Advanced Real Analysis

Digital Second Editions
By Anthony W. Knapp

Basic Algebra

Advanced Algebra

Basic Real Analysis,
with an appendix "Elementary Complex Analysis"

Advanced Real Analysis

Anthony W. Knapp

Advanced Real Analysis

Along with a Companion Volume *Basic Real Analysis*

Digital Second Edition
Corrected version, 2017

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To Susan

and

To My Children, Sarah and William,

and

To My Real-Analysis Teachers:

Salomon Bochner, William Feller, Hillel Furstenberg,
Harish-Chandra, Sigurdur Helgason, John Kemeny,
John Lamperti, Hazleton Mirkil, Edward Nelson,
Laurie Snell, Elias Stein, Richard Williamson

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CONTENTS OF *BASIC REAL ANALYSIS*

PREFACE TO THE SECOND EDITION

In the years since publication of the first editions of *Basic Real Analysis* and *Advanced Real Analysis*, many readers have reacted to the books by sending comments, suggestions, and corrections. They appreciated the overall comprehensive nature of the books, associating this feature with the large number of problems that develop so many sidelights and applications of the theory.

Along with the general comments and specific suggestions were corrections, and there were enough corrections to *Basic Real Analysis* to warrant a second edition. A second edition of *Advanced Real Analysis* was then needed for compatibility. As long as this volume was being changed, it seemed appropriate to add the two important topics that are listed below.

For the first editions, the author granted a publishing license to Birkhäuser Boston that was limited to print media, leaving the question of electronic publication unresolved. A major change with the second editions is that the question of electronic publication has now been resolved, and for each book a PDF file, called the "digital second edition," is being made freely available to everyone worldwide for personal use. These files may be downloaded from the author's own Web page and from elsewhere.

The two important additions to *Advanced Real Analysis* are as follows:

- Five sections on the Central Limit Theorem and an introduction to statistical inference. This material appears at the end of Chapter IX, "Foundations of Probability." Four sections deal with the Central Limit Theorem itself. The fifth section shows how the Central Limit Theorem arises in applications to statistics as a limiting case of the t distribution of W. S. Gosset, also known as Student's t distribution. Gosset's t distribution plays a fundamental role in statistical inference.
- A chapter on the theory of wavelets, with some commentary on applications. This material occupies Chapter X, "Introduction to Wavelets." Wavelets form another theory, beyond the frequency analysis of Fourier series and the Fourier transform, for decomposing functions of one or more variables into component parts that bring out hidden behavior of the functions. The theory was introduced in the 1980s and 1990s to bring together disparate applications in signal processing and related fields, and it has now reached a sufficient state of maturity that all mathematicians might benefit from some familiarity with

it. More information about the content of the chapter appears in the Guide to the Reader on pages xviii–xxii.

In addition, there were some minor changes to *Advanced Real Analysis*. First, Sections 1–4 of Chapter IX were revised and expanded to prepare for the Central Limit Theorem later in the chapter. Second, a third part was added to Problem 12 at the end of Chapter III to allow for an alternative approach to smoothness questions about wavelets. Third, a small number of minor corrections, fewer than a dozen, were made in the first eight chapters. And fourth, the References and Index were updated and expanded. No other changes were made to the body of the text.

As was true in the first edition, references in *Advanced Real Analysis* to the text of *Basic Real Analysis* abbreviate the title of the latter book as *Basic*.

Ann Kostant was the person who conceived the idea, about 2003, for Birkhäuser to have a series Cornerstones. Her vision was to enlist authors experienced at mathematical exposition who would write compatible texts at the early graduate level. The overall choice of topics was heavily influenced by the graduate curricula of major American universities. The idea was for each book in the series to explain what the young mathematician needs to know about a swath of mathematics in order to communicate well with colleagues in all branches of mathematics in the 21$^{\text{st}}$ century. Taken together, the books in the series were intended as an antidote for the worst effects of overspecialization. I am honored to have been part of her project.

It was Benjamin Levitt, Birkhäuser mathematics editor in New York as of 2014, who encouraged the writing of second editions of the real analysis books. He made a number of suggestions about pursuing them, and he passed along comments from several anonymous referees about the strengths and weaknesses of each book. I am especially grateful to those readers who have sent me comments over the years. The typesetting was done by the program Textures using $A_{M}S$-TEX, and the figures were drawn with Mathematica.

As with the first editions, I invite corrections and other comments about the second editions from readers. For as long as I am able, I plan to point to lists of known corrections from my own Web page, www.math.stonybrook.edu/∼aknapp.

A. W. KNAPP
August 2016

The corrected version issued in 2017 incorporates six small changes to Chapter III, one small change to Chapter IX, and approximately 80 small corrections to Chapter X. The ones for Chapter X were kindly pointed out by Esshan Khanmohammadi; one of them makes a correction to the formula in Theorem 10.10.

January 2017

PREFACE TO THE FIRST EDITION

This book and its companion volume *Basic Real Analysis* systematically develop concepts and tools in real analysis that are vital to every mathematician, whether pure or applied, aspiring or established. The two books together contain what the young mathematician needs to know about real analysis in order to communicate well with colleagues in all branches of mathematics.

The books are written as textbooks, and their primary audience is students who are learning the material for the first time and who are planning a career in which they will use advanced mathematics professionally. Much of the material in the books corresponds to normal course work. Nevertheless, it is often the case that core mathematics curricula, time-limited as they are, do not include all the topics that one might like. Thus the book includes important topics that are sometimes skipped in required courses but that the professional mathematician will ultimately want to learn by self-study.

The content of the required courses at each university reflects expectations of what students need before beginning specialized study and work on a thesis. These expectations vary from country to country and from university to university. Even so, there seems to be a rough consensus about what mathematics a plenary lecturer at a broad international or national meeting may take as known by the audience. The tables of contents of the two books represent my own understanding of what that degree of knowledge is for real analysis today.

Key topics and features of *Advanced Real Analysis* are that it:

- Develops Fourier analysis and functional analysis with an eye toward partial differential equations.
- Includes chapters on Sturm–Liouville theory, compact self-adjoint operators, Euclidean Fourier analysis, topological vector spaces and distributions, compact and locally compact groups, and aspects of partial differential equations.
- Contains chapters about analysis on manifolds and foundations of probability.
- Proceeds from the particular to the general, often introducing examples well before a theory that incorporates them.
- Includes many examples and almost 200 problems, and a separate section "Hints for Solutions of Problems" at the end of the book gives hints or complete solutions for most of the problems.

● Incorporates, both in the text and in the problems but particularly in the problems, material in which real analysis is used in algebra, in topology, in complex analysis, in probability, in differential geometry, and in applied mathematics of various kinds.

It is assumed that the reader has had courses in real variables and either is taking or has completed the kind of course in Lebesgue integration that might use *Basic Real Analysis* as a text. Knowledge of the content of most of Chapters I–VI and X of *Basic Real Analysis* is assumed throughout, and the need for further chapters of that book for particular topics is indicated in the chart on page xvii. When it is necessary in the text to quote a result from this material that might not be widely known, a specific reference to *Basic Real Analysis* is given; such references abbreviate the book title as *Basic*.

Some understanding of complex analysis is assumed for Sections 3–4 and 6 of Chapter III, for Sections 10–11 of Chapter IV, for Section 4 of Chapter V, for all of Chapters VII and VIII, and for certain groups of problems, but not otherwise. Familiarity with linear algebra and group theory at least at the undergraduate level is helpful throughout.

The topics in the first eight chapters of this volume are related to one another in many ways, and the book needed some definite organizational principle for its design. The result was a decision to organize topics largely according to their role in the study of differential equations, even if differential equations do not explicitly appear in each of the chapters. Much of the material has other uses as well, but an organization of topics with differential equations in mind provides a common focus for the mathematics that is presented. Thus, for example, Fourier analysis and functional analysis are subjects that stand on their own and also that draw on each other, but the writing of the chapters on these areas deliberately points toward the subject of differential equations, and toward tools like distributions that are used with differential equations. These matters all come together in two chapters on differential equations, Chapters VII and VIII, near the end of in the book.

Portions of the first eight chapters can be used as the text for a course in any of three ways. One way is as an introduction to differential equations within a course on Lebesgue integration that treats integration and the Fourier transform relatively lightly; the expectation in this case is that parts of at most two or three chapters of this book would be used. A second way is as a text for a self-contained topics course in differential equations; the book offers a great deal of flexibility for the content of such a course, and no single choice is right for everyone. A third way is simply as a text for a survey of some areas of advanced real analysis; again the book offers great flexibility in how such a course is constructed.

The problems at the ends of chapters are an important part of the book. Some

of them are really theorems, some are examples showing the degree to which hypotheses can be stretched, and a few are just exercises. The reader gets no indication which problems are of which type, nor of which ones are relatively easy. Each problem can be solved with tools developed up to that point in the book, plus any additional prerequisites that are noted.

This book seeks in part to help the reader look for and appreciate the unity of mathematics. For that reason some of the problems and sections go way outside the usual view of real analysis. One of the lessons about advanced mathematics is that progress is better measured by how mathematics brings together different threads, rather than how many new threads it generates.

Almost all of the mathematics in this book and *Basic Real Analysis* is at least forty years old, and I make no claim that any result is new. The two books are together a distillation of lecture notes from a 35-year period of my own learning and teaching. Sometimes a problem at the end of a chapter or an approach to the exposition may not be a standard one, but normally no attempt has been made to identify such problems and approaches.

I am grateful to Ann Kostant and Steven Krantz for encouraging this project and for making many suggestions about pursuing it, and to Susan Knapp and David Kramer for helping with the readability. The typesetting was by $A_{M}S$-TeX, and the figures were drawn with Mathematica.

I invite corrections and other comments from readers. I plan to maintain a list of known corrections on my own Web page.

A. W. KNAPP
June 2005

LIST OF FIGURES

DEPENDENCE AMONG CHAPTERS

The chart below indicates the main lines of logical dependence of sections of *Advanced Real Analysis* on earlier sections and on chapters in *Basic Real Analysis*. Starting points are the boxes with double ruling. All starting points take Chapters I–VI and X of *Basic Real Analysis* as known. Uses of complex analysis as in Appendix B of *Basic Real Analysis* are indicated in the Guide for the Reader.
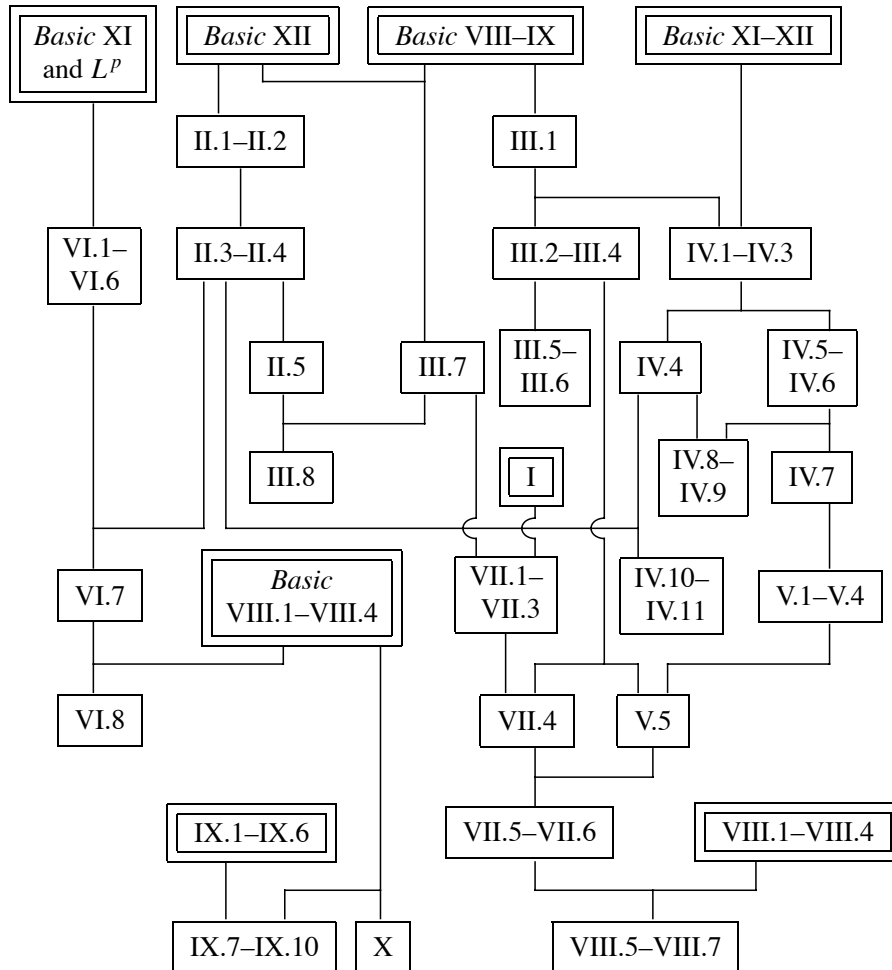
GUIDE FOR THE READER

This section is intended to help the reader find out what parts of each chapter are most important and how the chapters are interrelated. Further information of this kind is contained in the chart on page xvii and in the abstracts that begin each of the chapters.

Advanced Real Analysis deals with topics in real analysis that the young mathematician needs to know in order to communicate well with colleagues in all branches of mathematics. These topics include parts of Fourier analysis, functional analysis, spectral theory, distribution theory, abstract harmonic analysis, and partial differential equations. They tend to be ones whose applications and ramifications cut across several branches in mathematics. Each topic can be studied on its own, but the importance of the topic arises from its influence on the other topics and on other branches of mathematics. To avoid having all these relationships come across as a hopeless tangle, the book needed some organizational principle for its design. The principle chosen was largely to organize topics according to their role in the study of differential equations. This organizational principle influences what appears below, but it is certainly not intended to suggest that applications to differential equations are the only reason for studying certain topics in real analysis.

As was true also in *Basic Real Analysis*, several techniques that are used repeatedly in real analysis play a pivotal role. Examples are devices for justifying interchanges of limits, compactness and completeness as tools for proving existence theorems, and the approach of handling nice functions first and then passing to general functions. By the beginning of the present volume, these techniques have become sophisticated enough so as to account for entire areas of study within real analysis. The theory of weak derivatives illustrates this principle: The theory allows certain interchanges of limits involving weak derivatives to be carried out routinely, and the hard work occurs in translating the results into statements about classical derivatives. The main tool for this translation is Sobolev's Theorem, which in turn becomes the foundation for its own theory.

Each chapter is built around one or more important theorems. The commentary below tells the nature of each chapter and the role of some important theorems.

Chapter I marks two transitions—from concrete mathematics done by calculation to theorems established by functional analysis on the one hand, and from ordinary differential equations to partial differential equations on the other

hand. Section 2 about separation of variables is relatively elementary, introducing and illustrating a first technique for approaching partial differential equations. The technique involves a step of making calculations and a step of providing justification that the method is fully applicable. When the technique succeeds, the partial differential equation is reduced to two or more ordinary differential equations. Section 3 establishes, apart from one detail, the main theorem of the chapter, called Sturm's Theorem. Sturm's Theorem addresses the nature of solutions of certain kinds of ordinary differential equations with a parameter. This result can sometimes give a positive answer to the completeness questions needed to justify separation of variables, and it hints at a theory known as Sturm–Liouville theory that contains more results of this kind. The one detail with Sturm's Theorem that is postponed from Section 3 to Chapter II is the Hilbert–Schmidt Theorem.

Chapter II is a first chapter on functional analysis beyond Chapter XII of *Basic Real Analysis*, with emphasis on a simple case of the Spectral Theorem. The result in question describes the structure of compact self-adjoint operators on a Hilbert space. The Hilbert–Schmidt Theorem says that certain integral operators are of this kind, and it completes the proof of Sturm's Theorem as presented in Chapter I; however, Chapter I is not needed for an understanding of Chapter II. Section 4 of Chapter II gives several equivalent definitions of unitary operators and is relevant for many later chapters of the book. Section 5 discusses compact, Hilbert–Schmidt, and trace-class operators abstractly and may be skipped on first reading.

Chapter III is a first chapter on Fourier analysis beyond Chapters VIII and IX of *Basic Real Analysis*, and it discusses four topics that are somewhat independent of one another. The first of these, in Sections 1–2, introduces aspects of distribution theory and the idea of weak derivatives. The main result is Sobolev's Theorem, which tells how to extract conclusions about ordinary derivatives from conclusions about weak derivatives. Readers with a particular interest in this topic will want to study also Problems 8–12 and 25–34 at the end of the chapter. Sections 3–4 concern harmonic functions, which are functions annihilated by the Laplacian, and associated Poisson integrals, which relate harmonic functions to the subject of boundary-value problems. These sections may be viewed as providing an example of what to expect of the more general "elliptic" differential operators to be studied in Chapters VII–VIII. The main results are a mean value property for harmonic functions, a maximum principle, a reflection principle, and a characterization of harmonic functions in a half space that arise as Poisson integrals. Sections 5–6 establish the Calderón–Zygmund Theorem and give two applications to partial differential equations. The theorem generalizes the boundedness of the Hilbert transform, which was proved in Chapters VIII–IX of *Basic Real Analysis*. Historically the Calderón–Zygmund Theorem was a precursor to the theory of

pseudodifferential operators that is introduced in Chapter VII. Sections 7–8 gently introduce multiple Fourier series, which are used as a tool several times in later chapters. Chapter III makes connections with complex analysis in Sections 3–4 and 6, but complex analysis is needed only for examples and problems.

Chapter IV weaves together three lines of investigation in the area of functional analysis—one going toward spaces of smooth functions and distribution theory, another leading to fixed-point theorems, and a third leading to full-fledged spectral theory. The parts of the chapter relevant for spaces of smooth functions and distribution theory are Sections 1–2 and 5–7. This line of investigation continues in Chapters V and VII–VIII. The parts of the chapter relevant for fixed-point theorems are Sections 1, 3–6, and 8–9. Results of this kind, which have applications to equilibrium problems in economics and mathematical physics, are not pursued beyond Chapter IV in this book. The parts of the chapter relevant to spectral theory are Sections 1, 3–4, and 10–11, and spectral theory is not pursued beyond Chapter IV. Because the sections of the chapter have overlapping purposes, some of the main results play multiple roles. Among the main results are the characterization of finite-dimensional topological vector spaces as being Euclidean, the existence of "support" for distributions, Alaoglu's Theorem asserting weak-star compactness of the closed unit ball of the dual of a Banach space, the Stone Representation Theorem as a model for the theory of commutative C^* algebras, a separation theorem concerning continuous linear functionals in locally convex topological vector spaces, the construction of inductive limit topologies, the Krein–Milman Theorem concerning the existence of extreme points, the structure theorem for commutative C^* algebras, and the Spectral Theorem for commuting families of bounded normal operators. Spectral theory has direct applications to differential equations beyond what appears in Chapters I–II, but the book does not go into these applications. Section 10 makes serious use of elementary complex analysis, and complex analysis is used implicitly in Section 11.

Chapter V develops the theory of distributions, and of operations on them, without going into their connection with Sobolev spaces. The chapter includes a lengthy discussion of convolution. The main results are a structure theorem for distributions of compact support in terms of derivatives of measures, a theorem saying that the Fourier transforms of such distributions are smooth functions, and a theorem saying that the convolution of a distribution of compact support and a tempered distribution is meaningful and tempered, with its Fourier transform being the product of the Fourier transforms. One theorem in Section 4 observes that the Fourier transform of a distribution of compact support is an entire function, but complex analysis otherwise plays no role in the chapter.

Chapter VI introduces harmonic analysis using groups. Section 1 concerns general topological groups, Sections 2–5 are about invariant measures on locally compact groups and their quotients, and Sections 6–7 concern the representation

theory of compact groups. Section 8 indicates how representation theory simplifies problems concerning linear operators with a sizable group of symmetries. One main result of the chapter is the existence and uniqueness of Haar measure, up to a scalar factor, on any locally compact group. Another is the Peter–Weyl Theorem, which is a completeness theorem for Fourier analysis on a general compact group akin to Parseval's Theorem for Fourier series and the circle group. The proof of the Peter–Weyl Theorem uses the Hilbert–Schmidt Theorem.

Chapter VII is a first systematic discussion of partial differential equations, mostly linear, using tools from earlier chapters. Section 1 seeks to quantify the additional data needed for a differential equation or system simultaneously to have existence and uniqueness of solutions. The Cauchy–Kovalevskaya Theorem, which assumes that everything is holomorphic, is stated in general and gives a local result; for special kinds of systems it gives a global result whose proof is carried out in problems at the end of the chapter. Complex analysis of course plays a serious role in this section; it is used a little after Section 1, but its role thereafter is a minor one. Section 2 mentions some other properties and examples of differential equations, including the possibility of nonexistence of local solutions for linear equations $Lu = f$ when f is not holomorphic. Section 3 contains a general theorem asserting local existence of solutions for linear equations $Lu = f$ when L has constant coefficients; the proof uses multiple Fourier series. Section 5 concerns elliptic operators L with constant coefficients; these generalize the Laplacian. A complete proof is given in this case for the existence of a "parametrix" for L, which leads to control of regularity of solutions, and for the existence of "fundamental solutions." Section 6 introduces, largely without proofs, a general theory of pseudodifferential operators. To focus attention on certain theorems, the section describes how the theory can be used to obtain parametrices for elliptic operators with variable coefficients.

Chapter VIII in Sections 1–4 introduces smooth manifolds and vector bundles over them, particularly the tangent and cotangent bundles. Readers who are already familiar with this material may want to skip these sections. Sections 5–8 use this material to extend the theory of differential and pseudodifferential operators to the setting of smooth manifolds, where such operators arise naturally in many applications. Section 7 in particular describes how to adapt the theory of Chapter VII to obtain parametrices for elliptic operators on smooth manifolds.

Chapter IX is a stand-alone chapter on probability theory. Although partial differential equations interact with probability theory and have applications to differential geometry and financial mathematics, such interactions are too advanced to be addressed in this book. Instead four matters are addressed that are foundational and yet at the level of this book: how measure theory is used to model real-world probabilistic situations, how the Kolmogorov Extension Theorem constructs measure spaces that underlie stochastic processes, how probabilistic

independence and a certain indifference to the nature of the underlying measure space lead to a proof of the Strong Law of Large Numbers, and how the measure-theoretic techniques of the book yield a proof of the Central Limit Theorem. The material on the Central Limit Theorem occupies about half of the chapter and makes extensive use results from *Basic Real Analysis* about the Fourier transform. A final section introduces the subject of statistical inference and makes use of complex analysis in one calculation. In the simplest case the subject of statistical inference seeks to deduce the mean and variance of a whole population from those of a small sample. The probabilities in question are captured by the t distribution of W. S. Gosset. The main theorem calculates the t distribution, and it is seen under suitable hypotheses that the t distribution tends to the normal distribution as the sample size tends to infinity.

Chapter X introduces wavelets. In the simplest case a wavelet is a function of one variable such that the dilations by powers of 2 of the integer translates of the function form an orthogonal basis of the space of square integrable functions. Square integrable functions may be regarded as signals to be used as input. Associated to any wavelet is an analysis of all signals that is quite different from Fourier analysis via frequencies. Namely for each resolution, i.e., for each power of 2 determining a dilation, the function is decomposed into one function giving a rough approximation and another function providing some detail. The exact decomposition depends on the original wavelet and is regarded by people who do signal processing as the result of passing the signal through two complementary filters. After the decomposition the function giving the rough approximation is processed at successively higher resolutions, while the function giving the detail is retained as a component of the given function. The result is that the given signal is decomposed into infinitely many functions giving details. In practice, the detail functions for all the coarsest resolutions are lumped into a single function, a first approximation to the given signal, and the components giving the higher-order detail may be regarded as improving on the first approximation. Chapter X studies a mechanism called a multiresolution analysis for carrying out this decomposition systematically. Several of the main theorems construct concrete wavelets and families of wavelets, progressively more sophisticated, that have been found useful in applications. The last section of the chapter addresses the question of using wavelet analysis to decompose functions in practice, and it provides commentary on a number of applications. Chapter X intermittently makes use of complex analysis, mostly by directly applying results in Appendix B of *Basic Real Analysis*. Beyond that, Sections 7 and 9 prove and apply two further theorems in complex analysis, namely a formula for the sum over integers n of $(z - n)^{-2}$ and the classical Paley–Wiener Theorem.

NOTATION AND TERMINOLOGY

This section lists notation and a few unusual terms from elementary mathematics and from *Basic Real Analysis* that are taken as standard in the text without further definition. The items are grouped by topic. Complex analysis is not included.

Set theory

$\in$	membership symbol
$\#S$ or $\lvert S \rvert$	number of elements in S
$\varnothing$	empty set
$\{x \in E \mid P\}$	the set of x in E such that P holds
E^c	complement of the set E
$E \cup F, \ E \cap F, \ E - F$	union, intersection, difference of sets
$\bigcup_\alpha E_\alpha, \ \bigcap_\alpha E_\alpha$	union, intersection of the sets E_α
$E \subseteq F, \ E \supseteq F$	E is contained in F, E contains F
$E \times F, \ \bigtimes_{s \in S} X_s$	products of sets
$(a_1, \ldots, a_n)$	ordered n-tuple
$\{a_1, \ldots, a_n\}$	unordered n-tuple
$f : E \to F, \ x \mapsto f(x)$	function, effect of function
$f \circ g, \ f\big\vert_E$	composition of f following g, restriction to E
$f(\cdot, y)$	the function $x \mapsto f(x, y)$
$f(E), \ f^{-1}(E)$	direct and inverse image of a set
countable	finite or in one-one correspondence with integers
2^A	set of all subsets of A
B^A	set of all functions from B to A
card A	cardinality of A

Number systems

δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if $i \neq j$
$\binom{n}{k}$	binomial coefficient
n positive, n negative	$n > 0, n < 0$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integers, rationals, reals, complex numbers
$\mathbb{F}$	$\mathbb{R}$ or $\mathbb{C}$, the underlying field of scalars
max	maximum of finite subset of a totally ordered set
min	minimum of finite subset of a totally ordered set
$\sum$ or $\prod$	sum or product, possibly with a limit operation

$[x]$	greatest integer $\leq x$ if x is real
Re z, Im z	real and imaginary parts of complex z
$\bar{z}$	complex conjugate of z
$|z|$	absolute value of z

Linear algebra and elementary group theory

$\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{F}^n$	spaces of column vectors with n entries
$x \cdot y$	dot product
e_j	j^{th} standard basis vector of $\mathbb{R}^n$
1 or I	identity matrix or operator
det A	determinant of A
A^{tr}	transpose of A
$\text{diag}(a_1, \ldots, a_n)$	diagonal square matrix
Tr A	trace of A
$[M_{ij}]$	matrix with $(i, j)^{\text{th}}$ entry M_{ij}
dim V	dimension of vector space
0	additive identity in an abelian group
1	multiplicative identity in a group or ring
$\cong$	is isomorphic to, is equivalent to

Real-variable theory and calculus

$\mathbb{R}^*$	extended reals, reals with $\pm\infty$ adjoined
sup and inf	supremum and infimum in $\mathbb{R}^*$
(a, b), $[a, b]$	open interval in $\mathbb{R}^*$, closed interval
$(a, b]$, $[a, b)$	half-open intervals in $\mathbb{R}^*$
$\limsup_n$, $\liminf_n$	$\inf_n \sup_{k \geq n}$ in $\mathbb{R}^*$, $\sup_n \inf_{k \geq n}$ in $\mathbb{R}^*$
lim	limit in $\mathbb{R}$ or $\mathbb{R}^*$ or $\mathbb{R}^N$
$|x|$	$\left(\sum_{j=1}^N |x_j|^2\right)^{1/2}$ if $x = (x_1, \ldots, x_N)$, scalars in $\mathbb{R}$ or $\mathbb{C}$
e	$\sum_{n=0}^\infty 1/n!$
$\exp x$, $\sin x$, $\cos x$, $\tan x$	exponential and trigonometric functions
$\arcsin x$, $\arctan x$	inverse trigonometric functions
$\log x$	natural logarithm function on $(0, +\infty)$
$\frac{\partial f}{\partial x_j}$	partial derivative of f with respect to j^{th} variable
$C^k(V)$, $k \geq 0$	scalar-valued functions on open set $V \subseteq \mathbb{R}^N$ with all partial derivatives continuous through order k, no assumption of boundedness
$C^\infty(V)$	$\bigcap_{k=0}^\infty C^k(V)$
$f : V \to \mathbb{F}$ is smooth	f is scalar valued and is in $C^\infty(V)$
homogeneous of degree d	satisfying $f(rx) = r^d f(x)$ for all $x \neq 0$ in $\mathbb{R}^N$ and all $r > 0$ if f is a function $f : \mathbb{R}^N - \{0\} \to \mathbb{F}$

Metric spaces and topological spaces

d	typical name for a metric
$B(r; x)$	open ball of radius r and center x
A^{cl}	closure of A
A^{o}	interior of A
separable	having a countable base for its open sets
$D(x, A)$	distance to a set A in a metric space
$x_n \to x$ or $\lim x_n = x$	limit relation for a sequence or a net
S^{N-1}	unit sphere in $\mathbb{R}^N$
support of function	closure of set where function is nonzero
$\|f\|_{\sup}$	$\sup_{x \in S} |f(x)|$ if $f : X \to \mathbb{F}$ is given
$B(S)$	space of all *bounded* scalar-valued functions on S
$B(S, \mathbb{C})$ or $B(S, \mathbb{R})$	space of members of $B(S)$ with values in $\mathbb{C}$ or $\mathbb{R}$
$C(S)$	space of all *bounded* scalar-valued continuous functions on S if S topological
$C(S, \mathbb{C})$ or $C(S, \mathbb{R})$	space of members of $C(S)$ with values in $\mathbb{C}$ or $\mathbb{R}$
$C_{\mathrm{com}}(S)$	space of functions in $C(S)$ with compact support
$C_0(S)$	space of functions in $C(S)$ vanishing at infinity if S is locally compact Hausdorff
X^*	one-point compactification of X

Measure theory

$m(E)$ or $|E|$	Lebesgue measure of E
indicator function of set E	function equal to 1 on E, 0 off E
$I_E(x)$	indicator function of E at x
f^+	$\max(f, 0)$ for f with values in $\mathbb{R}^*$
f^-	$-\min(f, 0)$ for f with values in $\mathbb{R}^*$
$\int_E f\, d\mu$ or $\int_E f(x)\, d\mu(x)$	Lebesgue integral of f over E with respect to μ
dx	abbreviation for $d\mu(x)$ for μ=Lebesgue measure
$\int_a^b f\, dx$	Lebesgue integral of f on interval (a, b) with respect to Lebesgue measure
$(X, \mathcal{A}, \mu)$ or (X, μ)	typical measure space
a.e. $[d\mu]$	almost everywhere with respect to μ
$\nu = f\, d\mu$	complex measure ν with $\nu(E) = \int_E f\, d\mu$
$\mathcal{A} \times \mathcal{B}$	product of σ-algebras
$\mu \times \nu$	product of σ-finite measures
$\|f\|_p$	L^p norm, $1 \le p \le \infty$
p'	dual index to p with $p' = p/(p-1)$
$L^p(X, \mathcal{A}, \mu)$ or $L^p(X, \mu)$	space of functions with $\|f\|_p < \infty$ modulo functions equal to 0 a.e. $[d\mu]$

$f * g$	convolution
$f^*(x)$	Hardy–Littlewood maximal function, given by the supremum of the averages of $\|f\|$ over balls centered at x
$d\omega$	spherical part of Lebesgue measure on $\mathbb{R}^N$, measure on S^{N-1} with $dx = r^{N-1}\,dr\,d\omega$
Ω_{N-1}	"area" of S^{N-1} given by $\Omega_{N-1} = \int_{S^{N-1}} d\omega$
$\Gamma(s)$	gamma function with $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$
$\nu \ll \mu$	ν is absolutely continuous with respect to μ
Borel set in locally compact Hausdorff space X	set in σ-algebra generated by compact sets in X
$\mathcal{B}(X)$	σ-algebra of Borel sets if X is locally compact Hausdorff
compact G_δ	compact set equal to countable intersection of open sets
Baire set in locally compact Hausdorff space X	set in σ-algebra generated by compact G_δ's in X
$M(X)$	space of all finite regular Borel complex measures on X if X is locally compact Hausdorff
$M(X, \mathbb{C})$ or $M(X, \mathbb{R})$	$M(X)$ with values in $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$

Fourier series and Fourier transform

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$	Fourier coefficient
$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$	Fourier series of f, with c_n as above
$s_N(f; x) = \sum_{n=-N}^{N} c_n e^{inx}$	partial sum of Fourier series
$\widehat{f}(y) = \int_{\mathbb{R}^N} f(x)e^{-2\pi ix\cdot y}\,dx$	Fourier transform of an f in $L^1(\mathbb{R}^N)$
$f(x) = \int_{\mathbb{R}^N} \widehat{f}(y)e^{2\pi ix\cdot y}\,dy$	Fourier inversion formula
$\mathcal{F}$	Fourier transform as an operator
$\|\mathcal{F}f\|_2 = \|f\|_2$	Plancherel formula
$\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^N)$	Schwartz space on $\mathbb{R}^N$
$\frac{1}{\pi}\lim_{\varepsilon \downarrow 0} \int_{\|t\| \geq \varepsilon} \frac{f(x-t)}{t}\,dt$	Hilbert transform of function f on $\mathbb{R}^1$

Normed linear spaces and Banach spaces

$\|\cdot\|$	typical norm in a normed linear space
$(\cdot, \cdot)$	typical inner product in a Hilbert space, linear in first variable, conjugate linear in second
$M^\perp$	space of vectors orthogonal to all members of M
X^*	dual of normed linear space X
ι	canonical mapping of X into $X^{**} = (X^*)^*$
$\mathcal{B}(X, Y)$	space of bounded linear operators from X into Y

Advanced Real Analysis