{M}(\mathbb{Q})N(\mathbb{A})\backslash\overline{G}(\mathbb{A})} \big(f_{s_1}(g)(1 - I_T(h(g))) - (M(s_1)f)_{-s_1}(g) I_T(h(g))\big)\overline{E_P(g, f', \bar{s}_2)}\, dg \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 6.4} \\
&\quad = \int_{\overline{M}(\mathbb{Q})N(\mathbb{A})\backslash\overline{G}(\mathbb{A})} \big(f_{s_1}(g)(1 - I_T(h(g))) - (M(s_1)f)_{-s_1}(g) I_T(h(g))\big) \\
&\qquad\qquad \times \overline{\big(f'_{\bar{s}_2}(g) + (M(\bar{s}_2)f')_{-\bar{s}_2}(g)\big)}\, dg.
\end{aligned}$$

Now we substitute for g, reducing each function by its transformation rules to a function on $(\mathbb{A}^\times)^1 \times K$. The set of integration reduces to $A_\infty \times ((\mathbb{A}^\times)^1 \times K)$. The A_∞ integration can be done explicitly, and the $(\mathbb{A}^\times)^1 \times K$ integration gives inner products in the Hilbert space H. The result of this computation, initially valid for $\operatorname{Re} s_1 > \operatorname{Re} s_2 > 1$, extends by analytic continuation to be valid for all s_1 and s_2 where there is no singularity. We then put $s_2 = s$ and $s_1 = s + h$. Taking the limit as h tends to 0, we obtain the formula of the proposition.

We return to (7.4). Interchanging the order of integration yields

$$\begin{aligned}
&\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \Lambda_2^T K_{\mathrm{cont}}(x,x)\, dx \\
&\quad = \frac{1}{8\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty} \langle P^{\omega,it}(\varphi) f_\beta, f_\alpha\rangle \Big[\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} E(x, f_\alpha, it)\overline{\Lambda^T E(x, f_\beta, it)}\, dx\Big]\, dt.
\end{aligned}$$

The Hermitian property of Λ^T in Proposition 7.6 extends to this integral, and we can substitute from Proposition 7.13. Easy computation gives

$$\begin{aligned}
&\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \Lambda_2^T K_{\mathrm{cont}}(x,x)\, dx \\
&\qquad\qquad = \frac{1}{4\pi}\int_{-\infty}^{\infty} \operatorname{Tr}(M(-it)M'(it)P^{\omega,it}(\varphi))\, dt - \frac{1}{4}\operatorname{Tr}(M(0)\pi_0(\varphi)) \\
&\qquad\qquad\quad + (T)(\text{constant}) + (\text{term tending to 0 as } T \to \infty).
\end{aligned}$$

Finally the integral of $\Lambda_2^T K_{\mathrm{res}}(x,x)$ is just

$$\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \Lambda_2^T K_{\mathrm{res}}(x,x)\, dx \to \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} K_{\mathrm{res}}(x,x)\, dx = \sum_{\chi^2=\omega}\int_{\overline{G}(\mathbb{A})} \varphi(x)\chi(\det x)\, dx.$$

If we substitute all these results into (7.12), we obtain an equality for all T. Some terms have a coefficient T, and these all cancel (but not in an obvious way). The other terms tend to a finite limit as T tends to ∞. In the limit as T tends to ∞, we obtain the adelic form of the trace formula, as follows.

Theorem 7.14. *If φ is in $C^\infty_{\mathrm{com}}(G(\mathbb{A}),\omega^{-1})$, then*

$$\begin{aligned}
&\mathrm{Tr}(P_{\mathrm{cusp}}R_\omega(\varphi)P_{\mathrm{cusp}})\\
&\text{(i)} \qquad = \mathrm{vol}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))\varphi(1)\\
&\text{(ii)} \qquad + \sum_{\mathfrak{o}_\gamma \text{ elliptic}} \mathrm{vol}(\overline{G}(\mathbb{Q})^\gamma\backslash\overline{G}(\mathbb{A})^\gamma) \int_{\overline{G}(\mathbb{A})^\gamma\backslash\overline{G}(\mathbb{A})} \varphi(x^{-1}\gamma x)\,dx\\
&\text{(iii)} \qquad + \text{f.p.} \int_{\overline{N}(\mathbb{A})\backslash\overline{G}(\mathbb{A})} \varphi\Big(x^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x\Big)\,dx\\
&\text{(iv)} \qquad + \tfrac{1}{2}\mathrm{vol}(\mathbb{Q}^\times\backslash(\mathbb{A}^\times)^1) \sum_{\substack{q\in\mathbb{Q}^\times,\\ q\neq 1}} \int_{K\times N(\mathbb{A})} \varphi\big(k^{-1}n^{-1}\left(\begin{smallmatrix}q&0\\0&1\end{smallmatrix}\right)nk\big)\,(-\log h(wnk))\,dn\,dk\\
&\text{(v)} \qquad + \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(M(-it)M'(it)P^{\omega,it}(\varphi))\,dt\\
&\text{(vi)} \qquad - \tfrac{1}{4}\mathrm{Tr}(M(0)P^{\omega,0}(\varphi))\\
&\text{(vii)} \qquad - \sum_{\chi^2=\omega} \int_{\overline{G}(\mathbb{A})} \varphi(x)\chi(\det x)\,dx,
\end{aligned}$$

where the f.p. *term is computed as the value at $s=1$ of*

$$\left\{\int_{\mathbb{A}^\times\times K} \varphi\Big(k^{-1}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)k\Big)|a|^s_{\mathbb{A}}\,d^\times a\,dk - (\text{principal part at } s=1)\right\}$$

when Haar measures are normalized as in (6.7) *and the remarks following Lemma* 6.7.

On the right side of the trace formula above, the terms arise as follows. The first four come from $K(x,x)$, the next two come from $K_{\mathrm{cont}}(x,x)$, and the last one comes from $K_{\mathrm{res}}(x,x)$. The first four we may regard as **geometric terms**, and the others are **spectral terms**. Of the first four, (i) is from $\gamma=1$, (ii) is from elliptic γ, (iii) is from nontrivial unipotent γ, and (iv) is from hyperbolic regular γ.

There is an important special case in which the formula simplifies considerably.

Corollary 7.15. *Suppose that $\varphi\in C^\infty_{\mathrm{com}}(G(\mathbb{A}),\omega^{-1})$ decomposes into a product $\varphi(g)=\prod_v \varphi_v(g_v)$. If there are two places v such that*

$$\int_{M(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} \varphi_v\left(x^{-1}\left(\begin{smallmatrix}\alpha&0\\0&\beta\end{smallmatrix}\right)x\right)\,dx = 0$$

whenever α and β are distinct members of $\mathbb{Q}_v$, then terms (iii) *through* (vi) *vanish in the trace formula, so that*

$$\begin{aligned}
\mathrm{Tr}(P_{\mathrm{cusp}}R_\omega(\varphi)P_{\mathrm{cusp}}) &= \mathrm{vol}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))\varphi(1)\\
&\quad + \sum_{\mathfrak{o}_\gamma \text{ elliptic}} \mathrm{vol}(\overline{G}(\mathbb{Q})^\gamma\backslash\overline{G}(\mathbb{A})^\gamma) \int_{\overline{G}(\mathbb{A})^\gamma\backslash\overline{G}(\mathbb{A})} \varphi(x^{-1}\gamma x)\,dx\\
&\quad - \sum_{\chi^2=\omega} \int_{\overline{G}(\mathbb{A})} \varphi(x)\chi(\det x)\,dx.
\end{aligned}$$

REFERENCE. [Gb-Ja, §7].

PROOF OF VANISHING OF (iv). Without loss of generality, we may take Haar measure on K and $N(\mathbb{A})$ to be products of Haar measures from each place.

Let v_1 and v_2 be the places in question, let $\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, and write $k = \prod_v k_v$ and $n = \prod_v n_v$. Lemma 6.6 shows that $h(wnk)$ is a product $\prod_v h(wn_v k_v)$. Then

$$\begin{aligned}
&\int_{K\times N(\mathbb{A})} \varphi(k^{-1}n^{-1}\gamma nk)\log h(wnk)\,dn\,dk \\
&\qquad = \int_{K\times N(\mathbb{A})} \prod_v \Big(\varphi_v(k_v^{-1}n_v^{-1}\gamma n_v k_v)\Big)\Big(\sum_v \log h(wn_v k_v)\Big)\prod_v dn_v\,dk_v \\
&\qquad = \sum_u \Big(\prod_{v\neq u}\int_{K_v\times N(\mathbb{Q}_v)} \varphi_v(k_v^{-1}n_v^{-1}\gamma n_v k_v)\,dn_v\,dk_v\Big) \\
&\qquad\qquad \times \Big(\int_{K_u\times N(\mathbb{Q}_u)} \varphi_u(k_u^{-1}n_u^{-1}\gamma n_u k_u)\log h(wn_u k_u)\,dn_u\,dk_u\Big).
\end{aligned}$$

Consider the u^{th} term of the sum on the right side. In the product over $v \neq u$, either v_1 or v_2 must be one of the v's, and then the corresponding factor is 0 because of the hypothesis. Hence the u^{th} term is 0, and this happens for each u.

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