INTRODUCTION TO LIE GROUPS

Anthony W. Knapp

Sophus Lie (1842-1899)

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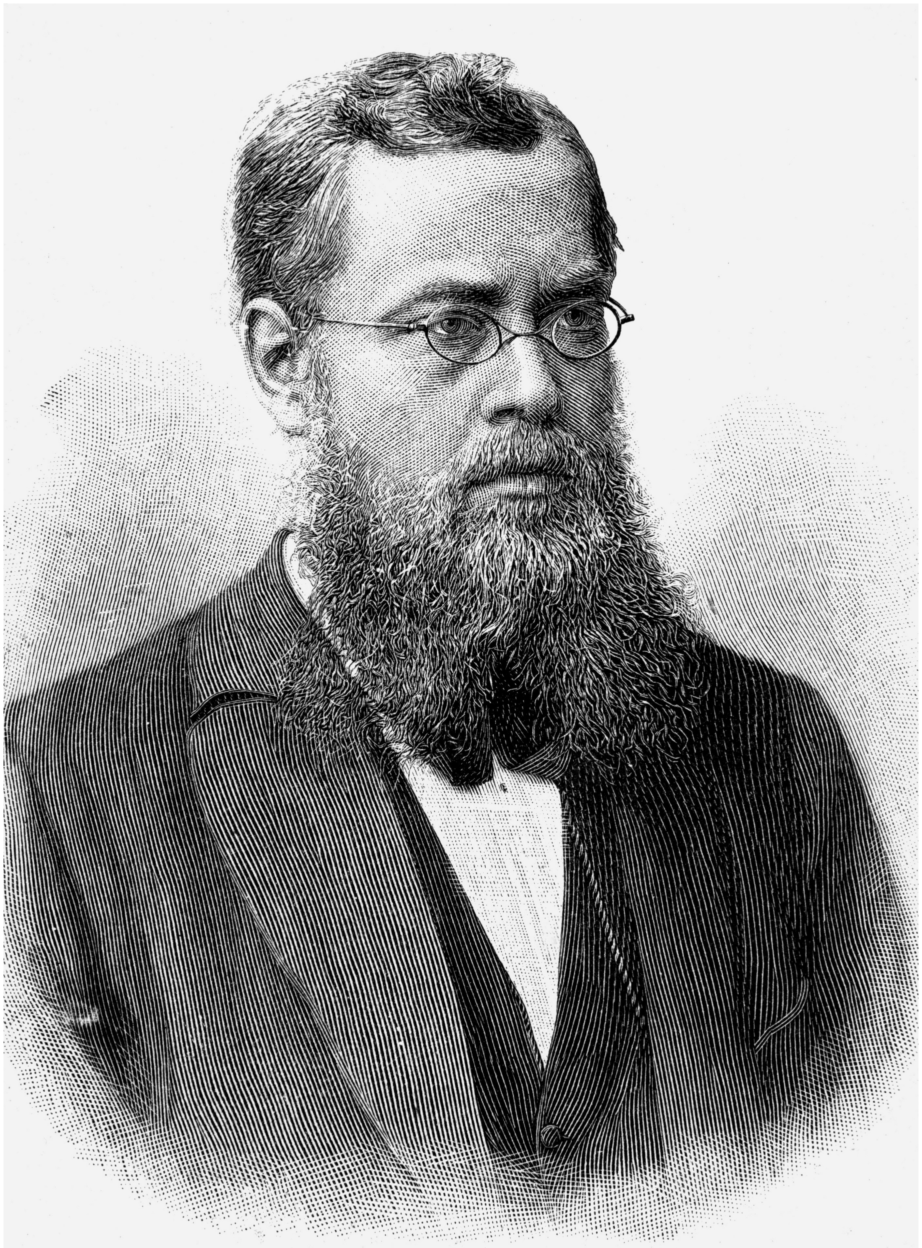
Preliminary Version 6



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A Prequel to *Lie Groups Beyond an Introduction*

Preliminary Version 6

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Frontispiece and cover: Sophus Lie, born December 17, 1842, died February 18, 1899. Wood engraving, 19th century, image thought to have been created about 1875, public domain image obtained from Alamy Inc., Brooklyn, NY 11217, USA.

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To Susan

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PREFACE

Sophus Lie (1842–1899) was a Norwegian geometer who in the 1870s and 1880s developed a theory of continuous transformation groups to use in analyzing partial differential equations. These transformation groups were local, operating in Euclidean space in a neighborhood of a point, and they related to finite-dimensional group actions, consisting of both a group and a space on which the group acts. The groups in question have come to be known as Lie groups in recognition of Lie's work. Lie groups nowadays play an important role in many areas of pure and applied mathematics, and this book develops the foundations of Lie theory in present-day language.

A feature of Lie's work was his examination of the behavior of the group infinitesimally at the group identity through a structure that was ultimately called the Lie algebra of the Lie group. Lie stated and proved three theorems about the relationship between his local Lie groups and the Lie algebras in question, showing in particular that the Lie algebra determines the Lie group completely in a neighborhood of the identity.[1] Lie's student F. Engel took up the job of putting Lie's work into writings. Despite Lie's hopes, continuation of this work was not taken up immediately by others. Instead, work on Lie algebras became paramount. Some of the most fundamental work on Lie algebras was carried out systematically by Lie's student Élie Cartan, and discussion of it is deferred to the sequel of the present volume.

Work of H. Weyl in 1924–25 and E. Cartan in 1930 began a change in focus of the theory from local to global, and C. Chevalley in 1946 completed the change by incorporating into the theory an extension of some nineteenth century work on foliations in differential topology. Chevalley used the term "analytic group" for connected Lie group. In that framework an analytic group

[1]See Cohn [1957] for precise statements.

becomes a group with a topological structure and a smooth manifold structure with all structures compatible. A subgroup of an analytic group is called an "analytic subgroup" if it has the structure of an analytic group for which the inclusion map is smooth. The four Fundamental Theorems of Lie Theory, which incorporate global versions of Lie's three local theorems and a fourth theorem that was not even stated until the 1920s, are then as follows:

(A) Every Lie group has a real Lie algebra of the same finite dimension as the group. The analytic subgroups of a given Lie group stand in one-one correspondence with the Lie subalgebras of the Lie algebra of the given Lie group. Under the correspondence each Lie subalgebra is the Lie algebra of its corresponding analytic group, the elements of the analytic subgroup are the ones generated by the exponentials of the members of the Lie subalgebra, the topology on the analytic subgroup is the *natural* topology defined in terms of the exponential map, and the manifold structure on the analytic subgroup is determined by exponential coordinates.

(B) The derivative at the identity of any smooth homomorphism of one analytic group into another is a homomorphism of the Lie algebra of the first into the Lie algebra of the second, and any homomorphism of the Lie algebras of two analytic groups lifts to a smooth homomorphism of the analytic groups if the domain group is simply connected.

(C) Every finite dimensional real Lie algebra is the Lie algebra of some analytic group.

(D) Every analytic group has a simply connected covering group that is itself an analytic group, and any two simply connected analytic groups with isomorphic Lie algebras are isomorphic analytic groups.

Largely the present volume is devoted to stating and proving all but the third of these four Fundamental Theorems. Fundamental Theorem C is too complicated to prove in the present volume. Some remarks about its proof appear in the section Summary of Chapter IV, and the full proof appears in an appendix "Lie's Third Theorem" of the author's volume entitled *Lie Groups Beyond an Introduction* that was published earlier.

In 1897 J. E. Campbell proved a formula for small matrix solutions Z of the equation $e^Z = e^X e^Y$ in terms of X and Y, matrix addition, scalar multiplication, and matrix multiplication. It was understood that if the expression for Z could be written using matrix commutators like $[X, Y] = XY - YX$ instead of matrix products like XY, then one would be close to having an explicit way of passing from a Lie algebra back to the analytic group from which it arose. In other words the hard part of the Fundamental Theorem A could be proved directly.

Over the years many people worked on aspects of "Campbell's Problem," as it was called, including H. F. Baker in 1905 and F. Hausdorff in 1906, and the result has come to be known as the Campbell-Baker-Hausdorff Theorem. Hausdorff gave an affirmative answer to the question whether Z could indeed be written using matrix commutators rather than matrix products, but a usable formula of this kind was not proved until E. Dynkin obtained such a result in 1947 and published it in Russian.

Meanwhile Chevalley's book appeared in English in 1946, and his argument for constructing analytic subgroups out of Lie subalgebras made use of his extension of an 1877 result of G. Frobenius about foliations in differential topology. That work on foliations had demanding prerequisites by the standards of the time.

The 1962 book by S. Helgason gave a different argument, exchanging demanding prerequisites in differential topology for demanding prerequisites in differential geometry. A 1982 book by R. Godement constructed analytic subgroups out of Lie subalgebras by a long argument using neither differential topology nor differential geometry.

Finally some books appeared that put together all the details, including Dynkin's Formula, for developing Lie theory economically without using foliations. One of these is a textbook in German from 1991 by Hilgert and Neeb. This author's favorite is the one from 2000 by Duistermaat and Kolk. With the Duistermaat–Kolk book in hand, it became possible to write an exposition of Lie theory that required less background than ever before. W. Rossmann wrote such a book in 2002, and B. Hall wrote another such book in 2003. Both these authors emphasized Lie groups that are subgroups of the multiplicative group of nonsingular real or complex matrices of some size—*linear groups* is what such Lie groups are called here. Rossmann gave also a few indications of how his approach might be extended to handle Lie groups that are not linear groups.

Rossmann and Hall were not the first authors to work especially with linear groups. For example, H. Freudenthal and H. de Vries had written a book about linear groups in 1969. But Rossmann and Hall were the first to work especially with linear groups and make use of Dynkin's Formula rather than extensive differential topology or geometry.

Rossmann and Hall each incorporated some additional simplifications of their own into their treatments of the foundational theory of Lie groups, and some of those have been included here.

My own experience with Lie groups began with courses taught at M.I.T. by B. Kostant and S. Helgason in 1965–67, and parts of those courses continue to affect my thinking about the subject. Some of the present book is based on various courses I taught at Cornell University and Stony Brook University between 1971 and 1995. Other parts of the book have been influenced by the more recent developments of Duistermaat, Kolk, Rossmann, and Hall.

The book has four chapters, and there are three appendices. Chapters I through III deal with linear analytic groups, and Chapter IV shows how to adjust the theory so that it applies to all analytic groups. The point of initially restricting the exposition to linear groups is that doing so cuts down on the prerequisites, making it unnecessary to know more than a minimal amount of differential topology or differential geometry for a long time.

The mathematical prerequisites for reading this book amount to undergraduate courses in real analysis and algebra, but greater sophistication is expected of the reader than those words suggest. Here is more detail.

For Chapters I and II, the mathematical prerequisites are the elements of linear algebra, group theory, uniform convergence, metric spaces, and vector calculus, including the statement of the Inverse Function Theorem. In addition, one needs to know something about topological groups as in the two sections of Appendix A and something about smooth manifolds as in the first section of Appendix B. Two optional examples at the end of Chapter III make use of covering spaces as in Appendix C. Chapter IV has further prerequisites that are included in Appendices A, B, and C. Further discussion of prerequisites appears on the page that follows this Preface.

There are several possibilities for study after one reads this book. One is this book's sequel, *Lie Groups Beyond an Introduction*. Another is any of several books by other authors on compact Lie groups and their representations.

Each chapter has an abstract at the beginning and a summary near the end. The latter indicates what progress has been made in that chapter toward establishing the four Fundamental Theorems of Lie Theory as listed above. Each chapter ends with a selection of problems, and hints for solving many of the problems appear near the end of the book. The back matter contains some historical commentary and references.

I am grateful to Brian Hall, Kiyoshi Sogo, Robert Stanton, and Po-Lam Yung for providing me with helpful comments about earlier versions of this book.

Anthony W. Knapp
July 2026

PREREQUISITES BY CHAPTER

Chapters I, II, and III

Algebra: unique factorization for $\mathbb{Z}$, unique factorization for polynomials over $\mathbb{C}$, row reduction of matrices over $\mathbb{C}$, matrix algebra, vector spaces, spanning and linear independence, linear maps and rank, dual spaces, quotients, isomorphism theorems, direct sums, determinants, characteristic polynomials, inner products, Gram–Schmidt orthogonalization process, orthogonal and unitary maps, elementary group theory, including homomorphisms and quotients, commutative rings with identity and determinants over them, quotients and homomorphisms, polynomials, substitution homomorphism, group actions, Jordan form.

Real analysis: theory of calculus in one and several variables including statement of Inverse Function Theorem, Taylor series in several variables with integral remainder, absolute and uniform convergence of series of functions of several variables, metric spaces, normed spaces, uniqueness theorem for linear systems of first-order ordinary differential equations.

Point-set topology: base for a topology, Hausdorff space, compactness, connectedness, countability properties. One problem in Chapter I uses the Baire Category Theorem.

The following material from Appendices A, B, and C:

A: topological groups and their quotients from Sections A.1 and A.2.

B: smooth manifolds, functions, germs, and maps from Section B.1.

C: the notion of "simply connected" and some related notions, which play a role in Sections III.6–III.8 but not otherwise in Chapters I, II, and III.

Chapter IV

All of the above plus the following: Remainder of Appendix B and Appendix C, including derivative of a smooth mapping and integral curves in Appendix B and existence and uniqueness of covering spaces in Appendix C. The material on integral curves in Appendix C assumes knowledge of the general existence and uniqueness theorems for solutions of first order systems of ordinary differential equations.

STANDARD NOTATION

Item	**Meaning**
$\lvert S \rvert$	number of elements in S
$\varnothing$	empty set
$\{x \in E \mid P\}$	the set of x in E such that P holds
E^c	complement of the set E
$E \cup F,\ E \cap F,\ E - F$	union, intersection, difference
$\bigcup_\alpha E_\alpha,\ \bigcap_\alpha E_\alpha$	union, intersection of the sets E_α
$E \subseteq F,\ E \supseteq F$	E is contained in F, E contains F
$E \times F$	product of sets E and F
$(a_1, \dots, a_n),\ \{a_1, \dots, a_n\}$	ordered n-tuple, unordered n-tuple
$f \circ g$ or fg	composition of g followed by f
$f\big\rvert_E$	restriction of f to E
$f(\,\cdot\,, y)$	the function $x \mapsto f(x, y)$
$f(E),\ f^{-1}(E)$	direct and inverse image of a set
δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if not
$\binom{n}{k}$	binomial coefficient
x positive, x negative	$x > 0, x < 0$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integers, rationals, reals, complex numbers
countable	finite or in one-one correspondence with $\mathbb{Z}$
$\lfloor x \rfloor$	greatest integer $\le x$ if x is real
$\mathbb{Q}^n, \mathbb{R}^n, \mathbb{C}^n$	spaces of column vectors
$(\,\cdot\,)^t$	transpose
$(\,\cdot\,)^*$	adjoint
Tr	trace
det	determinant

continued →

Item	**Meaning**
$\mathrm{Hom}_{\mathbb{R}}(U, V)$	space of linear maps of real vector space U to real vector space V
$\mathrm{End}_{\mathbb{R}}(V)$	space of linear maps of real vector space V to itself
$\sum$	sum, possibly with a limit
$\mathrm{Re}\, z$, $\mathrm{Im}\, z$	real, imaginary parts of complex z
$\bar{z}$	complex conjugate of z
$\lvert z \rvert$	absolute value of z
i	one particular square root of -1
1	multiplicative identity
1	identity matrix or operator
1_X	identity function on X
$\cong$	is isomorphic to, is equivalent to
$O(f(t))$ as $t \to 0$	quantity bounded as $t \to 0$ even when divided by $f(t)$

CHAPTER I

Closed Linear Groups and Lie Groups

Abstract. Chapter I concerns closed linear groups and their relation to Lie groups. Sections 1–5 are a leisurely introduction to the subject of Lie groups, Lie groups themselves being defined in Section 1. Linear groups are defined in Section 4 as *arbitrary* subgroups, not necessarily closed, of any general linear group of all nonsingular real or complex square matrices of a particular size under multiplication. Each linear group has a linear Lie algebra defined in Section 4 in terms of derivatives of curves extending from the identity matrix. Sections 6–7 represent a transition to the deeper theory, proving von Neumann's theorem that closed linear subgroups are Lie groups.

Section 1 defines Lie groups as groups that have also the structure of a smooth manifold in such a way that multiplication and inversion are smooth. Connected Lie groups are of special interest; these are often called analytic groups for historical reasons. Only a few examples of Lie groups are immediately at hand. These include the additive group $\mathbb{R}^n$ of a Euclidean space, the circle S^1 and its higher-dimensional version called a torus T^n, direct products of Lie groups, and the general linear groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. The section concludes by showing how complex matrices can be viewed as real matrices of twice the size.

Section 2 examines abelian Lie groups, connected or disconnected, particularly those of dimension 0, 1, or 2. Special attention is paid to lines on the torus T^2, which are analytic subgroups of T^2. An analytic subgroup of an analytic group is an abstract subgroup that is itself an analytic group for which the inclusion mapping is smooth.

Section 3 defines the notion of a semidirect product of Lie groups by a construction that produces a new Lie group out of two old ones. Four examples are given.

Section 4 works with linear groups as defined above. Of special note are some groups of matrices that are defined by polynomial equations relating entries to one another; such groups include rotation groups, unitary groups, and special linear groups. All these examples are closed subgroups in the relative topology. By contrast the line on T^2 with an irrational slope, as in Section 2, is not closed in the

relative topology even though it can be viewed as a linear group in a variety of ways. The linear Lie algebra of a linear group G is the real vector space of derivatives at $t = 0$ of smooth curves $t \mapsto c(t)$ that are in G for every t; this vector space has a natural "Lie bracket" product operation given by $[X, Y] = XY - YX$. Section 4 concludes by beginning to identify the linear Lie algebras of the rotation groups, the unitary groups, and the special linear groups.

Section 5 reviews from advanced calculus the exponential mapping $X \mapsto \exp X$ for real and complex square matrices, and then it completes the identification of the linear Lie algebras of the rotation groups, the unitary groups, and the special linear groups.

Section 6 shows for any closed linear group G, i.e., for any linear group that is closed in the relative topology from nonsingular matrices, that the linear Lie algebra as defined in Section 4 consists of all matrices X such that $\exp tX$ is in G for all real t.

Section 7 uses the exponential mapping to show that any closed linear group G admits consistently with its relative topology the structure of a smooth manifold such that G becomes a Lie group.

1. Examples of Lie groups

A **Lie group** G is a topological group that at the same time is a smooth ($= C^\infty$) manifold[1] in which multiplication, as a mapping from $G \times G$ into G, and inversion, as a mapping from G into G, are both smooth. The **dimension** of the Lie group is the dimension of the underlying smooth manifold. The fact that G is a manifold implies that G is locally connected as a topological space, and therefore its connected components are open. In particular the connected component of the group identity is an open normal subgroup, and the remainder of the group consists of the other cosets of the identity component. These other cosets offer nothing new for the topological structure or the manifold structure, and most of the effort in this book will be in dealing with the identity component.

The following definition takes this fact into account. For reasons that are explained in the Historical Commentary at the end of the book, a connected Lie group will often be called an **analytic group.**

When are two Lie groups the same? A Lie group G is said to be **isomorphic** to another Lie group G' if there is a group isomorphism $\Phi\colon G \to G'$ that is also a diffeomorphism of G onto G'.

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[1]Smooth manifolds are built from overlapping open subsets of a Hausdorff topological space that are identified with subsets in a Euclidean space. The actual definitions and elementary properties of smooth manifolds, smooth mappings, and diffeomorphisms are not needed now and will be treated in more detail in Section 7 when the need arises.

Only a few examples of Lie groups are immediately available, and for the most part, they are not particularly illuminating. A list appears below. Later in this chapter some further work will show that "closed linear groups," to be defined below, form a rich additional class of examples. Here are some immediately available examples.

EXAMPLE 1. Euclidean space $\mathbb{R}^m$ is a group with addition as the operation, and the usual metric on $\mathbb{R}^m$ makes $\mathbb{R}^m$ into a locally compact abelian topological group. This group is also a smooth manifold of dimension m, with only one chart necessary, namely $(1, \mathbb{R}^m)$ with 1 denoting the identity function from the manifold $\mathbb{R}^m$ to the Euclidean space $\mathbb{R}^m$. Addition is smooth from $\mathbb{R}^m \times \mathbb{R}^m$ into $\mathbb{R}^m$, and passage to the negative is smooth from $\mathbb{R}^m$ into $\mathbb{R}^m$. Thus $\mathbb{R}^m$ is an abelian Lie group, noncompact if $m > 0$.

EXAMPLE 2. The **circle** S^1, defined in one way as the subset of all complex numbers $z \in \mathbb{C}$ having $|z| = 1$, is a group under multiplication. With the relative topology from $\mathbb{C}$, S^1 is a compact metric space, being closed and bounded in a Euclidean space. In this topology, S^1 is a compact topological group; the continuity of multiplication and inversion follow from the facts that the limit of a product in $\mathbb{C}$ is the product of the limits and that the (nonzero) limit of a reciprocal is the reciprocal of the limit. One way of making S^1 into a smooth manifold of dimension 1 is to use two charts $(S^1 - \{i\}, \varphi)$ and $S^1 - \{-i\}, \psi)$ with φ and ψ given by

$$\varphi(x + iy) = \big(x(1 - y)^{-1}\big) \qquad \text{and} \qquad \psi(x + iy) = \big(x(1 + y)^{-1}\big).$$

Multiplication and inversion are smooth, and thus S^1 is a compact abelian Lie group.

EXAMPLE 3. The **direct product**[2] of a finite number of Lie groups is a Lie group. It is to be given the product group structure, the product topological structure, and the product smooth-manifold structure. The n fold product of copies of the circle group S^1 is called an n dimensional **torus** and is often written as T^n. In particular T^1 is just another name for S^1.

EXAMPLE 4. The **general linear group** $GL(n, \mathbb{R})$ of nonsingular n-by-n real matrices, with multiplication as the group operation and with the topology and smooth structure inherited as an open subset of the Euclidean space $\mathfrak{m}_{nn}(\mathbb{R})$ of all n-by-n real matrices, becomes a Lie group. More particularly let us write $e_{ij} \colon \mathfrak{m}_{nn}(\mathbb{R}) \to \mathbb{R}$ for the numerical-valued function defined on all n-by-n matrices that extracts the $(i, j)^{\text{th}}$ entry of a matrix. We can identify

[2]Sometimes the (finite) direct product is called the **direct sum**, particularly when the individual groups are all abelian.

$\mathfrak{m}_{nn}(\mathbb{R})$ with $\mathbb{R}^{n^2}$ by listing the n^2 functions e_{ij} in a convenient order and then identifying each matrix x with the member of $\mathbb{R}^{n^2}$ whose n^2 coordinates are the numbers $e_{ij}(x)$. With this identification we see that matrix multiplication is given, entry-by-entry, by certain polynomials of total degree 2 in the entries. The determinant function itself is a polynomial in the e_{ij}'s, and the set where the determinant is nonzero is an open set, polynomials being continuous functions. Thus we can regard $GL(n,\mathbb{R})$ as an open set in $\mathbb{R}^{n^2}$, and it becomes a smooth manifold. Multiplication, being given by polynomial functions, and inversion, being given by polynomial functions divided by the determinant function, are smooth functions on this open set. Thus $GL(n,\mathbb{R})$ is a Lie group.

EXAMPLE 5. In the same way, the **general linear group** $GL(n,\mathbb{C})$ of nonsingular n-by-n complex matrices, with multiplication as the group operation and with the topology and smooth structure inherited as an open subset of the Euclidean space $\mathfrak{m}_{nn}(\mathbb{C})$ of all n-by-n complex matrices, becomes a Lie group. Identifying $\mathfrak{m}_{nn}(\mathbb{C})$ with a Euclidean space involves one extra wrinkle: the Euclidean space in question has dimension $2n^2$, and the n^2 coordinate functions e_{ij} used for $\mathfrak{m}_{nn}(\mathbb{R})$ get replaced by the $2n^2$ functions $\operatorname{Re} e_{ij}$ and $\operatorname{Im} e_{ij}$.

A few further examples of Lie groups come from taking a general linear group and requiring that some entries be 0. For example, one might require that all entries below the main diagonal be 0. Such examples will not give us further insight at this point, and we shall skip over them.

It is understood that any linear group is a subgroup of some $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$. We shall usually write $\boldsymbol{M}$ for this **underlying general linear group.** The corresponding full matrix space, either $\mathfrak{m}_{nn}(\mathbb{R})$ or $\mathfrak{m}_{nn}(\mathbb{C})$, will be called $\boldsymbol{\mathfrak{m}}$.

It will be convenient to pin down the relationship between real and complex matrices a bit. Real matrices can be regarded as complex matrices of the same size, of course, but it is also true that complex matrices can be regarded as real matrices of twice the size. Moreover, the correspondence can be set up in such a way that we can easily keep track of what happens under familiar operations such as addition, multiplication, adjoint, trace, and determinant. Here are some details.

For v in $\mathbb{C}^n$, write $v = a + ib$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, and define functions Re and Im from $\mathbb{C}^n$ to $\mathbb{R}^n$ by $\operatorname{Re} v = a$ and $\operatorname{Im} v = b$. Then set up the $\mathbb{R}$ isomorphism $\mathbb{C}^n \to \mathbb{R}^{2n}$ given in block form by

$$v \mapsto \begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix}. \tag{1.1}$$

Next let U be an n-by-n matrix over $\mathbb{C}$ and write $U = \operatorname{Re} U + i \operatorname{Im} U$. Under the isomorphism (1.1), left multiplication by U on $\mathbb{C}^n$ corresponds to left multiplication on $\mathbb{R}^{2n}$ by

$$Z(U) = \begin{pmatrix} \operatorname{Re} U & -\operatorname{Im} U \\ \operatorname{Im} U & \operatorname{Re} U \end{pmatrix}. \tag{1.2}$$

This identification has the following properties:

(a) $Z(UV) = Z(U)Z(V)$,
(b) $Z(U^*) = Z(U)^*$,
(c) $\operatorname{Tr} Z(U) = 2 \operatorname{Re} \operatorname{Tr} U$,
(d) $\det Z(U) = |\det U|^2$.

For the proof, only (d) requires comment. Because the function $U \mapsto Z(U)$ is real linear and satisfies (a), use of row reduction shows that it is enough to check (d) for elementary matrices (the ones that implement a single step of row reduction). Matters come down to U of size 1-by-1, where the argument is that

$$(z) = (x + iy) \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \qquad \text{with} \qquad \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2 = |z|^2.$$

2. Analytic subgroups in abelian examples of low dimension

In topics related to group theory, it is natural to consider subgroups, quotients, and homomorphisms. In this section we take note of some of the considerations in dealing with the corresponding notion of subgroups for Lie groups. We shall return to these matters later.

First consider subgroups in the abelian case. The one-dimensional examples in the previous section were the additive group $\mathbb{R} = \mathbb{R}^1$ of the real line and the circle group $S^1 = T^1$. In the case of $\mathbb{R}$ as a Lie group under addition, most subgroups of $\mathbb{R}$ have nothing to do with the topological structure of $\mathbb{R}$, much less the manifold structure. The fact is that the only subgroups of $\mathbb{R}$ are $\{0\}$, $\{$all multiples of a nonzero real $\alpha\}$, and subgroups that are dense in $\mathbb{R}$. To see this fact, it is enough to see that any subgroup of $\mathbb{R}$ that contains the real number 1 and some irrational α is dense.

Proposition 1.1. Any additive subgroup of $\mathbb{R}$ that contains 1 and some irrational number is dense in $\mathbb{R}$.

Proof.[3] Let an integer $n \geq 1$ be given, and let α be a fixed irrational number. Consider the $n+1$ real numbers $\{k\alpha\}$ defined to be the fractional parts of $k\alpha$ for $0 \leq k \leq n$, specifically

$$\{k\alpha\} = k\alpha - \lfloor k\alpha \rfloor \qquad k = 0, 1, \dots, n,$$

where $\lfloor \cdot \rfloor$ is the floor function.[4] Each of these $n+1$ numbers is ≥ 0 and < 1, and thus each fits into exactly one of the n intervals given by

$$\Big[0, \frac{1}{n}\Big), \ \Big[\frac{1}{n}, \frac{2}{n}\Big), \ \dots, \Big[\frac{n-1}{n}, 1\Big).$$

By the pigeon-hole principle, there must be at least one of the intervals containing two of the numbers. Say that $\{k_1\alpha\}$ and $\{k_2\alpha\}$, with $k_1 \neq k_2$, both fit into the interval $[k/n, (k+1)/n)$ and that $\{k_1\alpha\} \leq \{k_2\alpha\}$. Because of the inequality $1 > \{k_2\alpha\} - \{k_1\alpha\} \geq 0$, we have

$$\begin{aligned} \{k_2\alpha\} - \{k_1\alpha\} &= \{\{k_2\alpha\} - \{k_1\alpha\}\} \\ &= \{k_2\alpha - \lfloor k_2\alpha \rfloor - k_1\alpha + \lfloor k_1\alpha \rfloor\} \\ &= \{k_2\alpha - k_1\alpha + \text{integer}\} \\ &= \{(k_2 - k_1)\alpha\}. \end{aligned}$$

The right side cannot be 0, since $(k_2 - k_1)\alpha$ is not an integer, α being irrational. Since k_1 and k_2 were chosen so that $\{k_1\alpha\}$ and $\{k_2\alpha\}$ fit into the same subinterval of $[0, 1)$, the above computation allows us to conclude that $0 < \{(k_2 - k_1)\alpha\} < 1/n$.

Let S be the additive subgroup of $\mathbb{R}$ consisting of all integer combinations of 1 and α. Briefly what we have just seen is that S contains arbitrarily small positive numbers. It will be enough to show that any $x > 0$ is a limit point of S. Thus let $x > 0$ be given in $\mathbb{R}$, and let $\epsilon > 0$ be given. Choose $y > 0$ in S with $0 < y < \epsilon$, and let $n = \lfloor x/y \rfloor$. From the inequality $n \leq x/y < n+1$, we see that $yn \leq x < yn + y$ and thus $0 \leq x - yn < y < \epsilon$. Thus the member yn of S is exhibited as within ϵ of x. $\square$

We turn to the circle group. Instead of considering it as S^1, let us consider it as $\mathbb{R}/\mathbb{Z}$, the set of real numbers modulo 1. Proposition 1.1 shows that any

[3]This argument was contributed to StackExchange by Pedro on July 23, 2013. The idea of using the pigeon-hole principle in this way in this argument is due to G. L. Dirichlet.

[4]The **floor function** is defined so that the error term $\{k\alpha\}$ is ≥ 0 and < 1. Historically the floor function often used to be called the "greatest integer" function, but the term "greatest integer" function has come nowadays to be used for something else.

irrational α in $\mathbb{R}$, when considered in $\mathbb{R}/\mathbb{Z}$, generates a dense subgroup. For rational α, say $\alpha = p/q$ in lowest terms with $q > 0$, the number α generates the cyclic subgroup $\{0, 1/q, 2/q, \ldots, (q-1)/q)\}$. Thus the subgroups of the circle group are 0, the cyclic subgroup of each finite order, and various dense subgroups.

Since the interest in this book is in Lie groups, not general groups, it is appropriate to single out subgroups with interesting features in their topological or manifold properties. For our abelian examples in dimension 1, the only way to bring interesting manifold structure into the study is to look for manifolds of dimension 1, since single points do not contribute anything of interest. That requirement leads us to discard all the dense subgroups for $\mathbb{R}$ and S^1 except for the whole groups themselves, saying that the excluded subgroups are of little interest for current purposes. For dimension ≥ 2, however, matters are more subtle. We make a definition.

A subset H of a Lie group G is called an **analytic subgroup** if

(i) it is a subgroup under the group operation inherited from G,
(ii) it has the structure of an analytic group, and
(iii) the inclusion mapping of H into G is smooth.

Right away, we find an interesting example to study. Take the two-dimensional torus T^2, and consider it as the set $\mathbb{R}^2/\mathbb{Z}^2$, the Euclidean plane modulo the lattice pairs with integer coordinates. Any line through the origin is a subgroup of $\mathbb{R}^2$, and it projects under the quotient mapping $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ to a subgroup in T^2. Let us parametrize it as $t \mapsto (\alpha t, \beta t)$ for $t \in \mathbb{R}$, where α and β are real numbers not both 0.

This subgroup is certainly an analytic subgroup of $\mathbb{R}^2$ because when we follow the parametrization mapping $\mathbb{R}^1 \to \mathbb{R}^2$ with the quotient mapping $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, the three properties of an analytic subgroup are still valid. The following result describes the two possibilities for the behavior of this analytic subgroup.

Corollary 1.2. View the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the unit square in $\mathbb{R}^2$ with opposite sides identified. If α and β are not both 0, then the image in T^2 of the line $t \mapsto (\alpha t, \beta t)$ is one of the following:

(a) a finite set of parallel line segments when viewed within the unit square with opposite sides identified, topologically a circle when viewed within T^2, or
(b) a countable dense infinite set of parallel line segments when viewed within the unit square with opposite sides identified.

The first case occurs when $\alpha = 0$ or the slope β/α is rational, and the second case occurs when the slope β/α is irrational.

PROOF. If α or β is 0, then the image of $t \mapsto (\alpha t, \beta t)$ is the segment $[0, 1]$ on one of the axes. For the remaining cases we may normalize matters and assume that $\alpha = 1$ and $0 \leq \beta \leq 1$. Then the first line segment that arises is the line segment from $(0, 0)$ to $(1, \beta)$, the next one is $(0, \beta)$ to $(1,\ 2\beta \bmod 1)$, the third one is $(0, 2\beta \bmod 1)$ to $(1,\ 3\beta \bmod 1)$, and so on. Evidently the numbers that occur as second coordinates of the endpoints of the segments are the real numbers $n\beta \bmod 1$ with $n \in \mathbb{Z}$. If β is rational, this set of these numbers is finite, and the image of $t \mapsto (t, \beta t)$ is topologically a circle. If β is irrational, then Proposition 1.1 says that this set is dense; Proposition 1.1 even enumerates the set and shows that it is countable. $\square$

In the first case the image is a circle, not only as a topological space but actually as a topological group. In the second case, the image is again a subgroup of T^2 and at first appears to be topologically a line in its relative topology. However, the image is not locally compact in the relative topology. Indeed, a locally compact dense subset of a Hausdorff space is open,[5] and an open subgroup of a topological group has to be closed by Proposition A.6. Under these circumstances the image would have to be all of T^2, and we know from Corollary 1.2 that it is not.

For the Lie group $\mathbb{R}^2$, the analytic subgroups are $\{0\}$, each line through the origin, and $\mathbb{R}^2$ itself. The only other example that we have of an abelian analytic group of dimension 2 is $\mathbb{R}^1 \times T^1$, and its analytic subgroups and their behavior tell us nothing new. A little care is needed in identifying analytic subgroups, however: the group $\mathbb{R}^2$ has a connected dense subgroup H with no inherited manifold structure. This example is discussed more at the end of Section II.3.

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[5]A proof that a locally compact dense subset of a Hausdorff space is open runs as follows: Let Y be a locally compact dense subset of the Hausdorff space X. If y is in Y, let N be a relatively open neighborhood of Y such that $N \subseteq K$ with K compact in Y. Since N is relatively open, $N = U \cap Y$ for some open $U \subseteq X$. It will be proved that $N = U$, so that each point of Y has an X open neighborhood, and then Y will be open. The set K is compact in X and must be closed since X is Hausdorff. The points of $U \cap K$ are in Y since $K \subseteq Y$, and hence $U \cap K \subseteq U \cap Y = N$. Consider a point x of the open set $U - K$. Arguing by contradiction, suppose x is not in Y. Then x is a limit point of Y since Y is dense. Hence the open neighborhood $U - K$ of y contains a point y_0 of Y. Then y_0 is in $U \cap Y = N \subseteq K$ and cannot be in $U - K$, contradiction. We conclude that x is in Y. Then x is in $U \cap Y = N$, and $U = N$.

For higher dimensions we shall see in Chapter IV that the problem of identifying abelian analytic groups comes down to understanding "discrete" abelian subgroups of $\mathbb{R}^n$ for each n. A subgroup is **discrete** if each element has an open neighborhood that meets no other member of the set, i.e., if its relative topology is the discrete topology. Right now we can identify such subgroups completely.

Proposition 1.3. If D is a discrete subgroup of $\mathbb{R}^n$, then there exists a basis $x_1, \dots, x_n$ of $\mathbb{R}^n$ such that D consists exactly of those vectors of the form $k_1 x_1 + \cdots + k_p x_p$ with all k_j in $\mathbb{Z}$. Here p is some integer with $0 \le p \le n$.

Proof. The proof will proceed by a kind of induction. We start from any vector subspace W of $\mathbb{R}^n$ for which $D \cap W$ is known to consist exactly of all integer combinations $k_1 x_1 + \cdots + k_p x_p$ of p vectors $x_1, \dots, x_p$ that form a basis of W over $\mathbb{R}$. The base case of the induction has $W = 0$ and $p = 0$. For the inductive case we take p as large as possible so that $D \cap W$ has the required form. Certainly $p \le n$, since $x_1, \dots, x_p$ are assumed to be linearly independent over $\mathbb{R}$ in $\mathbb{R}^n$. Arguing by contradiction, suppose that there is some member u of D that is not an integer combination of $x_1, \dots, x_p$. Consider the set S of all members of u of W of the form

$$x = c_1 x_1 + \cdots + c_p x_p + cu \tag{1.3}$$

with all c_j satisfying $0 \le c_j \le 1$ with $0 \le c \le 1$. This set S is compact. Since D is discrete, $S \cap D$ is finite. Since u is of the form (1.3) with $c = 1$, $S \cap D$ contains at least one element with $c > 0$. Choose x_0 in $S \cap D$ whose coefficient of u is as small as possible but nonzero.

Any other member x' of $S \cap D$ will have a multiple of c as coefficient of u. In fact if x' has coefficient c' that is not a multiple of c, then for some integer m, we can write $c' = mc + d$ with $0 < c' < c$. Subtraction of suitable integral multiples of the various elements x_j from x' will result in an element like the elements in (1.3) but with c' in place of c, in contradiction to the minimal choice of c.

As a consequence, $D \cap (W + \mathbb{R}u)$ consists exactly of all elements

$$(D \cap W) + \mathbb{Z}u = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_p + \mathbb{Z}u.$$

The conclusion is that $W' = W + \mathbb{R}u$ had $D \cap W'$ of the proper form, in contradiction to the maximality of p. Thus the element u in (1.3) could not have existed, and D must have consisted of those vectors $k_1 x_1 + \cdots + k_p x_p$ with all k_j in $\mathbb{Z}$ and no others. $\square$

3. Semidirect products of Lie groups

"Semidirect product" is a construction in abstract group theory that produces a new group out of two old ones. The construction has a meaning also in the subject of Lie groups, and thus it gives us a routine way of obtaining new Lie groups from old ones.

The prototype is the group of rotations and translations of the plane, the rotations being taken about an arbitrary center. What is happening here is that we have two groups and that one (the group of rotations about the origin) acts on the other by automorphisms, each rotation being an automorphism of the (group of translations of the) plane. Out of the these ingredients we can define a single group that captures all the structure. Let us write out the details for this example and then give the general definition.

Let G be the group of all rotations about the origin in $\mathbb{R}^2$, which we can view as the group of 2-by-2 matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and let H be $\mathbb{R}^2$, which we can view as the space two-dimensional column vectors with group operation given by addition of column vectors as usual.[6] The group G acts on the space H by automorphisms, the automorphism corresponding to g being written as $h \mapsto \tau(g,h) = gh$, the matrix product of a 2-by-2 matrix in G by a 2-dimensional column vector in H. The fact that each $\tau(g,\,\cdot\,)$ is an automorphism of $\mathbb{R}^2$ additively amounts to the identity $g(h_1 + h_2) = gh_1 + gh_2$, in combination with the invertibility of the operations. The semidirect product $G \times_\tau H$ is defined to be the group defined on the set-theoretic product $G \times H$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1 g_2, (\tau(g_2^{-1})h_1) + h_2)$.

In the general case with two groups G and H written multiplicatively and with G operating on H by automorphisms through τ, the semidirect product $G \times_\tau H$ is taken to be the set $G \times H$ with group law written as

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, \tau(g_2^{-1}, h_1)h_2). \tag{1.4}$$

[6] It will often be convenient to identify $\mathbb{R}^n$ with the vector space of all n dimensional real column vectors $\big($even if the column vectors are written horizontally as tuples $(c_1, \ldots, c_n)\big)$ and to identify the space of linear maps of $\mathbb{R}^n$ into itself with the space of all n dimensional real matrices. Under this identification the action of a linear map on a vector in $\mathbb{R}^n$ corresponds to the product of the matrix and the column vector. Normally we make these identifications without particular comment.

Associativity of this multiplication comes down to the equality of

$$((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1 g_2 g_3, \tau(g_3^{-1}, \tau(g_2^{-1}, h_1)h_2)h_3)$$

and

$$(g_1, h_1)((g_2, h_2)(g_3, h_3)) = (g_1 g_2 g_3, \tau(g_3^{-1} g_2^{-1}, h_1)\tau(g_3^{-1}, h_2)h_3),$$

which follows from the computation

$$\begin{aligned}\tau(g_3^{-1}, \tau(g_2^{-1}, h_1)h_2) &= \tau(g_3^{-1}, \tau(g_2^{-1}, h_1))\tau(g_3^{-1}, h_2)\\ &= \tau(g_3^{-1} g_2^{-1}, h_1)\tau(g_3^{-1}, h_2).\end{aligned}$$

One readily verifies that the ordered pair $(1, 1)$ is a two-sided group identity and that the inverse of (g, h) is given by

$$(g, h)^{-1} = (g^{-1}, \tau(g, h^{-1})). \tag{1.5}$$

In other words, $G \times_\tau H$ is indeed a group.

The other two structures needed for $G \times_\tau H$ to be a Lie group are a topological structure and a manifold structure, with everything defined compatibly. Both structures, the topological structure and the manifold structure, are taken to be product structures. Under the assumption that the group action $\tau: G \times H \to H$ is smooth, it is apparent from (1.4) and (1.5) that multiplication and inversion in $G \times_\tau H$ are smooth, and $G \times_\tau H$ is thus a Lie group.

Here are three further concrete examples.

Example 1. Affine group of the line. Let G be the group of dilations of additive $\mathbb{R}^1$ by positive real numbers. Each dilation can be viewed as a multiplication operator on $\mathbb{R}^1$ by a positive number e^t. The group G acts on the additive group $\mathbb{R}^1$, and the semidirect product is the group of all dilations of $\mathbb{R}^1$ relative to an arbitrary origin.

Example 2. Group of rigid motions of $\mathbb{R}^n$. The orthogonal group $G = O(n)$ of rotations and reflections about the origin acts on $H = \mathbb{R}^n$ by linear functions, and the semidirect product of G and H is the group of all rigid motions of $\mathbb{R}^n$ about an arbitrary center.

Example 3. Let G be any group of linear mappings of $\mathbb{R}^n$ into itself, and let $H = \mathbb{R}^n$. Then the semidirect product of G and H is defined and is a generalization of the two previous examples. One particular case of interest in physics has G equal to the group of all linear mappings of $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ into itself that leave the expression $x_1^2+x_2^2+x_3^2-c^2x_4^2$ invariant. The group H is $\mathbb{R}^4$ with G acting by the corresponding linear mappings. With the constant c taken as the speed of light and x_4 treated as time, the group

G is called the **homogeneous Lorentz group** in space-time. The semidirect product of G with H is called the **inhomogeneous Lorentz group**.

4. Introduction to linear groups

Let us turn now to $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. A **linear group** in this book is a subgroup of some general linear group, either $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ for some n, with matrix multiplication as group operation. No topological restriction is imposed whatsoever. The **underlying general linear group** will usually be called $\boldsymbol{M}$. Groups of linear transformations on a finite-dimensional real or complex vector space, with composition as the group operation, will be treated as linear groups also. We regard two linear groups as the same only when there is a Lie group isomorphism of the underlying general linear groups carrying the one linear group onto the other.[7]

Let us observe that all of the Lie groups we have encountered so far can be viewed as linear groups. Whether the scalars for the general linear group are from $\mathbb{R}$ or from $\mathbb{C}$ usually makes no difference. In fact we certainly have the inclusion $GL(n,\mathbb{R}) \subseteq GL(n,\mathbb{C})$. Also we saw in Section 1 that $GL(n,\mathbb{C})$ is isomorphic as a Lie group to a subgroup of $GL(2n,\mathbb{R})$.

Let us see some ways of viewing Examples 1 through 3 of Section 1 as linear groups. With Example 1 one way of viewing the additive group $\mathbb{R}^n$ as a linear group is to identify the member $(x_1,\ldots,x_n)$ of $\mathbb{R}^n$ with the real diagonal matrix with diagonal entries $e^{x_1},\ldots,e^{x_n}$. Another way of viewing the group $\mathbb{R}^n$ as a linear group is as the linear group of all real square matrices of size $n+1$ with 1's down the diagonal, with the coordinates of $\mathbb{R}^n$ listed down the last column except in the diagonal entry, and with 0 in all remaining entries. In block form the matrix corresponding to a column vector $x \in \mathbb{R}^n$ is thus

$$\begin{array}{cc} \begin{array}{cc} n & 1 \end{array} & \\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & \begin{array}{c} n \\ 1 \end{array} \end{array}$$

with the upper left entry 1 standing for the n-by-n identity matrix. By either identification the linear group in question can be given the structure of a smooth manifold, and the embedded copy of $\mathbb{R}^n$ becomes an analytic subgroup.

For Example 2 the Lie group S^1 is isomorphic to a subgroup of $GL(1,\mathbb{C})$ and thus to a subgroup if $GL(2,\mathbb{R})$. The identification is

$$e^{ix} \longleftrightarrow \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.$$

[7] This is a small point, and we shall not develop it.

For Example 3 if G_1 and G_2 are linear subgroups of $GL(m,\mathbb{R})$ and $GL(n,\mathbb{R})$, respectively, then $G_1 \times G_2$ can be viewed as a block diagonal subgroup of $GL(m+n,\mathbb{R})$, namely

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}.$$

Here are a few more examples of linear groups:

$$\begin{aligned} SO(n) &= \{x \in GL(n,\mathbb{R}) \mid xx^t = 1 \text{ and } \det x = 1\} \\ U(n) &= \{x \in GL(n,\mathbb{C}) \mid xx^* = 1\} \\ SU(n) &= \{x \in U(n) \mid \det x = 1\} \\ SL(n,\mathbb{R}) &= \{x \in GL(n,\mathbb{R}) \mid \det x = 1\} \\ SL(n,\mathbb{C}) &= \{x \in GL(n,\mathbb{C}) \mid \det x = 1\}. \end{aligned} \tag{1.6}$$

The first three are called the **rotation group**, the **unitary group**, and the **special unitary group**; the last two are the **special linear groups** over $\mathbb{R}$ and $\mathbb{C}$. More examples appear in the Problems at the end of the chapter.

Checking that any of the sets listed in (1.6) is a subgroup under matrix multiplication is merely a question of checking closure under multiplication and inversion. For example, if x and y are in $SO(n)$, the definitions give $xx^t = yy^t = 1$ and $\det x = \det y = 1$. Then

$$\begin{gathered} (xy)(xy)^t = xyy^tx^t = x(yy^t)x^t = xx^t, \\ (x^{-1})(x^{-1})^t = x^{-1}(x^t)^{-1} = ((x^t)x)^{-1} = 1, \\ \det xy = \det x \det y = 1, \quad \text{and} \quad \det(x^{-1}) = (\det x)^{-1} = 1. \end{gathered}$$

Thus xy and x^{-1} satisfy the defining conditions for being in $SO(n)$.

A **closed linear group** is a linear group that is topologically closed in the relative topology that it inherits from the ambient general linear group. For groups of complex matrices, the condition "closed" is unaffected by whether the matrices are considered as in $GL(n,\mathbb{C})$ or in $GL(2n,\mathbb{R})$.

All the examples listed in (1.6) above—namely $SO(n)$, $U(n)$, $SU(n)$, $SL(n,\mathbb{R})$, and $SL(n,\mathbb{C})$—are closed linear groups. The reason that these subgroups are closed is that they are defined as the simultaneous zero loci of some polynomial equations; polynomials being continuous, their zero loci are closed. The polynomials in question are the entry-by-entry relations in the real and imaginary parts of the matrix entries that come from the defining equations for the groups. And of course, $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are themselves closed linear groups, practically as a matter of definition.[8]

An example of a connected linear group that is not a closed subgroup is obtained as follows: regard the torus T^2 as a linear subgroup of $GL(2,\mathbb{C})$, embedded as diagonal matrices with both diagonal entries of absolute value 1, and consider any line in Corollary 1.2 that winds around T^2 with an irrational slope. For convenience we shall refer to any such line as a **line on the torus T^2 with an irrational slope.**

The study of linear groups as potential Lie groups begins from the fact that a kind of differentiation is available in any linear group. For any linear group G, we can speak of **smooth curves** in G, namely C^∞ functions defined on an open interval of $\mathbb{R}$ taking values in the **underlying matrix space $\mathfrak{m}$**, either $\mathfrak{m} = \mathfrak{m}_{nn}(\mathbb{R})$ or $\mathfrak{m} = \mathfrak{m}_{nn}(\mathbb{C})$, whose image is in G for each t.[9] We are especially interested in smooth curves in G with the property that $c(0)$ is the identity 1. An example is the curve

$$c(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the group $G = SO(3)$. For any such curve we can form the derivative $c'(0)$ at 0, which will be some 3-by-3 matrix. With c as in this example, $c'(t)$ is to be formed entry by entry. Thus

$$c'(t) = \begin{pmatrix} -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad c'(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $c'(0)$ tells us the direction in G in which the curve extends from the identity matrix 1. We may think of it as telling how a parameter describing c is to start out from the identity. Let

$$\mathfrak{g} = \left\{ c'(0) \;\middle|\; \begin{array}{l} c : \mathbb{R} \to G \text{ is a curve with} \\ c(0) = 1 \text{ that is smooth as} \\ \text{a function into matrices} \end{array} \right\}.$$

Then $\mathfrak{g}$ is a subset of matrices of the same size as the underlying general linear group $\boldsymbol{M}$ and matrix space $\mathfrak{m}$; the members of $\mathfrak{g}$ are not necessarily invertible.

[8] $GL(n,\mathbb{R})$ is an open subset of $M_{nn}(\mathbb{R})$, being the subset where the determinant function is nonzero, and this set is not a closed subset of $M_{nn}(\mathbb{R})$. However, $GL(n,\mathbb{R})$ is a closed linear group tautologically because it is the closed subset $GL(n,\mathbb{R})$ of $GL(n,\mathbb{R})$.

[9] For some purposes it will be convenient to allow smooth curves to be defined on closed intervals. These curves will be understood to be smooth functions defined on open intervals containing the given closed interval.

The above set $\mathfrak{g}$ has a certain amount of structure. First, $\mathfrak{g}$ is a real vector space. To see closure under addition, let $c'(0)$ and $b'(0)$ be in $\mathfrak{g}$. Forming $c(t)b(t)$, we have

$$\frac{d}{dt}\big[c(t)b(t)\big] = c(t)b'(t) + c'(t)b(t),$$
$$\frac{d}{dt}\big[c(t)b(t)\big]\Big|_{t=0} = c(0)b'(0) + c'(0)b(0) = b'(0) + c'(0).$$

Thus $\mathfrak{g}$ is closed under addition. To see closure under multiplication by real scalars, let $c'(0)$ be in $\mathfrak{g}$ and let k be in $\mathbb{R}$. Forming $c(kt)$, we have

$$\frac{d}{dt}\big[c(kt)\big] = c'(kt)k \quad \text{and} \quad \frac{d}{dt}\big[c(kt)\big]\Big|_{t=0} = kc'(0).$$

Thus $\mathfrak{g}$ is closed under scalar multiplication.

Second, G is closed under group conjugation $x \mapsto gxg^{-1}$. Thus if $c(t)$ is a smooth curve in G passing through 1 at $t = 0$, then so is $gc(t)g^{-1}$. Differentiating at $t = 0$, we see that $gc'(0)g^{-1}$ is in $\mathfrak{g}$. Thus $\mathfrak{g}$ is closed under the linear mappings $\operatorname{Ad}(g)$, for $g \in G$, that are defined by

$$\operatorname{Ad}(g)X = gXg^{-1}.$$

Third, let X be in $\mathfrak{g}$ and let $c(t)$ be a smooth curve in G with $c(0) = 1$. Then $t \mapsto \operatorname{Ad}(c(t))X$ is a smooth function from $\mathbb{R}$ into the vector space $\mathfrak{g}$, as we see by substituting $g = c(t)$ into the equation $\operatorname{Ad}(g)X = gXg^{-1}$ and writing matters out. By definition we have

$$\frac{d}{dt}\operatorname{Ad}(c(t))X\Big|_{t=0} = \lim_{t\to 0}\frac{1}{t}\big[\operatorname{Ad}(c(t))X - X\big], \tag{1.7}$$

and we know that the left side exists. The right side involves vector space operations and a passage to the limit, all on members of $\mathfrak{g}$. Since $\mathfrak{g}$, as a real vector subspace of matrix space, is closed topologically, the limit is in $\mathfrak{g}$. Let us calculate this limit.

We need a preliminary formula, namely

$$\frac{d}{dt}\,c(t)^{-1} = -c(t)^{-1}c'(t)c(t)^{-1}. \tag{1.8}$$

To see this, we differentiate $c(t)c(t)^{-1} = 1$ by the product rule, being careful about the order in which we write down products. We obtain

$$c'(t)c(t)^{-1} + c(t)\Big(\frac{d}{dt}\,c(t)^{-1}\Big) = 0,$$

and our preliminary formula (1.8) follows.

Therefore

$$\begin{aligned}\frac{d}{dt}\operatorname{Ad}(c(t))X &= \frac{d}{dt}\Big(c(t)Xc(t)^{-1}\Big)\\ &= c'(t)Xc(t)^{-1} + c(t)X\Big(\frac{d}{dt}\,c(t)^{-1}\Big)\\ &= c'(t)Xc(t)^{-1} - c(t)Xc(t)^{-1}c'(t)c(t)^{-1} \qquad \text{by (1.8).}\end{aligned}$$

Putting $t = 0$ and taking (1.7) into account, we see that

$$c'(0)X - Xc'(0)$$

is in $\mathfrak{g}$. We conclude that $\mathfrak{g}$ is closed under the **Lie bracket** operation

$$[X,Y] = XY - YX. \tag{1.9}$$

Any real vector space of square matrices (or linear maps from a finite-dimensional real vector space to itself) that is closed under the Lie bracket operation (1.9) will be called a **linear Lie algebra**. Any vector subspace of a linear Lie algebra that is closed under Lie brackets will be called a **linear Lie subalgebra.**

This definition will be codified below and will be formulated more generally in Chapter IV to provide a full definition of "Lie algebra." For the moment we work with only linear Lie algebras.

The Lie bracket operation $[X,Y] = XY - YX$ is $\mathbb{R}$ linear in each variable and has the properties that

(a) $[X,X] = 0$ for all $X \in \mathfrak{g}$ (and hence $[X,Y] = -[Y,X]$) and
(b) the **Jacobi identity** holds:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0. \tag{1.10}$$

Property (a) is clear, and property (b) follows from the calculation

$$\begin{aligned}&[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]\\ &\qquad = [XY-YX,\ Z] + [YZ-ZY,\ X] + [ZX-XZ,\ Y]\\ &\qquad = (XYZ-YXZ) - (ZXY-ZYX) + (YZX-ZYX)\\ &\qquad\quad - (XYZ-XZY) + (ZXY-XZY) - (YZX-YXZ)\\ &\qquad = 0.\end{aligned}$$

A real vector space with an operation $[X,Y]$ that is linear in each variable and satisfies (a) and (b) will later be called a real **Lie algebra.** Any vector subspace that is closed under the operation $[X,Y]$ will be called a **Lie subalgebra.** Accordingly we define

$$\mathfrak{g} = \left\{ c'(0) \;\middle|\; \begin{array}{l} c : \mathbb{R} \to G \text{ is a curve with} \\ c(0) = 1 \text{ that is smooth as} \\ \text{a function into matrices} \end{array} \right\}. \tag{1.11}$$

to be the **linear Lie algebra** of the linear group G. Observe that if the linear Lie algebra consists of complex matrices, then $\mathfrak{g}$, defined as in (1.11) is still asserted only to be a *real* Lie algebra.

Returning to the linear group $G = SO(3)$, we can compute its full linear Lie algebra explicitly. We have already seen that the matrix $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is one member of the linear Lie algebra $\mathfrak{g}$ of $SO(3)$. Using different curves involving cosines and sines, we can see that the matrices $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$ lie in $\mathfrak{g}$ also. Taking into account the vector space operations, we see that

$$\mathfrak{g} \supseteq \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \right\} = \{\text{skew-symmetric real 3-by-3 matrices}\}.$$

Actually equality holds in this inclusion. In fact if c is any smooth curve of matrices in $SO(3)$ with $c(0) = 1$ and $c(t)c(t)^t = 1$, then we calculate that

$$\begin{aligned} c'(t)c(t)^t + c(t)c'(t)^t &= 0, \\ c'(0)1^t + 1c'(0)^t &= 0, \\ c'(0) + c'(0)^t &= 0, \end{aligned} \tag{1.12}$$

and we see that $c'(0)$ is skew symmetric. Thus the linear Lie algebra of $SO(3)$ is exactly

$$\begin{aligned} \mathfrak{so}(3) &= \{\text{skew-symmetric real 3-by-3 matrices}\} \\ &= \{X \in \mathfrak{gl}(3, \mathbb{R}) \mid X + X^t = 0\}. \end{aligned}$$

Proposition 1.4. For the five classes of examples $SO(n)$, $U(n)$, $SU(n)$, $SL(n, \mathbb{R})$, and $SL(n, \mathbb{C})$ of closed linear groups, the respective linear Lie algebras are given by

$$\begin{aligned} \mathfrak{so}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X + X^t = 0\} \\ \mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^* = 0\} \\ \mathfrak{su}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^* = 0 \text{ and } \operatorname{Tr} X = 0\} \\ \mathfrak{sl}(n, \mathbb{R}) &= \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{Tr} X = 0\} \\ \mathfrak{sl}(n, \mathbb{C}) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \operatorname{Tr} X = 0\}. \end{aligned} \tag{1.13}$$

First part of proof. At this time we shall prove in each of the five cases that the appropriate linear Lie algebra is contained in the set on the right side of (1.11). The proof of the reverse inclusion requires our knowing some explicit smooth curves in each of the five cases, and this part of the proof is deferred to the next section after we have assembled properties of the exponential mapping on square matrices.

In the case of $SO(n)$, the argument is exactly like the one in (1.12): working with a smooth curve $c(t)$ in G with $c(0) = 1$, we start from the identity $xx^t = 1$ in (1.6), substitute $x = c(t)$ and differentiate, following the steps of (1.12) and writing X for $c'(0)$.

In the case of $U(n)$, we argue similarly, starting from $xx^* = 1$ with $x = c(t)$ and differentiating. The result is

$$c'(t)c(t)^* + c(t)c'(t)^* = 0,$$

from which we obtain $c'(0)1^* + 1c'(0)^* = 0$ and then $c'(0) + c'(0)^* = 0$. Thus a necessary condition is that $X + X^* = 0$. This handles $U(n)$.

To complete this direction of the proof in all cases, what is needed is that the condition $\det x = 1$ for all $x \in G$ forces $\operatorname{Tr} X = 0$ for all X in the linear Lie algebra of G. In symbols we shall verify that if $c(t)$ is a smooth curve of matrices with $c(0) = 1$, then

$$\left(\frac{d}{dt} \det c(t)\right)\Big|_{t=0} = \operatorname{Tr}\, c'(0). \tag{1.14}$$

To verify this formula, we regard the outer expression in parentheses on the left side as the derivative of a sum of n-fold products, and we apply the product rule to each n-fold product. If the matrices are of size n-by-n, the derivative of the determinant is in effect the sum of n determinants; in each of these n determinants, one row of $c(t)$ has been differentiated, and the others have been left alone. Evaluation at $t = 0$ of one of these determinants results in the determinant of a matrix that equals the identity in all but one row; hence the evaluated term yields a diagonal entry of $c'(0)$. Then (1.14) follows. □

5. Exponential of a matrix

Let us prove the reverse inclusions in Proposition 1.4, showing in each case that any matrix in one of the Lie algebras on the right side of (1.13) is indeed $c'(0)$ for some smooth curve $c(t)$ that lies in in the corresponding group G and has $c(0) = 1$. The tool for doing so is the **exponential of a matrix**.[10]

[10]This material if often part of an advanced calculus course. It is reproduced here for the reader's convenience.

If A is an n-by-n complex matrix, then we define

$$e^A = \exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{1.15}$$

This definition makes sense, according to the following proposition.

Proposition 1.5. For any n-by-n complex matrix A, e^A is given by a convergent series (entry by entry). Moreover

(a) $e^X e^Y = e^{X+Y}$ if X and Y commute,
(b) e^X is nonsingular,
(c) $e^{X^t} = (e^X)^t$ and $e^{X^*} = (e^X)^*$,
(d) $t \mapsto e^{tX}$ is a smooth curve into $GL(n, \mathbb{C})$ that is 1 at $t = 0$,
(e) $\frac{d}{dt}(e^{tX}) = Xe^{tX}$,
(f) $\det e^X = e^{\operatorname{Tr}(X)}$,
(g) $X \mapsto e^X$ is a C^∞ mapping from matrix space ($= \mathbb{R}^{2n^2}$) into itself.

Remarks.

(1) If the matrix A in the proposition has real entries, more can be said. In the first place, e^A is real. Conclusions (a), (b), (c), (e), and (f) are still valid. The conclusion of (d) can be strengthened to say that $t \mapsto e^{tX}$ is a smooth curve into $GL(n, \mathbb{R})$ that is 1 at $t = 0$, and (g) becomes the assertion that $X \mapsto e^X$ is a C^∞ mapping from matrix space ($= \mathbb{R}^{n^2}$) into itself.

(2) For any n-by-n complex matrix B, define the **operator norm** of B by

$$\|B\| = \sup_{|x| \le 1} |Bx|, \tag{1.16}$$

where $|x|$ and $|Bx|$ refer to the Euclidean norm.

Proof. The tail of the exponential series satisfies

$$\Big\| \sum_{N=N_1}^{N_2} \frac{1}{N!} A^N \Big\| \le \sum_{N=N_1}^{N_2} \frac{1}{N!} \|A^N\| \le \sum_{N=N_1}^{N_2} \frac{1}{N!} \|A\|^N,$$

and the right side tends to 0 as N_1 and N_2 tend to infinity. Hence the series for e^A is Cauchy, entry by entry, and it must be convergent. This convergence is good enough to justify the manipulations in the remainder of the proof.

For (a), under the assumption that $XY = YX$, we have

$$\begin{aligned} e^X e^Y &= \Big(\sum_{r=0}^{\infty} \frac{1}{r!}\, X^r\Big)\Big(\sum_{s=0}^{\infty} \frac{1}{s!}\, Y^s\Big) = \sum_{r,s} \frac{1}{r!s!}\, X^r Y^s \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{X^k Y^{N-k}}{k!(N-k)!} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{k=0}^{N} \binom{N}{k} X^k Y^{N-k} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} (X+Y)^N = e^{X+Y}. \end{aligned}$$

Conclusion (b) follows by taking $Y = -X$ in (a) and using $e^0 = 1$. Conclusion (c) follows by substituting from the series definition of the exponential and observing that the desired equality is valid term by term. For (d) and (e) we have

$$\begin{aligned} \frac{d}{dt}\big(e^{tX}\big) &= \frac{d}{dt} \sum_{N=0}^{\infty} \frac{1}{N!}\,(tX)^N = \sum_{N=0}^{\infty} \frac{d}{dt}\Big(\frac{1}{N!}\,(tX)^N\Big) \\ &= \sum_{N=1}^{\infty} \frac{N}{N!}\, t^{N-1} X^N = X \sum_{N=1}^{\infty} \frac{1}{(N-1)!}\,(tX)^{N-1} = Xe^{tX}. \end{aligned}$$

In (f) if X is upper triangular, then so is e^X. Moreover $\det e^X$ in this case depends only on the diagonal entries of e^X, which depend only on the diagonal entries of X. Thus $\det e^X = e^{\operatorname{Tr} X}$ in this case. A general complex matrix is of the form gXg^{-1} with X upper triangular, and then we have

$$\det(e^{gXg^{-1}}) = \det(ge^X g^{-1}) = \det e^X = e^{\operatorname{Tr} X} = e^{\operatorname{Tr} gXg^{-1}}.$$

Conclusion (g) follows from standard facts about term-by-term differentiation of series of functions. This completes the proof. □

Let us return to the problem of identifying the linear Lie algebras for the various groups in (1.6), i.e., of verifying that the linear Lie algebras are given by the formulas in (1.13).

Proof of reverse inclusions in Proposition 1.4. Suppose that X is a matrix in one of the five Lie algebras

$$\mathfrak{so}(n),\ \mathfrak{u}(n),\ \mathfrak{su}(n),\ \mathfrak{sl}(n,\mathbb{R}),\ \mathfrak{sl}(n,\mathbb{C})$$

listed in (1.13). Then we are to exhibit a smooth curve $c(t)$ in the corresponding linear group

$$SO(n),\ U(n),\ SU(n),\ SL(n,\mathbb{R}),\ SL(n,\mathbb{C})$$

in (1.6) such that $c(0) = 1$ and $c'(0) = X$. In every case the smooth curve will be $c(t) = e^{tX}$. In view of parts (d) and (e) of Proposition 1.5, this curve is smooth, has $c(0) = 1$, and has $c'(0) = X$. Thus we will be done if, for each of the five Lie algebras, each point $c(t)$ on the curve satisfies the defining conditions of the appropriate one of the five kinds of linear groups.

First consider $SO(n)$ and $\mathfrak{so}(n)$. If X has $X + X^t = 0$, then $x = e^X$ satisfies

$$xx^t = (e^X)(e^X)^t = (e^X)(e^{X^t}) = e^{X+X^t} = e^0 = 1$$

by parts (c), (a), and (d) of Proposition 1.5. Since X has trace 0, Proposition 1.5f shows that x has determinant 1.

The arguments for $U(n)$ and $\mathfrak{u}(n)$, as well as for $SU(n)$ and $\mathfrak{su}(n)$, are completely similar except that $(\,\cdot\,)^*$ replaces $(\,\cdot\,)^t$.

The conclusions for $SL(n,\mathbb{R})$ and $\mathfrak{sl}(n,\mathbb{R})$, as well as for $SL(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$, follow from Proposition 1.5f. $\square$

The above arguments used that for any square matrix X, $c(t) = e^{tX}$ is a smooth curve whose value at $t = 0$ is 1 and whose derivative at $t = 0$ is X. We checked manually that in each case, if X is in the linear Lie algebra $\mathfrak{g}$ of the linear group G, then the smooth curve $t \mapsto e^{tX}$ lies in G. Remarkably this conclusion remains valid for all linear groups, but its proof lies deeper in the general case. In the next section we shall prove this conclusion for all linear groups that are topologically *closed*.

6. Closed linear groups

According to (1.11) in Section 4, the linear Lie algebra $\mathfrak{g}$ of a closed linear group G consists of the set of matrices $X = c'(0)$ for smooth curves $c(t)$ in G that pass through 1 at $t = 0$. As with the examples in Sections 4–5, it will turn out that there is a "best" such curve, namely $t \mapsto e^{tX}$, which we are allowing ourselves to write also as $t \mapsto \exp tX$. Proposition 1.5 shows that $t \mapsto e^{tX}$ is a smooth curve with the correct derivative at $t = 0$, but we need to see that e^{tX} lies in G for all real t. We saw that this was so for the five classes of examples in listed in (1.6), and we establish it for all closed linear groups in Proposition 1.9 below. In Section 7 we shall review the definition of smooth manifold,[11] and then we shall see that any closed linear group has a smooth manifold structure consistent with the relative topology of the group so that the closed linear group becomes a Lie group.

[11]For a more detailed treatment than in Section 7, see Appendix B.

To fix the ideas, we shall work with a closed linear group that is a subgroup of $GL(n,\mathbb{C})$. With really minor modifications the proofs work also for any closed linear group that is a subgroup of $GL(n,\mathbb{R})$. The key tool is the Inverse Function Theorem from multivariable calculus; we need its statement but never its proof.

Lemma 1.6. There exist an open cube U about 0 in $\mathfrak{gl}(n,\mathbb{C})$ and an open neighborhood V of 1 in $GL(n,\mathbb{C})$ such that the exponential mapping $X \mapsto e^X$ is smooth and one-one onto from U onto V, with a smooth inverse.

REMARK. A similar conclusion is valid for $\mathfrak{gl}(n,\mathbb{R})$ and $GL(n,\mathbb{R})$. The only change in the proof below is that the basis of $\mathfrak{gl}(n,\mathbb{R})$ has n^2 members rather than $2n^2$.

PROOF. Proposition 1.4 says that exp is smooth. By the Inverse Function Theorem, it is enough to prove that the derivative matrix at $X=0$ of $X \mapsto \exp X$ is nonsingular. Let $\{E_i\}$ be the standard $2n^2$-member basis over $\mathbb{R}$ of $\mathfrak{gl}(n,\mathbb{C})$, and let x_i be the corresponding coordinate or coordinate function. Then

$$\begin{aligned}\frac{\partial(x_i(X \mapsto e^X))}{\partial x_j}\bigg|_{X=0} &= \frac{d}{dt}\, x_i(e^{tE_j})\Big|_{t=0} \\ &= x_i(E_j e^{tE_j})\Big|_{t=0} \quad \text{since } x_i(\,\cdot\,) \text{ is } \mathbb{R} \text{ linear} \\ &= x_i(E_j) \\ &= \delta_{ij}.\end{aligned}$$

Thus the derivative matrix is the identity matrix, and the lemma follows. $\square$

Though we do not need it just yet, let us point out right now a variant of Lemma 1.6 that we shall need starting in the next section.

Lemma 1.7. Let $\mathfrak{a}$ and $\mathfrak{b}$ be real subspaces of $\mathfrak{gl}(n,\mathbb{C})$ with $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{a} \oplus \mathfrak{b}$. Then there exist open balls U_1 about 0 in $\mathfrak{a}$ and U_2 about 0 in $\mathfrak{b}$, as well as an open neighborhood V of 1 in $GL(n,\mathbb{C})$, such that

$$(a,b) \mapsto \exp a \exp b$$

is a diffeomorphism of $U_1 \times U_2$ onto V.

REMARK. Other variants of Lemma 1.6 are valid and have similar proofs. For example, if $\mathfrak{gl}(n,\mathbb{C})$ is the direct sum of three subspaces $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$, then there exists open balls U_1, U_2, U_3 about 0 in each of the subspaces, as well as an open neighborhood V of 1 in $GL(n,\mathbb{C})$ such that $(a_1,a_2,a_3) \mapsto \exp a_1 \exp a_2 \exp a_3$ is a diffeomorphism of $U_1 \times U_2 \times U_3$ onto V.

Proof. Let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ be bases of $\mathfrak{a}$ and $\mathfrak{b}$, and consider the map

$$(u_1, \ldots, u_r, v_1, \ldots, v_s) \mapsto \exp^{-1}\big\{\big(\exp \sum u_i X_i\big)\big(\exp \sum v_j Y_j\big)\big\} \quad (1.17)$$

defined in an open neighborhood of 0, with the result written out as a linear combination of $X_1, \ldots, X_r, Y_1, \ldots Y_s$ with coefficients depending on $u_1, \ldots, u_r, v_1, \ldots, v_s$. We shall apply the Inverse Function Theorem to see that this map is locally invertible. Since exp is invertible in a neighborhood of 0 by Lemma 1.6, the present lemma will follow.

Thus we are to compute the derivative matrix at 0 of (1.17). In computing each of the required partial derivatives, we can set all variables but one equal to 0 before differentiating, and then we see that the expression to be differentiated reduces to a polynomial of degree 1. The derivative matrix is thus seen to be the identity matrix, the Inverse Function Theorem applies, and the proof is complete. $\square$

Lemma 1.8. If $c(t)$ is a smooth curve in $GL(n, \mathbb{C})$ with $c(0) = 1$ and $c'(0) = X$, then

$$\lim_{k \to +\infty} c\Big(\frac{t}{k}\Big)^k = e^{tX}$$

for all t in the domain of c.

Remark. This lemma has a familiar prototype in real-variable theory: We identify the additive reals with a closed linear group of 1-by-1 matrices by $t \mapsto (e^t)$. One curve in this group is $c(t) = (1 + t)$, and we have $c(t/k)^k = (1 + t/k)^k$, which is well known to converge to (e^t).

Proof. Choose U and V as in Lemma 1.6, and choose $\delta > 0$ so that $c(t)$ is in V for all t with $|t| < 2\delta$. Using Lemma 1.6, we can form a smooth curve

$$Z(t) = \exp^{-1} c(t) \quad \text{for } |t| < 2\delta.$$

This curve has $Z(0) = 0$, and the Chain Rule gives

$$Z'(0) = (\exp'(0))^{-1} c'(0) = X.$$

Thus Taylor's formula gives

$$Z(t) = tX + O(t^2),$$

where $O(t^2)$ is a term that is bounded for $|t| < \delta$ and remains bounded near $t = 0$ when divided by t^2. Replacing t by t/k and regarding t as fixed gives

$$Z\Big(\frac{t}{k}\Big) = \frac{t}{k} X + O\Big(\frac{t^2}{k^2}\Big) = \frac{t}{k} X + O\Big(\frac{1}{k^2}\Big)$$

and thus

$$kZ\Big(\frac{t}{k}\Big) = tX + O\Big(\frac{1}{k}\Big) \quad \text{for any } |t| < \delta.$$

Therefore

$$c\Big(\frac{t}{k}\Big)^k = \Big(\exp Z\Big(\frac{t}{k}\Big)\Big)^k = \exp kZ\Big(\frac{t}{k}\Big) = \exp\Big(tX + O\Big(\frac{1}{k}\Big)\Big).$$

Letting k tend to infinity and using the continuity of exp, we obtain the conclusion of the lemma for $|t| < \delta$.

For other values of t in the domain of c, let us modify the above argument. First we choose and fix a positive integer N with $|t/N| < \delta$. The above estimates apply to t/N. Instead of replacing t by t/k, we replace t/N by $t/(Nk+l)$ with $0 \le l \le N-1$. Again regarding t as fixed, we obtain

$$Z\Big(\frac{t}{Nk+l}\Big) = \frac{t}{Nk+l} + O\Big(\frac{t^2}{(Nk+l)^2}\Big) = \frac{t}{Nk+l}\,X + O\Big(\frac{1}{k^2}\Big)$$

and thus
$$(Nk+l)Z\Big(\frac{t}{Nk+l}\Big) = tX + O\Big(\frac{1}{k}\Big).$$

Therefore

$$\begin{aligned} c\Big(\frac{t}{Nk+l}\Big)^{Nk+l} &= \Big(\exp Z\Big(\frac{t}{Nk+l}\Big)\Big)^{Nk+l} \\ &= \exp\Big((Nk+l)Z\Big(\frac{t}{Nk+l}\Big)\Big) \\ &= \exp\Big(tX + O\Big(\frac{1}{k}\Big)\Big) \end{aligned}$$

for $0 \le l \le N-1$. Letting k tend to infinity, we obtain the conclusion of the lemma for this value of t. $\square$

Proposition 1.9. If G is a closed linear group and X is in its linear Lie algebra $\mathfrak{g}$, then $\exp X$ is in G. Consequently its linear Lie algebra is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \exp tX \text{ is in } G \text{ for all real } t\}.$$

PROOF. The containment $\supseteq$ is immediate from Proposition 1.5, parts (d) and (e). For the containment $\subseteq$, let X be in $\mathfrak{g}$, and let $c(t)$ be a smooth curve in G with $c(0) = 1$ and $c'(0) = X$. Then $c(t/n)^n$ is in G for $n \ge 1$, and since G is closed, so is the limit on n if the limit exists and is nonsingular. Lemma 1.8 says that the limit indeed exists for all t in the domain of c and is equal to $\exp tX$. Thus $\exp tX$ is in G for $|t|$ small, and it follows by raising to powers and using Proposition 1.5a that $\exp tX$ is in G for all real t. $\square$

Actually exp maps $\mathfrak{g}$ onto a neighborhood of 1 in a closed linear group G, and this fact indicates a strong connection between $\mathfrak{g}$ and G. To get at this fact, however, requires making more serious use of the definitions concerning smooth manifolds and smooth mappings.

7. Closed linear groups as Lie groups

The main result of this section is that any closed linear group G can be made into a smooth manifold in a canonical way such that G becomes a Lie group. The terms "smooth manifold" and "Lie group" have already been used in Section 1. They have a variety of meanings in books and papers, and we begin by giving their definitions as we shall understand them in this book.

The terms "smooth" and "C^∞" will be used interchangeably throughout. For us the underlying topological space of a smooth manifold may be assumed to be a metric space, not necessarily connected.[12] The connected component of the group identity will need to be open and separable, and the remaining components topologically will be translates of the identity component. A metric will usually not be specified, but a topological space can be recognized as admitting the structure of a separable metric space if it is separable, regular, and Hausdorff.[13]

Let M be such a topological space. We shall want M to have a well defined dimension, and the possible disconnectedness means that we have to specify the dimension in advance. Let the specified dimension be d. The manifold structure consists of a system of **charts** (U, φ) with U an open subset of M and φ a homeomorphism of U onto an open subset $\varphi(U)$ of $\mathbb{R}^d$. These charts are to have the following two properties:

(a) each pair of charts (U_1, φ_1) and (U_2, φ_2) is **smoothly compatible** in the sense that $\varphi_2 \circ \varphi_1^{-1}$ from the open set $\varphi_1(U_1 \cap U_2)$ of $\mathbb{R}^d$ to the open set $\varphi_2(U_1 \cap U_2)$ of $\mathbb{R}^d$ is smooth and has a smooth inverse, and

(b) the system of compatible charts (U, φ) is a C^∞ **atlas** in the sense that the sets U together cover M.

The topological space M, together with the C^∞ atlas as above, is said to be a **smooth manifold** of **dimension** d. We can then speak of **smooth functions** $f : E \to \mathbb{R}$ on an open subset E of M as functions that are C^∞ when referred back to functions on open subsets of $\mathbb{R}^d$. There is a small technical point here:

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[12] There exist connected smooth manifolds that are not separable, but such manifolds never admit the structure of a topological group.

[13] This result is called the Urysohn Metrization Theorem.

A different C^∞ atlas that leads to the same smooth functions on open sets is to yield the same smooth manifold. To handle this matter, one can observe that the set of all charts smoothly compatible with a particular C^∞ atlas is a maximal C^∞ atlas. Two C^∞ atlases lead to the same smooth functions exactly if their corresponding maximal atlases are the same. So, technically, M is a smooth manifold when it is endowed with a maximal C^∞ atlas.

The existence of an atlas implies that every point has an open neighborhood that is topologically connected. Briefly we say that the topological space is **locally connected**. In any locally connected space every connected component is open. In particular the connected components are open in any smooth manifold.

Now we return to the definition of "Lie group" as the term is to be understood in this book. A **Lie group** G is a topological group with the additional structure of a smooth manifold compatible with the given topology in such a way that multiplication and inversion are smooth. Here multiplication is a mapping from $G \times G$ into G, and we understand $G \times G$ to be a smooth manifold whose charts are products of charts in the factors.

The term **analytic group** is used for a connected Lie group. In a Lie group G, the identity component G_0, which is necessarily open because G is locally connected, is an open closed normal subgroup and is an analytic group.

Here is the main result of this section. We continue to work with a closed linear group that is a subgroup of $GL(n, \mathbb{C})$. With really minor modifications the proofs work also for any closed linear group arising as a subgroup of $GL(n, \mathbb{R})$.

Theorem 1.10 (von Neumann). If G is a closed linear group, then G with its relative topology becomes a Lie group in a unique way such that

(a) the restrictions from $GL(n, \mathbb{C})$ to G of the real and imaginary parts of each entry function are smooth and
(b) whenever $\Phi : M \to GL(n, \mathbb{C})$ is a smooth function on a smooth manifold M such that $\Phi(M) \subseteq G$, then $\Phi : M \to G$ is smooth.

Moreover, the dimension of the linear Lie algebra $\mathfrak{g}$ equals the dimension of the manifold G. In addition, there exist open neighborhoods U of 0 in $\mathfrak{g}$ and V of 1 in G such that $\exp\colon U \to V$ is a homeomorphism onto and such that $(V, \exp^{-1})$ is a compatible chart.

Remarks.

(1) The theorem is equally true if the general linear group over $\mathbb{C}$ is replaced in the statement with a general linear group over $\mathbb{R}$.

(2) Part (b) shows why it was enough in Section 1 for our definition of smooth curve to assume smoothness into the total underlying matrix space.

PROOF OF UNIQUENESS IN THEOREM 1.10. Suppose that G and G' are two versions of G as a smooth manifold. Let $j: G \to G'$ be the function that identifies G and G' as groups. The inclusion $\iota: G \to GL(n,\mathbb{C})$ is smooth because smoothness of this map is detected by smoothness of each real or imaginary part of an entry function, which is known from (a). By (b), $\iota : G \to G'$ is smooth. By the same argument, $\iota^{-1} : G' \to G$ is smooth. $\square$

PROOF OF EXISTENCE IN THEOREM 1.10. Choose a real vector subspace $\mathfrak{s}$ of $\mathfrak{gl}(n,\mathbb{C})$ such that $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{g} \oplus \mathfrak{s}$, and apply Lemma 1.7 to this decomposition, obtaining balls U_1 and U_2 about 0 in $\mathfrak{g}$ and $\mathfrak{s}$ and an open neighborhood V of 1 in $GL(n,\mathbb{C})$ such that $(X,Y) \mapsto \exp X \exp Y$ is a diffeomorphism from $U_1 \times U_2$ onto V. Let 2ϵ be the radius of U_2. For each integer $k \geq 1$, form the set $(k+1)^{-1}U_2$. The claim is that for large k, $\exp X \exp Y$ cannot be in G if X is in U_1 and Y is in $(k+1)^{-1}U_2$ unless $Y = 0$.

Assume the contrary. Then for every $k \geq 1$ we can find X_k in U_1 and Y_k in $(k+1)^{-1}U_2$ with $Y_k \neq 0$ and with $\exp X_k \exp Y_k$ in G. By Proposition 1.9, $\exp X_k$ is in G. Thus $\exp Y_k$ is in G for all k. Since $Y_k \neq 0$, we can choose an integer $n_k \geq 0$ such that $\epsilon/2 \leq n_k|Y_k| \leq 2\epsilon$. Passing to a subsequence if necessary, we may assume that $n_k Y_k$ converges, say to Y. Then Y is in $\mathfrak{s}$ and $Y \neq 0$. Since $\exp n_k Y_k = (\exp Y_k)^{n_k}$ is in G and G is closed, $\exp Y$ is in G.

Let us show that $\exp \frac{p}{q} Y$ is in G for all integers p and q with $q > 0$. Write $n_k p = m_k q + r_k$ with $0 \leq r_k \leq q - 1$. Then $\frac{r_k}{q} Y_k \to 0$ since $Y_k \to 0$. Also $\frac{n_k p}{q} Y_k \to \frac{p}{q} Y$, so that

$$\Big(\exp m_k Y_k\Big)\Big(\exp \frac{r_k}{q} Y_k\Big) = \exp \frac{n_k p}{q} Y_k \longrightarrow \exp \frac{p}{q} Y.$$

Since $\exp \frac{r_k}{q} Y_k \to 1$, $\lim_k \exp m_k Y_k$ exists and equals $\exp \frac{p}{q} Y$. However, $\exp m_k Y_k = (\exp Y_k)^{m_k}$ is in G and G is closed, and thus $\exp \frac{p}{q} Y$ is in G.

In other words, $\exp tY$ is in G for all rational t. Since G is closed and exp is continuous, $\exp tY$ is in G for all real t. By Proposition 1.9, Y is in $\mathfrak{g}$. Thus Y is a nonzero member of $\mathfrak{g} \cap \mathfrak{s}$, and this fact contradicts the directness of the sum $\mathfrak{g} \oplus \mathfrak{s}$. We conclude that for large k, $\exp X \exp Y$ cannot be in G if X is in U_1 and Y is in $(k+1)^{-1}U_2$ unless $Y = 0$.

Changing notation, we have found open balls U_1 and U_2 about 0 in $\mathfrak{g}$ and $\mathfrak{s}$ and an open neighborhood V of 1 in $GL(n,\mathbb{C})$ such that the function e with $e(X,Y) = \exp X \exp Y$ carries $U_1 \times U_2$ diffeomorphically onto V and $e(X,Y)$

is in G only if $Y = 0$. In view of Proposition 1.9, $V \cap G$ is an open set in G such that exp is a homeomorphism from U_1 onto $V \cap G$.

Let $L_g \colon G \to G$ be left translation by $g \in G$, given by $L_g(x) = gx$, and then take $(L_g(V \cap G), \exp^{-1} \circ L_g^{-1})$ as a chart about the element g of G. The image of this chart in Euclidean space is $\exp^{-1}(V \cap G) = U_1 \subseteq \mathfrak{g}$, and we are identifying U_1 with the subset $U_1 \times \{1\}$ of $U_1 \times U_2 \subseteq \mathfrak{g} \oplus \mathfrak{s} = \mathfrak{gl}(n, \mathbb{C})$. These charts cover G. Let us show that they are smoothly compatible. Take two of the charts $(L_g(V \cap G),\ \exp^{-1} \circ L_g^{-1})$ and $(L_h(V \cap G),\ \exp^{-1} \circ L_h^{-1})$ that overlap, and let

$$W = L_g(V \cap G) \cap L_h(V \cap G) \quad \text{and} \quad W^{\#} = L_g(V) \cap L_h(V).$$

Let U and U' be the images of W in the real vector space $\mathfrak{g}$,

$$U = \exp^{-1} \circ L_g^{-1}(W) \quad \text{and} \quad U' = \exp^{-1} \circ L_h^{-1}(W),$$

and let $U^{\#}$ and $U'^{\#}$ be the images of $W^{\#}$ in the real vector space $\mathfrak{gl}(n, \mathbb{C})$,

$$U^{\#} = e^{-1} \circ\ L_g^{-1}(W^{\#}) \quad \text{and} \quad U'^{\#} = e^{-1} \circ\ L_h^{-1}(W^{\#}).$$

The sets U and U' are open subsets of $U_1 \subseteq \mathfrak{g}$, and the sets $U^{\#}$ and $U'^{\#}$ are open subsets of $U_1 \times U_2 \subseteq \mathfrak{gl}(n, \mathbb{C})$. We are to check that

$$(\exp^{-1} \circ L_h^{-1}) \circ (\exp^{-1} \circ L_g^{-1})^{-1} \tag{1.18}$$

is smooth as a map of U onto U', i.e., as a map of $U \times \{1\}$ onto $U' \times \{1\}$. Since $U \subseteq U^{\#}$, the map (1.18) is the restriction to an open subset of a lower-dimensional Euclidean space of the map (1.18) from $U^{\#}$ onto $U'^{\#}$, and the latter map is known to be smooth. Thus the map of U onto U' is smooth, and we conclude that G is a smooth manifold.

The same reasoning shows that multiplication and inversion are smooth. For example, near the identity, what needs checking in order to see that multiplication is smooth is that, for a small open neighborhood U about 0 in $\mathfrak{g}$,

$$(X, X') \in U \times U \mapsto \exp^{-1}(\exp X \exp X')$$

is smooth into $\mathfrak{g}$. But this map is a restriction of the same map on a suitable $U^{\#} \times U^{\#}$, where we know it to be smooth.

Finally let us check (a) and (b). For (a), near 1 in $GL(n, \mathbb{C})$, we are writing coordinates in terms of $\exp X \exp Y$ with $X \in \mathfrak{g}$ and $Y \in \mathfrak{s}$, and smoothness near 1 of a numerical-valued function means smoothness as a function of the pair (X, Y). Smoothness near 1 of a function on G is measured by smoothness as a function of X alone, and this holds because the restriction to a vector subspace of a smooth function on Euclidean space is smooth. For (b) we check smoothness of Φ by following Φ with coordinate functions and checking smoothness of the composition. With coordinates again built from $(X, Y) \mapsto \exp X \exp Y$, we see that smoothness of Φ into G is controlled by the coordinates X alone, and (b) follows.

Since G as a manifold has open subsets of $\mathfrak{g}$ as charts, we have $\dim G = \dim \mathfrak{g}$. Thus the theorem is completely proved. □

Corollary 1.11. If G is a closed linear group and $f\colon G \to M$ is a function into a smooth manifold that is the restriction of a smooth map $F\colon U \to M$, where U is open in some $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ and where $G \subseteq U$, then $F\colon G \to M$ is smooth.

Proof. We can write $f = F \circ \iota$, where ι is the inclusion of G into $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Then ι is smooth by Theorem 1.10a, and hence f is a composition of smooth maps. □

Corollary 1.12. If G is a closed linear group and $\mathfrak{g}$ is its linear Lie algebra, then $\exp \mathfrak{g}$ generates the identity component G_0.

Proof. Proposition 1.9 shows that $\exp \mathfrak{g}$ is contained in G. By continuity, $\exp \mathfrak{g}$ is a connected subset of G. Thus $\exp \mathfrak{g} \subseteq G_0$. The final statement of Theorem 1.10 implies that $\exp \mathfrak{g}$ contains a neighborhood of 1 in G_0. Since the smallest subgroup of G_0 containing a nonempty open neighborhood of 1 in G_0 is all of G_0, the result follows. □

Corollary 1.13. If G and G' are closed linear groups with the same linear Lie algebra as sets of matrices, then the identity components of G and G' coincide.

Proof. Apply Corollary 1.12. □

8. Summary

A Lie group is a separable topological group with the additional structure of a smooth manifold consistent with the topology such that multiplication and inversion are smooth. An analytic group is a connected Lie group. Only a few examples of Lie groups are readily at hand, and of those examples, the richest are the general linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$. This chapter concentrates on linear groups, which are *arbitrary* subgroups of such a general linear group.

The first objective is to associate a "linear Lie algebra" to each linear Lie group to reflect the behavior of the group near the group identity. For linear groups that are closed subgroups of their underlying general linear groups, the theory is rather tidy and appears in Chapter I. One makes use of all smooth curves of matrices extending within G and passing through the identity at $t = 0$. The set of derivatives of such curves at $t = 0$ is a subset of matrices that forms a real vector subspace $\mathfrak{g}$. The space $\mathfrak{g}$ is closed under the Lie bracket operation $[X, Y] = XY - YX$ and is called the linear Lie algebra of G. From the fact that G is topologically closed, it is not too hard to see that the exponential map for matrices carries $\mathfrak{g}$ into G. This fact allows one to introduce the structure of a smooth manifold on G that is compatible with the topology and makes G into a Lie group.

The upshot of all this development is that one can associate to each closed linear group a linear Lie algebra by differentiation at $t = 0$ of curves through the identity and that the exponential map for matrices carries the Lie algebra back into the group.

9. Problems

1. Using Jordan form, prove that exp carries $\mathfrak{gl}(n, \mathbb{C})$ onto $GL(n, \mathbb{C})$.
2. (a) Show for all real square matrices X that $\exp X$ commutes with X.
   (b) Show that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not of the form $\exp X$ for any 2-by-2 real matrix X.
   (c) Show nevertheless that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is of the form $\exp X \exp Y$ for a certain pair of 2-by-2 real matrices X and Y.
   (d) Show that there is a smooth curve of nonsingular 2-by-2 real matrices from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$.
3. Identify the linear Lie algebra of
$$G = \left\{ \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a > 0,\ z \in \mathbb{C} \right\}.$$
4. With $\|\cdot\|$ defined as in (1.16), prove that every square matrix A has $\|e^A\| \le e^{\|A\|}$.
5. Consider the set of nonzero rational numbers as a linear group G of 1-by-1 real matrices. What is the linear Lie algebra of G?
6. Let G_1 and G_2 be separable topological groups whose topologies are locally compact, and let $\pi : G_1 \to G_2$ be a continuous one-one homomorphism onto. Taking for granted that the Baire Category Theorem is valid for locally compact Hausdorff spaces, prove that π is a homeomorphism. Give an example to show that this conclusion can fail if G_1 and G_2 are not assumed to be separable.

Problems 7–9 list some closed linear Lie groups. For each one, (a) prove that the group is connected (and is therefore a linear analytic group), (b) find the corresponding linear Lie algebra, and (c) give its dimension.

7. **(Heisenberg group)** The group of all real matrices $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.
8. **(Affine group of the line)** The group of all affine transformations of the line of the form $x \mapsto ax + b$ with $a > 0$ and $b \in \mathbb{R}$. In matrix form this group is given by all real matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$.
9. **(Upper triangular group)** The group of all real n by n matrices with positive entries on the diagonal and 0s below the diagonal.

Problems 10-13 introduce the algebra $\mathbb{H}$ of **quaternions** over $\mathbb{R}$. The space in question is a vector space of dimension 4 over $\mathbb{R}$ with a basis $1, i, j, k$ and with an $\mathbb{R}$ bilinear multiplication in which 1 is a two-sided identity and in which the rules for multiplying other basis vectors are

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i, \text{ and } ik = -j.$$

10. Another algebra of dimension 4 over $\mathbb{R}$ is the space $\mathfrak{m}_2(\mathbb{R})$ of all two-by-two real matrices, with matrix multiplication as operation. Prove that the linear extension of the map of $\mathbb{H}$ into $\mathfrak{m}_2(\mathbb{R})$ given by
$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
respects multiplication, and conclude that the algebra $\mathbb{H}$ is associative.
11. An $\mathbb{R}$ linear operation of conjugation of $\mathbb{H}$ into itself is given by
$$\overline{a + bi + cj + dk} = a - bi - cj - dk,$$
Show for x in $\mathbb{H}$ that $x\bar{x} = \bar{x}x = |x|^2$ is the square of the usual Euclidean norm on $\mathbb{R}^4$, and conclude that every nonzero member of $\mathbb{H}$ has a two-sided multiplicative inverse. Thus $\mathbb{H}$ is a division algebra. Why does it follow that $|xy| = |x||y|$ for all x and y in $\mathbb{H}$?
12. The real part of a quaternion is given by $\operatorname{Re}(a + bi + cj + dk) = a$. Despite the noncommutativity of $\mathbb{H}$, show that
    (a) the real part satisfies
    $$\operatorname{Re} xy = \operatorname{Re} yx \qquad \text{for all } x \text{ and } y \text{ in } \mathbb{H},$$
    (b) the operations of conjugation and real part on $\mathbb{H}$ commute with one another, and
    (c) the operations of conjugation and real part satisfy
    $$\operatorname{Re} x\bar{y} = \operatorname{Re} y\bar{x}. \qquad \text{for all } x \text{ and } y \text{ in } \mathbb{H}.$$

13. Show how to conclude that the unit sphere S^3 in $\mathbb{H}$ is a compact connected Lie group of dimension 3 under the multiplication operation in $\mathbb{H}$.

Problems 14-16 establish the **polar decomposition** of a matrix in $GL(n, \mathbb{C})$, saying that $GL(n, \mathbb{C}) = U(n) \exp \mathfrak{p}$ is the sense that each member of $GL(n, \mathbb{C})$ is uniquely the product of a unitary matrix by a positive definite Hermitian matrix that is the exponential of a Hermitian matrix. This set of problems claims only that the mapping in question is smooth, is one-one onto, and has a continuous inverse. An argument that the continuous inverse is smooth lies deeper and is not made at this time.

14. Start with any member g of $GL(n, \mathbb{C})$, form $q = gg^*$, and show that q has a unique positive definite square root p. Show how to write $p = \exp X$ for a unique Hermitian matrix X.

15. With g and q as in the previous problem, put $k = p^{-1}g$. Show that k is unitary.

16. Let $\mathfrak{p}$ be the vector space of Hermitian matrices. Deduce from the previous two problems that the continuous mapping $(k, X) \mapsto k \exp X$ of $U(n) \times \mathfrak{p}$ into G is one-one and onto.

Problems 17-20 give a technique for reducing certain questions about $GL(n, \mathbb{R})$ to questions about $O(n)$. The technique uses the **Gram–Schmidt orthogonalization process,** which takes any vector-space basis of $\mathbb{R}^n$ as input and yields a suitable orthonormal basis of $\mathbb{R}^n$ as output. Specifically the process is uniquely defined by the requirements that it takes any ordered basis $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ and produces an ordered basis $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$ satisfying the equalities $|x_j| = 1$ and

$$\operatorname{span}(x_1, \ldots, x_j) = \operatorname{span}(y_1, \ldots, y_j)$$

for every j with $1 \leq j \leq n$.

17. Write an explicit formula for x_j in terms of $y_1, \ldots, y_j$.

18. Fix g in $GL(n, \mathbb{R})$, and apply the Gram–Schmidt orthogonalization process to the basis $(ge_1, \ldots, ge_n)$ of $\mathbb{R}^n$, where $(e_1, \ldots, e_n)$ is the standard ordered basis of $\mathbb{R}^n$. Let k be the n-by-n matrix defined by $k^{-1}y_j = e_j$. Prove that $k^{-1}g$ is upper triangular with positive entries on the diagonal.

19. Let S be the subgroup of all matrices in $GL(n, \mathbb{R})$ that are upper triangular and have positive diagonal entries. How does it follow that $GL(n, \mathbb{R}) = O(n)S$?

20. **(Iwasawa decomposition)** Prove that the multiplication map $O(n) \times S \to GL(n, \mathbb{R})$ is a homeomorphism onto. (Educational note: It will be observed in Chapter IV that $SO(n)$ is connected, and it follows from this problem that $GL(n, \mathbb{R})$ has two components, one consisting of those matrices of positive determinant and the other consisting of those matrices of negative determinant.)

Problems 21–23 concern a class of examples of Lie groups that can be defined as semidirect products. Fix a 2-by-2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\mathbb{R}$ with real entries, and let $G = G_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ be the semidirect product Lie group $\mathbb{R} \times_\tau \mathbb{R}^2$ with $\tau(t)$ equal to the linear map on the space of column vectors $\mathbb{R}^2$ given for $t \in \mathbb{R}$ by left multiplication by $\exp tM$. A concrete formula for the multiplication rule in G is

$$(t_1, X_1)(t_2, X_2) = (t_1 + t_2, \exp(-t_2 M)X_1 + X_2)$$

for t_1 and t_2 in $\mathbb{R}$ and X_1 and X_2 in $\mathbb{R}^2$.

21. Let G' be the linear group

$$G' = \left\{ \begin{pmatrix} \exp tM & \begin{pmatrix} x \\ y \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix} \right\}$$

of all matrices of the indicated shape contained in $GL(3, \mathbb{R})$. Define a smooth map Φ from G into G' by the formula

$$\Phi\left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \exp tM & (\exp tM)\begin{pmatrix} x \\ y \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix}$$

Verify that $\Phi\colon G \to G'$ is a homomorphism of groups.

22. Calculate the linear Lie algebra of G'.

23. Show directly that $\Phi\colon G \to G'$ is smooth, one-one, and onto.

Problems 24–25 expand beyond $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ the list of examples of closed linear groups defined by polynomial equations that was provided in Section 4. The notation x^t refers to the transpose of a real or complex square matrix x, and the notation x^* refers to the conjugate transpose of the complex square matrix x. Further examples of closed linear groups may be found in Chapter I of *Lie Groups Beyond an Introduction*. The matrix $I_{m,n}$ is a diagonal matrix of size $m + n$ by $m + n$ with the first m diagonal entries equal to 1 and the last n diagonal entries equal to -1. The matrix $J_{n,n}$ is a matrix of size $2n$ by $2n$ that in blocks of size n by n is of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with -1 and 1 indicating minus the identity matrix and plus the identity matrix, respectively.

24. Verify that each of the following sets is a closed linear group. Some of the sets were already listed and checked in the text for this chapter:

    Compact groups
$$\begin{aligned} O(n) &= \{x \in GL(n,\mathbb{R}) \mid xx^t = 1\} \\ SO(n) &= \{x \in GL(n,\mathbb{R}) \mid xx^t = 1 \text{ and } \det x = 1\} \\ U(n) &= \{x \in GL(n,\mathbb{C}) \mid xx^* = 1\} \\ SU(n) &= \{x \in GL(n,\mathbb{C}) \mid xx^* = 1 \text{ and } \det x = 1\}. \end{aligned}$$

    Complex groups (with Lie algebras defined over $\mathbb{C}$)
$$\begin{aligned} SL(n,\mathbb{C}) &= \{g \in GL(n,\mathbb{C}) \mid \det g = 1\} \\ SO(n,\mathbb{C}) &= \{g \in SL(n,\mathbb{C}) \mid g^t g = 1\} \\ Sp(n,\mathbb{C}) &= \{g \in SL(2n,\mathbb{C}) \mid g^t J_{n,n} g = J_{n,n}\}. \end{aligned}$$

    Other noncompact groups
$$\begin{aligned} SL(n,\mathbb{R}) &= \{g \in GL(n,\mathbb{R}) \mid \det g = 1\} \\ SO(m,n) &= \{g \in SL(m+n,\mathbb{R}) \mid g^* I_{m,n} g = I_{m,n}\} \\ O(m,n) &= \{g \in GL(m+n,\mathbb{R}) \mid g^* I_{m,n} g = I_{m,n}\} \\ SU(m,n) &= \{g \in SL(m+n,\mathbb{C}) \mid g^* I_{m,n} g = I_{m,n}\} \\ Sp(n,\mathbb{R}) &= \{g \in SL(2n,\mathbb{R}) \mid g^t J_{n,n} g = J_{n,n}\}. \end{aligned}$$

25. Verify that the following are linear Lie algebras and that they are the linear Lie algebras of the respective linear Lie groups in Problem 24:
$$\begin{aligned} \mathfrak{o}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^t = 0\} \\ \mathfrak{so}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^t = 0 \text{ and } \operatorname{Tr} X = 0\} \\ \mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^* = 0\} \\ \mathfrak{su}(n) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^* = 0 \text{ and } \operatorname{Tr} X = 0\} \\ \mathfrak{sl}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} X = 0\} \\ \mathfrak{so}(n,\mathbb{C}) &= \{X \in \mathfrak{sl}(n,\mathbb{C}) \mid X^t + X = 0\} \\ \mathfrak{sp}(n,\mathbb{C}) &= \{X \in \mathfrak{sl}(2n,\mathbb{C}) \mid X^t J_{n,n} + J_{n,n} X = 0\} \\ \mathfrak{sl}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} X = 0\} \\ \mathfrak{so}(m,n) &= \{X \in \mathfrak{sl}(m+n,\mathbb{R}) \mid X^* I_{m,n} + I_{m,n} X = 0\} \\ \mathfrak{su}(m,n) &= \{X \in \mathfrak{sl}(m+n,\mathbb{C}) \mid X^* I_{m,n} + I_{m,n} X = 0\} \\ \mathfrak{sp}(n,\mathbb{R}) &= \{X \in \mathfrak{sl}(2n,\mathbb{R}) \mid X^t J_{n,n} + X J_{n,n} = 0\}. \end{aligned}$$

CHAPTER II

Arbitrary Linear Groups

Abstract. Chapter II develops a general theory of linear groups as Lie groups, beginning with the deep result Theorem 2.1 that associates a linear Lie algebra to an *arbitrary* subgroup of real or complex matrices.

Sections 1–3 examine arbitrary linear groups, attempting to extend to this case as much as possible of the theory of Chapter I. The first and most difficult step, which is carried out in Section 1, is to show for an arbitrary linear group G that if X is in its linear Lie algebra, then $\exp tX$ is in G for all real t. In Section 2 one introduces the "natural topology" on G. This is finer than the relative topology and is shown in Section 3 to be distinct from it unless G is a closed subgroup. The natural topology is generated by "basic open neighborhoods" defined in terms of the exponential map.

In Section 3 it is shown that when G is given its natural topology, the combined structure of the group and the topology makes G into a topological group. Moreover the identity component G_0 is separable. Unless G is a closed subgroup, only the identity component G_0 will be of any interest. One makes G_0 into an analytic group, which is called the "linear analytic group" associated with G. It is an analytic subgroup of the full general linear group of matrices in which the given linear group sits.

Section 4 treats homomorphisms of linear analytic groups, associating to any smooth homomorphism of linear analytic groups a homomorphism of their respective linear Lie algebras.

1. Lie algebra of an arbitrary linear group

Strikingly, much of the theory for closed linear groups as given in Sections 6–7 of Chapter I goes over for arbitrary linear groups, but this fact lies much deeper than the presentation in those sections. As we shall see in this section and the next, an arbitrary linear group has a natural topology and its connected component of the identity element canonically admits the structure of an analytic group. Analytic groups that arise in this way will be called "linear analytic groups." The linear Lie algebra of the linear analytic group will be the same as the linear Lie algebra of the given linear group. We must, however, abandon any expectation that

(i) the natural topology of a linear group is the same as the relative topology acquired from the underlying general linear group, or that
(ii) the manifold structure of the linear analytic group can be obtained from the manifold structure of the underlying general linear group.

For an arbitrary linear group G, closed or not, we have defined the linear Lie algebra of G in (1.11) as

$$\mathfrak{g} = \left\{ c'(0) \;\middle|\; \begin{array}{l} c : \mathbb{R} \to G \text{ is a curve with} \\ c(0) = 1 \text{ that is smooth as} \\ \text{a function into matrices} \end{array} \right\}.$$

Here G is a subgroup of some underlying general linear group, either some $GL(n, \mathbb{R})$ or some $GL(n, \mathbb{C})$, and we shall be writing $\boldsymbol{M}$ in either case for this general linear group. The corresponding underlying full matrix space of n-by-n matrices with entries in $\mathbb{R}$ or $\mathbb{C}$ will be called $\mathfrak{m}$; under the definition (1.11), $\mathfrak{m}$ *is the linear Lie algebra of* M. In any event the matrices $c'(0)$ are members of $\mathfrak{m}$.

The remarkable fact is that three results about closed linear groups extend to be valid for all linear groups G, closed or not, namely

(a) if X is in the linear Lie algebra $\mathfrak{g}$ of G, then $\exp tX$ is in G for all $t \in \mathbb{R}$,
(b) the linear Lie algebra $\mathfrak{g}$ of G consists exactly of those X in the underlying full matrix space $\mathfrak{m}$ such that $\exp tX$ is in G for all real t, and
(c) in any sufficiently small neighborhood of 1 in the underlying general linear group $\boldsymbol{M}$, every element of G that can be connected to 1 by a smooth curve that stays in G and in that neighborhood is of the form $\exp X$ for some X in the linear Lie algebra $\mathfrak{g}$.

We shall prove these results in this section.

Example. Consider a line on the torus T^2 with irrational slope. In terms of the torus as $\mathbb{R}^2/\mathbb{Z}^2$, the subgroup of interest is the line $t \mapsto (\alpha t, \beta t)$, with $\alpha \neq 0$ and β/α irrational. We can cast the situation in terms of linear groups by regarding T^2 as the subgroup of $GL(2,\mathbb{C})$ of all matrices

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}.$$

Here T^2 is a closed linear group in $GL(2,\mathbb{C})$, and its linear Lie algebra consists of all matrices

$$\begin{pmatrix} ix & 0 \\ 0 & iy \end{pmatrix} \quad \text{with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}. \tag{2.1}$$

This linear Lie algebra is abelian in the sense that all Lie brackets are 0. In terms of matrices the subgroup G of interest becomes the line given by

$$t \mapsto \begin{pmatrix} e^{i\alpha t} & 0 \\ 0 & e^{i\beta t} \end{pmatrix}.$$

According to (b), its linear Lie algebra consists of all matrices

$$\begin{pmatrix} i\alpha t & 0 \\ 0 & i\beta t \end{pmatrix} \quad \text{with } t \in \mathbb{R}. \tag{2.2}$$

Conclusions (a) and (b) mentioned above are not difficult in this example, but (c) needs examination. If we imagine T^2 again as the points of the unit square in $\mathbb{R}^2$ with opposite sides identified, then G consists of a dense set of line segments with slope β/α, one of the lines being the line through the origin. The identity element of G corresponds to the point $(0,0)$ in the unit square. If we view a disk of radius 1/2 about (0,0), then the only elements of G that can be connected to the identity by a path that stays in G and in that neighborhood are the points on the line segment that passes through $(0,0)$ with slope β/α. The elements of G on the other parallel line segments are not involved.

Theorem 2.1. Let G be any linear group, let $\mathfrak{g}$ be its linear Lie algebra as defined in (1.11), let $\boldsymbol{M}$ be the underlying general linear group, either some $GL(n,\mathbb{R})$ or some $GL(n,\mathbb{C})$, and let $\mathfrak{m}$ be the linear Lie algebra of M, either $\mathfrak{m}_{nn}(\mathbb{R})$ or $\mathfrak{m}_{nn}(\mathbb{C})$ in the two cases. Then $\exp tX$ is in G for every X in $\mathfrak{g}$ and t in $\mathbb{R}$, and $\mathfrak{g}$ consists exactly of those X in $\mathfrak{m}$ such that $\exp tX$ is in G for all real t.

Remark. Regardless of whether the underlying general linear group $\boldsymbol{M}$ is $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$, we regard $\mathfrak{m}$ as a vector space over $\mathbb{R}$.

Proof. We begin by constructing a certain function u from $\mathfrak{g}$ into G. This function will be noncanonical but, once constructed, will be fixed throughout the proof. Fix an ordered basis $(X_1,\ldots,X_k)$ of $\mathfrak{g}$ over $\mathbb{R}$. For each j with

$1 \le j \le k$, let $t \mapsto c_j(t)$ be a smooth curve in G with $c_j(0) = 1$ and $c_j'(0) = X_j$, and define $u\colon \mathfrak{g} \to G \subseteq \boldsymbol{M}$ by

$$u(t_1X_1 + t_2X_2 + \cdots + t_kX_k) = c_1(t_1)c_2(t_2)\cdots c_k(t_k).$$

Choose a vector subspace $\mathfrak{s}$ of $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{g} \oplus \mathfrak{s}$, and define a function $v\colon \mathfrak{s} \to \boldsymbol{M}$ by $v(Y) = \exp Y$ for $Y \in \mathfrak{s}$. The function that will interest us is $w\colon \mathfrak{m} \to \boldsymbol{M}$ given by $w(X,Y) = u(X)v(Y)$. This has $w(0,0) = 1$, $w(X,0) = u(X)$, and $w(0,Y) = v(Y) = \exp Y$. We can compute its Jacobian matrix at $(0,0)$. The vector partial derivative of u with respect to t_j when all t_ℓ are 0 for $1 \le \ell \le k$ is

$$\frac{\partial u}{\partial t_j}\bigg|_{\text{all } t_\ell=0} = \frac{\partial\big(c_1(t_1)\cdots c_k(t_k)\big)}{\partial t_j}\bigg|_{\text{all } t_\ell=0} = \frac{\partial c_j(t_j)}{\partial t_j}\bigg|_{t_j=0} = X_j, \tag{2.3}$$

and thus the Jacobian matrix of w in block form when all t_ℓ are 0 is

$$D_0w = \begin{array}{cc} \begin{array}{cc} \mathfrak{g} & \mathfrak{s} \end{array} & \\ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} & \begin{array}{c} \mathfrak{g} \\ \mathfrak{s} \end{array} \end{array}$$

with the first k entries corresponding to $\mathfrak{g}$ and the last $n - k$ entries corresponding to $\mathfrak{s}$. This matrix is nonsingular.

The Inverse Function Theorem applies and shows that there exist an open neighborhood N_0 of 0 in $\mathfrak{m}$ and an open neighborhood N_1 of 1 in $\boldsymbol{M}$ such that w is a diffeomorphism of N_0 onto N_1. The remainder of the proof will involve only the neighborhoods N_0 of 0 in $\mathfrak{m}$ and N_1 of 1 in $\boldsymbol{M}$. If we write $W\colon N_1 \to N_0$ for the inverse function of $w\colon N_0 \to N_1$, then every $a \in N_1$ has $W(a) = (U(a), V(a))$, where $U(a)$ and $V(a)$ are the respective components of $W(a)$ in $\mathfrak{g}$ and $\mathfrak{s}$. Here U and V are functions with

$$U\colon N_1 \to \mathfrak{g} \quad \text{and} \quad V\colon N_1 \to \mathfrak{s}.$$

In particular, for $a = w(X,Y)$, we have

$$(U(a), V(a)) = W(a) = W(w(X,Y)) = (X,Y).$$

The upshot is the following:

$$\begin{array}{l} \text{Every } a \text{ in } N_1 \text{ is of the form } a = u(X)v(Y) \text{ for a unique} \\ \text{pair } (X,Y) \text{ in } N_0, \text{ namely } X = U(a) \text{ and } Y = V(a). \end{array} \tag{2.4}$$

From this statement we read off the conclusion that if $V(a) = 0$, then $Y = 0$, $v(Y) = 1$, and $a = u(X)$. But $u(X)$ is in G. We summarize as follows:[1]

$$V(a) = 0 \quad \text{for some } a \in N_1 \quad \text{implies} \quad a \text{ is in } G. \tag{2.5}$$

Fix $a = u(X)v(Y)$ in N_1. Let Z be any element of $\mathfrak{g}$, and consider the product $a' = u(X+tZ)v(Y)$ for real t small enough so that $(X+tZ, Y)$ is in N_0. From (2.4) we have $X + tZ = U(a')$ and $Y = V(a')$ for some $a' \in N_1$. From above we have $(U(a'), V(a')) = W(a') = W(w(X + tZ, Y)) = (X + tZ, Y)$. Consequently

$$V(u(X + tZ)v(Y)) = Y, \tag{2.6}$$

both sides being equal to $V(a')$.

Differentiation of this identity with respect to t at $t = 0$ and use of the Chain Rule gives

$$(D_aV)\Big(\frac{d}{dt}\, u(X + tZ)\Big|_{t=0}\, v(Y)\Big) = 0.$$

Here D_aV is a linear mapping of $\mathfrak{m}$ into $\mathfrak{s}$ that depends on a, and the product of it and the expression in large parentheses has to be 0. Since $v(Y) = \exp Y$ is invertible and so is $u(X)$, it is the same to say

$$(D_aV)\Big(\frac{d}{dt}\, u(X + tZ)\Big|_{t=0}\, u(X)^{-1}\Big) = 0. \tag{2.7}$$

The function u was constructed to take values in G. Therefore the expression $\frac{d}{dt}\, u(X+tZ)\big|_{t=0}\, u(X)^{-1}$ in large parentheses in (2.7) is the tangent vector at $t = 0$ to the smooth curve in G given by

$$t \mapsto u(X + tZ)u(X)^{-1},$$

Consequently it is in $\mathfrak{g}$.

In other words we have constructed a function $A\colon \mathfrak{g} \to \mathfrak{g}$ by the definition

$$A(Z) = \frac{d}{dt}\, u(X + tZ)\Big|_{t=0}\, u(X)^{-1}. \tag{2.8}$$

This function A is a linear mapping of $\mathfrak{g}$ into itself that depends on the initial point $a = u(X)v(Y)$ in N_1. The dependence of A on a is continuous.

Let us see that A is the identity mapping on $\mathfrak{g}$ if $a = 1$. The point $a = 1$ of N_1 results from the choice $(X, Y) = (0, 0)$. Expanding $X = 0$ in terms of our noncanonical basis $X_1, \dots, X_k$ of $\mathfrak{g}$ shows that $t_1 = \cdots =$

[1] In the case that the linear group G is closed within the set of invertible matrices, the converse of (2.5) is valid for suitably small N_0 and N_1, i.e., a in G implies that $V(a) = 0$. This fact makes the proof of Theorem 1.10 succeed more quickly than the proof of the present theorem. Unfortunately the converse of (2.4) fails for arbitrary G, and we have to proceed in a more roundabout fashion here.

$t_k = 0$. Then $u(0) = c_1(0)c_2(0)\cdots c_k(0) = 1$. For the vector of partial derivatives, (2.3) gives $(\partial u/\partial t_j)\big|_{\text{all } t_\ell = 0} = X_j$. For the individual partial derivative $(\partial u_i/\partial t_j)\big|_{\text{all } t_\ell = 0}$, we want the i^{th} component of this, hence the i^{th} component of X_j, which is δ_{ij}. The matrix of $D_X u$ for $X = 0$ is thus the identity matrix, and $((D_X u)(Z))u(X)^{-1}$ reduces to Z. In other words $A(Z) = Z$ for every Z when $a = 1$, and A in this case is the identity mapping of $\mathfrak{g}$ into itself.

Being nonsingular at $a = 1$, the continuously varying A must be nonsingular for all a in some open neighborhood N_2 of 1 that is contained in N_1, and $A(Z)$ takes on all values Z' in $\mathfrak{g}$ for such values of a. Substituting (2.8) into (2.7), we see that

$$(D_a V)(Z') = 0$$

for all Z' in $\mathfrak{g}$ as long as a is in the subneighborhood N_2 of 1 in $\boldsymbol{M}$. Since Z' is arbitrary, we obtain

$$D_a V = 0 \tag{2.9}$$

for all nonsingular a in the subneighborhood of N_2 in $\boldsymbol{M}$. By continuity, (2.9) extends to be valid for all a in the subneighborhood N_2 of 1 in $\boldsymbol{M}$.

For $X \in \mathfrak{g}$, we now take $a(t) = \exp tX$ with t small enough so that $a(t)$ is in N_2. Multiplying (2.9) on the right by $a'(t)$, we obtain

$$\frac{d}{dt} V(a(t)) = 0, \tag{2.10}$$

from which we see that $V(a(t))$ is constant. The constant is $V(a(0)) = V(1) = 0$, and thus (2.5) says that $a(t)$ is in G as long as t is small enough so that $a(t)$ is in N_2. Since G is a group and $(\exp sX)^r = \exp rsX$ for every positive integer r, it follows that $\exp tX$ is in G for all real t. $\square$

Theorem 2.1 establishes results (a) and (b) mentioned at the beginning of this section. It also allows us to use the special function $u(X) = \exp X$ in the set-up of the proof of the theorem and to mine the consequences. A first consequence is the following.

Corollary 2.2. Let G be a linear group that is a subgroup of an underlying general linear group $\boldsymbol{M}$, let $\mathfrak{m}$ be the underlying full space of matrices regarded as a real vector space, and let $\mathfrak{g}$ be the linear Lie algebra of G. If $a(t)$ is a smooth curve in $\mathfrak{m}$ with $a(0) = 1$ and with $a'(t)$ in the matrix product $\mathfrak{g}a(t)$ for t in an interval about 0, then there exists a smooth curve $X(t)$ in $\mathfrak{g}$ defined for t in an interval about 0 such that $a(t) = \exp X(t)$ for those values of t.

Proof. Taking the function $u : \mathfrak{g} \to G$ to be $u(X) = \exp X$, we go over the proof of Theorem 2.1. This choice of u is permissible by an application of Theorem 2.1. The definition of w becomes $w(X, Y) = (\exp X)(1 + Y)$, the Inverse Function Theorem applies to w about 0 in $\mathfrak{m}$, and we obtain a diffeomorphism $w : N_0 \to N_1$ from an open neighborhood of 0 in $\mathfrak{m}$ onto an open neighborhood N_1 of 1 in $\boldsymbol{M}$. For $a = w(X, Y)$, we are led to functions $U \colon N_1 \to \mathfrak{g}$ and $V \colon N_2 \to \mathfrak{s}$ such that $(U(a), V(a)) = (X, Y)$. Once again we obtain (2.4) and (2.5).

We are given a smooth curve $t \mapsto a(t)$ in G defined for t near 0. Fix a value of t in the domain close enough to 0, and put $b = a(t)$. We can write $b = (\exp X)v(Y)$ for some $X \in \mathfrak{g}$ and $Y \in \mathfrak{s}$. Let Z be an element of $\mathfrak{g}$, and form the product $\exp(X + tZ)v(Y)$ for real values of t close to 0. As in (2.6) we find that

$$V(\exp(X + tZ)v(Y)) = Y.$$

Imitating the argument after (2.6), we are led to consider the member $((D_X \exp X)(Z)(\exp X)^{-1}$ of $\mathfrak{m}$, which is the tangent vector at $s = 0$ of $s \mapsto \exp(X + sZ)(\exp X)^{-1}$ and is therefore in $\mathfrak{g}$. Arguing as in (2.8) through (2.9), we obtain $(D_b V)(b) = 0$. Restoring $b = a(t)$ and multiplying by $a'(t)$, we see that $\dfrac{d}{dt} V(a(t)) = 0$, from which we see that $V(a(t))$ is constant. At $t = 0$, the constant is 0, and thus $V(a(t)) = 0$ for all t near 0. Consequently $a(t) = \exp(U(a(t)))$, and the corollary follows with $X(t) = U(a(t))$. □

Now we can address result (c) that was stated at the beginning of this section.

Corollary 2.3. Let G be any linear group, let $\mathfrak{g}$ be its linear Lie algebra, let $\boldsymbol{M}$ be the underlying general linear group, and let $\mathfrak{m}$ be the corresponding full space of matrices regarded as a real vector space. In a sufficiently small open neighborhood of 1 in $\boldsymbol{M}$, every element a of G that can be connected to the identity by a smooth curve that lies in G and stays in this neighborhood is of the form $a = \exp X$ for some X in $\mathfrak{g}$.

Remark. The example of a line of irrational slope on the torus is illuminating. The content of this corollary for that example was worked out before the statement of Theorem 2.1.

Proof. Let p be a point of $\boldsymbol{M}$ near 1, and let a be a smooth curve from 1 to p that lies in G and stays in the neighborhood of 1 produced in Corollary 2.2. The smooth curve $s \mapsto a(t + s)a(t)^{-1}$ lies in G and for each t, has tangent vector $a'(t)a(t)^{-1}$ in $\mathfrak{g}$ by definition of linear Lie algebra. Thus $a'(t)$ lies in

$\mathfrak{g}a(t)$ for each t. Then Corollary 2.2 shows that $a(t) = \exp X(t)$ with $X(t)$ in $\mathfrak{g}$. $\square$

2. Natural topology on an arbitrary linear group

Let G be a linear group, let $\mathfrak{g}$ be its linear Lie algebra, and let d be the dimension of $\mathfrak{g}$. We shall introduce canonically a Hausdorff locally connected topology on G and then the structure of a smooth manifold of dimension d on G such that the connected component G_0 of G containing 1 becomes an analytic group of dimension d. We call this topology the **natural topology** on G or G_0. If G is a closed linear group, then the natural topology will coincide with the relative topology. But if G is not closed, the two topologies will necessarily be different. Here are some examples. A more dramatic example appears at the end of Section 3.

EXAMPLE 1. If G is a line on the torus T^2 with irrational slope (viewed as a linear group in one of the ways suggested in Section I.4), then the natural topology turns out to be the usual topology on the line. The dimension is 1. As pointed out in connection with Corollary 1.2, the relative topology is not locally compact and therefore differs from the usual topology on the real line.

EXAMPLE 2. If G is the additive group of rational numbers, considered as a subgroup of the real line (which is to be viewed as a linear group in one of the ways suggested near the beginning of Section I.4), then the linear Lie algebra is 0 and the natural topology on G is the discrete topology. The dimension is 0.

EXAMPLE 3. If G is an uncountable but proper subgroup of the additive group of real numbers (which is again to be viewed as a linear group), then the linear Lie algebra is still 0 and the natural topology on G is again the discrete topology. Observe that G is not separable. However, the identity component G_0 of G is a one-element group and is a perfectly good analytic group.

For the given linear group G, let $\boldsymbol{M}$ be the underlying general linear group, let $\boldsymbol{\mathfrak{m}}$ be the corresponding full space of matrices regarded as a real vector space, let $\mathfrak{g} \subseteq \boldsymbol{\mathfrak{m}}$ be the linear Lie algebra of G, and let $\|\cdot\|$ be the operator norm in $\boldsymbol{\mathfrak{m}}$ as defined in (1.16). We come to the definition of the "natural topology" on G. The definition is in terms of basic open neighborhoods. According to Lemma 1.6, we can find open neighborhoods N_0 of 0 in $\boldsymbol{\mathfrak{m}}$ and N_1 of 1 in $\boldsymbol{M}$ such that $\exp\colon \boldsymbol{\mathfrak{m}} \to M$ is a diffeomorphism of N_0 onto N_1. Fix a number $\varepsilon_0 > 0$ small enough so that every X in $\mathfrak{m}$ with $\|X\| < \varepsilon_0$ lies in N_0 and therefore has $\exp X$ in N_1. Without loss of generality, we may take ε_0 to satisfy $e^{2\varepsilon_0} \le 2$. This number ε_0 will occur repeatedly in the arguments below.

If a is in G and X is in $\mathfrak{g}$, then $a \exp X$ is in G by Theorem 2.1, G being a group. For $a \in G$ and $\varepsilon \le \varepsilon_0$, the **basic open neighborhood** $N_{a,\varepsilon}$ in G of the element a of G is defined as the subset of G given by

$$N_{a,\varepsilon} = \{a \exp X \mid X \in \mathfrak{g} \text{ and } \|X\| < \varepsilon\}.$$

We shall show that the basic open neighborhoods in G define a topology on G, and this topology will be what we call the **natural topology** on G. We require a lemma.

Lemma 2.4. For a linear group G, if a and b are elements of G such that b lies in some basic open neighborhood $N_{a,\varepsilon}$ with $\varepsilon < \frac{1}{2}\varepsilon_0$, then there exists a basic open neighborhood $N_{b,\varepsilon'}$ in G with $\varepsilon' \le \varepsilon$ such that $N_{b,\varepsilon'} \subseteq N_{a,\varepsilon}$.

Proof. Since the element b lies in $N_{a,\varepsilon}$, we have $b = a \exp X$ for some $X \in \mathfrak{g}$ with $\|X\| < \varepsilon$. Thus $a^{-1}b = \exp X$. Suppose that some $\varepsilon' \le \varepsilon$ has been specified. If c is in $N_{b,\varepsilon'}$, then $b^{-1}c = \exp X'$ with $\|X'\| < \varepsilon'$, and hence

$$a^{-1}c = (a^{-1}b)(b^{-1}c) = (\exp X)(\exp X') \tag{2.11}$$

with X and X' in $\mathfrak{g}$ and with $\|X\| < \varepsilon$ and $\|X'\| < \varepsilon'$.

We shall make use of two facts connecting the exponential mapping and the operator norm $\|\cdot\|$, namely

$$\|e^X\| \le e^{\|X\|}, \tag{2.12}$$

$$\|e^X - 1\| \le \|X\| e^{\|X\|}. \tag{2.13}$$

Inequality (2.12) is proved by writing out the series expansion of e^X, recognizing that the operator norm of e^X is $\le$ the sum of the series of operator norms, and reading off the result. Inequality (2.13) follows similarly by writing out the series expansion of $e^X - 1$, applying the operator norm to both sides, and comparing the resulting series term by term with the series for $\|X\| e^{\|X\|}$.

From (2.11) we have

$$a^{-1}c = (\exp X)(\exp X') = (\exp X - 1)(\exp X' - 1) + \exp X + \exp X' - 1$$

and hence

$$\begin{aligned}
\|a^{-1}c - 1\| &\le \|\exp X - 1\|\|\exp X' - 1\| + \|\exp X - 1\| + \|\exp X' - 1\| \\
&\le \|X\|e^{\|X\|}\|X'\|e^{\|X'\|} + \|X\|e^{\|X\|} + \|X'\|e^{\|X'\|} \\
&\le \varepsilon\varepsilon' e^{\varepsilon+\varepsilon'} + \varepsilon e^{\varepsilon} + \varepsilon' e^{\varepsilon'} \\
&\le 2\varepsilon\varepsilon' + e^{\varepsilon}(\varepsilon + \varepsilon'),
\end{aligned}$$

the last inequality holding since $\varepsilon' \le \varepsilon$ and $e^{2\varepsilon_0} \le 2$. The right member of the chain of inequalities has $\limsup_{\varepsilon'\downarrow 0}\big(\varepsilon\varepsilon' + e^{\varepsilon}(\varepsilon+\varepsilon')\big) = e^{\varepsilon}\varepsilon$.

Now $e^{\varepsilon} < e^{\varepsilon_0/2} < e^{2\varepsilon_0} \le 2$ and $\varepsilon < \frac{1}{2}\varepsilon_0$, and hence $e^{\varepsilon}\varepsilon < \varepsilon_0$. Thus for ε' sufficiently small,

$$\varepsilon\varepsilon' + e^{\varepsilon}(\varepsilon + \varepsilon') < \varepsilon_0.$$

For any such ε', we see that $N_{b,\varepsilon'} \subseteq N_{a,\varepsilon}$, and the proof is complete. □

Proposition 2.5. If G is a linear group, then the collection of all unions of basic open neighborhoods is a topology on G. This topology is locally compact Hausdorff and locally connected, and it makes G into a topological group. Moreover this topology is at least as fine as the relative topology from the underlying matrix space $\mathfrak{m}$.

Remarks. The topology in the statement of the proposition is the **natural topology** on G. It need not be separable, as we have already seen in Example 3 above. The connected component containing the identity 1 will, however, be separable, as we shall see in Theorem 2.6, and it will yield an analytic group. We shall make use of the positive number ε_0 defined earlier in this section. It has the properties that $e^{2\varepsilon_0} \le 2$ and that every X in $\mathfrak{m}$ with $\|X\| < \varepsilon_0$ lies in N_0 and therefore has $\exp X$ in N_1.

Proof. The collection of all unions of basic open neighborhoods $N_{a,\varepsilon}$ in G such that $\varepsilon < \frac{1}{2}\varepsilon_0$ is closed under finite intersections by virtue of Lemma 2.4, and it covers G. Therefore it is a topology on G. We shall show that this topology, the natural topology on G, is Hausdorff, and we shall compare compare it with the topology on $\mathfrak{m}$.

On any basic open neighborhood $N_{1,\varepsilon}$ with $\varepsilon < \frac{1}{2}\varepsilon_0$, where ε_0 is defined earlier in this section just before "basic open neighborhood," the Inverse Function Theorem shows that $N_{1,\varepsilon}$ has the same topology as the subset

$$\{X \in \mathfrak{g} \mid \|X\| < \varepsilon\},$$

where $\mathfrak{g} \subseteq \mathfrak{m}$ is the linear Lie algebra of G. It follows readily that for any point a of G and any neighborhood $U_{a,\mathfrak{m}}$ of a in $\mathfrak{m}$, there exists a neighborhood $U_{a,\mathfrak{g}}$ in G with $a \in U_{a,\mathfrak{g}}$ and $U_{a,\mathfrak{g}} \subseteq U_{a,\mathfrak{m}}$. In other words the natural topology on

$N_{a,\varepsilon}$ is at least as fine as the relative topology from $\mathfrak{m}$. The latter topology is Hausdorff, and therefore the natural topology on G is Hausdorff.

We see that $N_{a,\varepsilon}$ is behaving like the Euclidean space $\mathfrak{g}$ topologically. In particular it is locally compact and connected. Since $N_{a,\varepsilon}$ is open in the natural topology on G, the natural topology on G is locally compact and locally connected. □

3. Linear analytic group associated to a linear group

Now we concentrate on the connected component G_0 of G containing the group identity 1.

Theorem 2.6. If a linear group G with linear Lie algebra $\mathfrak{g}$ is given its natural topology, then G becomes a topological group, and the connected component G_0 of the identity becomes a separable locally compact (Hausdorff) topological group. Each basic open neighborhood $N_{a,\varepsilon}$ with $a \in G$ and $\varepsilon < \frac{1}{2}\varepsilon_0$ defines a chart $(\varphi_{a,\varepsilon}, N_{a,\varepsilon})$ on G of dimension $\dim \mathfrak{g}$, and the collection of all such charts makes G into a smooth manifold of dimension $\dim \mathfrak{g}$ in such a way that multiplication as a map of $G \times G$ into G and inversion as a map of G into itself are smooth.

Remarks.

(1) In the terminology that we introduced in Section I.1, G_0 becomes an analytic group. We call it the **linear analytic group** associated with G. Although the passage from a given linear group to its associated linear analytic group may not be transparent (for example, the passage from a line on the torus with irrational slope to the analytic group $\mathbb{R}$ is not transparent), it is nevertheless canonical.

(2) If a linear group G is a closed subgroup of matrices, then it follows from Theorem 1.10 that the natural topology on the identity component of G coincides with the relative topology.

Proof. We begin by defining the charts $(\varphi_{a,\varepsilon}, N_{a,\varepsilon})$ for each $a \in G$ and each positive $\varepsilon < \frac{1}{2}\varepsilon_0$. Lemma 2.4 shows that the function

$$\varphi_{a,\varepsilon} : N_{a,\varepsilon} \to \{X \in \mathfrak{g} \mid \|X\| < \varepsilon\}$$

given by

$$\varphi_{a,\varepsilon}(a \exp X) = X$$

is a homeomorphism of $N_{a,\varepsilon}$ onto an open subset of the Euclidean space $\mathfrak{g}$ as long as a is in G and $\varepsilon < \frac{1}{2}\varepsilon_0$, the number ε_0 having been defined earlier.

So $(\varphi_{a,\varepsilon}, N_{a,\varepsilon})$ is a chart on G of dimension $\dim \mathfrak{g}$. The set of such charts covers G. Later we shall show that these charts are smoothly compatible.

First we use these charts to see that multiplication and inversion are continuous. In both cases continuity is a local question. For inversion we consider a point $x = a \exp X$ and its inverse $x^{-1} = (\exp(-X))a^{-1} = a^{-1}(\exp(-aXa^{-1}))$, and we regard X as moving and a as fixed. The map $\exp X \mapsto X$ for $a \exp X \in N_{a,\varepsilon}$ is continuous since $\varepsilon < \varepsilon_0$. It is followed in succession by

$$\begin{aligned}
&X \mapsto -X \text{ for } X \in \mathfrak{g},\\
&Y \mapsto aYa^{-1} \text{ for } Y \in \mathfrak{g} \text{ when } a \text{ is in } G,\\
&Z \mapsto \exp Z \text{ for } Z \in \mathfrak{g}, \text{ and}\\
&\exp Z \mapsto a^{-1} \exp Z \text{ for } Z \in \mathfrak{g} \text{ when } a \text{ is in } G.
\end{aligned}$$

which are all continuous. Hence the composition $x \mapsto x^{-1}$ is continuous. Thus inversion is continuous.

Next we consider two points $x = a \exp X$ and $y = b \exp Y$. Their product is $xy = a(\exp X)b(\exp Y) = (ab)(\exp(b^{-1}Xb))(\exp Y)$. Here we regard X and Y in $\mathfrak{g}$ as moving but small, and a and b are fixed in G. We start with the functions $\exp X \mapsto X$ and $\exp Y \mapsto Y$ for $a \exp X \in N_{a,\varepsilon}$ and $b \exp Y \in N_{b,\varepsilon}$. These are continuous since $\varepsilon < \varepsilon_0$. Then we set up the ingredients for the next step, namely

$$\begin{aligned}
&\exp X \mapsto X \text{ for } a \exp X \in N_{a,\varepsilon},\\
&\exp Y \mapsto Y \text{ for } b \exp Y \in N_{b,\varepsilon}, \text{ and}\\
&X \mapsto b^{-1}Xb \text{ for } X \in \mathfrak{g} \text{ when } b \text{ is in } G,
\end{aligned}$$

and we combine them by composing

$$(\exp Z, \exp Y) \mapsto (Z, Y) \mapsto Z + Y \mapsto \exp(Z + Y)$$

and $\exp W \mapsto ab \exp W$. All the steps involve continuous functions, and hence the composition $(x, y) \mapsto xy$ is continuous.

Thus G is a topological group, locally compact (and Hausdorff) by Proposition 2.5. That proposition shows also that G is locally connected, and hence the identity component G_0 is an open connected subgroup.

To prove that G_0 is separable, the first step is to prove that G_0 is σ-compact. For this purpose write $N^{(1)}$ for the basic open neighborhood $N_{1,\varepsilon}$, where ε is fixed as some positive number less than $\frac{1}{2}\varepsilon_0$. For each integer $n \geq 2$, define inductively $N^{(n+1)}$ to be the set of products of a member of $N^{(n)}$ and a member of $N^{(1)}$. The set $N^{(1)}$ is closed under inverses, since $\|\exp(-X)\| = \|\exp X\|$, and it is open and connected. Its closure is compact. For each n, the set $N^{(n+1)}$ is the union over the members a of $N_{1,\varepsilon}$ of the sets $N^{(n)}a$. If we assume inductively that $N^{(n)}$ is open and connected and contains 1, then we see that the same thing is true of $N^{(n+1)}$. Consequently $N^{(n+1)}$ is open and connected. Taking the union on n, we see that the union of all these connected open sets is an open connected subgroup of G and is necessarily G_0. Meanwhile, the closure of $N^{(n+1)}$ contains the set of products of the closure of $N^{(n)}$ and the closure of $N_{1,\varepsilon}$ since the product of two compact subsets of a topological group is compact.[2] Thus the closure of $N^{(n)}$ is compact for every n, and G_0 is exhibited as being σ-compact.

Let $G_0 = \bigcup_{n=1}^{\infty} K_n$ exhibit G_0 as the union of countably many compact sets. The charts cover G_0 and hence cover K_n for each n. By compactness finitely many charts cover each K_n, and hence G_0 is covered by countably many charts. Each chart has a countable base, and the union of these countable bases is a countable base for G_0. Thus G_0 is separable.

All that is left is to prove that the charts that we have defined are smoothly compatible and that multiplication and inversion are smooth.

First we address the compatibility of the charts. For $\varepsilon \le \varepsilon_0$, let $\widetilde{B}_\varepsilon = \{X \in \mathfrak{m} \mid \|X\| < \varepsilon\}$. We are to consider $\varphi_{b,\varepsilon} \circ \varphi_{a,\varepsilon}^{-1}$ as a mapping taking the subset of the ball $B_\varepsilon = \widetilde{B}_\varepsilon \cap \mathfrak{g}$ on which $\varphi_{a,\varepsilon}^{-1}$ is defined, carrying it to the subset of the same ball B_ε in the image of $\varphi_{b,\varepsilon}$. That is, we consider $\varphi_{b,\varepsilon} \circ \varphi_{a,\varepsilon}^{-1}$ as a mapping from the open subset $\varphi_{a,\varepsilon}(N_{a,\varepsilon} \cap N_{b,\varepsilon})$ of $\mathfrak{g}$ onto the open subset $\varphi_{b,\varepsilon}(N_{a,\varepsilon} \cap N_{b,\varepsilon})$ of $\mathfrak{g}$. To understand $\varphi_{b,\varepsilon} \circ \varphi_{a,\varepsilon}^{-1}$ better, we put $x = a \exp X$ in the formula $\varphi_{a,\varepsilon}(a \exp X) = X$. Then we see that $\varphi_{a,\varepsilon}(x) = (\exp^{-1} \circ L_a^{-1})(x)$, where L_a is the function G given by $L_a(y) = ay$. In other words, our chart about the element a of G is

$$((L_a \circ \exp)(\widetilde{B}_\varepsilon \cap \mathfrak{g}), \exp^{-1} \circ L_a^{-1}),$$

and we have

$$N_{a,\varepsilon} = (L_a \circ \exp)(\widetilde{B}_\varepsilon \cap \mathfrak{g})$$

and

$$\varphi_{a,\varepsilon} = (L_a \circ \exp)^{-1} \quad \text{on } \widetilde{B}_\varepsilon \cap \mathfrak{g} \quad \text{with image } N_{a,\varepsilon}.$$

The composition $\varphi_{b,\varepsilon} \circ \varphi_{a,\varepsilon}^{-1}$ is thus given by

$$\begin{aligned} \varphi_{b,\varepsilon} \circ \varphi_{a,\varepsilon}^{-1} &= (L_b \circ \exp)^{-1} \circ (L_a \circ \exp) \\ &= \exp^{-1} \circ L_b^{-1} \circ L_a \circ \exp \end{aligned}$$

[2]The set of products of the members of two compact sets is the continuous image in G under the multiplication mapping of a compact set in $G \times G$. It is therefore compact.

from the subset $\varphi_{a,\varepsilon}(N_{a,\varepsilon}\cap N_{b,\varepsilon})$ of $\widetilde{B}_\varepsilon\cap\mathfrak{g}$ to the subset $\varphi_{b,\varepsilon}(N_{a,\varepsilon}\cap N_{b,\varepsilon})$ of $\widetilde{B}_\varepsilon\cap\mathfrak{g}$. Is it smooth? Yes, it is, because it is the restriction to the Euclidean subspace $\mathfrak{g}$ of the smooth function $\exp^{-1}\circ L_b^{-1}\circ L_a\circ\exp$ on $\widetilde{B}_\varepsilon$ defined on an open subset of $\mathfrak{m}$. Thus the charts are smoothly compatible.

The last step is to show that multiplication and inversion are smooth functions. To handle this step, we go back to the arguments earlier in this proof that multiplication and inversion are continuous. Each step in those arguments showed the continuity of some constituent mapping, and we observe that each of those constituent mappings is in fact smooth. Consequently multiplication and inversion are smooth. □

Corollary 2.7. If G is a linear group with Lie algebra $\mathfrak{g}$ and if G_0 is the associated linear analytic group, then $\exp\mathfrak{g}$ generates G_0.

Proof. Since G_0 is connected, any $N_{1,\varepsilon}$ with $\varepsilon<\frac{1}{2}\varepsilon_0$ generates G_0. Hence the subset $\exp\mathfrak{g}$ generates G_0. □

Corollary 2.8. If a linear group G has $\mathfrak{m}$ as its underlying vector space of matrices and if G_0 is the associated linear analytic group, then the inclusion of G_0 into $\mathfrak{m}$ is smooth; consequently the restrictions to G_0 of the real and imaginary parts of each entry function are smooth. In the reverse direction if f is any smooth function from a smooth manifold into $\mathfrak{m}$ whose image lies in G_0, then f is smooth as a function into G_0.

Proof. The inclusion of G_0 into $\mathfrak{m}$ is given near a by $i_a(\exp X)=\exp X$, which for $\|X\|<\varepsilon<\frac{1}{2}\varepsilon_0$ is the composition of the smooth maps $\varphi_{a,\varepsilon}$, which carries $\exp X$ into X, followed by $X\mapsto\exp$. Thus it is smooth. Smoothness of a map into $\mathfrak{m}$ can be detected entry by entry, and the conclusion about smoothness of the entry functions follows.

For the reverse direction let D be the domain of f. Without loss of generality we may assume that D is an open subset of a Euclidean space. To check smoothness of f into $\mathfrak{m}$ or G_0, we refer the image space to charts. Define balls $\widetilde{B}_\varepsilon$ in $\mathfrak{m}$ and B_ε in $\mathfrak{g}$ by

$$\widetilde{B}_\varepsilon=\{X\in\mathfrak{m}\mid\|X\|<\varepsilon\}\qquad\text{and}\qquad B_\varepsilon=\widetilde{B}_\varepsilon\cap\mathfrak{g}=\{X\in\mathfrak{g}\mid\|X\|<\varepsilon\}.$$

To handle the image $f(x)$ of f near a point $f(a)$ of $\boldsymbol{M}$, we follow f with the translation $L_{f(a)^{-1}}$ to center matters at 1 and then use $\exp^{-1}$ to pass to $\widetilde{B}_\varepsilon$ or B_ε. Thus we are to compare the smoothness of $x\mapsto\exp^{-1}\circ L_{f(a)^{-1}}\circ f(x)$ into $\widetilde{B}_\varepsilon$ with the smoothness of the same expression into B_ε. The second one is just the same expression viewed as in a Euclidean subspace, and the second one is smooth if the first one is smooth. The result follows. □

Remarks.

(1) The first conclusion of Corollary 2.8 is an extension of Theorem 1.10a from closed linear groups to arbitrary linear groups. Similarly the second conclusion is an extension of Theorem 1.10b.

(2) The critical fact for the above proof is that the values $f(x)$ for x near a all lie in G, so that left translation of all of them by $f(a)^{-1}$ moves them into the image of exp on B_ε, not merely the image of exp on $\widetilde{B}_\varepsilon$.

(3) The group G_0 is an analytic group by Theorem 2.6, and it is also a subgroup of the Lie group $\boldsymbol{M}$. The first conclusion of the corollary says that the inclusion of G_0 into $\boldsymbol{M}$ is smooth. Therefore G_0 is an analytic subgroup of $\boldsymbol{M}$. This conclusion establishes part of Fundamental Theorem A of the Preface: we have succeeded in passing from an arbitrary subgroup G of $\boldsymbol{M}$ to its Lie algebra $\mathfrak{g}$ in $\mathfrak{m}$, and then to an analytic subgroup G_0 of $\boldsymbol{M}$.

Proposition 2.9. If the natural topology on a linear group G coincides with the relative topology, then G is closed as a subgroup of invertible matrices.

Proof. The natural topology on G and on its identity component G_0 have to be locally compact Hausdorff. A footnote in Section I.2 contains a proof that a locally compact dense subspace of a Hausdorff space is necessarily open, i.e., open in its closure. If the natural topology on G_0 coincides with the relative topology, then G_0 must be open in its closure in the relative topology. The closure being a subgroup, the union of the remaining cosets must be open, and it follows that G must itself be closed. □

Corollary 2.10. If G is a linear group, then the following conditions on G are equivalent:

(a) G is connected in its *natural* topology,
(b) any two points of G can be connected by a smooth curve in G,
(c) any two points of G can be connected by a smooth curve of matrices that lies in G, and
(d) G is generated by $\exp \mathfrak{g}$.

Remarks. A linear group will be said to be **connected** if the equivalent conditions of the corollary are satisfied. In this case G is automatically connected in the *relative* topology because any open closed subset of G in the natural topology is open closed in the relative topology. A pathological example after the proof of Corollary 2.10 refers to a linear group in the literature that is connected in the relative topology but is not connected in the natural topology.

Proof. If (a) holds, then Corollary 2.7 shows that (d) holds. If (d) holds and if x and y are given elements of G, write $x^{-1}y = \exp X_1 \exp X_2 \cdots \exp X_k$. Then $t \mapsto x \exp tX_1 \exp tX_2 \cdots \exp tX_k$ is a smooth curve in G that is x at $t = 0$ and is y at $t = 1$. Thus (b) holds. If (b) holds, then (c) holds by Corollary 2.8. If (c) holds, then (a) holds because any pathwise connected topological space is connected. □

Example. Connectedness in the relative topology does not imply connectedness in the natural topology. In fact Jones [1942] constructed some really pathological examples of additive subgroups of $\mathbb{R}^2$, and Theorem 5 of his paper gives an example of a connected dense one of these that is not a manifold and is therefore not an analytic subgroup of $\mathbb{R}^2$. That subgroup G of $\mathbb{R}^2$ has Lie algebra 0, and it follows that $G_0 = 1$. By contrast any *pathwise* connected subgroup of $\mathbb{R}^2$ has to be an analytic subgroup, according to Yamabe [1950] and Goto [1969].

4. Smooth homomorphisms between linear analytic groups

Suppose that G and G' are linear analytic groups endowed with their natural topologies and natural smooth structures in the sense of the second remark after Theorem 2.6. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the linear Lie algebras of G and G', and suppose that π is a smooth homomorphism of G into G'. For G and G' connected Corollary 2.8 has shown that we can test for smoothness of a homomorphism by testing whether it is smooth into the underlying matrix space for G', i.e., whether each entry $g \mapsto \pi(g)_{ij}$ is a smooth numerical-valued function on G.

The notion of smoothness of π makes sense when G and G' are possibly disconnected, since G and G' with their natural topologies still have smooth structures: the mapping π is to be continuous, and its restriction to each (open) component of G is to be a smooth mapping into a component of G'.

With this understanding, let $\mathfrak{g}$ and $\mathfrak{g}'$ be the linear Lie algebras of G and G'. If $\pi\colon G \to G'$ is a smooth homomorphism, we shall associate to π a linear mapping $D\pi : \mathfrak{g} \to \mathfrak{g}'$ that suitably respects the Lie bracket multiplication in $\mathfrak{g}$ and $\mathfrak{g}'$.

Before proceeding, let us comment on the case that G or G' is the Lie group $\mathbb{R}$. We can always regard $\mathbb{R}$ as a linear group, say as the set $\{(e^t)\}$ of 1-by-1 matrices. We use this convention throughout the remainder of this section. Here are three examples of smooth homomorphisms of linear groups.

EXAMPLE 1. If H is any linear group and X is in $\mathfrak{h}$, then $t \mapsto \exp tX$ is a smooth homomorphism of $\mathbb{R}$ into H. As a homomorphism between groups of matrices, this map is $(e^t) \mapsto \exp tX$.

EXAMPLE 2. The circle group $S^1 = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ is a closed linear group within $GL(1, \mathbb{C})$, and $t \mapsto e^{it}$ is a smooth homomorphism of $\mathbb{R}$ into S^1.

EXAMPLE 3. The triangular group of real matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ is a closed linear group G, and the map that sends the indicated matrix into x is a smooth homomorphism of G into $\mathbb{R}$.

We return to the general setting of smooth homomorphisms π of linear analytic groups G and G' endowed with their natural topologies and smooth structures. If $X \in \mathfrak{g}$ is given, let $c(t)$ be a smooth curve in G with $c(0) = 1$ and $c'(0) = X$. (For example, we can take $c(t) = \exp tX$.) Then the composition $t \mapsto \pi(c(t))$ is a smooth curve in G' with $\pi(c(0)) = 1$, and we define $D\pi$ by

$$D\pi(X) = (\pi \circ c)'(0). \tag{2.14}$$

Let us see that this definition is independent of the choice of c. If $c_1(t)$ and $c_2(t)$ both have starting value 1 in G and starting derivative X in $\mathfrak{g}$, then we compute

$$\begin{aligned}
(\pi \circ c_2)'(0) &= \frac{d}{dt}\,\pi(c_2(t))\Big|_{t=0} \\
&= \frac{d}{dt}\,\pi(c_2(t)c_1(t)^{-1} \cdot c_1(t))\Big|_{t=0} \\
&= \frac{d}{dt}\,\pi(c_2(t)c_1(t)^{-1})\pi(c_1(t))\Big|_{t=0} \quad \text{since } \pi \text{ is a homomorphism} \\
&= \frac{d}{dt}\,\pi(c_2(t)c_1(t)^{-1})\Big|_{t=0} \cdot\ \pi(c_1(0)) \\
&\quad + \pi(c_2(0)c_1(0)^{-1})\ \cdot\ \frac{d}{dt}\,\pi(c_1(t))\Big|_{t=0} \\
&= \frac{d}{dt}\,\pi(c_2(t)c_1(t)^{-1})\Big|_{t=0} + (\pi \circ c_1)'(0).
\end{aligned}$$

Thus it is enough to prove that the curve $c(t) = c_2(t)c_1(t)^{-1}$, which has $c(0) = 1$ and $c'(0) = 0$, has $(\pi \circ c)'(0) = 0$. We now refer matters to local coordinates, using the exponential map, taking advantage of Theorem 2.6. The local expression for $\pi \circ c$ near $t = 0$ is $\exp^{-1} \circ \pi \circ c$, which we write as

$$\exp^{-1} \circ \pi \circ c = (\exp^{-1} \circ \pi \circ \exp) \circ (\exp^{-1} \circ c).$$

Here the two factors on the right side are the local expressions for π and c. However, the second factor on the right is a curve in $\mathfrak{g}$, and it has $(\exp^{-1} \circ c)'(0) = 0$ by the Chain Rule, since $c'(0) = 0$. Still another application of the Chain Rule gives $(\exp^{-1} \circ \pi \circ c)'(0) = 0$. Applying the exponential map and using the Chain Rule once more, we see that $(\pi \circ c)'(0) = 0$. Thus our definition (2.14) is independent of the choice of c.

Now we can imitate some of the development earlier of the properties of linear Lie algebras. First of all, $D\pi : \mathfrak{g} \to \mathfrak{g}'$ is linear. In fact let $c_1(t)$ and $c_2(t)$ correspond to X and Y, and let k be in $\mathbb{R}$. Then

$$\begin{aligned} D\pi(kX) &= \frac{d}{dt}\,\pi(c_1(kt))\Big|_{t=0} \\ &= \frac{d}{ds}\,\pi(c_1(s))\Big|_{s=0} \cdot \frac{d}{dt}(kt)\Big|_{t=0} \\ &= k\,D\pi(X) \end{aligned}$$

and

$$\begin{aligned} D\pi(X+Y) &= \frac{d}{dt}\,\pi(c_1(t)c_2(t))\Big|_{t=0} \\ &= \frac{d}{dt}\,\pi(c_1(t))\pi(c_2(t))\Big|_{t=0} \\ &= D\pi(X) + D\pi(Y) \end{aligned}$$

by the product rule for derivatives.

To see the effect of $D\pi$ on the Lie bracket of two elements of the linear Lie algebra $\mathfrak{g}$, we proceed somewhat as in the middle part of Section I.4, beginning by introducing Ad. Since the group G is closed under conjugation, it follows that G acts on the linear Lie algebra $\mathfrak{g}$ by conjugation. Specifically let g be in G. If X is in $\mathfrak{g}$ and $c(t)$ is a smooth curve in G with $c(0) = 1$ and $c'(0) = X$, then $b(t) = gc(t)g^{-1}$ is another smooth curve in G, and it has $b(0) = 1$ and $b'(0) = gXg^{-1}$. Hence gXg^{-1} is in $\mathfrak{g}$.

We define $\mathrm{Ad}(g)$ on $\mathfrak{g}$ to be conjugation by g. If we write $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ for the vector space of all mappings from $\mathfrak{g}$ into itself that are linear over $\mathbb{R}$, then what we have just seen is that $\mathrm{Ad}(g)$ is a member of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. From the fact that $\mathrm{Ad}(g)$ is the operation of conjugation by g on $\mathfrak{g}$, it follows immediately that

$$\mathrm{Ad}(g_1g_2) = \mathrm{Ad}(g_1)\mathrm{Ad}(g_2) \qquad \text{for } g_1 \text{ and } g_2 \text{ in } G. \tag{2.15}$$

One calls Ad the **adjoint representation** of G on $\mathfrak{g}$.

We return to the question of how $D\pi$ interacts with Lie brackets. Again let $c(t)$ be a smooth curve in G with $c(0) = 1$ and $c'(0) = X$. If g is in G, then

$gc(t)g^{-1}$ has derivative $\mathrm{Ad}(g)X$ at $t = 0$. Hence our smooth homomorphism $\pi\colon G \to G'$ has

$$\begin{aligned} D\pi(\mathrm{Ad}(g)X) &= \frac{d}{dt}\,\pi(gc(t)g^{-1})\Big|_{t=0} \\ &= \frac{d}{dt}\,\pi(g)\pi(c(t))\pi(g)^{-1}\Big|_{t=0} \\ &= \pi(g)\,\frac{d}{dt}\,\pi(c(t))\Big|_{t=0}\,\pi(g)^{-1} \\ &= \pi(g)D\pi(X)\pi(g)^{-1}. \end{aligned}$$

If also Y is in $\mathfrak{g}$, then this formula similarly says that

$$D\pi(\mathrm{Ad}(c(t))Y) = \pi(c(t))D\pi(Y)\pi(c(t))^{-1}.$$

Differentiating at $t = 0$ and using (1.8) and the fact that $D\pi$ is a linear mapping, we obtain

$$D\pi[X, Y] = D\pi(X)D\pi(Y) - D\pi(Y)D\pi(X). \tag{2.16}$$

The right side of (2.16) is by definition the Lie bracket $[D\pi(X), D\pi(Y)]$ of the two linear mappings of $\mathfrak{g}'$ into itself given by $D\pi(X)$ and $D\pi(Y)$. Thus (2.16), in the presence of the linearity of $D\pi$, says that $D\pi$ is a **Lie algebra homomorphism**, i.e., $D\pi$ is $\mathbb{R}$ linear and maps Lie brackets to Lie brackets. A **Lie algebra isomorphism** is a Lie algebra homomorphism with a two-sided inverse. Anyway, our smooth homomorphism $\pi : G \to G'$ leads to a Lie algebra homomorphism $D\pi : \mathfrak{g} \to \mathfrak{g}'$. We summarize as follows.

Theorem 2.11. If $\pi\colon G_1 \to G_2$ is a smooth homomorphism of one linear analytic group to another with their natural topologies and manifold structures, then π induces a homomorphism $D\pi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of their linear Lie algebras by the following formula: whenever $c(t)$ is a smooth curve in G_1 with $c(0) = 1$ and $c'(0)$ equal to a member X of $\mathfrak{g}_1$, then $D\pi(X)$ is the member Y of $\mathfrak{g}_2$ such that $\dfrac{d}{dt}\,\pi(c(t))\Big|_{t=0} = Y$.

The discussion of examples earlier in this section continues.

Example 1. If $\pi((e^t)) = \exp tX$, then $D\pi((1)) = X$. To see this, we use the curve $c(t) = (e^t)$, which has $c'(0) = (1)$, and we compute $(\pi \circ c)'(0)$ from Proposition 1.5e.

Example 2. If $\pi((e^t)) = (e^{it})$, then $D\pi((1)) = (i)$ as a 1-by-1 matrix.

Example 3. If $\pi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (e^x)$, then $D\pi\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = (a)$.

A particularly important instance of Theorem 2.11 arises when π is Ad. If G has linear Lie algebra $\mathfrak{g}$, then each $\mathrm{Ad}(g)$ for $g \in G$ is an invertible linear mapping of $\mathfrak{g}$ into itself. For brevity let us say that each $\mathrm{Ad}(g)$ is in $GL(\mathfrak{g})$. Thus Ad is a smooth homomorphism from G into $GL(\mathfrak{g})$. It is customary to abbreviate $D(\mathrm{Ad})$ as ad. The corresponding homomorphism of linear Lie algebras is then written as

$$\mathrm{ad}\colon \mathrm{End}_{\mathbb{R}}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g}).$$

The effect of $\mathrm{ad}\, X$ on $\mathfrak{g}$ is "left bracket by X." To see this fact, let $c(t)$ be a smooth curve in G with $c(0) = 1$ and $c'(0) = X$. Then

$$\begin{aligned} \mathrm{ad}(X)(Y) &= \frac{d}{dt}\,\mathrm{Ad}(c(t))(Y)\Big|_{t=0} \\ &= \frac{d}{dt}\,c(t)Yc(t)^{-1}\Big|_{t=0} \\ &= c'(0)Y + Y\frac{d}{dt}(c(t)^{-1})\Big|_{t=0} \\ &= XY - YX \qquad\qquad \text{by (1.8).} \end{aligned} \tag{2.17}$$

Since ad is D of a smooth homomorphism of linear groups, ad must be a homomorphism of Lie algebras. In other words, ad must satisfy

$$\mathrm{ad}[X,Y](Z) = (\mathrm{ad}\, X)(\mathrm{ad}\, Y)(Z) - (\mathrm{ad}\, Y)(\mathrm{ad}\, X)(Z).$$

for all Z in $\mathfrak{g}$. Indeed, one readily checks that this equation is just a rewritten version of the Jacobi identity.

The key tool for dealing with smooth homomorphisms of linear groups is the following formula, which relates π, $D\pi$, and the exponential map.

Theorem 2.12. If $\pi : G \to G'$ is a smooth homomorphism of one linear analytic group to another, both with their natural topologies and smooth structures, then $\pi \circ \exp = \exp \circ D\pi$.

Proof. Fix X in the linear Lie algebra of G, and let $c_1(t)$ and $c_2(t)$ be the smooth curves of matrices in G' given by

$$c_1(t) = \exp(t\, D\pi(X)) \quad \text{and} \quad c_2(t) = \pi(\exp tX).$$

Then $c_1(0) = c_2(0) = 1$ and

$$\frac{d}{dt}\,c_1(t) = D\pi(X)\exp(t\, D\pi(X)) = D\pi(X)c_1(t)$$

by Proposition 1.5e. Also

$$\begin{aligned}\frac{d}{dt}c_2(t) &= \frac{d}{dh}\pi(\exp(t+h)X)\Big|_{h=0} = \frac{d}{dh}\pi(\exp hX)\pi(\exp tX)\Big|_{h=0}\\ &= \frac{d}{dh}\pi(\exp hX)\Big|_{h=0}\pi(\exp tX) = D\pi(X)c_2(t)\end{aligned}$$

by definition of $D\pi(X)$. Since the j^{th} columns of both $c_1(t)$ and $c_2(t)$ solve the initial-value problem for the linear system of differential equations

$$\frac{dy}{dt} = D\pi(X)y \quad \text{with} \quad y(0) = (j^{\text{th}} \text{ column of } 1),$$

Theorem B.16 (the uniqueness theorem for linear systems of first-order ordinary differential equations) says that $c_1(t) = c_2(t)$ for all t. The theorem follows by taking $t = 1$. $\square$

Corollary 2.13. If G is a linear analytic group and $\mathfrak{g}$ is its linear Lie algebra, then $\mathrm{Ad}\circ\exp = \exp\circ\,\mathrm{ad}$.

Proof. This is the special case of Theorem 2.12 in which $\pi = \mathrm{Ad}$ and $D\pi = \mathrm{ad}$. $\square$

Corollary 2.14. Let $\pi : G \to G'$ be a smooth homomorphism of linear analytic groups, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be the linear Lie algebras. If the map $x \mapsto \pi(x)$ near $x = 1$ is referred to local coordinates relative to the exponential maps of G and G', then the corresponding homomorphism of linear Lie algebras is exactly $D\pi : \mathfrak{g} \to \mathfrak{g}'$, which is linear. Hence $D\pi$ is also the derivative of π at the identity when π is referred to local coordinates by the exponential maps.

Proof. The map in local coordinates is $Y = \exp^{-1}(\pi(\exp(X))$ near $X = 0$, and Theorem 2.12 says that this is the same as $Y = \exp^{-1}(\exp D\pi(X)) = D\pi(X)$, which is linear. A linear map is its own derivative, and the corollary follows. $\square$

Corollary 2.15. If π_1 and π_2 are smooth homomorphisms from one linear analytic group G to another G' such that $D\pi_1 = D\pi_2$, then $\pi_1 = \pi_2$.

Proof. For X in $\mathfrak{g}$, Theorem 2.12 gives

$$\pi_1(\exp X) = \exp D\pi_1(X) = \exp D\pi_2(X) = \pi_2(\exp X).$$

Corollary 2.7 shows that $\exp\mathfrak{g}$ generates G. Therefore $\pi_1 = \pi_2$. $\square$

Corollary 2.16. Let $\pi : G \to G'$ be a smooth homomorphism of linear analytic groups, and let $D\pi : \mathfrak{g} \to \mathfrak{g}'$ be the corresponding homomorphism of linear Lie algebras. Then

(a) $D\pi$ onto implies π is onto, and
(b) $D\pi$ one-one implies π is one-one in some neighborhood of 1 in G,

Proof. We start from the formula of Theorem 2.12, namely

$$\pi(\exp X) = \exp D\pi(X) \qquad \text{for } X \in \mathfrak{g}. \tag{2.18}$$

For (a), $D\pi$ is assumed to be onto $\mathfrak{g}'$, and thus the image of $\exp \circ D\pi$ contains $\exp \mathfrak{g}'$, which contains a neighborhood of 1 in G'. On the other hand, the image $\pi(G)$ is a subgroup of G'. Any subgroup of G' containing a neighborhood of 1 must contain the identity component of G'. Thus $\pi(G) = G'$.

For (b), suppose $D\pi$ is one-one. Since $D\pi$ is linear, there exists a constant C such that $\|D\pi(X)\} \le C\|X\|$ for all $x \in \mathfrak{g}$. Let X_1 and X_2 be distinct members of $\mathfrak{g}$ with $\|X_1\| \le C^{-1}\varepsilon$ and $\|X_2\| \le C^{-1}\varepsilon$, with ε chosen so that $C^{-1}\varepsilon < \frac{1}{2}\varepsilon_0$ for the number ε_0 defined earlier for the group H. Then we have $\|D\pi(X_1)\| \le \varepsilon$ and $\|D\pi(X_2)\| \le \varepsilon$. Since $D\pi(X_1) \ne D\pi(X_2)$ and since the exponential map from $\mathfrak{g}'$ to G' is one-one for $\|Y\| \le \varepsilon < \frac{1}{2}\varepsilon_0$, we see that $\exp(D\pi(X_1)) \ne \exp(D\pi(X_2))$. In view of (2.18), $\pi(\exp X_1) \ne \pi(\exp X_2)$. $\square$

5. Summary

Chapter II continues the effort begun in Chapter I to attach a linear Lie algebra to each subgroup of a given linear group and to use it to understand the subgroup better. Chapter I dealt particularly with the case of a closed subgroup. In that case the closed subgroup admitted the structure of a Lie group in such a way that the identity component was an analytic subgroup with the relative topology.

Somewhat surprisingly a considerable amount of this theory extends to arbitrary linear groups that are not necessarily closed subgroups of matrices. An arbitrary linear group G comes initially with neither a useful topology nor a manifold structure, only with a specific embedding as an abstract subgroup of some general linear group $\boldsymbol{M}$. In turn this underlying general linear group is a specific open dense set in the full matrix space $\mathfrak{m}$ of real or complex square matrices of that same size. The space $\mathfrak{m}$ is of course a real vector space under addition and multiplication by real scalars.

To construct the topology of interest on a linear group G, one makes use of all smooth curves in $\mathfrak{m}$ extending within G and passing through the identity at $t = 0$. Just as in the case of closed subgroups, the set of derivatives of such curves at $t = 0$ is a subset of $\mathfrak{m}$ that forms a real vector subspace $\mathfrak{g}$ of $\mathfrak{m}$, and the space $\mathfrak{g}$ is closed under the Lie bracket operation $[X, Y] = XY - YX$. Again the space $\mathfrak{g}$ with this operation is called the linear Lie algebra of G. Remarkably the exponential mapping for matrices still carries $\mathfrak{g}$ into G. From this fact one can construct the "natural topology" on G. The natural topology is locally compact Hausdorff and is locally connected, and it makes G into a topological group and also a smooth manifold for which multiplication and inversion are smooth. Topologically the identity component G_0 is separable. Thus G_0 becomes an analytic group. This is the linear analytic group associated to the given untopologized linear group G.

The chapter concludes by associating to any smooth homomorphism of one linear analytic group to another, a Lie algebra homomorphism of their respective linear Lie algebras, i.e., a linear map that respects Lie brackets. If the homomorphism of Lie algebras is onto, then the smooth homomorphism of analytic groups is onto. If the homomorphism of Lie algebras is one-one, then the smooth homomorphism of analytic groups is one-one in some open neighborhood of the identity.

The upshot of all this development is that one can associate to each linear group a linear Lie algebra by differentiation at $t = 0$ of curves through the identity, the exponential map for matrices carries the Lie algebra back into the group, and the exponential map allows one to define a natural topology on the given linear group. The identity component of this topologized group is taken to be the associated linear analytic group for the given linear group. In addition this association carries smooth homomorphisms of linear analytic groups into linear maps of the corresponding Lie algebras that respect Lie brackets.

Chapters I and II have made some progress toward establishing for linear groups Fundamental Theorems A and B of Lie Theory as formulated in the Preface. Indeed Chapter I has associated a real Lie algebra $\mathfrak{g}$ to each linear analytic group G, and that real Lie algebra has $\dim \mathfrak{g} = \dim G$. Also for any smooth homomorphism of G into some linear analytic group H, there is a corresponding Lie algebra homomorphism of $\mathfrak{g}$ into the linear Lie algebra $\mathfrak{h}$ of H. Taking the smooth homomorphism to be an inclusion shows that the Lie algebra of a linear analytic subgroup is a Lie subalgebra of the linear Lie algebra of the whole group.

6. Problems

Problems 1–5 construct a specific smooth homomorphism of $SU(2)$ onto $SO(3)$.

1. The linear Lie algebra of $SU(2)$ is the real vector space $\mathfrak{su}(2)$ of all 2-by-2 skew Hermitian matrices of trace 0, and the elements

$$u_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis of this 3 dimensional vector space. For $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2+|\beta|^2 = 1$, compute $\mathrm{Ad}(g)u_1 = gu_1g^{-1}$, $\mathrm{Ad}(g)u_2, = gu_2g^{-1}$, and $\mathrm{Ad}(g)u_3 = gu_3g^{-1}$, and use the results to write out the matrix of $\mathrm{Ad}(g)$ relative to the basis u_1, u_2, u_3 of $\mathfrak{su}(2)$. How could we have seen in advance that this matrix has to be an orthogonal matrix?

2. For the three basis vectors u_1, u_2, u_3 of $\mathfrak{su}(2)$ listed in Problem 1, compute $\exp tu_1$, $\exp tu_2$, and $\exp tu_3$ for t real.
   (a) Substitute these elements for g into the formula for $\mathrm{Ad}(g)$ from Problem 1, and check in each case that the matrix $\mathrm{Ad}(g)$ is in $SO(3)$, not merely $GL(3, \mathbb{R})$.
   (b) Why do the matrices $\mathrm{Ad}(g)$ for $g = \exp tu_1$, $\exp tu_2$, and $\exp tu_3$ generate all of $\mathrm{Ad}(G)$?
   (c) How can one conclude that this mapping carrying g in $SU(2)$ into $\mathrm{Ad}(g)$ in $O(3)$ is a smooth homomorphism of linear analytic groups?

3. Prove that the smooth homomorphism $\mathrm{Ad}\colon SU(2) \to O(3)$ of Problem 2 is in fact onto $SO(3)$ and has a two-element kernel $\{\pm 1\}$.

4. Using the definitions of u_1, u_2, and u_3 in Problem 1, compute ad of each of u_1, u_2, and u_3. Compute the matrix of ad relative to the basis u_1, u_2, u_3 of $\mathfrak{su}(2)$.

5. Using the 3-by-3 matrix in Problem 1 obtained as $\mathrm{Ad}\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, compute the matrix of $D(\mathrm{Ad})$ relative to the basis u_1, u_2, u_3 of $\mathfrak{su}(2)$ by differentiating $\mathrm{Ad}(\exp tX)$ at $t = 0$ for each of the 2-by-2 matrices u_1, u_2, u_3.

Problems 6–7 introduce the **Pauli matrices** of mathematical physics as a means for working with quaternions without going outside the complex numbers. There are three such matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They occur in the "Pauli equation," which takes into account the interaction of the spin of a particle with an external electromagnetic field. Sometimes the 2-by-2 identity matrix is introduced as a fourth Pauli matrix σ_0.

6. Check that the Pauli matrices σ_1, σ_2, σ_3 have trace 0, have square the identity, and are Hermitian and unitary. Check also that when σ_0 in included, they form a vector space basis of the real vector space $i\mathfrak{su}(2)$ of all 2-by-2 Hermitian matrices.
7. Show that the three-dimensional vector space of all quaternions q satisfying $\overline{q} = -q$ is closed under the bracket multiplication operation $[a, b] = ab - ba$, and check that the bracket operation is skew symmetric and satisfies the Jacobi identity.

Problems 8–11 give another construction of a smooth homomorphism of $SU(2)$ onto $SO(3)$. This one involves the algebra $\mathbb{H}$ of quaternions, which was discussed in detail in the Problems for Chapter I.

8. Let V be the three-dimensional real vector space of all quaternions z such that $\overline{z} = -z$. Show for each $q \in \mathbb{H}$ that the linear mapping of $\mathbb{H}$ into $\mathbb{H}$ given by $L_q(z) = qz\overline{q}$ carries V into itself and satisfies $L_{q_1q_2} = L_{q_1}L_{q_2}$.
9. Identify V with $\mathbb{R}^3$ by the formula

$$ai + bj + ck \longleftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and prove that each of the linear mappings L_q of Problem 8 for $|q| = 1$ is an orthogonal transformation of $\mathbb{R}^3$ to itself.
10. Show that the mapping $q \mapsto L_q$ of Problem 8, when restricted to the multiplicative group of all $q \in \mathbb{H}$ satisyfing $\overline{q} = -q$ and $|q| = 1$ is a continuous homomorphism into $SO(3)$.
11. Explain how to interpret Problem 10 as providing a smooth homomorphism of the analytic group S^3 onto $SO(3)$ with a two-element kernel.

Problems 12–16 concern further isomorphisms of linear analytic groups and of linear Lie algebras.

12. Prove that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ are not isomorphic.
13. Prove that $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2, \mathbb{R})$ are isomorphic as Lie algebras by proving that $SU(1, 1)$ and $SL(2, \mathbb{R})$ are isomorphic as linear analytic groups.
14. Let $\mathfrak{g}$ be the **Heisenberg Lie algebra** over $\mathbb{R}$, consisting of all real matrices $\left(\begin{smallmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Verify that $\mathfrak{g}$ is isomorphic to the linear Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ it & \bar{w} & 0 \end{pmatrix} \;\middle|\; z \in \mathbb{C},\ t \in \mathbb{R} \right\}.$$

15. This problem exhibits an isomorphism of the linear analytic group $\mathrm{Ad}(SL(2, \mathbb{R}))$ to the linear analytic group $SO(2, 1)$. The conclusion is that $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(2, 1)$ are isomorphic as Lie algebras.

(a) Put $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, and let $math\mathfrak{k}$ and $\mathfrak{p}$ be the vector subspaces

$$math\mathfrak{k} = \left\{ X \in \mathfrak{g} \text{ of the form } \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \right\}, \quad \mathfrak{p} = \left\{ X \in \mathfrak{g} \text{ of the form } \begin{pmatrix} y & z \\ z & -y \end{pmatrix} \right\}.$$

Check that $\mathfrak{g} = math\mathfrak{k} \oplus \mathfrak{p}$ and that when X is in $math\mathfrak{k}$, ad X maps $math\mathfrak{k}$ into itself and $\mathfrak{p}$ into itself. Check also that when X is in $\mathfrak{p}$, ad X maps $math\mathfrak{k}$ into $\mathfrak{p}$ and $\mathfrak{p}$ into $math\mathfrak{k}$.

(b) Introduce a symmetric real-valued binear function $T(X, Y)$ on $\mathfrak{g} \times \mathfrak{g}$ by $T(X, Y) = \mathrm{Tr}(XY)$. Check for nonzero X in $math\mathfrak{k}$ and nonzero Y in $\mathfrak{p}$ that

$$T(X, Y) \begin{cases} < 0 & \text{if } X = Y \text{ and } X \text{ is a nonzero member of } math\mathfrak{k}, \\ = 0 & \text{if } X \text{ is in } math\mathfrak{k} \text{ and } Y \text{ is in } \mathfrak{p}, \\ > 0 & \text{if } X = Y \text{ and } X \text{ is a nonzero member of } \mathfrak{p}. \end{cases}$$

(The expression $T(X, Y)$ is called the **trace form** on $\mathfrak{g}$. Conclusion (b) says that the trace form on $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ is a nondegenerate real symmetric bilinear form with signature (2,1).)

(c) Check that $T(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = T(X, Y)$ for all g in G and all X and Y in $\mathfrak{g}$. (This conclusion says that the trace form on $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ is invariant under the group $\mathrm{Ad}(SL(2, \mathbb{R}))$.)

(d) Explain how (c) yields a smooth homomorphism of $SL(2, \mathbb{R})$ into $O(2, 1)$.

(e) Show that the kernel of the homomorphism in (d) is $\{\pm 1\}$ and the image is the identity component of $SO(2, 1)$.

16. The closed linear group $SU(2)$ obtains the structure of the smooth manifold S^3 in at least two ways. One comes from the identification of S^3 with the unit sphere in $\mathbb{R}^4$ (as in Section B.1) because

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{C},\ \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The other comes about from Theorem 1.10 because $SU(2)$ is a closed subgroup of $GL(2, \mathbb{C})$. Prove that these two structures of smooth manifolds on $SU(2)$ are the same, i.e., that the identity map from $SU(2)$ with one structure to $SU(2)$ with the other structure is a diffeomorphism.

Problems 17–18 introduce real representations of Lie groups on finite-dimensional vector spaces over $\mathbb{R}$. If V is such a vector space and G is a compact Lie group, then a

real representation of G on V is a smooth homomorphism Φ of G into $GL(V)$. For brevity one can refer to the real representation as (Φ, V). If (Φ_1, V_1) and (Φ_2, V_2) are finite dimensional real representations, then an **intertwining operator** T between them is a linear $T\colon V_1 \to V_2$ such that $(T\Phi_1(g))(v_1) = (\Phi_2(g)T)(v_1)$ for all $g \in G$ and $v_1 \in V_1$. An **invariant subspace** for (Φ, V) is a vector subspace U such that $\Phi(g)(U) \subseteq U$ for all $g \in G$. A real representation of G on V is said to be **irreducible** if its only invariant subspaces are 0 and V itself. The notion of isomorphism of real representations of G on real vector spaces is captured by the word **equivalence**: A representation (Φ_1, V_1) is equivalent to a representation (Φ_2, V_2) if there exists an invertible linear map $A\colon V_1 \to V_2$ such that $(A\Phi_1(g) = \Phi_2 A(g)$ for all g in G. The **direct sum** of finitely many real representations $\Phi_1, \ldots, \Phi_k$ of G is the representation on the direct sum of the vector spaces in which $\Phi(g)$ for each $g \in G$ is the operator that acts as $\Phi_1(g) \oplus \cdots \oplus \Phi_k(g)$ on the direct sum.

17. **(Schur's Lemma,** named for I. Schur**)**
    (a) Prove that if $T\colon V_1 \to V_2$ is an intertwining operator for two irreducible finite dimensional real representations (Φ_1, V_1) and (Φ_2, V_2) of a Lie group G, then either $T = 0$ or T is one-one onto.
    (b) Prove that if (Φ, V) is an irreducible finite-dimensional real representation of a Lie group G, then the vector space of intertwining operators of (Φ, V) with itself, with composition as multiplication law, is a division algebra over $\mathbb{R}$, necessarily isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.
18. This problem concerns real representations of the compact group $G = O(2)$. Set aside the requirement of finite dimensionality for the moment. The group action of $O(2)$ on column vectors in $\mathbb{R}^2$ by multiplication of matrices times column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ yields a real representation of $O(2)$ on real-valued functions F on $\mathbb{R}^2$ by $(\Phi(g)F)\begin{pmatrix} x \\ y \end{pmatrix} = F(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix})$. The polynomial functions on $\mathbb{R}^2$ form an invariant subspace, and the polynomial functions of degree $\leq k$ form a real finite dimensional representation of G in the sense above. The same is true of the polynomial functions on $\mathbb{R}^2$ homogeneous of degree k, i.e., those polynomial functions in which every monomial term is of total degree k.
    (a) Prove that the real representation of all polynomial functions of degree $\leq k$ is the direct sum for ℓ satisfying $0 \leq \ell \leq k$ of all polynomial functions homogeneous of degree ℓ.
    (b) What is the dimension of the space of all polynomial functions on $\mathbb{R}^2$ homogeneous of degree n?
    (c) According to (a), the space in (b) carries a representation of $O(2)$. This representation is actually a direct sum of irreducible representations. For which values of n is the representation in (b) irreducible? Explain what happens in the reducible cases.

CHAPTER III

Inversion of Constructions for Linear Groups

Abstract. Chapter III continues the development of Lie theory for linear groups. Chapter II showed how to associate a linear Lie algebra and a linear analytic group to each linear group and how to associate a Lie algebra homomorphism to each smooth homomorphism from one linear analytic group to another. The present chapter concerns running these constructions backwards. First it shows how to associate a linear analytic subgroup to each Lie subalgebra of a linear Lie algebra. Second it shows under an additional hypothesis, how to lift a given Lie algebra homomorphism of linear Lie algebras to the corresponding linear analytic groups.

Section 1 addresses situations in which convergent power series with complex coefficients are replaced by power series with matrix coefficients. For a matrix A, the series definition of $\exp A$ appeared in Chapter I. The present section works with a version of $\log A$ and pays particular attention to the identity $\log(\exp X) = X$.

Section 2 proves the Campbell–Baker–Hausdorff Theorem, which shows how to obtain a series expansion for the small solution Z of the matrix equation $\exp Z = \exp X \exp Y$ when X and Y are small. The solution is in the nature of a recursion and involves matrix additions, scalar multiplications, and iterated Lie brackets of X and Y.

Sections 3–4 derive Dynkin's Formula. This formula again seeks the small solution Z of the matrix equation $\exp Z = \exp X \exp Y$ when X and Y are small, but this time the solution is given by a convergent infinite series involving only matrix additions and scalar multiplications, iterated Lie brackets of X and Y, and a passage to the limit. Thus it is available for any linear Lie algebra and gives a usable form of the limit of the series. Section 3 gives F. Schur's power series formula for the derivative of a smooth curve of matrices $\exp X(t)$ in terms of dX/dt, and Section 4 derives Dynkin's Formula from the result in Section 3.

Section 5 establishes the main result of the chapter that if $\boldsymbol{M}$ is a linear analytic group with linear Lie algebra $\mathfrak{m}$ and if $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{m}$, then there exists a linear analytic subgroup G of $\boldsymbol{M}$ with linear Lie algebra $\mathfrak{g}$.

Section 6 shows that if G and G' are linear analytic groups with G simply connected, then every homomorphism of the linear Lie algebra of $\mathfrak{g}$ into the linear Lie algebra of G' lifts uniquely to a smooth homomorphism of G into G'. This is the first section in which any use is made of material in Appendix C on covering spaces.

Section 7 shows that nontrivial covering groups of the linear group $SL(2, \mathbb{R})$ exist but are never linear groups. It makes crucial use of the theorem of Section 5.

Section 8 produces a linear analytic group that covers a group that is not linear. The linear analytic group in this example is the group of 3-by-3 upper triangular matrices with real entries and with 1s on the diagonal. This group covers the quotient by the subgroup consisting of the members of the one-dimensional center with integer entries.

1. Use of power series with matrices

The previous chapter showed how to work with any abstract group of real or complex matrices as if it were a Lie group and in particular how to associate a linear Lie algebra to each such group. It showed also how to associate a Lie algebra homomorphism to any suitably smooth homomorphism between two such groups. This chapter concerns running these associations backwards. We shall see that there is a stunningly close connection between the formula for multiplication of elements in a Lie group and the formula for bracket multiplication in the corresponding linear Lie algebra. Making this connection will involve manipulations with infinite series of square matrices, and in this section we make some preliminary remarks about such manipulations.

We take $\mathfrak{m}$ to be the full vector space of n-by-n matrices over $\mathbb{R}$ or $\mathbb{C}$, and we make use of the **operator norm,** which is given in terms of the Euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ by

$$\|A\| = \sup_{|x| \leq 1} |Ax|. \tag{3.1}$$

Already in defining the exponentiation for matrices in Section I.5, we began with the traditional power series expansion $e^z = \sum_{k=0}^{\infty} \frac{1}{n!} z^n$ familiar from the theory of one complex variable. Then we replaced the complex number z in the formula by a square matrix A and interpreted the sum of the series as $\exp A$.

We shall study solutions of the matrix equation

$$\exp Z = (\exp X)(\exp Y) \tag{3.2}$$

in $\mathfrak{m}$ with X, Y, and Z all small in norm. In doing so, we shall work with power series of matrices beyond just the one for the exponential function.

Suppose we have a power series expansion $\sum_{k=0}^{\infty} a_k z^k$ with numerical coefficients and with z complex. The series is absolutely convergent within a region $|z| < R$ in $\mathbb{C}$ for some radius of convergence R with $0 \le R \le \infty$, and then $C = \sum_{k=0}^{\infty} |a_k| r^k$ is finite whenever $0 \le r < R$; here C depends on r. If we substitute a matrix $X \in \mathfrak{m}$ for z and use the operator norm, then we find that $\sum_{k=0}^{\infty} |a_k| \|X\|^k \le C$ whenever $\|X\| \le r$. Consequently when $\|X\| \le r$, each entry of the sequence of partial sums of $\sum_{k=0}^{\infty} a_k X^k$ forms a Cauchy sequence and must be convergent. Since r is any positive number less than R, entry-by-entry convergence and convergence in norm both take place whenever $\|X\| < R$, In short, $\sum_{k=0}^{\infty} a_k X^k$ is meaningful as a member of $\mathfrak{m}$ whenever $\|X\| < R$. Sums and products of scalar power series go into the corresponding power series of members of $\mathfrak{m}$, and there are no difficulties.

Arguments for dealing with compositions are more subtle. We need to make repeated use of the fact that any absolutely convergent series of complex numbers converges independently of the order of the terms. It is immediate that a series of matrices converges independently of the order of the terms if the series of norms of the terms is absolutely convergent.

The case that will be of interest to us begins with the two power series of scalars

$$\exp z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad \text{for } |z| < \infty \tag{3.3}$$

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k \qquad \text{for } |w| < 1.$$

Term-by-term integration of the second formula yields

$$\log(1-w) = -\sum_{k=1}^{\infty} \frac{1}{k} w^k \quad \text{for } |w| < 1,$$

and this formula becomes

$$\log s = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (s-1)^k \quad \text{for } |s-1| < 1 \tag{3.4}$$

upon replacing $1 - w$ by s. According to our convention, the definition of $\log A$ for a matrix A is thus

$$\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (A-1)^k \quad \text{for } \|A-1\| < 1. \tag{3.5}$$

Meanwhile if we apply (3.4) with $s = e^z$, we see that

$$\log(e^z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (e^z-1)^k \quad \text{for } |e^z-1| < 1. \tag{3.6}$$

In particular (3.6) holds for $|z| < \log 2$. The left side of (3.6) is just z by complex analysis, and thus the right side of (3.6) must expand out to equal z when $|z| < \log 2$.

Going over our steps, let us use the formula (3.4), with the complex number s replaced by a matrix $\exp X$, to define $\log(\exp X)$ of the matrix, and then let us calculate $\log(\exp X)$ by substituting the series for $\exp X$ into (3.6) without paying attention to convergence initially. We wonder whether the result will be just X. We have

$$\begin{aligned} \log(\exp X) &= \big(X + \tfrac{1}{2!}X^2 + \dots\big) - \tfrac{1}{2}\big(X + \tfrac{1}{2!}X^2 + \dots\big)^2 + \dots \\ &= X + \big(\tfrac{1}{2!}X^2 - \tfrac{1}{2}X^2\big) + \big(\tfrac{1}{3!}X^3 - \tfrac{1}{2}X^3 + \tfrac{1}{3}X^3\big) + \dots \\ &= X + 0 + 0 + \dots . \end{aligned} \tag{3.7}$$

It is not transparent how the algebraic manipulations will work out on the right side of (3.7), but what we can see immediately is that they must be the same algebraic manipulations that would be used in examining (3.6). We know with (3.6) that all the terms in the expansion with z to a power greater than 1 have to have coefficient 0, and therefore all the terms after the first on the right side of (3.7) are 0. The conclusion is that

$$\log(\exp X) = X \qquad \text{for } X \in \mathfrak{m} \text{ with } \|X\| < \log 2. \tag{3.8}$$

2. Campbell–Baker–Hausdorff Theorem in the linear case

It was part of the original theory of Sophus Lie that a Lie algebra captures not only the infinitesimal behavior of a Lie group at the identity, but also the actual multiplication law in the group in a full neighborhood of the identity. In 1897, J. E. Campbell undertook to find an actual formula for the multiplication law, at least in the setting of matrices. His efforts toward solving **Campbell's Problem**, as H. Poincaré called it, were soon followed by relevant work of E. Pascal (1865–1940), Poincaré himself, H. F. Baker, and H. Hausdorff that made progress but did not give a fully satisfactory answer to the question. Their combined progress through 1906 fairly or unfairly has come to be known as the Campbell–Baker–Hausdorff Theorem. The text isolates a preliminary step in this work as Proposition 3.1 below and gives the general theorem as Theorem 3.2.

In our notation we can phrase the question this way: We operate within the space $\mathfrak{m}$ of all n-by-n matrices, seeking to understand the solutions of $\exp Z = \exp X \exp Y$ near where $X = Y = Z = 0$. Since the mapping exp is everywhere smooth and since multiplication is smooth within $\mathfrak{m}$, we want to consider the effect of applying $\exp^{-1}$ to a product $\exp X \exp Y$ that is near 1. In order to get $\exp^{-1}$ defined, we have merely to apply the Inverse Function Theorem, invoking the conclusion of Lemma 1.6 or 1.7 that the appropriate Jacobian determinant is nonzero. The resulting mapping $(X, Y) \mapsto \exp^{-1}(\exp X \exp Y)$ will be written as $Z = C(X, Y)$. It is defined in a neighborhood of $(0, 0)$ in $\mathfrak{m} \times \mathfrak{m}$, and its image lies in a neighborhood of 0 in $\mathfrak{m}$. The answer is to be expressed as a convergent series in terms of the Lie algebra operations in $\mathfrak{m}$, i.e., addition, multiplication by real scalars, and the Lie bracket multiplication $[A, B] = AB - BA$.

For the preliminary step toward solving Campbell's Problem, we shall allow matrix multiplication as well as the Lie algebra operations. Even then, the resulting formula is messy. Fortunately we will never need its exact details.

Proposition 3.1 (preliminary step toward solving Campbell's Problem). If X and Y are members of $\mathfrak{m}$ such that $\|X\| + \|Y\| < \log 2$, then the series

$$Z = C(X, Y) = \sum_{k=1}^{\infty} \sum \frac{(-1)^{k-1}}{k} \frac{X^{i_1} Y^{j_1} \cdots X^{i_k} Y^{j_k}}{i_1! j_1! \cdots i_k! j_k!},$$

with the inner sum taken over all finite sequences $(i_1, j_1, \ldots, i_k, j_k)$ of integers ≥ 0 satisfying $i_r + j_r \geq 1$ for all r with $1 \leq r \leq k$, converges in $\mathfrak{m}$ absolutely and independently of the order in which the terms are added, and the sum Z is a solution of $\exp Z = \exp X \exp Y$.

Proof. Because of the hypothesis $\|X\| + \|Y\| < \log 2$, the calculation

$$\| \exp X \exp Y - 1 \| = \Big\| \Big(\sum_{i=0}^{\infty} \frac{X^i}{i!} \Big) \Big(\sum_{j=0}^{\infty} \frac{Y^j}{j!} \Big) - 1 \Big\|$$

$$
\begin{aligned}
&= \Big\| \sum_{i\ge 0,\ j\ge 0,\ i+j>0} \frac{X^i Y^j}{i!\,j!} \Big\| \\
&\le \sum_{i\ge 0,\ j\ge 0,\ i+j>0} \frac{\|X\|^i \|Y\|^j}{i!\,j!} \\
&= \Big(\sum_{i=0}^{\infty} \frac{\|X\|^i}{i!} \Big) \Big(\sum_{j=0}^{\infty} \frac{\|Y\|^j}{j!} \Big) - 1 \\
&= e^{\|X\|+\|Y\|} - 1
\end{aligned}
$$

shows that $\|\exp X \exp Y - 1\| < 1$. Consequently the power series expansion for log of a matrix as given in (3.5) shows that it is meaningful to expand $\log(\exp X \exp Y)$ in a power series, and we obtain

$$
\begin{aligned}
Z &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\exp X \exp Y - 1)^k \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \bigg(\Big(\sum_{i=0}^{\infty} \frac{X^i}{i!} \Big) \Big(\sum_{j=0}^{\infty} \frac{Y^j}{j!} \Big) - 1 \bigg)^k \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \bigg(\sum_{i,j\ge 0,\ 1\le i+j<\infty} \frac{X^i Y^j}{i!\,j!} \bigg)^k .
\end{aligned}
$$

Rearranging the terms within the large parentheses on the right side yields the formula in the statement of the proposition. □

Now let us see what really has been proved and how much it falls short of the goal. To examine matters, it is instructive to group the terms of the formula for Z by total degree $i_1 + j_1 + \cdots + i_k + j_k$, there being only finitely many terms of each total degree. We can write $C_n(X, Y)$ for the sum of the terms of total degree n for $n \ge 1$. This sum is **homogeneous of degree** n in the sense that $C_n(aX, aY) = a^n C(X, Y)$ for all real a.

The terms of total degree 1 come from $k = 1$ and correspond to $(i_1,\ j_1)$ equal to $(1, 0)$ and $(0, 1)$. They are

$$X \quad \text{and} \quad Y,$$

and thus $C_1(X,Y) = X+Y$. The terms of total degree 2 come partly from $k=1$ with coefficient 1 and partly from $k=2$ with coefficient $-\frac{1}{2}$. Those terms are

$$XY + \tfrac{1}{2}X^2 + \tfrac{1}{2}Y^2 \quad \text{with coefficient } 1$$

and

$$XY + YX + X^2 + Y^2 \quad \text{with coefficient } -\tfrac{1}{2}.$$

From this computation we can see the shortcoming of Proposition 3.1. The quadratic terms involve matrix multiplication, not Lie brackets or iterated Lie brackets of matrices. But if we regroup the terms, with their coefficients in place, as

$$C_2(X,Y) = \tfrac{1}{2}XY - \tfrac{1}{2}YX,$$

then we see that the quadratic terms contribute a total of $C_2(X,Y) = \frac{1}{2}[X,Y]$, an expression written in terms of only vector space operations and Lie brackets. We say that the homogeneous polynomial $C_2(X,Y)$ has been written as a **Lie polynomial** in X and Y, i.e., a finite linear combination of X, Y, and iterated brackets of X and Y. Observe that the term $\frac{1}{2}[X,Y]$ contributing to $C_2(X,Y)$ counts as having degree 2 even though it may be tempting to count $[X,Y]$ as having degree 1. More generally, the rewriting of a polynomial as a Lie polynomial does not change the degree of any its terms. With the terms of the series for $C(X,Y)$ grouped by degree of homogeneity, we have a formal equality

$$C(X,Y) = X + Y + \sum_{n\geq 2} C_n(X,Y),$$

where $C_n(X,Y)$ is the sum of all the terms of $C(X,Y)$ homogeneous of degree n. Taking into account the known absolute convergence of the series in Proposition 3.1, we see that the above formal equality is actually an equality of $C(X,Y)$ with the sum of an absolutely convergent series.

In fact if one wanted, one could check by tedious calculation that regroupings similar to the one for $C_2(X,Y)$ are possible in degrees 3 and 4 and that the homogeneous polynomials $C_3(X,Y)$ and $C_4(X,Y)$ are therefore Lie polynomials. The miracle is that such a regrouping is possible in all higher degrees.

Theorem 3.2 (Campbell–Baker–Hausdorff Theorem). For every $n \geq 2$, the homogenous polynomial $C_n(X,Y)$ of degree n is a Lie polynomial.

Remarks. This theorem has applications to many areas of mathematics and physics, and correspondingly it has many different proofs, beginning with the ones by Baker and Hausdorff in 1905 and 1906 that were meant to complete Campbell's argument. Each such proof has its own lines of motivation from mathematics or physics. The proof below, published in 1968, is due to M. Eichler, and it is one of the shortest. The key idea behind it is that the group operation being defined by the sum of an infinite series has to be associative.

PROOF. The argument is by induction on n. The base cases are $n = 1$ and $n = 2$, and we have seen that $C_1(X,Y) = X + Y$ and $C_2(X,Y) = \frac{1}{2}[X,Y]$. The proof will make calculations not just with partial sums of the series $\sum_{n\geq 1} C_n(X,Y)$, which are noncommuting polynomials,[1] but also with the limit of the convergent series of partial sums, which is in the full space of matrices $\mathfrak{m}$. The passage to the limit may be taken as given by norm convergence or entry-by-entry convergence; it makes no difference which method is used.

The associative law for matrix multiplication implies that

$$(\exp R \exp S)(\exp T) = (\exp R)(\exp S \exp T).$$

Thus the matrix W with $(\exp R)(\exp S)(\exp T) = \exp W$ is given by the two equal expressions for W:

$$\sum_{i=1}^{\infty} C_i\left(\sum_{j=1}^{\infty} C_j(R,S), T\right) = \sum_{i=1}^{\infty} C_i\left(R, \sum_{j=1}^{\infty} C_j(S,T)\right). \tag{3.9}$$

To apply this equality, we shall substitute various linear expressions in X and Y for each of R, S, and T.

Within the sum on i on either side, we regard each term as a polynomial expression in three noncommuting variables, and the term with indices i and j on each side has homogeneity $i + j$. Because the mode of convergence of the limits is a standard one (either norm convergence or entry-by-entry convergence), (3.9) is just a system of infinitely many equalities of finite linear combinations of homogeneous expressions $C(\,\cdot\,,\cdot\,)$ in the various possible degrees ≥ 1.

By inductive hypothesis, C_j is a Lie polynomial for $1 \leq j < n$. Again this condition means that $C_j(X,Y)$ is a finite linear combination of X, Y, and iterated brackets of X and Y. We are to prove that C_n is a Lie polynomial. The equation for the terms of (3.9) that are homogeneous of degree n has two indices adding to n, and if both of those indices are $< n$, then the inductive hypothesis shows that their respective contributions to (3.9) are known to be Lie polynomials. For homogeneous polynomials P and Q of the same degree,

[1]For references to the formalism of noncommuting polynomials, see the Historical Commentary at the end of book. This formalism is not really needed for understanding the present argumenr.

let us write $P \equiv Q$ if the difference $P - Q$ is known to be a Lie polynomial. Discarding from the two sides of (3.9) those polynomials that are known to be Lie polynomials, we are left with the relation

$$C_1(C_n(R,S),T) + C_n(C_1(R,S),T) \equiv C_1(R,C_n(S,T)) + C_n(R,C_1(S,T)).$$

Since $C_1(R,S) = R + S$ and $C_1(S,T) = S + T$ and since T and R are Lie polynomials, this congruence simplifies to

$$C_n(R,S) + C_n(R+S,T) \equiv C_n(S,T) + C_n(R,S+T). \tag{3.10}$$

Meanwhile we know that $\exp X \exp Y = \exp(X+Y)$ if X and Y commute. Thus $C_n(X,Y) = 0$ for all $n \geq 2$ if X and Y commute, and consequently

$$C_n(\alpha X, \beta X) = 0 \qquad \text{if } n \geq 2 \text{ and if } \alpha \text{ and } \beta \text{ are real scalars.} \tag{3.11}$$

We shall see that the conclusion $C_n(X,Y) \equiv 0$ of the inductive step of the proof is a formal consequence of (3.10) and (3.11) if $n > 2$, and then the proof of the theorem will be complete.

First we replace (R,S,T) by $(X,Y,-Y)$ in (3.10) and use (3.11) to obtain

$$C_n(X,Y) \equiv -C_n(X+Y,-Y). \tag{3.12}$$

Next we replace (R,S,T) by $(-X,X,Y)$ in (3.10). The result is

$$C_n(X,Y) \equiv -C_n(-X,X+Y). \tag{3.13}$$

Then we have

$$\begin{aligned} C_n(X,Y) &\equiv -C_n(-X,X+Y) && \text{by (3.13)} \\ &\equiv -(-C_n(-X+X+Y,-X-Y)) && \text{by (3.12)} \\ &= C_n(Y,-X-Y) \\ &\equiv -C_n(-Y,-X) && \text{by (3.13),} \end{aligned}$$

and it follows by the homogeneity of $C_n(X,Y)$ of degree n that

$$C_n(X,Y) \equiv -(-1)^n C_n(Y,X). \tag{3.14}$$

If we replace (R,S,T) by $(X,Y,-Y/2)$ in (3.10) and use that $C_n(Y,-Y/2) = 0$, then we find that

$$C_n(X,Y) \equiv C_n(X,Y/2) - C_n(X+Y,-Y/2). \tag{3.15}$$

Similarly if we replace (R,S,T) by $(-X/2,X,Y)$ in (3.10) and use that $C_n(-X/2,X) = 0$, then we find that

$$C_n(X,Y) \equiv C_n(X/2,Y) - C_n(-X/2,X+Y). \tag{3.16}$$

Congruence (3.15) allows us to rewrite the two terms on the right side of (3.16) as follows:

$$\begin{aligned} C_n(X/2, Y) &\equiv C_n(X/2, Y/2) - C_n(X/2 + Y, -Y/2) && \text{by (3.15)} \\ &\equiv C_n(X/2, Y/2) + C_n(X/2 + Y/2, Y/2) && \text{by (3.12)} \\ &\equiv 2^{-n}C_n(X, Y) + 2^{-n}C_n(X + Y, Y) && \text{by homogeneity,} \end{aligned}$$

and

$$\begin{aligned} C_n(-X/2, X + Y) \equiv{} & C_n(-X/2, X/2 + Y/2) \\ & - C_n(X/2 + Y, -X/2 - Y/2) && \text{by (3.15)} \\ \equiv{} & -C_n(X/2, Y/2) - C_n(X/2 + Y, -X/2 - Y/2) && \text{by (3.13)} \\ \equiv{} & -C_n(X/2, Y/2) + C_n(Y/2, X/2 + Y/2) && \text{by (3.12)} \\ \equiv{} & -2^{-n}C_n(X, Y) + 2^{-n}C_n(Y, X + Y) && \text{as above.} \end{aligned}$$

Substituting into (3.16) from the two expressions just given yields

$$C_n(X, Y) \equiv 2^{1-n}C_n(X, Y) + 2^{-n}C_n(X + Y, Y) + 2^{-n}C_n(Y, X + Y).$$

Application of (3.14) to the last term on the right makes this congruence become

$$\begin{aligned} (1 - 2^{1-n})C_n(X, Y) &\equiv 2^{-n}\big(C_n(X + Y, Y) + (-1)^n C_n(X + Y, Y)\big) \\ &= 2^{-n}\big(1 + (-1)^n\big)C_n(X + Y, Y). \end{aligned} \tag{3.17}$$

For odd $n > 1$, (3.17) gives $C_n(X, Y) \equiv 0$ as required. For even $n \geq 2$, we have

$$2^{1-n}C_n(X + Y, Y) \equiv (1 - 2^{1-n})C_n(X, Y)$$

Replacing X by $X - Y$ gives

$$2^{1-n}C_n(X, Y) \equiv (1 - 2^{1-n})C_n(X - Y, Y) \equiv -(1 - 2^{1-n})C_n(X, -Y).$$

So

$$C_n(X, -Y) \equiv -2^{1-n}(1 - 2^{1-n})^{-1}C_n(X, Y).$$

Applying this formula twice gives

$$C_n(X, Y) \equiv 2^{2-2n}(1 - 2^{1-n})^{-2}C_n(X, Y).$$

Since $n > 2$, we conclude that $C_n(X, Y) \equiv 0$, as required. □

3. Derivative of exp in the linear case

We already saw in Section I.4 a hint that differentiation along smooth curves of matrices is subtle because matrices do not necessarily commute under multiplication. For example we saw in (1.8) that if $c(t)$ is a smooth curve of matrices, then $\frac{d}{dt}c(t)^{-1}$ equals $-c(t)^{-1}c'(t)c(t)^{-1}$, not $-c(t)^{-2}c'(t)$ or $-c'(t)c(t)^{-2}$. Differentiation along curves of matrices involving the exponential function is even more complicated. In this section we shall establish the following result.

Theorem 3.3 (F. Schur). Let $t \mapsto \exp X(t)$ be a smooth curve of matrices. Then

$$\frac{d}{dt}(\exp X(t)) = \exp X(t)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\operatorname{ad} X(t))^k\right)\frac{dX(t)}{dt}.$$

Remark. In this equality, $\operatorname{ad} X(t)$ means "left bracket by $X(t)$," as in Section II.4. The expression in large parentheses can be written formally as

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\operatorname{ad} X(t))^k = \frac{1-\exp(-\operatorname{ad} X(t))}{\operatorname{ad} X(t)}.$$

Rewriting the series in this way provides an indicator and a reminder where the series comes from, but $\operatorname{ad} X(t)$ is not invertible, and thus the right side is meaningful only when interpreted as being given by the left side. With this note of caution, we shall retain the formal expression for use in calculations in the next section.

Proof. Introduce a second real variable u, and define

$$Y(t,u) = (\exp(-uX(t)))\frac{\partial}{\partial t}\exp(uX(t)). \tag{3.18}$$

If we abbreviate $X(t)$ as X, then partial differentiation with respect to u gives

$$\begin{aligned}\frac{\partial Y}{\partial u}(t,u) &= -X(\exp(-uX))\frac{\partial}{\partial t}\exp(uX) + (\exp(-uX))\frac{\partial}{\partial t}(X\exp(uX))\\ &= -X(\exp(-uX))\frac{\partial}{\partial t}\exp(uX) + \exp(-uX)\frac{dX}{dt}\exp(uX)\\ &\quad + \exp(-uX)X\frac{\partial}{\partial t}\exp(uX)\end{aligned}$$

$$\begin{aligned} &= \exp(-uX)\frac{dX}{dt}\exp(uX) \\ &= \mathrm{Ad}(\exp(-uX))\frac{dX}{dt} \\ &= \exp(\mathrm{ad}(-uX))\frac{dX}{dt}. \end{aligned}$$

Evidently $Y(t,0)=0$, and thus

$$\begin{aligned} Y(t,1) &= \int_0^1 \frac{\partial Y}{\partial u}(t,u)\,du \\ &= \int_0^1 \exp(\mathrm{ad}(-uX))\frac{dX}{dt}\,du \\ &= \int_0^1 \sum_{k=0}^{\infty}\frac{(-1)^k u^k(\mathrm{ad}\,X)^k}{k!}\frac{dX}{dt}\,du \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(\mathrm{ad}\,X)^k}{(k+1)!}\frac{dX}{dt}. \end{aligned} \tag{3.19}$$

Now (3.18) shows that $Y(t,1) = \exp(-X(t))\dfrac{d}{dt}\exp X(t)$. When combined with (3.19), this result yields the desired formula. □

4. Dynkin's Formula for linear analytic groups

We return to equation (3.2), seeking to understand the solutions near $X = Y = Z = 0$ of

$$\exp Z = \exp X \exp Y$$

in the real vector space $\mathfrak{m}$ of all n-by-n matrices over $\mathbb{R}$ or $\mathbb{C}$. Proposition 3.1 gave a preliminary answer to this question by producing a formula

$$Z = X + Y + \sum_{n=1}^{\infty} C_n(X,Y)$$

in which the various terms $C_n(X,Y)$ of the series are noncommuting polynomials in X and Y homogeneous of degree n. Unfortunately such an answer is not detailed enough to address fully the Lie-theoretic questions that one may have about this series.

What was needed just before the beginning of the twentieth century and continued to be needed 50 years later was a formula for Z as above in which the individual terms $C_n(X,Y)$ of the series are Lie polynomials in X and Y *and* the question of convergence can be addressed. Theorem 3.2 addressed the first question recursively, but the use of a recursion made it harder to address the second question. This need for the two questions to be addressed simultaneously was not met sufficiently until 1947 when E. Dynkin produced such a formula. He worked at first with formal series involving noncommuting variables, without regard to convergence, and later gave a second derivation that addressed convergence. We shall reproduce Dynkin's Formula as Theorem 3.4 below but shall do so following a method of Duistermaat and Kolk that uses Theorem 3.3 and is capable of exhibiting convergence of the series for a specific region without any use of formal series.

In this book we shall not need the precise formula of Theorem 3.4, only the qualitative conclusion given in Corollary 3.5. Thus the complexity of the formula in Theorem 3.4 will not be a concern for us.

Theorem 3.4 (Dynkin's Formula). If X and Y are members of the matrix space $\mathfrak{m}$ sufficiently close to 0, then the series

$$C(X,Y) = Y + X + \sum_{n\geq 1} \sum \frac{(-1)^n}{n+1} \frac{1}{i_1 + \cdots + i_n + 1} \Big(\frac{(\operatorname{ad} X)^{i_1}}{i_1!} \frac{(\operatorname{ad} Y)^{j_1}}{j_1!} \cdots \frac{(\operatorname{ad} X)^{i_n}}{i_n!} \frac{(\operatorname{ad} Y)^{j_n}}{j_n!} \Big)(X),$$

with the inner sum indexed as below, converges in $\mathfrak{m}$ independently of the order in which the terms are added, and the full expression $Z = C(X,Y)$ is a solution of $\exp Z = \exp X \exp Y$. Here the inner sum is taken over all sequences $(i_1, j_1, \dots, i_n, j_n)$ of integers ≥ 0 satisfying $i_r + j_r \geq 1$ for all $1 \leq r \leq n$.

Remarks. As we did with Proposition 3.1, we can group terms according to their total degree, and we write

$$C(X,Y) = \sum_{n=1}^{\infty} C_n(X,Y)$$

to reflect that grouping. Let us look at the cases of total degree 1 or 2 to get a sense of what happens. For total degree 1, the summation over n is not involved, and we get only the contribution

$$C_1(X,Y) = X + Y$$

just as before.

For total degree 2, we get contributions only from $n = 1$. Since $1 \leq r \leq n$, the only case to consider is $r = 1$. Terms of degree 2 come only from $i_1 + j_1 = 1$, and the coefficient is $= -\dfrac{1}{2}\dfrac{1}{i_1+1}$. Thus the only possibilties are

$$\begin{array}{llll} i_1 = 1 & \text{with} & (\operatorname{ad} X)X = 0 & \text{and with coefficient } -\tfrac{1}{4}, \\ i_1 = 0 & \text{with} & (\operatorname{ad} Y)(X) = -[X,Y] & \text{and with coefficient } -\tfrac{1}{2}. \end{array}$$

Thus

$$C_2(X,Y) = \frac{1}{2}[X,Y],$$

in agreement with what we obtained before the statement of Theorem 3.2.

PROOF OF THEOREM 3.4. Fix X and Y, and for each t, let $Z(t)$ be the unique small solution of

$$\exp Z(t) = \exp tX \exp Y. \tag{3.20}$$

The parameter t will vary from 0 to 1, and we have $Z(0) = Y$ and $Z(1) = Z$. Applying $\exp^{-1}$ to both sides of (3.20) and using Lemma 1.7, we see that $t \mapsto Z(t)$ is a smooth function of t. In fact, $(t, X, Y) \mapsto t \exp X \exp Y$ is in the open set where $\exp^{-1}$ is smooth as long as X and Y are small enough and t is in a small enough open set containing $[0, 1]$, and thus $t \mapsto \exp^{-1}(\exp tX \exp Y)$ is smooth. This is the function $t \mapsto Z(t)$.

If we differentiate (3.20) with respect to t and use Theorem 3.3, we get

$$\exp Z(t)\, \frac{1 - \exp(-\operatorname{ad} Z(t))}{\operatorname{ad} Z(t)}\, \frac{dZ(t)}{dt} = X \exp Z(t). \tag{3.21}$$

The key step is to multiply (3.21) on the left by $\exp(Z(t))^{-1}$ and make use of the identity

$$(\exp Z)^{-1} X (\exp Z) = \operatorname{Ad}(\exp Z)^{-1} X = \exp(-\operatorname{ad}(Z))(X)$$

from Corollary 2.13. The result is

$$\frac{1 - \exp(-\operatorname{ad} Z(t))}{\operatorname{ad} Z(t)}\, \frac{dZ(t)}{dt} = \exp(-\operatorname{ad} Z(t))X. \tag{3.22}$$

In (3.22) we see that we have eliminated all occurrences of the undifferentiated $Z(t)$ from (3.21) except where it occurs as $\operatorname{ad} Z(t)$. Consequently we are going to succeed in obtaining some formula for $C(X,Y)$ that involves only $X + Y$ plus the iterated effects of $\operatorname{ad} X$ and $\operatorname{ad} Y$.

Thus we solve for $dZ(t)/dt$ and simplify to get first

$$\frac{dZ(t)}{dt} = \frac{\operatorname{ad} Z(t) \exp(-\operatorname{ad} Z(t))}{1 - \exp(-\operatorname{ad} Z(t))}\, X$$

and then

$$\frac{dZ(t)}{dt} = \frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1}\, X. \tag{3.23}$$

Put (3.23) aside for a moment and return to (3.20), applying Ad to both sides. If we use the formula $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$ from Corollary 2.13 on each of the three members of the result, we get

$$\exp \mathrm{ad}\, Z(t) = (\exp \mathrm{ad}\, tX)(\exp \mathrm{ad}\, Y). \tag{3.24}$$

Now we can proceed somewhat as in the proof of Proposition 3.1, using the series expansion

$$A = \log(\exp A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (\exp A - 1)^{k+1}$$

with $A = \mathrm{ad}\, Z(t)$. We know from (3.8) that the series expansion is valid for the matrix A whenever $\|A\| < \log 2$. The condition $\|\mathrm{ad}\, Z(t)\| < \log 2$ is certainly satisfied for all t with $0 \leq t \leq 1$ as long as $\|\mathrm{ad}\, X\|$ and $\|\mathrm{ad}\, Y\|$ are sufficiently small. Assuming X and Y to be chosen that way, we see from (3.23) and then (3.24) that

$$\begin{aligned}
\frac{dZ(t)}{dt} &= \frac{\mathrm{ad}\, Z(t)}{\exp(\mathrm{ad}\, Z(t)) - 1} X \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \big(\exp \mathrm{ad}\, Z(t) - 1\big)^k X \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (\exp \mathrm{ad}\, tX \exp \mathrm{ad}\, Y - 1)^k X \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \Bigg(\sum_{i \geq 0,\, j \geq 0,\, i+j>0} \frac{t^i (\mathrm{ad}\, X)^i (\mathrm{ad}\, Y)^j}{i!\, j!} \Bigg)^k X \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{i_k \geq 0,\, j_k \geq 0,\, i_k+j_k>0} t^{i_1+\cdots+i_k} \qquad\qquad (3.25) \\
&\qquad \Big(\frac{(\mathrm{ad}\, X)^{i_1}}{i_1!}\Big)\Big(\frac{(\mathrm{ad}\, Y)^{j_1}}{j_1!}\Big) \cdots \Big(\frac{(\mathrm{ad}\, X)^{i_k}}{i_k!}\Big)\Big(\frac{(\mathrm{ad}\, Y)^{j_k}}{j_k!}\Big)(X).
\end{aligned}$$

Meanwhile since $Z(0) = Y$ and $Z(1) = Z$, we have

$$Z = Y + \int_0^1 \frac{dZ(t)}{dt}\, dt.$$

If we substitute (3.25) into this expression, then the term with $k = 0$ yields X, the integration of the power of t yields $(i_1 + \cdots + i_k + 1)^{-1}$, and the rest matches the formula for $C(X, Y)$ in the statement of the theorem. $\square$

In this book we do not really need Dynkin's exact formula as in Theorem 3.4, only the qualitative form of it given in Corollary 3.5 below. However, no proof of Corollary 3.5 is known that does not involve proving something like Dynkin's Formula completely.[2]

Corollary 3.5. Let X, Y, and Z be matrices in $\mathfrak{m}$ that are sufficiently close to 0, and suppose that $(\exp X)(\exp Y) = \exp Z$. If X and Y are both in a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{m}$, then Z lies in that same Lie subalgebra $\mathfrak{g}$.

PROOF. In a set sufficiently close to 0, Theorem 3.4 gives a formula for $Z = C(X, Y)$ that involves a convergent infinite series whose terms are $X + Y$ and iterated brackets of X and Y. Thus the partial sums of the infinite series are in $\mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, it is closed under the limits in question. □

5. Construction of linear analytic subgroups from Lie subalgebras

Now we come to the main result of this chapter.

Theorem 3.6. Let $\boldsymbol{M}$ be a general linear group, and let $\mathfrak{m}$ be its linear Lie algebra. If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{m}$, then there exists a linear analytic subgroup G of $\boldsymbol{M}$ with linear Lie algebra $\mathfrak{g}$. Specifically the group of matrices generated by all $\exp X$ with $X \in \mathfrak{g}$ becomes an analytic subgroup in its natural topology, and its linear Lie algebra is $\mathfrak{g}$.

PROOF. For definiteness let us take $\mathfrak{m}$ to consist of all n-by-n complex matrices. We can then write $\boldsymbol{M} = GL(n, \mathbb{C})$.

Let G be the subgroup of $\boldsymbol{M}$ generated by all elements $\exp X$ with $X \in \mathfrak{g}$. The subgroup G coincides with the set of all finite products $\exp X_1 \cdots \exp X_k$ with each X_j in $\mathfrak{g}$. The reason G is the desired subgroup is that this set of products is evidently closed under multiplication, the formula $1 = \exp 0$ shows that this set of products contains the identity, and the formula $(\exp X)^{-1} = \exp(-X)$ shows that this set of products is closed under inverses.

Being an abstract subgroup of $\boldsymbol{M} = GL(n, \mathbb{C})$, G has a linear Lie algebra $\mathfrak{g}'$ as defined in (1.11). If X is in the given Lie subalgebra $\mathfrak{g}$, then so is tX for every real t, and hence G contains $\exp tX$ for all real t. Consequently (1.11) shows that X is in the linear Lie algebra $\mathfrak{g}'$ of G. Thus $\mathfrak{g} \subseteq \mathfrak{g}'$. We are to prove that equality holds.

[2]See the Historical Commentary for remarks on the problem of associating analytic subgroups of Lie groups to Lie subalgebras of Lie algebras. The use by G. Frobenius of "involutive distributions" in an analytic approach to this problem may be compared with the requirement in Corollary 3.5 that $\mathfrak{g}$ be closed under brackets.

Choose a real vector subspace $\mathfrak{s}$ of $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{g} \oplus \mathfrak{s}$. By Lemma 1.7 there exist open balls U_1 about 0 in $\mathfrak{g}$ and U_2 about 0 in $\mathfrak{s}$, as well as an open neighborhood V of 1 in $\boldsymbol{M} = GL(n, \mathbb{C})$ such that

$$(a, b) \mapsto \exp a \exp b \tag{3.26}$$

is a diffeomorphism of $U_1 \times U_2$ onto V. Fix attention on a member Z of $\mathfrak{g}'$. For small positive t, $\exp tZ$ lies in V, and it therefore decomposes according to (3.26) as

$$\exp tZ = \exp X(t) \exp Y(t) \quad \text{with } X(t) \in \mathfrak{g} \text{ and } Y(t) \in \mathfrak{s}.$$

Since $X(t)$ is in $\mathfrak{g}$, $\exp X(t)$ is in G.

On the other hand, Z is in $\mathfrak{g}'$, which is the linear Lie algebra of G, and thus Theorem 2.1 shows that $\exp tZ$ is in G for all t. Hence $\exp Y(t)$ lies in G for all small positive t. We shall prove that

$$\begin{array}{c}\text{The set of } Y \in U_2 \text{ such that} \\ \exp Y \text{ lies in } G \text{ is countable.}\end{array} \tag{3.27}$$

Assuming (3.27), let us complete the proof of Theorem 3.6. Being a smooth curve, $t \mapsto \exp Y(t)$ either is constant for t in an interval about 0, or it takes on uncountably many values. Because of (3.27) it must be constant, and thus $\exp Y(t) = \exp Y(0) = 1$ in an interval about 0. Then it follows that $\exp tZ = \exp X(t) \exp Y(t) = \exp X(t)$ in an interval about 0. Thus $\exp tZ$ for t near 0 is a smooth curve in G, and its derivative at $t = 0$, which is Z, must be in the linear Lie algebra $\mathfrak{g}$ of G. □

To address Theorem 3.6 fully, we are thus left with proving (3.27). We formulate that statement as a lemma.

Lemma 3.7. With notation as in the proof of Theorem 3.6, the set of $Y \in U_2$ such that $\exp Y$ lies in G is countable.

Proof. It is only in this lemma that we make use of the hypothesis that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{m}$ and not merely a vector subspace. If X and Y are members of $\mathfrak{m}$ with $\|X\| + \|Y\| < 2$ and if there exists a matrix Z with $\|Z\| < 1$ such that $(\exp X)(\exp Y) = \exp Z$, then we have seen that Z is unique and we can write $C(X, Y)$ for the matrix Z. By Lemma 1.6, $\exp\colon \mathfrak{m} \to \boldsymbol{M}$ is invertible near the identity of $\boldsymbol{M}$; we write $\exp^{-1}$ for the inverse function. Since this inverse function is continuous at 1, there exists an $\varepsilon > 0$ such that $\|W - 1\| < \varepsilon$ implies $\|\exp^{-1} W\| < 1$. By continuity of $(X, Y) \mapsto \exp X \exp Y$ near $(X, Y) = (0, 0)$, choose and fix $\delta > 0$ such that

$$\|X\| < \delta \text{ and } \|Y\| < \delta \text{ together imply } \|\exp X \exp Y - 1\| < \varepsilon. \tag{3.28}$$

Then $\|\exp^{-1}(\exp X \exp Y)\| < 1$.

If we assume as we may that $\delta < 1$ and if we take any X and Y with $\|X\| < \delta$ and $\|Y\| < \delta$, then $Z = \exp^{-1}(\exp X \exp Y)$ is a matrix with $\|Z\| < 1$ such that $(\exp X)(\exp Y) = \exp Z$. According to the previous paragraph, Z is unique in the sense that no other matrix Z' with $\|Z'\| < 1$ has $(\exp X)(\exp Y) = \exp Z'$. We write $Z = C(X, Y)$ accordingly. If $B_\delta = \{X \in \mathfrak{g} \mid \|X\| < \delta\}$, then

$$\exp C(X, Y) = \exp X \exp Y \tag{3.29}$$

for a smooth function C on the subset $B_\delta \times B_\delta$ of $\mathfrak{g} \times \mathfrak{g}$ with $C(0, 0) = 0$. According to Corollary 3.5, if X and Y are in the subset B_δ of $\mathfrak{g}$, then $C(X, Y)$ is in $\mathfrak{g}$. This fact is crucial in the proof.

To prove the lemma, we are going to count the set of $Y \in U_2$ such that $\exp Y$ lies in G. Thus we study an element a of G. The element a is a product of exponentials of members of $\mathfrak{g}$. Using the identity $\exp X = (\exp X/n)^n$, we can expand some of the factors to see that our given element $a \in G$ can be written as a product

$$a = \exp X_1 \exp X_2 \cdots \exp X_m \tag{3.30}$$

of members X_j of $\mathfrak{g}$ with $\|X_j\| < \delta$. Fix a vector space basis of $\mathfrak{g}$, and call a member of $\mathfrak{g}$ **rational** if it is a rational linear combination of the members of this vector space basis. Let us see recursively[3] for $m \geq 2$ that

For every $a \in G$ as in (3.30) and every $\eta > 0$, there exist rational elements $R_1, \dots, R_{m-1}$ of $\mathfrak{g}$ such that $a = \exp R_1 \cdots \exp R_{m-1} \exp \widetilde{X}$ for some element $\widetilde{X} \in \mathfrak{g}$ with $\|\widetilde{X}\| < \eta$.
(3.31)

We begin with the case $m = 2$. Elements X_1 and X_2 of $\mathfrak{g}$ are given with $\|X_1\| < \delta$ and $\|X_2\| < \delta$, where δ is as in (3.28). Then (3.28) and (3.29) show that $\|\exp C(X_1, X_2) - 1\| < \varepsilon$. Using a rational element R_1 to be specified, let us write

$$\begin{aligned}\exp X_1 \exp X_2 &= \exp C(X_1, X_2)\\ &= \exp R_1 \exp(-R_1) \exp C(X_1, X_2)\\ &= \exp R_1 \exp \widetilde{X}_2,\end{aligned}$$

where $$\widetilde{X}_2 = C(-R_1, C(X_1, X_2)) \quad \text{lies in } \mathfrak{g}.$$

[3] The base case of the induction here will be $m = 2$. If it should happen that an expansion (3.30) for a particular element a has $m = 1$, then we simply rewrite the expansion with $m = 2$ by inserting a factor $\exp 0$ at the left end.

The elements X_1 and X_2 are fixed, and we have R_1 at our disposal in order to make $\|\widetilde{X}_2\| < \eta$. Continuity of C in the first variable implies that if the rational element R_1 is chosen close enough to $C(X_1, X_2)$, then the element $\widetilde{X}_2 = C(-R_1, C(X_1, X_2))$ will have

$$\|C(-R_1, C(X_1, X_2)) - C(-C(X_1, X_2), C(X_1, X_2))\| < \eta.$$

However,

$$C(-C(X_1, X_2), C(X_1, X_2)) = -C(X_1, X_2) + C(X_1, X_2) = 0$$

since the element $C(X_1, X_2)$ commutes with itself. Thus

$$\|\widetilde{X}_2\| = \|C(-R_1, C(X_1, X_2)) - 0\| < \eta,$$

and we have shown that $\exp X_1 \exp X_2 = \exp R_1 \exp \widetilde{X}_2$ with $\|\widetilde{X}_2\| < \eta$ as required.

Turning to the case of $m \geq 3$, we can use the case $m = 2$ twice to write

$$\begin{aligned} a &= \exp X_1 \exp X_2 \exp X_3 \cdots \exp X_m \\ &= \exp R_1 \exp \widetilde{X}_2 \exp X_3 \cdots \exp X_m \\ &= \exp R_1 \exp R_2 \exp \widetilde{X}_3 \exp X_4 \cdots \exp X_m \end{aligned}$$

with R_1 and R_2 rational and $\|\widetilde{X}_3\| < \eta$. Continuing in this way, we arrive at an expression

$$a = \exp R_1 \exp R_2 \cdots \exp R_{m-1} \exp \widetilde{X}_m$$

with $R_1, \ldots, R_{m-1}$ rational, $\widetilde{X}_m$ in $\mathfrak{g}$, and $\|\widetilde{X}_m\| < \eta$. This proves (3.31).

In (3.31) we shall take $\eta = \delta$ as defined in (3.28), and then we can complete the proof of the lemma. Let us see for each $m \geq 1$ and for each set of $m - 1$ rational elements of $\mathfrak{g}$ that there is at most one element X of $\mathfrak{g}$ with $\|X\| < \delta$ such that $\exp R_1 \cdots \exp R_{m-1} \exp X$ lies in V. Indeed, suppose that

$$\exp Y_1 = \exp R_1 \exp R_2 \cdots \exp R_{m-1} \exp X_1$$

and

$$\exp Y_2 = \exp R_1 \exp R_2 \cdots \exp R_{m-1} \exp X_2$$

are in V with $X_1 \in \mathfrak{g}$, $X_2 \in \mathfrak{g}$, $\|X_1\| < \delta$, and $\|X_2\| < \delta$. Then

$$(\exp Y_2)(\exp 0) = \exp Y_1 \exp(-X_1) \exp X_2 = (\exp Y_1) \exp C(-X_1, X_2). \tag{3.32}$$

Here Y_2 and Y_1 are in U_2, and 0 and $C(-X_1, X_2)$ are in U_1. In our construction each element of $\exp U_2 \exp U_1$ has a unique decomposition as $\exp Y \exp X$ with $X \in U_1$ and $Y \in U_2$. Thus (3.32) implies that $Y_2 = Y_1$. Consequently $\exp X_2 = \exp X_1$. Since exp is one-one on U_1, $X_2 = X_1$. □

6. Construction of homomorphisms of analytic groups from homomorphisms of Lie algebras

Let us move beyond linear groups and linear Lie algebras for now and turn to a consideration of smooth homomorphisms between linear groups. If $\Phi\colon G \to G'$ is a smooth homomorphism between two linear groups, then Theorem 2.11 showed how to associate to Φ a homomorphism $D\Phi\colon \mathfrak{g} \to \mathfrak{g}'$ between their linear Lie algebras. The formula given in that theorem was that if $c(t)$ is a smooth curve in G with $c(0) = 1$ and $c'(0) = X$, then $D\Phi(X)$ is the member Y of $\mathfrak{g}$ such that $\dfrac{d}{dt}\,\Phi(x(t))\big|_{t=0} = Y$.

In the spirit of the present chapter, let us try to invert this construction, passing from a homomorphism between the linear Lie algebras of two linear groups to a smooth homomorphism between the groups themselves. Discussion of this topic depends somewhat on the subject of covering spaces, and we shall take the content of Appendix C as known for the present section.

It turns out that there is a topological obstruction to lifting the Lie algebra homomorphism to a Lie group homomorphism. The following example illustrates the problem.

Example 1. Let G and G' be the one-dimensional linear groups given by $G = S^1 = \{(e^{i\theta}) \mid 0 \le \theta \le 2\pi\}$ and $G' = \mathbb{R} = \{(e^t) \mid -\infty < t < \infty\}$. We can take $\mathfrak{g} = \{i\theta \mid \theta \in \mathbb{R}\}$ and $\mathfrak{g}' = \{t \mid t \in \mathbb{R}\}$. The homomorphisms between $\mathfrak{g}$ and $\mathfrak{g}'$ are parametrized by a real parameter c and are given by $\varphi_c(i\theta) = \theta c$. But the only smooth homomorphism Φ from G into G' is the one with $\Phi((e^{i\theta})) = 1$ for all θ since the image of the compact group S^1 has to be a compact subgroup of $\mathbb{R}$. Thus for all choices of c other than $c = 0$, $\varphi_c : \mathfrak{g} \to \mathfrak{g}'$ does not lift to a smooth homomorphism $\Phi : G \to G'$.

If we were to look at Example 1 more closely, first by taking $G = G' = S^1$ and then by taking $G = \mathbb{R}$ and $G' = S^1$, we might see that the problem appears to be coming from a nontrivial loop in the domain group G, specifically from the fundamental group of G.

Theorem 3.8. Let G and G' be linear Lie groups with G connected and simply connected, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be their linear Lie algebras. If $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism between the linear Lie algebras, then there exists a unique smooth homomorphism $\Phi :\ G \to G'$ between the groups such that $D\Phi = \varphi$.

PROOF. Form the group $G \times G'$. To view $G \times G'$ as a linear group, we can regard it as contained in the product of the two underlying general linear groups. In $G \times G'$, one multiplies coordinate by coordinate, and one uses the product manifold structure. The corresponding linear Lie algebra is $\mathfrak{g} \oplus \mathfrak{g}'$ with the vector space and bracket operations taking place coordinate by coordinate.

The graph of φ, given by $\text{Graph}(\varphi) = \{(X, \varphi(X)) \in \mathfrak{g}\oplus\mathfrak{g}' \mid X \in \mathfrak{g}\}$, is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}'$, and Theorem 3.6 produces a unique analytic subgroup F of $G\times G'$ with linear Lie algebra $\text{Graph}(\varphi)$. Let $p : F \to G$ be the composition of the inclusion $F \to G \times G'$ followed by the projection to the first coordinate $(x, y) \mapsto x$. On the Lie algebra level, Dp carries $\text{Graph}(\varphi)$ into $\mathfrak{g} \oplus \mathfrak{g}'$ and then projects to the first coordinate. The formula is $Dp(X, \varphi(X)) = X$. Since the domain and image have equal dimension, namely $\dim \mathfrak{g}$, the subgroup $p(F)$ contains an open neighborhood of the identity in G. But G is connected by hypothesis, and thus $p(F) = G$. Since F is connected by construction, Proposition C.19d shows that $p\colon F \to G$ is a covering map. By hypothesis, G is simply connected and hence has no nontrivial coverings. Consequently p is a homeomorphism and has an inverse p^{-1}. Let $q : F \to G'$ be the composition of the inclusion $F \to G \times G'$ followed by the projection to the second coordinate $(x, y) \mapsto y$. On the Lie algebra level, Dq carries $\text{Graph}(\varphi)$ into $\mathfrak{g} \oplus \mathfrak{g}'$ and then projects to the second coordinate. The formula is $Dq(X, \varphi(X)) = \varphi(X)$, and the composition is $q \circ p^{-1} = \varphi$. We can therefore take Φ to be $q \circ p^{-1}$. Uniqueness follows since a smooth homomorphism of an analytic group is determined by its values on a set that generates the group. $\square$

EXAMPLE 2. Let $G = SU(2)$ and $G' = SO(3)$ with linear Lie algebras $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathfrak{g}' = \mathfrak{so}(3)$. Problems 1–5 at the end of Chapter II construct explicitly a smooth homomorphism $\text{Ad} : SU(2) \to SO(3)$. The homomorphism Ad is onto $SO(3)$ and has a two-element kernel. The corresponding homomorphism of linear Lie algebras is $\text{ad} : \mathfrak{su}(2) \to \mathfrak{so}(3)$, and it is one-one onto. On the other hand, $\text{ad}^{-1}\colon \mathfrak{so}(3) \to \mathfrak{su}(2)$ is a homomorphism between linear Lie algebras, and there is no corresponding homomorphism of linear Lie groups.

The question of when it is possible to lift a homomorphism of linear Lie algebras to a smooth homomorphism of the corresponding linear Lie groups is complicated and will not be pursued further here.

The remainder of this chapter concerns the interaction between the theory of linear groups and the theory of covering spaces. We shall give examples to show that

(a) a covering group of a connected linear group need not be linear. An example appears in Section 7.
(b) a group covered by a linear group need not be linear. An example appears in Section 8.

7. Coverings of linear analytic groups need not be linear

In this section we give an example of a linear analytic group that is not simply connected yet is covered nontrivially by no linear analytic group.

The example is $G = SL(2,\mathbb{R})$, realized as a subgroup of 2-by-2 real matrices. We shall make use of the following subgroups and notation concerning those subgroups. Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad - bc = 1 \right\}, \qquad K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\},$$

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ x \in \mathbb{R} \right\}.$$

Proposition 3.9. In $G = SL(2,\mathbb{R})$, the multiplication mapping is a diffeomorphism of $K \times A \times N$ onto G.

Proof. The product formula

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t\cos\theta & e^t x\cos\theta - e^{-t}\sin\theta \\ e^t\sin\theta & e^t x\sin\theta + e^{-t}\cos\theta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

shows that each entry a, b, c, d is a smooth function of θ, t, x. Hence the map $K \times A \times N \to G$ is smooth. The formulas

$$e^{2t} = a^2 + c^2, \quad \cos\theta = ae^{-t}, \quad \sin\theta = ce^{-t},$$

$$\text{and} \qquad x = (ab + cd)(a^2 + c^2)^{-1}$$

show that t, $\sin\theta$, $\cos\theta$, and x are smooth functions of a, b, c, d. Therefore the inverse function is well defined and smooth. □

Corollary 3.10. $SL(2,\mathbb{R})$ is connected and has fundamental group $\mathbb{Z}$.

Proof. The groups A and N are isomorphic to $\mathbb{R}$ and are connected and have fundamental group 1. Also K is isomorphic to S^1 and thus by Proposition C.15 is connected and has fundamental group $\mathbb{Z}$. Consequently the topological product, which is homeomorphic to $SL(2,\mathbb{R})$ by Proposition 3.9, is connected and has fundamental group equal to the product $1 \times 1 \times \mathbb{Z}$, which is $\mathbb{Z}$. □

Consequently $SL(2,\mathbb{R})$ has infinitely many nontrivial covering groups. We shall show that none of them is linear. To do so, we make use of the following analog of Proposition 3.9 for $SL(2,\mathbb{C})$.

Proposition 3.11. In $G = SL(2,\mathbb{C})$, the multiplication mapping is a diffeomorphism of $U \times A \times N^+$ onto G, where

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \det = 1 \right\}, \qquad U = \text{special unitary group } SU(2),$$

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\}, \qquad N^+ = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},\ z \in \mathbb{C} \right\}.$$

Proof. Following the line of argument of Proposition 3.9, we set $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equal to a product from UAN^+. The result shows that $e^{2t} = |a|^2 + |c|^2$ and that the first column of the member of U has entries ae^{-t} and ce^{-t}. Then we can solve for the remaining parameter z and see that all the parameters in U, A, and N depend smoothly on a, b, c, d. □

Corollary 3.12. $SL(2,\mathbb{C})$ is connected and simply connected, and so is U.

Proof. A member of U is determined by its first column, which is any unit vector in $\mathbb{C}^2$. Thus U is topologically the sphere S^3. By Proposition C.16, U is simply connected, and also it is connected. Since A and N^+ are homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2$, the result follows from Proposition 3.11. □

Proposition 3.13. The linear analytic group $SL(2,\mathbb{R})$ has no nontrivial linear covering group.

Proof. Let G be a linear covering group of $SL(2,\mathbb{R})$. Without loss of generality we may assume that G is a subgroup of some $GL(n,\mathbb{R})$. Let $\mathfrak{g}$ be its linear Lie algebra, which is a 3-dimensional Lie algebra of n-by-n real matrices and is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Form $\mathfrak{g} + i\mathfrak{g}$ as a 6-dimensional real Lie algebra of n-by-n complex matrices. Applying Theorem 3.6, we can produce a linear analytic subgroup $\widetilde{G}$ of $GL(n,\mathbb{C})$ with linear Lie algebra

$\mathfrak{g} + i\mathfrak{g}$. According to Corollary 3.12, $SL(2, \mathbb{C})$ is a simply connected linear analytic group whose linear Lie algebra is isomorphic to $\mathfrak{g} + i\mathfrak{g}$, and it follows from Theorem 3.8 that $\widetilde{G}$ is a homomorphic image of $SL(2, \mathbb{C})$. In case there is any doubt, the homomorphism in question is onto $SL(2, \mathbb{C})$ by Corollary 2.16a and the connectedness of $SL(2, \mathbb{C})$. Its kernel has dimension 0 and is thus discrete. Referring to Proposition C.19d, we see that the homomorphism is a covering map. Since $SL(2, \mathbb{C})$ is simply connected, the covering map is an isomorphism. In other words, $\widetilde{G}$ is isomorphic to $SL(2, \mathbb{C})$, and hence G is isomorphic to $SL(2, \mathbb{R})$. □

8. Groups covered by linear analytic groups need not be linear

In this section we give an example of a linear analytic group that covers another analytic group G that has no realization as a linear group.

We start from the three dimensional group Heisenberg group H_3 of real matrices

$$H_3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, t \in \mathbb{R} \right\}. \tag{3.33}$$

The center C of the group H_3 is the 1-dimensional subgroup of all matrices with $x = y = 0$. Every subgroup of the center C is normal, and we focus on the subgroup D consisting of those members of C for which t is an integer. This is a discrete subgroup of H_3, and the group of interest will be

$$H_3' = H_3/D.$$

According to Proposition C.19d, the quotient map $H_3 \to H_3' = H_3/D$ is a covering map. Therefore H_3' is an analytic group of dimension 3 that is covered by a linear analytic group.

Proposition 3.14. The group H_3' has no realization as a linear group in the sense that there is no continuous one-one homomorphism $\Phi : H_3' \to GL(n, \mathbb{C})$ for any integer $n \geq 1$.

Proof. For x, y, and t in $\mathbb{R}$, define

$$a_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad c_t = \begin{pmatrix} 1 & 0 & t \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Direct computation gives

$$b_y a_x b_y^{-1} a_x^{-1} = c_{-xy}. \tag{3.34}$$

Let us put this computation aside for the moment. Arguing by contradiction, let $n \geq 1$ be the smallest integer such that H_3' has a continuous one-one homomorphism into some $GL(n, \mathbb{C})$. Let $\Phi\colon H_3' \to GL(n, \mathbb{C})$ be such a homomorphism, and let $\widetilde{\Phi}$ be the lift of Φ from H_3' to H_3. Since $\widetilde{\Phi}$ is trivial on the discrete subgroup D of H_3, $\widetilde{\Phi}(c_t)$ depends only on the value of t modulo 1. Moreover, $\widetilde{\Phi}(c_t)$ is a continuous function of t mod 1 with values in $GL(n, \mathbb{C})$, and thus its image is compact. The norms of these matrices are bounded, and therefore all the eigenvalues of all the matrices $\widetilde{\Phi}(c_t)$ are uniformly bounded.

Temporarily let us introduce a new Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on the vector space $\mathbb{C}^n$ where the members of $\Phi(G_3)$ act. The new Hermitian inner product is given in terms of the standard Hermitian inner product $(\,\cdot\,,\,\cdot\,)$ by the formula

$$\langle v_1, v_2\rangle = \int_0^1 (\Phi(c_t)v_1, \Phi(c_t)v_2)\,dt,$$

and we readily check that

$$\langle \Phi(c_{t'})v_1, \Phi(c_{t'})v_2\rangle = \langle v_1, v_2\rangle \tag{3.35}$$

for all t' mod 1.

Fix a number t_0 mod 1. Let λ be any eigenvalue of $\widetilde{\Phi}(c_{t_0})$, and let V_λ be the corresponding eigenspace in $\mathbb{C}^n$. Putting $t' = t_0$ and taking $v_1 = v_2$ in (3.35) to be in V_λ, we see that $|\lambda| = 1$.

From (3.35) it follows that the orthogonal complement relative to $\langle\,\cdot\,,\,\cdot\,\rangle$ of an invariant subspace in $\mathbb{C}^n$ under $\Phi(c_{t_0})$ is an invariant subspace, and consequently $\mathbb{C}^n$ is the direct sum of the eigenspaces under $\Phi(c_{t_0})$.

Since c_{t_0} is in the center of H_3, every g in H_3 has $\widetilde{\Phi}(c_{t_0})(\widetilde{\Phi}(g)v) = \widetilde{\Phi}(g)\widetilde{\Phi}(c_{t_0})v = \lambda\widetilde{\Phi}(g)v$ for all v in V_λ. In other words, each $\widetilde{\Phi}(g)$ carries each eigenspace V_λ into itself.

Recall, however, that we arranged that n is the smallest positive integer for which H_3' has a continuous one-one homomorphism into some $GL(n, \mathbb{C})$ and that Φ is such a homomorphism. Consequently there is only one value of λ, say $\lambda = \lambda_0$, for which $V_\lambda \neq 0$, and that value has $V_{\lambda_0} = \mathbb{C}^n$. In short, $\widetilde{\Phi}(c_{t_0})$ acts by the scalar λ_0 on $\mathbb{C}^n$, and $\det\widetilde{\Phi}(c_{t_0}) = \lambda_0^n$.

Now we can make use of (3.34). Consider the action of $\widetilde{\Phi}(H_3)$ on $\mathbb{C}^n$. Application of $\det\widetilde{\Phi}$ to both sides of (3.34) shows that

$$1 = \det\widetilde{\Phi}(c_{xy})$$

for all x and y in $\mathbb{R}$. Consequently $\det\widetilde{\Phi}(c_t) = 1$ on $\mathbb{C}^n$ for all t. In particular, $\lambda_0^n = 1$.

Thus $\widetilde{\Phi}(t_0)$ is a scalar operator equal to an n^{th} root of unity, no matter what t_0 is. The operator varies continuously with t_0 and is the identity when $t_0 = 0$. Thus $\widetilde{\Phi}(t) = 1$ for all t, and it follows that Φ is not one-one. This conclusion contradicts the choice of Φ as a continuous one-one homomorphism and concludes the proof. $\square$

9. Summary

Chapter III continues the development of Lie theory for linear groups. Chapters I and II showed how to associate a linear analytic group and a linear Lie algebra to each linear group and how to associate a Lie algebra homomorphism to each smooth homomorphism from one linear analytic group to another. This chapter concerns running these constructions backwards.

The key tool for successfully getting these constructions to run backwards is Dynkin's Formula, which gives an explicit convergent infinite series expansion for a square matrix Z near 0 in terms of square matrices X and Y near 0 when $\exp Z = \exp X \exp Y$. The terms of the infinite series involve only vector space operations and iterated Lie brackets of X and Y. This result shows that the linear Lie algebra of a Lie group captures not just the infinitesimal behavior of the group operation at the identity but in fact captures a convergent infinite series expansion for the multiplication rule in a neighborhood of the identity.

Dynkin's Formula improves on the earlier Baker–Campbell–Hausdorff Theorem (Theorem 3.2), whose pre-Dynkin proofs made it difficult to address the convergence of the series that arises. The main new ingredient in the proof of Dynkin's Formula as given here is an application of a nineteenth century formula of F. Schur for differentiating along a curve of square matrices of the form $\exp(Z(t))$.

With Dynkin's Formula in hand, one can see how to pass from the Lie subalgebra determined by a linear analytic subgroup to the subgroup itself. In fact the subgroup is the one generated by all elements $\exp X$ with X in the given Lie subalgebra.

Dynkin's Formula shows that the multiplication rule in a linear analytic group is determined near the identity by the linear Lie algebra, and this fact indicates that a homomorphism on the Lie algebra level can be lifted near the identity to a smooth Lie group homomorphism between the corresponding linear analytic groups. With further argument one can see that this smooth local homomorphism extends to be defined globally if the domain is simply connected.

The remaining sections of the chapter give examples to show that the class of linear groups is not closed under a couple of desirable operations. Section 7 shows that even though the linear analytic group $SL(2,\mathbb{R})$ has infinitely many proper covering groups, none of the proper covering groups can be realized as a linear group. Section 8 gives an example of a linear analytic group that covers an analytic group having no realization as a linear group.

In particular Chapters I through III have completely proved the results announced in the Preface as Fundamental Theorems A and B of Lie Theory under the hypothesis that the given Lie group is a general linear group.

10. Problems

1. Use Dynkin's Formula directly as it is stated in Theorem 3.4 to calculate $C_3(X,Y)$, showing that it is given by
$$C_3(X,Y) = \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]].$$

Problems 2–7 lead to a different approach to the theorem of F. Schur (Theorem 3.3) and its application to Dynkin's Formula (Theorem 3.4) from the one given in the text. One objective is to differentiate a matrix-valued function $\exp C(t)$, and another is to obtain the terms in Dynkin's Formula in a relatively systematic manner. Some of the argument will be carried out here only formally, leaving aside rigorous attention to certain convergence questions. The work dates to the paper of Baker [1905] and was elucidated by Sogo [2016]. If P and Q are two n-by-n matrices that are not scalar multiples of one another, the interest is in writing $\exp P \exp Q$ as an explicit exponential. Consider a linear operator $\mathcal{D}$ on the space $S(P,Q)$ of all linear combinations of monomials whose terms are of the form $P^{i_1}Q^{j_1}P^{i_2}Q^{j_2}\cdots P^{i_k}Q^{j_k}$. If F is in $S(P,Q)$, the value of the operator $\mathcal{D}$ on F is defined as

$$\mathcal{D}(F) = \text{result of replacing } Q \text{ by } X \text{ at one occurrence of } Q \text{ in } F$$
$$\text{and summing the result over all occurrences,}$$

where X is an expression in P and Q to be specified presently. Thus for example,

$$\mathcal{D}(1) = 0, \quad \mathcal{D}(Q) = X, \quad \mathcal{D}(Q^2) = XQ + QX$$
$$\mathcal{D}(P) = 0, \quad \mathcal{D}(Q^2PQ) = XQPQ + QXPQ + Q^2PX.$$

Take as given that $\mathcal{D}F$ is well defined independently of the way a member of $S(P,Q)$ is expanded in terms of monomials. Observe from the definition that $\mathcal{D}(QF) = XF + Q\mathcal{D}(F)$ for all F in $S(P,Q)$.

2. Prove that
$$\mathcal{D}(FG) = (\mathcal{D}F)G + F(\mathcal{D}G)$$
for all F and G in $S(P, Q)$. (Educational note: Because of this formula, which has the same flavor as the rule in calculus for differentiation of a product, one says that $\mathcal{D}$ is a **derivation.** This notion can be defined more generally, and further instances of derivations will appear in the Problems for Chapter IV.)

3. For $0 \le r \le m-1$, establish the scalar identity
$$(1+x)^r + (1+x)^{r+1} + \cdots + (1+x)^{m-1} = \frac{(1+x)^m}{x} - \frac{(1+x)^r}{x}$$
for x positive, and use it to prove that
$$\binom{m-1}{r} + \binom{m-2}{r} + \cdots + \binom{r}{r} = \binom{m}{r+1}.$$

4. Define operators L_Q, standing for left-by-Q, and R_Q, standing for right-by-Q, by $L_Q F = QF$ and $R_Q F = FQ$, and put $\operatorname{ad} Q = L_Q - R_Q$. Note that L_Q, R_Q, and $\operatorname{ad} Q$ commute with one another. Starting from the identity
$$\begin{aligned}\mathcal{D}(Q^m) &= (L_Q^{m-1} + L_Q^{m-2}R_Q + L_Q^{m-3}R_Q^2 + \cdots + R_Q^{m-1})X \\ &= (R_Q + \operatorname{ad} Q)^{m-1} + (R_Q + \operatorname{ad} Q)^{m-2}R_Q + \cdots + R_Q^{m-1})X\end{aligned}$$
and using the result of the previous problem and the fact that L_Q and R_Q commute with each other, apply the Binomial Theorem to prove for $m \ge 1$ that
$$\mathcal{D}(Q^m) = \binom{m}{1}XQ^{m-1} + \binom{m}{2}((\operatorname{ad} Q)X)Q^{m-2} + \cdots + \binom{m}{m}((\operatorname{ad} Q)^{m-1}X).$$

5. It will be necessary to make use of some power series.
   (a) Verify that the Taylor series expansion of the analytic function $z^{-1}(e^z - 1)$ about $z = 0$ is
   $$1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots$$
   and that the expansion of the reciprocal $z/(e^z - 1)$ about $z = 0$ is
   $$\begin{aligned}\frac{z}{e^z - 1} &= -\frac{z}{2} + \frac{z}{2}\,\frac{e^z + 1}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \cdots \\ &= 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{B_n}{(2n)!}z^{2n}.\end{aligned}$$

(b) Explain why the latter series is convergent for $|z| < 2\pi$. (Educational note: The numbers B_n are called **Bernoulli numbers.** The first few are $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$; some authors index them differently. With this indexing, one of their remarkable properties is that $\sum_{k=1}^{\infty} k^{-2n} = \dfrac{2^{2n-1}}{(2n)!} B_n \pi^{2n}$. In particular all the B_n are positive.)

6. Let us extend the domain of $\mathcal{D}$ from $S(P,Q)$ to include any series like the one for $\exp Q$. Specifically let us take the domain of $\mathcal{D}$ to be all **formal power series** in Q with coefficients in $S(P,Q)$, i.e., all sequences of coefficients indexed by $n \geq 0$, each coefficient being in $S(P,Q)$. Calculate $\mathcal{D}(\exp Q)$ formally (i.e., calculate the coefficients without regard to convergence of the resulting series and without regard to questions of associativity of multiplication), arriving at the result
$$\mathcal{D}(\exp Q) = \Big[\Big(1 + \frac{\operatorname{ad} Q}{2!} + \frac{(\operatorname{ad} Q)^2}{3!} + \frac{(\operatorname{ad} Q)^3}{4!} + \cdots\Big)X\Big] \exp Q.$$

7. Recall that the definition of $\mathcal{D}$ depends on an unspecified expression X to be given in terms of P and Q. That expression X will now be specified. Take X to be the product of a certain formal power series and P, namely
$$X = \Big(1 - \frac{(\operatorname{ad} Q)}{2} + \frac{B_1}{2!}(\operatorname{ad} Q)^2 - \frac{B_2}{4!}(\operatorname{ad} Q)^4 + \frac{B_3}{6!}(\operatorname{ad} Q)^6 - \cdots\Big)P,$$
where the numbers B_n for $n \geq 1$ are the Bernoulli numbers defined in Problem 5. Explain why this choice of X makes
$$\mathcal{D}(\exp Q) = P \exp Q.$$
as an equality of formal power series.

Problems 8–9 continue the development begun in Problems 2–7. The first step is to enlarge the space $S(P,Q)$ with which we work. The new space will be called $\widetilde{S}(P,Q)$. In the new space a "monomial" will be any finite product whose factors are P and any formal power series in Q, and a general term will be any finite linear combination of such "monomials." Take it as given that $\mathcal{D}$ extends consistently to be defined on all of $\widetilde{S}(P,Q)$ in such a way that the derivation formula of Problem 2 remains valid. Take it as a further given that the same kinds of manipulations that are valid with $S(P,Q)$ remain valid with $\widetilde{S}(P,Q)$.

8. Apply $\mathcal{D}$ repeatedly to $\mathcal{D}(\exp Q) = P \exp Q$ to obtain $\mathcal{D}^n(\exp Q) = P^n \exp Q$. Sum on n and introduce a real parameter t to obtain the formula
$$(\exp t\mathcal{D})(\exp Q) = (\exp tP)(\exp Q). \tag{$*$}$$

Define $C(t)$ by $\exp C(t) = \exp tP \exp Q$, and set $C = C(1)$. Interchange the two sides of $(*)$ with each other in order to rewrite $(*)$ as

$$\exp C(t) = (\exp t\mathcal{D})(\exp Q).$$

Differentiate both sides with respect to t term by term, and show that

$$\frac{d}{dt} \exp C(t) = \mathcal{D} \exp(C(t)).$$

9. With $C(t)$ defined as in the previous problem, use the derivation formula of Problem 2 to deduce that
$$\frac{dC(t)}{dt} = \mathcal{D}C(t).$$
10. Bearing in mind that $C(0) = Q$ and $C(1) = C$, show how it follows from the result of the previous problem that
$$C = (\exp \mathcal{D})Q.$$
This is the fundamental formula of the Baker–Sogo method. It gives an explicit expression for $C = (\exp P)(\exp Q)$.

Problems 11–12 show how to use the fundamental formula of the Baker–Sogo method to give manageable expansions of the terms $C_k(X, Y)$ in Dynkin's Formula as long as k is not too large.

11. Show from the fundamental formula of the Baker–Sogo method that
$$C = Q + X + \frac{1}{2}\,\mathcal{D}(X) + \frac{1}{6}\,\mathcal{D}^2(X) + \cdots,$$
with X as in Problem 7:
$$X = P + \frac{1}{2}[P, Q] + \frac{1}{12}(\operatorname{ad} Q)^2 P - \frac{1}{720}(\operatorname{ad} Q)^4 P + \cdots.$$

12. Show from the result of the previous problem that
$$\begin{aligned} C_1 &= P + Q, \\ C_2 &= \frac{1}{2}\,[P, Q], \\ C_3 &= \frac{1}{12}\,[P, [P, Q]] + \frac{1}{12}\,[Q, [Q, P]] \\ C_4 &= \frac{1}{24}\,[Q, [P, [Q, P]]]. \end{aligned}$$

Problems 13–15 indicate another approach to proving Theorem 3.8 besides the one given above in Section 6. Let two linear analytic groups G and G' be given with respective linear Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, and let $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism between their linear Lie algebras. One first constructs in some way a "local homomorphism" of a neighborhood of 1 in G into a neighborhood of 1 in G', and then, as Chevalley [1946] does in his Theorem 3, p. 49, one makes use of the assumption that G is simply connected to extend the local homomorphism to be globally defined. These problems address the construction of the local homomorphism.

13. A **local homomorphism** $\Phi\colon G \to G'$ is a smooth function defined on an open neighborhood $U = U^{-1}$ of 1 in G and taking values in an open neighborhood V of 1 in G' such that whenever a, b, and ab are in U, then $\Phi(a)$, $\Phi(b)$, and $\Phi(ab)$ are in V and satisfy $\Phi(ab) = \Phi(a)\Phi(b)$. Show that any such local homomorphism Φ gives rise to a homomorphism of the linear Lie algebra of $\mathfrak{g}$ into the linear Lie algebra $\mathfrak{g}'$.
14. Prove that if $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of linear Lie algebras and if X and Y are in a sufficiently small neighborhood of 0 in $\mathfrak{g}$, then $\varphi(C(X,Y)) = C(\varphi(X), \varphi(Y))$.
15. Let G and G' be local Lie groups, and let $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of their linear Lie algebras. Let U be a neighborhood of 0 in $\mathfrak{g}$ small enough so that $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism of U onto an open subset of G and so that the series for $C(X,Y)$ in the statement of Theorem 3.4 converges for X and Y in U and satisfies
$$\exp C(X,Y) = \exp X \exp Y.$$
Assume in addition that the series for $C(\varphi(X), \varphi(Y))$ converges for X and Y in U and satisfies
$$\exp C(\varphi(X), \varphi(Y)) = \exp \varphi(X) \exp \varphi(Y).$$
Define
$$\Phi(\exp X) = \exp \varphi(X)$$
for X in U. Prove that Φ is a local homomorphism of G into G'.

Problems 16–17 establish symmetry relations for the coefficients in the Campbell–Baker–Hausdorff formula of Theorem 3.2.

16. Prove that the formal series expansion for $C(X,Y)$ in the Campbell–Baker-Hausdorff formula satisfies $C(-Y,-X) = -C(X,Y)$.
17. Deduce from Problem 16 that $C_n(Y,X) = (-1)^{n-1} C_n(X,Y)$ for every $n \geq 1$.

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CHAPTER IV

Lie Algebras, Lie Groups, and Homomorphisms

Abstract. Chapter IV turns from the subject of linear groups to a consideration of arbitrary Lie groups. For arbitrary Lie groups, even though no underlying matrix space is present, there is still to be a correspondence between analytic subgroups and Lie subalgebras. To get this correspondence, this chapter uses any Lie group M and its Lie algebra $\mathfrak{m}$ as a frame of reference. These replace the general linear group and the corresponding full matrix space that were ubiquitous in Chapters II and III. The matrix exponential is lost in this generalization, and it needs to be replaced. Introducing the replacement involves making fuller use of the material in Appendix B concerning smooth manifolds and smooth mappings between smooth manifolds.

Section 1 begins the description of the structure that will be used. The section introduces abstract Lie algebras defined over a field, normally taken to be $\mathbb{R}$ in this book, and it gives a number of examples. The section concludes with a brief discussion of Lie algebra homomorphisms.

Section 2 introduces the Lie algebra of an arbitrary Lie group, which can be defined as either the Lie algebra of left invariant vector fields on the group or as the tangent space at the identity, the two being isomorphic as real vector spaces. In the latter case the bracket multiplication of tangent vectors is obtained by extending the given vectors to left invariant vector fields, forming the commutator $XY - YX$ of the vector fields, and passing back to a tangent vector at the identity. In the case of $GL(n, \mathbb{R})$, the Lie algebra of the Lie group reduces to the linear Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, and this sort of identification extends to other Lie groups that can be realized as linear groups.

Section 3 constructs the exponential mapping for an arbitrary Lie group out of integral curves for the left invariant vector fields on the group. In the case of $GL(n, \mathbb{R})$, the exponential mapping coincides with exponentiation of matrices as defined by the usual power series. This identification extends to other Lie groups that can be realized as linear groups.

Section 4 works within the framework of a given Lie group M and its associated Lie algebra $\mathfrak{m}$ of left invariant vector fields. The goal is to associate a Lie subalgebra

of $\mathfrak{m}$ to each "analytic subgroup" of M, i.e., to each subgroup G of M that has the structure of an analytic group for which the inclusion of G into M is smooth. The difficulty is that examples of analytic subgroups of M other than 1 and M itself are not readily apparent. The main theorem of the section relates the inclusion of the analytic subgroup into M and the inclusion of the Lie subalgebra into $\mathfrak{m}$, and it relates the exponential mappings for M and the analytic subgroup in various ways.

Section 5 generalizes to arbitrary Lie groups the construction of Section II.4, which shows how to associate a homomorphism of linear Lie algebras to a smooth homomorphism of linear Lie groups. In the general setting, instead of investigating how curves are mapped by a smooth homomorphism, one uses the derivative mapping on the tangent space at the identity.

Section 6 establishes a formula for the derivative of the exponential map that generalizes the formula in Section III.3 for differentiating an expression $\exp X(t)$ in a matrix group, and then it goes on to prove Dynkin's Formula for general analytic groups.

Section 7 resumes the investigation of an arbitrary subgroup G of a given Lie group M by introducing the "natural topology" on the subgroup, finer than the relative topology as a subgroup of the given Lie group M. The natural and relative topologies are different unless G is a closed subgroup. The natural topology is generated by "basic open neighborhoods" defined in terms of the exponential map. It is shown that when G is given its natural topology, the combined structure of the group and topology is a topological group in which the identity component is open and separable. Then one makes the identity component of G into an analytic group, which is called the "analytic subgroup" associated with G.

Section 8 states the main result of the chapter, which gives a one-one correspondence between the analytic subgroups of M and the Lie subalgebras of the Lie algebra of M. Much of the proof has been given earlier. Section 8 briefly reviews the part handled by earlier sections, and then it gives the final details.

Section 9 extends to arbitrary analytic groups the construction of Section III.6, which shows for linear analytic groups how to lift any homomorphism of their linear Lie algebras uniquely to a smooth homomorphism of the groups provided that the domain is simply connected. A consequence is a uniqueness theorem for simply connected covering groups.

Section 10 proves the existence of a simply connected covering group for any analytic group. In effect the section supplements material in Appendix C by examining the effect of smoothness on universal covering spaces and on topological groups simultaneously.

Section 11 classifies abelian analytic groups up to isomorphism. The simply connected ones are the additive Euclidean groups $\mathbb{R}^n$ for each $n \geq 0$, and the general ones are exactly the products of an additive Euclidean group and a torus.

Section 12 begins by establishing the Closed Subgroup Theorem, which exhibits for each closed subgroup G of the analytic group M a diffeomorphism of the product of G and an open set in a Euclidean space onto an open neighborhood of G in $\boldsymbol{M}$. This result allows for the introduction of a smooth manifold structure on the quotient of an analytic group by a closed subgroup. As a consequence, whenever a Lie group G acts by a smooth transitive action on a smooth manifold $\boldsymbol{M}$, then the natural map of $\boldsymbol{M}/G$ onto $\boldsymbol{M}$ is a diffeomorphism.

Section 13 relates the center and the commutator subgroup of an analytic group to the corresponding objects at the the Lie algebra level. In particular the commutator subgroup is always an analytic subgroup, and its Lie algebra is the commutator subalgebra of the Lie algebra.

1. Lie algebras

Regard $\mathbb{R}$ as a field. There are situations in this book where it might be handy to work with Lie algebras over $\mathbb{C}$ rather than $\mathbb{R}$, but we shall usually downplay them. Some of the situations arise in the examples below and in the Problems at the end of the chapter. An **algebra** $\mathfrak{g}$ over $\mathbb{R}$, not necessarily associative, is a vector space over $\mathbb{R}$ with a product operation $(x, y) \mapsto [x, y]$ carrying $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ that is $\mathbb{R}$ bilinear (i.e., linear in each variable when the other variable is held fixed). An algebra $\mathfrak{g}$ is a **Lie algebra** if the product satisfies

(i) $[x, x] = 0$ for all x in $\mathfrak{g}$, and
(ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all x, y, z in $\mathfrak{g}$. This condition is known as the **Jacobi identity.**

As a consequence of (i), we have $[x, y] = -[y, x]$ for all x and y in $\mathfrak{g}$ because $[x, y] + [y, x] = [x + y, x + y] - [x, x] - [y, y] = 0 - 0 - 0 = 0$.

For any algebra $\mathfrak{g}$, we get a linear transformation $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}_{\mathbb{R}}(\mathfrak{g})$ from the definition

$$(\operatorname{ad} x)(y) = [x, y].$$

In the presence of condition (i) above, condition (ii) is equivalent with the condition

$$[z, [x, y]] = [x, [z, y]] + [[z, x], y]$$

for all x, y, z. In turn this condition is is equivalent with the condition

$$(\operatorname{ad} z)[x, y] = [(\operatorname{ad} z)x, y] + [x, (\operatorname{ad} z)y]$$

for all x, y, z. Any $\alpha \in \operatorname{End}_{\mathbb{R}} \mathfrak{g}$ for which $\alpha[x, y] = [\alpha x, y] + [x, \alpha y]$ for all x and y in $\mathfrak{g}$ is called a **derivation** of $\mathfrak{g}$. What we have just seen is that in the presence of (i), the validity of the Jacobi identity in $\mathfrak{g}$ is equivalent with the fact that every $\operatorname{ad} x$ for x in $\mathfrak{g}$ is a derivation.

Lie algebras often arise from first-order differentiations in some way, and $[x, y]$ represents the commutator of the product under composition. In this case it is 0 if and only if the two differentiations commute. Example 4 of Lie algebras below will illustrate. The list of examples of Lie algebras below contains eight classes of examples.

EXAMPLE 1. Square matrices. For $\mathbb{k}$ equal to $\mathbb{R}$ or $\mathbb{C}$, let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{k})$ be the $\mathbb{k}$ vector space of all n-by-n matrices with entries in $\mathbb{k}$, and define $[X, Y] = XY - YX$. Condition (i) is certainly satisfied, and condition (ii) follows by expanding everything out and canceling terms in pairs:

$$\begin{aligned}
&[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \\
&\qquad = [X, Y]Z - Z[X, Y] + [Y, Z]X - X[Y, Z] + [Z, X]Y - Y[Z, X] \\
&\qquad = XYZ - YXZ - ZXY + ZYX + YZX - ZYX \\
&\qquad\quad - XYZ + XZY + ZXY - XZY - YZX + YXZ \\
&\qquad = 0.
\end{aligned}$$

In Chapters I and II we have referred to the product rule $[X, Y] = XY - YX$ for matrices as **Lie bracket**.

EXAMPLE 2. Associative algebras. Let $\mathfrak{g}$ be any associative algebra, and let $[X, Y] = XY - YX$. This is a generalization of Example 1, and the verification of (i) and (ii) proceeds in the same way.

EXAMPLE 3. Vector product algebra. Let $\mathfrak{g} = \mathbb{R}^3$ with $[x, y] = x \times y$, the **vector product** or **cross product**. The bracket operation on the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is determined by the formulas $[\mathbf{i}, \mathbf{j}] = \mathbf{k}$, $[\mathbf{j}, \mathbf{k}] = \mathbf{i}$, and $[\mathbf{k}, \mathbf{i}] = \mathbf{j}$.

EXAMPLE 4. Smooth vector fields.[1] Let U be a nonempty open subset of $\mathbb{R}^n$, and let $\mathfrak{g}$ be the real vector space of all smooth vector fields on U. If coordinates for $\mathbb{R}^n$ are given as $(x_1, \dots, x_n)$, then the most general element of $\mathfrak{g}$ is a linear combination of the operators $\partial/\partial x_1, \dots, \partial/\partial x_n$ with smooth real-valued functions on U as coefficients. If $X = \sum_{j=1}^{n} a_j(x_1, \dots, x_n) \dfrac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^{n} b_j(x_1, \dots, x_n) \dfrac{\partial}{\partial x_j}$ are two such vector fields, define $[X, Y] = XY - YX$. Use of the product rule for first derivatives gives

[1] Appendix B discusses vector fields on smooth manifolds. A nonempty open subset of $\mathbb{R}^n$, which is what is considered here, is an example of a smooth manifold.

$$\begin{aligned} XYf(x_1,\dots,x_n) &= X\sum_{j=1}^{n} b_j(x_1,\dots,x_n)\,\frac{\partial f}{\partial x_j}(x_1,\dots,x_n) \\ &= \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i,j}\Big(a_i \frac{\partial b_j}{\partial x_i}\Big)\frac{\partial f}{\partial x_j}. \end{aligned}$$

Thus the expression for XYf involves second derivatives. Since $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$, we find that the second derivatives in $XY - YX$ cancel, and we get

$$[X,Y]f = XYf - YXf = \sum_{i,j}\Big(a_i \frac{\partial b_j}{\partial x_i}\Big)\frac{\partial f}{\partial x_j} - \sum_{i,j}\Big(b_i \frac{\partial a_j}{\partial x_i}\Big)\frac{\partial f}{\partial x_j}.$$

The expression for $[X,Y]f$ involves only first partial derivatives, and thus the commutator $[X,Y]$ is indeed a smooth vector field. Condition (i) above is clearly satisfied, and the same computation as with Example 1 shows that condition (ii) is satisfied. Thus the space of all linear combinations of first partial derivatives with coefficients equal to smooth functions on U is a Lie algebra over $\mathbb{R}$.

Example 5. Skew-symmetric matrices. For $\Bbbk$ equal to $\mathbb{R}$ or $\mathbb{C}$, let $\mathfrak{g}$ be the vector space of all skew-symmetric matrices over $\Bbbk$, i.e.,

$$\mathfrak{g} = \{x \in \mathfrak{gl}(n,k) \mid x + x^t = 0\},$$

with the bracket operation inherited from $\mathfrak{gl}(n,\Bbbk)$. To see that $\mathfrak{g}$ is closed under the bracket operation, we simply compute $[x,y]^t = (xy - yx)^t = y^t x^t - x^t y^t = yx - xy = -[x,y]$. For $\Bbbk = \mathbb{R}$, this $\mathfrak{g}$ is written as $\mathfrak{so}(n)$ or $\mathfrak{o}(n)$, and we know it is the Lie algebra of the rotation group $SO(n)$ or the orthogonal group $O(n)$.

Example 6. Generalization of skew-symmetric matrices. For any matrix j in $\mathfrak{gl}(n,\Bbbk)$, let

$$\mathfrak{g} = \{x \in \mathfrak{gl}(n,\Bbbk) \mid jx + x^t j = 0\}.$$

This $\mathfrak{g}$ generalizes Example 5. To check closure under bracket multiplication, observe that

$$[x,y]^t j = (y^t x^t - x^t t^t)j = jyx - jxy = -j[x,y].$$

For $\Bbbk = \mathbb{R}$ and j written in block form as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with p indices in the first block followed by q indices in the second block, the resulting Lie algebra is written $\mathfrak{so}(p,q)$ or $\mathfrak{o}(p,q)$ and equals the Lie algebra of the orthogonal group relative to the bilinear form

$$\langle (x_i),(y_j)\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_j y_j .$$

For $p = 3$ and $q = 1$, the corresponding Lie group is called the **homogeneous Lorentz group**. For $\mathbb{k} = \mathbb{R}$ and n even and $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with each of the four blocks of size $n/2$ by $n/2$, $\mathfrak{g}$ is the Lie algebra of the **real symplectic group**.

EXAMPLE 7. Matrices of trace 0. For $\mathbb{k}$ equal to $\mathbb{R}$ or $\mathbb{C}$, let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{k})$ be the set of all matrices x in $\mathfrak{gl}(n,\mathbb{k})$ with $\mathrm{Tr}(x) = 0$, $\mathrm{Tr}(x)$ being the trace. This $\mathfrak{g}$ is closed under Lie brackets because for any two matrices x and y, $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and consequently $\mathrm{Tr}([x,y]) = 0$. This $\mathfrak{g}$ is the Lie algebra of the group of n-by-n matrices of determinant 1 having entries in $\mathbb{R}$ or $\mathbb{C}$.

EXAMPLE 8. One-dimensional examples of Lie algebras. If $\mathfrak{g}$ is one-dimensonal, then $\mathfrak{g}$ consists of all scalar multiples of a nonzero element, and all brackets are 0 by condition (i) above. Motivated by the examples with $[x,y] = xy - yx$, we say that a Lie algebra is **abelian** if all brackets are 0.

An analysis of Lie algebras in dimensions 2 and 3 appears in the Problems at the end of this chapter.

We introduce the following definitions. A **homomorphism** $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras (over $\mathbb{R}$) is a linear map over $\mathbb{R}$ such that $\varphi([x,y]) = [\varphi(x),\varphi(y)]$ for all x and y in $\mathfrak{g}$. An **isomorphism** is a one-one homomorphism onto. If $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of an algebra whose product is given by $(x,y) \mapsto [x,y]$, then $[\mathfrak{a},\mathfrak{b}]$ means the linear span of all $[x,y]$ with x in $\mathfrak{a}$ and y in $\mathfrak{b}$. A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ with $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. An **ideal** in a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ for which $[\mathfrak{h},\mathfrak{g}] \subseteq \mathfrak{h}$. Again a Lie algebra $\mathfrak{g}$ is **abelian** if $[\mathfrak{g},\mathfrak{g}] = 0$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a Lie algebra, then so are

$\mathfrak{a} + \mathfrak{b}$, the set of sums from the two ideals,
$\mathfrak{a} \cap \mathfrak{b}$, the intersection of the two ideals, and
$[\mathfrak{a},\mathfrak{b}]$, the set of linear combinations of brackets.

The proof that $[\mathfrak{a},\mathfrak{b}]$ is an ideal runs as follows:

$$[\mathfrak{g},[\mathfrak{a},\mathfrak{b}]] \subseteq [[\mathfrak{g},\mathfrak{a}],\mathfrak{b}] + [\mathfrak{a},[\mathfrak{g},\mathfrak{b}]] \subseteq [\mathfrak{a},\mathfrak{b}] + [\mathfrak{a},\mathfrak{b}] = [\mathfrak{a},\mathfrak{b}],$$

the first inclusion following since each $\operatorname{ad} x$ for $x \in \mathfrak{g}$ is a derivation.

Here are three standard examples of ideals in a Lie algebra $\mathfrak{g}$.

Example 1. The **center** $Z(\mathfrak{g})$ of $\mathfrak{g}$, the set of all $x \in \mathfrak{g}$ for which $[x, y] = 0$ for all $y \in \mathfrak{g}$.

Example 2. The **commutator ideal** $[\mathfrak{g}, \mathfrak{g}]$. (This is an ideal since the bracket of any two ideals is an ideal.)

Example 3. The kernel of any homomorphism of Lie algebras, the kernel being the subset of the domain that is mapped into 0. We write $\ker \varphi$ for the kernel of a homomorphism φ. More generally if $\varphi \colon \mathfrak{g} \to \mathfrak{a}$ is a homomorphism of Lie algebras and if $\mathfrak{h}$ is an ideal in $\mathfrak{a}$, then $\varphi^{-1}(\mathfrak{h})$ is an ideal in $\mathfrak{g}$.

If $\mathfrak{g}$ is a Lie algebra, let us define inductively $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0]$, and $\mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$ for $j \geq 0$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \cdots$$

is a decreasing sequence of ideals in $\mathfrak{g}$ and is called the **commutator series** of $\mathfrak{g}$. We say that $\mathfrak{g}$ is **solvable** if the commutator series ends in 0.

2. Lie algebra of a Lie group

The Lie algebra of a Lie group will be the same as the Lie algebra of the analytic group that is its identity component, and we define the Lie algebra of an analytic group now. In doing so, we make use of material in Appendix B on vector fields and integral curves, and we shall take that material as known.

Let G be an analytic group. For each $g \in G$, let $L_g \colon G \to G$ be left translation by g, namely $L_g(x) = gx$. From the definition of analytic group, L_g is a diffeomorphism of G.

A vector field X on G is said to be **left invariant** if it has the property that

$$(DL_g)_x(X_x) = X_{gx} \tag{4.1}$$

for all g and x in G. Let us unpack this definition a little, starting from the definitions as given in Section B.2. It will be helpful to avoid introducing coordinates, since writing out the condition of left invariance in coordinates in any nontrivial example is likely to be complicated and not very enlightening.

Germs are defined in Section B.3. According to the definition at the beginning of Section B.3, the left side of (4.1) when applied to a germ h at gx, is given by

$$((DL_g)_x(X_x))(h) = X_x(h \circ L_g). \tag{4.2}$$

Meanwhile the right side of (4.1) applied to h is just $X_{gx}(h)$. Thus the question of left invariance for the vector field X is whether

$$X_x(h \circ L_g) \stackrel{?}{=} X_{gx}(h) \tag{4.3}$$

for all group elements g and x and all germs h at gx.

In more detail the vector field X has two variables to specify, a germ variable and a group variable. For the left side of (4.3), the germ variable is hL_g, and the group variable is x. For the right side of (4.3), the germ variable is h, and the group variable is $L_g x$. The question asked by (4.3) is whether a certain associativity formula holds, saying that (hL_g, x) and $(h, L_g x)$ lead to the same result.

The role of L_g in (4.3) is as a diffeomorphism of G. One can ask more generally, what is the condition for a smooth vector field X on G to be invariant under a given diffeomorphism F of G. The answer similarly is the condition that $X(f \circ F) = X(f) \circ F$ for all smooth functions f on G.

Proposition 4.1. If G is an analytic group, then the map $X \mapsto X_1$ is a vector space isomorphism of the vector space of left invariant vector fields of G onto the tangent space $T_1(G)$ at the identity, and the inverse map is given by $X_1 \mapsto (DL_g)_1(X_1)$. Every left invariant vector field on G is smooth, and the bracket of two left invariant vector fields is left invariant.

Proof. The map $X \mapsto X_1$ is certainly linear. If X is any left invariant vector field, then (4.1) with $x = 1$ says that X_g must equal $(DL_g)_1(X_1)$. Thus if X_1 is given, we use the formula $X_g = (DL_g)_1(X_1)$ to define the vector field at all points g. To check left invariance of this X, we check that (4.1) holds. We have

$$\begin{aligned}
(DL_g)_h(X_h) &= (DL_g)_h(DL_h)_1(X_1) \\
&= (D(L_g L_h))_1(X_1) && \text{by the Chain Rule} \\
&= (DL_{gh})_1(X_1) && \text{since } L_g L_h = L_{gh} \\
&= X_{gh},
\end{aligned}$$

and indeed (4.1) holds.

For the smoothness of any given left invariant vector field X, let f be in $C^\infty(G)$, and let $c\colon (-\varepsilon, \varepsilon) \to G$ be a smooth curve with $c(0) = 1$ and $c'(0) = X$. Then the value of X on f at an arbitrary point x of G is given by

$$\begin{aligned}
X_x(f) &= (DL_x)_1(X_1) = X_1(f \circ L_x) \\
&= \frac{d}{dt}\bigg|_{t=0} f(xc(t)) = \frac{d}{dt}\bigg|_{t=0} f(\mu(x, c(t))),
\end{aligned}$$

where $\mu\colon G\times G\to G$ is the multiplication map. Thus $x\mapsto X_x(f)$ is exhibited as the composition of smooth functions and is smooth. For closure under brackets suppose that X and Y are left invariant. By (4.3), $X_x(h\circ L_g)=X_{gx}(h)$ and $Y_x(h\circ L_g)=Y_{gx}(h)$. Hence $[X,Y]$ satisfies

$$\begin{aligned}[X,Y](h\circ L_g) &= X(h\circ L_g)Y(h\circ L_g)-Y(h\circ L_g)X(h\circ L_g)\\ &= X_{gx}(h)Y_{gx}(h)-Y_{gx}(h)X_{gx}(h)\\ &= [X_{gx},Y_{gx}](h)\\ &= [X,Y]_{gx}(h),\end{aligned}$$

and $[X,Y]$ is left invariant, again in view of (4.3). □

Consequently the space of left invariant vector fields on an analytic group G is a Lie algebra. This Lie algebra is called the **Lie algebra of the analytic group**.

For the case of a linear analytic group, we now have two ways of associating a Lie algebra to the group—one from the theory in Section I.4 concerning linear groups and the other from just now concerning all analytic groups. The following result shows how those two descriptions for a linear group come to the same thing. Matters come down to writing the isomorphism explicitly for $GL(n,\mathbb{R})$ or for its identity component.

Proposition 4.2. Let $G=GL(n,\mathbb{R})$, let $\mathfrak{g}$ be its Lie algebra of left invariant vector fields, and let e_{ij} be the $(i,j)^{\text{th}}$ entry function. Then the map $\varphi\colon \mathfrak{g}\to\mathfrak{gl}(n,\mathbb{R})$ given by $\varphi(X)_{ij}=Xe_{ij}$ is a Lie algebra isomorphism of the Lie algebra of left invariant vector fields onto the linear Lie algebra.

Remark. Similarly the Lie algebra of left invariant vector fields for $GL(n,C)$ is isomorphic as a real Lie algebra to $\mathfrak{gl}(n,\mathbb{C})$.

Proof. The map φ is linear. It is one-one bcause $\{e_{ij}\}$ is a local coordinate system near the identity matrix. Since φ is one-one and $\dim\mathfrak{g}=\dim\mathfrak{gl}(n,\mathbb{R})$, φ is onto. We need to check that the brackets correspond. Let $\widetilde{X}$ be the left invariant vector field with $\widetilde{X}_1=X$. Then we have

$$e_{ij}(x)=x_{ij},$$

$$L_xe_{ij}(y)=e_{ij}(xy)=\sum_k e_{ik}(x)e_{kj}(y),$$

and

$$\widetilde{X}e_{ij}(x) = X(L_x e_{ij}) = \sum_k e_{ik}(x) X e_{kj} = \sum_k e_{ik}(x)\varphi(X)_{kj}.$$

Hence

$$\begin{aligned}\widetilde{Y}\widetilde{X}e_{ij}(x) &= Y\big(L_x(\widetilde{X}e_{ij})\big)\\ &= Y\Big(\sum_{k,\ell} e_{i\ell}(x)e_{\ell k}(y)\varphi(X)_{kj}\Big)\\ &= \sum_{k,\ell} e_{i\ell}(x)\varphi(Y)_{\ell k}\varphi(X)_{kj}.\end{aligned}$$

Interchanging X and Y, subtracting the results, and evaluating expressions at $x = 1$, we obtain

$$\begin{aligned}\sum_k \delta_{ik}\varphi([X,Y])_{kj} &= \widetilde{[X,Y]}e_{ij}(1)\\ &= [\widetilde{X},\widetilde{Y}]e_{ij}(1)\\ &= \sum_{k,\ell}\delta_{i\ell}\big(\varphi(X)_{\ell k}\varphi(Y)_{kj} - \varphi(Y)_{\ell k}\varphi(X)_{kj}\big).\end{aligned}$$

In other words, $\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) = [\varphi(X),\varphi(Y)]$. □

3. Exponential mapping for a Lie group

Our frame of reference for the remainder of this chapter will be a fixed analytic group M and its Lie algebra $\mathfrak{m}$ as defined in Section 2. Within that frame of reference, we shall study subgroups of M and their Lie algebras. The prototype is the situation in Chapters I to III, in which M was a general linear group (or its identity component) of a particular size. Then $\mathfrak{m}$ was the full matrix space of that size. Our interest will be in analytic subgroups G of M and their Lie algebras $\mathfrak{g}$, viewed as Lie subalgebras of $\mathfrak{m}$. Accordingly we shall first develop the exponential map for an analytic group M with Lie algebra $\mathfrak{m}$.

Let M be an analytic group, and let $\mathfrak{m}$ be its Lie algebra. In accordance with Proposition 4.1, we can regard the members of $\mathfrak{m}$ either as tangent vectors at the identity or as left invariant vector fields on M, whichever is more convenient. Often we shall write X and Y for tangent vectors and $\widetilde{X}$ and $\widetilde{Y}$ for the corresponding left invariant vector fields.

Theorem 4.3. If M is an analytic group with Lie algebra $\mathfrak{m}$, then there exists a unique mapping $\exp\colon \mathfrak{m} \to M$ such that for each X in $\mathfrak{m}$, the function $t \mapsto \exp tX$ for $-\infty < t < \infty$ is an integral curve for the left invariant vector field $\widetilde{X}$ with $\exp 0 = 1$. Moreover exp is smooth, exp satisfies

$$\exp(s+t)X = \exp sX \exp tX \qquad \text{for all real } s \text{ and } t,$$

and exp is a diffeomorphism from any sufficiently small open neighborhood of 0 in $\mathfrak{m}$ onto an open neighborhood of 1 in M. Consequently $\exp^{-1}$ defines a chart about the identity of M.

Remark. The last sentence of the statement, the one about the chart, means the following: If $X_1, \dots, X_n$ is a vector space basis of $\mathfrak{m}$, then

$$x_j(\exp(\lambda_1 X_1 + \cdots + \lambda_n X_n)) = \lambda_j$$

is a system of local coordinates about 1 in M. These are called **canonical coordinates**.

Proof. Fix a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ and a bounded open ball V about 0 in $\mathbb{R}^n$. For λ in V, put $X_\lambda = \sum_{j=1}^{n} \lambda_j X_j$, and apply the second variant of Corollary B.16 of Appendix B to obtain the following conclusion: Let U be the ball about 0 in $\mathfrak{m}$ given by $U = \{X_\lambda \mid \lambda \in V\}$. Then there exist an $\varepsilon > 0$ and a system of curves $c(t,x)$ defined for $-\varepsilon < t < \varepsilon$ and X in U such that $c(\,\cdot\,, X)$ is an integral curve for $\widetilde{X}$ with $c(0,X) = 1$. Each curve $t \mapsto c(t,X)$ is unique, and $c\colon (-\varepsilon,\varepsilon) \times U \to M$ is smooth. If $0 < \delta < \varepsilon$, then $c(\delta, \cdot)$ is a diffeomorphism from an open subneighborhood of 0 in $\mathfrak{m}$ onto an open subset of M. Let

$$t_0 = \sup_{\varepsilon > 0} \big\{\varepsilon \,\big|\, \mapsto c(t,X) \text{ is an integral curve for } \widetilde{X} \text{ for } |t| < \varepsilon \text{ for all } X \text{ in } U\big\}.$$

First we shall show that $t_0 = +\infty$ and that

$$c(t,X)c(s,X) = c(s+t,X) \qquad \text{for } s \text{ and } t \text{ in } \mathbb{R} \text{ and } x \text{ in } U. \tag{4.4}$$

The defining condition on c is that

$$Dc_{t,X}\Big(\frac{d}{dt}\Big) = \widetilde{X}_{c(t,X)}.$$

Let $|s| < t_0$ and $|t| < t_0$. Then with t fixed, the map of $s \mapsto c(t,X)c(s,X)$ is a curve with

$$\begin{aligned} D\big(c(t,X)c(\,\cdot\,,X)\big)_{s,X}\Big(\frac{d}{ds}\Big) &= D\big(L_{c(t,X)}c(\,\cdot\,,X)\big)\Big(\frac{d}{ds}\Big) \\ &= (DL_{c(t,X)})_{c(s,X)}Dc_{s,X}\Big(\frac{d}{ds}\Big) \\ &= (DL_{c(t,X)})_{c(s,X)}\widetilde{X}_{c(s,X)} \\ &= \widetilde{X}_{c(s,X)} \qquad\qquad \text{by left invariance} \end{aligned}$$

and with

$$c(t,X)c(s,X)\big|_{s=0} = c(t,X).$$

But $s \mapsto c(s+t,X)$ is another such curve if $|s+t| < t_0$, and so (4.4) holds for such s and t. If we define $c(s+\frac{1}{2}t_0,X)$ to be $c(\frac{1}{2}t_0,X)c(s,X)$ for $|s| < t_0$ and if we define $c(s-\frac{1}{2}t_0,X)$ to be $c(-\frac{1}{2}t_0)c(s,X)$, then we see that $s \mapsto c(s,X)$ is an integral curve for $|s| < \frac{3}{2}t_0$. Hence $t_0 = +\infty$. In addition, the two sides of (4.4), as functions of t, are integral curves for all t, and thus (4.4) holds for all s and t.

Now with (4.4) completely established, we shall show that if X is in U and if $0 < \delta < 1$, then

$$c(\delta t,X) = c(t,\delta X). \tag{4.5}$$

Write $\delta(\,\cdot\,)$ for the operation of multiplication by the number δ. Then $c_X \circ \delta$ is a smooth curve with $c_X \circ \delta(0) = 1$ and

$$D(c_X\circ\delta)_t\Big(\frac{d}{dt}\Big) = Dc_{\delta t,X}D\delta_t\Big(\frac{d}{dt}\Big) = \delta Dc_{\delta t,X}\Big(\frac{d}{dt}\Big) = \delta\widetilde{X}_{c(\delta t,X)} = \widetilde{\delta X}_{c(\delta t,X)}.$$

In short, $c_X \circ \delta$ is an integral curve for $\widetilde{\delta X}$. Since $c_{\delta X}$ is another such curve and both curves have initial value 1, we conclude that $c_X \circ \delta = c_{\delta X}$. In other words, (4.5) follows.

Finally we can define exp. Let X be in $\mathfrak{m}$ and choose a positive integer K large enough so that $\frac{1}{K}X$ is in U. Define $\exp X = c(K,\frac{1}{K}X)$. This definition is unambiguous by (4.5), and $\exp 0 = 1$. Also $\exp tX = c_{\frac{1}{K}X} \circ K(t)$ is an integral curve for $\widetilde{X}$ by the same computation as above, and

$$\exp(s+t)X = \exp sX \exp tX$$

as a consequence of (4.4). In addition, exp is smooth on KU because it is a composition involving the three maps $X \in KU \mapsto \frac{1}{K}X$, $Y \in U \mapsto c(1,Y)$, and $x \in \boldsymbol{M} \mapsto x^K$. Specifically

$$\exp X = c\Big(K,\frac{1}{K}X\Big) = c\Big(1,\frac{1}{K}X\Big)^K$$

by (4.5). Hence exp is smooth on $\boldsymbol{M}$. The map exp is a diffeomorphism onto an open set for some ball about 0 in $\mathfrak{g}$ since for small X,

$$\exp X = c\left(\frac{\varepsilon}{2}, \frac{2}{\varepsilon}X\right)$$

is a diffeomorphism. □

Proposition 4.4 (Taylor's Theorem). Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$. If X is in $\mathfrak{m}$, if $\widetilde{X}$ is the left invariant vector field corresponding to X, and if f is in $C^\infty(\boldsymbol{M})$, then

$$(\widetilde{X}^n f)(g \exp tX) = \frac{d^n}{dt^n}(f(g\exp tX)) \qquad \text{for all } g \in \boldsymbol{M}.$$

Moreover for X in any bounded set,

$$f(\exp X) = \sum_{k=0}^{n} \frac{1}{k!}(\widetilde{X}^k f)(1) + R_n(X),$$

where $|R_n(X)| \leq C_n |X|^{n+1}$.

Proof. For the first statement it is enough (by induction) to handle the case $n = 1$. Since $t \mapsto \exp tX$ is an integral curve for $\widetilde{X}$,

$$(\widetilde{X} f)(\exp tx) = \widetilde{X}_{\exp tX} f = D(\exp tX)_t\left(\frac{d}{dt}\right) f = \frac{d}{dt}(f(\exp tX)).$$

For the statement about the remainder, we expand $t \mapsto f(\exp tX)$ in classical Taylor series with integral remainder about $t = 0$ and evaluate at $t = 1$. Then

$$\begin{aligned} f(\exp X) &= \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{d}{dt}\right)^k (f(\exp tX))\Big|_{t=0} \\ &\qquad + \frac{1}{n!}\int_0^1 (1-s)^n \left(\frac{d}{ds}\right)^{n+1} (f(\exp sX)\, ds \\ &= \sum_{k=0}^{n} \frac{1}{k!}(\widetilde{X}^k f)(1) + \frac{1}{n!}\int_0^1 (1-s)^n (\widetilde{X}^{n+1} f)(\exp sX)\, ds. \end{aligned}$$

In the second term on the third line, set $X = \sum \lambda_j X_j$ and expand $\widetilde{X}^{n+1}$. Since X lies in a bounded set, so does $\exp sX$, and the integral is dominated by $|\lambda|^{n+1}$ times a harmless integral. This proves the statement about the remainder. □

Corollary 4.5. If X is in $\mathfrak{m}$ and f is in $C^\infty(\boldsymbol{M})$, then the left invariant vector field $\widetilde{X}$ corresponding to X has

$$(\widetilde{X} f)(g) = \left(\frac{d}{dt}\right) f(g \exp tX)\Big|_{t=0}.$$

PROOF. Take $n = 1$ in Proposition 4.4 applied to $x \mapsto f(gx)$, and then put $t = 0$. $\square$

For the case of a linear analytic group, we now have two ways of associating an exponential mapping to the group—one from the theory in Section I.4 concerning linear groups and the other from just now concerning all analytic groups. Proposition 4.6 below shows how the two ways come to the same thing. The proposition describes the correspondence explicitly for $GL(n, \mathbb{R})$ (or its identity component). The correspondence then follows for subgroups of $GL(n, \mathbb{R})$ because the formula for the matrix exponential is unchanged and the way that the group exponential is determined by integral curves and left invariant vector fields is unchanged.

Proposition 4.6. In $GL(n, \mathbb{R})$ if the Lie algebra of the group is identified with the Lie algebra of matrices $\mathfrak{gl}(n, \mathbb{R})$, then the effect of the exponential mapping as defined by integral curves in Theorem 4.3 matches the effect of the mapping $X \mapsto e^X$ on matrices.

REMARKS. A similar conclusion is valid for $GL(n, \mathbb{C})$ with substantially the same proof.

PROOF. If X is in $\mathfrak{gl}(n, \mathbb{R})$, then $\widetilde{X} e_{ij} = X_{ij}$ by definition, and

$$\widetilde{X} e_{ij}(x) = \sum_{k=1}^{n} x_{ik} X_{kj}.$$

So by Proposition 4.4 with f taken to be e_{ij},

$$\frac{d}{dt}\big(\exp tX\big)_{ij} = \widetilde{X} e_{ij}(\exp tX) = \sum_k (\exp tX)_{ik} X_{kj} = ((\exp tX)X)_{ij}.$$

Consequently $Y(t) = \exp tX$ satisfies

$$\frac{dY(t)}{dt} = Y(t)X \quad \text{with} \quad Y(0) = 1.$$

This equation and initial condition are satisfied also by e^{tX}, and hence $e^{tX} = \exp tX$ by the standard existence-uniqueness theorem for systems of first-order linear differential equations (Theorem B.16). To complete the proof, one has only to put $t = 1$. $\square$

4. Lie algebra of an analytic subgroup

Bear in mind that even in the linear case, an analytic subgroup typically does not get its topology and analytic structure by restriction of the structure of the whole analytic group. This circumstance argues for being able to work with analytic subgroups before knowing how to create them. For linear groups we were able to accomplish this miracle because of Theorem 2.1, which worked with arbitrary linear groups. A suitable generalization of Theorem 2.1 has not been proved without the assumption of linearity, and we shall have to develop a theory for objects that we do not initially do not know how to identify.

How then shall we approach the notion of an analytic subgroup? A clue lies in Fundamental Theorem A of Lie Theory as announced in the Preface. According to that theorem, we expect that an analytic subgroup corresponds to a unique Lie subalgebra and that the elements of the analytic subgroup are generated by the exponentials of the members of the Lie subalgebra. With the elements of an analytic subgroup known, it becomes a question of introducing a topology and a manifold structure.

Thus in order to work with analytic subgroups, we want first to identify their Lie algebras. In turn, before we introduce the Lie algebra of an analytic subgroup, two further tools are needed. One is for working with smooth vector fields, and the other is for working with one-parameter subgroups.

For the first tool the underlying question concerning smooth vector fields is how they behave under smooth mappings. Although smooth mappings carry tangent vectors to tangent vectors, they do not necessarily carry smooth vector fields to smooth vector fields. There is no problem if the smooth mapping is a diffeomorphism, but our interest will be in a different class of smooth mappings. Let us capture a certain kind of behavior in a definition.

Let $F : A \to B$ be any smooth map of one smooth manifold to another. If X and Y are smooth vector fields on A and B, respectively, one says that X and Y are F **related** if

$$F_p(X_p) = Y_{F(p)} \qquad \text{for all } p \in A. \tag{4.6}$$

This equation says that two particular vector fields are equal, one on A and one on B; one way of checking it is to verify that the two sides yield the same result on all smooth functions h of compact support on B. Acting on h, the left side of this equation is $X_p(h \circ F)$, and the right side is $Y_{F(p)}h$. The definition of F related thus asks that they be equal:

$$X_p(h \circ F) = Y_{F(p)}h. \tag{4.7}$$

The result we need, stated in the proposition below, is that this equality respects brackets.

Proposition 4.7. Let F be a smooth map of a smooth manifold A into a smooth manifold B, let X_1 and X_2 be smooth vector fields on A, and let Y_1 and Y_2 be smooth vector fields on B. If X_1 and Y_1 are F related and X_2 and Y_2 are F related, then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F related.

PROOF. For each p in A, equation (4.7) insists that

$$X_i(h \circ F)(p) = Y_i h(F(p)) = Y_i h \circ F(p)$$

for all $h \in C^\infty(B)$ and for i equal to 1, 2. Hence

$$Y_1 Y_2 h(F(p)) = X_1(Y_2 h \circ F)(p) = X_1(X_2(h \circ F))(p).$$

A similar computation works for $Y_2 Y_1$, and subtraction gives

$$([Y_1, Y_2]h) \circ F = [X_1, X_2](h \circ F).$$

Taking into account the equivalence of (4.6) and (4.7), we obtain $[Y_1, Y_2]_p = (DF)_p [X_1, X_2]_p$, as required. □

Our chief interest will be in the case that $F : A \to B$ is the inclusion i of a submanifold S in the manifold B. The smooth mapping to consider is thus $i : S \to B$. The tangent space to S at a point p is $T_p(S)$, and the condition for smooth vector fields X on S and Y on B to be i related is that $i_p(X_p) = Y_{i(p)}$ for all $p \in S$. In this case the vector field Y on S is called the **contraction** of X to S. The following result, which is what we need, is immediate from Proposition 4.7.

Corollary 4.8. If S is a submanifold of a smooth manifold B, then the contraction to S of the bracket of two smooth vector fields on B is the bracket of their contractions.

Corollary 4.8 is the first tool that we need in order to introduce the Lie algebra of an analytic subgroup. The second tool is a classification of smooth homomorphisms of $\mathbb{R}$ into analytic groups. That classification is as follows.

Proposition 4.9. Any continuous homomorphism φ of $\mathbb{R}$ into an analytic group H with Lie algebra $\mathfrak{h}$ is of the form $\varphi(t) = \exp tX$ for some $X \in \mathfrak{h}$. In particular, φ is smooth from $\mathbb{R}$ into H.

PROOF. Let U be any convex open neighborhood of 0 in $\mathfrak{h}$ small enough so that exp is a diffeomorphism of U onto an open subset V of H. We show that every element of V has one and only one square root in V. If x is in V, then $x = \exp X$ for some unique X in U. Thus x has $\exp\left(\frac{1}{2}X\right)$ as a square root, and $\frac{1}{2}X$ is in U since U is assumed convex. This square root of x lying in V is in fact unique. Indeed if r is any square root of x lying in the open set V, then write $r = \exp Y$ with $Y \in U$. Squaring gives $\exp X = x = r^2 = \exp 2Y$. Since exp is one-one on U, $X = 2Y$. Therefore $r = \exp Y = \exp\left(\frac{1}{2}X\right)$, and $\exp\left(\frac{1}{2}X\right)$ is the unique square root of x in V.

By continuity of φ, fix $r_0 > 0$ small enough so that $\varphi(r_0)$ is in V and therefore is of the form $\exp Y$ for some Y in U. Let us see by induction on n that for every positive integer n, $\varphi(2^{-n}r_0) = \exp 2^{-n}Y$. The base case of the induction is $n = 0$, and that case has been settled by construction. For the inductive step with $n \geq 1$, $\varphi\big(2^{-n}r_0\big)$ has square $\varphi\big(2^{-n+1}r_0\big)$, and $\exp 2^{-n}Y$ has square $\exp 2^{-(n-1)}Y$. By induction and the uniqueness of square roots proved in the previous paragraph, $\varphi\big(2^{-n}r_0\big) = \exp 2^{-n}Y$ for all integers $n \geq 0$.

Since φ is a homomorphism, raising both sides of the equality $\varphi\big(2^{-n}r_0\big) = \exp 2^{-n}Y$ to any positive power shows that $\varphi(sr_0) = \exp sY$ for every rational $s > 0$ whose denominator is a power of 2. Taking inverses yields the same conclusion $\varphi(sr_0) = \exp sY$ for every rational s, positive or negative, whose denominator is a power of 2. Using continuity of φ and passing to the limit, we see that

$$\varphi(sr_0) = \exp sY \tag{4.8}$$

for every real s.

Finally if we set $Y = r_0X$ in (4.8), then we find that

$$\varphi(sr_0) = \exp sY = \exp sr_0X.$$

Putting $t = sr_0$ shows that $\varphi(t) = \exp tX$ for all real r, as required. □

Now we shall use these two devices to make a definition and begin work with the Lie algebra of an analytic subgroup.

We continue with the notation that $\boldsymbol{M}$ is an analytic group and $\mathfrak{m}$ is its Lie algebra, namely the tangent space at the identity of $\boldsymbol{M}$. If X is in $\mathfrak{m}$, we write $\widetilde{X}$ for the corresponding left invariant vector field on $\boldsymbol{M}$.

An **analytic subgroup** G of $\boldsymbol{M}$ is a subgroup of $\boldsymbol{M}$ with the structure of an analytic group such that the inclusion mapping $i : G \to \boldsymbol{M}$ is smooth. Its **Lie algebra** is the Lie algebra of all left invariant vector fields on G.

Theorem 4.10.

(a) If $i : G \to \boldsymbol{M}$ is the inclusion of an analytic subgroup G into the analytic group $\boldsymbol{M}$ and if G has Lie algebra $\mathfrak{g}$, then the induced mapping of tangent spaces

$$\mathfrak{g} = T_1(G) \longrightarrow \widetilde{T}_1(G) \subseteq \widetilde{T}_1(\boldsymbol{M}) = \mathfrak{m}$$

is a one-one Lie algebra homomorphism. Consequently $\widetilde{T}_1(G)$ can be regarded as the Lie algebra of G.

(b) If G is an analytic subgroup of $\boldsymbol{M}$ with Lie algebra $\mathfrak{g}$, if $i_G : G \to \boldsymbol{M}$ and $i_\mathfrak{g} : \mathfrak{g} \to \mathfrak{m}$ are the respective inclusions of groups and Lie algebras, and if $\exp_G$ and $\exp_{\boldsymbol{M}}$ are the respective exponential maps for G and $\boldsymbol{M}$, then $i_G \circ \exp_G = \exp_{\boldsymbol{M}} \circ i_\mathfrak{g}$ as mappings of $\mathfrak{g}$ onto $\boldsymbol{M}$.

(c) If G is an analytic subgroup of $\boldsymbol{M}$ with Lie algebra $\mathfrak{g}$ and if X is in $\mathfrak{m}$, then X is in $\mathfrak{g}$ if and only if $\exp_{\boldsymbol{M}} tX$ is in G for all $t \in \mathbb{R}$.

Proof of (a). Let $i : G \to \boldsymbol{M}$ be inclusion, let ℓ_g be left translation by g in G, and let L_g be left translation by g in $\boldsymbol{M}$. Since ℓ_g is a diffeomorphism of G, $T_g(G) = (D\ell_g)_1(T_1(G))$. Therefore

$$\begin{aligned}\widetilde{T}_g(G) &= (Di)_g(D\ell_g)_1T_1(G) = D(i \circ \ell_g)_1T_1(G) = D(L_g \circ i)_1T_1(G)\\ &= (DL_g)_1(Di)_1T_1(G) = (DL_g)_1\widetilde{T}_1(G).\end{aligned}$$

If X is in $\widetilde{T}_1(G)$, then it follows that $\widetilde{X}_g$ is in $\widetilde{T}_g(G)$, and hence $\widetilde{X}$ has a contraction to G. The contraction is left invariant, and brackets contract to brackets by Corollary 4.8. Then (a) follows. □

Proof of (b). Let X be in $\mathfrak{g}$, let $\widetilde{X}$ be the corresponding vector field on $\boldsymbol{M}$, and let $\widetilde{X}^G$ be its contraction to G. Then

$$\begin{aligned}D(i_G \circ \exp_G tX)_t(d/dt) &= (Di_G)D(\exp_G tX)(d/dt)\\ &= (Di_G)\widetilde{X}^G_{\exp_G tX} = \widetilde{X}_{\exp_G tX},\end{aligned}$$

and thus $i_G \circ \exp_{\boldsymbol{M}} tX$ is an integral curve for $\widetilde{X}$ with value at 0 equal to 1. Also $\exp_{\boldsymbol{M}} \circ i_\mathfrak{g}$ is another such integral curve, and they must be equal by the uniqueness of integral curves. This proves (b). □

Proof of (c). Apply the formula of (b) to tX with X in $\mathfrak{g}$. The left side, namely $i_G \circ \exp_G(tX)$, is in the subgroup G of $\boldsymbol{M}$; hence so is the right side, namely $\exp_{\boldsymbol{M}} \circ i_\mathfrak{g}(tX)$. Thus $\exp_{\boldsymbol{M}} tX$ is in G. Conversely suppose that an element X of $\mathfrak{m}$ has the property that $\exp_{\boldsymbol{M}} tX$ is in G for all $t \in \mathbb{R}$. Then the mapping $t \mapsto \exp_{\boldsymbol{M}}(tX)$ is a continuous homomorphism of $\mathbb{R}$ into G. By Proposition 4.9 it is of the form $t \mapsto \exp_G tZ$ for some $Z \in \mathfrak{g}$. Thus $\exp_G tZ = \exp_{\boldsymbol{M}}(tX)$, and $X = i_G(Z)$, as required. □

5. Passage from homomorphisms of analytic subgroups to homomorphisms of Lie subalgebras

The passage from smooth homomorphisms of linear analytic groups to homomorphisms of their Lie algebras was treated in Section II.4. In that section we proceeded by composing smooth curves through the identity in the domain with the smooth homomorphism of groups.

In this section we consider the corresponding question for Lie groups that are not necessarily linear. To handle the enlarged framework, we shall make serious use of the material in Appendix B concerning smooth manifolds and smooth mappings between smooth manifolds. The investigation of how curves are mapped by a smooth homomorphism will be replaced by an investigation of the derivative of the smooth homomorphism at the identity, which is a linear map between the tangent spaces at the identity.

Thus let G and G' be Lie groups, let $\mathfrak{g}$ and $\mathfrak{g}'$ be their respective Lie algebras, and let $\pi\colon G \to G'$ be a smooth homomorphism. Then the derivative of π at the identity, which Appendix B might write as $(D\pi)_1$, is a linear mapping from $\mathfrak{g}$ into $\mathfrak{g}'$, i.e., a linear map between the tangent spaces of G and G' at the identity. We write $\operatorname{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{g}')$ for the vector space of all such linear maps. In this section we shall prove that $(D\pi)_1$ is in fact a Lie algebra homomorphism.

The proof that $(D\pi)_1$ is a Lie algebra homomorphism comes after two preliminary steps. The first step is to handle the linear case. Thus we suppose that G and G' are subgroups of respective general linear groups $\boldsymbol{M}$ and $\boldsymbol{M}'$ with Lie algebras $\mathfrak{m}$ and $\mathfrak{m}'$. We are given a smooth homomorphism $\pi\colon G \to G'$. According to the linear theory of Chapters I–III, whenever we have a smooth curve $c(t)$ in G with $c(0) = 1$, it gets mapped to the smooth curve $\pi(c(t))$, and the version of π on the Lie algebra level in the linear theory, call it $E\pi$, has to carry $c'(0)$ to $(\pi \circ c)'(0)$. Thus

$$(E\pi)(c'(0)) = (\pi \circ c)'(0).$$

On the other hand, the Euclidean Chain Rule gives

$$(\pi \circ c)'(0) = (D\pi)_{c(0)} c'(0) = (D\pi)_1(c'(0)),$$

and thus $(E\pi)(c'(0)) = (D\pi)_1(c'(0))$. Since $c'(0)$ is arbitrary, $E\pi = (D\pi)_1$. In other words taking the derivative at the identity of a smooth homomorphism between Lie groups leads to the same result in the linear case as the recipe in Section II.4.

The second preliminary step is to handle the "adjoint representation." Each g in G gives rise to a group isomorphism $\mathbf{Ad}(g)$ of G into itself given by $\mathbf{Ad}(g)(x) = gxg^{-1}$. This isomorphism is smooth, and it therefore has a derivative mapping at the identity, which is a member of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. We write $D(\mathbf{Ad}(g)) = \mathrm{Ad}(g)$ for this derivative mapping. It is immediate from the associative law in G that

$$\mathbf{Ad}(gh) = \mathbf{Ad}(g)\mathbf{Ad}(h),$$

and it follows from the Chain Rule that

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h),$$

i.e., Ad is a smooth homomorphism into the general linear group on $\mathfrak{g}$, which we refer to as $GL(\mathfrak{g})$. The smooth homomorphism $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is called the **adjoint representation** of G on $\mathfrak{g}$. After some preparation we shall prove that $(D\,\mathrm{Ad})_1 = \mathrm{ad}$, where $\mathrm{ad}\,X$ means "left bracket by X," as usual.

Proposition 4.11. If $\pi\colon G \to G'$ is a smooth homomorphism between Lie groups, then $\pi \circ \exp = \exp \circ (D\pi)_1$.

PROOF. If X is in $\mathfrak{g}$, then $t \mapsto \pi(\exp tX)$ is a smooth curve in G' whose tangent at 0 is $(D\pi)_1(X)$. It is also a one-parameter subgroup of G' since π is a homomophism. However, $t \mapsto \exp t(D\pi)_1(X)$ is the unique one-parameter subgroup of G' whose tangent at 0 is $(D\pi)_1(X)$, by Theorem 4.3. Thus $\pi(\exp tX) = \exp t(D\pi)_1(X)$ for all t. The proposition follows by taking $t = 1$. $\square$

Corollary 4.12. If G is a Lie group with Lie algebra $\mathfrak{g}$, then

$$g(\exp X)g^{-1} = \exp \mathrm{Ad}(g)X \tag{4.9}$$

for all $X \in \mathfrak{g}$ and $g \in G$.

PROOF. Apply Proposition 4.11 with $\pi = \mathbf{Ad}$. $\square$

We shall identify $(D\,\mathrm{Ad})_1$ as the operation $X \mapsto \mathrm{ad}\,X$ of left-bracket-by in $\mathfrak{g}$. The argument is a little bit subtle. To get started, we make use of how some curves in $\mathfrak{g}$ behave near 0.

Lemma 4.13. If G is a Lie group with Lie algebra $\mathfrak{g}$ and dimension d, if $\exp\colon \mathfrak{g} \to G$ is the exponential mapping, and if X and Y are members of $\mathfrak{g}$, then for $|t|$ small and for $O(t^3)$ representing a quantity that remains bounded when divided by t^3 and t tends to 0, the following equations are valid:

$$\exp tX \exp tY = \exp(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3), \tag{4.10}$$

$$\exp(-tX)\exp(-tY)\exp tX\exp tY = \exp\big(t^2[X,Y] + O(t^3)\big), \tag{4.11}$$

and

$$\exp tX\exp tY\exp(-tX) = \exp\big(tY + t^2[X,Y] + O(t^3)\big). \tag{4.12}$$

Proof. If f is a smooth function on G, then Taylor's Theorem with remainder (Proposition 4.4) gives

$$(\widetilde{X}^n f)(g\exp tX) = \frac{d^n}{dt^n}(f(g\exp tX)) \qquad \text{for all } g\in G. \tag{4.13}$$

Taylor's Theorem says further that for X in any bounded set of $\mathfrak{g}$,

$$f(\exp X) = \sum_{k=0}^{N}\frac{1}{n!}(\widetilde{X}^n f)(1) + R_N(X),$$

where $|R_N(X)| \le C_N|X|^{N+1}$. Use of the formula (4.13) twice gives

$$(\widetilde{X}^n\widetilde{Y}^m f)(1) = \Big(\frac{d^n}{dt^n}\,\frac{d^m}{ds^n}f(\exp tX\exp sY)\Big)_{s=t=0},$$

and the Taylor expansion of $f(\exp tX\exp sY)$ with X and Y fixed is therefore

$$f(\exp tX\exp sY) = \sum_{n=0}^{N}\sum_{m=0}^{M}\frac{t^n}{n!}\frac{s^m}{m!}(\widetilde{X}^n\widetilde{Y}^m)(1) + R_{N,M}(t,s), \tag{4.14}$$

where $|R_{N,M}(t,s)| \le C_{N,M}|t|^N|s|^M$ for small $|t|$ and $|s|$.

To prove (4.10), (4.11), and (4.12), we are going to apply (4.14) with $s=t$ and with f equal to any of the coordinate functions $\exp(x_1X_1+\ldots x_dX_d)\mapsto x_i$. For small $|t|$, the Inverse Function Theorem allows us to write $\exp tX\exp tY = \exp Z(t)$ for a function $t\mapsto Z(t)$ defined and smooth near $t=0$ and taking values in $\mathfrak{g}$. Accordingly for small $|t|$, let us write $Z(t) = tZ_1 + t^2Z_1 + O(t^3)$ with Z_1 and Z_2 in $\mathfrak{g}$. With f equal to one of the coordinate functions and with $s=t$, (4.14) gives

$$\begin{aligned} f(\exp Z(t)) &= f(\exp(tZ_1 + t^2Z_2 + O(t^3))) \\ &= f(\exp(tZ_1 + t^2Z_2) + O(t^3)) \\ &= \sum_{n=0}^{2}\frac{1}{n!}\big((t\widetilde{Z}_1 + t^2\widetilde{Z}_2)^n f\big)(1) + O(t^3) \\ &= f(1) + t\widetilde{Z}_1f(1) + \tfrac12 t^2\widetilde{Z}_2f(1) + \tfrac12 t^2\widetilde{Z}_1^2 + O(t^3). \end{aligned} \tag{4.15}$$

Meanwhile putting $s = t$ in (4.14) gives us the expansion

$$f(\exp Z(t)) = \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{t^{n+m}}{n!m!}(\widetilde{X}^n\widetilde{Y}^m f)(1) + O(|t|^{N+M}). \tag{4.16}$$

The respective terms in (4.15) and (4.16) involving t must match, and so must the terms involving t^2. Therefore

$$\widetilde{Z}_1 = \widetilde{X} + \widetilde{Y} \quad \text{and} \quad \tfrac{1}{2}\widetilde{Z}_1^2 + \widetilde{Z}_2 = \tfrac{1}{2}\widetilde{X}^2 + \tfrac{1}{2}\widetilde{Y}^2 + \widetilde{X}\widetilde{Y}.$$

Consequently

$$Z_1 = X + Y \quad \text{and} \quad \tfrac{1}{2}Z_1^2 + Z_2 = \tfrac{1}{2}X^2 + \tfrac{1}{2}Y^2 + XY.$$

Eliminating Z_1 in the second equation shows that $Z_2 = XY - YX = [X, Y]$, and thus (4.10) is proved. Equation (4.11) follows by using (4.10) twice.

To prove (4.12), we argue similarly to write

$$f(\exp tX \exp tY \exp(-t)X) \sum_{m,n,p\geq 0} \frac{t^{n+n+p}}{m!n!p!}\big(\widetilde{X}^m\widetilde{Y}^n(-\widetilde{X})^p f\big)(1)$$

and

$$\exp tX \exp tY \exp(-tX) = \exp W(t)$$

with $W(t) = tW_1 + t^2W_2 + O(t^3)$ and with W_1 and W_2 in $\mathfrak{g}$. For any of the above coordinate functions f, we then have

$$\begin{aligned} f(\exp W(t)) &= f(\exp(tW_1 + t^2W_2)) + O(t^3) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\big((t\widetilde{W}_1 + t^2\widetilde{W}_2)^n f\big)(1) + O(t^3), \end{aligned}$$

and we find that $W_1 = Y$ and $W_2 = [X, Y]$. Then (4.12) follows. □

Proposition 4.14. If G is a Lie group with Lie algebra $\mathfrak{g}$, then the derivative of the member Ad of $GL(\mathfrak{g})$ at the identity is the member ad of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. In symbols, $(D\,\mathrm{Ad})_1 = \mathrm{ad}$.

PROOF. Corollary 4.12, read from right to left with X taken as tY and g taken as $\exp tX$, gives

$$\begin{aligned} \exp(\mathrm{Ad}(\exp tX)tY) &= \exp tX \exp tY (\exp tX)^{-1} \\ &= \exp tX \exp tY \exp(-tX) \\ &= \exp(tY + t^2[X, Y] + O(t^3)), \end{aligned}$$

the last equality following from (4.12) for small t. In a neighborhood of 1 in G small enough for $\exp^{-1}$ to be well defined, we then have

$$\operatorname{Ad}(\exp tX)tY = tY + t^2[X, Y] + O(t^3)$$

and thus also

$$\operatorname{Ad}(\exp tX)Y = Y + t[X, Y] + O(t^2)$$

for those same values of t. Consequently

$$\frac{d}{dt}\operatorname{Ad}(\exp tX)Y\Big|_{t=0} = [X, Y]. \tag{4.17}$$

On the other hand, Proposition 4.11 gives $\pi \circ \exp = \exp \circ (D\pi)_1$ for any homomorphism π of G, and hence

$$\operatorname{Ad}(\exp tX) = \exp((D\operatorname{Ad})_1(tX)) = \exp(t(D\operatorname{Ad})_1(X)).$$

We apply both sides to Y and obtain

$$\operatorname{Ad}(\exp tX)(Y) = (\exp(t(D\operatorname{Ad})_1(X)))(Y).$$

Finally Taylor's Theorem (Proposition 4.4) allows us to take the derivative of both sides with respect to t, put $t = 0$, and use (4.17) to arrive at

$$[X, Y] = (D\operatorname{Ad})_1(X)(Y).$$

The left side is $(\operatorname{ad} X)(Y)$, and thus $(D\operatorname{Ad})_1 = \operatorname{ad}$. □

Corollary 4.15. If G is a Lie group with Lie algebra $\mathfrak{g}$, then

$$(\operatorname{Ad})(\exp X) = \exp \operatorname{ad} X \tag{4.18}$$

for all $X \in \mathfrak{g}$.

Proof. Apply Proposition 4.11 with $\pi = \operatorname{Ad}$ and use the result of Proposition 4.14 that $(D\operatorname{Ad})_1 = \operatorname{ad}$. □

Theorem 4.16. If $\pi : G \to G'$ is a smooth homomorphism of Lie groups and if $(D\pi)_1 : \mathfrak{g} \to \mathfrak{g}'$ is the derivative of π at the identity of G, then the mapping $(D\pi)_1 : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism. Thus any smooth homomorphism between Lie groups leads to a homomorphism between their Lie algebras.

Proof. The question is how $(D\pi)_1$ interacts with brackets. Let $c(t)$ be the smooth curve in G given by $c(t) = \exp tX$. If g is in G, then $gc(t)g^{-1}$ has derivative $\mathrm{Ad}(g)X$ at $t = 0$. Hence our smooth homomorphism $\pi\colon G \to G'$ has

$$\begin{aligned}(D\pi)_1(\mathrm{Ad}(g)X) &= \frac{d}{dt}\,\pi(gc(t)g^{-1})\Big|_{t=0} \\ &= \frac{d}{dt}\,\pi(g)\pi(c(t))\pi(g)^{-1}\Big|_{t=0} \\ &= \pi(g)\,\frac{d}{dt}\,\pi(c(t))\Big|_{t=0}\,\pi(g)^{-1} \\ &= \pi(g)(D\pi)_1(X)\pi(g)^{-1}.\end{aligned}$$

If also Y is in $\mathfrak{g}$, then we have similarly

$$(D\pi)_1(\mathrm{Ad}(c(t))Y) = \pi(c(t))(D\pi)_1(Y)\pi(c(t))^{-1}. \tag{4.19}$$

We want to differentiate both sides and set $t = 0$.

For the derivative of the left side of (4.19), we use Proposition 4.14. Then

$$\begin{aligned}\frac{d}{dt}(D\pi)_1(\mathrm{Ad}(c(t))Y)\Big|_{t=0} &= (D\pi)_1\Big(\frac{d}{dt}\,\mathrm{Ad}(c(t))Y\Big)\Big|_{t=0} \\ &= (D\pi)_1\big((\mathrm{ad}X)(Y)\big) \\ &= (D\pi)_1([X,Y]).\end{aligned}$$

For the derivative of the right side of (4.19), we use (1.8) to deduce from $\frac{d}{dt}\,\pi(c(t)\big|_{t=0} = (D\pi)_1(X)$ that $\frac{d}{dt}\,\pi(c(t)^{-1}\big|_{t=0} = -(D\pi)_1(X)$. Then

$$\begin{aligned}&\frac{d}{dt}\big(\pi(c(t))(D\pi)_1(Y)\pi(c(t))^{-1}\big)\Big|_{t=0} \\ &\qquad\qquad = (D\pi)_1(X)(D\pi)_1(Y) - (D\pi)_1(Y)(D\pi)_1(X).\end{aligned}$$

Thus the differentiated form of (4.19) is

$$(D\pi)_1([X,Y]) = (D\pi)_1(X)(D\pi)_1(Y) - (D\pi)_1(Y)(D\pi)_1(X),$$

and $(D\pi)_1$ indeed respects brackets. $\square$

6. Dynkin's Formula for general analytic groups

At this stage in the development for linear groups in Chapters I through III, we introduced the natural topology on each arbitrary linear group, starting from what were called basic open neighborhoods. The construction made use of the operator norm $\|\cdot\|$ on the underlying matrix space. For a general analytic group, the operator norm is no longer available to us, and we shall proceed a little differently. Here instead of immediately introducing basic open neighborhoods, we shall first establish Dynkin's Formula for general analytic groups, generalizing the treatment of that formula given in Chapter III for linear groups.

For this purpose we need two further tools. The first of these is a generalization of Theorem 3.3, which tells how to differentiate the exponential of a smooth curve in a linear group. The second of these is a generalization of Theorem 3.4, which gives Dynkin's Formula itself. This result expresses small solutions Z of the equation $\exp Z = \exp X \exp Y$ near $(X, Y) = (0, 0)$ in a group G in terms of vector space operations, Lie brackets, and passages to the limit in the Lie algebra $\mathfrak{g}$ of G.

The setting for both tools is as follows. Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$, and consider the exponential map exp as a smooth mapping of $\mathfrak{m}$ into $\boldsymbol{M}$. Let $|\cdot|$ be any norm on $\mathfrak{m}$. As at the very end of Section 4, we chose open neighborhoods N_0 of 0 in $\mathfrak{m}$ and N_1 of 1 in $\boldsymbol{M}$ such that $\exp\colon \mathfrak{m} \to \boldsymbol{M}$ is a diffeomorphism of N_0 onto N_1.

For the generalization of Theorem 3.3, we are interested in the exponential map only insofar as it carries N_0 into N_1. The underlying spaces in question are $\mathfrak{m}$ for N_0 and $\boldsymbol{M}$ for N_1. The derivative[2] $D(\exp)$ of exp at the point X of N_0, which is written as $D_X(\exp)$, is a linear mapping of the tangent space $T_X(\mathfrak{m})$ of $\mathfrak{m}$ at X to the tangent space $T_{\exp X}(\boldsymbol{M})$ of $\boldsymbol{M}$ at $\exp X$. The tangent space $T_X(\mathfrak{m})$ of $\mathfrak{m}$ at X we can identify with the tangent space $T_0(\mathfrak{m})$, as is customary for a tangent space to a Euclidean space at a point. Here then is the statement to be proved that generalizes Theorem 3.3.

Theorem 4.17 (F. Schur). For the analytic group $\boldsymbol{M}$ with Lie algebra $\mathfrak{m}$, the derivative $D_X(\exp)$ of the exponential map at $X \in N_0$, namely the map given by

$$\exp\colon N_0 \subseteq \mathfrak{m} \to N_1 \subseteq \boldsymbol{M},$$

is the linear map from $T_0(\mathfrak{m})$ to $T_{\exp X}(\boldsymbol{M})$ given by

$$D_X(\exp)(Y) = (D_1 L_{\exp X}) \circ \Big(\int_0^1 e^{-s \operatorname{ad} X}\, ds \Big)(Y), \tag{4.20}$$

where $L_{\exp X}(u) = (\exp X)u$ and where $\circ$ indicates composition of linear maps.

Remarks.

[2]Called "differential" by some authors but not called "differential" in this book. See Section B.3 and especially the first footnote within it.

(1) We can make (4.20) easier to understand by evaluating the integral. Formally the integral equals

$$(-\operatorname{ad} X)^{-1}\big[e^{-s\operatorname{ad} X}\big]_{s=0}^{s=1} = \frac{1-e^{-\operatorname{ad} X}}{\operatorname{ad} X},$$

and the result is to be applied to Y. The quotient on the right is indeterminate because $\operatorname{ad} X$ is singular. However, if we expand the integrand in power series, then we can integrate term by term and get a valid calculation. In more detail,

$$\int_0^1 e^{-s\operatorname{ad} X}\,ds = \sum_{k=0}^{\infty}\int_0^1 \frac{-s^k(\operatorname{ad} X)^k}{k!}\,ds = \sum_{k=0}^{\infty}\frac{(-1)^k(\operatorname{ad} X)^k}{(k+1)!}.$$

Thus

$$\Big(\int_0^1 e^{-s\operatorname{ad} X}\,ds\Big)(Y) = \sum_{k=0}^{\infty}\frac{(-1)^k(\operatorname{ad} X)^k(Y)}{(k+1)!},$$

and (4.20) says that

$$D_X(\exp)(Y) = (D_1 L_{\exp X})\sum_{k=0}^{\infty}\frac{(-1)^k(\operatorname{ad} X)^k(Y)}{(k+1)!}.$$

The derivative $D_1 L_{\exp X}$ maps the tangent space at 1 in $\boldsymbol{M}$ to the tangent space at $\exp X$, and it is just scalar multiplication by $\exp X$. Thus

$$D_X(\exp)(Y) = (\exp X)\sum_{k=0}^{\infty}\frac{(-1)^k(\operatorname{ad} X)^k(Y)}{(k+1)!}.$$

(2) Theorem 4.17 generalizes Theorem 3.3. Indeed Theorem 3.3 gives the formula

$$\frac{d}{dt}(\exp X(t)) = \exp X(t)\,\Big(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\operatorname{ad} X(t))^k\Big)\,\frac{dX(t)}{dt} \tag{4.21}$$

for differentiating along a curve of matrices, and this formula results by applying the Chain Rule to the composition of $t\mapsto X(t)$ followed by the map $X\mapsto \exp X$. Specifically the Chain Rule gives

$$D(t\mapsto \exp X(t)) = D(X\mapsto \exp X)_{X=X(t)}D(t\mapsto X(t)). \tag{4.22}$$

The left side equals the left side of (4.21), $D(t\mapsto X(t))$ equals $dX(t)/dt$, and $D(X\mapsto \exp X)\big|_{X=X(t)}$ equals the product of the remaining two factors on the right side of (4.21), namely

$$\exp X(t)\,\Big(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\operatorname{ad} X(t))^k\Big).$$

(3) Formula (4.21) is valid in every analytic group, not just in matrix groups, and invoking this formula is the way that we shall make use of Theorem 4.17. The proof is the same as in the previous remark: we use the Chain Rule for the composition (4.22).

Proof. It is enough to prove that (4.21) holds for every Y in the tangent space $T_0(\mathfrak{m})$, and for this purpose it is enough to prove that (4.21) holds for every smooth curve $X(t)$ in $\mathfrak{m}$ with $X(0) = 0$ since we can always find such a curve with $X'(0) = Y$. In other words, the proof of Theorem 4.17 reduces to the proof of Theorem 3.3, which has already been given in Chapter III. $\square$

We turn to the question of generalizing Dynkin's Formula (Theorem 3.4) in such a way that the Lie groups and Lie algebras in question need not be linear. Looking at the statement of Theorem 3.4, we see that most of the objects that are involved are located in $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$, the finite-dimensional vector space of all real linear maps of $\mathfrak{m}$ into itself. This space *is* a matrix algebra, and the idea is to formulate as much as possible of a generalization within $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$.

Theorem 4.18 (Dynkin's Formula). Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$, let X and Y be members of $\mathfrak{m}$ sufficiently close to 0, and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{m}$ generated by X and Y. Then the series $S(X, Y)$ in $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ equal to

$$\sum_{k\geq 1}\sum \frac{(-1)^k}{k+1}\,\frac{1}{i_1+\cdots+i_k+1}\Big(\frac{(\operatorname{ad}X)^{i_1}}{i_1!}\,\frac{(\operatorname{ad}Y)^{j_1}}{j_1!}\cdots\frac{(\operatorname{ad}X)^{i_k}}{i_k!}\,\frac{(\operatorname{ad}Y)^{j_k}}{j_k!}\Big),$$

with the inner sum indexed as below, converges in the operator norm topology in $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ independently of the order in which the terms are added, and the full expression

$$Z = C(X, Y) = Y + X + S(X, Y)(X)$$

is a solution in $\boldsymbol{M}$ of $\exp Z = \exp X \exp Y$. Here the inner sum is taken over all sequences $(i_1, j_1, \ldots, i_k, j_k)$ of integers ≥ 0 satisfying $i_r + j_r \geq 1$ for all $1 \leq r \leq k$.

Proof. The continuity of exp and the Inverse Function Theorem together say that for some $\eta > 0$, as long as X and Y have norm $< \eta$ and t is a real number with $0 \leq t \leq 1$, the equation

$$\exp Z(t) = \exp tX \exp Y \tag{4.23}$$

has a unique solution $Z(t)$ in $\mathfrak{m}$ with norm $< \eta$. The function $t \mapsto Z(t)$ is smooth by the Inverse Function Theorem.

The curve $Z(t)$ is a smooth curve in $\mathfrak{m}$, and $\exp Z(t)$ is a smooth curve in the analytic group $\boldsymbol{M}$, which is not necessarily a matrix group. Referring to Theorem 4.17 and its third remark, we see that $dZ(t)/dt$ satisfies the same equation as in (3.23) even though $\boldsymbol{M}$ may not be a matrix group, namely

$$\frac{dZ(t)}{dt} = \frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1}\, X. \tag{4.24}$$

We apply Ad to both sides of (4.23) and then use the formula $\operatorname{Ad} \circ \exp = \exp \circ \operatorname{ad}$ of Corollary 4.15 to obtain

$$\exp \operatorname{ad} Z(t) = \exp\,(t \operatorname{ad} X) \exp\,(\operatorname{ad} Y). \tag{4.25}$$

Then we expand the right side of (4.24) as a member of $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ operating on X, and we work in that space with convergence relative to the operator norm. The computation is the same as in (3.25) except that we drop X throughout, and the series expansion is valid as long as the operator norm of $\exp \operatorname{ad} Z(t) - 1$ satisfies $\|\exp \operatorname{ad} Z(t) - 1\| < 1$:

$$\begin{aligned}
\frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \big(\exp \operatorname{ad} Z(t) - 1\big)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (\exp \operatorname{ad} tX \exp \operatorname{ad} Y - 1)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left(\sum_{i \geq 0,\, j \geq 0,\, i+j>0} \frac{t^i (\operatorname{ad} X)^i (\operatorname{ad} Y)^j}{i!\, j!} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{i_k \geq 0,\, j_k \geq 0,\, i_k + j_k > 0} t^{i_1 + \cdots + i_k} \\
&\qquad \Big(\frac{(\operatorname{ad} X)^{i_1}}{i_1!}\Big)\Big(\frac{(\operatorname{ad} Y)^{j_1}}{j_1!}\Big) \cdots \Big(\frac{(\operatorname{ad} X)^{i_k}}{i_k!}\Big)\Big(\frac{(\operatorname{ad} Y)^{j_k}}{j_k!}\Big).
\end{aligned} \tag{4.26}$$

On each line of the display (4.26), we interpret the $k = 0$ term on the right side as 1. Since (4.24) gives

$$\frac{dZ(t)}{dt} = \frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1}\, X,$$

we compute the integral from 0 to 1 of (4.26), considering the result as a member of $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$. The result after interchange of sum and integral is the following identity of members of $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$, the limit on the right being interpreted in the operator norm topology on $\mathrm{End}_{\mathbb{R}}(\mathfrak{m})$ and remaining valid when both sides act on a member of $\mathfrak{m}$:

$$\int_0^1 \frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{i_k \geq 0,\, j_k \geq 0,\, i_k + j_k > 0} \frac{1}{i_1 + \cdots + i_k + 1} \Big(\frac{(\operatorname{ad} X)^{i_1}}{i_1!}\Big)\Big(\frac{(\operatorname{ad} Y)^{j_1}}{j_1!}\Big) \cdots \Big(\frac{(\operatorname{ad} X)^{i_k}}{i_k!}\Big)\Big(\frac{(\operatorname{ad} Y)^{j_k}}{j_k!}\Big). \tag{4.27}$$

Since $Z(0) = Y$ and $Z(1) = Z$, we therefore have

$$Z = Y + \int_0^1 \frac{dZ(t)}{dt}\, dt = Y + \Big(\int_0^1 \frac{\operatorname{ad} Z(t)}{\exp(\operatorname{ad} Z(t)) - 1}\Big)(X).$$

We can substitute into this equality from (4.27). The term with $k = 0$ yields the identity operator acting on X, the integration of the power of t is responsbile for the factor $(i_1 + \cdots + i_k + 1)^{-1}$, and the terms with $k \geq 1$ match the formula for $S(X, Y)(X)$ in the statement of the theorem.

Going back over the proof, we can check on the meaning of the hypothesis that X and Y are sufficiently close to 0. One condition is that X and Y have norm $< \eta$, in order for (4.23) to be valid. The other condition is that $\|\exp \operatorname{ad} Z(t) - 1\| < 1$ for $0 \leq t \leq 1$, in order to make the computation (4.26) be valid, and for this it is sufficient that $\|\operatorname{ad} X\| + \|\operatorname{ad} Y\| < \log 2$, by the same kind of calculation as in the proof of Proposition 3.1. □

7. Natural topology on an arbitrary untopologized subgroup

Let $\boldsymbol{M}$ be an analytic group, let $\mathfrak{m}$ be its Lie algebra, and let $|\cdot|$ be any norm in $\mathfrak{m}$. We worked with analytic subgroups of $\boldsymbol{M}$ in Section 4, and now we generalize those considerations to *arbitrary* untopologized subgroups G of $\boldsymbol{M}$. Just before Theorem 4.10 we defined the Lie algebra $\mathfrak{g}$ of each analytic subgroup G of $\boldsymbol{M}$. We now extend the definition of the Lie algebra $\mathfrak{g}$ of a group to all subgroups G of $\boldsymbol{M}$ without reference to any topology. Then we use the exponential map of $\boldsymbol{M}$ to introduce the "natural topology" on each such G. This topology will make G into a topological group. When we ignore all but the identity component of G, we shall see that the result G_0 becomes an analytic group that is an analytic subgroup of $\boldsymbol{M}$. We shall see also that this process leads us to our original analytic subgroup of $\boldsymbol{M}$ if G had started out as an analytic subgroup.

Proposition 4.19. For any subgroup G of $\boldsymbol{M}$, the linear span of all $X \in \mathfrak{m}$ such that $\exp tX$ is in G for all real t is a Lie subalgebra of $\mathfrak{m}$.

PROOF. Let X and Y be in $\mathfrak{g}$. If g is in G, then $\exp t\mathrm{Ad}(g)Y = g(\exp tY)g^{-1}$ lies in G for all t, and the definition above shows that $\mathrm{Ad}(g)Y$ is in $\mathfrak{g}$ for each $g \in G$. Taking $g = \exp tX$ shows that $\mathrm{Ad}(\exp tX)Y$ is in $\mathfrak{g}$ for every real t. Since $\mathfrak{g}$ is topologically closed as a vector subspace of $\mathfrak{m}$, $\frac{d}{dt}\mathrm{Ad}(\exp tX)Y\big|_{t=0}$ lies in $\mathfrak{g}$. But this expression is just $[X, Y]$ by (4.17), and thus $\mathfrak{g}$ is closed under brackets. $\square$

REMARKS. No analog of Theorem 2.1 is available for nonlinear groups. Proposition 4.19 serves as a substitute.

The Lie algebra described in Proposition 4.19. is called the **Lie algebra of the untopologized subgroup** G of $\boldsymbol{M}$. If G happens to be analytic subgroup, then Theorem 4.10c shows that this notion of Lie algebra coincides with the one defined just before the statement of Theorem 4.10.

We shall define a "natural topology" on G in terms of "basic open neighborhoods." By the next-to-last conclusion of Theorem 4.3, we can find open neighborhoods N_0 of 0 in $\mathfrak{m}$ and N_1 of 1 in $\boldsymbol{M}$ such that $\exp\colon \mathfrak{m} \to \boldsymbol{M}$ is a diffeomorphism of N_0 onto N_1. These open neighborhoods will be fixed throughout.

Fix a number $\varepsilon_0 > 0$ small enough so that every X in $\mathfrak{m}$ with $|X| < \varepsilon_0$ lies in N_0 and therefore has $\exp X$ in N_1. This number ε_0 will occur repeatedly in the arguments below.

If a is in G and X is in $\mathfrak{g}$, then $a \exp X$ is in G, G being a group. For $a \in G$ and $\varepsilon < \varepsilon_0$, the **basic open neighborhood** $N_{a,\varepsilon}$ in G of the element a of G is defined as the subset of G given by

$$N_{a,\varepsilon} = \{a \exp X \bigm| X \in \mathfrak{g} \text{ and } |X| < \varepsilon\}.$$

We shall show that the basic open neighborhoods in G form a topology[3] on G, and this topology will be what we call the **natural topology** on G. We require a lemma.

Lemma 4.20. For a subgroup G of $\boldsymbol{M}$, if a and b are elements of G such that b lies in some basic open neighborhood $N_{a,\varepsilon}$ with $\varepsilon < \varepsilon_0$, then there exists a basic open neighborhood $N_{b,\varepsilon'}$ in G with $\varepsilon' \le \varepsilon$ such that $N_{b,\varepsilon'} \subseteq N_{a,\varepsilon}$.

[3] In this topology the sets $N_{a,\varepsilon}$ with $\varepsilon < \varepsilon_0$ will indeed be open, but no assertion is made about $N_{a,\varepsilon}$ when $\varepsilon \ge \varepsilon_0$.

PROOF. Since the element b lies in $N_{a,\varepsilon}$, we have $b = a \exp X$ with $X \in \mathfrak{g}$ and $|X| < \varepsilon$. Thus $a^{-1}b = \exp X$. Suppose that some $\varepsilon' \le \varepsilon$ has been specified. If c is in $N_{b,\varepsilon'}$, then $c = b \exp X'$ with $|X'| < \varepsilon'$, and hence

$$a^{-1}c = (a^{-1}b)(b^{-1}c) = (\exp X)(\exp X')$$

with X and X' in $\mathfrak{g}$ and with $|X| < \varepsilon$ and $|X'| < \varepsilon'$. Application of Theorem 4.18 shows that the small solution Z of the equation $\exp Z = (\exp X)(\exp X')$ satisfies

$$Z = X + X' + S(X, X')X,$$

where $S(X, X') = \frac{1}{2}(\operatorname{ad} X)(X') +$ "error terms", with "error terms" referring to a higher order combination of brackets applied to X'. With $\|\operatorname{ad} X\|$ standing for the operator norm of $\operatorname{ad} X$ relative to $|\cdot|$, it follows that

$$\begin{aligned} |Z| &\le |X| + |X'| + \tfrac{1}{2}\|\operatorname{ad} X\||X'| + \dots \\ &\le |X| + (1 + K\varepsilon)|X'| \\ &\le |X| + (1 + K\varepsilon)\varepsilon', \end{aligned}$$

where K is a bound for the operator in $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ appearing in the display in Theorem 4.18. The right side is $< \varepsilon$ if ε' is sufficiently small, and the lemma follows. □

Proposition 4.21. For the abstract subgroup G of the given analytic group M with Lie algebra $\mathfrak{m}$, the collection of all unions of basic open neighborhoods is a topology on G. This topology is locally compact Hausdorff and locally connected. Moreover this topology is at least as fine as the relative topology from the underlying vector space $\mathfrak{m}$ of matrices.

REMARKS. Once again the topology in the statement of the proposition is defined to be the **natural topology** on G. The topology need not be separable, as we have already seen in the linear case. The connected component containing the group identity will, however, be separable, as we shall see in Theorem 4.22, and it is the space that we shall make into a smooth manifold.

PROOF. The proof is similar to the proof of Proposition 2.5. The collection of all unions of basic open neighborhoods $N_{a,\varepsilon}$ in G such that $\varepsilon < \varepsilon_0$ is closed under finite intersections by virtue of Lemma 4.20, and it covers G. Therefore it is a topology on G. We are to show that this topology is Hausdorff and to compare it with the topology on $\mathfrak{m}$.

On any basic open neighborhood $N_{1,\varepsilon}$ with $\varepsilon < \varepsilon_0$, where ε_0 is defined earlier in this section just before "basic open neighborhood," the Inverse

Function Theorem shows that $N_{1,\varepsilon}$ has the same topology as the subset

$$\{X \in \mathfrak{g} \bigm| |X| < \varepsilon\},$$

where $\mathfrak{g} \subseteq \mathfrak{m}$ is the Lie algebra of the untopologized group G. It follows for any point a of G and any neighborhood $U_{a,\mathfrak{m}}$ of a in $\mathfrak{m}$ that there exists a neighborhood $U_{a,\mathfrak{g}}$ in G with $a \in U_{a,\mathfrak{g}}$ and $U_{a,\mathfrak{g}} \subseteq U_{a,\mathfrak{m}}$. In other words the natural topology on $N_{a,\varepsilon}$ is at least as fine as the relative topology from $\mathfrak{m}$. The latter topology is Hausdorff, and therefore the natural topology on G is Hausdorff.

We see that $N_{a,\varepsilon}$ is behaving like the Euclidean space $\mathfrak{g}$ topologically. In particular it is locally compact and connected. Since $N_{a,\varepsilon}$ is open in the natural topology on G, the natural topology on G is locally compact and locally connected. □

Theorem 4.22. If the untopologized subgroup G of the given analytic group $\boldsymbol{M}$ with Lie algebra $\mathfrak{m}$ is given its natural topology and if $\mathfrak{g}$ is its Lie algebra, then G becomes a locally connected topological group, and the connected component G_0 of the identity becomes a separable locally compact (Hausdorff) topological group. Each basic open neighborhood $N_{a,\varepsilon}$ with $a \in G$ and $\varepsilon < \varepsilon_0$ defines a chart $(\varphi_{a,\varepsilon}, N_{a,\varepsilon})$ on G of dimension $\dim \mathfrak{g}$, and the collection of all such charts makes G into a smooth manifold of dimension $\dim \mathfrak{g}$ in such a way that multiplication as a map of $G \times G$ into G and inversion as a map of G into itself are smooth.

PROOF. The proof is the same as for Theorem 2.6. □

Thus G_0 becomes an analytic group. We call it the **analytic subgroup** of ***M*** **associated** with G.

Theorem 4.22 has the same kinds of consequences at its level of generality that Theorem 2.6 had in the linear case. Here are some.

Corollary 4.23. If an untoplogized subgroup G of a given analytic group $\boldsymbol{M}$ has Lie algebra $\mathfrak{g}$, and if G_0 is the analytic subgroup of $\boldsymbol{M}$ associated with G, then $\exp \mathfrak{g}$ generates G_0.

PROOF. Since G_0 is connected, any $N_{1,\varepsilon}$ with $\varepsilon < \frac{1}{2}\varepsilon_0$ generates G_0. Hence the subset $\exp \mathfrak{g}$ generates G_0. □

Corollary 4.24. Let $\boldsymbol{M}$ be an analytic group, let $\mathfrak{m}$ be its Lie algebra, and let G be an untopologized subgroup of $\boldsymbol{M}$. If G_0 is the analytic subgroup of $\boldsymbol{M}$ associated with G, then the inclusion of G_0 into $\boldsymbol{M}$ is smooth. In the reverse direction if f is any smooth function from a smooth manifold into $\boldsymbol{M}$ whose image lies in G_0, then f is smooth as a function into G_0.

PROOF. The inclusion of G_0 into $\boldsymbol{M}$ is given near a by $i_a(\exp X) = \exp X$, which for $|X| < \varepsilon < \frac{1}{2}\varepsilon_0$ is the composition of the smooth map $\varphi_{a,\varepsilon}$, which carries $\exp X$ into X, followed by $X \mapsto \exp$. Thus it is smooth.

For the reverse direction let D be the domain of f. Without loss of generality we may assume that D is an open subset of a Euclidean space. To check smoothness of f into $\boldsymbol{M}$ or G_0, we refer the image space to charts. Define balls $\widetilde{B}_\varepsilon$ in $\mathfrak{m}$ and B_ε in $\mathfrak{g}$ by

$$\widetilde{B}_\varepsilon = \{X \in \mathfrak{m} \mid X| < \varepsilon\} \qquad \text{and} \qquad B_\varepsilon = \widetilde{B}_\varepsilon \cap \mathfrak{g} = \{X \in \mathfrak{g} \mid |X| < \varepsilon\}.$$

To handle the image $f(x)$ of f near a point $f(a)$ of $\boldsymbol{M}$, we follow f with the translation $L_{f(a)^{-1}}$ to center matters at 1 and then use $\exp^{-1}$ to pass to $\widetilde{B}_\varepsilon$ or B_ε. Thus we are to compare the smoothness of $x \mapsto \exp^{-1} \circ L_{f(a)^{-1}} \circ f(x)$ into $\widetilde{B}_\varepsilon$ with the smoothness of the same expression into B_ε. The second one is just the same expression viewed as in a Euclidean subspace, and the second one is smooth if the first one is smooth. The result follows. $\square$

REMARKS.

(1) The first conclusion of Corollary 4.24 is an extension of Theorem 1.10a from closed linear groups to arbitrary analytic groups. Similarly the conclusion about the "reverse direction" is an extension of Theorem 1.10b.

(2) The critical fact for the above proof is that the values $f(x)$ for x near a all lie in G, so that left translation of all of them by $f(a)^{-1}$ moves them into the image of exp on B_ε, not merely the image of exp on $\widetilde{B}_\varepsilon$.

(3) The group G_0 is an analytic group by Theorem 4.22, and it is also a subgroup of the Lie group $\boldsymbol{M}$. The first conclusion of the corollary says that the inclusion of G_0 into $\boldsymbol{M}$ is smooth. Therefore G_0 is an analytic subgroup of $\boldsymbol{M}$. This conclusion establishes part of Fundamental Theorem A of the Preface: we have succeeded in passing from an arbitrary subgroup G of $\boldsymbol{M}$ to its Lie algebra $\mathfrak{g}$ in $\mathfrak{m}$, and then to an analytic subgroup G_0 of $\boldsymbol{M}$.

Proposition 4.25. Let $\boldsymbol{M}$ be an analytic group, let $\mathfrak{m}$ be its Lie algebra, and let G be a subgroup of $\boldsymbol{M}$ considered in its natural topology. If the identity component G_0 of G is not closed as a subgroup of $\boldsymbol{M}$, then the natural topology on G is different from the relative topology.

PROOF. The natural topology on G and on its identity component G_0 have to be locally compact Hausdorff. A footnote in Section I.2 contains a proof that a locally compact dense subspace of a Hausdorff space is necessarily open, i.e., open in its closure. If the natural topology on G_0 coincides with the relative topology, then G_0 must be open in its closure in the relative topology. The closure being a subgroup, the union of the remaining cosets must be open, and it follows that G must itself be closed. $\square$

8. Construction of general analytic subgroups from Lie subalgebras

We continue with our underlying analytic group $\boldsymbol{M}$ and its Lie algebra $\mathfrak{m}$. The main theorem of this chapter, Theorem 4.26, which will be stated momentarily, will associate an analytic subgroup G of $\boldsymbol{M}$ to each Lie subalgebra $\mathfrak{g}$ of $\mathfrak{m}$. The construction will proceed in such a way that Theorem 4.26, together with the association of Lie subalgebras to analytic subgroups given in Theorem 4.10a, gives a one-one correspondence of the Lie subalgebras of $\mathfrak{m}$ to the analytic subgroups of $\boldsymbol{M}$.

Theorem 4.26. If $\boldsymbol{M}$ is an analytic group with Lie algebra $\mathfrak{m}$ and if $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{m}$, then there exists an analytic subgroup G of $\boldsymbol{M}$ with Lie algebra $\mathfrak{g}$. Specifically the subgroup of $\boldsymbol{M}$ generated by all $\exp X$ with $X \in \mathfrak{g}$ is the analytic subgroup G, and its Lie algebra is $\mathfrak{g}$. In particular if X is in $\mathfrak{m}$, then the subset $\{\exp tX \mid t \in \mathbb{R}\}$ is an analytic subgroup of $\boldsymbol{M}$ and its Lie algebra is $\mathbb{R}X$.

Remark. The subgroups of $\boldsymbol{M}$ of the form $\{\exp tX \mid t \in \mathbb{R}\}$ are called the **one-parameter subgroups** of $\boldsymbol{M}$.

Proof. The proof of Theorem 3.6 works in the present setting as soon as its internal references are updated from Chapters I through III and the operator norm for $\mathfrak{m}$ is replaced by a general norm $|\cdot|$. Let us go over some of the details.

Define G as a set be the subgroup of $\boldsymbol{M}$ generated by all $\exp X$ with X in $\mathfrak{g}$. Proposition 4.19 says that the linear span of all such X is a Lie algebra $\mathfrak{g}'$. We have $\mathfrak{g} \subseteq \mathfrak{g}'$ and we are to prove that equality holds.

Choose a real vector subspace $\mathfrak{s}$ of $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{g} \oplus \mathfrak{s}$. By the Inverse Function Theorem, there exist open balls U_1 about 0 in $\mathfrak{g}$ and U_2 about 0 in $\mathfrak{s}$, as well as an open neighborhood V of 1 in $\boldsymbol{M}$ such that

$$(a, b) \mapsto \exp a \exp b$$

is a diffeomorphism of $U_1 \times U_2$ onto V. Fix attention on a member Z of $\mathfrak{g}'$. For small positive t, $\exp tZ$ lies in V, and it therefore decomposes as

$$\exp tZ = \exp X(t) \exp Y(t) \quad \text{with } X(t) \in \mathfrak{g} \text{ and } Y(t) \in \mathfrak{s}.$$

Since $X(t)$ is in $\mathfrak{g}$, $\exp X(t)$ is in G.

Meanwhile since Z is in the Lie algebra $\mathfrak{g}'$ of the subgroup G of $\boldsymbol{M}$, $\exp tZ$ is in G for all t. Hence $\exp Y(t)$ lies in G for all small positive t. Arguing as in Lemma 3.7, we can show that the set of $Y \in U_2$ such that $\exp Y$ lies in G is countable. This is the most difficult step.

With this countability in hand, we can complete the proof of Theorem 4.26. Being a smooth curve, $t \mapsto \exp Y(t)$ either is constant for t in an interval about 0, or it takes on uncountably many values. Because of the countability it must be constant, and thus $\exp Y(t) = \exp Y(0) = 1$ in an interval about 0. Then it follows that $\exp tZ = \exp X(t) \exp Y(t) = \exp X(t)$ in an interval about 0. Thus $\exp tZ$ for t near 0 is a smooth curve in G, and its derivative at $t = 0$, which is Z, must be in the Lie algebra $\mathfrak{g}$ of the subgroup G of $\boldsymbol{M}$. □

This completes the construction of an analytic subgroup of the group $\boldsymbol{M}$ corresponding to each Lie subalgebra of the Lie algebra $\mathfrak{m}$ of $\boldsymbol{M}$.

9. Passage from homomorphisms of Lie subalgebras to homomorphisms of analytic subgroups

We continue the investigation of homomorphisms begun in Section 5. The main result of that section was Theorem 4.16, which associates to any smooth homomorphism of Lie groups a homomorphism of their Lie algebras. Specifically if $\pi\colon G \to G'$ is a smooth homomorphism of Lie groups, then the derivative of π at the identity is a homomorphism of the corresponding Lie algebras.

In this section we attempt to invert this construction, passing from a homomorphism between the Lie algebras of two analytic groups to a smooth homomorphism between the groups themselves. The discussion makes use of material on covering spaces, and we shall take the content of Appendix C as known for the present section.

As we found in Section III.6 in the linear case, there is a topological obstruction to lifting the Lie algebra homomorphism to a Lie group homomorphism. We can get around this obstruction by assuming that the domain group is connected and simply connected.

Theorem 4.27. Let G and G' be analytic groups with G simply connected, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be their Lie algebras. If $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of the Lie algebras, then there exists a unique smooth homomorphism $\Phi\colon G \to G'$ of the groups such that $(D\Phi)_1 = \varphi$.

REMARKS. The proof is almost word-for-word the same as the proof of Theorem 3.9. It builds on the main theorem of the chapter, which was Theorem 3.6 in the earlier case and is Theorem 4.26 now.

PROOF. Form the group $G \times G'$. In $G \times G'$, one multiplies coordinate by coordinate, and one uses the product manifold structure defined in Section B.5. The corresponding Lie algebra is $\mathfrak{g} \oplus \mathfrak{g}'$ with the vector space and bracket operations taking place coordinate by coordinate.

The graph of φ, given by $\mathrm{Graph}(\varphi) = \{(X, \varphi(X)) \in \mathfrak{g} \oplus \mathfrak{g}'\}$, is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}'$, and Theorem 4.26 produces a unique analytic subgroup F of $G \times G'$ with Lie algebra $\mathrm{Graph}(\varphi)$. Let $p : F \to G$ be the composition of the inclusion $F \to G \times G'$ followed by the projection to the first coordinate $(x, y) \mapsto x$. On the Lie algebra level, $(Dp)_1$ carries $\mathrm{Graph}(\varphi)$ into $\mathfrak{g} \oplus \mathfrak{g}'$ and then projects to the first coordinate. The formula is $(Dp)_1(X, \varphi(X)) = X$. Since the domain and image have equal dimension, namely $\dim \mathfrak{g}$, the subgroup $p(F)$ contains an open neighborhood of the identity in G. But G is connected by hypothesis, and thus $p(F) = G$. Since F is connected by construction, Proposition C.19d shows that $p\colon F \to G$ is a covering map. By hypothesis, G is simply connected; thus Corollary C.9 shows that G has no nontrivial coverings. Consequently p is a homeomorphism and has an inverse p^{-1}. Let $q : F \to G'$ be the composition of the inclusion $F \to G \times G'$ followed by the projection to the second coordinate $(x, y) \mapsto y$. On the Lie algebra level, $(Dq)_1$ carries $\mathrm{Graph}(\varphi)$ into $\mathfrak{g} \oplus \mathfrak{g}'$ and then projects to the second coordinate. The formula is $(Dq)_1(X, \varphi(X)) = \varphi(X)$, and the composition is $q \circ p^{-1} = \varphi$. We can therefore take Φ to be $q \circ p^{-1}$. Uniqueness follows since a smooth homomorphism of an analytic group is determined by its values on a set that generates the group. □

Corollary 4.28. Any two simply connected analytic groups with isomorphic Lie algebras are isomorphic as Lie groups.

PROOF. Let G and G' be simply connected analytic groups with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, and let $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ be an isomorphism between their Lie algebras. Application of Theorem 4.27 produces a homomorphism $\Phi\colon G \to G'$ such that $(D\Phi)_1 = \varphi$. Reversing the roles of G and G' produces a homomorphism $\Phi'\colon G' \to G$ such that $(D\Phi')_1 = \varphi^{-1}$. From the equality $(D\Phi')_1(D\Phi)_1 = \varphi^{-1} \circ \varphi = 1_{\mathfrak{g}}$ and the uniqueness in Theorem 4.27, we see that $\Phi' \circ \Phi = 1_G$. Similarly $\Phi \circ \Phi' = 1_{G'}$. □

10. Simply connected analytic groups

Corollary 4.28 goes in the direction of a classification theorem for simply connected analytic groups, but it does not address the extent to which analytic groups actually have simply connected covering groups that are analytic groups. Appendix C provides only a simply connected covering space (via Theorems C.8 and C.12) that can be made into a simply connected topological group (via Proposition C.22). To address this deficiency, we include here two results stating how the theory of covering spaces makes allowance for smooth manifolds and analytic groups.

Proposition 4.29. Let Y be a smooth manifold of dimension n, and let X be a covering space with a covering map $e\colon X \to Y$. Then X uniquely admits the structure of a smooth manifold of dimension n in such a way such that e is smooth and everywhere regular. Moreover if P is another smooth manifold, if $g\colon P \to Y$ is a smooth mapping, and if $f\colon P \to X$ is a continuous function such that $ef = g$, then f is smooth.

Remarks. The second statement will be what we will use in this section. It says that in any set-up for the Map-lifting Theorem in which the base mapping is smooth and the necessary condition on fundamental groups is satisfied, the lifted map is smooth.

Proof. Begin with a maximal atlas of smoothly compatible charts (V, φ) of Y, and retain only those such that V is a connected open subset of Y that is evenly covered by e. To each connected component U of $e^{-1}(V)$, associate a chart $(U, \varphi \circ e)$ of X. The system of all such charts covers X and forms an atlas of smoothly compatible charts. The result is that X becomes a smooth manifold, and e is smooth and regular. All such charts have to be in the maximal atlas for X, and hence the smooth structure on X is unique.

The final statement of the proposition concerns the situation of the Map-lifting Theorem, Theorem C.7, except that the map f is assumed to exist and we are to prove that it is smooth. It is enough to prove smoothness in a neighborhood of any given point p of P. Fix an open neighborhood U of $f(p)$ in X. Without loss of generality, we may assume that U is a connected component of $e^{-1}(V)$ for some connected open subset V of Y that is evenly covered by e. By continuity of f, we can find an open neighborhood W of p in P such that $f(W) \subseteq U$. On W, we have $f = e \circ (e^{-1} \circ g)$, and therefore f is smooth on W. $\square$

Theorem 4.30. If G is any analytic group, then G has a simply connected covering group that is unique up to equivalence, it is an analytic group, and the covering map is smooth.

PROOF. Theorems C.8 and C.12 show that G has a simply connected covering space $\widetilde{G}$, and Proposition 4.29 shows that $\widetilde{G}$ canonically becomes a smooth manifold in such a way that the covering map e is smooth.

If we choose any member of the inverse image $e^{-1}(1)$ and call it $\widetilde{1}$, then Proposition C.22 shows that there is a unique operation of multiplication on $\widetilde{G}$ such that $\widetilde{G}$ becomes a separable topological group, e is a homomorphism, and $\widetilde{1}$ acts as the identity.

Examining the proof, we see that the argument for Proposition C.22 starts from the multiplication mapping $m: G \times G \to G$, forms the continuous function $\varphi: \widetilde{G} \times \widetilde{G} \to G$ defined by $\varphi = m \circ (e, e)$, lifts it by the Map-lifting Theorem to a continuous function $\widetilde{\varphi}: \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, and takes $\widetilde{\varphi}$ as the multiplication mapping in $\widetilde{G}$. Applying Proposition 4.29, we see that multiplication is smooth in $\widetilde{G}$.

Similarly let $j: G \to G$ be group inversion in G, and form the function $\varphi: \widetilde{G} \to G$ given by $\varphi = j \circ e$. The Map-lifting Theorem shows that φ lifts to a continuous function $\widetilde{\varphi}: \widetilde{G} \to \widetilde{G}$ carrying $\widetilde{1}$ to $\widetilde{1}$. Proposition C.22 takes this to be group inversion in $\widetilde{G}$, and Proposition 4.29 shows that it is smooth. Therefore $\widetilde{G}$ is a Lie group. It was given as connected and is therefore an analytic group. □

Corollary 4.28 and Theorem 4.30 together say that the real Lie algebras that arise as Lie algebras of Lie groups are classified up to isomorphism by the simply connected analytic groups. Fundamental Theorem C of Lie Theory in the Preface, whose proof is deferred to the sequel *Lie Groups Beyond an Introduction*, says that every real Lie algebra is isomorphic to the Lie algebra of some analytic group. Thus real Lie algebras are classified up to isomorphism by simply connected analytic groups.

11. The abelian case

Theorem 4.30 is a powerful tool for classifying analytic groups. Let us illustrate this fact now by classifying those analytic groups that are abelian. Recall from Section 2 that a Lie algebra $\mathfrak{g}$ is **abelian** if $[X, Y] = 0$ for all X and Y in $\mathfrak{g}$.

Proposition 4.31. An analytic group G is abelian if and only if its Lie algebra $\mathfrak{g}$ is abelian.

Proof. Corollary 4.12 gives
$$g(\exp Y)g^{-1} = \exp(\mathrm{Ad}(g)Y)$$
for all $g \in G$ and $Y \in \mathfrak{g}$. Taking $g = \exp X$ and applying Corollary 4.15 gives
$$\begin{aligned}(\exp X)(\exp Y)(\exp X)^{-1} &= \exp(\mathrm{Ad}(\exp X)Y)\\ &= \exp\big((\exp \mathrm{ad}\, X)(Y)\big) \qquad (4.28)\end{aligned}$$
for all X and Y in $\mathfrak{g}$.

If $\mathfrak{g}$ is abelian, then $\mathrm{ad}\, X = 0$ and $\exp(\mathrm{ad}\, X) = 1$ for all X. Thus the right side of (4.28) reduces to $\exp Y$, and (4.28) says that $\exp X$ commutes with $\exp Y$ for all X and Y. Since G is connected, $\exp \mathfrak{g}$ generates G, and we conclude that G is abelian.

In the reverse direction if G is abelian, then (4.28) shows that $\exp Y = \exp\big((\exp \mathrm{ad}\, X)(Y)\big)$ for all X and Y. Near 0, exp is one-one, and therefore $Y = (\exp\, \mathrm{ad}\, X)(Y)$ for all small X and Y. Then it follows that $\exp\, \mathrm{ad}\, X = 1$ for all small X. Since exp is one-one near 0, $\mathrm{ad}\, X = 0$ for all small X and therefore for all X. Consequently $\mathfrak{g}$ is abelian. □

We know what all the abelian Lie algebras are, up to isomorphism: being just real vector spaces with all brackets 0, they are classified by their dimension. In dimension n, the additive group $\mathbb{R}^n$ is a simply connected analytic group with Lie algebra $\mathbb{R}^n$, and Corollary 4.28 leads us to the following conclusion.

Corollary 4.32. Up to isomorphism the only simply connected abelian analytic groups are the additive groups $\mathbb{R}^n$ for each $n \geq 0$.

Because every analytic group has a simply connected covering group, every abelian analytic group is the quotient of some additive group $\mathbb{R}^n$ by a discrete subgroup (the group of deck transformations as in Theorem C.18). Those discrete subgroups are classified in Proposition 1.3, whose statement we recall now.

Proposition 1.3. If D is a discrete subgroup of $\mathbb{R}^n$, then there exists a basis $x_1, \ldots, x_n$ of $\mathbb{R}^n$ such that D consists exactly of those vectors of the form $k_1x_1 + \cdots + k_px_p$ with all k_j in $\mathbb{Z}$. Here p is some integer with $0 \leq p \leq n$.

This result allows us to classify the connected abelian Lie groups as follows.

Corollary 4.33. Any connected abelian Lie group is isomorphic to the product of a Euclidean group and a torus, thus to $\mathbb{R}^k \times T^\ell$ for some integers $k \geq 0$ and $\ell \geq 0$.

Proof. Corollary 4.32 shows that the given group has to be isomorphic to the quotient of the additive group of some Euclidean space $\mathbb{R}^m$ by a discrete subgroup. Proposition 1.3 classifies the discrete subgroups of $\mathbb{R}^m$ up to isomorphism as of the form $\mathbb{Z}^p$, the set of points in $\mathbb{R}^p$ with all coordinates in $\mathbb{Z}$, with $0 \leq p \leq m$. The quotient of the two is $\mathbb{R}^{m-p}$. □

12. Quotients of Lie groups

For quotients of Lie groups, the general theory of Lie groups shows a versatility that the theory of linear Lie groups does not have. Toward the end of Chapter III, we saw an example of a linear analytic group with a nonlinear quotient group, and we saw an example in which the simply connected covering group of a linear group had no realization as a linear group. These kinds of bad behavior do not occur when we allow nonlinear groups into the discussion.

The starting point of our investigation is the fact that the quotient of an analytic group $\boldsymbol{M}$ by a subgroup G is Hausdorff if and only if the subgroup G is a closed subgroup. Proposition A.2 establishes that $\boldsymbol{M}/G$ is Hausdorff if G is closed, and the converse is addressed in a remark. Thus it makes sense to concentrate in this section on the case that G is closed.

Theorem 4.34. (Closed Subgroup Theorem). Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$, and let G be a closed subgroup with Lie algebra $\mathfrak{g} \subseteq \mathfrak{m}$. Choose a vector subspace $\mathfrak{s}$ of $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{g}$. Then there is an open neighborhood U of 0 in $\mathfrak{s}$ such that the mapping $F\colon U \times G \to \boldsymbol{M}$ given by

$$F(X, g) = (\exp X)g$$

is a diffeomorphism of $U \times G$ onto an open neighborhood of G in $\boldsymbol{M}$.

Proof. We write $|\cdot|$ for a norm on $\mathfrak{m}$. Form the derivative $(DF)_{(0,1)}$. In each variable $X \in \mathfrak{s}$ and $g \in G$, when the other variable is held fixed, F is the identity map, and thus $(DF)_{(0,1)}$ is invertible. By the Inverse Function Theorem, F is a diffeomorphism of some open neighborhood of $(0, 1)$ in $U \times G$ onto an open neighborhood of 1 in $\boldsymbol{M}$. Right translation by an element g_0 of G allows us to translate this conclusion to be valid instead about the point $(0, g_0)$ of $U \times G$. Thus about the point $(0, g_0)$, there is an open neighborhood on which F is one-one. Taking the union of the open neighborhoods of the points

$(0, g_0)$ produces an open set in $U \times G$ such that F is locally one-one from that open set into $\boldsymbol{M}$. The claim is that there is an open ball $U_\varepsilon = \{X \in \mathfrak{s} \bigm| |X| < \varepsilon\}$ such that F is globally one-one on $U_\varepsilon \times G$.

Assume the contrary. Then for any $\varepsilon > 0$, there exist distinct pairs (X_1, g_1) and (X_2, g_2) with X_1 and X_2 in $\mathfrak{s}$, g_1 and g_2 in G, $|X_1| < \varepsilon$, $|X_2| < \varepsilon$, and

$$(\exp X_1)g_1 = (\exp X_2)g_2.$$

Then

$$\exp(-X_2)(\exp X_1) = g_2 g_1^{-1}$$

For sufficiently small ε, as a consequence of the Inverse Function Theorem, the left side is a member of an open neighborhood of 1 in G where each member decomposes uniquely as a product $(\exp X_\mathfrak{s})(\exp X_\mathfrak{g})$ with $X_\mathfrak{s} \in \mathfrak{s}$ and $X_\mathfrak{g} \in \mathfrak{g}$, and thus we have

$$\exp(-X_2)(\exp X_1) = (\exp X_\mathfrak{s})(\exp X_\mathfrak{g}).$$

Since the left side and also $\exp X_\mathfrak{g}$ are in G, so is $\exp X_\mathfrak{s}$. In this equation we cannot have $X_\mathfrak{s} = 0$ because otherwise $\exp X_1 = (\exp X_2)(\exp X_\mathfrak{g})$ would give two distinct decompositions of a member of G close to 1 as a product of an exponential of a small member of $\mathfrak{s}$ and an exponential of a member of $\mathfrak{g}$. Thus $X_\mathfrak{s}$ is a *nonzero* member of $\mathfrak{s}$ for which $\exp X_\mathfrak{s}$ is in G.

Let ε run through a sequence $\{\varepsilon_k\}$ tending to 0, and repeat the above argument for ε_k, obtaining in place of $X_\mathfrak{s}$ a sequence $\{X_k\}$ of nonzero members of $\mathfrak{s}$ tending to 0, all of whose exponentials are in G. Passing to a subsequence if necessary, we may assume that the sequence $\{|X_k|^{-1}X_k\}$ converges (in the unit sphere of $\mathfrak{s}$) to a member X of $\mathfrak{s}$ with $|X| = 1$.

We shall show that X is in $\mathfrak{g}$. Let a positive real t be given, remember that $\{|X_k|\}$ is tending to 0, and choose integers n_k such that $\{n_k|X_k|\}$ converges to t. Then

$$(\exp X_k)^{n_k} = \exp(n_k X_k) = \exp\Big(n_k|X_k|\frac{X_k}{|X_k|}\Big),$$

and the right side tends to $\exp tX$ as t tends to infinity. The left side is in G for each k. Since G is closed, we conclude that $\exp tX$ is in G for all positive t. Since G is a group, the elements $\exp(-tX)$ and 1 are in G also, and we conclude from Theorem 4.10c that X is in the Lie algebra of G, which is $\mathfrak{g}$.

This conclusion contradicts the facts that X is a unit vector of $\mathfrak{s}$ and that $\mathfrak{s} \cap \mathfrak{g} = 0$. We thus find that the assumption that F fails to be globally one-one on some $U_\varepsilon \times G$ has led to a contradiction. Theorem 4.33 follows. □

As mentioned above, Proposition A.2 of Appendix A shows that when G is closed in $\boldsymbol{M}$, $\boldsymbol{M}/G$ is a Hausdorff space in the quotient topology. Also the quotient map $\boldsymbol{M} \to \boldsymbol{M}/G$ is continuous and open, and the group action $\boldsymbol{M} \times \boldsymbol{M}/G \to \boldsymbol{M}/G$ given by $(m_1, m_2G) \mapsto (m_1m_2)G$ is continuous. We form charts on the Hausdorff space $\boldsymbol{M}/G$ as follows: Write $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{g}$ for some vector subspace $\mathfrak{s}$, and fix an open set U of $\mathfrak{s}$ as in Theorem 4.34. We define one chart for each point m_0 of $\boldsymbol{M}$. The chart indexed by $m_0 \in \boldsymbol{M}$ is (U, α_{m_0}), where α_{m_0} is equal to the inverse of

$$X \in U \quad \mapsto \quad m_0(\exp X)G \in \boldsymbol{M}/G.$$

This construction leads to a manifold structure on $\boldsymbol{M}/G$, according to the following result.

Corollary 4.35. Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$, and let G be a closed subgroup with Lie algebra $\mathfrak{g} \subseteq \mathfrak{m}$. With the vector subspace $\mathfrak{s}$ chosen so that $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{g}$ and with an open neighborhood U of 0 in $\mathfrak{s}$ chosen to make the mapping $F(x, g) = (\exp X)g$ of $U \times G$ into $\boldsymbol{M}$ be a diffeomorphism, the system of charts defined on $\boldsymbol{M}/G$ is an atlas for $\boldsymbol{M}/G$, and the group action $\boldsymbol{M} \times \boldsymbol{M}/G \to \boldsymbol{M}/G$ defined by $m_1(m_2G) = (m_1m_2)G$ is smooth. If the subgroup G is normal in $\boldsymbol{M}$, then the smooth manifold structure on $\boldsymbol{M}/G$ makes $\boldsymbol{M}/G$ into an analytic group.

Proof. The charts for $\boldsymbol{M}/G$ are restrictions of charts for $\boldsymbol{M}$, and the latter are compatible. Then it follows that the former are compatible. The group action of $\boldsymbol{M}$ on $\boldsymbol{M}/G$ comes from the composition of multiplication in $\boldsymbol{M}$ followed by passage to the quotient in $\boldsymbol{M}/G$:

$$\boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{M} \to \boldsymbol{M}/G \quad \text{given by} \quad (m_1, m_2) \to m_1m_2 \to (m_1m_2)G.$$

Passage to the quotient $\boldsymbol{M} \times \boldsymbol{M}/G \to \boldsymbol{M}/G$ is then well defined and continuous. The individual ingredients here are all smooth, and hence the group action on $\boldsymbol{M}/G$ is smooth.

If the subgroup G is normal, then the action $(a, bG) \to abG$ descends to a multiplication on $\boldsymbol{M}/G$ by

$$(ag, bG) \to (ag)bG = ab(b^{-1}gb)G = abG.$$

The result is smooth. Similarly inversion descends to an operation on $\boldsymbol{M}/G$, and thus $\boldsymbol{M}/G$ becomes an analytic group. $\square$

Corollary 4.36. Let $\boldsymbol{M}$ be an analytic group with Lie algebra $\mathfrak{m}$, let G be a closed subgroup with Lie algebra $\mathfrak{g}$, and let $p\colon \boldsymbol{M} \to \boldsymbol{M}/G$ be the quotient mapping. Then a numerical-valued function f on $\boldsymbol{M}/G$ is smooth if and only if the numerical-valued function $f \circ p$ on $\boldsymbol{M}$ is smooth.

Proof. If $f\colon \boldsymbol{M}/G \to \mathbb{R}$ is smooth, then $f \circ p\colon \boldsymbol{M} \to \mathbb{R}$ is smooth as the composition of smooth maps. Conversely let $f \circ p$ be smooth, fix $m_0 \in \boldsymbol{M}$, and form the chart (U, α_{m_0}) on $\boldsymbol{M}/G$ about m_0 in which α_{m_0} is the inverse of

$$X \in U \qquad \mapsto \qquad m_0(\exp X)G \in \boldsymbol{M}/G.$$

Then for $m_0(\exp X)G$ near m_0G in $\boldsymbol{M}/G$, $(f \circ p)(m_0 \exp X)$ is the same function of X as $f(m_0 \exp X)$. Since $(f \circ p)(m_0 \exp X)$ is smooth, so is $f(m_0 \exp X)$. $\square$

Example. Spheres in dimension ≥ 2. Let us see that the unit sphere S^n in $\mathbb{R}^{n+1}$ can be viewed as a quotient $SO(n+1)/SO(n)$. This identification will be valid both algebraically and topologically.

To make everything concrete, regard the rotation group $SO(n+1)$ as acting on the space $\mathbb{R}^{n+1}$ of $n+1$ dimensional column vectors by left multiplication. To see that the action is transitive,[4] let $v_1 = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}$ be any member of the unit sphere S^n, i.e., any column vector of Euclidean norm 1. Then we can extend this vector to a basis of $\mathbb{R}^n$, apply the Gram-Schmidt orthogonalization process, and obtain an orthonormal basis $v_1, \dots, v_{n+1}$ of $\mathbb{R}^{n+1}$ whose first member is v_1. We form the square matrix whose first column is v_1, whose second column is v_2, and so on, and the result will be a matrix in $O(n+1)$ because the columns are orthonormal. If this matrix is not in $SO(n+1)$, i.e., if its determinant is -1 rather than $+1$, we can adjust it by replacing v_n by $-v_n$ in the construction,[5] and then the matrix will have determinant $+1$ and will therefore be in $SO(n+1)$.

In any event what we see is that there exists a member of $SO(n+1)$ carrying $e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$ into v_1. Thus $SO(n+1)$ acts transitively on the unit sphere S^n of $\mathbb{R}^{n+1}$. The subset of $SO(n+1)$ that fixes e_1 is the subset of $SO(n+1)$ with entry 1 on the upper left, and this subset is a subgroup isomorphic to $SO(n)$. Thus we have an identification of S^n set theoretically as $SO(n+1)/SO(n)$.

To bring the topology into the discussion, we refer to Proposition A.2 of Appendix A. The map q of $SO(n+1)$ into $S^n \subseteq \mathbb{R}^{n+1}$ given by $r \mapsto re_1$ is

[4]A group action on a set is **transitive** if for any two points p and q of the set, there exists a member g of the group such that $gp = q$. It is enough to show for some point p_0 that each point of the set is the image of p_0 under some member of the group.

[5]This adjustment is possible under the assumption that $n \geq 2$.

continuous and descends to a continuous map q' of $SO(n+1)/SO(n)$ into S^n. This descended map q' is one-one and onto. Since $SO(n+1)/SO(n)$ is compact and S^n is Hausdorff, q'^{-1} is continuous.

$$\begin{tikzcd} SO(n+1) \arrow[d, "p"'] \arrow[dr, "q"] & \\ SO(n+1)/SO(n) \arrow[r, "q'"'] & S^n \end{tikzcd}$$

The manifold structure on $SO(n+1)/SO(n)$ in the example is specified in Corollary 4.35, and the manifold structure on S^n is specified in Section B.1 of Appendix B. Comparing these structures, we find that the map of $SO(n+1)/SO(n)$ onto S^n is a diffeomorphism.

Parenthetically let us observe that $SO(n)$ is connected as a consequence of the above argument. This connectedness follows by induction from the above example and Proposition A.3.

The ease of handling the above example resulted from the compactness of $SO(n+1)$. However the compactness turns out not to be essential to the conclusion that the map q' is a diffeomorphism, as we shall see in Theorem 4.38 below. First let us observe in Theorem 4.37 that any closed subgroup of a Lie group is a Lie group whose natural topology on its identity component is its relative topology; this result generalizes Theorem 1.10 from the linear case to the general case.

Theorem 4.37. If M is an analytic group with Lie algebra $\mathfrak{m}$ of dimension m and if G is a closed subgroup, then G with its relative topology becomes a Lie group in a unique way such that

(a) the restrictions from M to G of any smooth numerical-valued function are smooth and
(b) whenever $\Phi : A \to M$ is a smooth function on a smooth manifold A such that $\Phi(A) \subseteq G$, then $\Phi : A \to G$ is smooth.

Moreover there exist open neighborhoods U of 0 in the Lie algebra $\mathfrak{g}$ of G and V of 1 in G such that $\exp\colon U \to V$ is a homeomorphism onto and such that $(V, \exp^{-1})$ is a compatible chart.

REMARK. This theorem asserts that the natural topology on G_0 coincides with the relative topology as a subset of M. It says by implication that G has only countably many components and that they are open.

PROOF. The argument is the same as for Theorem 1.10 or Corollary 2.8. $\square$

Theorem 4.38. Let G be a Lie group, and let $\boldsymbol{M}$ be a smooth manifold on which there is a smooth *transitive* group action $G \times \boldsymbol{M} \to \boldsymbol{M}$. Fix m_0 in $\boldsymbol{M}$, let H be the subgroup of all $h \in G$ such that $hm_0 = m_0$, and define $q\colon G \to \boldsymbol{M}$ by $q(g) = gm_0$. Then the function q' that carries $\boldsymbol{M}/G$ onto $\boldsymbol{M}$, given by $q'(gH) = gm_0$, is well defined and is a diffeomorphism.

Remarks. For example, $\boldsymbol{M}$ can be S^n, G can be $SO(n+1)$, and H can be $SO(n)$, embedded in $SO(n+1)$ as in the discussion above.

Proof. Let $p\colon G \to G/H$ be the quotient map. The function $q = q'p$ is one-one onto as a consequence of abstract set theory. We shall prove that q' is smooth and that Dq' is nonsingular at each point of G/H, and then the theorem will follow.

Since q is the restriction to the G variable of the smooth group action $G \times \boldsymbol{M} \to \boldsymbol{M}$ when the $\boldsymbol{M}$ variable equals m_0, q is smooth. Corollary 4.36 shows that q' is smooth on $\boldsymbol{M}/G$ if and only if q is smooth on $\boldsymbol{M}$, and thus q' is smooth.

We are left with proving that Dq' is everywhere nonsingular. For this purpose let us examine $Dq = (Dq')(Dp)$ at each point g of G. At g, we shall examine the equation

$$(Dq)_g = (Dq')_{gH}(Dp)_g \tag{4.29}$$

in order to prove that $(Dq')_{gH}$ is nonsingular.

First we consider a special case. Write $\mathfrak{g}$ for the Lie algebra of G, and let $\mathfrak{h}$ be the Lie algebra of H. The special case in question is that $g = 1$. In this case the equation under study is

$$(Dq)_1 = (Dq')_H(Dp)_1. \tag{4.30}$$

The dimension $\dim \mathfrak{g}$ of the domain $\mathfrak{g} = T_1(G)$ of the linear mapping $(Dq)_1$ equals the sum of the dimension of the kernel and the dimension of the image. As to the kernel and the image, the kernel of $(Dq)_1$ includes the kernel of $(Dp)_1$. The image of $(Dp)_1$ is $\mathfrak{g}/\mathfrak{h}$, and $\mathfrak{g}/\mathfrak{h}$ is also the domain of (Dq'_H). Showing that $(Dq')_H$ is nonsingular thus comes to the same thing as showing that the kernel of $(Dq)_1$ is no larger than $\mathfrak{h}$.

Arguing by contradiction, suppose that the inclusion

$$\operatorname{kernel}(Dq)_1 \supseteq \mathfrak{h} \tag{4.31}$$

is strict. Let X be a member of $(Dq)_1$ that is not in $\mathfrak{h}$. This means that $X(m_0) = 0$ but X is not in $\mathfrak{h}$. Then $q(\exp tX)$ is given by

$$q(\exp tX) = (\exp tX)m_0 = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n(m_0) = m_0$$

for every real t, and we deduce for every t that $\exp tX$ is a member of the subgroup of G that fixes m_0, namely H. Thus X is in $\mathfrak{h}$, in contradiction to the hypothesis about X. We conclude therefore that the inclusion (4.31) is not strict and hence that $(Dq')_{gH}$ is nonsingular at the point $1H$ of G/H with $g = 1$.

This completes the argument in the special case. We return to a proof of the nonsingularity of $(Dq')_{gH}$ for general g, suitably adapting the argument for $g = 1$. The starting point is (4.29). The image of $(Dp)_g$ is $\mathfrak{g}/\mathrm{Ad}(g^{-1})\mathfrak{h}$, and $\mathfrak{g}/\mathrm{Ad}(g^{-1})\mathfrak{h}$ is also the domain of $(Dq')_{gH}$. Showing that $(Dq')_{gH}$ is nonsingular comes to the same thing as showing that the kernel of $(Dq)_g$ is no larger than $\mathrm{Ad}(g)^{-1}\mathfrak{h}$. Arguing by contradiction, suppose that the inclusion $\mathrm{kernel}(Dq)_g \supseteq \mathrm{Ad}(g)^{-1}\mathfrak{h}$ is strict. Let X be an element in $\mathrm{kernel}(Dq)_g$ but not in $\mathrm{Ad}(g)^{-1}\mathfrak{h}$. Then X has $X(gm_0) = 0$, and $(\mathrm{Ad}(g)^{-1}X)(m_0) = 0$. Summing the series for the exponential gives $\exp(\mathrm{Ad}(g)^{-1}tX)(m_0) = m_0$ for all real t. Hence $g^{-1}(\exp tX)g)$ is in the subgroup fixing m_0, namely H, and X must have been in the corresponding Lie algebra, namely $\mathrm{Ad}(g)^{-1}\mathfrak{h}$. This conclusion contradicts our assumption about X. Thus the kernel of $(Dq)_g$ must equal $\mathrm{Ad}(g)^{-1}\mathfrak{h}$, and $(Dq')_{gH}$ must be nonsingular. □

13. Centralizers and commutators

Two examples of closed subgroups of a Lie group G are the **center** Z_G of G and the **centralizer**[6] $Z_G(S)$ in G of any subset S of G. These are defined by the formulas

$$Z_G = \{x \in G \mid xy = yx \text{ for all } y \in G\},$$
$$Z_G(S) = \{x \in G \mid xy = yx \text{ for all } y \in S\}.$$

The center Z_G and the centralizer $Z_G(S)$ are indeed closed subgroups, as we see by examining what happens to convergent sequences in each case. The center is of course a special case of the centralizer, since $Z_G = Z_G(G)$.

Let G be a Lie group, and let S be a nonempty subset of G. Within the Lie algebra $\mathfrak{g}$ of G, the Lie subalgebras of Z_G and $Z_G(S)$ are given by

$$Z_{\mathfrak{g}}(G) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(y)X = X \text{ for all } y \in G\},$$
$$Z_{\mathfrak{g}}(S) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(y)X = X \text{ for all } y \in S\},$$

[6]Authors who are writing about finite groups often write $C_G(S)$ instead of $Z_G(S)$ for the centralizer of S in G. In the subject of Lie groups, the symbol Z is often used as an acknowledgment of contributions to the subject that were written in German, such as those by F. G. Frobenius.

as we readily check by considering curves $\exp tX$ that lie in G. The center $Z_{\mathfrak{g}}$ of $\mathfrak{g}$ is

$$Z_{\mathfrak{g}} = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

This equals $Z_{\mathfrak{g}}(G)$ if G is connected.

Proposition 4.39. Let G be an analytic group, and let H be an analytic subgroup. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h}$ be the Lie subalgebra corresponding to H. Then $H \subseteq Z_G$ if and only if $\mathfrak{h} \subseteq Z_{\mathfrak{g}}$.

Proof. Suppose that H is contained in the center of G. If X is in $\mathfrak{h}$, then the map $g \in G \mapsto (\exp tX)g(\exp tX)^{-1}$ is the identity, and so $\operatorname{Ad}(\exp tX) = 1$. Hence $\operatorname{ad} X = \frac{d}{dt} \operatorname{Ad}(\exp tX)\big|_{t=0} = 0$, and X is in the center of $\mathfrak{g}$. In the reverse direction if $\mathfrak{h}$ is contained in the center of $\mathfrak{g}$, then $\operatorname{ad} \mathfrak{h} = 0$ and therefore $\operatorname{Ad}(\exp \mathfrak{h}) = e^{\operatorname{ad}(\mathfrak{h})} = 1$. Since Ad is a homomorphism, $\operatorname{Ad}(h) = 1$ for all h in $\mathfrak{h}$. Finally if X is in $\mathfrak{g}$ and h is in $\mathfrak{h}$, then $h(\exp X)h^{-1} = \exp \operatorname{Ad}(h)X = \exp X$, and so h commutes with $\exp \mathfrak{g}$. Hence h commutes with G, and H lies in the center of G. □

In a group G, a commutator is any element $xyx^{-1}y^{-1}$ with x and y in G. The **commutator subgroup** of G is the subgroup $[G, G]$ generated by all commutators. The inverse of a commutator is a commutator because

$$(xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1},$$

and any conjugate of a commutator is a commutator because

$$a(xyx^{-1}y^{-1})a^{-1} = (axa^{-1})(aya^{-1})(ax^{-1}a^{-1})(aya^{-1}).$$

It follows that $[G, G]$ is always normal and that the quotient of G by $[G, G]$ is always abelian as an abstract group.

Theorem 4.40. If G is an analytic group with Lie algebra $\mathfrak{g}$, then the commutator subgroup $[G, G]$ of G is an analytic subgroup and the corresponding Lie subalgebra of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$.

Proof. Applying Proposition 4.21 to the subgroup $[G, G]$ of G, we form the natural topology on $[G, G]$. Let us see that this natural topology is pathwise connected. In fact let x and y be in G. Since G is assumed to be a connected Lie group, x and y can be connected to the identity by paths $x(t)$ and $y(t)$ with $x(0) = y(0) = 1$, $x(1) = x$, and $y(1) = y$. Then $xyx^{-1}y^{-1}$ is connected to the identity by the path $x(t)y(t)x(t)^{-1}y(t)^{-1}$ with $0 \leq t \leq 1$. Consequently $[G, G]$ is pathwise connected in its natural topology, and the corresponding group $[G, G]_0$ is all of $[G, G]$. Theorem 4.21 allows us to conclude that $[G, G]$ is an analytic subgroup of G.

If X and Y are in $\mathfrak{g}$, then $\exp sX$ and $\exp tY$ are in G, and each element $\exp(-sX)\exp(-tY)\exp sX\exp tY$ is in $[G, G]$. Taking $s = t > 0$ and applying (4.11) allows us to see that the image for $t > 0$ of $t \mapsto \exp\big(t[X, Y] + O(t^{3/2})\big)$ lies in the commutator subgroup $[G, G]$. Replacing t by $-t$ shows that the image for $t < 0$ of $t \mapsto \exp\big(t[X, Y] + O(t^{3/2})\big)$ lies in $[G, G]$ also. From Theorem 4.10c we conclude that $[X, Y]$ lies in the Lie subalgebra corresponding to $[G, G]$. Thus $[\mathfrak{g}, \mathfrak{g}]$ lies in the Lie subalgebra corresponding to $[G, G]$.

For the reverse inclusion it is enough to show that $[G, G]$ lies in the analytic subgroup corresponding to $[\mathfrak{g}, \mathfrak{g}]$. Thus it is enough to show for X and Y small that each commutator $c(t) = \exp(X)\exp(tY)\exp(-X)\exp(-tY)$ with t small lies in the analytic subgroup corresponding to $[\mathfrak{g}, \mathfrak{g}]$. As long as t, X, and Y are sufficiently small, we can use Dynkin's Formula (Theorem 4.18) to write

$$c(t) = \big(\exp(X)\exp(tY)\big)\big(\exp(-X)\exp(-tY)\big) = \exp(Z_1)\exp(Z_2)$$

with Z_1 equal to $X + tY + tW_1$, with Z_2 equal to $= -X - tY + tW_2$, and with W_1 and W_2 each equal to the sum of a convergent series whose terms have a polynomial in t as one factor and a monomial in X and Y involving a factor of $[X, Y]$. Applying Theorem 4.18 similarly to the product $\exp(Z_1)\exp(Z_2)$, we see that $c(t)$ equals the sum of a convergent series whose terms each have a polynomial in t as one factor and a monomial in X and Y with a factor of $[X, Y]$. Since $[\mathfrak{g}, \mathfrak{g}]$ is closed as a vector subspace of $\mathfrak{g}$, the sum of the series is in $[\mathfrak{g}, \mathfrak{g}]$. In other words, as long as t, X, and Y are sufficiently small, $c(t)$ is the exponential of a member of $[\mathfrak{g}, \mathfrak{g}]$. Thus $[G, G]$ lies in the analytic subgroup corresponding to $[\mathfrak{g}, \mathfrak{g}]$. $\square$

14. Summary

Chapter IV extends the development of Lie theory to groups that are not necessarily linear. Chapters I through III showed how in a linear group one can pass back and forth between analytic subgroups of a given general linear group and Lie subalgebras of the corresponding full matrix algebra. They showed also how to associate a Lie algebra homomorphism to each smooth homomorphism from one linear analytic group to another. The reverse association—of a homomorphism between analytic groups to a homomorphism of their Lie algebras—was proved only under the assumption that the domain is simply connected. Chapter IV concerns extending these same conclusions to all analytic groups, whether or not they are linear. It makes extensive use of Appendix B.

Two tools are needed to make any progress at all in generalizing the theory to groups that are not linear: a broader definition of Lie algebra and a generalization of the exponential map. In its early sections Chapter IV gives an abstract definition of real Lie algebra and shows that the set of left invariant vector fields on the group meets the conditions of the definition. Accordingly the Lie algebra of left invariant vector fields is taken to be the Lie algebra of the given Lie group. The exponential mapping from the Lie algebra to the Lie group can then be formed by piecing together integral curves for all the left invariant vector fields.

With these tools in hand, the chapter shows how to modify the arguments in the linear case so that the theory applies to all Lie groups, whether linear or not. These modifications may be subtle, but they are not complicated. For example one applies the linear version of Dynkin's Formula to the adjoint representation of the Lie algebra in order to make the convergent series occur in a matrix space, and then the arguments with Dynkin's Formula in Chapter III go through. In constructing the analytic subgroup corresponding to a Lie subalgebra, one again forms the union of the one-parameter groups that the analytic subgroup must contain, introduces the natural topology, and passes to the identity component.

Just as in Chapter III, if a smooth homomorphism is given between analytic subgroups, the corresponding map of Lie algebras is given by the derivative at the identity. Dynkin's Formula ensures that the smooth homomorphism is determined in a full neighborhood of the identity by the derivative at the identity. A supplementary argument shows that the derivative at the identity lifts to a smooth homomorphism on the group level if the domain is simply connected.

Sections 11 to 13 handle loose ends. The loose end in Section 11 is the classification of abelian analytic groups. The one in Section 12 is the identification as quotient spaces of analytic groups those smooth manifolds on which an analytic group acts smoothly and transitively. The one in Section 13 is the correspondence between Lie groups and Lie algebras in centralizers and commutator subgroups.

Chapter IV has completed for all analytic groups the proofs of Fundamental Theorems A and B of Lie Theory that were announced in the Preface. Fundamental Theorem D is an immediate consequence of results here and in Appendix C of the present volume, as follows: For existence of the simply connected covering group, Theorem C.12 establishes the existence of a universal covering space, Proposition C.22 shows how to make it into a group, and Proposition 4.28 shows how to give it the structure of a smooth manifold. For uniqueness Corollary 4.27 asserts that any two simply connected analytic groups with isomorphic Lie algebras are isomorphic as Lie groups.

Fundamental Theorem C is not addressed in this book but instead is addressed in an appendix of the author's volume entitled *Lie Groups Beyond an Introduction* that was published earlier. A key step in the proof is to show that any finite dimensional real Lie algebra is the semidirect product of its radical and a semisimple Lie algebra.

15. Problems

1. Why is there only one Lie algebra of dimension 1 over $\mathbb{R}$ up to isomorphism?
2. For a nonabelian Lie algebra of dimension 2 over $\mathbb{R}$, show that there exists an ordered basis (X, Y) such that $[X, Y] = Y$. Conclude that up to isomorphism there are exactly two Lie algebras of dimension 2 over $\mathbb{R}$, that one is abelian, and that the nonabelian one is solvable with a basis (X, Y) such that $[X, Y] = Y$.
3. Find the cardinality of the centers of the groups $SU(n)$, $SO(n)$, $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, and $Sp(n, \mathbb{C})$.
4. Let $G = \{g \in SL(2n, \mathbb{C}) \mid g^t I_{n,n} g = I_{n,n}\}$, with $I_{n,n}$ defined the same way as in Problems 24–25 at the end of Chapter I. Prove that G is isomorphic to $SO(2n, \mathbb{C})$.
5. The Lie algebra $\mathfrak{u}(n)$ is isomorphic to the direct sum $\mathfrak{su}(n) \oplus \mathbb{R}$, where $\mathbb{R}$ is the center. Let Z be the analytic subgroup of $U(n)$ with Lie algebra the center $\mathbb{R}$. Identify Z as an explicit group of matrices, and explain whether $U(n)$ is isomorphic with the direct product of $SU(n)$ and Z.

Problems 6–8 introduce quotients of Lie algebras by ideals.

6. Let $\mathfrak{a}$ be an ideal in a Lie algebra $\mathfrak{g}$. Define a product operation on the quotient vector space $\mathfrak{g}/\mathfrak{a}$ by $[X + \mathfrak{a}, Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$.
   (a) Show that this product makes $\mathfrak{g}/\mathfrak{a}$ into a Lie algebra.
   (b) Show that the quotient mapping $X \mapsto X + \mathfrak{a}$ of $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{a}$ is a Lie algebra homomorphism onto and that its kernel is $\mathfrak{a}$.
7. Prove the **First Isomorphism Theorem** for Lie algebras: If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then the correspondence $\mathfrak{h} \to \mathfrak{h}/\mathfrak{a}$ is a one-one inclusion-preserving correspondence of the set of Lie subalgebras that contain $\mathfrak{a}$ onto the set of subalgebras of $\mathfrak{g}/\mathfrak{a}$. Furthermore any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ if and only if $\mathfrak{h}/\mathfrak{a}$ is an ideal of $\mathfrak{g}/\mathfrak{a}$.
8. Prove the **Second Isomorphism Theorem** for Lie algebras: If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$, then $\mathfrak{g}/\mathfrak{a} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$.

Problems 9–11 concern solvable Lie algebras. In each problem, all the Lie algebras that appear are assumed to be finite dimensional.

9. Prove that any Lie subalgebra or homomorphic image of a solvable Lie algebra is solvable.

10. Prove that if α is a solvable ideal in $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{a}$ solvable, then $\mathfrak{g}$ is solvable.

11. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Define $\operatorname{rad}\mathfrak{g}$, the **radical** of $\mathfrak{g}$, to be the sum of all solvable ideals in $\mathfrak{g}$. Show that $\operatorname{rad}\mathfrak{g}$ is a solvable ideal and that it contains all solvable ideals of $\mathfrak{g}$.

Problems 12–19 concern nilpotent Lie algebras. In each case, $\mathfrak{g}$ is a finite-dimensional Lie algebra. For such a $\mathfrak{g}$, let us define inductively $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}]$ for $j \geq 0$. Then

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \mathfrak{g}_3 \supseteq \cdots$$

is a decreasing sequence of ideals in $\mathfrak{g}$ and is called the **lower central series** of $\mathfrak{g}$. One says that $\mathfrak{g}$ is **nilpotent** if the lower central series ends in 0.

12. Prove that every nilpotent finite-dimensional Lie algebra is solvable.

13. Prove that any Lie subalgebra or homomorphic image of a nilpotent Lie algebra is nilpotent.

14. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Define $\operatorname{nil}\mathfrak{g}$, the **nilradical** of $\mathfrak{g}$, to be the sum of all nilpotent ideals in $\mathfrak{g}$. Show that $\operatorname{nil}\mathfrak{g}$ is a nilpotent ideal and that it contains all nilpotent ideals of $\mathfrak{g}$.

15. Prove that the member $[\mathfrak{g}, \mathfrak{g}]_n$ of the lower central series of $[\mathfrak{g}, \mathfrak{g}]$ and the member $\mathfrak{g}^{2n+1}$ of the commutator series of $\mathfrak{g}$ are related by $[\mathfrak{g}, \mathfrak{g}]_n \subseteq \mathfrak{g}^{2n+1}$ for $n \geq 0$.
    (a) Deduce that if $\mathfrak{g}$ is a solvable Lie algebra, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.
    (b) Deduce that if $\mathfrak{g}$ is a solvable Lie algebra, then $\mathfrak{g}/\operatorname{nil}\mathfrak{g}$ is abelian.

16. In the full matrix space $\mathfrak{m}_{nn}(\mathbb{R})$, let $\mathfrak{g}$ be the vector subspace of all matrices X having $X_{ij} = 0$ whenever $i \geq j$. Prove that $\mathfrak{g}$ is a Lie subalgebra and is nilpotent.

17. The special case of Problem 16 when $n = 3$ is known as the three dimensional **Heisenberg algebra.** Show that the three dimensional Heisenberg algebra is the linear Lie algebra of the Heisenberg group defined in Problem 7 for Chapter I.

18. The real Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} i\theta & 0 & 0 \\ z & -2i\theta & 0 \\ it & \bar{z} & i\theta \end{pmatrix} \,\middle|\, z \in \mathbb{C},\ \theta \in \mathbb{R},\ t \in \mathbb{R} \right\}$$

is the **oscillator Lie algebra**, namely the linear Lie algebra of the "oscillator group."
    (a) The oscillator group is known to be connected. What matrices are in the oscillator group?

(b) Show that $[\mathfrak{g}, \mathfrak{g}]$, the Lie algebra of all linear combinations of brackets of two members of the oscillator Lie algebra, is isomorphic to the three dimensional Heisenberg algebra.

19. The **Heisenberg algebra** $\mathfrak{h}_n$ of dimension $2n+1$ is a Lie algebra with an ordered basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n, c\}$ such that

$$[p_i, p_j] = [q_i, q_j] = 0, \quad [p_i, q_j] = \delta_{ij} c, \quad [c, p_i] = [c, q_j] = 0$$

for all i and j with $1 \le i \le n$ and $1 \le j \le n$. Take for granted that $\mathfrak{h}_n$ is a Lie algebra, and prove that it is nilpotent.

Problems 20–22 concern simple and semisimple Lie algebras. In these problems let $\mathfrak{g}$ be a finite dimensional Lie algebra. The Lie algebra $\mathfrak{g}$ is **simple** if it is nonabelian and it has no proper nonzero ideals. It is **semisimple** if $\operatorname{rad} \mathfrak{g} = 0$.

20. Prove that
 (a) every simple Lie algebra $\mathfrak{g}$ has $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$,
 (b) every simple Lie algebra is semisimple, and
 (c) every semisimple Lie algebra has 0 center.

21. If $\mathfrak{g}$ is a finite dimensional Lie algebra, prove that $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is semisimple.

22. Prove that if $\mathfrak{g}$ has dimension 3, then either $\mathfrak{g}$ is solvable or $\mathfrak{g}$ is simple.

Problems 23–25 identify, possibly with repetitions, all solvable Lie algebras $\mathfrak{g}$ of dimension 3 over $\mathbb{R}$, up to isomorphism. Let $\mathfrak{g}$ be such a Lie algebra. The argument proceeds by examining each of the possibilities for $\dim[\mathfrak{g}, \mathfrak{g}]$, which is an integer from 0 to 3. If this number is 0, $\mathfrak{g}$ is abelian. If it is 3, then the commutator series has all terms equal to $\mathfrak{g}$; thus $\mathfrak{g}$ is not solvable, and Problem 22 shows that it must be simple.

23. Prove that if $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, then $\mathfrak{g}$ is isomorphic to either the three dimensional Heisenberg Lie algebra of Problem 17 or to the direct sum of an abelian Lie algebra of dimension 1 and a nonabelian Lie algebra of dimension 2.
24. Suppose that $\dim[\mathfrak{g}, \mathfrak{g}] = 2$.
 (a) Prove that $[\mathfrak{g}, \mathfrak{g}]$ is abelian.
 (b) Extend the basis $\{X, Y\}$ of $[\mathfrak{g}, \mathfrak{g}]$ to a basis $\{X, Y, Z\}$ of $\mathfrak{g}$. Define α, β, γ, δ by

$$[X, Z] = \alpha X + \beta Y$$
$$[Y, Z] = \gamma X + \delta Y.$$

 Show that $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is nonsingular.
 (c) Show that α is characterized as follows: Let $\{X, Y\}$ be any basis of $[\mathfrak{g}, \mathfrak{g}]$, let Z be any vector not in $[\mathfrak{g}, \mathfrak{g}]$, and define $\alpha, \beta, \gamma, \delta$ as in (b). Then α is the ratio of the larger absolute value of an eigenvalue of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ to the smaller one.

(d) Conclude that there are uncountably many solvable Lie algebras in dimension 3 that are mutually nonisomorphic.

25. Conclude that the only nilpotent Lie algebras of dimension 3 over $\mathbb{R}$, up to isomorphism, are the abelian one and the three dimensional Heisenberg algebra. Conclude that the only other solvable Lie algebras of dimension 3 up to isomorphism are those given by Problem 24 and the one that is a direct sum of an abelian Lie algebra of dimension 1 and a nonabelian Lie algebra of dimension 2.

Problems 26–30 show that the only simple Lie algebras $\mathfrak{g}$ of dimension 3 over $\mathbb{R}$, up to isomorphism, are $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$. Let $\mathfrak{g}$ be a such a Lie algebra. In view of the result in Problem 22, Problems 23–30 together classify all the Lie algebras of dimension 3 over $\mathbb{R}$. (Note that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$, are isomorphic as a consequence of the analysis done in Chapter II, Problems 1–2.)

26. Prove that $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ are not isomorphic.
27. Show that $\operatorname{Tr}(\operatorname{ad} X) = 0$ for all X in $\mathfrak{g}$ because $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.
28. Take for granted that because $\mathfrak{g}$ is not nilpotent, there exists some $X_0 \in \mathfrak{g}$ such that $\operatorname{ad} X_0$ is not nilpotent. Show that the one dimensional space $\mathbb{R}X_0$ has a complementary subspace stable under $\operatorname{ad} X_0$. (Educational note: The existence of X_0 is a consequence of a result known as Engel's Theorem that is not proved until the sequel *Lie Groups Beyond an Introduction*.)
29. Show by linear algebra that some real multiple X of X_0 is a member of a basis $\{X, Y, Z\}$ of $\mathfrak{g}$ in which $\operatorname{ad} X$ has matrix realization either

$$\operatorname{ad} X = \begin{array}{c} \begin{matrix} X & Y & Z \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array} \begin{matrix} X \\ Y \\ Z \end{matrix} \quad \text{or} \quad \operatorname{ad} X = \begin{array}{c} \begin{matrix} X & Y & Z \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} X \\ Y \\ Z \end{matrix}.$$

30. Writing $[Y, Z]$ in terms of the basis and applying the Jacobi identity, show that Y can be multiplied by a constant so that the first case of Problem 29 leads to an isomorphism with $\mathfrak{sl}(2,\mathbb{R})$ and the second case of Problem 29 leads to an isomorphism with $\mathfrak{so}(3)$.

Problems 31–32 introduce automorphisms and derivations of Lie algebras. In these problems, let $\mathfrak{g}$ be a finite dimensional Lie algebra. An **automorphism** of $\mathfrak{g}$ is a member Φ of $GL(\mathfrak{g})$ such that $[\Phi(X), \Phi(Y)] = \Phi([X, Y])$ for all X and Y in $\mathfrak{g}$; the group of all such automorphisms is written as $\operatorname{Aut}_{\mathbb{R}} \mathfrak{g}$. A **derivation** of $\mathfrak{g}$ is a member φ of $\operatorname{End}_{\mathbb{R}}(\mathfrak{g})$ such that $\varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)]$ for all x and Y in $\mathfrak{g}$. The vector space of all such derivations is written as $\operatorname{Der}_{\mathbb{R}} \mathfrak{g}$.

31. Let $\mathfrak{g}$ be a finite dimensional Lie algebra.
    (a) Why is $\operatorname{Aut}_{\mathbb{R}} \mathfrak{g}$ a closed linear group?
    (b) Show that $\operatorname{Der}_{\mathbb{R}} \mathfrak{g}$ is a Lie subalgebra of $\operatorname{End}_{\mathbb{R}}(\mathfrak{g})$ under Lie bracket and that it is in fact the linear Lie algebra of $\operatorname{Aut}_{\mathbb{R}} \mathfrak{g}$.

(c) Show that if G is an analytic group with Lie algebra $\mathfrak{g}$, then each $\mathrm{Ad}(g)$ for g in G is in $\mathrm{Aut}_{\mathbb{R}}\,\mathfrak{g}$.

32. Let $\mathfrak{g}$ be a finite dimensional Lie algebra.
   (a) Show that each $\mathrm{ad}\,X$ for $X \in \mathfrak{g}$ is in $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{g}$.
   (b) For some $\mathfrak{g}$, exhibit an example of a member of $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{g}$ that is not in $\mathrm{ad}\,\mathfrak{g}$.

Problems 33–35 concern Sections 12–13.

33. Prove the following result about local coordinates in quotients by closed subgroups: if $\boldsymbol{M}$ is an analytic group with Lie algebra $\mathfrak{m}$ of dimension m and if G is a closed subgroup with Lie algebra $\mathfrak{g}$ of dimension k, then about any point g_0 of G within $\boldsymbol{M}$, there exists a compatible chart (U, φ) with $\varphi = (x_1, \dots, x_m)$ such that the elements of G lying in U are precisely those members p of U for which $x_j(p) = 0$ for $m - k + 1 \le j \le m$.

34. Let $G = GL(n, \mathbb{R})$, let D be the diagonal subgroup, and let S be the subgroup of scalar multiples of the identity. Find
   (a) $Z_G(S)$ and $Z_G(D)$, and
   (b) $Z_G(Z_G(S))$ and $Z_G(Z_G(D))$.

35. Let G be a simply connected analytic group. Apply Theorem 4.27 to lift the Lie algebra homomorphism from $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to a group homomorphism $G \to G/[G, G]$, and conclude that the commutator subgroup of any simply connected analytic group is a closed subgroup.

Problem 36 and Proposition 4.9 address smoothness of certain homomorphisms that are assumed only to be continuous. An important branch of the subject of representation theory works with continuous group homomorphisms of topological groups into some $GL(n, \mathbb{C})$. When the topological group G is an analytic group, one gets a handle on the situation if the homomorphism is smooth. Problem 36 and Proposition 4.9 lead to the conclusion that smoothness is automatic in this situation. Thus in studying a continuous homomorphism of an analytic group into $GL(n, \mathbb{C})$, one may make use of the corresponding Lie algebra homomorphisms of the Lie algebra of G into $\mathfrak{gl}(n, \mathbb{C})$.

36. Using the result of Proposition 4.9 in combination with canonical coordinates as in Section 3, prove that any continuous homomorphism φ of one analytic group into another is smooth.

Problems 37–39 below construct a smooth homomorphism of $SU(2) \times SU(2)$ onto $SO(4)$ with two-element kernel and thereby exhibit $SU(2) \times SU(2)$ as a universal covering group of $SO(4)$. These problems make use of material in Appendix C on universal covering groups and also Problems 1–5 of Chapter II, which construct a smooth homomorphism of $SU(2)$ onto $SO(3)$, and Problems 8–11 of Chapter II, which relate $SU(2)$ and $SO(3)$ to the algebra $\mathbb{H}$ of quaternions.

37. Exhibit the Lie algebra $\mathfrak{so}(4)$ as the direct sum of two ideals, both of which are isomorphic as Lie algebras to $\mathfrak{so}(3)$. To do so, let A_1, A_2, A_3 be the standard basis of the upper left copy of $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$. With suitable choices of signs, introduce B_1, B_2, B_3 as the result of operating on A_1, A_2, A_3, respectively, by interchange of indices 2 and 3 with 1 and 4, by interchange of indices 1 and 3 with 2 and 4, and by interchange of indices 1 and 2 with 3 and 4. Using suitable signs, let $X_i = \frac{1}{2}(\pm A_i + \pm B_i)$ and $Y_i = \frac{1}{2}(\pm A_i - \pm B_i)$ in such a way that $[X_i, Y_j] = 0$ for all i and j. Show that the consequence is that X_1, X_2, X_3 span one copy of $\mathfrak{so}(3)$ and Y_1, Y_2, Y_3 span the other copy of $\mathfrak{so}(3)$.
38. Deduce that the universal covering group $\widetilde{SO}(4)$ of $SO(4)$ is isomorphic with the product $SU(2) \times SU(2)$ and that the center of $\widetilde{SO}(4)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
39. Explain how the diagonal matrix of $SO(4)$ with diagonal entries -1 fits into this picture.

The remaining problems work with the linear Lie groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(n,\mathbb{C})$, which were defined in Problem 24 of Chapter I, and their linear Lie algebras $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(n,\mathbb{C})$, which were defined in Problem 25 of Chapter I. Each of these groups and Lie algebras plays a role in classical geometries of one kind or another. These linear Lie algebras are each **complex Lie algebras** in the sense that the bracket product is bilinear over $\mathbb{C}$, not merely over $\mathbb{R}$. The present interest, which is part of what this block of problems addresses, is in isomorphisms of low-dimensional cases of the Lie algebras. We shall be interested as well in isomorphisms and covering groups for the corresponding analytic groups. A **blinear form** on a finite-dimensional complex vector space V is defined to be a function $\langle \cdot, \cdot \rangle$ carrying $V \times V$ into $\mathbb{C}$ that is linear in each variable when the other variable is held fixed. It is **symmetric** if $\langle u_1, u_2\rangle = \langle u_2, u_1\rangle$ for all u_1 and u_2 in V. It is or **skew symmetric** if $\langle u_1, u_2\rangle = -\langle u_2, u_1\rangle$ for all u_1 and u_2 in V. In either case it is **nondegenerate** if no $u \neq 0$ has $\langle u, V\rangle = 0$. The **orthogonal space** $U^\perp$ to a vector subspace U of V is the space of all vectors $w \in V$ with $\langle u, w\rangle = 0$. If a bilinear form is symmetric or skew symmetric and if it is nondegnerate, then $\dim U + \dim U^\perp = \dim V$ for any vector subspace U.

Problems 40–43 prepare for computations with the linear Lie groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(n,\mathbb{C})$ by addressing the question of when their Lie algebras are simple.

40. Prove that the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ is simple for $n \geq 2$ in the sense of having no ideals other than 0 and $\mathfrak{g}$. Let $\mathfrak{h}$ be the (commutative) diagonal subalgebra, and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{i\neq j} \mathbb{C}E_{ij}$ be the decomposition of $\mathfrak{g}$ into simultaneous eigenspaces under $\mathrm{ad}(\mathfrak{h})$. Here E_{ij} is the matrix that is 1 in position (i, j) and is 0 elsewhere. The corresponding eigenvalue for $\mathrm{ad}(\mathfrak{h})$ comes from the equation
$$(\mathrm{ad}\, H)E_{ij} = (H_{ii} - H_{jj})E_{ij} = (e_i - e_j)(H)$$

and is thus the linear functional $e_i - e_j$ on $\mathfrak{h}$. Such linear functionals are called **roots**. The corresponding eigenspace $\mathfrak{g}_{e_i-e_j} = \mathbb{C}(E_{ij})$ is called a **root space**, and $\mathfrak{g} = \mathfrak{h} \oplus \sum_{i\neq j} \mathfrak{g}_{e_i-e_j}$ is called the **root space decomposition** of $\mathfrak{g}$. Observe that all the root spaces are one dimensional over $\mathbb{C}$.

To prove that $\mathfrak{g}$ is simple, argue by contradiction, assuming that $\mathfrak{g}$ is not simple. First dispose of the easy case that $\mathfrak{a}$ is a nonzero ideal in $\mathfrak{g}$ that is contained in $\mathfrak{h}$. The remaining case is that a nonzero ideal $\mathfrak{a}$ is not contained in $\mathfrak{h}$. Let $X = H + \sum X_\alpha$ be in $\mathfrak{a}$ with each X_α in $\mathfrak{g}_\alpha$ and with some $X_\alpha \neq 0$. Order the roots so that $e_1 > \cdots > e_n > 0 > -e_n > \cdots > -e_1$ and the sum of positive elements is positive. For the moment assume that there is some root $\alpha < 0$ with $X_\alpha \neq 0$, and let β be the smallest such α. Say $X_\beta = cE_{ij}$ with $i > j$ and $c \neq 0$. Form the iterated bracket product $u = [E_{1i}, [[X, E_{jn}]]$. Show that u is a nonzero multiple of E_{1n}. Complete the argument by using the presence of E_{1n} in $\mathfrak{a}$ to prove that $\mathfrak{a} = \mathfrak{g}$.

41. Prove that the Lie algebra $\mathfrak{sp}(2, \mathbb{C})$ is simple by forming a root space decomposition like that in the previous problem and then arguing with root vectors as in that problem. There are eight roots; they are of the form $\pm e_1 \pm e_2$, $\pm 2e_1$, $\pm 2e_2$.

42. Prove that the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ is simple for $n \geq 2$. The roots are of the form $\pm e_i \pm e_j$ with $1 \leq i < j \leq n$ and $\pm 2e_i$ for $1 \leq i \leq n$.

43. Prove that the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ is simple for $n \geq 5$. The roots are of the form $\pm e_i \pm e_j$ with $1 \leq i < j \leq n$ and $\pm e_i$ for $1 \leq i \leq n$.

Problems 44–48 construct a one-one complex-linear Lie algebra homomophism of $\mathfrak{sp}(2, \mathbb{C})$ onto $\mathfrak{so}(5, \mathbb{C})$, and then they interpret the result in terms of analytic groups.

44. A 4-by-4 complex matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is **symplectic**, i.e., is in the complex symplectic group $Sp(2, \mathbb{C})$, if $g^t J g = J$, where $J = J_{2,2}$ is the 4-by-4 matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since J^2 is minus the identity, it is the same to require that $g^t J^{-1} g = J^{-1}$. Write $g^\# = J g^t J^{-1}$. Check that
    (a) g is symplectic if and only if $g^\# g = 1$.
    (b) $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $x^\# = x$ if and only if b and c are skew symmetric and also $a = d^t$.
    (c) the group $Sp(2, \mathbb{C})$ acts on $\mathfrak{m}_{4,4}(\mathbb{C})$ by the formula $g \cdot x = g x g^\#$. In other words, x is in $\mathfrak{m}_{4,4}(\mathbb{C})$ and $g \in Sp(2, \mathbb{C})$ implies $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

45. Define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}_{4,4}(\mathbb{C})$ by $\langle x, y \rangle = \mathrm{Tr}(xy)$. Show that the action in Problem 44c of $Sp(2, \mathbb{C})$ on $\mathfrak{m}_{4,4}(\mathbb{C})$ preserves $\langle \cdot, \cdot \rangle$ and that each member of $Sp(2, \mathbb{C})$ fixes the 4-by-4 identity matrix 1_4.

46. As a consequence of Problem 45, the action of $Sp(2,\mathbb{C})$ maps the subspace V of $\mathfrak{m}_{4,4}(\mathbb{C})$ given by
$$V = \{x \in \mathfrak{m}_{4,4}(\mathbb{C}) \mid x^{\#} = x \text{ and } \langle x, 1_4\rangle = 0\}$$
into itself. Prove that $\dim V = 5$.

47. Show that the five members of $\mathfrak{m}_{4,4}(\mathbb{C})$ given by
$$\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & \\ -1 & 0 & & \end{pmatrix}, \quad \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix}$$
are members of V that are mutually orthogonal relative to $\langle \cdot, \cdot \rangle$, and explain how it follows that $Sp(2,\mathbb{C})$ has been mapped homomorphically into a subgroup of $SO(5,\mathbb{C})$. (The omitted entries in the five matrices are understood to be 0.)

48. How does it follow that the smooth homomorphism of $Sp(2,\mathbb{C})$ into $SO(5,\mathbb{C})$ constructed above is actually onto $SO(5,\mathbb{C})$. What is the kernel of this homomorphism?

Problems 49–53 construct a one-one complex-linear Lie algebra homomorphism of $\mathfrak{sl}(4,\mathbb{C})$ onto $\mathfrak{so}(6,\mathbb{C})$. and then they interpret the result in terms of analytic groups. They assume some knowledge of how to use the exterior algebra of a finite-dimensional complex vector space. See Chapter VI of Knapp [2016a] for background on exterior algebras.

49. Take for granted the validity of the universal mapping property of exterior algebras, a special case of which says the following: If ι is a function that embeds a complex vector space E in its exterior algebra as $\bigwedge^1(E) \subseteq \bigwedge(E)$, then whenever ℓ is any linear map of E into an associative algebra A with identity such that $\ell(v)^2 = 0$ for all $v \in E$, there exists a unique associative algebra homomorphism $L : \bigwedge(E) \to A$ with $L(1) = 1$ such that $L \circ \iota = \ell$.
   (a) Apply this universal mapping property to prove that the operation of $SL(4,\mathbb{C})$ on $\mathbb{C}^4$ extends to an operation of $SL(4,\mathbb{C})$ on $\bigwedge(\mathbb{C}^4)$ such that $g \cdot (x \wedge y) = gx \wedge gy$ for all $g \in SL(4,\mathbb{C})$ and all x and y in $\mathbb{C}^4$.
   (b) Check that the result is s group action in the sense that $(g_1g_2)\cdot w = g_1\cdot(g_2\cdot w)$ for every $w \in \wedge^2\mathbb{C}^4$.

50. Define a bilinear form $\langle u, v\rangle$ on $\wedge^2\mathbb{C}^4$ by
$$u \wedge v = \langle u, v\rangle(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \quad \text{for } u, v \in \wedge^2\mathbb{C}^4.$$

(a) Why is this form symmetric?

(b) Use the facts that

$$\langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle = 1, \quad \langle e_1 \wedge e_3, e_2 \wedge e_4 \rangle = -1, \quad \langle e_1 \wedge e_4, e_2 \wedge e_3 \rangle = 1,$$
$$\langle e_i \wedge e_j, e_k \wedge e_\ell \rangle = 0 \quad \text{whenever } \{i, j\} \cap \{k, \ell\} \neq \varnothing,$$

to check that

$$(e_1 \wedge e_2) \pm (e_3 \wedge e_4), \quad (e_1 \wedge e_3) \pm (e_2 \wedge e_4), \quad (e_1 \wedge e_4) \pm (e_2 \wedge e_3)$$

form an orthogonal basis of $\wedge^2 \mathbb{C}^4$.

(c) Conclude that $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric bilinear form on $\wedge^2 \mathbb{C}^4$.

51. Using that $gx \wedge gy \wedge gz \wedge gw = (\det g)(x \wedge y \wedge z \wedge w)$, prove that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is invariant under the action of $SL(4, \mathbb{C})$.

52. Explain how the previous problems show that the group action of $SL(4, \mathbb{C})$ factors through to a group action of $SO(6, \mathbb{C})$ on $\wedge^2 \mathbb{C}^4$.

53. How does it follow that the smooth homomorphism of $SL(4, \mathbb{C})$ into $SO(6, \mathbb{C})$ constructed above is actually onto $SO(6, \mathbb{C})$. What is the kernel of this homomorphism?

APPENDIX A

Topological Groups

Abstract. This appendix is a brief introduction to the subject of topological groups.

Section 1 defines topological groups, gives some examples, and establishes some standard properties.

Section 2 considers the quotient space of a topological group by a closed subgroup and relates properties of the quotient to properties of the original group. When the closed subgroup is normal, the quotient is again a topological group.

1. Topological groups

A **topological group** G is a Hausdorff space that is a group in such a way that multiplication is a continuous function from $G \times G$ into G and inversion is a continuous function from G into G.

EXAMPLES.

(1) Euclidean space under addition. If $\mathbb{R}^n = \{x = (x_1, \ldots, x_n)\}$ is given the topology associated with the Euclidean norm $|x|^2 = x_1^2 + \cdots + x_n^2$ and if vector addition is taken as the group operation, then $\mathbb{R}^n$ becomes a topological group. The continuity of the group operation amounts to the fact that the limit of a sum equals the sum of the limits. The continuity of inversion amounts to the continuity of multiplication by -1 on members of $\mathbb{R}^n$.

(2) $GL(n, \mathbb{R})$. Write $G = GL(n, \mathbb{R})$ for the set of all real n-by-n nonsingular matrices. Under the operation of matrix multiplication, G is a group. With the usual topology on n^2 dimensional space, G is the open subset where the determinant function is nonzero. The operation of group multiplication is given entry-by-entry by polynomial functions of the entries. These functions are all continuous, and thus multiplication is continuous. The continuity of inversion is a consequence of Cramer's Rule for computing determinants.

Proposition A.1. Every topological group G has the following properties:

(a) All left and right translations are homeomorphisms.
(b) To each neighborhood V of the identity 1 corresponds a neighborhood U of 1 such that $UU^{-1} \subseteq V$.
(c) G is regular as a topological space.

PROOF. For (a) the continuity of multiplication $(x, y) \mapsto xy$ implies the continuity of each translation, namely $x \mapsto ax$ or $x \mapsto xa$, because a translation is just the restriction of multiplication to one of the variables with the other variable held fixed. Translation by an element has a continuous inverse because the inverse function is another translation, specifically translation by the inverse element.

Statement (b) is a restatement of the continuity of the map $(x, y) \mapsto xy^{-1}$ at $(x, y) = (1, 1)$.

In (c) if a point x and a closed set F are given with $x \notin F$, we may assume that the point is 1, by (a). Find U by (b) for $V = G - F$. The claim is that $U \subseteq V$. In fact, if x is in $\overline{U} - U$, then xU is a neighborhood of x and so meets U. If y is in $xU \cap U$, then $y = xu$, and we see that $x = yu^{-1}$ is in $UU^{-1} \subseteq V$. □

2. Quotients

In the situation of a topological group G and a closed subgroup H, we shall work with the quotient space G/H, whose elements are all cosets gH for $g \in G$. We give G/H the **quotient topology**, as follows. A subset V of G/H is defined to be open if its inverse image in G under the quotient map is open, i.e., if $\{x \in G \mid x \in gH \text{ for some } g \in V\}$ is open in G.

Proposition A.2. Let H be a closed subgroup of a topological group G, and let G/H have the quotient topology. Then the quotient mapping $p\colon G \to G/H$ is open, and G/H is a Hausdorff regular space such that the action of G on G/H is jointly continuous. If G is separable, i.e., has a countable base for its topology,[1] then so does G/H.

REMARK. Conversely if the subgroup H is not closed, then G/H is not Hausdorff. Indeed, if H is not closed, then the one-point set $1H$ in G/H is not closed.

[1]Throughout this book, the word **separable** means "having a countable base."

Proof. The set $E \subseteq G/H$ is open if and only if $p^{-1}(E)$ is open. Let U be open in G. Then $p^{-1}(p(U)) = \bigcup_{h\in H} Uh$, which is open. So $p(U)$ is open, and p is an open map. The action of G on G/H is the composition

$$G \times G \xrightarrow{\text{multiplication}} G \xrightarrow{\text{quotient}} G/H$$

which is continuous.

If x is in G/H and F is a disjoint closed subset to be separated from x, we may assume $x = 1$ by this continuity. Choose a neighborhood U of 1 in G and a neighborhood N of 1 in G/H such that $UN \subseteq G/H - F$. Let us observe that $\overline{N} \subseteq UN$. In fact, if y is in $\overline{N} - N$, then $U^{-1}y$ is a neighborhood of y since we have established that p is an open map. Hence $U^{-1}y \cap N$ is not empty. Let z be a member. Then $z = u^{-1}y$, and $y = uz$ lies outside UN. Consequently G/H is regular as a topological space. Next, $p^{-1}(\{x\}) = xH$ is closed, and so $\{x\}$ is closed. Thus G/H is a regular T_1 space and must be Hausdorff. If $\mathcal{U}$ is a base for the topology on G, then $p\mathcal{U}$ is a base for the topology on G/H, since p is open. Hence if G has a countable base, so does G/H. □

Proposition A.3. Let H be a closed subgroup of a topological group G. If H and G/H are connected, then G is connected.

Proof. Suppose $G = U \cup V$ with U and V nonempty, open, and disjoint. For each x in G, xH is connected by Proposition A.1a and the connectedness of H. Since $xH \subseteq U \cup V$, we must have $xH \subseteq U$ or $xH \subseteq V$. Let

$$A = \{x \in G \mid xH \subseteq U\} \quad \text{and} \quad B = \{x \in G \mid xH \subseteq V\}.$$

We have seen that $G = A \cup B$. Since U and V are complements of each other, $A = \{x \in G \mid xH \cap V = \varnothing\}$. It follows that

$$A = \{x \in G \mid xH \cap p^{-1}pV = \varnothing\}.$$

Consequently Proposition A.2 shows that A is open in G. Similarly B is open in G. Since $p^{-1}pA = A$ and $p^{-1}pB = B$, we see that $G/H = pA \cup pB$ is a disjoint decomposition of G/H into open sets. By connectedness of G/H, one of pA and pB is empty, say pB. Then B is empty, $A = G$, and $U = G$. We conclude that G is connected. □

Proposition A.4. Let H be a closed subgroup of a topological group G. If H and G/H are compact, then G is compact.

Proof. Let $\mathcal{U}$ be an open cover of G. We seek a finite subcover. For each x in G, $\mathcal{U}$ is an open cover of xH, which is compact. Let $\mathcal{V}_x \subseteq \mathcal{U}$ be a finite subcover and let

$$V_x = \{y \in G \mid yH \text{ is covered by } \mathcal{V}_x\}.$$

We prove V_x is open. Fix y in V_x. For each $h \in H$, we can find $U_h \in \mathcal{V}_x$ with $yh \in U_{yh}$. By continuity of multiplication, we can then find open neighborhoods M_h of y and N_h of h such that $M_h N_h \subseteq Uh$. The open sets N_h together form a cover of H, and we let $\{N_{h_j}\}_{j=1}^k$ be a finite subcover. Then $M = \bigcap_{j=1}^k M_{h_j}$ is an open neighborhood of y. If z is in M, we show that z is in V_x. Let h be given and suppose h is in N_{h_j}. Then zh is in $M_{h_j}N_{h_j} \subseteq U_{h_j}$, and this is a member of $\mathcal{V}_x$. Hence z is in V_x, and we conclude that V_x is open. The open sets pV_x cover G/H. Since G/H is compact, we can extract a finite subcover $\{pV_{x_1}, \ldots, pV_{x_n}\}$. Since $X_{x_j} = p^{-1}pV_{x_j}$ for all j, $\{V_{x_1}, ..., V_{x_n}\}$ covers G. Then $\bigcup_{j=1}^n \mathcal{V}_{x_j}$ is the required finite subcover of $\mathcal{U}$. □

Proposition A.5. If a closed subgroup H of a topological group G is normal as a subgroup, then G/H is a topological group.

Proof. Let V be a neighborhood of 1 in G/H. Choose by Proposition A.2 a neighborhood U of 1 in G and a neighborhood N of 1 in G/H such that $UN \subseteq V$. Then pU and N are neighborhoods of 1 in G/H such that $(pU)N \subseteq V$. Hence multiplication is continuous at $(1, 1)$, therefore everywhere. If V is a neighborhood of 1 in G/H and U is an open neighborhood of 1 in G with $U^{-1} \subseteq p^{-1}(V)$, then $pU^{-1} \subseteq V$, and so inversion is continuous at 1, hence everywhere. Finally G/H is Hausdorff by Proposition A.2. □

Proposition A.6. Any open subgroup of a topological group is closed.

Proof. If G is the group and H is the open subgroup, then the set-theoretic equality $H = G - \bigcup_{x \notin H} xH$ shows that H is closed. □

Proposition A.7. In a topological group G, the identity component, i.e., the connected component G_0 of the identity l, is a closed normal subgroup. If G is locally connected, then its identity component G_0 is open, and G/G_0 has the discrete topology.

Proof. The image of $G_0 \times G_0$ under multiplication is a connected set containing 1. Hence $G_0G_0 \subseteq G_0$. Similarly $G_0^{-1} \subseteq G$. Thus G_0 is a group. It is closed because connected components are closed in any topological space. It is normal because the same argument shows $xG_0x^{-1} \subseteq G_0$ for each x in G.

If G is locally connected, then its connected components are open, and G_0 in particular is open. Hence so are the cosets of G_0. From Proposition A.2,

$p\colon G \to G/G_0$ is an open mapping, and thus G/G_0 has every subset open. $\square$

Proposition A.8. In a connected topological group G, any neighborhood of 1 generates G.

PROOF. Let V be a neighborhood of 1, and choose an open neighborhood U of 1, by continuity of inversion, such that $U = U^{-1} \subseteq V$. Set $H = \bigcup_{n=1}^{\infty} UU\cdots U$, with n factors in the n^{th} term. Then H is nonempty, is open, is closed under products, and is closed under inversion since $U = U^{-1}$. Thus H is an open subgroup. By Proposition A.6, H is closed. Since G is connected, $H = G$. Then V must generate G since $U \subseteq V$. $\square$

A subgroup H of a topological group G is said to be **discrete** if every subset of H is relatively open.

Proposition A.9. If G is a topological group and if a subgroup H is discrete, then H is a closed subgroup.

PROOF. Choose a neighborhood V of 1 in G so that $H \cap V = \{1\}$, and choose an open neighborhood U of 1 with $UU \subseteq V$, by Proposition A.1b. Arguing by contradiction, suppose that H is not closed. If x is in $\overline{H} - H$, then $U^{-1}x$ is a neighborhood of x and so must contain a member h of H. Write $u^{-1}x = h$. Then $u = xh^{-1}$ is in $\overline{H} - H$ and is in U. Since x is a limit point of H, we can find $h' \neq 1$ such that h' is in Uxh^{-1}. Then h' is in $U(xh^{-1}) \subseteq UU \subseteq V$. Since $H \cap V = \{1\}$, we conclude that $h' = 1$, contradiction. $\square$

Proposition A.10. If G is a connected topological group, then any discrete normal subgroup H of G lies in the center of G.

PROOF. Fix h in H. The map $g \mapsto ghg^{-1}$ carries G onto the subset $\{ghg^{-1} \mid g \in G\}$ of G. Since the continuous image of the connected set G has to be connected, the subset $C(h) = \{ghg^{-1} \mid g \in G\}$ of G is connected. On the other hand, $C(h) \subseteq H$ since H is normal. By assumption, H is discrete. Thus the connected set $C(h)$ must lie in a single connected component of H, and $C(h) = \{h\}$. In other words, h lies in the center of G. $\square$

APPENDIX B

Smooth Manifolds, Vector Fields, and Integral Curves

Abstract. This appendix introduces the differential topology that is needed as background for an introduction to Lie groups. The subject matter is the elementary structure of smooth manifolds, which is a topic in real analysis that makes use of metric spaces and point-set topology. It will sometimes be handy to have specific references, and this appendix will point to the relevant sections of the author's own books, *Basic Real Analysis* and *Advanced Real Analysis* for that purpose.

Section 1 presents the beginning definitions and results about smooth manifolds, smooth functions, and smooth maps. Only this one section from Appendix B is needed as background for Chapters I and II.

Section 2 continues the introduction of smooth manifolds by discussing tangent vectors and vector fields on smooth manifolds.

Section 3 introduces derivatives of smooth mappings. Some authors refer to derivatives of smooth mappings as differentials, but the use of the term "differential" in this connection runs the risk of confusing two objects that are isomorphic but not canonically isomorphic. A footnote in Section 3 provides more detail. Thus the term "differential" will not otherwise be used in this book.

Section 4 examines changes of coordinates on smooth manifolds in some detail. One result tells when a system of numerical-valued functions can define a compatible chart, two results examine consequences for coordinate systems in the domain or range when the derivative of a smooth mapping is either one-one or onto, and a fourth result says that when the derivative of a smooth mapping is one-one onto at a point, then the smooth mapping locally has a smooth inverse. The section goes on to examine two situations under which a smooth mapping has a useful effect on smooth vector fields.

Section 5 defines the product of two smooth manifolds, and it observes that the operation of a smooth vector field in one variable on a smooth function of two variables results in a smooth function of one variable.

Section 6 defines submanifolds for smooth manifolds to be subsets for which the inclusion mapping is smooth and everywhere regular. An infinite line of irrational

slope on a torus in $\mathbb{R}^2$ supplies an important example. In some fields of mathematics, such lines are not considered to be submanifolds, but the subject of Lie groups requires that one be able to deal with them, whatever they are called.

Section 7 defines integral curves for smooth vector fields, states an existence-uniqueness theorem for systems of ordinary differential equations, and examines the consequences of this theorem for the subject of integral curves.

1. Smooth manifolds, smooth functions, and smooth maps

This section introduces smooth manifolds, and it briefly develops the notions of smooth function and smooth map.

"Manifolds" in our treatment are built from "charts," each manifold has a uniform dimension, and each manifold will be the disjoint union of open sets with each one separable in the sense of having a countable base for its topology. The term "smooth" is used interchangeably with the term C^∞. The prototype for a manifold is the surface of a sphere in three dimensions. Let us discuss this case informally first and then return to develop the formal mathematics.

In the real world one describes the surface of the earth by means of "charts," with each chart containing a likeness of part of the earth's surface and with all the charts together describing the whole surface. The collection of charts is an "atlas." The sense in which a chart contains a likeness of part of the surface is that there is an understood one-one function ("map") from the one onto the other. In mathematics this function goes from a part of the surface into a likeness; in the real world it tends to go in the opposite direction, namely from the likeness into the surface.

Let M be a Hausdorff topological space that is the disjoint union of separable subspaces,[1] and fix an integer $m \geq 0$. A **chart** (M_α, α) on M of **dimension** m is a homeomorphism $\alpha : M_\alpha \to \alpha(M_\alpha)$ of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{R}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. We say that M is a **manifold** if there is an integer $m \geq 0$ such that each point of M has a chart of dimension m about it.

A **smooth structure** of dimension m on a manifold M is a family $\mathcal{F}$ of m dimensional charts with the following three properties:

[1]The treatment in Sections VIII.1–4 of *Advanced Real Analysis* does not assume separability of manifolds, but in anybody's definition, a Lie group has to be the disjoint union of separable manifolds.

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are smoothly **compatible** in the sense that $\beta\circ\alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha\cap M_\beta)$ of $\mathbb{R}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$, is smooth and has a smooth inverse,

(ii) the system of compatible charts (M_α, α) is an **atlas** in the sense that the domains M_α together cover M,

(iii) $\mathcal{F}$ is maximal among families of compatible charts on M.

A **smooth manifold** of **dimension** m is a manifold together with a smooth structure of dimension m. In the presence of an understood atlas, a chart will be said to be **compatible** if it is smoothly compatible with all the members of the atlas.

Once we have an atlas of compatible m dimensional charts for a manifold M, i.e., once (i) and (ii) are satisfied, then the family of *all* compatible charts satisfies (i) and (iii), as well as (ii), and therefore is a smooth structure. In other words, an atlas of compatible charts determines one and only one smooth structure. As a practical matter we can thus construct a smooth structure for a manifold by finding an atlas satisfying (i) and (ii), and the extension of the atlas for (iii) to hold is automatic.

Example. The unit sphere $M = S^n$ in $\mathbb{R}^{n+1}$, the set of vectors of Euclidean norm 1, can be made into a smooth manifold of dimension n by using two charts defined as follows. One of these charts is (M_φ, φ) with

$$\varphi(x_1, \dots, x_{n+1}) = \Big(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\Big)$$

and with domain $M_\varphi = S^n - \{(0, \dots, 0, 1)\}$, and the other is (M_ψ, ψ) with

$$\psi(x_1, \dots, x_{n+1}) = \Big(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}\Big)$$

and with domain $M_\psi = S^n - \{(0, \dots, 0, -1)\}$. We need to check that the two charts are smoothly compatible. The set $M_\varphi\cap M_\psi$ is $S^n-\{(0, \dots, 0, \pm1)\}$, and the image of this under φ and ψ is $\mathbb{R}^n - \{(0, \dots, 0)\}$. Put $y_j = x_j/(1-x_{n+1})$, so that $\varphi^{-1}(y_1, \dots, y_n) = (x_1, \dots, x_{n+1})$. Then $\psi \circ \varphi^{-1}(y_1, \dots, y_n)$ is

$$\begin{aligned} &= \big(x_1/(1+x_{n+1}), \dots, x_n/(1+x_{n+1})\big) \\ &= \big(y_1(1-x_{n+1})/(1+x_{n+1}), \dots, y_n(1-x_{n+1})/(1+x_{n+1})\big). \end{aligned}$$

To compute $(1-x_{n+1})/(1+x_{n+1})$, we take $\sum_{j=1}^{n+1} x_j^2 = 1$ into account and write $1 = \sum_{j=1}^{n+1} x_j^2 = x_{n+1}^2 + \sum_{j=1}^{n} y_j^2(1-x_{n+1})^2$. Then $\sum_{j=1}^{n} y_j^2 = (1-x_{n+1}^2)/(1-x_{n+1})^2 =$

$(1 + x_{n+1})/(1 - x_{n+1})$, and

$$\psi \circ \varphi^{-1}(y_1, \dots, y_n) = \Big(y_1 \Big/ \sum_{j=1}^n y_j^2, \dots, y_n \Big/ \sum_{j=1}^n y_j^2\Big).$$

The entries on the right are smooth functions of y since $y \neq 0$. Similarly if we put $z_j = x_j/(1 + x_{n+1})$, we calculate that

$$\varphi \circ \psi^{-1}(z_1, \dots, z_n) = \Big(z_1 \Big/ \sum_{j=1}^n z_j^2, \dots, z_n \Big/ \sum_{j=1}^n z_j^2\Big).$$

Again the entries on the right are smooth functions of z since $z \neq 0$. Thus the two charts are smoothly compatible, and S^n is a smooth manifold.

Euclidean space $\mathbb{R}^m$ itself is of course a smooth manifold of dimension m, with an atlas consisting of the single chart $(\mathbb{R}^m, 1)$, where 1 is the identity function on $\mathbb{R}^m$. Real projective spaces, which can be defined in much the same way as spheres, give further straightforward examples. A number of interesting manifolds arise as a part of the space of simultaneous solutions of some equations, often polynomial equations in several variables. The technical device that shows the solution space to be part of a smooth manifold is normally the Implicit Function Theorem (Theorem 3.16 of *Basic Real Analysis*).

Another simple example of a smooth manifold M of dimension m is any nonempty open subset U of M. The subset U becomes a smooth manifold of dimension m if we define an atlas for it to consist of all restrictions $(U \cap M_\alpha, \alpha\big|_{U \cap M_\alpha})$ of members of the atlas $\{(M_\alpha, \alpha)\}$ for M; then we must discard occurrences of the empty set. We shall often use this observation without special notice, in effect making definitions and deducing conclusions for nonempty open subsets of a manifold M from the corresponding definitions and conclusions about all manifolds.

Most manifolds, however, are constructed globally out of other manifolds or are pieced together from local data. The Hausdorff condition often has to be checked, is often subtle, and is always important. The first place that the Hausdorff condition plays a role is in Lemma B.2 below.

Any manifold M is a locally compact Hausdorff space. If M is separable, then there exists an **exhausting sequence** in M, i.e., an increasing sequence of compact sets with union all of M and with each set contained in the interior of the next member of the sequence. This is Proposition 10.25 of *Basic Real Analysis*.

Let us mention that because of the separability and Theorem 10.45 of *Basic Real Analysis*, the topology of a separable manifold can always be realized by a metric; this fact turns out to be comforting but not particularly useful.

Although manifolds have a global definition, it is often convenient to work with them by referring matters to local coordinates. If p is a point of the smooth manifold M of dimension m, then a compatible chart (M_α, α) about p can be viewed as giving a **local coordinate system** near p. Specifically if the Euclidean coordinates in $\alpha(M_\alpha)$ are $(u_1, \dots, u_m)$, then $q = \alpha^{-1}(u_1, \dots, u_m)$ is a general point of M_α, and we define m real-valued functions $q \mapsto x_j(q)$ on M_α by $x_j(q) = u_j$, $1 \le j \le m$. Then $\alpha = (x_1, \dots, x_m)$. To refer the functions x_j to Euclidean space $\mathbb{R}^m$, we use $x_j \circ \alpha^{-1}$, which carries $(u_1, \dots, u_m)$ to u_j.

The way that the functions x_j are referred to Euclidean space mirrors how a more general real-valued function on an open subset of M may be referred to Euclidean space, and then we can define a real-valued function on M to be smooth if it is smooth in the sense of Euclidean differential calculus when referred to Euclidean space.

Therefore a **smooth function** $f : M \to \mathbb{R}$ on the smooth manifold M is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{R}^m$ into $\mathbb{R}$. A smooth function is necessarily continuous.

In verifying that a real-valued function f on M is smooth, it is sufficient, for each point in M, to check smoothness within only one compatible chart about that point. The reason is the compatibility of the charts: if (M_α, α) and (M_β, β) are two compatible charts about p, then $f \circ \beta^{-1}$ is the composition of the smooth function $\alpha \circ \beta^{-1}$ followed by $f \circ \alpha^{-1}$.

We write $C^\infty(U)$ for the space of smooth real-valued functions on the nonempty open set U of M. The space $C^\infty(U)$ is an associative algebra over $\mathbb{R}$ under the pointwise operations, and it contains the constants. The **support** of a real-valued function is the *closure* of the set where the function is nonzero. We write $C^\infty_{\text{com}}(U)$ for the subset of $C^\infty(U)$ of functions whose support is a compact subset of U.

The space $C^\infty_{\text{com}}(U)$ is not 0. As we shall see in Lemma B.2 below, this fact is a consequence of the following result for Euclidean space that appears as Proposition 8.12 in *Basic Real Analysis*. We omit the proof of Lemma B.1 here.

Lemma B.1. If K and U are subsets of $\mathbb{R}^m$ with K compact, U open, and $K \subseteq U$, then there exists $\varphi \in C^\infty_{\text{com}}(U)$ with values in $[0, 1]$ such that φ is identically 1 on K.

Lemma B.2. If U is a nonempty open subset of a smooth manifold M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U,

PROOF. The set $S = \text{support}(f)$ is a compact subset of U and is compact as a subset of M since the fact that U gets the relative topology means that the inclusion of U into M is continuous. *Since M is by assumption Hausdorff*, S is closed in M. The function F is smooth at all points of U and in particular at all points of S, and we need to prove that it is smooth at all points of the open complement V of S in M. If p is in V, we can find a compatible chart (M_α, α) about p with $M_\alpha \subseteq V$. The function F is 0 on $M_\alpha \cap U \subseteq V \cap U = S^c \cap U$ because it equals f on U and f is 0 on the complement of S in U. The function F is 0 on $M_\alpha \cap U^c$ since it is 0 everywhere on U^c. Therefore F is identically 0 on M_α and is exhibited as smooth in a neighborhood of p. Thus F is smooth. □

Lemma B.3. Suppose that p is a point in a smooth manifold M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in $[0, 1]$ and f is identically 1 on K.

PROOF. The set $\alpha(K)$ is a compact subset of the open subset $\alpha(M_\alpha)$ of Euclidean space, and Lemma B.1 produces a smooth function g in $C^\infty_{\text{com}}(\alpha(M_\alpha))$ with values in $[0, 1]$ that is identically 1 on $\alpha(K)$. If f is defined to be $g \circ \alpha$ on M_α, then f is in $C^\infty_{\text{com}}(M_\alpha)$. Extending f to be 0 on the complement of M_α in M and applying Lemma B.2, we see that the extended f satisfies the required conditions. □

Proposition B.4. Let p be a point of a smooth manifold M, let U be an open neighborhood of p, and let f be in $C^\infty(U)$. Then there is a function g in $C^\infty(M)$ such that $g = f$ in a neighborhood of p.

PROOF. Possibly by shrinking U, we may assume that U is the domain of some compatible chart (M_α, α) about p. Let K be a compact neighborhood of p contained in U, and use Lemma B.3 to find h in $C^\infty(M)$ with compact support in U such that h is identically 1 on K. Define g to be the pointwise product hf on U and to be 0 off U. Then g equals f on the neighborhood K of p, and Lemma B.2 shows that g is everywhere smooth. □

In the same way that we defined smoothness of real-valued functions on smooth manifolds by means of local coordinates, we define smoothness for a *continuous* function from an m dimensional manifold M into an n dimensional manifold N. Namely let p be in M, so that $F(p)$ is in N. Assuming that F is continuous at p, let a local coordinate system be given at $F(p)$ by means of a chart (N_β, β), and choose a local coordinate system at p given by a chart (M_α, α) such that $F(M_\alpha) \subseteq N_\beta$. The local version of F is the function $\beta \circ F \circ \alpha^{-1}$, which carries $\alpha(M_\alpha)$ into $\beta(N_\beta)$. If we write $\alpha = (x_1, \dots, x_m)$ and $\beta = (y_1, \dots, y_n)$, then we obtain an expression of the form

$$(y_1, \dots, y_n) = \beta \circ F \circ \alpha^{-1}(x_1, \dots, x_m),$$

and we see that $\beta \circ F \circ \alpha^{-1}$ is the local function in the Euclidean setting that corresponds to F in the manifold setting. The function $F : M \to N$ is said to be **smooth** if it is continuous and all the functions $\beta \circ F \circ \alpha^{-1}$ are smooth, more precisely if it is continuous and for each p in E and each compatible chart β about $F(p)$, there is some compatible chart α about p such that $\beta \circ F \circ \alpha^{-1}$ is defined and smooth. In this case we often call F a **smooth map**. A smooth map between smooth manifolds with a smooth inverse is called a **diffeomorphism**.

In this way all questions about smoothness in the manifold setting can be translated into questions about smoothness in the Euclidean setting. One consequence, by means of the Inverse Function Theorem,[2] is that the dimension of a smooth manifold is well defined. More specifically the same underlying topological space cannot have two compatible atlases of distinct dimensions.

2. Tangent spaces and vector fields

We turn to a discussion of tangent spaces and vector fields. For a more thorough presentation of this material, the reader may wish to consult the author's *Advanced Real Analysis*, particularly Sections VIII.1–4.

Let M be a smooth manifold of dimension m. The idea is that the tangent space to M at p is the space of all first-order derivatives at p. To make this notion precise, one introduces the space of **germs** $C_p(M)$ of smooth functions at p. These are equivalence classes formed from pairs (f, U), each pair consisting of an open set U containing p and a smooth real-valued function f defined on that open set, two such being equivalent if their restrictions are equal on some subneighborhood of p. The set $C_p(M)$ of equivalence classes inherits arithmetic operations that make it into an associative algebra over $\mathbb{R}$. Evaluation at p is a well defined linear functional e on $C_p(M)$. A

[2] Theorem 3.17 of *Basic Real Analysis*.

derivation of $C_p(M)$ is a linear function $L : C_p(M) \to \mathbb{R}$ such that $L(fg) = L(f)e(g) + e(f)L(g)$. Each such L annihilates constant functions because

$$L(1) = L(1 \cdot 1) = L(1)e(1) + e(1)L(1) = 2L(1)$$

forces $L(1) = 0$. The set of derivations of $C_p(M)$ forms a real vector space $T_p(M)$ that is called the **tangent space** of M at p. If a local coordinate system at p is given by means of a chart (M_α, α) with $\alpha = (x_1, \ldots, x_m)$, then m examples of members of $T_p(M)$ are given by the derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ defined by

$$\Big[\frac{\partial f}{\partial x_j}\Big]_p = \frac{\partial (f \circ \alpha^{-1})}{\partial u_j}\Big|_{(u_1,\ldots,u_m)=(x_1(p),\ldots,x_m(p))} \qquad \text{for} \quad j = 1, \ldots, m.$$

These derivations satisfy

$$\Big[\frac{\partial x_i}{\partial x_j}\Big]_p = \frac{\partial u_i}{\partial u_j}\Big|_{(u_1,\ldots,u_m)=(x_1(p),\ldots,x_m(p))} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. It follows that the m derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ are linearly independent. Actually these m derivations form a vector-space basis of $T_p(M)$, as is shown in the following proposition. Spanning follows from an expansion formula established by the proposition for all members of $T_p(M)$.

Proposition B.5. If M is a smooth manifold and if a compatible chart (M_α, α) about a point p in M is given by $\alpha = (x_1, \ldots, x_m)$, then each member L of $T_p(M)$ is given on $C_p(M)$ by

$$L = \sum_{j=1}^{m} L(x_j) \Big[\frac{\partial}{\partial x_j}\Big]_p.$$

Consequently the m derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ form a vector-space basis of $T_p(M)$.

PROOF. Let L be a derivation of $C_p(M)$, and let (f, U) represent a member of $C_p(M)$, U being an open neighborhood of p in M. Without loss of generality, we may assume that $U \subseteq M_\alpha$ and that $\alpha(U)$ is an open ball in $\mathbb{R}^m$. Put $u_0 = (u_{0,1}, \ldots, u_{0,m}) = \alpha(p)$, let q be a variable point in U, and define $u = (u_1, \ldots, u_n) = \alpha(q)$. Taylor's Theorem[3] applied to $f \circ \alpha^{-1}$ on

[3]In the form of Theorem 3.11 of *Basic Real Analysis*.

$\alpha(U)$ gives

$$f \circ \alpha^{-1}(u) = f \circ \alpha^{-1}(u_0) + \sum_{j=1}^{m} (u_j - u_{0,j})\, \partial(f \circ \alpha^{-1})/\partial u_j(u_0)$$
$$+ \sum_{i,j} (u_i - u_{0,i})(u_j - u_{0,j}) R_{ij}(u)$$

with each R_{ij} in $C^\infty(\alpha(U))$. Referring this formula to M, we obtain

$$f(q) = f(p) + \sum_{j=1}^{m} (x_j(q) - x_j(p)) \Big[\frac{\partial f}{\partial x_j}\Big]_p$$
$$+ \sum_{i,j} (x_i(q) - x_i(p))(x_j(q) - x_j(p)) r_{ij}(q)$$

on U, where $r_{ij} = R_{ij} \circ \alpha$ on U. Because L annihilates constant functions and has the derivation property and satisfies $e(x_j) = x_j(p)$ for $1 \le j \le m$, application of L yields

$$L(f) = \sum_{j=1}^{m} L(x_j) \Big[\frac{\partial f}{\partial x_j}\Big]_p + \sum_{i,j} \big(L(x_i)(e(x_j) - x_j(p))e(r_{ij})$$
$$+ (e(x_i) - x_i(p))L(x_j)e(r_{ij}) + (e(x_i) - x_i(p))(e(x_j) - x_j(p))L(r_{ij})\big)$$
$$= \sum_{j=1}^{m} L(x_j) \Big[\frac{\partial f}{\partial x_j}\Big]_p,$$

as asserted. □

Still with M as a smooth manifold, form the set $T(M)$ of all pairs (p, L) such that p is in M and L is in $T_p(M)$. The set $T(M)$ can be topologized and given a smooth manifold structure in a natural way, and then the pair consisting of $T(M)$ together with the projection-to-the-first-component function is called the **tangent bundle** of M. For current purposes we do not need to know what the topology and manifold structure on $T(M)$ are, and we shall ignore them.[4] A **vector field** X on M is a function from M into $T(M)$ that selects a member of $T_p(M)$ for each p in M; in other words, a vector field is any right inverse to the projection-to-the-first-component function under composition.[5] An

[4]The detailed construction of the tangent bundle is a chore to carry out. Not needing it, we skip the details. The reader who would like to see a careful construction of the tangent bundle may wish to look at Proposition 8.14 and the remarks after it in Section VIII.4 of *Advanced Real Analysis*.

[5]In the terminology of tangent bundles, a vector field is any section of the tangent bundle.

immediate consequence of Proposition B.5 is the following expansion of any vector field.

Corollary B.6. Let M be a smooth manifold of dimension m. If (M_α, α) is any compatible chart for M, say with $\alpha = (x_1, \dots, x_m)$, and if X is a vector field on M_α, then

$$Xf(p) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p)\,(Xx_i)(p)$$

for all p in M_α and f in $C^\infty(M_\alpha)$.

For vector fields we content ourselves with the following definition of smoothness: the vector field X on M is defined to be **smooth** on M if Xx_i is smooth for each coordinate function x_i of each compatible chart[6] on M. From Corollary B.6 it is apparent that the set of smooth vector fields on M is closed under addition and scalar multiplication and is also closed under multiplication by members of $C^\infty(M)$. It is therefore a $C^\infty(M)$ module.

The set of smooth vector fields is not closed under composition because composition introduces second-order derivatives. However, it is closed under the bracket multiplication $[X, Y] = XY - YX$, according to the following proposition.

Proposition B.7. If X and Y are smooth vector fields on a smooth manifold M, then $[X, Y] = XY - YX$ is also a smooth vector field.

Proof. For any smooth function g on M, Corollary B.6 gives

$$Xg(p) = \sum_{i=1}^{m} \frac{\partial g}{\partial x_i}(p)(Xx_i)(p).$$

This equation applied with $g = Yf$ gives

$$\begin{aligned} XYf(p) = (X(Yf))(p) &= \sum_{i=1}^{m} \frac{\partial (Yf)}{\partial x_i}(p)(Xx_i)(p) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f(p)}{\partial x_i \partial x_j}(Xx_i)(p)(Yx_j)(p) \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial f(p)}{\partial x_j}(Xx_i)(p)(Yx_j)(p). \end{aligned}$$

[6]Section VIII.4 of *Advanced Real Analysis* shows that the smooth vector fields are exactly the sections of the tangent bundle that are smooth. We never need to use this fact in this book.

In this identity we reverse the roles of X and Y and then subtract to get

$$(XY - YX)f(p) = \sum_{i=1}^{m}\sum_{j=1}^{m}\Big(\frac{\partial f(p)}{\partial x_j} - \frac{\partial f(p)}{\partial x_i}\Big)(Xx_i)(p)(Yx_j)(p).$$

The terms with second-order derivatives have canceled, and what is left is the local form of a smooth vector field. $\square$

3. Derivatives of smooth maps

Next we discuss derivatives. Let $F : M \to N$ be a smooth map from a smooth manifold M of dimension m into a smooth manifold N of dimension n. For any p in M, the function F carries any tangent vector L in $T_p(M)$ into a tangent vector $(DF)_p(L)$ in $T_{F(p)}(N)$ by the formula $(DF)_p(L)(g) = L(g \circ F)$ for g in the space $C_{F(p)}(N)$ on which a tangent vector in $T_{F(p)}(N)$ operates. The result is a linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ called the **derivative** of F at p.

WARNING. The name "derivative" and the notation $(DF)_p$ are a change from *Advanced Real Analysis*.[7]

Proposition B.8. Let M and N be smooth manifolds of respective dimensions m and n, and let $F : M \to N$ be a smooth map. Fix p in M, let $\alpha = (x_1, \dots, x_m)$ be a compatible chart in M about p, and let $\beta = (y_1, \dots, y_n)$ be a compatible chart in N about $F(p)$. Define $F_i = y_i \circ F$ for $1 \le i \le n$. Relative to the bases $\Big[\dfrac{\partial}{\partial x_j}\Big]_p$ of $T_p(M)$ and $\Big[\dfrac{\partial}{\partial y_i}\Big]_{F(p)}$ of $T_{F(p)}(N)$, the matrix of the linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ has size n by m, and its $(i, j)^{\text{th}}$ entry is $\Big[\dfrac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}\Big]$.

[7] In *Advanced Real Analysis* the name was "differential," and the notation was $(dF)_p$. The need for a change may be seen in the case that F is a real-valued function, i.e., $N = \mathbb{R}$. In this case, $(DF)_p$ is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), T_{F(p)}(\mathbb{R}))$, and in this book it is being called the "derivative." With many authors the word "differential" acquires a different standard meaning in such a way that $(dF)_p$ is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$. The two range spaces, $T_{F(p)}(\mathbb{R})$ and $\mathbb{R}$, are isomorphic, but confusion easily arises when the isomorphism is not made explicit. Some authors use the term "push-forward" in referring to what is being called the the derivative here, but we shall use the term "push-forward" to refer to the effect of a diffeomorphism on a smooth vector field.

Remarks. In other words the matrix in question is the usual derivative matrix or Jacobian matrix of the set of coordinate functions of the function obtained by referring F to Euclidean space. Hence the derivative at a point is the object for smooth manifolds that generalizes the multivariable derivative at a point for Euclidean space. Accordingly, let us make the definition

$$\left[\frac{\partial F_i}{\partial x_j}\right]_p = \left[\frac{\partial F_i}{\partial u_j}\bigg|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\right].$$

Proof. Application of the definitions gives

$$\begin{aligned}(DF)_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big)(y_i) &= \Big[\frac{\partial}{\partial x_j}\Big]_p(y_i\circ F)\\ &= \frac{\partial(y_i\circ F\circ\alpha^{-1})}{\partial u_j}\,(x_1(p),\dots,x_n(p))\\ &= \frac{\partial F_i}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}.\end{aligned}$$

The formula in Proposition B.5 allows us to express any member of $T_{F(p)}(N)$ in terms of its values on the local coordinate functions y_i, and therefore

$$(DF)_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big) = \sum_{i=1}^n \frac{\partial F_i}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}\Big[\frac{\partial}{\partial y_i}\Big]_p$$

for $1\le j\le m$. Thus the matrix is as asserted. □

Proposition B.9 (Chain Rule for smooth maps). Let M, N, and R be smooth manifolds, and let $F: M\to N$ and $G: N\to R$ be smooth maps. If p is in M, then

$$(D(G\circ F))_p = (DG)_{F(p)}\circ(DF)_p.$$

Proof. If L is in $T_p(M)$ and h is in $C_{G(F(p))}(R)$, then the definitions give

$$\begin{aligned}(D(G\circ F))_p(L)(h) &= L(h\circ G\circ F)\\ &= (DF)_p(L)(h\circ G)\\ &= (DG)_{F(p)}(DF)_p(L)(h),\end{aligned}$$

as asserted. □

4. Changes of coordinates

In the subject of Lie groups, it is important to be able freely to change local coordinates in use on manifolds. The main tool for making such a change is the following.

Proposition B.10. Let (U, α) be a compatible chart of dimension n on a smooth manifold, and let p be in U. In order for there to exist an open neighborhood V of p such that $(V, (f_1, \ldots, f_m))$ is a compatible chart, it is necessary and sufficient that

(a) $m = n$ and

(b) $\det\left\{\dfrac{\partial(f_i \circ \alpha)}{\partial u_j}\right\} \neq 0$ for $u = (x_1(p), \ldots, x_n(p))$.

Proof of necessity. Let $f = (f_1, \ldots, f_m)$. If (V, f) is a chart, then $f \circ \alpha^{-1}$ and $\alpha \circ f^{-1}$ are smooth and inverse to each other. By the Chain Rule for Euclidean space, the products of their derivatives in the two orders are the identity of size m and the identity of size m. Therefore $m = n$, and the determinant in (b) of the derivative is not 0 at the Euclidean point corresponding to p. □

Proof of sufficiency. Let $m = n$. By the Inverse Function Theorem we can find a cubical open neighborhood V' of $f(p)$ and an open neighborhood $U' \subseteq \alpha(U)$ of $\alpha(p)$ such that $f \circ \alpha^{-1}$ has a smooth inverse g mapping V' one-one onto U'. Let $V = \alpha^{-1}(U')$, and define $F = \alpha^{-1} \circ g$. Then F maps V' one-one onto V, and

$$Ff = Ff\alpha^{-1}\alpha = \alpha^{-1}(g(f\alpha^{-1})\alpha) = \alpha^{-1}\alpha = 1.$$

So $F = f^{-1}$, and (V, f) is a chart. If (W, β) is another chart and $V \cap W \neq \varnothing$, then

$$f\beta^{-1} = (f\alpha^{-1})(\alpha\beta^{-1})$$

is smooth, and $\beta f^{-1} = \beta F = (\beta\alpha^{-1})g$ is smooth. Thus the chart (V, F) is compatible. □

A smooth map $F\colon M \to N$ is **regular** at the point p of M if the derivative $(DF)_p\colon T_p(M) \to T_{F(p)}(N)$ is one-one.

Proposition B.11. Suppose that $F\colon M \to N$ is a smooth map regular at a point p of M. Let (N_β, β) be a compatible chart at $F(p)$ with $\beta = (y_1, \ldots, y_n)$. Then it is possible to select from the set of functions $y_1 \circ F, \ldots, y_n \circ F$ a subset containing m functions that form a system of coordinates at p on M.

Moreover if (M_α, α) is a chart at p on M, say $\alpha = (x_1, \dots, x_m)$, then there exists a system of coordinates $(z_1, \dots, z_n)$ at $F(p)$ on N such that x_j coincides in a neighborhood of p with $z_j \circ F$ for $1 \le j \le m$.

Proof. Since $(DF)_p$ is one-one, the matrix $\left[\dfrac{\partial F_i}{\partial u_j}\right]\Bigg|_{u_k = x_k(p)}$ has rank m. Choose m independent rows of this matrix, say the ones for rows numbered $i_1, \dots, i_m$. Since $F_i = (y_i \circ F) \circ \alpha^{-1}$, Proposition B.10 shows that the corresponding functions $y_i \circ F$ form a coordinate system at p.

With these functions $y_{i_1} \circ F, \dots, y_{i_m} \circ F$, which we write in vector form as $\widetilde{y} \circ F$, we can write

$$x_j = \psi_j \circ \widetilde{y} \circ F$$

in a neighborhood of p, with each ψ_j smooth near $\widetilde{y}(F(p))$. Since the x_j's form a coordinate system, Proposition B.10 shows that

$$\det\left[\frac{\partial \psi_i}{\partial v_j}\right]_{x = \widetilde{y}(F(p))} = \det\left[\frac{\partial(\psi_i \circ \widetilde{y} \circ F) \circ (\tilde{y} \circ F)^{-1}}{\partial v_j}\right]_{v = \widetilde{y}(F(p))} \ne 0.$$

Set $x_i = \psi_i \circ \widetilde{y}$ and take for $z_{m+1}, \dots, z_n$ those functions y_j whose indices do not occur among $i_1, \dots, i_m$. Then $\{z_1, \dots, z_n\}$ is a system of coordinates at $F(p)$ on N by Proposition B.10, and $z_j \circ F = x_j$ for $1 \le j \le m$. □

In the situation of Proposition B.11, sufficiently small open neighborhoods of p are mapped by F homeomorphically onto their images.

Proposition B.12. Suppose that $F\colon M \to N$ is a smooth map, and let p be a point in M such that $(DF)_p$ maps $T_p(M)$ onto $T_{F(p)}(N)$. Let (V, β) be chart at $F(p)$, say with $\beta = (y_1, \dots, y_n)$. Then the functions $y_j \circ F$ can be taken as the first n functions in a coordinate system at p.

Proof. The matrix $\left\{\left[\dfrac{\partial F_i}{\partial u_j}\right]_{u_k = x_k(p)}\right\}$ has rank n. Choose n independent columns. Without loss of generality we may take them to be the first n. Then the Jacobian matrix formed from the functions $y_1 \circ F, \dots, y_n \circ F, x_{n+1}, \dots, x_m$ has determinant of the form

$$\det\begin{pmatrix} \dfrac{\partial F_i}{\partial u_j} & 0 \\ * & 1 \end{pmatrix} \ne 0$$

and therefore the functions form a system of coordinates at p, by Proposition B.10. □

In the situation of Proposition B.12, the image under F of an neighborhood of p contains a neighborhood of $F(p)$.

Proposition B.13. Suppose that $F\colon M \to N$ is a smooth map such that $(DF)_p$ is an isomorphism of $T_p(M)$ onto $T_{F(p)}(N)$. Then there is a neighborhood V of p that is mapped homeomorphically by F onto a neighborhood W of $F(p)$ in N, and the inverse mapping $F^{-1}\colon W \to V$ is smooth in a neighborhood of $F(p)$.

Proof. This follows by combining Propositions B.11 and B.12. □

Proposition B.14. If $F\colon M \to N$ is a smooth map such that $(DF)_p = 0$ for all p in M, then F is a locally constant map.

Proof. Proposition B.8 shows that the entries of the matrix of $(DF)_p$ when computed in local coordinates are equal to the first partial derivatives of each of the components of F. By hypothesis these are 0 everywhere. Then it follows that F is constant in a neighborhood of p. □

When $F\colon M \to N$ is a smooth map, its derivative DF at p carries tangent vectors at p to tangent vectors at $F(p)$ by means of the formulas

$$(DF)_p\colon T_p(M) \to T_{F(p)}(N) \quad \text{and} \quad ((DF)_p(L))(g) = L(g \circ F)$$

whenever L is in $T_{F(p)}(N)$. However, the derivative DF does not carry vector fields defined in a neighborhood of p to vector fields defined in a neighborhood of $F(p)$ because there may be points near $F(p)$ are not image points of F. Instead an extra hypothesis is needed.

We mention two ways of proceeding. The first way assumes that the smooth map is a diffeomorphism. An important example of this situation arises with left translation L_g in a Lie group. Anyway, in case $F\colon M \to N$ is a diffeomorphism, F maps smooth vector fields on M to smooth maps on N by a function F_* called the **push-forward** of F. The defining formula[8] is

$$F_*(X)(h)(y) = X(h \circ F) \circ F^{-1}(y)$$

for any smooth vector field X on M, smooth function h on N, and element y of N.

The second way allows F to be any smooth map, not necessarily a diffeomorphism, but imposes a condition on the vector fields in question. This condition will not play a role in the text of this book but appears in the

[8]Terminology varies from one author to another. The definition here is that the push-forward is a version of F in another setting. Other authors would say that the thing being pushed forward is a vector field X, and thus the push-forward is $F_*(X)$.

Problems for Chapter III. It has various names, all of which involve the word "related." An example of this situation arises with the inclusion mapping $\iota\colon S \to M$ of a submanifold into a manifold.

5. Products of smooth manifolds

Let M_1 and M_2 be smooth manifolds, with respective atlases $\mathcal{A}_1$ and $\mathcal{A}_2$. Their **product** $M_1 \times M_2$ has the product of the topological spaces as underlying topological space and has atlas

$$\mathcal{A} = \{(U_1 \times U_2) \mid (U_1, \alpha_1) \in \mathcal{A}_1 \text{ and } (U_2, \alpha_2) \in \mathcal{A}_2\}$$

The projections $\pi_1(m_1, m_2) = m_1$ and $\pi_2(m_1, m_2) = m_2$ are smooth, and the injections $f_{m_2}(m_1) = (m_1, m_2)$ and $f_{m_1}(m_2) = (m_1, m_2)$ are smooth. If N is another smooth manifold, then a map $F\colon N \to M_1 \times M_2$ is smooth if $\pi_1 \circ F$ and $\pi_2 \circ F$ are smooth.

Proposition B.15. Let M and N be smooth manifolds of respective dimensions m and n, let p be in M, and let X_p be in $T_p(M)$. Let $\widetilde{X}_{(p,y)}$ be the member $(X_p, 0)$ of $T_{(p,x)}(M \times N)$. If F is a smooth function on $M \times N$, then $\widetilde{X}_{(p,\,\cdot\,)}F$ is a smooth function on N.

Proof. Let us work in local coordinates. Suppose that (M_α, α) and (N_β, β) are compatible charts on M and N, respectively, with (M_α, α) a chart about $p \in M$. Say that $\alpha = (x_1, \ldots, x_m)$ and $\beta = (y_1, \ldots, y_n)$. Using Corollary B.6, write the smooth vector field X locally as

$$Xf(q) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(q)(Xx_i)(q)$$

for all q in M_α and f in $C^\infty(M_\alpha)$. Define functions z_k on $M_\alpha \times N_\beta$ by

$$z_k = \begin{cases} x_k & \text{if } 1 \le k \le m \\ y_{k-m} & \text{if } m+1 \le k \le m+n. \end{cases}$$

The corresponding product chart in the product space $M \times N$ is $(M_\alpha \times N_\beta, \gamma)$ with $\gamma = (z_1, \ldots, z_{m+n})$.

We compute $\widetilde{X}z_k$ for $1 \le k \le m+n$. For $1 \le k \le m$, we have $\widetilde{X}z_k = Xx_k$. For $m+1 \le k \le m+n$, we have $\widetilde{X}z_k = 0$. Thus for $1 \le k \le m$,

$$\widetilde{X}h(p, y) = \sum_{i=1}^{k} \frac{\partial h}{\partial x_i}(p, y)(\widetilde{X}x_i)(p, y),$$

and this is smooth as a function of y. □

6. Submanifolds

Let M be a smooth manifold. A smooth manifold S is a **submanifold** of M if

(i) $S \subseteq M$ as a set, and
(ii) the inclusion mapping of S into M is smooth and everywhere regular.

A nonempty open subset of a manifold gives an easy example of a submanifold. In any case since the inclusion of a submanifold in a manifold is smooth, the inclusion has to be continuous. However, the inclusion does not have to yield a homeomorphism of the submanifold with its image (except in the case of an open submanifold).

EXAMPLE. A line through the identity on the torus T^2 and having irrational slope is a perfectly good example of a submanifold. In more detail one way of seeing the circle $T^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as a smooth manifold is to use two charts, say $U_1 = T^1 - \{1\}$ with $\alpha_1^{-1} = e^{ix}\big|_{(-\pi,\pi)}$ and $U_2 = T^1 - \{-1\}$ with $\alpha_2^{-1} = e^{ix}\big|_{(0,2\pi)}$. In the smooth manifold $\mathbb{C}^2$, let

$$S = \{(e^{it}, e^{it\sqrt{2}}) \subseteq T^2.$$

We make S into a manifold by identifying it with $\mathbb{R}$ through the parameter t. Then the inclusion is smooth since when followed by each of the projections it is smooth. To see regularity, we are to show that the derivative of the inclusion mapping at t is not 0. It is enough that the composition of the inclusion and the one of the projections has nonzero derivative. In fact, with the first projection the composition is just $t \mapsto e^{it} \mapsto t - 2n\pi$, where $t - 2n\pi$ is in $(-\pi, \pi)$. This has derivative 1, not 0. The argument with the second projection is similar. Thus the inclusion is everywhere regular.

WARNING. This definition of "submanifold" is not uniform among all authors in the subject of Lie groups, but the definition given above is the easiest to use because the correspondence between Lie subgroups and Lie subalgebras is tidiest in this case. Its use dates to the book Chevalley [1946]. In books on differential topology, authors tend to limit the use of the term "submanifold" to subsets that are locally closed, referring to the general case by using a word like "immersed."

EXAMPLE. Let M be $\mathbb{R}^2$, and let S be the subset of points (x, y) with $y^2 = x^3$. The subset S is a manifold in its own right with a single chart for which α of (x, y) equals y. The cusp at $(0,0)$ prevents the inclusion mapping from being regular at $(0,0)$.

Let us apply the results of the previous section to submanifolds and see what is being said. Let $S \subseteq M$ be a submanifold with $\dim S = s$ and $\dim M = m$m

(a) Let (U, α) be an M chart at $p \in S$, say $\alpha = (y_1, \ldots, y_m)$. Then the restrictions to S of some subset of s of the functions y_i define an S chart near p. This is an instance of Proposition B.11.

(b) Let (U, α) be an S chart at $p \in S$, say $\alpha = (x_1, \ldots, x_s)$. Then there exists an M chart $(z_1, \ldots, z_m)$ at p such that x_j near p is the restriction to S of z_j for $1 \leq j \leq s$. Again this follows from Proposition B.11.

(c) If $U \subseteq M$ is a nonempty open subset and $f\colon U \to N$ is smooth, then the restriction of f to $U \cap S$ is smooth. In fact, the restriction is just the composition of the inclusion followed by f.

(d) If $V \subseteq S$ is open and p is in V and $f\colon V \to N$ is smooth, then there is a (suitably small) S neighborhood of p such that f is the restriction of a smooth function on M. In fact, we can apply (1), setting $f = f(y_1, \ldots, y_s)$ and taking $F(y_1, \ldots, y_m) = f(y_1, \ldots, y_s)$.

7. Integral curves

The classical role for vector fields is as a mathematical model for time-independent fluid flows in mechanics and for various force fields in electromagnetic theory. The underlying space is an open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, and a vector is attached to each point, telling in the case of mechanics the instantaneous velocity of an element of fluid at that point. What one wants to deduce from this model is an answer to the question: How does the fluid flow? Mathematically this answer is obtained through "integral curves," the curves along which the instantaneous elements of fluid are predicted to move. The integral curves are obtained in principle from solving a certain system of ordinary differential equations in which the sole independent variable is time.

The mathematical problem in the case of mechanics is to find the trajectories along which the fluid is flowing. In $\mathbb{R}^2$ with the vector field given by $X = a_1(x, y)\dfrac{\partial}{\partial x} + a_2(x, y)\dfrac{\partial}{\partial y}$, the condition on a such a curve $c(t)$ for it to be part of the solution is that $c'(t) = \begin{pmatrix} a_1(x, y) \\ a_2(x, y) \end{pmatrix}_{(x,y)=c(t)}$ for each t. A curve satisfying this condition for each t is called an integral curve for the fluid flow.

In this section let M be smooth manifold of dimension m. A **curve** in this context is a smooth function $t \mapsto c(t)$ from an open interval of $\mathbb{R}^1$ into M. Let X be a smooth vector field on M. A curve $t \mapsto c(t)$ is an **integral curve** for X if $c(t)$ is a smooth function and if $X_{c(t)} = (Dc)_t\Big(\dfrac{d}{dt}\Big)$ for all t in the domain of c. Let us see what this condition means in terms of local coordinates. Let (U, φ) be a chart with $\varphi = (x_1, \dots, x_m)$, and use Corollary 1.6 to write $X = \sum_{i=1}^m a_i(x) \dfrac{\partial}{\partial x_i}$ within the chart, with each $a_i(x)$ defined on U. Define

$$\begin{aligned} b_i(y) &= a_i(\alpha^{-1}(y)) \qquad \text{for } y \in \alpha(U), \\ y(t) &= (y_1(t), \dots, y_m(t)) = \varphi(c(t)). \end{aligned}$$

We have

$$\begin{aligned} X(c(t)f &= \sum_{i=1}^m \Big[a_i(x) \frac{\partial f}{\partial x_i}\Big]_{c(t)} \\ &= \sum_{i=1}^m (a_i \varphi^{-1}(\varphi(c(t)) \Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)} \\ &= \sum_{i=1}^m b_i(y(t)) \Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)}. \end{aligned}$$

If $c(t)$ is to be an integral curve, then for all smooth functions f defined on U, this expression must equal

$$\begin{aligned} (Dc)_t\Big(\frac{d}{dt}\Big)(f) &= \frac{d}{dt}(f \circ c)(t) \\ &= \frac{d}{dt}(f \circ \varphi^{-1} \circ y)(t) \\ &= \sum_{i=1}^m \Big[\frac{\partial(f \circ \varphi^{-1})}{\partial u_i}\Big]_{u=y(t)} \Big[\frac{dy_i(t)}{dt}\Big]_t \\ &= \sum_{i=1}^n \Big[\frac{dy_i(t)}{dt}\Big]_t \Big[\frac{\partial f}{\partial x_i}\Big]_{c(t)}. \end{aligned}$$

The equality of the two expressions for $f = x_i$ allows us to conclude that $y(t)$ is a solution of the system of ordinary differential equations

$$\frac{dy_i}{dt} = b_i(y), \quad 1 \le i \le m.$$

Conversely if $y(t)$ solves this system, then $X_{c(t)} = (Dc)_t\Big(\frac{d}{dt}\Big)(f)$ for all f, and so $c(t) = \varphi^{-1}(y(t))$ is an integral curve for X.

To address existence and uniqueness of solutions of this system, we appeal to the standard existence-uniqueness theorem for first-order systems of ordinary differential equations that are not necessarily linear. We state this theorem below as Theorem B.16; a proof may be found in Section IV.2 of *Basic Real Analysis*. The statement makes use of the notion that a function $F\colon \mathbb{R}^1 \to \mathbb{R}^n$ satisfies a **Lipschitz condition** in its y variable. For our purposes this condition means that there exists a real number k such that any two pairs of points (t, y_1) and (t, y_2) in the domain of F can be connected by a line in the domain and that

$$|F(t, y_1) - F(t, y_2)| \le k|y_1 - y_2|$$

for all such pairs of points. Continuous differentiability on a compact subset is sufficient.

Theorem B.16. Let D be a nonempty open set in $\mathbb{R}^1 \times \mathbb{C}^n$, let (t_0, y_0) be in D, and suppose that $F\colon D \to \mathbb{C}^n$ is a continuous function such that $F(t, y)$ satisfies a Lipschitz condition in the y variable and has $|F(t, y)| \le M$ on D. Let R be a compact set in $\mathbb{R}^1 \times \mathbb{C}^n$ of the form

$$R = \big\{(t, y) \bigm| |t - t_0| \le a \text{ and } |y - y_0| \le b\big\},$$

and suppose that R is contained in D. If a' is the smaller of a and b/M, then there exists a solution $y(t)$ of the system

$$y' = F(t, y)$$

on the open interval $|t - t_0| < a'$ satisfying the initial condition

$$y(t_0) = y_0.$$

If for a number a'' with $a' \le a''$, there exists a solution of the equation on the open interval $|y - t_0| < a''$ with the same initial condition satisfied, then it is unique.

Remarks. The last sentence in the statement of the theorem concerns uniqueness. Existence is asserted on the interval centered at t_0 and having width $2a'$. Uniqueness works on that interval, but it works as well on any larger interval centered at t_0 for which existence is known.

In view of our calculations immediately before the statement of Theorem B.16, the following corollary is an immediate consequence.

Corollary B.17. Let $X\colon U \to \mathbb{R}^n$ be a smooth vector field within a chart (U, φ) of a smooth manifold M, and let p be in U. Then there exist an $\varepsilon > 0$ and an integral curve $c : (-\varepsilon, -\varepsilon) \to U \subseteq M$ such that $c(0) = p$. Any two integral curves c and d having $c(0) = d(0) = p$ coincide on the intersection of their domains.

Theorem B.16 has two variants that have bearing on our use of vector fields. They relate to the general construction of the exponential map for a Lie group:

(1) The first variant allows for the initial data (t_0, y_0) to vary continuously, and it asserts that the unique solution depends continuously on the triple (t, t_0, y_0) consisting of the domain variable t and the initial data (t_0, y_0). If F is smooth, the dependence of the unique solution on the triple is smooth. Handling this variant involves going over the proof of Theorem B.16 carefully to check what happens to the constructed solution.

(2) The second variant allows for the function F to depend also on finitely many parameters $\lambda_1, \ldots, \lambda_n$, as well as (t, y). Handling this variant involves the trick of introducing new dependent variables $z_1, \ldots, z_n$ whose corresponding equations are $z_1' = 0, \ldots, z_n' = 0$ and whose initial values are $z_1(t_0) = \lambda_1, \ldots, z_n(t_0) = \lambda_n$.

APPENDIX C

Covering Spaces

Abstract. This appendix introduces the subject of covering spaces in the context of a connected, locally pathwise connected separable regular Hausdorff space. In applications the space will be a connected smooth manifold, always assumed to be separable.

Section 1 defines the fundamental group of such a space, first introducing the necessary concepts of base point, paths, loops, and equivalence of paths. Mappings between such spaces are generally assumed to carry base point to base point. A space in which any two loops based at a point are equivalent is called simply connected.

Section 2 defines notions of evenly covered and covering map for two such spaces, and then it establishes the Path-lifting Theorem, the Covering Homotopy Theorem, and the Map-lifting Theorem. A consequence of these results is the uniqueness theorem for covering spaces, which characterizes "equivalent coverings" in terms of how the fundamental group of each covering space maps under the corresponding covering map.

Section 3 gives the existence theorem for covering spaces under the assumption that the base space for a covering is locally simply connected. With base points aligned, the covering spaces of a base space, up to equivalence, stand in one-one correspondence with the subgroups of the fundamental group of the base space.

Section 4 gives techniques for computing fundamental groups. It handles the product of two spaces, as well as Euclidean n dimensional space, the unit circle in $\mathbb{R}^2$, and the unit sphere S^n in $\mathbb{R}^{n+1}$ when $n \geq 2$. The case of the unit sphere makes use of a lemma saying that the union of two simply connected open subsets of a space is simply connected if the intersection of the two open sets is nonempty and connected. The section concludes with a discussion of deck transformations.

Section 5 reviews elementary facts about topological groups beyond those given in Appendix A. It then examines what can be said about covering spaces when the spaces in question are topological groups. Of particular interest for this book is the construction of a universal covering group for a topological group with certain properties.

1. Fundamental group

Let X be a separable regular Hausdorff space. The Urysohn Metrization Theorem shows that X can be given a metric so as to become a separable metric space. An explicit metric will play no role in the results of this appendix, and we shall usually ignore its existence.

A **path** in X is a continuous function $a : [0, s] \to X$ for some number $s \geq 0$ that depends on a. We write s_a for s whenever the presence of more than one path might lead to ambiguity. For such a path a, s will be called the **stopping time**, $a(0)$ will be called the **initial point**, and $a(s)$ will be called the **final point**.

A **loop** in X is a path with $a(0) = a(s)$; in this case, $a(0)$ is called the **base point** for the loop. An **identity path** is a path with $s = 0$, a **constant path** is a path a with $a(t) = a(0)$ for all t, and the **inverse path** a^{-1} to a path a is defined by $a^{-1}(t) = a(s - t)$ for $0 \leq t \leq s$. In other words, a^{-1} is just a traversed in the reverse order. Observe that $s_{a^{-1}} = s_a$.

The **product** of two paths a and b, written $c = a \cdot b$, is defined whenever $a(s_a) = b(0)$, has $s_{a \cdot b} = s_a + s_b$, and is given by

$$c(t) = \begin{cases} a(t) & \text{for } 0 \leq t \leq s_a, \\ b(t - s_a) & \text{for } s_a \leq t \leq s_a + s_b. \end{cases}$$

In other words, $a \cdot b$ is just a followed by b whenever this makes sense.

The above notion of product has the following properties:

(a) The product c is a path with stopping time $s_a + s_b$.
(b) If $a \cdot b$ and $b \cdot c$ are defined, then $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$ are defined and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(c) If i is an identity path, then $i \cdot a = a$ whenever $i \cdot a$ is defined, and $b \cdot i = b$ whenever $b \cdot i$ is defined.
(d) $a \cdot a^{-1}$ and $a^{-1}a$ are always defined.

A path a is said to be **equivalent** to a path b, written $a \simeq b$, if $a(0) = b(0)$, $a(s_a) = b(s_b)$, and there are continuous functions

$$s : [0, 1] \mapsto [0, \infty)$$

and

$$h : \{(u, t) \mid u \in [0, 1] \text{ and } t \in [0, s(u))\} \mapsto X$$

such that, for all u in $[0, 1]$,

$$h(0, t) = a(t), \qquad h(u, 0) = a(0) = b(0),$$
$$h(1, t) = b(t), \qquad h(u, s(u)) = a(s_a) = b(s_b).$$

Informally the condition is that a is equivalent to b if the path $t \mapsto a(t)$ can be deformed continuously into the path $t \mapsto b(t)$. The parameter u is the **deformation parameter**. Such a continuous deformation is called a **homotopy**.

The notion of equivalence of paths has the following properties:

(a) "Equivalent" is an equivalence relation. We shall write $[a]$ for the class of a.
(b) If $a \simeq a'$ and $b \simeq b'$ and if $a \cdot b$ is defined, then $a' \cdot b'$ is defined and $a \cdot b \simeq a' \cdot b'$. Also $a^{-1} \simeq a'^{-1}$.
(c) The constant path with $a(t) = a(0)$ for all t is equivalent to the identity path c with $c(0) = a(0)$.
(d) $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are equivalent to constant paths.

Multiplication and inversion of classes of paths are defined as follows: Whenever two paths a and b are such that $a \cdot b$ is defined, then their product and inverses of their classes are defined by

$$[a][b] = [ab], \quad [a]^{-1} = [a^{-1}], \quad \text{and} \quad [b]^{-1} = [b^{-1}].$$

As is easily checked, the notion of multiplication of classes of paths has the following properties:

(a) If $[a][b]$ and $[b][c]$ are defined, then $([a][b])[c]$ and $[a]([b][c])$ are defined and are equal.
(b) $[1][a] = [a]$, where 1 is the identity path $a(0)$, and
$[a][1] = [a]$, where 1 is the identity path $a(s_a)$.
(c) $[a][a]^{-1} = [1]$, where 1 is the identity path $a(0)$, and
$[a]^{-1}[a] = [1]$, where 1 is the identity path $a(s_a)$.

Proposition C.1. Fix p in X. The set of classes of loops with base point p is a group under class multiplication, written $\pi(X, p)$. If also q is in X, then any path ξ from p to q canonically defines an isomorphism of $\pi(X, p)$ onto $\pi(X, q)$.

Proof. In the properties above, all the products are now defined. Then the properties exhibit $\pi(X, p)$ as a group. If a is a loop based at p, then $\xi^{-1} \cdot a \cdot \xi$ is a loop based at q, and this correspondence defines the required isomorphism. □

The group $\pi(X, p)$ of classes of loops in X based at p is called the **fundamental group** of X based at p.

A pathwise connected regular Hausdorff space X is said to be **simply connected** if $\pi(X, p) = 1$ for some p in X. In view of Proposition C.1, this condition is independent of p.

A loop in X based at x is said to be **contractible** if it is equivalent to a constant path at x.

If $f : X \to Y$ is a continuous function between two separable regular Hausdorff spaces and if a is a loop in X based at x_0 in X, then the composition $t \mapsto f(a(t))$ is a loop in Y based at $f(x_0)$. Moreover, it is easy to see that equivalent loops map to equivalent loops in this way and that this mapping respects multiplication of paths. Consequently f induces a group homomorphism of $\pi(X, p)$ into $\pi(Y, f(p))$. This group homomorphism will be called f_*. It will be of considerable interest to us.

We conclude this opening section with two remarks about alternative definitions.

Remark 1. Readers who find the auxiliary function $u \mapsto s(u)$ in the definition of equivalence of paths to be inelegant may wish to change the definition of path to insist that the domain of a path a is always $[0, 1]$ rather than an interval $[0, s_a]$ that depends on the path. Such a definition would occasionally simplify some later calculations, but already it would force a change in the definition of $a \cdot b$. Then the "associativity" formula stated above as (ii) in the list of properties of product would be problematic, and associativity would emerge only after passage to equivalence classes.

Remark 2. In the next section we shall encounter a situation where it is inconvenient to have different stopping times for different paths, and we shall want to make an adjustment to have all the stopping times be 1. For this purpose observe that any path with stopping time > 0 is equivalent to a path whose stopping time is 1, simply by making a linear change in the t scale. Moreover any path with stopping time 0 is an identity path and is equivalent to a constant path that has stopping time > 0, by property (iii) of equivalence. Thus in working with equivalence classes of paths, we can whenever necessary choose a member of the class whose stopping time is 1. Moreover in working with homotopies, we may assume that the function $u \mapsto s(u)$ has $s(u) = 1$ for all u. We shall make use of these observations in the proof of Proposition C.6 in the next section.

2. Properties of covering spaces

Throughout this section, let X and Y be separable regular Hausdorff spaces that are pathwise connected and locally pathwise connected. **Locally pathwise connected** by definition means that each point has arbitrarily small pathwise connected open connected neighborhoods. In this case every open set has all of its connected components open.

Let $e\colon X \to Y$ be continuous, and let V be open in Y. We say that V is **evenly covered** by e if each connected component of $e^{-1}(V)$ is mapped by e homeomorphically *onto* V. For this condition to be satisfied, V must be connected.

Let $e\colon X \to Y$ be continuous. We say that e is a **covering map** if each y in Y has an open neighborhood V_y that is evenly covered by e. For this condition to be satisfied, e must map X onto Y. If $e\colon X \to Y$ is a covering map, then Y is called the **base space** and X is called the **covering space**.

EXAMPLE 1. Let X be the open subset of $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ for which $0 < x < 2$ and $-\infty < y < \infty$, and let $e\colon X \to Y$ be the map $z \mapsto \exp z$. Then $Y = \{u + iv \mid u \in \mathbb{R},\ v \in \mathbb{R},\ 1 < u^2 + v^2 < e^4\}$. If V is the connected open set about $(e, 0)$ given by $\{(x, y) \mid 0 < x < 2 \text{ and } -\pi < y < \pi\}$, then $e^{-1}(V)$ has infinitely many components, parametrized by the integers, the n^{th} component being $\{x + iy \mid 0 < x < 2 \text{ and } -\pi(2n-1) < y < \pi(2n+1)\}$. Each component is homeomorphic to V via the mapping e, and thus the point $(e, 0)$ has an open neighborhood that is evenly covered by e. This condition persists for every point of Y, and e is a covering map.

EXAMPLE 2. Let X be the open subset of $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ for which $0 < x < 2$ and $0 < y < 3\pi$, and let $e\colon X \to Y$ be the map $z \mapsto \exp z$. Again $Y = \{u + iv \mid u \in \mathbb{R},\ v \in \mathbb{R},\ 1 < u^2 + v^2 < e^4\}$. However, the inverse image under e of any small disk about the point $(e, 0)$ of Y has two components, one for small positive y and one for y near 2π. The first of these is not homeomorphic via e to the small disk in Y because the part of the disk with y small and ≤ 0 is missing. Thus the small disk about $(e, 0)$ is not evenly covered, and e is not a covering map.

Proposition C.2 (Path-lifting Theorem). Suppose that $e : X \to Y$ is a covering map. If $t \mapsto y(t)$, $0 \leq t \leq 1$, is a path in Y and if x_0 is in $e^{-1}(y(0))$, then there exists a unique path $t \mapsto x(t)$, $0 \leq t \leq 1$, in X with $x(0) = x_0$ and $e(x(t)) = y(t)$.

PROOF. Let T be the set of points t in $[0, 1]$ such that $y\big|_{[0,t]}$ lifts to a path from x_0. The set T is nonempty since 0 is in T. Let us see that T is open. In fact, let t_0 be in T. Form the connected component U_0 of $e^{-1}(V_{y(t_0)})$ containing $x(t_0)$, so that $e^{-1} : V_{y(t_0)} \to U_0$ is continuous. Extend $x(t)$ by the definition $x(t) = e^{-1}(y(t))$. The existence of this extension shows that T

is open.[1] Next let us see that T is closed. Form $V_{y(t_0)}$ and choose $\delta > 0$ so that $y(t)$ is in $V_{y(t_0)}$ for $t_0 - \delta \le t \le t_0$. Find the component of $e^{-1}(V_{y(t_0)})$ containing $x(t_0 - \delta)$ and lift y to this component. As above y_0 is in T. Thus T is closed. Since $[0,1]$ is connected and T is nonempty, open, and closed, we conclude that $T = [0,1]$.

For uniqueness let T' be the set of all t in $[0,1]$ such that all lifts of $y\big|_{[0,t]}$ starting at x_0 agree. Then T' is nonempty, and it is closed by continuity. It is open by the same argument as for T above. By connectedness of $[0,1]$, $T' = [0,1]$. ☐

Lemma C.3. Let $e\colon X \to Y$ be a covering map and let V be an open subset of Y that is evenly covered. Then the connected components of $e^{-1}(V)$ are open.

Proof. Let U_0 be a component of $e^{-1}(V)$, and let x_0 be in U_0. Since X is assumed locally connected, there exists a connected open neighborhood U of x_0 contained in the open set $e^{-1}(V)$. Then $U \subseteq U_0$ since U_0 is a connected component, and thus U_0 is open. ☐

Theorem C.4 (Covering Homotopy Theorem). Let $e : X \to Y$ be a covering map, let K be a compact topological space, and let $f_0\colon K \to X$ be continuous. If $g\colon K \times [0,1] \to Y$ is continuous and satisfies $g(\,\cdot\,,0) = ef_0$, then there is a unique continuous $f\colon K \times [0,1] \to X$ such that $f(\,\cdot\,,0) = ef_0$.

Proof. For each k in K, Proposition C.2 shows that there is a unique path $f\big|_{k\times[0,1]}$ starting at $f_0(k)$ and covering the path $g\big|_{k\times[0,1]}$. This collection of paths defines f and proves uniqueness. We shall prove that f is continuous as a function of two variables. Let T be the set of all t_0 in $[0,1]$ such that $(k,t) \mapsto f(k,t)$ is continuous at (k,t) for all k in K and all $t \le t_0$.

We show t_0 is in T. In fact, fix k_0 in K. Form the connected component U_0 of $e^{-1}(V_{g(k_0,0)})$ containing $f_0(k_0)$. Then $e^{-1}\colon V_{g(k_0,0)} \to U_0$ is continuous, and Lemma C.3 shows that U_0 is open. Choose an open neighborhood N_{k_0} of k_0 so that $f_0(N_{k_0}) \subseteq U_0$ and $g(N_{k_0} \times [0,\varepsilon)) \subseteq V_{g(k_0,0)}$ for some $\varepsilon > 0$. Then $f(N_{k_0\times[0,\varepsilon)}) \subseteq U_0$. Hence $f = e^{-1}g$ on $N_{k_0} \times [0,\varepsilon)$, and f is exhibited as continuous.

Next we show that T is open. In fact, let t_0 be in T. Fix k_0 in K. Form the component U_0 of $e^{-1}(V_{g(k_0,t_0)})$ containing $f(k_0,t_0)$. Then $e^{-1}\colon V_{g(k_0,t_0)} \to U_0$ is continuous. Choose an open neighborhood N_{k_0} of k_0 and an $\varepsilon_{k_0} > 0$ so that $f(N_{k_0} \times (t_0 - \varepsilon_{k_0}, t_0 + \varepsilon_{k_0})) \subseteq U_0$. Then f equals $e^{-1}g$

[1]This definition is forced since $x(t)$ for t near t_0 must be a connected subset of $e^{-1}(V_{y(t_0)})$ containing $x(t_0)$ and so must be in the component U_0.

on $N_{k_0} \times (t_0 - \varepsilon_{k_0}, t_0 + \varepsilon_{k_0})$ and so is continuous on this set. As k_0 varies, these sets N cover K. By compactness of K, we can extract a finite subcover. Use of the minimum of the ε's shows that T is open.

Finally we show that T is closed. In fact, let t_0 be a limit point of T not in T. Fix k_0 in K. Form $V = V_{g(k_0,t_0)}$, and choose a neighborhood N' of k_0 and a number $\varepsilon > 0$ so that $g(N' \times [t_0 - \varepsilon, t_0 + \varepsilon]) \subseteq V$. If U_0 is the connected component of $e^{-1}(V)$ containing $f(k_0, t_0 - \varepsilon)$, then $e^{-1} : V \to U_0$ is continuous. Choose an open neighborhood $N \subseteq N'$ such that $f(N, t_0 - \varepsilon) \subseteq U_0$. Then $f(N \times [t_0 - \varepsilon, t_0 + \varepsilon]) \subseteq U_0$, $f = e^{-1}g$ on this set, and f is continuous at (k_0, t_0). Thus T is closed. Since T is nonempty, open, and closed in $[0, 1]$, $T = [0, 1]$. □

Proposition C.5. If $e : X \to Y$ is a covering map, then a path in X is a contractible loop if and only if the projected path $ex(t)$ is a contractible loop. Consequently e_* is one-one.

Proof. If a path in X is a contractible loop, then its projection to Y by e is certainly contractible. We are to prove the converse direction. Put $f_0(t) = x(t)$, and suppose $t \mapsto ex(t)$ is contractible. Using u as deformation parameter, we can find a continuous $g(t, u)$ for $u \in [0, 1]$ such that $g(\,\cdot\,, 0) = ex(\,\cdot\,)$, $g(0, u) = ex(0)$, $g(1, u) = ex(1) = ex(0)$, and $g(\,\cdot\,, 1) = ex(0)$. Find f as in Theorem C.4. Then $f(\,\cdot\,, 0) = x(\,\cdot\,)$. Also $u \mapsto f(0, u)$ is continuous into the discrete space $e^{-1}(ex(0))$ and so is constant, hence constantly equal to $x(0)$. Similarly $f(\,\cdot\,, 1) = x(0)$, and then $f(1, s) = x(0)$. So $x(0) = x(1)$, and x is equivalent to the constant path at $x(0)$. □

In terms of the notation introduced at the end of the previous section, a covering map $e : X \to Y$ induces a group homomorphism $e_* : \pi(X, x_0) \to \pi(Y, y_0)$ provided that base points x_0 and y_0 are chosen so that $e(x_0) = y_0$. We shall examine this homomorphism more closely.

Proposition C.6. If $e : X \to Y$ is continuous, if y_0 is in Y, and if x_0 is in $e^{-1}(y_0)$, then the lift of a loop $t \mapsto y(t)$ based at y_0 to a path in X based at x_0 is a loop if and only if $[y(\,\cdot\,)]$ is in $e_*(\pi(X, x_0))$. Consequently e_* is one-one.

Proof. If the loop $t \mapsto y(t)$ in Y based at $y_0 \in Y$ lifts to a loop $t \mapsto x(t)$ based at x_0 in X, then $y(t) = ex(t)$ and the class $[t \mapsto y(t)] = [t \mapsto ex(t)] = e_*([t \mapsto x(t)])$ is in $e_*\pi(X, x_0)$.

Conversely suppose that $t \mapsto y(t)$ is given with $y(0) = y_0$ and that there exists a loop $t \mapsto z(t)$ with $z(0) = x_0$ and $[t \mapsto ez(t)] = [t \mapsto y(t)]$. We are to prove that $[t \mapsto y(t)]$ is a loop. By Remark 2 at the end of Section 1, we may assume without loss of generality that y and ez have stopping time 1 and that a given homotopy between them has $s(u)$ identically equal to 1. Let the homotopy $g : [0,1] \times [0,1] \to Y$ exhibit ez and y as equivalent, the first variable being the deformation parameter. Then $g(u,t)$ has $g(0,t) = ez(t)$ for $0 \le t \le 1$, $g(1,t) = y(t)$ for $0 \le t \le 1$, and $g(u,0) = y_0$ for all $u \in [0,1]$. By the Covering Homotopy Theorem (Theorem C.4), there exists a continuous function $f : [0,1] \times [0,1] \to X$ with $ef = g$, $f(0,t) = z(t)$ and $f(u,0) = x_0$ for all $u \in [0,1]$. Since $ef = g$, we see that $y_0 = g(u,1) = ef(u,1)$. Thus $u \mapsto f(u,1)$ is a continuous function into the discrete space $e^{-1}(y_0)$ and must be constant. From the known value $f(0,0) = z(0) = x_0$, we deduce that $f(u,1) = x_0$ for $0 \le u \le 1$. In particular, $f(1,1) = x_0$. Then $t \mapsto f(1,t)$ is a loop in X based at x_0, and e carries it to $t \mapsto y(t)$. □

Theorem C.7 (Map-lifting Theorem) Suppose that $e : X \to Y$ is a covering map and x_0 and y_0 are members of X and Y with $e(x_0) = y_0$. If P is a pathwise connected, locally pathwise connected separable regular Hausdorff space, if $g\colon P \to Y$ is a continuous function, and if p_0 is a point in $g^{-1}(y_0)$, then there exists a continuous function $f : P \to X$ with $f(p_0) = x_0$ and $g = ef$ if and only if $g_*(\pi(P,p_0)) \subseteq e_*(\pi(X,x_0))$. When f exists, it is unique.

Remark. More informally the theorem gives a necessary and sufficient condition for a map into a space to factor through to a map into a covering space. It is assumed that base points line up appropriately, and the condition is that the fundamental groups map in the expected way.

Proof. If f exists, then

$$g_*\pi(P,p_0) = (ef)_*\pi(P,p_0) = e_*f_*\pi(P,p_0) \subseteq \pi(X,x_0).$$

Conversely suppose that this inclusion holds. If p is given in P, then the pathwise connectedness of P says that there is a path w in P from p_0 to p. Then gw is a path in Y from y_0 to $y = g(p)$. By the Path-lifting Theorem (Proposition C.2), there exists a path u in X starting from x_0 such that $eu = gw$. If x is the final point of u, we define $f(p)$ to be x. We need to see that f is well defined. If we make another choice of path w from y_0 to $y = g(p)$, then the two paths together make a loop in P that is mapped under g to a loop in Y. Since the class of the loop in Y is in $g_*\pi(P,p_0)$, which is $\subseteq e_*\pi(X,x_0)$ by hypothesis, Proposition C.6 shows that the loop lifts to a loop in X. As a result, x is uniquely determined by this definition.

To see that f is continuous at p, choose an open neighborhood $V_{g(p)}$ of $g(p)$ in Y that is evenly covered by e, and let U_0 be the component of $e^{-1}(V_{g(p)})$ in X to which $f(p)$ belongs. Then $e^{-1}\colon V_{g(p)} \to U_0$ is continuous. Choose a pathwise connected neighborhood N of p in P so that

$g(N) \subseteq V_{g(p)}$. Then a path from p within N is mapped by g and lifts to a path from $f(p)$ within U_0. Hence $f(N) \subseteq U_0$. Then $f = e^{-1}g$ on N, and f is continuous on N.

If f exists, then the conditions in the definition above must be satisfied, and it follows that f is unique. This completes the proof. □

The theory of covering spaces has associated existence and uniqueness theorems. We give first the theorem that asserts uniqueness of covering spaces. Let $e\colon X \to Y$ and $e'\colon X' \to Y$ be coverings. We say that e and e' are **equivalent coverings** if there is a homeomorphism ι of X onto X' such that $e'\iota = e$.

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "e"'] & & X' \arrow[dl, "e'"] \\ & Y & \end{tikzcd}$$

Theorem C.8 (Uniqueness theorem). Let $e\colon X \to Y$ and $e'\colon X' \to Y$ be coverings, and let y_0 be in Y. Then e and e' are equivalent coverings if and only if base points x_0 in X and x_0' in X' can be chosen so that $e(x_0) = e'(x_0) = y_0$ and

$$e_*(\pi(X, x_0)) = e'_*(\pi(X', x_0')).$$

Proof. If e and e' are equivalent coverings via the homeomorphism $\iota : X \to X'$, choose x_0 arbitrarily in $e^{-1}(y_0)$ and define $x_0' = \iota(x_0)$. Then $e_*\pi(X, X_0) = (e'\iota)_*\pi(X, x_0) = e'_*\iota_*\pi(X, x_0) = e'_*\pi(X', x_0')$.

Conversely suppose that $e_*\pi(X, x_0) = e'_*\pi(X', x_0')$. By the Map-lifting Theorem (Theorem C.7), we can lift $e\colon X \to Y$ to $\iota : X \to X'$. and we can lift $e'\colon C' \to Y$ by $j\colon X' \to X$. Now $j\iota$ leaves x_0 fixed and $e(j\iota) = (ej)\iota = e'\iota = e$. So $j\iota$ covers the identity map of Y into itself . By uniqueness in Theorem C.7, $j\iota$ is the identity map of Y into itself. Similarly ιj is the identity, and therefore ι is a homeomorphism. By construction, $e'\iota = e$, and therefore ι defines an equivalence. □

Corollary C.9. Let $e\colon X \to Y$ be a covering, and suppose that Y is simply connected. Then e is one-one and X is simply connected.

Proof. Fix a base point x_0 in X and let $y_0 = e(x_0)$. Apply Theorem C.8 with $X' = Y$, $e' = 1$, and $x_0' = y_0$. Then $e_*(\pi(X, x_0))$ and $e'_*(\pi(X', x_0'))$ are both $\{1\}$, and the theorem produces a homeomorphism $\iota : X \to Y$ with $e = e'\iota$. Since e' and ι are one-one, so is e, and the result follows. □

3. Existence of covering spaces

In this section, X and Y continue to be separable regular Hausdorff spaces, pathwise connected, and locally pathwise connected. The objective is to prove the existence of covering spaces for a space Y corresponding to each subgroup of its fundamental group.

In proving this existence, we need a supply of open sets in Y that will be evenly covered. In particular, we need an assumption that guarantees that such open sets in Y exist. Lemma C.10 points to such an assumption.

Lemma C.10. If $e\colon X \to Y$ is a covering map, then any pathwise connected open subset Q of Y such that any loop in Q is contractible in Y is evenly covered.

Example. An example of such a set Q may be helpful even if it relies on the calculation of a fundamental group to be carried out in the next section. If $Y = S^2$ is the two-dimensional unit sphere in $\mathbb{R}^3$, take Q to be an open band about the equator. Then Q is not simply connected, but any loop in Q is a loop in $Y = S^2$, which will be seen to be contractible in Proposition C.16 in the next section. Lemma C.10 thus says that the open subset Q of S^2 is always evenly covered in any covering of S^2.

Proof. Fix y_0 in Q and x_0 in $e^{-1}(y_0)$. Using the Path-lifting Theorem (Proposition C.2), lift all possible paths in Q from y_0 to paths in $e^{-1}(Q)$ from x_0. Let P_{x_0} be the union of the images of these paths, and let e' be the restriction of e to P_{x_0}. We show that $e'\colon P_{x_0} \to Q$ is a homeomorphism.

Certainly e' is continuous from P_{x_0} onto Q. Let us see that e' in one-one. Thus let p_1 and p_2 be in $e'^{-1}(Q)$, and suppose that $e'(p_1) = e'(p_2)$, i.e., $e(p_1) = e(p_2)$. Connect x_0 to p_1 and to p_2 by paths a_1 and a_2 in $e^{-1}(Q)$. The image under e of $a_1^{-1}a_2$ is a path in Q from $e(p_1)$ to $e(p_2)$, hence a loop in Q based at $e(p_1)$. Its image in Q is contractible in Y by hypothesis, and Proposition C.5 allows us to conclude that $a_1^{-1}a_2$ is a contractible loop (based at p_1) in $e^{-1}(Q)$. Its initial point and endpoint are p_1 and p_2, and thus $p_1 = p_2$, as required.

Let us see that e' carries P_{x_0} onto Q. If q is given in Q, join y_0 to q by a path, and lift that path to a path from x_0 to some x in P_{x_0}. Then $e'(x) = q$, and hence e' carries P_{x_0} onto Q.

Next let us see that e'^{-1} is continuous from Q into P_{x_0}. If $y \in Q$ is given, let $V \subseteq Q$ be a pathwise connected evenly covered neighborhood of y. This exists since e is assumed to be a covering map. Let x be in $P_{x_0} \cap e^{-1}(y)$, and let U be the component of $e^{-1}(V)$ containing x. By construction $e^{-1}\colon V \to U$

is continuous. Then $e'^{-1} = e^{-1}$ on V, and e'^{-1} is thus continuous at y. We conclude that e' is a homeomorphism of P_{x_0} onto Q.

Certainly P_{x_0} is connected, being pathwise connected. If x_1 is another point in $e^{-1}(y_0)$, then P_{x_0} and P_{x_1} coincide or are disjoint, since a point of intersection yields a path from x_0 to x_1.

Also P_{x_0} is open. In fact, if x is in P_{x_0} and $U \subseteq e^{-1}(Q)$ is a pathwise connected open neighborhood of x, then U is path connected to P_{x_0} and so $U \subseteq P_{x_0}$. Thus P_{x_0} is open.

Finally the union of all P_x for x in $e^{-1}(Q)$ is $e^{-1}(Q)$, and thus P_{x_0} is a connected component of $e^{-1}(Q)$. We conclude that Q is evenly covered. □

Following Chevalley [1946], we say that the space Y is **locally simply connected** if each y in Y has an open pathwise connected, simply connected neighborhood.

Remark. In a locally simply connected space Y, each y in Y has arbitrarily small open pathwise connected neighborhoods with the following property: Any loop in the neighborhood is contractible in Y. For our purposes this is the property of a locally simply connected space that we shall use.

At this point two negative examples may be helpful.

Example 1. Let Y be countable product of circles T_1, T_1 being the unit circle of $\mathbb{C}$. The circle T_1 is a compact metric space, and the countable product of such spaces is again a compact metric space. It is easy to see that the product is pathwise connected and locally pathwise connected. However, one checks that it is not locally simply connected.

Example 2. Let Y be the subset of $\mathbb{R}^2$ given as the countable union on integers $n \geq 1$ of all circles centered at points $(0, \frac{1}{n})$. This is a compact metric space and is pathwise connected and locally pathwise connected. However, it is not locally simply connected because each neighborhood of $(0, 0)$ contains a circle that prevents the set from being simply connected.

Lemma C.11. If the space Y is locally simply connected and y_0 is in Y, then $\pi(Y, y_0)$ is countable.

Proof. In view of the remark above, the space Y has a countable base $\mathcal{V}$ of open sets that are pathwise connected and contain only loops that are contractible in Y. If V_j and V_k are in $\mathcal{V}$, then $V_j \cap V_k$ has a countable number of connected components, since Y is separable and the components are open. Let $\mathcal{V}^*$ be the collection of all components of all $V_j \cap V_k$. By the separability we can cover each V_j by countably many open set V'_l such that $V'_l \subseteq V_j$. Let $t \mapsto f(t)$, with $0 \leq t \leq 1$, be a loop based at y_0 in Y. Cover $[0, 1]$ with open intervals such that f of each interval is in V'_l, and by compactness of $[0,1]$ extract a finite subcover. If $V'_l \subseteq V_j$ and if f maps some open interval in $[0,1]$ to V'_l, then f maps the closure of the interval into V_j, by continuity. As a result, we can choose points t_i and sets V_i with $0 \le i \le n$ such that $t_0 = 0$, $t_n = 1$, $f(t_i) \in V_i \cap V_{i-1}$ for $i \ge 1$, and $f(t) \in V_i$ for $t_i \le t \le t_{i+1}$. Let V_i^* be the component of $V_i \cap V_{i-1}$ to which $f(t_i)$ belongs. To f we can associate the finite sequence

$$V_1, V_1^*, V_1, V_2^*, \ldots, V_{n-1}^*, V_n,$$

and the claim is that this sequence determines the class $[f]$ of the loop f.

To verify this claim, suppose that g is given with the same sequence. We may assume without loss of generality that $g(t_i) = f(t_i)$ for all i since V_i^* is pathwise connected. Then f is homotopic to g for each interval $t_i \le t \le t_{i+1}$, because taken in succession the components of f and of g give a loop in V_i.

The set of finite sequences as above is countable, and so $\pi(Y, y_0)$ is countable. □

Theorem C.12 (Existence Theorem). If Y is locally simply connected, if y_0 is in Y, and if H is a subgroup of $\pi(Y, y_0)$, then there exists a covering space X with covering map $e\colon X \to Y$ and with point x_0 in X such that $e(x_0) = y_0$ and $e_*(\pi(X, x_0)) = H$.

Proof. Let X be the set of equivalence classes of paths in Y from y_0 under the equivalence relation that $f \sim f'$ if

(i) f and f' have the same final point and
(ii) $[f \cdot f^{-1}]$ is in H.

This is an equivalence relation because H is a group. Typical equivalence classes will be called x or $\{f\}$. Let x_0 be the class of the constant path at y_0. For x in X, let $e(x)$ be the endpoint of a path in the class x; then $e(x_0) = y_0$.

Let $\mathcal{V}$ be a base of open sets in Y that are pathwise connected and contain only loops that are contractible in Y. A base of simply connected open sets will do fine. For V in $\mathcal{V}$ and x in $e^{-1}(V)$, define $U(x, V) \subseteq X$ by

$$U(x,V) = \left\{ x' \in e^{-1}(V) \;\middle|\; x' = \left\{ \begin{array}{l} \text{class of some } f \cdot h, \text{ where} \\ \{f\} = x \text{ and } h \text{ is a path starting} \\ \text{at } e(x) \text{ and remaining within } V \end{array} \right\} \right\}.$$

Let us verify that the sets $U(x, V)$ have the following three properties:

(a) A class x' in $U(x, V)$ is not affected by using a different representative f, nor is it affected by using a different h as long as the endpoint of h is not changed. In fact, $\{f \cdot h\} = \{f' \cdot h\}$ because $f \cdot h$ and $f' \cdot h$ have the same endpoint and $[f \cdot h \cdot h^{-1} \cdot f'^{-1}] = [f \cdot f'^{-1}]$ is in H. Also, if h and h' both have the same endpoint, then so do $f \cdot h$ and $f \cdot h'$. Since $h \cdot h'^{-1}$ is contractible in Y, we see that

$$[f \cdot h \cdot h'^{-1} \cdot f^{-1}] = [f \cdot f^{-1}] = [1]$$

is in H, and hence $\{f \cdot h\} = \{f \cdot h'\}$. This proves (a).

(b) If x' is in $U(x, V)$, then $U(x', V) = U(x, V)$. In fact, let x'' be in $U(x, V)$, and write $x' = \{f_0 \cdot h_0\}$ and $x'' = \{f \cdot h\}$. Applying property (a), we obtain

$$x'' = \{f \cdot h\} = \{f \cdot h_0 \cdot h_0^{-1} \cdot h\} = \{(f \cdot h_0) \cdot (h_0^{-1} \cdot h)\}. \qquad (*)$$

On the right side, $\{f \cdot h_0\} = x'$ because $f \cdot h_0$ has endpoint the same as for $f_0 \cdot h_0$ and because $x = \{f\} = \{f_0\}$ implies that

$$\{f_0 \cdot h_0 \cdot h_0^{-1} \cdot f^{-1} = \{f_0 \cdot f\}$$

is in H. Also $h_0^{-1} \cdot h$ is a path within V starting at $e(x')$. Thus $(*)$ shows that x'' is in $U(x', V)$. Hence $U(x, V) \subseteq U(x', V)$. In particular, x is in $U(x', V)$. Then we can repeat the above argument with x and x' interchanged to conclude that $U(x', V) \subseteq U(x, V)$. This proves (b).

(c) If x is in $U(x_1, V_1) \cap U(x_2, V_2)$ and V in a member of $\mathcal{V}$ so that $e(x)$ is in V and $V \subseteq V_1 \cap V_2$, then

$$x \text{ is in } U(x, V) \quad \text{and} \quad U(x, V) \subseteq U(x_1, V_1) \cap U(x_2, V_2).$$

In fact, $U(x, V) \subseteq U(x, V_1) = U(x_1, V_1)$, with the equality holding by (b). Similarly $U(x, V) \subseteq U(x_2, V_2)$, and (c) follows.

Now let $\mathcal{U} = \{U(x, V)\}$ be the collection of all sets $U(x, V)$ as defined above. The set $\mathcal{U}$ is a base for a topology on X, according to (c), and e is continuous since $e^{-1}(V)$ is the union of all $U(x, V)$ for x in $e^{-1}(V)$. Let us see that e is one-one on $U(x, V)$. In fact, suppose that x and x' are in $U(x, V)$, $e(x) = e(x')$, and $\{f\} = x$. The path h that exhibits x' as in $U(x, V)$ is then a loop in V, hence is contractible. Thus $x = \{f\} = \{f \cdot h\} = x'$, and e is indeed one-one.

Let us see that e maps $U(x, V)$ onto V. In fact, let v be given in V, and join $e(x)$ to v by a path g in V. Then g defines a point x' of X with $e(x') = v$ and exhibits x' as in $U(x, V)$. Hence e maps $U(x, V)$ onto V.

Moreover, $e^{-1} : V \to U(x, V)$ is continuous because $(e^{-1})^{-1}(U(x', V)) = V'$ is open. Thus $e : U(x, V) \to V$ is a homeomorphism. Since V is connected, $U(x, V)$ is connected. The set $U(x, V)$ is open by definition, and $e^{-1}(V)$ is the union of the $U(x, V)$ for x in $e^{-1}(V)$, with the $U(x, V)$ disjoint or equal, by (b) above. Consequently the sets $U(x, V)$ are the connected components of $e^{-1}(V)$, and V is evenly covered by e.

Now we prove the appropriate topological properties of X. Since $U(x, V)$ is open in X and is homeomorphic with V, X is locally pathwise connected. To see that X is pathwise connected, let x in X be given. Then x is a class of paths in Y starting at y_0. Pick such a path f. For t between 0 and the stopping time of f, the formula $\widetilde{f}(t) = f\big|_{[0,t]}$ defines a path in X from x_0 to x. Thus X is pathwise connected. The space X is Hausdorff and regular because these properties are local properties and they hold in Y. To complete the argument that X is a separable regular Hausdorff space, it is enough to prove that X has a countable base.

To prove that X has a countable base, we may assume that $\mathcal{V}$ is countable. For each V in $\mathcal{V}$, select one y in V. Then the number of sets $U(x, V)$ with $e(x) = y$ is the same as the number of elements of $e^{-1}(y)$, which is the number of classes of paths from y_0 to y, modulo H. Since Lemma C.11 shows $\pi(Y, Y_0)$ to be countable, $\mathcal{U}$ is countable.

Finally we are to show that $e_*(\pi(X, x_0)) = H$. Let $\widetilde{f}$ be a loop in X based at x_0, and let $e = e\widetilde{f}$ be the image of $\widetilde{f}$ in Y. If x is the point in X corresponding to f, then $f'(t) = f\big|_{[0,t]}$ is a path from x_0 to x covering f, and so $f' = \widetilde{f}$. Thus x is the endpoint of $\widetilde{f}$, which is x_0. Consequently f and the constant path at x_0 represent the same point in X, and $[f]$ must be in H. In other words, $e_*(\pi(X, x_0)) \subseteq H$. For the reverse inclusion let $[f]$ be in H, and lift f to a path $\widetilde{f}$ in X. Again f represents x_0 since $[f]$ is in H, and it represents the endpoint of $\widetilde{f}$. Thus $\widetilde{f}$ is a loop, and Proposition C.6 shows that $[f]$ is in $e_*(\pi(X, x_0))$. Hence $e_*(\pi(X, x_0)) = H$, and the proof is complete.□

By Theorems C.8 and C.12, if the pathwise connected, locally pathwise connected, regular Hausdorff space is locally simply connected, then it has a simply connected covering space and that space is unique up to equivalence. Such a space is called a **universal covering space** of Y.

4. Computation of fundamental groups

In this section we shall establish formulas for the fundamental group $\pi(X)$ for some basic spaces X, and we shall show that $\pi(Y)$ can often be computed when a covering $e\colon X \to Y$ is given and $\pi(X)$ is known. These simple results will allow us to compute $\pi(X)$ in all cases of interest to us in the theory of Lie groups.

Proposition C.13. $\mathbb{R}^n$ is simply connected.

Proof. Without loss of generality, let the base point by 0. If $t \mapsto f(t)$ is a loop based at 0, then $h(u,t) = (1-u)f(t)$, for $0 \le u \le 1$, exhibits $f(t)$ as equivalent to a constant path. □

Proposition C.14. If X and Y are pathwise connected separable regular Hausdorff spaces and x_0 and y_0 are points with $x_0 \in X$ and $y_0 \in Y$, then $\pi(X \times Y, (x_0, y_0))$ is canonically isomorphic with the direct product $\pi(X, x_0) \times \pi(Y, y_0)$.

Proof. We map $\pi(X \times Y, (x_0, y_0))$ to $\pi(X, x_0) \times \pi(Y, Y_0)$ by mapping a loop $(f(t), g(t))$ based at (x_0, y_0) to the pair of classes (f, g). If p_X and p_Y denote the projection maps of $X \times Y$ onto X and Y, respectively, then this map can be written as $((p_X)_*, (p_Y)_*)$. Hence it is a well defined map of the fundamental group and is a group homomorphism. To see that it is onto, let $[f]$ be in $\pi(X, x_0)$ and $[g]$ be in $\pi(Y_0, y_0)$. Without loss of generality, we may assume that f and g both have domain $0 \le t \le 1$. Then $[t \mapsto (f(t), g(t))]$ maps to $([t \mapsto f(t)], [t \mapsto g(t)])$. Thus the map is onto. To see that the map is one-one, suppose that $[t \mapsto f(t)] = 1$ and $[t \mapsto g(t)] = 1$. Again suppose that $0 \le t \le 1$ in both cases. Find $h_f(u,t)$ with $h(u,0) = h(u,1) = x_0$, $h(0,t) = f(t)$, and $h(1,t) = x_0$. Find $h_g(u,t)$ similarly. Then $h_{(f,g)}(u,t) = (h_f(u,t), h_g(u,t))$ exhibits $[t \mapsto f(t,g(t))]$ as equal to 1. □

Proposition C.15. Let $X = \mathbb{R}^1$ be the line, let $Y = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle, and let $e \colon X \to Y$ be the map $e(x) = e^{ix}$. Then e is a covering map, $\pi(Y) = \mathbb{Z}$, and $t \mapsto y(t) = e^{it}$ for $0 \le t \le 2\pi$ is a generator of $\pi(Y)$.

Proof. Let $e^{i\theta}$ be given with $0 \le \theta \le 2\pi$, and define

$$V_\theta = \{e^{i\varphi} \,\big|\, |\varphi - \theta| < \pi/2\}.$$

Then

$$e^{-1}(V_\theta) = \bigcup_{n=-\infty}^{+\infty} \{\varphi \in X \,\big|\, |\varphi - \theta = 2\pi n| < \pi/2\}$$

disjointly, with each set homeomorphic to V_θ. This equality shows that V_θ is evenly covered. Since θ is arbitrary, e is a covering map.

We use $x_0 = 0$ as base point in X. By Proposition C.13, $\pi(X, 0) = \{1\}$. For each integer n, let y_n be the path in Y given by e^{it} for $0 \le \theta \le 2\pi$. The lift x_n of y_n to X has $x_n(t) = t$ for $0 \le t \le 2\pi n$, and this is not a loop. By Proposition C.6, $\{t \mapsto y_n(t)\}$ is not in $e_*\pi(X, 0) = \{1\}$. From its definition, $[y_n] = [y_1]^n$. Hence y_1 generates an infinite cyclic group contained in $\pi(Y, 1)$.

Now suppose that $t \mapsto y^*(t)$ for $0 \le t \le 1$ is any loop based at 1 in Y. Lift $t \mapsto y^*(t)$ to a path based at 0 given by $t \mapsto x^*(t)$ for $0 \le t \le 1$. Then $e(x^*(1)) = 1$ implies that $x^*(1) = 2\pi n$ for some integer n. Form $y^* \cdot (y_n)^{-1}$. This path lifts to a loop in X (necessarily contractible) and so is contractible in Y, by Proposition C.5. Thus $[y^*] = [y_n] = [y_1]^n$. Hence y_1 generates all of $\pi(Y, 1)$. $\square$

Proposition C.16. The n sphere S^n in $\mathbb{R}^{n+1}$, given by

$$S^n = \{(x_1, \ldots, x_{n+1} \in \mathbb{R}^{n+1} \Big|, \sum_{j=1}^{n+1} x_j^2 = 1\}$$

is simply connected if $n > 1$.

Proof. This follows from Lemma C.17 below if we take

$$X_1 = S^n - \{1, 0, \ldots, 0\} \quad \text{and} \quad X_2 = S^n - \{-1, 0, \ldots, 0\}. \qquad \square$$

Lemma C.17. Let X be a pathwise connected, locally pathwise connected separable regular Hausdorff space. If there exist connected, simply connected open subsets X_1 and X_2 of X with $X = X_1 \cup X_2$ and with $X_1 \cap X_2$ connected, then X is simply connected.

Proof. If $X_1 \cap X_2$ is empty, then the connectedness of $X = X_1 \cup X_2$ forces X_1 or X_2 to be empty, and the lemma follows. Thus we may assume that the intersection is nonempty. Fix x_0 in $X_1 \cap X_2$. Let $t \mapsto a(t)$ be a loop based at x_0. Without loss of generality, we may assume that a has domain $[0, 1]$. To each point t in $(0, 1)$, we can associate an open interval centered at t such that a maps the closure of this interval completely into X_1 or completely into X_2. For $t = 0$ similarly, a maps the closure of some relatively open interval $[0, \epsilon)$ into one of X_1 and X_2, and for $t = 1$, we similarly obtain a relatively open interval $(\epsilon', 1]$ that maps into one of X_1 and X_2. By compactness of $[0, 1]$, we can extract a finite subcover and obtain a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $a([t_{i-1}, t_i]) \subseteq X_{k_i}$ for $1 \le i \le n$. Construct a path b_i, $0 \le i \le n$, from x_0 to $a(t_i)$ such that b_i remains in whatever X_j's the point $a(t_i)$ is in. This construction is possible since $X_1 \cap X_2$ is pathwise connected. Let $a_i(t) = a(t + t_{i-1})$ for $0 \le t \le t_i$ and $1 \le i \le n$. Then

$$[a] = [b_0 a_1 b_1^{-1}][b_1 a_2 b_2^{-1}] \cdots [b_{n-1} a_n b_n^{-1}]. \qquad (*)$$

With i fixed, we see that $a_i(t) \subseteq X_{k_i}$, and this conclusion holds in particular for $t = 0$ and for $t = t_i - t_{i-1}$. By construction of b_i, $b_i^{-1} \subseteq X_{k_i}$. Thus $b_{i-1} a_i b_i^{-1}$

lies in X_{k_i}. Since X_{k_i} is simply connected by hypothesis, $[b_{i-1}a_ib_i^{-1}] = 1$. Therefore $[a] = 1$. □

Let X be a universal covering space of a pathwise connected, locally pathwise connected, separable regular Hausdorff space Y, and let $e\colon X \to Y$ be the covering map. A **deck transformation** of X is a homeomorphism f of X that satisfies $ef = e$.

Theorem C.18. Let $e\colon X \to Y$ be a *universal* covering map, and let $e(x_0) = y_0$. Then

(a) $\pi(Y, y_0)$ is in one-one correspondence with $e^{-1}(y_0)$, the correspondence being that $x_1 \in e^{-1}(y_0)$ corresponds to
$$[e(\text{any path from } x_0 \text{ to } x_1)].$$
(b) the group of deck transformations H of X acts simply transitively on $e^{-1}(y_0)$.

(c) the correspondence that associates to a deck transformation f in H the member of $\pi(Y, y_0)$ corresponding to $f(x_0)$ is a group isomorphism of H onto $\pi(Y, y_0)$.

Proof of (a). Let $[t \mapsto y(t)]$, $y(0) = y_0$ be in $\pi(Y, y_0)$. Let $x(t)$ be its lift to X with $x(0) = x_0$, and let x_y be its endpoint. We define a function from Y into X by $y \mapsto x_y$. This map is independent of the representative because a contractible loop based at y_0 lifts to a loop in X, by Proposition C.5. For the inverse correspondence let x_1 be given. Then $[e(\text{any path from } x_0 \text{ to } x_1)]$ is well defined because any loop based at x_0 is contractible, X being simply connected. The result is in $\pi(Y, y_0)$ and defines the inverse map. □

Proof of (b). If x_1 is in $e^{-1}(y_0)$, we apply Theorem C.7 to the diagram

$$\begin{tikzcd} & X \arrow[d, "e"] \\ X \arrow[ur, dashed, "f"] \arrow[r, "e=(g)"'] & Y \end{tikzcd} \qquad \text{with } f(x_0) = x_1.$$

Since $g_*(\pi(X, x_0)) = \{1\}$ when $g = e$, the theorem shows that f exists with $ef = e$ and $f(x_0) = x_1$. Arguing similarly with the diagram

$$\begin{tikzcd} & X \arrow[d, "e"] \\ X \arrow[ur, dashed, "f^{\#}"] \arrow[r, "e"'] & Y \end{tikzcd} \qquad \text{with } f^{\#}(x_1) = x_0,$$

we see that $f^\#$ exists with $ef^\# = e$ and $f^\#(x_1) = x_0$. Combining these results, we obtain $ef^\# f = e$ and $f^\# f(x_0) = x_0$. By uniqueness in Theorem C.7, $f^\# f$ is the identity. Similarly $ff^\#$ is the identity, and f is a homeomorphism. This proves transitivity of the group of deck transformations. That the transitive action is simply transitive follows by the uniqueness in Theorem C.7. $\square$

PROOF OF (c). The map in question is one-one onto by (a) and (b). We show that is is a group homomorphism. Let f and g be in H. Then $f \cdot g$ corresponds to $[ex(t)]$, where x is any path from x_0 to $f(g(x_0))$. If we choose $x(t)$ to pass through $f(x_0)$ on the way, we see that the problem is to show that $[eu(t)] = [ev(t)]$ if

$$u(t) \text{ is a path from } x_0 \text{ to } g(x_0)$$
$$v(t) \text{ is a path from } f(x_0) \text{ to } f(g(x_0)).$$

Now $[ef(u(t))] = [eu(t)]$ since f is a deck transformation, and $f(u(t))$ is a path from $f(x_0)$ to $f(g(x_0))$. Since X is simply connected, $[ef(u(t))] = [ev(t)]$. Therefore $[eu(t)] = [ev(t)]$. $\square$

5. Topological groups

A **topological group** G is a Hausdorff space that is a group in such a way that multiplication is a continuous function from $G \times G$ into G and inversion is a continuous function from G into G. Elementary properties of topological groups were established in Appendix A. The new material about topological groups in Appendix C will concern how topological groups relate to the topics of pathwise connectedness, local pathwise connectedness, and covering spaces.

Proposition C.19. Let G be a connected, locally pathwise connected separable topological group, and let H be a closed subgroup of G that is locally pathwise connected.

(a) The quotient G/H is pathwise connected and locally pathwise connected.
(b) If H_0 is the identity component of H, then the natural map of G/H_0 onto G/H is a covering map.
(c) If G/H is simply connected, then H is connected.
(d) If H is discrete, then the quotient map of G onto G/H is a covering map.

PROOF.

(a) Let $p : G \to G/H$ be the quotient map. If x and y are given in G/H, take members of $p^{-1}(x)$ and $p^{-1}(y)$ in G, connect them by a path, and map them back down to G/H by p to see that G/H is pathwise connected. If x is in an open subset V of G/H, take $\widetilde{x}$ in $p^{-1}(x)$, and choose a connected open neighborhood U of $\widetilde{x}$ that lies in $p^{-1}(V)$. Since G is locally pathwise connected, U is pathwise connected. Thus any $\widetilde{x}'$ in U can be connected to $\widetilde{x}$ by a path in U. It follows that $p(U)$ is a pathwise connected open neighborhhod of x that lies in V. So G/H is locally pathwise connected.

(b) Let $p_0 : G \to G/H_0$, $p : G \to G/H$, and $q : G/H_0 \to G/H$ be the quotient maps. For any open set V in G/H, $q^{-1}(V) = p_0(p^{-1}(V))$ is open, and thus q is continuous. If U is open in G/H_0, then $q(U) = p(p_0^{-1}(U))$ shows that $q(U)$ is open. Thus q is an open map. Since H is locally connected, Proposition A.7 shows that there is an open neighborhood U of 1 in G with $U \cap H = H_0$. By local connectivity of G and continuity of the group operations, find an open connected neighborhood V of 1 in G with $V^{-1}V \subseteq U$. Then $V^{-1}V \cap H \subseteq H_0$. Form the sets VhH_0 in G/H_0 for $h \in H$. These sets are open connected, and their union is $q^{-1}(VH)$. If $Vh_1H_0 \cap Vh_2H_0$ is not empty, then the same thing is true first of $VH_0h_1 \cap VH_0h_2$ because H_0 is normal in H, then of $V^{-1}VH_0 \cap H_0h_2h_1^{-1}$, and finally of $V^{-1}V \cap H_0h_2h_1^{-1}$. Since $V^{-1}V \cap H \subseteq H_0$, $h_2h_1^{-1}$ is in H_0, and so $Vh_1H_0 = Vh_2H_0$. In short, distinct sets VhH_0 are disjoint. Finally let us see that q is one-one from each VhH_0 onto VH. If $q(v_1hH_0) = q(v_2hH_0)$, then $v_1h = v_2hh'$ for some $h' \in H$. The equation $v_2^{-1}v_1 = hh'h^{-1}$ exhibits $v_2^{-1}v_1$ as in $V^{-1}V \cap H \subseteq H_0$. Hence $v_1 = v_2h_0$ for some $h_0 \in H_0$, and the fact that H_0 is normal in H implies that $v_1hH_0 = v_2h_0hH_0 = v_2hH_0$. Thus q is one-one continuous open from each VhH_0 onto VH, and VH is evenly covered. By translation we see that each gVH is evenly covered. Hence q is a covering map.

(c) We apply (b) to see that $G/H_0 \to G/H$ is a covering map. Since G/H is simply connected, Corollary C.9 says that $G/H_0 \to G/H$ is one-one. Therefore $H = H_0$, and H is connected.

(d) This is the special case of (b) in which $H_0 = \{1\}$. □

Proposition C.20. Let G be a connected, locally pathwise connected, simply connected separable topological group, and let H be a discrete subgroup of G, so that the quotient map $p : G \to G/H$ is a covering map. Then the group of deck transformations of G is exactly the group of right translations in G by members of H. Consequently $\pi_1(G/H, 1 \cdot H)$ is canonically isomorphic with H.

PROOF. Let $f_h(g) = gh$ for g in G and h in H. Then $pf_h(g) = ghH = gH = p(g)$, so that $pf_h = p$ and f_h is a deck transformation. Since $p^{-1}(1 \cdot H) = H$, the group of translations f_h is simply transitive on $p^{-1}(1 \cdot H)$. By Theorem C.18b the group of translations f_h is the full group of deck transformations. By Theorem C.18c, $\pi_1(G/H, 1 \cdot H) \cong H$. □

Corollary C.21. Let G be a connected, locally pathwise connected, simply connected separable topological group, and let H be a discrete normal subgroup of G, so that the quotient homomorphism $p\colon G \to G/H$ is a covering map. Then

(a) H is contained in the center of G,
(b) $\pi_1(G/H, 1) \cong H$ is abelian, and
(c) the end-to-end multiplication of loop classes in $\pi_1(G/H, 1)$ coincides with pointwise multiplication of loop classes in G/H.

PROOF. Part (a) is immediate from Proposition A.10, and (b) follows from this conclusion and Proposition C.20. For (c) we observe that elements h_1, h_2, and h_1h_2 of H correspond under the isomorphism of Proposition C.20 to the classes of loops $[e(\text{path in } G \text{ from } 1 \text{ to } h_1)]$, $[e(\text{path in } G \text{ from } 1 \text{ to } h_2)]$, and $[e(\text{path in } G \text{ from } 1 \text{ to } h_1h_2)]$. But one particular path from 1 to h_1h_2 is the group product of the path from 1 to h_1 by the path from 1 to h_2. This proves (c). □

Proposition C.22. Let G be a pathwise connected, locally connected, locally simply connected separable topological group, let $\widetilde{G}$ be a universal covering space with covering map $e : \widetilde{G} \to G$, and let $\widetilde{1}$ be in $e^{-1}(1)$. Then there exists a unique multiplication on $\widetilde{G}$ that makes $\widetilde{G}$ into a separable topological group in such a way that e is a group homomorphism.

REMARK. The group $\widetilde{G}$ is called a **universal covering group** of G. The proposition asserts that it is unique up to isomorphism.

PROOF. Let $m\colon G \times G \to G$ be multiplication, and let $\varphi\colon \widetilde{G} \times \widetilde{G} \to G$ be the composition $m \circ (e, e)$. Since

$$\{1\} = \varphi_*(\pi_1(\widetilde{G} \times \widetilde{G}, \widetilde{1} \times \widetilde{1})) \subseteq e_*(\pi_1(\widetilde{G}, \widetilde{1})),$$

the Map-lifting Theorem (Theorem C.7) provides a unique continuous $\widetilde{\varphi} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\varphi = e\widetilde{\varphi}$ and $\widetilde{\varphi}(\widetilde{1}, \widetilde{1}) = \widetilde{1}$. This $\widetilde{\varphi}$ is the multiplication for $\widetilde{G}$.

It is associative by uniqueness of map lifting because

$$\begin{aligned} e\widetilde{\varphi}(\widetilde{\varphi}(x,y),z) &= \varphi(\widetilde{\varphi}(x,y),z) = \varphi(x,y)(ez) \\ &= ((ex)(ey))ez = ex((ey)(ez)) = e\widetilde{\varphi}(x,\widetilde{\varphi}(y,z)) \end{aligned}$$

and $\widetilde{\varphi}(\widetilde{\varphi}(\widetilde{1},\widetilde{1}),\widetilde{1}) = \widetilde{1} = \widetilde{\varphi}(\widetilde{1},\widetilde{\varphi}(\widetilde{1},\widetilde{1}))$.

It has $\widetilde{1}$ as identity because $\widetilde{\varphi}(\widetilde{1},\,\cdot\,)$ and $\widetilde{\varphi}(\,\cdot\,,\widetilde{1})$ cover the identity and send $\widetilde{1}$ to $\widetilde{1}$. To obtain existence of inverses, we lift the composition of e followed by inversion, as a map of $\widetilde{G} \to G$, to a map of $\widetilde{G}$ to $\widetilde{G}$ sending $\widetilde{1}$ to $\widetilde{1}$. Finally e is a group homomorphism because

$$e(\widetilde{\varphi}(x,y)) = \varphi(x,y) = (ex)(ey).$$

□

HINTS FOR SOLUTIONS OF PROBLEMS

Chapter I

1. For seeing that exp is onto $GL(n,\mathbb{C})$, use of Jordan form shows that it is enough to handle a single Jordan block B and that the diagonal entries of the block may be taken to be 1. Set up an upper-triangular matrix X with 0 on the diagonal, a_1 in all $(i,i+1)^{\text{st}}$ entries, a_2 in all $(i,i+2)^{\text{nd}}$ entries, and so on. The equation $\exp X = B$ leads to equations $a_1 = 1$, $a_2 = f_2(a_1)$, $a_3 = f_3(a_1, a_2)$, $\dots$, and all these equations can be solved in turn.

2. In (b), if $\exp X = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$, then (a) says that X commutes with $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$, and it follows that X is diagonal. In this case the diagonal entries of X must each be in $\exp\mathbb{R} = (0,\infty)$, and they cannot match the negative numbers -2 and -1. In (c), we have $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, and both factors are exponentials of real matrices. In (d), we can take $c(t) = \exp tX \exp tY$ for $0 \le t \le 1$, where $\exp X = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\exp Y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ are as in (c).

3. $\left\{ \begin{pmatrix} a & z \\ 0 & -a \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ z \in \mathbb{C} \right\}$.

4. Aside from the norm properties of $\|\cdot\|$, one uses the facts that $\|1\| = 1$ and $\|AB\| \le \|A\|\|B\|$.

5. 0.

6. For the first part of the problem, what needs to be shown is that every sufficiently small open neighborhood N of 1 in G_1 is mapped to an open set by π. Since G_1 is locally compact and has a countable base, there exist open neighborhoods U_k of 1 such that $\overline{U}_k$ is compact and $G_1 = \bigcup_k U_k$. The Baire Category Theorem shows that $\pi(U_n)$ has nonempty interior V for some n. Let y be in V, and put $U = \pi^{-1}(y^{-1}V)$. Then U is an open neighborhood of 1 in G_1, $\pi(U)$ is open in G_2, and $\overline{U}$ is compact. Let N be any open neighborhood of 1 in G_1 that is contained in

U. Since $\overline{U}$ is compact, π is a homeomorphism from $\overline{U}$ with the relative topology to $\pi(\overline{U})$ with the relative topology. Thus $\pi(N)$ is relatively open in $\pi(\overline{U})$. Hence $\pi(N) = \pi(\overline{U}) \cap W$ for some open set W in G_2. Since $\pi(N) \subseteq \pi(U)$, we can intersect both sides with $\pi(U)$ and get $\pi(N) = \pi(\overline{U}) \cap W \cap \pi(U) = W \cap \pi(U)$. Since $W \cap \pi(U)$ is open in G_2, $\pi(N)$ is open in G_2.

For the example that is sought, one can take G_1 to be $\mathbb{R}^1$ with the discrete topology, G_2 to be $\mathbb{R}^1$ with the usual topology, and π to be the inclusion.

7. The group is homeomorphic to $\mathbb{R}^3$, which is connected. The linear Lie algebra consists of all real matrices $\begin{pmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix}$, and the dimension is 3.

8. The group is homeomorphic to $\mathbb{R}^2$, which is connected. The linear Lie algebra consists of all real matrices $\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$, and the dimension is 2.

9. The group is homeomorphic to $\mathbb{R}^{n(n+1)/2}$, which is connected. The linear Lie algebra consists of all real matrices whose entries below the main diagonal are 0, and the dimension is $\frac{1}{2}n(n+1)$.

10. Routine.

11. The operation is $\mathbb{R}$ linear, and we can multiply out the product

$$\begin{aligned}
(\overline{a + bi + cj + dk})&(a + bi + cj + dk) \\
&= (a + b(-i) + c(-j) + d(-k))(a + bi + cj + dk) \\
&= a^2 + b^2 + c^2 + d^2 + \text{terms like } abi + ab(-i) \text{ that cancel in pairs} \\
&= a^2 + b^2 + c^2 + d^2.
\end{aligned}$$

The two-sided multiplicative inverse of $q \in \mathbb{H}$ is thus $|q|^-2\bar{q}$. Finally if x and y are given, then we have

$$|xy|^2 = (xy)(\overline{xy}) = xy\bar{x}\bar{y} = (x\bar{x})(y\bar{y}) = |x|^2|y|^2.$$

Taking the (real) square root gives $|xy| = |x||y|$.

12.

(a) If $x = x_0 + x_1 i + x_2 j + x_3 k$ and $y = y_0 + y_1 i + y_2 j + y_3 k$, then $\operatorname{Re} xy = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$ by inspection, and this equals $\operatorname{Re} yx$.

(b) Apply the operations in the two orders to $x = x_0 + x_1 i + x_2 j + x_3 k$. Then $\overline{\operatorname{Re} x} = \bar{x}_0 = x_0$, and $\operatorname{Re}(\bar{x}) = \operatorname{Re} x_0 = x_0$. The two are equal.

(c) The formula for extracting the real part gives $\operatorname{Re}\ x\bar{y} = \frac{1}{2}x\bar{y} + \frac{1}{2}\overline{x\bar{y}} = \frac{1}{2}(x\bar{y} + y\bar{x})$. which is a symmetric expression in x and y.

13. The identity $|xy| = |x||y|$ shows closure under multiplication, and we know that $|x| = 1$ implies $|x^{-1}| = 1$ because $|x^{-1}| = |x|^{-1}$. Hence S^3, which Appendix C shows to be compact and connected, is a group under the multiplication within $\mathbb{H}$.

14.–15. If g is in $GL(n, \mathbb{C})$, then $q = gg^*$ is positive definite Hermitian since $(gg^*u, u) = (g^*u, g^*u) > 0$ for each $u \neq 0$. By the finite-dimensional Spectral Theorem, $q = ada^{-1}$ with a unitary and d diagonal, the latter with positive diagonal entries. Under the circumstances the notation $d^{1/2}$ has a natural meaning. Put

$p = ad^{1/2}a^{-1}$ and $k = p^{-1}g$. Then $p^2 = ada^{-1} = q$ is positive definite Hermitian and $k = p^{-1}q$ has $k^* = g^*p^{-1}$. So $kk^* = p^{-1}gg^*p^{-1} = p^{-1}qp^{-1} = p^{-1}p^2p^{-1} = 1$, and k is unitary.

16. From Problems 14 and 15 we know that the map $(k, X) \mapsto k \exp X$ carries $K \times \mathfrak{p}$ onto G. To see that it is one-one, suppose that an element g of G can be written as $g = k \exp X = k' \exp X'$. Write $P = \exp X$ and $P' = \exp X'$. Then we have $g^*g = \exp 2X = P^2$ and also $g^*g \exp 2X' = P'^2$. We conclude that P and P' are both positive semidefinite square roots of the positive definite matrix g^*g. We know such an element to be unique, and thus $P = P'$.

17. Define $u_1, \dots, u_n$ inductively by

$$\begin{aligned}
u_1 &= |v_1|^{-1}v_1 \\
u_2' &= v_2 - (v_2 \cdot u_1)u_1 \\
u_2 &= |u_2'|^{-1}u_2' \\
&\dots \\
u_j' &= v_j - \sum_{i=1}^{j-1}(v_j \cdot u_i)u_i \\
u_j &= |u_j'|^{-1}u_j' \\
&\dots,
\end{aligned}$$

and stop after the formula for u_n.

18. If one applies the Gram–Schmidt orthogonalization process to the ordered basis $(ge_1, \dots, ge_n)$, then the resulting orthonormal basis $(u_1, \dots, u_n)$ has $\operatorname{span}(ge_1, \dots, ge_j) = \operatorname{span}(u_1, \dots, u_j)$ for each j and u_j is in

$$\mathbb{R}^+ ge_j + \operatorname{span}(u_1, \dots, u_{j-1}).$$

Define a matrix k by $k^{-1}u_j = e_j$ for all j. Since $(u_1, \dots, u_n)$ and $(e_1, \dots, e_n)$ are orthonormal bases, k is in $O(n)$.

To see that $k^{-1}g$ is upper triangular with positive diagonal entries, we compute $(k^{-1}ge_j)\cdot e_i = ge_j \cdot ke_i = ge_j \cdot u_i$ for all i. Since ge_j is in $\mathbb{R}^+ge_j + \operatorname{span}(u_1, \dots, u_{j-1})$, we can write $ge_j = cu_j + \sum_{i=1}^{j-1} a_iu_i$ with $c > 0$. Then $ge_j \cdot u_i = 0$ if $i > j$, and $ge_j \cdot u_j = c > 0$. Hence ge_j has its i^{th} entry equal to 0 for $i > j$, and its j^{th} entry is positive. So $k^{-1}g$ is upper triangular with positive diagonal entries.

19. In the notation of Problem 18, $k^{-1}g$ is in S. Consequently the factorization $g = k(k^{-1}g)$ exhibits g as in $O(n)S$. Thus $G \subseteq O(n)S$. The reverse containment is clear.

20. Problem 19 shows that multiplication carries $O(n)S$ onto G and is continuous. The map of $O(n) \times S$ to G is invertible because $O(n) \cap S = 1$. For continuity of the inverse, suppose that $g_n = k_ns_n$ and $\{g_n\}$ converges to some g. We know

that we can write $g = ks$ uniquely. We are to show that $\{k_n\}$ converges to k and $\{s_n\}$ converges to $k^{-1}g$. Because $O(n)$ is compact, it is enough to prove that every convergent subsequence of $\{k_n\}$ has limit k. Arguing by contradiction, suppose along some subsequence that $\{k_n\}$ converges to some $k_0 \in O(n)$ different from k. On that subsequence, $\{s_n\}$ converges to $k_0^{-1}g$, and then we have two decompositions for g, namely $g = k_0(k_0^{-1}g)$ and $g = k(k^{-1}g)$. By uniqueness of the decomposition $O(n)S$, we obtain $k_0 = k$, and we have arrived at a contradiction. We conclude that every convergent subsequence of $\{k_n\}$ has limit k.

21. The product in G is given by

$$\left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) \left(t', \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = \left(t + t', (\exp t'M)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right),$$

and the product of the values of Φ at the two points on the left is

$$\begin{pmatrix} \exp tM & (\exp tM) \begin{pmatrix} x \\ y \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix} \begin{pmatrix} \exp t'M & (\exp t'M) \begin{pmatrix} x' \\ y' \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \exp((t+t')M) & (\exp tM) \begin{pmatrix} x \\ y \end{pmatrix} + \exp(t+t')M \begin{pmatrix} x' \\ y' \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix}.$$

This is to be compared with Φ of the product, which is

$$\begin{pmatrix} \exp(t+t')M & (\exp tM) \begin{pmatrix} x \\ y \end{pmatrix} + \exp(t+t')M \begin{pmatrix} x' \\ y' \end{pmatrix} \\ (0 \quad 0) & 1 \end{pmatrix}.$$

The two expressions match, and therefore Φ is a homomorphism of groups.

22. The linear Lie algebra consists of all real 3-by-3 matrices of the form

$$\begin{pmatrix} M & \begin{pmatrix} X \\ Y \end{pmatrix} \\ (0 \quad 0) & 0 \end{pmatrix},$$

where M is the given 2-by-2 real matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

23. Each entry of $\Phi(t, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$ is smooth, and therefore Φ is smooth. The fact that Φ is one-one and onto follows by direct calculation.

24–25. These examples are handled in much the same way that the examples in Section 4 are handled.

Chapter II

1. Take each of the three basis vectors u of $\mathfrak{su}(2)$ and compute the 2-by-2 matrix $\mathrm{Ad}(g)u = gug^{-1}$ for $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{smallmatrix}\right)$. The results are

$$\begin{aligned}
\mathrm{Ad}(g)u_1 &= \begin{pmatrix} i(|\alpha|^2-|\beta|^2) & -2i\alpha\beta \\ -2i\bar\alpha\bar\beta & -i(|\alpha|^2-|\beta|^2) \end{pmatrix}, \\
\mathrm{Ad}(g)u_2 &= \begin{pmatrix} -\bar\alpha\beta+\alpha\bar\beta & \alpha^2+\beta^2 \\ -\bar\alpha^2+\bar\beta^2 & \bar\alpha\beta-\alpha\bar\beta \end{pmatrix}, \\
\mathrm{Ad}(g)u_3 &= \begin{pmatrix} (\bar\alpha\beta-\alpha\bar\beta)i & (\alpha^2-\beta^2)i \\ (\bar\alpha^2+\bar\beta^2)i & -(\bar\alpha\beta+\alpha\bar\beta)i \end{pmatrix}.
\end{aligned}$$

Use these results to obtain an expansion of each $\mathrm{Ad}(g)u$ in terms of u_1, u_2, u_3. Those expansions are

$$\begin{aligned}
\mathrm{Ad}(g)u_1 &= (|\alpha|^2-|\beta|^2)u_1 + \mathrm{Re}(-2\alpha\beta i)u_2 + \mathrm{Im}(-2\alpha\beta i)u_3, \\
\mathrm{Ad}(g)u_2 &= -i(-\bar\alpha\beta+\alpha\bar\beta)u_1 + \mathrm{Re}(\alpha^2+\beta^2)u_2 + \mathrm{Im}(\alpha^2+\beta^2)u_3, \\
\mathrm{Ad}(g)u_3 &= (\bar\alpha\beta-\alpha\bar\beta)u_1 + \mathrm{Re}((\alpha^2-\beta^2)i)u_2 + \mathrm{Im}((\alpha^2-\beta^2)i)u_3.
\end{aligned}$$

The coefficients of u_1 on the right sides are lined up as the first column of the desired matrix, etc. The resulting 3-by-3 matrix for $\mathrm{Ad}(g)$ is

$$\mathrm{Ad}(g) = \begin{pmatrix} |\alpha|^2-|\beta|^2 & -i(-\bar\alpha\beta+\alpha\bar\beta) & (\bar\alpha\beta-\alpha\bar\beta) \\ \mathrm{Re}(-2\alpha\beta i) & \mathrm{Re}(\alpha^2+\beta^2) & \mathrm{Re}((\alpha^2-\beta^2)i) \\ \mathrm{Im}(-2\alpha\beta i) & \mathrm{Im}(\alpha^2+\beta^2) & \mathrm{Im}((\alpha^2-\beta^2)i) \end{pmatrix}.$$

2. The basis vectors u_1, u_2, u_3 are given by

$$u_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and thus

$$\exp tu_1 = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad \exp tu_2 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \exp tu_3 = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}.$$

For (a) we get

$$\begin{aligned}
\mathrm{Ad}(\exp tu_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & \sin 2t \\ 0 & -\sin 2t & \cos 2t \end{pmatrix}, \\
\mathrm{Ad}(\exp tu_2) &= \begin{pmatrix} \cos 2t & 0 & \sin 2t \\ 0 & 1 & 0 \\ -\sin 2t & 0 & \cos 2t \end{pmatrix}, \\
\mathrm{Ad}(\exp tu_3) &= \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

In each case the matrix is a rotation in a plane about a fixed axis. Thus each matrix is in $SO(3)$. For (b) we simply apply Corollary 2.7. In (c) the smoothness is known for the functions in question, and Ad is a homomorphism. Nothing else needs proof.

3. From Problem 2, $\mathrm{Ad}(G)$ contains all rotations about the coordinate axes. Such rotations are known from geometry to generate the full rotation group $SO(3)$, and thus Ad maps $SU(2)$ onto $SO(3)$.

To compute the kernel, suppose that $\mathrm{Ad}(g) = 1$. Using the computation of $\mathrm{Ad}(g)$ in Problem 1, we see that any g in the kernel must have $-i(\bar{\alpha}\beta + \alpha\bar{\beta}) = 0$ and $\bar{\alpha}\beta - \alpha\bar{\beta} = 0$. The second of these of these says that $\bar{\alpha}\beta$ is real, and the first of these says that $\bar{\alpha}\beta$ is imaginary. Therefore $\bar{\alpha}\beta = 0$ and one of α or β is 0. If $\beta = 0$, then $\alpha = e^{it}$ for some t, and we see that α has to be ± 1. If $\alpha = 0$, then the upper left entry $|\alpha|^2 - |\beta|^2 = 1$ forces $|\beta| = -1$, which is impossible. So the only possibilities are $g = \pm 1$.

4. The basis vectors u_1, u_2, u_3 are given by

$$u_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and direct computation yields

$$\begin{aligned}
(\mathrm{ad}\, u_1)(u_2) &= [u_1, u_2] = 2u_3 \\
(\mathrm{ad}\, u_2)(u_3) &= [u_2, u_3] = 2u_1 \\
(\mathrm{ad}\, u_3)(u_1) &= [u_3, u_1] = 2u_2.
\end{aligned}$$

The other expressions $(\mathrm{ad}\, u_i)(u_j)$ follow from these and from the skew symmetry of the bracket. Thus the matrix for ad in the basis u_1, u_2, u_3 of $\mathfrak{su}(2)$ is

$$\begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

5. For $\pi = \mathrm{Ad}$, Theorem 2.11 and (2.17) together say that $D(\mathrm{Ad})$ is the homomorphism of $\mathfrak{g}$ into itself given by ad. Problem 1 says that

$$\begin{aligned}
\mathrm{Ad}(g)u_1 &= \begin{pmatrix} i(|\alpha|^2 - |\beta|^2) & -2i\alpha\beta \\ -2i\bar{\alpha}\bar{\beta} & -i(|\alpha|^2 - |\beta|^2) \end{pmatrix}, \\
\mathrm{Ad}(g)u_2 &= \begin{pmatrix} -\bar{\alpha}\beta + \alpha\bar{\beta} & \alpha^2 + \beta^2 \\ -\bar{\alpha}^2 + \bar{\beta}^2 & \bar{\alpha}\beta - \alpha\bar{\beta} \end{pmatrix}, \\
\mathrm{Ad}(g)u_3 &= \begin{pmatrix} (\bar{\alpha}\beta - \alpha\bar{\beta})i & (\alpha^2 - \beta^2)i \\ (\bar{\alpha}^2 + \bar{\beta}^2)i & -(\bar{\alpha}\beta + \alpha\bar{\beta})i \end{pmatrix}.
\end{aligned}$$

These results can be used to obtain an expansion of each $\mathrm{Ad}(g)u$ in terms of u_1, u_2, u_3. Those expansions are

$$\begin{aligned}
\mathrm{Ad}(g)u_1 &= (|\alpha|^2 - |\beta|^2)v_1 + \mathrm{Re}(-2\alpha\beta i)v_2 + \mathrm{Im}(-2\alpha\beta i)v_3, \\
\mathrm{Ad}(g)u_2 &= -i(-\bar{\alpha}\beta + \alpha\bar{\beta})v_1 + \mathrm{Re}(\alpha^2 + \beta^2)v_2 + \mathrm{Im}(\alpha^2 + \beta^2)v_3, \\
\mathrm{Ad}(g)u_3 &= (\bar{\alpha}\beta - \alpha\bar{\beta})v_1 + \mathrm{Re}((\alpha^2 - \beta^2)i)v_2 + \mathrm{Im}((\alpha^2 - \beta^2)i)v_3.
\end{aligned}$$

The coefficients of u_1 on the right sides are lined up as the first column of the desired matrix, etc. The resulting 3-by-3 matrix for $\mathrm{Ad}(g)$ is

$$\mathrm{Ad}(g) = \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -i(-\bar\alpha\beta + \alpha\bar\beta) & (\bar\alpha\beta - \alpha\bar\beta) \\ \mathrm{Re}(-2\alpha\beta i) & \mathrm{Re}(\alpha^2 + \beta^2) & \mathrm{Re}((\alpha^2 - \beta^2)i) \\ \mathrm{Im}(-2\alpha\beta i) & \mathrm{Im}(\alpha^2 + \beta^2) & \mathrm{Im}((\alpha^2 - \beta^2)i) \end{pmatrix}.$$

6. Routine.

7. For the identity let q and r satisfy $\overline{q} = -q$ and $\overline{r} = -r$. Then $\overline{[q, r]} = \overline{qr - rq} = \overline{qr} - \overline{rq} = \overline{r}\,\overline{q} - \overline{q}\,\overline{r} = (-r)(-q) - (-q)(-r) = rq - qr = -[r, q]$. For the skew symmetry and the Jacobi identity, the computation is the same as with matrices under matrix multiplication.

8. Let z satisfy $\bar z = -z$. For $q \in \mathbb{H}$, we are first to see that $\overline{L_q(z)} = -L_q(z)$. The computation is $\overline{L_q(z)} = \overline{qz\bar q} = \bar{\bar q}\bar z\bar q = -qz\bar q = -L_q(z)$.

Then we are to see that $L_{q_1q_2} = L_{q_1}L_{q_2}$ on any z with $\bar z = -z$. The computation is $L_{q_1q_2}(z) = q_1q_2z\overline{q_1q_2} = q_1q_2z\overline{q_2}\,\overline{q_1}$ and $L_{q_1}(L_{q_2}(z)) = L_{q_1}(q_2z\overline{q_2}) = q_1q_2z\overline{q_2}\,\overline{q_1}$.

9. The given identification of V with $\mathbb{R}^3$ carries the orthonormal basis (i, j, k) of V to the standard basis of $\mathbb{R}^3$, which is orthonormal in $\mathbb{R}^3$. We are to show that the matrix of each L_q relative to these bases is in $O(3)$.

For $q = i$, the formula is $L_i(z) = iz\bar i$. Thus L_i carries i to i, j to $ij\bar i = k(-i) = -j$, and k to $ik\bar i = (-j)(-i) = -k$. Thus the matrix of L_i relative to the ordered basis (i, j, k) is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For $q = j$ and $L_j(z) = jz\bar j$, L_j carries i to $(-k)(-j) = -i$, j to j, and k to $i(-k) = i(-j) = -k$. Thus the matrix of L_j relative to the ordered basis (i, j, k) is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For $q = k$ and $L_k(z) = kz\bar k$, L_k carries i to $-i$, j to $-j$, and k to k. Thus the matrix of L_k relative to the ordered basis (i, j, k) is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus each of the three matrices in $O(3)$, as required.

10. The homomorphism property was shown in Problem 8. The images of generators were actually in $SO(3)$, according to the computation in Problem 9.

11. We have just seen that the map in question is a homomorphism, evidently smooth, of the group of quaternions of norm 1, which is a version of the analytic group S^3, into a subgroup of $SO(3)$. Moreover Problem 9 shows that the image

includes three matrices that generate $SO(3)$. Hence the homomorphism is onto. For the kernel we are to find those elements q in S^3 with L_q equal to the identity. If $L_q = 1$ for some q with $|q| = 1$, then $L_q(z) = z$ for all z and $qz\bar{q} = z$ for all z. Right multiplying by q gives $qz = zq$ for all $z \in \mathbb{H}$ with $|z| = 1$ and shows q has to be a real multiple of 1. To have norm 1, q must be ± 1. So the kernel is at most ± 1.

12. From Problem 1, a certain basis of $\mathfrak{su}(2)$ has $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$, and $[u_3, u_1] = u_2$. In $\mathfrak{sl}(2, \mathbb{R})$, the elements $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ have $[h, x] = 2x$. Let us see that no two elements of $\mathfrak{su}(2)$ have $[h, x] = 2x$, and then it will follow that $\mathfrak{su}(2)$ cannot be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Thus suppose that

$$[au_1 + bu_2 + cu_3, du_1 + eu_2 + fu_3] = 2du_1 + 2eu_2 + 2fu_3. \tag{$*$}$$

Equating coefficients of u_1, u_2, u_3 separately gives

$$\begin{aligned} 2d &= bf - ce \\ 2e &= cd - af \\ 2f &= ae - bd. \end{aligned}$$

Multiplying these equations by b, e, f respectively gives

$$\begin{aligned} 2d^2 &= bdf - cde \\ 2e^2 &= cde - aef \\ 2f^2 &= aef - bdf. \end{aligned}$$

If we add these three equations, we obtain

$$2d^2 + 2e^2 + 2f^2 = 0.$$

Since d, e, f are all real, they are all 0, and we conclude that equation $(*)$ above has only trivial solutions. Therefore $\mathfrak{su}(2)$ is not isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

13. Write $SU(1, 1)$ as the set of all matrices $\left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$ with α and β complex and $|\alpha|^2 - |\beta|^2 = 1$, and take $SL(2, \mathbb{R}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ \text{real entries, det } = 1\right\}$. Note that

$$\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right)\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Direct computation shows that

$$\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right)\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right)^{-1}$$

has real entries, and of course its determinant is 1. Therefore this group of matrices is contained in $SL(2, \mathbb{R})$. It has to be 3-dimensional, and we conclude that it is all of $SL(2, \mathbb{R})$.

14. An isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras is

$$\varphi\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x+iy & 0 & 0 \\ 2iz & x-iy & 0 \end{pmatrix}.$$

15. For (a), let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the linear map $\theta(x) = -x^t$. Then we see directly that θ has order 2 as a linear map and satisfies $[\theta(x), \theta(y)] = [x, y]$. Everything in (a) then follows from the fact that the eigenspaces for θ in $\mathfrak{g}$ for eigenvalues $+1$ and -1 are $math\mathfrak{k}$ and $\mathfrak{p}$. Conclusion (b) is elementary, and (c) follows from the computation

$$\begin{aligned} T(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) &= \mathrm{Tr}((\mathrm{Ad}(g)X)(\mathrm{Ad}(g)Y)) \\ &= \mathrm{Tr}((gXg^{-1})(gYg^{-1})) \\ &= \mathrm{Tr}(gXYg^{-1}) = \mathrm{Tr}(XY) = T(X, Y). \end{aligned}$$

For (d), Ad is a smooth homomorphism defined on $SL(2, \mathbb{R})$ and having image in the group $GL(\mathfrak{g})$ with $\dim \mathfrak{g} = 3$. Conclusions (b) and (c) already said that the trace form T on $\mathfrak{g}$ has signature (2,1), and (c) says that every member of $\mathrm{Ad}(SL(2, \mathbb{R})$ acts as an isometry relative to the trace form. Hence Ad is a smooth homomorphism of $SL(2, \mathbb{R})$ into the group $O(2, 1)$ of isometries of $\mathbb{R}^3$ with a symmetric invariant bilinear form of signature (2,1).

In (e), the question about the kernel asks when $\mathrm{Ad}(g) = 1$, i.e, when $gXg^{-1} = X$ for all X, and the answer is that g has to be scalar, thus ± 1. The image under Ad has to be connected, since $SL(2, \mathbb{R})$ is connected, and thus it is contained in $SO(2, 1)$. Corollary 2.16a shows that the image is exactly the identity component of $SO(2, 1)$. Parenthetically $SO(2, 1)$ can be shown to be connected, and thus the image is all of $SO(2, 1)$.

16. Form the map $F\colon S^3 \to SU(2)$ given by $F(x_1, x_2, x_3, x_4) = (\alpha, \beta) = (x_1 + ix_2, x_3 + ix_4)$. To see that F is smooth into $GL(2, \mathbb{C})$, we form $F \circ \varphi_1^{-1}$ or $F \circ \varphi_2^{-1}$ as appropriate, say the former.

In this case we have

$$\varphi_1(x_1, x_2, x_3) = \Big(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\Big) \qquad \text{with} \qquad x_4 = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}.$$

For clarity[1] we can introduce

$$s_1 = \frac{x_1}{1-x_4}, \quad s_2 = \frac{x_2}{1-x_4}, \quad s_3 = \frac{x_3}{1-x_4},$$

and we can put

$$|s|^2 = s_1^2 + s_2^2 + s_3^2 = \frac{x_1^2 + x_2^2 + x_3^2}{(1-x_4)^2} = \frac{1 - x_4^2}{(1-x_4)^2} = \frac{1+x_4}{1-x_4}.$$

[1]The local form of F is $F \circ \varphi_1^{-1}$, which is genuinely a function of three variables. Although it is not logically necessary to do so, it may be helpful psychologically to write $F \circ \varphi_1^{-1}$ out explicitly as a function of three variables, and that is what the next few steps accomplish.

Solving for $1 - x_4$ gives $1 - x_4 = 2/(1 + |s|^2)$. So $x_j = 2s_j/(1 + |s|^2)$ for $j \le 3$, and one finds that

$$(F \circ \varphi_1^{-1})(s_1, s_2, s_3) = (x_1, x_2, x_3).$$

The composition of this expression followed by the inclusion into $\mathbb{C}^2$ via the functions $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$ is smooth, and thus F is smooth into $GL(2, \mathbb{C})$. By Theorem 1.10b, F is smooth into $SU(2)$. Theorem 1.10b says that the real and imaginary parts of α and β are smooth functions on $SU(2)$, and this is just the statement that F^{-1} is smooth.

17. For (a) since T is an intertwining operator, one readily checks that $\ker T$ has to be an invariant subspace of V_1 and that $\operatorname{image} T$ has to be an invariant subspace of V_2. The assumption of irreducibility on V_1 forces $\ker T$ to be 0 or V_1, and the assumption of irreducibility on V_2 forces $\operatorname{image} T$ to be 0 or V_2. The result follows.

For (b) the set of intertwining operators is closed under addition, scalar multiplication, and composition. Hence it forms a ring. To see that it is a division ring, one needs to show that each of its members has a two-sided inverse. For each of its members T, the kernel and image of T are invariant subspaces. If T is not 0, then T has to be one-one onto and therefore has a two-sided inverse in the algebra of intertwining operators.

18.

(a) This is elementary.

(b) $n - 1$.

(c) Irreducibility holds for $n = 0$ and $n = 1$. For larger n, the homogeneous polynomials of degree n that are harmonic in the sense of being annihilated by the Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ form an invariant subspace.

Chapter III

1. In the statement of Dynkin's Formula (Theorem 3.4), the interest is in terms of total degree 3, and nonzero contributions come only from $k = 1$ and $k = 2$.

For $k = 1$, the only possible nonzero terms come from $(i_1, j_1) = (1, 1)$ and $(i_1, j_1) = (0, 2)$. The outside coefficient is $-\frac{1}{2}$ in both cases, the coefficient $1/(i_1 + 1)$ is $\frac{1}{2}$ in the first case and 1 in the second case. The reciprocals of factorials contribute 1 in the first case and $1/2!$ in the second case. Thus the terms for $k = 1$ yield $-\frac{1}{4}[X, [Y, X]]$ in the first case and $-\frac{1}{4}[Y, [Y, X]]$ in the second case.

For $k = 2$, the only possible nonzero terms come from $(i_1, j_1, i_2, j_2) = (1, 0, 0, 1)$ and $(i_1, j_1, i_2, j_2) = (0, 1, 0, 1)$. The outside coefficient is $\frac{1}{3}$ in both cases, the coefficient $1/(i_1 + i_2 + 1)$ is $\frac{1}{2}$ in the first case and 1 in the second case, and the reciprocals of factorials contribute 1 in both cases. Thus these terms contribute $\frac{1}{6}[X, [Y, X]]$ in the first case and $\frac{1}{3}[Y, [Y, X]]$ in the second case.

The total contribution from $k = 1$ and $k = 2$ is thus $(-\frac{1}{4} + \frac{1}{6})[X, [Y, X]]$ plus $(-\frac{1}{4} + \frac{1}{3})[Y, [Y, X]]$, for a total of $\frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]]$.

2. Both sides are linear, and it is therefore enough to prove the result for monomials in P and Q. We proceed by induction on the total degree of a monomial. Degree 1 is handled by inspection. Then under the assumption that we have handled F and G, we may assume that a new factor appears on the left, and then it is enough to handle the four products: PF times G, QF times G, F times PG, and F times QG. Since P behaves like a constant in the operation of $\mathcal{D}$, the derivation formula holds for PF times G and F times PG. Let us carry out the steps for QF times G and F times QG. We are given that $\mathcal{D}(FG) = (\mathcal{D}F)G + F(\mathcal{D}G)$. We are to check that the two equalities $\mathcal{D}(QFG) = \mathcal{D}(QF)G+QF(\mathcal{D}G)$ and $\mathcal{D}(FQG) = (\mathcal{D}F)QG+F\mathcal{D}(QG)$.

Using the observation just before the statement of Problem 2, we find that

$$\begin{aligned}\mathcal{D}(QFG) &= X(FG) + Q(\mathcal{D}(FG))\\ &= XFG + Q((\mathcal{D}F)G) + Q(F(\mathcal{D}G))\\ &= \mathcal{D}(QF)G + QF(\mathcal{D}G),\end{aligned}$$

and the first equality has been checked. The second equality is similar.

3. The identity concerns a finite geometric series, and the sum is the quotient of the first term minus the next term after the last, all divided by 1 minus the common ratio. Thus it is $((1+x)^r-(1+x)^m)/(1-(1+x)$, which equals $x^{-1}(1+x)^m-x^{-1}(1+x)^r$. If we expand out the terms of the identity, then the coefficient of x^r on the left side is

$$\binom{m-1}{r} + \binom{m-2}{r} + \cdots + \binom{r}{r},$$

and the coefficient of x^r on the right side is $\displaystyle\binom{m}{r+1}$. The equality follows.

4. For $m \geq 1$,

$$\begin{aligned}\mathcal{D}(Q^m) &= (R_Q + \operatorname{ad} Q)^{m-1}X + (R_Q + \operatorname{ad} Q)^{m-2}R_QX + (R_Q + \operatorname{ad} Q)^{m-3}R_Q^2X\\ &\qquad + \cdots + (R_Q + \operatorname{ad} Q)^2R_Q^{m-3}X + (R_Q + \operatorname{ad} Q)^1R_Q^{m-2}X + R_Q^{m-1}X\\ &= \sum_{k=0}^{m-1}\binom{m-1}{k}R_Q^{m-1}X + \sum_{k=0}^{m-2}\binom{m-2}{k}R_Q^{m-2+1}(\operatorname{ad} Q)^1X\\ &\qquad + \sum_{k=0}^{m-3}\binom{m-3}{k}R_Q^{m-3+2}(\operatorname{ad} Q)^2X\\ &\qquad + \cdots + \sum_{k=0}^{0}\binom{m-(m-1)}{k}R_Q^{m-1-m+m}(\operatorname{ad} Q)^{m-1}X\end{aligned}$$

$$= \sum_{k=0}^{m-1} \sum_{r=k}^{m-1} \binom{m-k}{r} R_Q^{m-1} (\operatorname{ad} Q)^k X$$

$$= \sum_{k=0}^{m-1} \binom{m}{k+1} R_Q^{m-1} (\operatorname{ad} Q)^k X,$$

the last equality holding by Problem 3.

5. The series $\sum_{k=1}^{\infty} z^k/k!$ converges with sum $e^z - 1$ and is written in powers of z. Thus it is the Taylor expansion of $e^z - 1$ about $z = 0$. Then it follows that $\sum_{k=1}^{\infty} z^{k-1}/k!$ is the Taylor expansion of $(e^z - 1)/z$ about $z - 0$. A change in indexing yields $\sum_{k=0}^{\infty} z^k/(k+1)!$ as the same series. The reciprocal $z/(e^z - 1)$ has singularities only when $e^z = 1$, hence only when $z = 2\pi i n$ for some integer $n \neq 0$. The closest such points to the origin are $\pm 2\pi i$, and therefore the power series representing the reciprocal has radius of convergence 2π. Since the function $z/(e^z - 1) + z/2$ is an even function, being equal to

$$(z/2)(e^z + 1)/(e^z - 1) = (z/2)(e^{z/2} + e^{-z/2})/(e^{z/2} - e^{-z/2}),$$

its Taylor series about $z = 0$ involves only even powers of z.

6. Because of Problem 4, the formal calculation is

$$\begin{aligned}
\mathcal{D}(\exp Q) &= \mathcal{D}(1 + Q + \frac{1}{2!}Q^2 + \dots) \\
&= X + \frac{1}{2!}(2XQ + (\operatorname{ad} Q)X) \\
&\quad + \frac{1}{3!}(3XQ^2 + 3((\operatorname{ad} Q)X)Q + (\operatorname{ad} Q)^2 X) + \dots \\
&= X\Big(1 + Q + \frac{Q^2}{2!} + \dots\Big) + \frac{(\operatorname{ad} Q)X}{2!}\Big(1 + Q + \frac{Q^2}{2!} + \dots\Big) + \dots \\
&= \Big[\Big(1 + \frac{\operatorname{ad} Q}{2!} + \frac{(\operatorname{ad} Q)^2}{3!} + \cdots\Big)X\Big] \exp Q.
\end{aligned}$$

7. From Problem 5

$$\Big(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^4}{4!} + \dots\Big)\Big(1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^3 + \frac{B_3}{6!}z^5 + \dots\Big) = 1.$$

Substituting $z = \operatorname{ad} Q$ gives

$$\Big(1 + \frac{\operatorname{ad} Q}{2!} + \frac{(\operatorname{ad} Q)^2}{3!} + \dots\Big)\Big(1 - \frac{(\operatorname{ad} Q)}{2} + \frac{B_1}{2!}(\operatorname{ad} Q)^2 - \frac{B_2}{4!}(\operatorname{ad} Q)^4 + \dots\Big) = 1.$$

Right multiplying by P and using associativity on the right side of the equality yields

$$\Big(1 + \frac{\operatorname{ad} Q}{2!} + \frac{(\operatorname{ad} Q)^2}{3!} + \dots\Big)X = P.$$

Multiplying through on the right by $\exp Q$ leads to

$$\Big[\Big(1 + \frac{\operatorname{ad} Q}{2!} + \frac{(\operatorname{ad} Q)^2}{3!} + \cdots\Big)X\Big] \exp Q = P \exp Q.$$

Problem 6 shows that the left side equals $\mathcal{D}(\exp Q)$, and the desired result follows.

8. Multiplying $\mathcal{D}^n(\exp Q) = P^n \exp Q$ by $t^n/n!$ and summing on n, we obtain

$$(\exp t\mathcal{D})(\exp Q) = (\exp tP)(\exp Q).$$

Then

$$\exp C(t) = (\exp t\mathcal{D})(\exp Q).$$

Hence

$$\frac{d}{dt} \exp C(t) = \frac{d}{dt}(\exp t\mathcal{D})(\exp Q) = \mathcal{D}(\exp t\mathcal{D})(\exp Q) = \mathcal{D} \exp(C(t)).$$

9. Iteration of the derivation formula of Problem 2 shows that

$$\mathcal{D}(C(t)^n) = \sum_{k=0}^{n-1} C(t)^k (\mathcal{D}C(t))C(t)^{n-k-1} = nC(t)^{n-1}\mathcal{D}C(t),$$

for $n \geq 1$, the second equality holding since the factors on the right side commute with one another. Summing on n and substituting gives

$$\mathcal{D} \exp(C(t)) = \frac{1}{n!}\, \mathcal{D}(C(t)^n) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, C(t)^{n-1}\mathcal{D}C(t) = (\exp C(t))\mathcal{D}C(t)$$

In combination with the result of Problem 8, this equality yields

$$\frac{d}{dt} \exp C(t) = (\exp C(t))\mathcal{D}C(t).$$

The left side is $(\exp C(t))\dfrac{d}{dt}C(t)$, and hence

$$\frac{d}{dt}\, C(t) = \mathcal{D}C(t),$$

as required.

10. Form $Z(t) = \exp(-t\mathcal{D})C(t)$. Then

$$Z'(t) = -\mathcal{D} \exp(-t\mathcal{D})C(t) + \exp(-t\mathcal{D})\mathcal{D}C(t) = 0$$

for all t. Thus $\exp(-t\mathcal{D})C(t)$ is constant. At $t = 0$, the constant is $C(0) = Q$. Thus $\exp(-t\mathcal{D})C(t) = Q$, and $C(t) = (\exp(t\mathcal{D}))Q$ as required.

11–12. The formula of Problem 10 gives $C = (\exp \mathcal{D})Q$, and thus

$$\begin{aligned} C &= Q + \mathcal{D}Q + \frac{1}{2!}\mathcal{D}^2 Q + \frac{1}{3!}\mathcal{D}^3 Q + \ldots \\ &= Q + X + \frac{1}{2!}\mathcal{D}X + \frac{1}{3!}\mathcal{D}^2 X + \ldots . \end{aligned}$$

With terms involving more than four factors of P or Q replaced by dots, X is defined in Problem 7 by

$$\begin{aligned} X &= \Big(1 - \frac{(\operatorname{ad} Q)}{2} + \frac{B_1}{2!}(\operatorname{ad} Q)^2 - \frac{B_2}{4!}(\operatorname{ad} Q)^4 + \dots\Big)P \\ &= P - \frac{1}{2}[Q,P] + \frac{1}{12}[Q,[Q,P]] - \dots . \end{aligned}$$

Substitution of X into $\mathcal{D}X$ and evaluation of $\mathcal{D}$ on each term gives

$$\begin{aligned} \mathcal{D}X &= \mathcal{D}\Big(P - \frac{1}{2}[Q,P] + \frac{1}{12}[Q,[Q,P]] - \dots\Big) \\ &= -\frac{1}{2}[X,P] + \frac{1}{12}[X,[Q,P]] + \frac{1}{12}[Q,[X,P]] - \dots . \end{aligned}$$

Again we substitute for X from above. When done by hand, this computation is hazardous, and we give some details. Number the three nonzero terms of X as 1, 2, 3, and label the three occurrences of X in $\mathcal{D}X$ as A, B, C. In making the substitution, we find that nonzero terms with three factors come from 1B and 2A, while nonzero terms with four factors come from 1A and 3B. The sum for $\mathcal{D}X$ is

$$\begin{aligned} &= \tfrac{1}{4}[[Q,P],P] + \tfrac{1}{12}[P,[Q,P]] - \tfrac{1}{24}[[[Q,[Q,P]]],P] - \tfrac{1}{24}[Q,[[Q,P],P]] + \dots \\ &= \tfrac{1}{6}[P,[P,Q]] + \tfrac{1}{12}[Q,[P,[Q,P]]] + \dots . \end{aligned}$$

Next we compute $\mathcal{D}^2X$ by applying $\mathcal{D}$ to this expression. The result is

$$\begin{aligned} \mathcal{D}^2X &= \tfrac{1}{6}[P,[P,X]] + \tfrac{1}{12}[X,[P,[Q,P]]] + \tfrac{1}{12}[Q,[P,[X,P]]] + \dots \\ &= \tfrac{1}{6}[P,[P,P - \tfrac{1}{2}[Q,P]]] + \tfrac{1}{12}[P,[P,[Q,P]]] + \tfrac{1}{12}[Q,[P,[P,P]]] + \dots \\ &= -\tfrac{1}{12}[P,[P,[Q,P]]] + \tfrac{1}{12}[P,[P,[Q,P]]] + 0 + \dots \\ &= 0 + 0 + \dots ! \end{aligned}$$

Finally we substitute for C above to get

$$\begin{aligned} C &= Q + X + \tfrac{1}{2}\mathcal{D}X + 0 + \dots \\ &= Q + P - \tfrac{1}{2}[Q,P] + \tfrac{1}{12}[Q,[Q,P]] + \tfrac{1}{6}[P,[P,Q]] + \tfrac{1}{12}[Q,[P,[Q,P]]] + \dots . \end{aligned}$$

From this formula we read off the values of C_1, C_2, C_3, and C_4 as stated in Problem 12.

13. The definition and steps are the same as in Section II.4. One uses the linear Lie algebra just as one does in Chapter II for homomorphisms of groups, using (2.14) as the definition of D, showing that D is well defined and linear. Then one introduces Ad, obtains a formula for $D\pi(\mathrm{Ad})$, and differentiates it to see that D is a a Lie algebra homomorphism.

14. The Lie algebra homomorphism φ respects addition, scalar multiplication, and bracket product, and thus it respects partial sums in the series for $C(X,Y)$. Since φ is linear and the space φ is finite-dimensional, φ extends continuously to

the smallest Lie subalgebra containing X, Y, and all partial sums in the series for $C(X,Y)$. The result follows.

15. We calculate

$$\begin{aligned}
\Phi(\exp X)\Phi(\exp Y) &= \exp(\varphi(X))\exp(\varphi(Y)) && \text{by how } \Phi,\ \varphi, \text{ and exp are related}\\
&= \exp C(\varphi(X),\varphi(Y)) && \text{by definition of } C\\
&= \exp \varphi(C(X,Y)) && \text{by Problem 14}\\
&= \Phi(\exp C(X,Y)) && \text{by how } \Phi,\ \varphi, \text{ and exp are related}\\
&= \Phi(\exp X \exp Y) && \text{by definition of } C,
\end{aligned}$$

and the result follows readily.

16. For small X and Y in the Lie algebra of a linear Lie group, we know that

$$\begin{aligned}
\exp(-C(-Y,-X)) &= \exp(C(-Y,-X))^{-1} = \big(\exp(-Y)\exp(-X)\big)^{-1}\\
&= (\exp(-X))^{-1}(\exp(-Y))^{-1}\\
&= \exp X \exp Y = \exp C(X,Y).
\end{aligned}$$

Hence

$$-C(-Y,-X) = C(X,Y).$$

17. Write the Campbell–Baker–Hausdorff series as $C(X,Y) = \sum_{n\geq 1} C_n(X,Y)$. To extract the homogeneous terms, we simply expand the function $t \mapsto C(tX,tY)$ in power series about $t=0$ and read off the coefficients of the various powers of t. In view of Problem 16, we are led from

$$-C(-tY,-tX) = C(tX,tY)$$

to

$$-C_n(-tY,-tX) = C_n(tX,tY)$$

for every n, and we conclude that $C_n(Y,X) = (-1)^{n-1}C_n(X,Y)$ as required.

Chapter IV

1. Up to isomorphism, everything is determined by a multiplication table for the vectors in a basis. Here if $\{X\}$ is a basis, the multiplication table must have just one entry, and that entry has $[X,X]=0$.

2. Start with any ordered basis (X',Y'). If $[X',Y']=0$, then the given Lie algebra is abelian. Otherwise we may assume that $[X',Y'] = uX'+vY' \neq 0$. We shall introduce a new ordered basis (X,Y) such that $[X,Y]=Y$. Write $X = aX'+bY'$ and $Y = cX'+dY'$ with a,b,c,d to be specified. The condition $[X,Y]=Y$ says that $[aX'+bY', cX'+dY'] = cX'+dY'$, hence that $(ad-bc)[X',Y'] = cX'+dY'$. Since u or v must be nonzero, we can find real numbers c and d such that $uc+vd=1$. Set $b=-u$ and $a=v$, so that $ad-bc=1$. With a,b,c,d defined, X and Y are defined, and their bracket is $[X,Y] = [aX'+bY', cX'+dY'] = (ad-bc)[X',Y'] = cX'+dY' = Y$, as required.

3. n for $SU(n)$, 1 for $SO(n)$ with n odd and 2 for $SO(n)$ with n even, n for $SL(n,\mathbb{C})$, 1 for $SO(n,\mathbb{C})$ with n odd and 2 for $SO(n,\mathbb{C})$ with n even, 2 for $Sp(n,\mathbb{C})$.

4. Let M be the diagonal matrix with n diagonal entries of i and then n diagonal entries of 1. An isomorphism $G \to SO(2n,\mathbb{C})$ is $x \mapsto y$, where $y = MxM^{-1}$.

5. Z is the group of scalar matrices whose entries are equal to some $e^{i\theta}$. Then $Z \cap SU(n)$ is the group of scalar matrices whose diagonal entries are all equal to some n^{th} root of 1, $e^{2\pi ik/n}$. So $Z \cap SU(n)$ is a finite group of order n and does not reduce to the identity unless $n = 1$.

6. For (a), it is to be shown that the bracket on $\mathfrak{g}/\mathfrak{a}$ is has $[X, X] = 0$ and satisfies the Jacobi identity. For the first, $[X + \mathfrak{a}, X + \mathfrak{a}] = [X, X] + \mathfrak{a} = \mathfrak{a}$ as required. For the second, $[X + \mathfrak{a}, [Y + \mathfrak{a}, Z + \mathfrak{a}]] + [Y + \mathfrak{a}, [Z + \mathfrak{a}], X + \mathfrak{a}]] + [Z + \mathfrak{a}, [X + \mathfrak{a}, Y + \mathfrak{a}]] = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] + \mathfrak{a} = 0 + \mathfrak{a}$ as required.

For (b), under the quotient mapping if X and Y are given, X goes to $X + \mathfrak{a}$, and Y goes to $Y + \mathfrak{a}$. Then $[X, Y] + \mathfrak{a} = [X + \mathfrak{a}, Y + \mathfrak{a}]$ shows that the quotient map is a Lie algebra homomorphism. If $X + \mathfrak{a}$ is in $\mathfrak{g}/\mathfrak{a}$, then X is a member of $\mathfrak{g}$ that maps to $X + \mathfrak{a}$. So the map is onto. The elements in the kernel are those members X of $\mathfrak{g}$ such that $X + \mathfrak{a} = \mathfrak{a}$, and these elements are exactly the members of $\mathfrak{a}$.

7. The proof of this result is the same as the proof of the corresponding result for vector spaces.

8. As with the previous problem, the proof of this result is the same as the proof of the corresponding result for vector spaces. Here is a little more detail. To form the map from $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ to $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$, one takes any member Z of $\mathfrak{g}$, decomposes it in any fashion into $X + Y$ in $\mathfrak{a} + \mathfrak{b}$ with $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$, and then maps it to the member $Y + (\mathfrak{a} \cap \mathfrak{b})$ of $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$. To see that this map is well defined, one has to check two things. The first is that a different decomposition $Z = X' + Y'$ leads to the same image. The second is that if Z is changed by adding an element X'' of $\mathfrak{a}$ to it, then the image is the same in $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$.

9. Each member of the commutator series for a Lie subalgebra is a Lie subalgebra of the corresponding member of the commutator series for the original Lie algebra. Thus any Lie subalgebra of a solvable Lie algebra is solvable.

For the question about homomorphic images, let φ be a Lie algebra homomorphism of $\mathfrak{g}$ onto $\mathfrak{h}$, and suppose that $\mathfrak{g}$ is solvable. Then $\varphi([\mathfrak{g}, \mathfrak{g}]) = [\varphi(\mathfrak{g}), \varphi(\mathfrak{g})]$, and more generally $\varphi(\mathfrak{g}^k) = \varphi(\mathfrak{g})^k$ for all $k \geq 1$. Thus $\mathfrak{g}^k = 0$ implies $\varphi(\mathfrak{g})^k = 0$.

10. From the previous problem, $(\mathfrak{g}/\mathfrak{a})^k$ is the image of $\mathfrak{g}^k$ under passage to the quotient modulo $\mathfrak{a}$. Since $\mathfrak{g}/\mathfrak{a}$ is solvable, $(\mathfrak{g}/\mathfrak{a})^k = 0$ for some k, and therefore $\mathfrak{g}^k \subseteq \mathfrak{a}$. If $\mathfrak{a}^\ell = 0$, then it follows that $\mathfrak{g}^{k+\ell} = 0$, and $\mathfrak{g}$ is solvable.

11. It is enough to show that the sum of two solvable ideals, which is known to be an ideal, is solvable. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then the ideal $\mathfrak{a} + \mathfrak{b}$ has $(\mathfrak{a} + \mathfrak{b})^1 = [\mathfrak{a} + \mathfrak{b}, \mathfrak{a} + \mathfrak{b}] \subseteq [\mathfrak{a}, \mathfrak{a}] + [\mathfrak{b}, \mathfrak{b}]$. Thus $(\mathfrak{a} + \mathfrak{b})^1 \subseteq \mathfrak{a}^1 + \mathfrak{b}^1$. In similar fashion, $(\mathfrak{a} + \mathfrak{b})^k \subseteq \mathfrak{a}^k + \mathfrak{b}^k$. Thus if $\mathfrak{a}^k = 0$ and $\mathfrak{b}^k = 0$, then $(\mathfrak{a} + \mathfrak{b})^k = 0$, and it follows that if $\mathfrak{a}$ and $\mathfrak{b}$ are both solvable ideals, then $\mathfrak{a} + \mathfrak{b}$ is solvable.

12. If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is solvable because $\mathfrak{g}_k \supseteq \mathfrak{g}^k$ for every $k \geq 0$.

13. This proved in the same way as in Problem 9.

14. In the same way as in Problem 11, one proves that the sum of two nilpotent ideals is nilpotent.

15. For $n = 0$, we have $[\mathfrak{g}, \mathfrak{g}]_0 = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^1$, i.e., equality holds. Assume inductively that $[\mathfrak{g}, \mathfrak{g}]_n \subseteq \mathfrak{g}^{2n+1}$. Then $[\mathfrak{g}, \mathfrak{g}]_{n+1} = [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]_n]$. The right side by the Jacobi identity is $\subseteq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]_n]] + [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]_n]]$, and this by inductive hypothesis is $\subseteq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}^{2n+1}]] + [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}^{2n+1}]] = \mathfrak{g}^{2n+3}$. The induction is complete, and the result follows.

16. The members X of $[\mathfrak{g}, \mathfrak{g}]$ have $X_{ij} = 0$ whenever $i \geq j - 1$. The members of $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ have $X_{ij} = 0$ whenever $i \geq j - 2$, and so on. The members of $\mathfrak{g}_k$ have $X_{ij} = 0$ whenever $i \geq j - k$. For $k = n$, all pairs (i, j) satisfy the condition $i \geq j - n$. Thus $\mathfrak{g}_k = 0$, and $\mathfrak{g}$ is nilpotent.

17. The linear Lie algebra is the set of derivatives at 0 of curves that equal the identity at $t = 0$. Using a curve that is $1 + t$ in one entry and is 0 in all other entries proves the desired result.

18. For (a), one answer is all matrices

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ \frac{1}{2}|z|^2 + it & \bar{z} & 1 \end{pmatrix}.$$

For (b), we have

$$\left[\begin{pmatrix} i\theta & 0 & 0 \\ 0 & -2i\theta & 0 \\ 0 & 0 & i\theta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ z & 0 & 0 \\ 0 & \bar{z} & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ -3i\theta z & 0 & 0 \\ 0 & -3i\theta\bar{z} & 0 \end{pmatrix}.$$

The special cases $z = 1$ and $z = i$ show that

$$\begin{pmatrix} 0 & 0 & 0 \\ -3i\theta & 0 & 0 \\ 0 & -3i\theta & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 3\theta & 0 & 0 \\ 0 & -3\theta & 0 \end{pmatrix}$$

are both in $[\mathfrak{g}, \mathfrak{g}]$. Another element in $[\mathfrak{g}, \mathfrak{g}]$ is

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix}.$$

The elements of $[\mathfrak{g}, \mathfrak{g}]$ have to be strictly lower triangular, and we have exhibited three linearly independent strictly lower triangular matrices in $[\mathfrak{g}, \mathfrak{g}]$. The dimension of $[\mathfrak{g}, \mathfrak{g}]$ is at most 3, and thus we have identified $[\mathfrak{g}, \mathfrak{g}]$ completely as the set of all the strictly lower triangular matrices in $\mathfrak{g}$. To prove the desired isomorphism, let $\{X, Y, Z\}$ be the basis of the three-dimensional Heisenberg algebra given by

$$X = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \ Y = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right), \ Z = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right),$$

and let $\{A, B, C\}$ be the basis of $[\mathfrak{g}, \mathfrak{g}]$ given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix}.$$

These bases have $[X, Y] = Z$ and $[A, B] = C$. Then it follows that the linear map φ with $\varphi(A) = X$, $\varphi(Y) = B$, and $\varphi(Z) = C$ can be used as the required isomorphism.

19. Direct computation shows that $[\mathfrak{h}_n, \mathfrak{g}]$ equals $\mathbb{R}c$ and that $[\mathfrak{h}_n, [\mathfrak{h}_n, \mathfrak{g}]] = 0$. Hence $\mathfrak{h}_n$ is nilpotent. (Observe that the vector c is not 0 because it is a member of a basis.)

20. For (a), $[\mathfrak{g}, \mathfrak{g}]$ is an ideal. By the hypothesis "simple," it must be 0 or $\mathfrak{g}$. Since "simple" by definition excludes $\mathfrak{g}$ abelian, the possibility $[\mathfrak{g}, \mathfrak{g}] = 0$ is ruled out, and only the possibility $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ remains.

For (b), $\dim \mathfrak{g} > 0$ since $\mathfrak{g}$ is nonabelian. Also $\operatorname{rad} \mathfrak{g}$ must be 0 or $\mathfrak{g}$ since $\mathfrak{g}$ is simple. If $\operatorname{rad} \mathfrak{g}$ is 0, then $\mathfrak{g}$ is semisimple by definition. If $\operatorname{rad} \mathfrak{g}$ is $\mathfrak{g}$, then $\mathfrak{g}$ is solvable, and $[\mathfrak{g}, \mathfrak{g}]$ must be properly contained in $\mathfrak{g}$ for the commutator series to end in 0.

For (c), let $\mathfrak{g}$ be semisimple, and let $\mathfrak{z}$ be the center. Then $\mathfrak{z}$ is a solvable ideal and must be contained in $\operatorname{rad} \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, the latter is 0, and thus $\mathfrak{z} = 0$.

21. The ideals in $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$ by the First Isomorphism Theorem (Problem 7) stand in one-one correspondence with the ideals in $\mathfrak{g}$ that contain $\operatorname{rad} \mathfrak{g}$, and moreover solvable ideals correspond to solvable ideals. The radical of $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$ is a solvable ideal and must correspond to a solvable ideal of $\mathfrak{g}$ containing $\operatorname{rad} \mathfrak{g}$. The only such ideal is $\operatorname{rad} \mathfrak{g}$. Thus $\operatorname{rad}(\mathfrak{g}/\operatorname{rad} \mathfrak{g}) = 0$, and $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$ is semisimple.

22. Let $\mathfrak{g}$ have dimension 3, and suppose $\mathfrak{g}$ is not simple. One possibility is that $\mathfrak{g}$ is abelian, in which case it is solvable. The only other possibilty is that it contains a proper nonzero ideal, say $\mathfrak{h}$. The classification in dimensions 1 and 2 shows that there are no simple Lie algebras in these dimensions, and thus $\mathfrak{h}$ must be solvable. Similarly $\mathfrak{g}/\mathfrak{h}$ is solvable. By Problem 10, $\mathfrak{g}$ is solvable.

23. Let Z be nonzero in $[\mathfrak{g}, \mathfrak{g}]$. Extend to a basis $\{X, Y, Z\}$. If $[\mathfrak{g}, Z] = 0$, then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}[X, Y]$ and hence $[X, Y] = cZ$ with $c \neq 0$; in this case we can easily set up an isomorphism with the Heisenberg algebra. Otherwise $[\mathfrak{g}, Z] = \mathbb{R}Z$. Since $[X, Z]$ and $[Y, Z]$ are multiples of Z, some nonzero linear combination of X and Y brackets Z to 0. Thus we can find a basis $\{X', Y', Z\}$ with $[X', Z] = 0$, $[Y', Z] = Z$, $[X', Y'] = cZ$. Then $\{X' + cZ, Y', Z\}$ has $[X' + cZ, Z] = 0$, $[Y', Z] = Z$, $[X' + cZ, Y'] = 0$. Consequently $\mathfrak{g} = \mathbb{R}(X' + cZ) \oplus \operatorname{span}\{Y', Z\}$ as required.

24. (a) Problem 15 implies that a 2-dimensional nilpotent Lie algebra is abelian. Consequently $[\mathfrak{g}, \mathfrak{g}]$ is abelian.

(b) The matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is nonsingular since otherwise $\dim[\mathfrak{g}, \mathfrak{g}] < 2$.

(c) The span of X and Y is characterized as $[\mathfrak{g}, \mathfrak{g}]$. The given Z can be any element not in $[\mathfrak{g}, \mathfrak{g}]$, and $\operatorname{ad} Z$ has eigenvalues 0 (from Z), α (from X), and 1 (from Y). If Z is replaced by $Z + sX + tY$, then the eigenvalues are unchanged. If we multiply by $c \in \mathbb{R}$, the eigenvalues are multiplied by c. Hence α is characterized as follows:

Let Z be any vector not in $[\mathfrak{g}, \mathfrak{g}]$. Then α is the ratio of the larger nonzero absolute value of an eigenvalue of Z to the smaller one.

(d) The description of α in the solution of (c) is invariant under isomorphisms of the group, and thus no two numbers $\alpha > 1$ lead to isomorphic Lie algebras. Since there are uncountably many such numbers α, the description applies to uncountably many Lie algebras that are mutually nonisomorphic.

25. The nilpotent Lie algebras come from the cases $[\mathfrak{g}, \mathfrak{g}] = 0$ and $[\mathfrak{g}, \mathfrak{g}] = 1$ in Problems 23–24. If $[\mathfrak{g}, \mathfrak{g}] = 2$, then Problem 24 handles matters. The remaining Lie algebras $\mathfrak{g}$ have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and $\mathfrak{g}$ is not solvable. Problem 22 shows that such a Lie algebra $\mathfrak{g}$ is simple.

26. Problem 12 in Chapter II shows that $\mathfrak{su}(2)$ is not isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. On the other hand, $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$ as a consequence of Problem 3 in Chapter II. The transitivity of "is isomorphic to" therefore implies that $\mathfrak{so}(3)$ cannot be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

27. If X is a commutator $[Y, Z]$, then $\operatorname{ad} X = \operatorname{ad} Y \operatorname{ad} Z - \operatorname{ad} Z \operatorname{ad}, Y$, and the trace is 0. Similarly the trace of $\operatorname{ad} X$ is 0 if X is any linear combination of commutators. This condition applies to all X in $\mathfrak{g}$ since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

28. One of the roots of the characteristic polynomial of $\operatorname{ad} X_0$ is 0, and the sum of the roots is 0 by Problem 27. Hence the roots are $0, \lambda, -\lambda$ with $\lambda \in \mathbb{C}$. The number λ cannot be 0 since $\operatorname{ad} X_0$ is by assumption not nilpotent. Since the characteristic polynomial of $\operatorname{ad} X_0$ has real coefficients, λ is real or purely imaginary. If λ is real, then the sum of the eigenspaces in $\mathfrak{g}$ for λ and $-\lambda$ is the required complement. If λ is purely imaginary, then the intersection of $\mathfrak{g}$ with the sum of the λ and $-\lambda$ eigenspaces in $\mathfrak{g} + i\mathfrak{g}$ is the required complement.

29. Let λ be as in Problem 28. If λ is real, then scale X_0 to X to make $\lambda = 2$ and show that $\operatorname{ad} X$ has the first form. If λ is purely imaginary, scale X_0 to X to make $\lambda = i$ and see that $\operatorname{ad} X$ has the second form.

30. The Jacobi identity gives $(\operatorname{ad} X)[Y, Z] = 0$. So $[Y, Z] = aX$, and a cannot be 0 since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Scale one of Y and Z to make $a = 1$, and then compare the bracket relations in $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$ in the two cases.

31. For (a) the members of $GL(\mathfrak{g})$ that are in $\operatorname{Aut}_{\mathbb{R}}(\mathfrak{g})$ are the ones satisfying certain polynomial equations in the entries.

For (b) let $t \mapsto c(t)$ be a curve in $\operatorname{Aut}_{\mathbb{R}}(\mathfrak{g})$ with $c(0) = 1$ and $c'(0) = D$, D being a member of $\operatorname{End}_{\mathbb{R}}(\mathfrak{g})$. If Y and Z are in $\mathfrak{g}$, then

$$c(t)[Y, Z] = [c(t)Y, c(t)Z] = (c(t)Y)(c(t)Z) - (c(t)Z(c(t)Y)$$

for all t. Differentiating with respect to t and putting $t = 0$ gives $c'(0)[Y, Z] = [(c'(0)Y), (c(0)Z)] + [(c(0)Y), c'(0)Z]$. Since $c(0) = 1$ and $c'(0) = D$, this equation gives

$$D[Y, Z] = [DY, Z] + [Y, DZ]$$

and shows that D is a derivation. Thus every member of the Lie algebra of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ is a derivation of $\mathfrak{g}$. In the reverse direction if D is a derivation of $\mathfrak{g}$, then e^{tD} satisfies

$$\begin{aligned}[e^{tD}Y, e^{tD}Z] &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\Big[\frac{t^kD^kY}{k!}, \frac{t^\ell D^\ell Z}{\ell!}\Big] = \sum_{N=0}^{\infty}\sum_{k=0}^{N}\frac{t^N}{N!}\binom{N}{k}[D^kY, D^{N-k}Z] \\ &= \sum_{N=0}^{\infty}\frac{t^N}{N!}D^N[Y,Z] = e^{tD}[Y,Z].\end{aligned}$$

Thus e^{tD} is an automorphism for every real t, and it follows that the linear analytic group corresponding to the Lie algebra of derivations includes the set of automorphisms.

For (c), the equality that needs proof is that $\mathrm{Ad}(g)[X,Y] = [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]$, where $\mathrm{Ad}(g)$ is conjugation by g, and this equality follows since $g[X,Y]g^{-1} = g(XY-YX)g^{-1} = (gXg^{-1})(gYg^{-1}) - (gYg^{-1})(gXg^{-1}) = [gXg^{-1}, gYg^{-1}]$.

32. For (a) the Jacobi identity gives $(\mathrm{ad}\,X)([Y,Z]) = [Y,(\mathrm{ad}\,X)Z]+[Y,(\mathrm{ad}X)Z]$, and this is just the derivation property for $\mathrm{ad}\,X$.

For (b) let $\mathfrak{g}$ be the 3-by-3 Heisenberg algebra, and define $D\left(\begin{smallmatrix}0&a&c\\0&0&b\\0&0&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&a&c\\0&0&0\\0&0&0\end{smallmatrix}\right)$. Put $A = \left(\begin{smallmatrix}0&a&c\\0&0&b\\0&0&0\end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix}0&x&z\\0&0&y\\0&0&0\end{smallmatrix}\right)$. Then $[A,X] = \left(\begin{smallmatrix}0&0&ay-bx\\0&0&0\\0&0&0\end{smallmatrix}\right)$ and $D[A,X] = \left(\begin{smallmatrix}0&0&ay-bx\\0&0&0\\0&0&0\end{smallmatrix}\right)$. Since $[DA,X] = \left(\begin{smallmatrix}0&0&ay\\0&0&0\\0&0&0\end{smallmatrix}\right)$ and $[A,DX] = \left(\begin{smallmatrix}0&0&-bx\\0&0&0\\0&0&0\end{smallmatrix}\right)$, we have $D[A,X] = [DA,X]+[A,DX]$, and D is a derivation. For any Y and Z, $(\mathrm{ad}\,Y)(Z)$ has only the upper right entry nonzero. But DY can have nonzero entry in the first row and second column. Thus this D is not given by ad of something.

33. As with Theorem 4.33, write $\mathfrak{m} = \mathfrak{s}\oplus\mathfrak{g}$. For g in a suitable open neighborhood of g_0, we can write $g = (\exp X)(\exp Y)g_0$ uniquely with $X \in \mathfrak{s}$ and $Y \in \mathfrak{g}$. Forming φ by using $x_1, \dots, x_{m-k}$ to parametrize U and $x_{m-k+1}, \dots, x_m$ to parametrize $\mathfrak{g}$, we obtain the desired chart.

34.
(a) $Z_G(S) = G$ and $Z_G(D) = D$.
(b) $Z_G(Z_G(S)) = S$ and $Z_G(Z_G(D)) = D$.

35. Theorem 4.26 is to be applied with G equal to the given group and φ equal to the Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian, it has all Lie brackets equal to 0 and is uniquely determined up to isomorphism by its dimension $d = \dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \geq 0$. The group G' is taken as the corresponding simply connected analytic group $\mathbb{R}^d$. What Theorem 4.26 supplies is a smooth group homomorphism $\Phi\colon G \to G'$ whose derivative at the identity of G is the quotient mapping $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.

The Lie algebra of the kernel of Φ is the kernel of φ, which is $[\mathfrak{g},\mathfrak{g}]$. Thus $\ker\Phi$ contains the analytic subgroup of G with Lie algebra $[\mathfrak{g},\mathfrak{g}]$, which is called $[G,G]$ in the text. Since Φ is continuous, its kernel is closed and must contain the closure

$\overline{[G,G]}$. This fact gives us an inequality concerning dimensions:

$$\dim[G,G] = \dim[\mathfrak{g},\mathfrak{g}] = \dim\ker\varphi = \dim\ker\Phi \ge \dim\overline{[G,G]}.$$

Since $[G,G]$ and $\overline{[G,G]}$ are analytic subgroups with the second containing the first, this inequality forces them to be equal. Consequently $[G,G]$ is closed.

36. Proposition 4.9 establishes the result if the domain of the given homomorphism is $\mathbb{R}$. Now let the domain of φ be an analytic group G with Lie algebra $\mathfrak{g}$, and let $X_1,\dots,X_n$ be a vector space basis of $\mathfrak{g}$. Put $Y_j = \varphi(X_j)$ for $1 \le j \le n$. In a neighborhood of g_0 in G, every element g is of the form

$$g = g_0(\exp u_1X_1)(\exp u_2X_2)\cdots(\exp u_nX_n).$$

Then the homomorphism property gives

$$\begin{aligned}\varphi(g) &= \varphi(g_0)\varphi(\exp(u_1X_1))\cdots\varphi(\exp(u_nX_n))\\ &= \varphi(g_0)(\exp u_1Y_1)\cdots(\exp u_nY_n).\end{aligned}$$

Smoothness of a function of g near g_0 is tested by smoothness of $u_1,\dots,u_n$, and this formula therefore exhibits φ as smooth near g_0.

37. The following three matrices generate $\mathfrak{so}(3)$:

$$A_1 = \begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\qquad A_2 = \begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},\qquad A_3 = \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

The following three matrices, together with the three above, generate $\mathfrak{so}(4)$.

$$B_1 = \begin{pmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\qquad B_2 = \begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\qquad B_3 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}.$$

Direct computation shows that bracketing these in pairs gives us the formulas

$$[A_i,A_j] = \epsilon_{ijk}A_k,\qquad [B_i,B_j] = \epsilon_{ijk}B_k,\qquad [A_i,B_j] = \epsilon_{ijk}B_k,$$

where ϵ_{ijk} is the **Levi-Civita symbol,** equal to 1 if i,j,k is an even permutation of $1,2,3$ or equal to -1 if i,j,k is an odd permutation of $1,2,3$.

Now we define

$$X_i = \frac{1}{2}(A_i+B_i)\qquad\text{and}\qquad Y_i = \frac{1}{2}(A_i-B_i)\qquad\text{for}\quad i=1,2,3.$$

Then we compute that

$$[X_i,X_j] = \epsilon_{ijk}X_k,\qquad [Y_i,Y_j] = \epsilon_{ijk}Y_k,\qquad [X_i,Y_j] = 0.$$

These relations show that the X's and the Y's bracket among themselves, defining ideals, and exhibiting $\mathfrak{so}(4)$ as a sum of two ideals. Each of the these ideals satisfies the relations for a copy of $\mathfrak{so}(3)$, and the computation is complete.

38. We know that $SO(4)$ has a universal covering group whose Lie algebra is isomorphic to $\mathfrak{so}(4)$. That group is the product of a group whose Lie algebra is one factor of $\mathfrak{so}(3)$ and a group whose Lie algebra is the other factor of $\mathfrak{so}(3)$. Each Lie algebra corresponds to a tower of covering groups that are homomorphic images of a simply connected one. From Chapter II we know that the tower for a single $\mathfrak{so}(3)$ consists of a simply connected group with Lie algebra $\mathfrak{so}(3)$ isomorphic with the group of unit quaternions and the quotient by a group with $\pi_1 = \mathbb{Z}/2\mathbb{Z}$, which must be $SO(3)$. The rest follows.

39. The tower of covering groups containing $SO(4)$ consists of the simply connected cover, $SO(4)$ itself, and the quotient of $SO(4)$ by its two-element center.

40. In the Lie algebra $\mathfrak{g}$, let $\mathfrak{h}$ be the subalgebra of diagonal matrices, and put $\mathfrak{g}_{e_i-e_j} = \mathbb{C}E_{ij}$ for $i \neq j$. The linear functions $e_i - e_j$ on $\mathfrak{h}$ for $i \neq j$ are called the roots. The direct-sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i\neq j} \mathfrak{g}_{e_i-e_j}$ is called the root-space decomposition of $\mathfrak{g}$. Among all members λ of $\mathfrak{h}'$ that vanish on scalar matrices, we say that the positive ones are those λ for which the first nonzero $\langle \lambda, e_j\rangle$ is positive.

Assuming that $\mathfrak{g}$ is not simple, let $\mathfrak{a}$ be a nonzero ideal in $\mathfrak{g}$. First suppose that $\mathfrak{a} \subseteq \mathfrak{h}$. Let $H \neq 0$ be in $\mathfrak{a}$. Since the roots span $\mathfrak{h}'$, some root is nonvanishing on H; say $\alpha(H) \neq 0$. If X is a nonzero member of the root space $\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$, then $\alpha(H)X = [H, X] \in [\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a} \subseteq \mathfrak{h}$ gives an immediate contradiction since the sum $\mathbb{C}X \oplus \mathfrak{h}$ is direct.

Now suppose that $\mathfrak{a}$ is not contained in $\mathfrak{h}$. Let $X = H + \sum X_\alpha$ be in $\mathfrak{a}$ with each X_α in $\mathfrak{g}_\alpha$ and with some $X_\alpha \neq 0$. For the moment assume that there is some root $\alpha < 0$ with $X_\alpha \neq 0$, and let β be the smallest such α. Say $X_\beta = cE_{ij}$ with $i > j$ and $c \neq 0$. Form the iterated bracket product $u = [E_{1i}, [[X, E_{jn}]]$. The claim is that u is a nonzero multiple of E_{1n}. In fact we cannot have $i = 1$ since $j < i$. If $i < n$, then $[E_{ij}, E_{jn}] = aE_{in}$ with $a \neq 0$, and also $[E_{1i}, E_{jn}] = bE_{1n}$ with $b \neq 0$. Thus u has a nonzero component in the term $\mathfrak{g}_{e_1-e_n}$ of the root-space decomposition. The other components of u must correspond to larger roots than $e_1 - e_n$ if they are nonzero, but $e_1 - e_n$ is the largest root. Hence the claim follows if $i < n$. If $i = n$, then u is

$$= [E_{1n}, [cE_{nj} + \dots, E_{jn}]] = c[E_{1n}, E_{nn} - E_{jj}] + \cdots = cE_{1n}.$$

Thus the claim follows if $i = n$.

In any case we conclude that E_{1n} is in $\mathfrak{a}$. For $i \neq j$, the formula

$$E_{kl} = c'[E_{kl}, [E_{1n}, E_{nl}]] \qquad \text{with } c' \neq 0$$

(with obvious changes if $k = 1$ or $l = n$) shows that E_{kl} is in $\mathfrak{a}$, and the formula

$$[E_{kl}, E_{lk}] = E_{kk} - E_{ll}$$

shows that a spanning set of $\mathfrak{h}$ lies in $\mathfrak{a}$. Hence $\mathfrak{a} = \mathfrak{g}$.

Thus an ideal $\mathfrak{a}$ that is not contained in $\mathfrak{h}$ has to be all of $\mathfrak{g}$ if there is some $\alpha < 0$ with $X_\alpha \neq 0$ as above. Similarly if there is some $\alpha > 0$ with $X_\alpha \neq 0$, let β be the largest such α, say $\alpha = e_i - e_j$ with $i < j$. Form $[E_{ni}, [X, E_{j1}]]$ and argue with E_{n-1} in the same way to get $\mathfrak{a} = \mathfrak{g}$. Thus $\mathfrak{g}$ is simple over $\mathbb{C}$.

41. Let $\mathfrak{h}$ be the diagonal subalgebra with entries $H = \left\{\begin{pmatrix} h_1 &&& \\ & h_2 && \\ && -h_1 & \\ &&& -h_2 \end{pmatrix}\right\}$,
and define $e_1(H) = h_1$ and $e_2(H) = h_2$. The eight roots are as in the statement of the problem, and the root vectors, which are unique up to multiplication by nonzero scalars, are

$$\begin{aligned} E_{e_1-e_2} &= E_{12} - E_{43}, & E_{2e_k} &= E_{k,k+2} \\ E_{e_1+e_2} &= E_{14} + E_{23}, & E_{-2e_k} &= E_{k+2,k}. \\ E_{-e_1-e_2} &= E_{32} + E_{41} \end{aligned}$$

with $k = 1, 2$ above. Order the roots with $e_1 > e_2 > 0 > -e_1 > -e_2$. Then the argument proceeds as in Problem 47 (40), pushing root vectors upward (by bracketing with the root vector for a positive root) until they get to the largest root and then pushing them downward until they get into $\mathfrak{h}$. The argument is completed as in Problem 40.

42. This is completely analogous to Problem 41.

43. Here the cases n odd and n even are slightly different. Let us write $n = 2m+1$ or $n = 2m$. For $n = 2m+1$, $\mathfrak{h}$ consists of m 2-by-2 blocks of the form $\begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix}$ down the diagonal and a final diagonal entry of 0. For $n = 2m$, $\mathfrak{h}$ consists of the m 2-by-2 blocks only. The root vectors are similar to those in $\mathfrak{sp}(n, \mathbb{C})$ and are omitted here. The argument with an ideal $\mathfrak{a}$ is similar to that in Problem 47.

44. (a) $g^\# g = Jg^tJ^{-1}g = J^{-1}g^tJg$, and this is $J^{-1}J = 1$ if and only if g is symplectic.

(b) $x^\# = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b^t & d^t \\ -a^t & -c^t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$.
Comparing the right side with x proves the result.

(c) Since $(g^1g_2)^\# = Jg_2^tg_1^tJ^{-1} = Jg_2^tJ^{-1}Jg_1^tJ^{-1} = g_2^\# g_1^\#$, we see that

$$(g_1g_2)\cdot x = g_1g_2x(g_1g_2)^\# = g_1g_2xg_2^\# g_1^\# = g_1(g_2\cdot x)g_1^\# = g_1\cdot(g_2\cdot x).$$

45. Making use of Problem 44a and the fact that g is assumed symplectic allows us to make the computation

$$\langle g\cdot x, g\cdot y\rangle = \mathrm{Tr}(gxg^\# gyg^\#) = \mathrm{Tr}(gxyg^\#) = \mathrm{Tr}(g^\# gxy) = \mathrm{Tr}(xy) = \langle x, y\rangle.$$

Also if g is symplectic, then $1_4 = gg^\# = g\cdot 1_4$, and thus every symplectic matrix fixes 1_4.

46. The space $\mathfrak{m}_{4,4}(\mathbb{C})$ has dimension 16, and V is the subspace where $x^\# = x$ and $\langle x, 1\rangle = 0$. The latter subspace is defined by linear conditions. Problem 44b says that the conditions for $x^\# = x$ are that b and c are skew symmetric and $d = a^t$ is arbitrary. The conditions on b and c are independent linear conditions with each

cutting down the dimension by 3, and the condition on d cuts it down by 4 more. Finally the condition $\langle x, 1\rangle = 0$ means that $\operatorname{Tr} a = 0$, and this cuts down the dimension 1 more. Thus the dimension is cut down from 16 by $10 + 1 = 11$, and the result is that $\dim V = 5$.

47. The trace of the product of any two distinct members of the five is 0, by direct computation. Hence the five are mutually orthogonal under $\langle\cdot,\cdot\rangle$.

The homomorphism in question is

$$g \mapsto (x \in V \mapsto gxg^{\#}).$$

The linear map on the right side carries $\mathbb{C}^5$ to itself, with $\mathbb{C}^5$ spanned by the matrices in the statement of this problem, and $\langle g\cdot x, g\cdot y\rangle = \langle x, y\rangle$. Thus the displayed map above is a homomorphism carrying $Sp(2,\mathbb{C})$ into $O(5,\mathbb{C})$. The image matrices must have determinant 1 because Problem 41 shows $\mathfrak{sp}(2,\mathbb{C})$ to be simple.

48. Problem 41 shows that $Sp(2,\mathbb{C})$ is simple. The map in Problem 47 not being the 0 map, the kernel of the derivative of the Lie group homomorphism in Problem 47 must be 0. Since $Sp(2,\mathbb{C})$ and $Sp(5,\mathbb{C})$ both have dimension 10, the Lie algebra homomorphism must be onto. Thus the homomorphism is a covering map. Since $Sp(2,\mathbb{C})$ has center ± 1 and $SO(5,\mathbb{C})$ has trivial center, the kernel is ± 1.

49. (a) With $g \in SL(4,\mathbb{C})$ fixed, take ℓ to be the map of E into $\bigwedge E$ with $\ell(x) = gx$ for $x \in E$. Then $\ell(x)\wedge\ell(x) = 0$ for all x, and the universal mapping property is applicable. The result is a definition of g on all of $\bigwedge E$, and $g(x\wedge y) = gx \wedge gy$ for all x and y in $\mathbb{C}^4$.

(b) The identity $(g_1g_2)\cdot w = g_1\cdot(g_2\cdot w)$ for every $w \in \wedge^2\mathbb{C}^4$ is an immediate consequence of the uniqueness in the statement of the universal mapping property.

50. (a) The bilinearity of $\wedge$ on $\bigwedge^2\mathbb{C}^4$ immediately reduces the question of symmetry to the case that u and v are of the form $e_i\wedge e_j$, and we get 0 unless we are considering $\langle e_i\wedge e_j, e_k\wedge e_\ell\rangle$ with $\{i,j\}\cap\{k,\ell\} = \varnothing$. Matters come down to three cases for u: $e_1\wedge e_2$, $e_1\wedge e_3$, and $e_1\wedge e_4$. The corresponding v's to consider may be taken to be $e_3\wedge e_4$, $e_2\wedge e_4$, and $e_2\wedge e_3$. For $u = e_1\wedge e_2$ and $v = e_3\wedge e_4$, the two sides are $u\wedge v = (e_1\wedge e_2)\wedge(e_3\wedge e_4)$ and $\langle u,v\rangle(e_1\wedge e_2\wedge e_3\wedge e_4)$, from which we obtain $(u,v) = +1$. When u and v are reversed, we have $v\wedge u = (e_3\wedge e_4)\wedge(e_1\wedge e_2)$ and $\langle v,u\rangle = (v,u)(e_1\wedge e_2\wedge e_3\wedge e_4)$, from which we obtain $(v,u) = +1$ since passing from $e_3\wedge e_4\wedge e_1\wedge e_2$ to $e_1\wedge e_2\wedge e_3\wedge e_4$ involves an even number of transpositions of indices. The other two cases are similar.

(b) This is routine.

(c) From (a) the bilinear form is symmetric, and from (b) it is nondegenerate.

51. We are to show that $(gu, gv) = (u,v)$ for u and v in $\bigwedge^2\mathbb{C}^4$ and g in $SL(4,\mathbb{C})$. It is enough to treat $u = e_i\wedge e_j$ and $v = e_k\wedge e_\ell$. We have

$$\begin{aligned}\langle gu, gv\rangle(e_1\wedge e_2\wedge e_3\wedge e_4) &= gu\wedge gv\\ &= g(u\wedge v) = g(\langle u,v\rangle(e_1\wedge e_2\wedge e_3\wedge e_4))\\ &= (\det g)\langle u,v\rangle(e_1\wedge e_2\wedge e_3\wedge e_4),\end{aligned}$$

and invariance is proved.

52. Problem 49 establishes the group homomorphism

$$g \in SL(4,\mathbb{C}) \mapsto g \in GL(6,\mathbb{C}),$$

and Problem 51 shows that the image is in $O(6,\mathbb{C})$ relative to the symmetric invariant bilinear form $\langle \cdot\, \cdot \rangle$. The members of the image have to have determinant one because the Lie algebra of $SL(4,\mathbb{C})$ is simple (according to Problem 40) and therefore has no one dimensional quotients. In other words the image has to be in $SO(6,\mathbb{C})$.

53. Passing to Lie algebras, we have a nonzero homomorphism of $\mathfrak{sl}(4,\mathbb{C})$ into $\mathfrak{so}(6,\mathbb{C})$. Since $\mathfrak{sl}(4,\mathbb{C})$ is simple, the Lie algebra homomorphism is one-one. Both $\mathfrak{sl}(4,\mathbb{C})$ and $\mathfrak{so}(6,\mathbb{C})$ have dimension 15 over $\mathbb{C}$ and therefore dimension 30 over $\mathbb{R}$. Thus the homomorphism of $SL(4,\mathbb{C})$ to $SO(6,\mathbb{C})$ has to be onto, and the map is a covering map. The kernel is contained in the group of scalar 4^{th} roots of unity in $SL(4,\mathbb{C})$. Thus it has order 4, 2, or 1. The kernel cannot be $\{1\}$ since $SO(4,\mathbb{C})$ and $SO(6,\mathbb{C})$ have nonisomorphic centers. To complete the argument, we show that the kernel cannot have order 4.

The map of $SO(6,\mathbb{C})$ to the quotient modulo its center (of order 2) is a covering map, with two sheets in the sense that the inverse image of any evenly covered open neighborhood has two components. Since a covering by $SL(4,\mathbb{C})$ has at most four sheets (because $SL(4,\mathbb{C})$ has center of order 4), the case that the kernel of $SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$ has one element is ruled out, and the kernel must be ± 1.

HISTORICAL COMMENTARY

Mathematicians are often poor historians and do not always attribute accurately results they use to their original sources.

SIGURDUR HELGASON

Despite the caution recommended by the Helgason quotation above, this Historical Commentary will include a brief description of how Sophus Lie was led to study local versions of the groups that bear his name. They give also some idea of the initial results that Lie obtained and some information about how the basic theory evolved in the twentieth century and for a time thereafter.

This evolution, like that of so many branches of mathematics, has not been in a straight line. New interpretations and new proofs of old results have led to branches and switch-backs that do not lead to a tidy complete exposition.

Lie's early work

A short summary of Lie's early scientific life appears in the article Helgason [1994] in connection with a memorial conference for Lie. Helgason says that Lie finished his university studies in the natural sciences in 1865 but did not find a calling to research in mathematics until 1868 at age 26. Helgason writes:

"During the three years after his university degree he gave private lessons in mathematics and once gave a series of popular lectures on astronomy. At the same time he speculated a good deal about geometry and in 1868 he came across the works of J. Poncelet and J. Plücker [from decades earlier]. These seem to have inspired Lie in a very decisive fashion."

Lie worked with equations that define sets in projective geometry, and following Poncelet, he allowed complex solutions of his equations, not just real ones. From this investigation he wrote his first paper, eight pages in a later German translation. That paper and his prior reputation were good enough so that a request for a fellowship was granted, and he was able to visit the capitals of Europe and meet other mathematicians. He started with Berlin in 1869. There he met Felix Klein, and, according to Helgason, "they became good friends." They traveled together to Paris in the spring of 1870 and took adjoining rooms in a hotel. In July 1870, according to Helgason, "Lie made his most famous geometric discovery, what is now called Lie's line-sphere transformation." Briefly the group $SL(4,\mathbb{C})$ acts via projective transformations on the space of lines in $\mathbb{C}^3$, and $SO(6,\mathbb{C})$ acts on the projective bundle associated with the tangent bundle of $\mathbb{C}^3$. The Lie line-sphere transformation is a realization of the local isomorphism of these two groups.

The line-sphere transformation of Lie is an example of a "contact transformation," and Lie's mathematical interest turned to those transformations. Back in Christiania (Oslo, Norway), he submitted his doctoral thesis in July 1871 on a general theory concerning contact transformations, and he became a professor at the University of Christiania in 1872. He continued to work on contact transformations for one year and in 1873 began a systematic study of transformation groups. As Helgason explains matters, the motivation was a question Lie stated explicitly in [1874a]: "How can the knowledge of a stability group for a differential equation be utilized for its integration?" In that 1874 paper, according to Helgason, Lie proved the following "now famous theorem." The example that follows its statement here appears in Helgason [1994].

Theorem. Let ϕ_t be a one-parameter group of diffeomorphisms of an open subset of $\mathbb{R}^2$, let Ψ be the induced vector field on $\mathbb{R}^2$ whose value at p is

$$\Psi_p = \left\{\frac{\partial \phi_t(p)}{\partial t}\right\}_{t=0} = \xi(p)\frac{\partial}{\partial x} + \eta(p)\frac{\partial}{\partial y},$$

and let

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}$$

be a given first-order differential equation. Then the transformations ϕ_t leave the differential equation stable if and only if the vector field

$$Z = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}$$

satisfies

$$[\Psi, Z] = \lambda Z$$

for some scalar-valued function λ. In this case, $(X\eta - Y\xi)^{-1}$ is an integrating factor for the equation

$$X\,dy - Y\,dx = 0.$$

Example. These considerations apply to the differential equation

$$\frac{dy}{dx} = \frac{y + x(x^2+y^2)}{x - y(x^2+y^2)}$$

with $X(x,y) = x - y(x^2+y^2)$ and $Y(x,y) = y + x(x^2+y^2)$. The given equation can be written as $X\,dy - Y\,dx = 0$ and rewritten as

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}\frac{dy}{dx}} = x^2 + y^2.$$

The rewritten left side equals $\tan\alpha$, where α is the angle at (x,y) between the integral curve through (x,y) and the radius from $(0,0)$ to (x,y). The right side depends only on the distance from the origin; thus the angle α is constant on the circle with center $(0,0)$ passing through (x,y). Consequently the rotation group

$$\phi_t(x,y)\colon\ (x,y) \longrightarrow (x\cos t - y\sin t, x\sin t + y\cos t),$$

for which

$$\Psi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

leaves the equation stable. The vector field Z in the theorem is given by

$$Z = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y},$$

and

$$[\Psi, Z] = -2xy^2\frac{\partial}{\partial x} - 2x^2y\frac{\partial}{\partial y} = -2xyZ.$$

The theorem applies because the condition $[\Psi, Z] = \lambda Z$ is satisfied with λ equal to the function $-2xy$. Since $\xi = -y$ and $\eta = x$, the theorem says that $(X\eta - Y\xi)^{-1} = (Xx + Yy)^{-1}$ is an integrating factor for the given equation $X\,dy - Y\,dx = 0$.

Helgason [1994] goes on to say, "Generalizations of the theorem above to systems led Lie to develop a theory for r-parameter local groups of transformations in $\mathbb{R}^n$:

$$Y_t\colon x_i' = f_i(x_1, \ldots, x_n; t_1, \ldots, t_r), \quad (t) = (t_1, \ldots, t_r).$$

[See Lie [1874b]]. Here it is assumed that the f_i are analytic functions, depending essentially on all the t_i, and that

$$T_{(0)} = 1 \qquad T_{(t)} \circ T_{(s)} = T_{(u)},$$

where $(u) = (u_1, \ldots, u_r)$ depends analytically on (t) and (s). The group ϕ_t above is a special case. Lie's big project (expressed in print in 1874) was to determine all such transformation groups $T_{(t)}$ and apply the results to the solving of differential equations."

Helgason [1994] includes a brief description of "Lie's fundamental results toward solving this problem," and the role of "infinitesimal groups", which were later to be called Lie algebras, comes more into view. The single vector field Ψ of the example gets replaced by a collection $X_1, \ldots, X_r$ of vector fields, r being the number of parameters. Lie proved that the bracket of any two these vector fields is a *scalar* linear combination of the other vector fields and that the resulting system of scalars satisfies the properties that we now recognize as defining a Lie algebra. He proved conversely that every system of scalars satisfying those properties arises in this way.

Helgason [1994] summarizes, "This was a major accomplishment and the original proofs were very complicated. Part of the problem is to establish one-parameter subgroups inside the original group [namely $T_{(t)}$]. These results reduce the internal study of local transformation groups to the study of Lie algebras."

"... [Although] Lie intended this as a tool in the study of differential equations, the influence in other fields has been much greater after the general theory of Lie groups had grown into a field of its own, independent of transformation group theory."

Lie took special notice of cases that he could reduce completely to the one-parameter case by induction. He defined a Lie algebra over $\mathbb{C}$ to be integrable if this kind of reduction was possible, and he showed that the condition on the Lie algebra is what today is called solvability. In more modern language he proved that any solvable Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a descending sequence of ideals, going from $\mathfrak{g}$ down to 0, with successive quotients each of dimension 1. This is the result that is now known as Lie's Theorem.

Work with Engel

Lie continued to work on these and related questions through 1883 but in an 1884 letter to Adolf Mayer expressed despair that he could not interest other mathematicians in transformation groups and their applications to differential equations. Helgason [1994] writes:

"Being aware of Lie's isolation in Christiania, F. Klein and A. Mayer arrange for Klein's student in Leipzig, Friedrich Engel (then 22), to go there to work with Lie. ... Lie decided time was ripe for a full exposition of his theory of transformation groups and Engel was the perfect collaborator. Lie and Engel met every day, in the morning in Engel's apartment, at Lie's in the afternoon. Lie produced a skeleton of each section, discussed contents in detail and left it to Engel to provide this with flesh and blood."

Engel returned to Leipzig after one year, but the project did not end at that time. Lie was appointed Professor of Geometry in Leipzig in 1886, and the joint work continued. Helgason says that it was finally "completed after nine years of labor and appeared in three large volumes between 1888 and 1893." These volumes are listed here as Lie–Engel [1888–90–93].

A detailed summary of this early theory, with extensive references, appears in Bourbaki [1972], pp. 286–308. Further historical information may be found in Section 1.16 of Duistermaat-Kolk [2000]. Lie's Theorem about solvable Lie algebras appeared originally in Lie–Engel [1888–90–93], Vol. III, Satz 2, p. 678, and it is reproduced in the present volume's sequel, *Lie Groups Beyond an Introduction*, as Theorem 1.25. A dictionary relating Lie's terminology (in German) with modern terminology appears on page 87 of Duistermaat–Kolk [2000].

The main results of Lie–Engel for current purposes are three theorems of Lie for passing back and forth between "finite continuous" transformation groups (locally defined Lie groups) and their families of "infinitesimal transformations" (their Lie algebras). Each theorem consisted of a statement and its converse. For precise statements, see Bourbaki [1972], pp. 294–296. For precise statements with proofs, see Cohn [1957], Chapter V. Since the emphasis in the present book is on how the subject evolved up to a certain point, the Preface skips a precise formulation of the local theory and recasts those three theorems of Lie's in a global form in which they are now understood. The Preface adds to the three a fourth result related to covering groups[1] and calls the four the Fundamental Theorems of Lie Theory.

Transition to a global theory

A number of people carried on Sophus Lie's work after Lie's move to Leipzig in 1888. First of all, there was Engel, who had received his doctoral degree from Universität Leipzig in 1883, with F. Klein and W. Scheibner as advisers. As to students of Lie himself, the Mathematics Genealogy Project[2] as of this

[1]The theory of covering groups is a development from the 1920s, long after Lie's death.

[2]https://genealogy.math.ndsu.nodak.edu

writing lists 19 successful doctoral students for Lie, of whom 18 finished in 1887 or later. Of the 18, all but two were from Leipzig. Those two were from Université de Paris, Arthur Tresse in 1893 and Elie Cartan in 1894. Evidently the move to Leipzig stimulated great interest in the area in which Lie was working. We return to Lie's doctoral students after mentioning contributions of three young German mathematicians who worked in related areas: F. G. Frobenius (1870), W. Killing (1872), and F. Schur (1879). All three had K. Weierstrass and E. Kummer as advisors.

F. G. Frobenius: Frobenius was only a few years younger than Lie, and his mathematics research began in partial differential equations and eventually moved into developing character theory for finite groups, exploiting field automorphisms in number theory, and advancing the theory of elliptic functions. His work in partial differential equations amounted to work on what are now called foliations, and his paper Frobenius [1877] contained a proof of the local result that Lie needed to pass from a Lie subalgebra to a local version of the left cosets of an analytic subgroup. Frobenius acknowledged in his paper that this result of his had appeared earlier in two papers, partly in Deahna [1840] and partly in Clebsch [1866]. More detail about this aspect of the history can be found in Samelson [2001]. In recent times the theorem often has only the name Frobenius attached to it, but some people prefer to call it the Deahna–Clebsch–Frobenius Theorem. The present book uses the shorter name.

A previously known special case of the setup for foliations concerned existence of integral curves locally for a nonvanishing smooth vector field. Such integral curves exist locally by the standard existence-uniqueness results for ordinary differential equations. These results are attributed to Picard, Lindelöf, and Lipschitz.

The Frobenius theorem addresses a d dimensional analog, asking when a system of d dimensional subspaces of $\mathbb{R}^n$ can match the system of tangent spaces in $\mathbb{R}^n$ to some d dimensional submanifold of $\mathbb{R}^n$, one for each point of the submanifold, at least at all points near a base point in $\mathbb{R}^n$. Such a system of d dimensional subspaces is nowadays called a d **distribution**. It is assumed that the distribution is **smooth** in the sense that for each point p near the base point, there is a finite set of smooth vector fields X such that the tangent vectors X_p span the d dimensional subspace of $\mathbb{R}^n$ associated to p. Frobenius [1877] contains a necessary and sufficient condition—that the given d distribution is **involutive** in the sense that the defining set of vector fields is closed under bracket. This was exactly the kind of result that Lie needed in order to construct local Lie groups out of Lie subalgebras. For more information about d distributions and foliations, see Lawson [1974].

W. Killing: For many years interest in the area in which Lie worked tended not to be in transformation groups but in the subject of Lie algebras, which were called "infinitesimal groups" until the mid 1930s. Killing, from Universität Berlin, came at "infinitesimal groups" from a completely different point of view from Lie. Killing was interested in classification questions, and he thought it was more reasonable to classify "infinitesimal groups" first and afterward to classify finite-dimensional continuous transformation groups, i.e., local versions of connected Lie groups. Accordingly he wanted to find all possible ways of imposing a multiplication on n dimensional space to make it behave like a Lie algebra. In modern terminology Killing introduced the radical of a Lie algebra, and he sought to discover and classify the simple Lie algebras over $\mathbb{C}$. In his work toward this classification, he introduced an object that amounted to a Cartan subalgebra, but he did not give a complete proof of existence. He used Cartan subalgebras to define roots, and he attempted to use bases of roots to carry out the classification. Ultimately he found the four countably infinite families of simple Lie algebras over $\mathbb{C}$, and he found indications of six other such algebras and no more; two of the six were later observed to be isomorphic to each other. He presented his work to Lie and Engel, but his work contained mistakes and other weaknesses, and much of the work needed to be redone. Lie dismissed Killing's efforts, but Engel worked with Killing to correct the mathematics, enlisting his own first doctoral student, K. Umlauf, to help. Despite his mistakes Killing must be acknowledged as the first person to point to the existence of simple Lie algebras over $\mathbb{C}$ other than the well known ones.

F. Schur: F. Schur, not the same person as the later and better known I. Schur, did two important pieces of work related to Lie theory. In F. Schur [1891] he in essence found the derivative of the exponential mapping from a Lie algebra to the corresponding Lie group. Poincaré [1899] elaborated on this formula later. An equivalent way of summarizing Schur's result is that Schur's found a differentiation formula for the function $t \mapsto \exp X(t)$ when $t \mapsto X(t)$ is a smooth curve in a Lie algebra. This result appears in Sections III.3 and IV.6 of the present volume.

F. Schur [1893] raised the question of the extent to which real analyticity of the various defining functions was really needed in Lie's theory. He proved that if the group operations were of class C^2, that would be enough, and the real analyticity could be inferred. The proof appears in Duistermaat–Kolk [2000] on page 29. The present volume works with groups having smooth (i.e., C^∞) multiplication and inversion. An assumption that the group operations are real analytic would produce the same groups and theorems as when they are assumed to be smooth, but there are pedagogical advantages in using smooth operations. Accordingly the present book works with group operations that are assumed to be smooth.

A. Tresse: Lie's student Arthur Tresse apparently sought and received permission from Lie to use the term "Lie group" in his 1893 Paris thesis, but the term did not fully catch on until his fellow student E. Cartan used it in print many years later (see Cartan [1930b]). Tresse's thesis dealt with an aspect of Lie's theory applied to second order ordinary differential equations, and Tresse's subsequent publications appear to have been on more elementary matters.

E. Cartan: The thesis of Cartan [1894] was far reaching and opened the subject of "infinitesimal groups" to extensive investigations. Cartan showed existence of the subalgebras that now bear his name, and he used also a certain bilinear form on Lie algebras related to but different from what came to be known as the Killing form. With the thesis work Umlauf [1891] in hand, Cartan succeeded in classifying the simple Lie algebras over $\mathbb{C}$, characterizing the structure of the five exceptional simple Lie algebras. For the final details of this mathematics as eventually carried out by Elie Cartan, see Chapter II of *Lie Groups Beyond an Introduction*.

In a remarkable advance Cartan [1914] classified the simple Lie algebras over $\mathbb{R}$. in particular Cartan must be given credit for the discovery of any of the simple Lie algebras over $\mathbb{R}$ that were not previously known from geometry.

Further commentary of the early development of the theory of Lie groups may be found in Cartan [1971], Hawkins [2000], Borel [2001], and Cogliati [2014].

Campbell's Problem

Except for the work of F. Schur and E. Cartan, research on Lie groups came largely to a halt from 1890 until the late 1890s. Toward the end of this period, J. E. Campbell (in Campbell [1897] and Campbell [1898]) found a formula for passing from a pair of square matrices X and Y near 0 to a square matrix Z near 0 such that

$$\exp Z = \exp X \exp Y,$$

and he made progress on showing that if X and Y are in a linear Lie algebra, then Z will be in the same linear Lie algebra. Such a formula is given in this book by Proposition 3.1. Writing matters out to see the effect on Lie brackets involves complicated calculations. Both Poincaré [1899] and E. Pascal [1901] made progress on these calculations, which were akin to working in a universal enveloping algebra and using a symmetrization operator. Poincaré called this effort "Campbell's Problem."

Beginning of the treatment of Lie groups as global objects

The idea of treating global groups systematically did not arise until H. Weyl, inspired by work of I. Schur [1924] that extended representation theory from finite groups to the orthogonal and unitary groups, began his study of compact connected groups (Weyl [1924] and [1925–26]). Schreier [1926] and [1927] defined topological groups and proved the existence of universal covering groups of connected global Lie groups, and von Neumann [1927] and [1929] investigated smoothness properties of matrix groups and of homomorphisms between them. Cartan [1930a] underlined the importance of global groups by giving a direct proof of a global version of Lie's third theorem—that every finite-dimensional real Lie algebra is the Lie algebra of a Lie group.

Cartan [1930b] wrote the first book on global Lie theory, in effect proving theorems about locally Euclidean groups, Lie groups, compact Lie groups, and various homogeneous spaces of Lie groups. This 60-page book is remarkable for transforming the emphasis in the subject. The first chapter axiomatizes locally Euclidean groups and makes them an object of study. The second chapter sketches proofs that a local Lie group can always be extended to a global Lie group and that a closed subgroup of a Lie group is a Lie group. The third chapter expounds in Cartan's own way the structure theory of Weyl [1925–26] for compact connected Lie groups.

Weyl [1934] gave a course on these matters, elaborating on Cartan [1930b] and including a segment on "foundations of a general theory of Lie algebras." Pontrjagin [1939] provided a systematic exposition of topological groups, carefully distinguishing local and global results for Lie groups and proving global results where he could. Finally Chevalley [1946] established a complete global theory, introducing analytic subgroups and establishing a one-one correspondence between Lie subalgebras and analytic subgroups.

As noted earlier the term "Lie group" was introduced by Arthur Tresse in his 1893 thesis, but it was not adopted widely until Cartan [1930b] used it systematically. The term "Lie algebra" appeared in the lecture notes Weyl [1934], which were written by N. Jacobson.

Global theory in the hands of Chevalley

The book Chevalley [1946] marked a turning point. The 1996 edition of the sequel *Lie Groups Beyond an Introduction* to the present book says in its preface:

> Fifty years ago Claude Chevalley revolutionized Lie theory by publishing his classic *Theory of Lie Groups I*. Before his book Lie theory was a mixture of local and global results. As Chevalley put it, "This limitation was probably necessary

> as long as general topology was not yet sufficiently well elaborated to provide a solid base for a theory in the large. These days are now passed."
>
> Indeed, they are passed because Chevalley's book changed matters. Chevalley made global Lie groups into the primary objects of study. In his third and fourth chapters he introduced the global notion of analytic subgroup, so that Lie subalgebras corresponded exactly to analytic subgroups. This correspondence is now taken as absolutely standard, and any introduction to general Lie groups has to have it at its core. Nowadays "local Lie groups" are a thing of the past; they arise only at one point in the development, and only until Chevalley's results have been stated and have eliminated the need for the local theory.

In the present book "Lie groups" are defined already in Section I.1. For historical continuity the book retains use of the term "analytic group" for connected Lie group despite working with smooth manifolds and smooth functions, as opposed to real analytic manifolds and real analytic functions. This fact explains the definition of "analytic group" given in Section I.1. A number of authors have attempted in vain to define "Lie subgroup" in some useful way. In response to those unsuccessful efforts, the present book leaves "Lie subgroup" as an undefined term. An **analytic subgroup** is a subgroup of a Lie group that has the structure of an analytic group for which the inclusion mapping is smooth.

The big advance in Chevalley [1946] was a global version of the Frobenius theorem on involutive distributions. With this theorem Chevalley was able to start with a connected Lie group G with Lie algebra $\mathfrak{g}$ and to construct an analytic subgroup H of G for each Lie subalgebra $\mathfrak{h}$. In this approach one uses $\mathfrak{h}$ to construct a global foliation of G, each leaf being a left coset gH. Once the foliation is in place, there is still a tricky point in the argument—proving continuity of group multiplication within the subgroup H. Since the topology on H is finer than the relative topology that H gets from G, one sees that multiplication is continuous from $H \times H$ into G. But it does not follow automatically that multiplication is continuous from $H \times H$ into H. Notably Chevalley carries along the whole foliation in his argument and is able to address continuity successfully.

In retrospect the development of the foundations of Lie theory paused for a time after Chevalley [1946] and then resumed in two directions—(1) by means of differential topology or geometry and (2) by means of the Campbell–Baker–Hausdorff Theorem.

Global theory using differential topology or geometry

Building on Chevalley [1946], Séminaire "Sophus Lie" [1955] sought to publicize some mathematical steps beyond the fundamental theorems that had been taken for the subject of Lie theory; those steps tended to relate to the early work of E. Cartan. Cohn [1957] sought to organize what was known about Lie's original three fundamental theorems, providing proofs with differential geometry.

A number of books on Lie theory from the 1960s either dealt with topics in algebra or else concentrated on closed linear groups, whose theory is addressed in the present book in Chapter I. The fundamental theorems of Lie theory do not play an important role with books of either kind, and details about such books are often omitted from the detailed discussion here. Among these books are Hausner–Schwartz [1968], Freudenthal–de Vries [1969], and Adams [1969].

The mathematical subject area containing the theory developed and exploited by Chevalley has come to be known as differential topology. In the early 1970s Spivak [1970] and Warner [1971] published well written books on differential topology at the beginning graduate level (in the United States) that were so successful that a certain amount of differential topology came to be part of the standard mathematics curriculum. Both books developed differential topology from the beginning, they were tightly reasoned, and the subject of Lie groups in effect could be regarded as an example.

Locked into place by this approach was the use of the Frobenius theorem on involutive distributions to pass from a Lie subalgebra of a Lie group to the corresponding analytic subgroup. Such a use runs into a pedagogical problem: the Frobenius theorem is completely out of character with everything else that is needed for a global theory of Lie groups and their Lie algebras. For someone who knows differential topology already or wants to learn it anyway, there is no loss in following the Spivak–Warner approach to learning the subject of Lie groups along with the subject of differential topology. For others, following the Spivak–Warner approach makes the subject harder than it needs to be.

An alternative to using differential topology and the Frobenius theorem to develop Lie theory is to use differential geometry. The groundbreaking book Helgason [1962], *Differential Geometry and Symmetric Spaces*, successfully handled the foundational questions about Lie theory by fully bringing differential geometry to bear, rather than the differential topology that Chevalley [1946] had developed and used. A subsequent edition of Helgason's book, Helgason [1978], showed further success at handling Lie groups with a differential geometry approach. But the educational effect of the change from differential topology to differential geometry is merely to trade the need of background in differential topology for a need of background in differential geometry.

Global theory using the Campbell–Baker–Hausdorff Theorem

Campbell's Problem was mentioned earlier in this Historical Commentary. For a linear Lie algebra the objective was to take the unique solution Z near 0 in a linear Lie algebra to the equation $\exp Z = \exp X \exp Y$ with X and Y near 0 in the Lie algebra and to establish the expansion

$$Z = X + Y + \sum_{n \geq 2} C_n(X, Y),$$

where $C_n(X, Y)$ is a recursively defined polynomial in X and Y that is homogeneous of total degree n and is expressible in terms of iterated brackets. Baker [1905] and Hausdorff [1906] gave the first solutions of Campbell's Problem in this form, and there were others later. The result is stated in the text as Theorem 3.2, the Campbell–Baker–Hausdorff Theorem.

Work related to the Campbell–Baker–Hausdorff Theorem continued internationally after Hausdorff's efforts, taking advantage of the many known points of view that one could bring to the subject. Some work sought improved insight into the formula and the recursion, and other work pursued a surprising number of generalizations. The book Bonfiglioli–Fulci [2012] is an account of many of the lines of investigation in mathematics and physics related to the Campbell–Baker–Hausdorff Theorem as of its date of publication in 2012.

Eventually one was going to be interested in unwinding the recursion and dealing with convergence. The problem in this form remained unsolved until the appearance of Dynkin [1947], but partial progress was made by a number of people. The proof given in the text for Theorem 3.2 is one of many that have appeared over the years. It originally appeared in Eichler [1968]. It was published after Dynkin's results were widely known but did not make use of those results. It has the advantage of being short and using only high school algebra.

Since an answer to all these questions would give a formula for group multiplication near the identity in terms of the Lie algebra, vector space operations, and passages to the limit, it was reasonable to hope that giving a full solution of Campbell's Problem would lead to a direct proof of Fundamental Theorem A as stated in the Preface. In particular, there would be no need for the Frobenius theorem or any similar advanced result in differential topology or geometry.

Dynkin's Approach to the Campbell–Baker–Hausdorff Theorem

Dynkin [1947] gave an explicit formula that implied the Campbell–Baker–Hausdorff Theorem, in essence giving the first fully satisfactory proof of that theorem. But Dynkin's first paper was in Russian, and it had a gap. The paper Dynkin [1949] was acknowledged as having repaired the error. Yet because the papers were in Russian, years passed before Dynkin's two papers came to be widely known.

Dynkin [1950] published a long paper with complete proofs that appeared in translation in 1953. A reprint of the translation that was more widely circulated appeared in 1962. This article described how the Campbell–Baker–Hausdorff Theorem, now an established theorem, could be used to generate a local Lie group from a Lie algebra. It worked even over fields more general than $\mathbb{R}$ and $\mathbb{C}$ and no doubt stimulated interest in this direction. It pointed toward a proof of Fundamental Theorem A of the Preface by suggesting that the local Lie group associated to a Lie subalgebra might generate the analytic subgroup corresponding to the Lie subalgebra.

In 1960 or a little before, a group of mathematicians who used the pseudonym Nicolas Bourbaki, most but not all of them from France, was working on a series of textbooks entitled *Éléments de mathématique*. It decided to start a "volume" *Groupes et algèbres de Lie*, and Dynkin's approach would play a great role in the development.

It could take years for the group to generate a single "chapter" of one volume. Chapter 1 on Lie algebras was published in 1960, and it was followed in 1968 by Chapters 4–6 on topics in discrete algebra. Chapters 2 and 3 did not appear until 1972, and their respective topics were free Lie algebras and Lie groups.

It is likely that certain other books published in the 1960s, 1970s , and 1980s by various authors were related to early efforts at writing Chapters 2 and 3 of Bourbaki. These books include Serre [1964], Hochschild [1965], Godement [1982], and Tits [1983]. Each of these approached the mathematics in Dynkin [1950] in its author's own particular way, but the books of Godement and Tits were delayed by some years before publication. Anyway, for Bourbaki the final result was Bourbaki [1972].

Because of Bourbaki's continual efforts "to go from the general to the particular," this author sometimes finds other people's writing more illuminating than Bourbaki's. In the present case this author prefers the exposition in Godement [1982] to that in Bourbaki [1972].

An important point in the treatment of Fundamental Theorem A in the Preface is to identify explicitly the analytic subgroup associated with a given Lie subalgebra. An analytic subgroup has three structures—one as an explicit set, one as a Hausdorff topological space, and one as a collection of smoothly compatible charts. The explicit set is easy enough. It consists of the smallest subgroup containing all exponentials of members of the given Lie subalgebra; the exponentials are accessible because of the Campbell–Baker–Hausdorff Theorem. The charts are straightforward; they are all the left[3] translates of exponentials by elements of the subgroup of a fixed small open set about the identity.

The key notion relates to the topological structure. That notion is what Godement [1982] called the "natural topology" for each subgroup, connected or not, of a given Lie group. The natural topology of a subgroup is finer than the usual relative topology, and the identity component in the natural topology yields the analytic subgroup associated to the given subgroup, complete with its correct manifold structure. Theorems 3.6 and 4.26 affirm that the natural topology is the correct topology to complete the proof of Fundamental Theorem A.

In a carefully written book Sagle–Walde [1973] gave their own proof (in their Theorem 6.6) of the existence of an analytic subgroup corresponding to any given Lie subalgebra. They too made use of the Campbell–Baker–Hausdorff Theorem. The analytic subgroup is defined to be the subgroup generated by exp of the given Lie subalgebra. But their argument comes across as incomplete in the same way that Chevalley's argument is incomplete; one really has no global sense of the topology of the constructed group without the natural topology available.

Varadarajan [1974] published an authoritative introduction to Lie theory shortly after this, but he reverted to using the Frobenius theorem on involutive distributions at the crucial point.

Since Godement's use of the natural topology had eliminated the need for differential topology or differential geometry in the construction of analytic subgroups from Lie subalgebras, the foundations of Lie theory could now be developed with mathematical tools that were more elementary than ever before. The notion was that Lie theory could be taught at the undergraduate level—or close to it. The books Curtis [1979], Howe [1983], Baker [2002], Tapp [2005], Sepanski [2007], Abbaspour-Moskowitz [2007], and Stillwell [2008] claim to be at this level. Except as discussed starting in the next paragraph, Lie theory books published after 1974 have been inclined to concentrate on more advanced topics and to skip lightly over Lie theory foundations. Such books include Ise–Takeuchi [1984], Bröcker–tom Dieck

[3]Or right, if one prefers. The text uses left translates.

[1985], Sattinger–Weaver [1986], Faraut [2008], Kirillov [2008], and Hilgert–Neeb [2012].

Let us return to Dynkin and the impact of his Lie theory papers of 1947 and 1949, as well as his survey of 1950. Dynkin's work showed that the Campbell–Baker–Hausdorff Theorem for matrices could be rewritten in terms of brackets *and still be manageable*. Bourbaki [1972] and Godement [1982] then showed how the Dynkin formula could be used in place of the Frobenius theorem to pass back from Lie subalgebras to their corresponding analytic subgroups.

Hilgert–Neeb [1991] and Duistermaat–Kolk [2000] completed this line of argument by showing how a nineteenth century formula of F. Schur [1891] was just the tool needed for giving a really tidy proof of the passage from Dynkin's formula to the construction of an analytic subgroup for each Lie subalgebra. The formula of Schur is tantamount to a formula for the derivative of the exponential mapping carrying a Lie algebra to the corresponding Lie group.

Unfortunately the book Hilgert–Neeb [1991], which was written in German, was regarded as a German textbook at the time of publication. As such, it received no listing in *MathSciNet*, and its review in *Zentralblatt* simply listed its chapter titles. Its impact on contemporary research was therefore zero.

The book Duistermaat–Kolk [2000] led to the publication of two further books, Rossmann [2002] and Hall [2003], that elaborate on this new approach via the Campbell–Baker–Hausdorff and Dynkin Formulas and introduce their own ideas about details. Each addresses only linear groups, though Rossmann [2002] makes a two-page digression midway through the book to give a few indications how one might extend the treatment from linear Lie groups to all Lie groups. Each book derives Schur's 1891 formula, uses it to obtain the Campbell–Baker–Hausdorff and Dynkin Formulas, and then goes to work on the three of the four Fundamental Theorems of Lie Theory that were mentioned in the Preface.[4] Each of Rossmann [2002] and Hall [2003] has its own really good idea about how to apply the formulas to obtain proofs of those fundamental theorems. These really good ideas are noted in the section "Chapter III" below of this Historical Commentary.

The two books have some differences. Rossmann gets to the Campbell–Baker–Hausdorff Formula after only 22 pages, whereas Hall gets to it after 75. Rossmann seeks a series expansion, whereas Hall finds it easier to work with an integral. Although both books include many examples, Rossmann's choices seem influenced by geometry, whereas Hall's choices seem influenced by mathematical physics. Both books take up aspects of representation theory to fill out the exposition to book length, Hall even more so than Rossmann.

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[4]As it happens, Duistermaat–Kolk [2000] proves all four fundamental theorems.

The remainder of this Historical Commentary relates specific sources to descriptions of developments in the individual chapters.

Chapter I

Sections 1–7 of Chapter I introduce Lie groups and linear groups. The idea of singling out linear groups for special study is that they quickly provide a large class of examples of Lie groups, interesting examples otherwise being hard to come by. The theory of closed linear groups may be said to have begun with von Neumann [1927] and [1929], who proved the result listed here as Theorem 1.10 and showed that continuous homomorphisms from closed linear groups into matrix groups are smooth. This fact about continuous homomorphisms appears here in Chapter IV as Problem 36 and Proposition 4.9. The proof of Proposition 1.3 was lifted from Rossmann [2002].

Section 2, which introduces analytic subgroups, contains the warning that the additive group $\mathbb{R}^2$ has a connected subgroup H with no inherited manifold structure. By way of background, any *pathwise* connected subgroup of $\mathbb{R}^2$ must be an analytic subgroup. This positive statement is a special case of a theorem of Yamabe [1950]; Goto [1969] gave a somewhat longer proof of Yamabe's result that is easier to understand. The question arises what can be said if the given subgroup of $\mathbb{R}^2$ is merely connected. Ahdout–Rothman [1997] pointed out that Theorem 5 of Jones [1942] gives an example of a connected dense subgroup of $\mathbb{R}^2$ that is not a manifold and is therefore not an analytic subgroup of $\mathbb{R}^2$. More generally Thomas [1987] proved under the assumption that the continuum hypothesis is valid, that any connected Lie group G of dimension greater than one has a connected subgroup that is dense in G but is not an analytic subgroup.

A person whose interest is in Lie groups will likely not want to pursue these matters. The point to remember is that topological connectedness is a subtle property in the theory; pathwise connectedness is a property that better reflects one's intuition.

Chapter II

Chapter II develops the theory of linear groups that are not necessarily closed. In fact the linear groups need have so special topological structure at all. The Lie algebra of such a group is defined as the set of all derivatives at 0 of curves into the group that are equal to the identity at 0. The result given in Theorem 2.1 that X in the Lie algebra implies $\exp tX$ is in the group for all real t appears as Theorem 3 on page 46 of Rossmann [2002] and seems in this generality to be new with Rossmann. It is a powerful result. Further development in the chapter follows Rossmann [2002]. Once Theorem 2.1 is available, one can introduce the natural topology on the subgroup, in the sense of Godement [1982], and then single out the linear analytic group associated to the original linear group. Rossmann gives no name to the natural topology; the name is from Godement [1982]. In preparation for Chapter III one considers smooth homomorphisms between analytic groups at this time.

Chapter II does one other thing. It prepares the notation for the generality in which the Fundamental Theorems of Lie Theory as stated in the Preface will be approached in the remainder of the book. There is to be an underlying analytic group $\boldsymbol{M}$ with Lie algebra $\mathfrak{m}$, and the theory works with subgroups G of $\boldsymbol{M}$ and Lie subalgebras $\mathfrak{g}$ of $\mathfrak{m}$. In Chapters II and III, $\boldsymbol{M}$ is a general linear group or its identity component, and $\mathfrak{m}$ is the corresponding full matrix space. To remove the restriction to linear groups for Chapter IV, one allows $\boldsymbol{M}$ and $\mathfrak{m}$ to be arbitrary.

Chapter III

Sections 1–5 of Chapter III address the inverse construction that goes from any Lie subalgebra of the Lie algebra of $\boldsymbol{M}$ back to the associated analytic subgroup of $\boldsymbol{M}$. Sections 1–3 in essence follow Rossmann [2002], but other material is included to give further background on the history that is involved. In particular the proof of Theorem 3.2 is taken from Eichler [1968]; the text takes advantage of some clarifying remarks about Eichler's proof that appear in Stillwell [2008]. To see Eichler's proof in a broader algebraic context, see Hofstätter [2021]. Section 4 follows Duistermaat–Kolk [2000]. It may look at first as if Sections 1–4 have addressed the inverse construction, but as both Rossmann [2002] and Hall [2003] observe: there is still a serious gap. The present book follows the approach of Hall [2003] in completing the argument; this step occupies Section 5. Hall uses a really clever counting argument; the author of the present book has no intuition why such an argument should succeed other than: "if not this, then what?". But it does succeed.

Section 6 of Chapter III deals with homomorphisms. Sections 7–8 produce two examples to show that a covering of a linear group need not be linear and a quotient of a linear group need not be linear. The example in Section 7 essentially carries out manually an instance of what is called "Weyl's unitary trick." The example in Section 8 is taken from Chapter 7 of Baker [2002].

Problems 2–12 at the end of Chapter III report on Sogo [2016], which gives an interpretation to a certain previously unidentified derivation operator in Baker [1905], uses the interpretation to follow through on Baker's method, and in principle obtains Dynkin's Formula. A certain amount of detail in the argument is only sketched.

Chapter IV

The exponential mapping as defined in Section IV.3 was already part of the work in Lie–Engel [1888–90–93], and Lie understood the exponential's behavior through quadratic terms.

Sections 4–9 extend many of the results of Chapters I, II, and III from linear groups to arbitrary Lie groups. This material has been influenced by Rossmann [2002], pp. 158–159; by Duistermatt–Kolk [2000], pp. 16–19; by Hall [2003], pp. 43–44; and by Woit [2003]. Lemma 4.13 and surrounding material are taken from Helgason [1962]. Theorem 4.34, the Closed Subgroup Theorem, is due to Cartan [1930b]. Cartan went on to define and discuss Lie groups generally, regarding von Neumann's earlier work on closed linear groups as a special case.

Problem 38 and its solution at the end of Chapter IV owe something to the lecture notes Conrad [2018], which were taken by A. Landsman. Problems 39–41 owe something to the course notes Haber [2013]. Some later problems make use of material on the exterior algebra of a finite dimensional vector space, as may be found in Chapter VI of Knapp [2016a]. The statements and solutions of problems for Chapter IV owe something to the sets of notes Haber [2013], Yung [2014], Devalapurkar [2023], and Garrett [2025].

The decomposition theorem mentioned at the end of the Summary of Chapter IV has an interpretation in terms of Lie algebra cohomology. More detail may be found in Duistermaat–Kolk [2000], Section 1.14.

Appendices A, B, and C

The exposition in Appendices A and C is based on notes from the lecture series Fox [1962]. For Appendix A and part of Appendix C, it was Schreier [1926] who introduced topological groups, and it was Poincaré [1895] who introduced fundamental groups and covering spaces. Schreier [1927] proved the existence of universal covering groups of connected (global) Lie groups.

The exposition in Appendix B is in part lifted from Knapp [2017] and refers back to material on differential equations in Knapp [2016b].

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INDEX OF NOTATION

This list indexes many of the recurring symbols introduced in Chapters I through IV and Appendices A, B, and C. For other recurring symbols see the Standard Notation on pages xv–xvi. For a list of some classical groups and their Lie algebras that occur only once, see page 34.

In the list below, each piece of notation is regarded as having a key symbol. The first group consists of those items for which the key symbol is a fixed Latin letter, and the items are arranged roughly alphabetically by that key symbol. The next group consists of those items for which the key symbol is a Greek letter. The final group consists of those items for which the key symbol is a variable or a nonletter, and these are arranged by type.

Greek

Operations on sets given by subscripts and superscripts

Specific functions

Isolated symbols

Operations on sets and classes

Miscellaneous

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