LIE GROUPS, LIE ALGEBRAS, AND COHOMOLOGY

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by

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To My Family

With Love

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PREFACE

This material is based on a one-semester course given at SUNY Stony Brook in Fall 1986. The audience consisted largely of graduate students knowledgeable about geometry, acquainted with tensor products of vector spaces, and having little or no background in Lie groups. The objective was to go in one semester from the beginnings of Lie theory to the frontier in algebraic constructions of group representations. The course consisted of much of the first seven chapters of the present book, done in a slightly different order. Actually the course was designed backwards from a key algebraic computation (7.79) that yields the Borel-Weil-Bott Theorem, and it ended up including whatever seemed appropriate as preliminary material. Chapter VIII was added to indicate how the computation (7.79) leads to the frontier.

The topic of interest here is the representation theory of compact Lie groups and of their natural noncompact analogs, the noncompact semisimple Lie groups. Special linear groups, symplectic groups, and isometry groups of quadratic forms give examples of noncompact semisimple Lie groups. Group representations are homomorphisms of a group into invertible linear transformations on a complex vector space, possibly infinite-dimensional and possibly with some continuity assumption. Understanding of group representations allows one to take advantage of symmetries in various problems in analysis and algebra.

For a compact group, the irreducible representations (those with no nontrivial closed subspaces invariant under the representation) are finite-dimensional. In the case of any compact connected Lie group, the Borel-Weil Theorem gives a way of realizing all such irreducible representations in spaces of holomorphic functions, and a generalization due to Bott gives alternative realizations in spaces of differential forms having only $d\bar{z}$'s present.

In 1966 Langlands conjectured that a version of the Borel-Weil-Bott Theorem should provide a realization of "discrete series" representations of noncompact semisimple Lie groups. "Discrete series" are certain irreducible infinite-dimensional representations that are known to play a fundamental role in the construction of all irreducible representations. Over a period of some years ending in the mid 1970's, Schmid proved the Langlands conjecture and several variants of it.

In 1978 Zuckerman found that he could bypass a number of difficult analytic problems of Schmid's by phrasing the Langlands conjecture in algebraic terms and using constructions in homological algebra. He found that similar algebraic constructions yielded other interesting representations for a variety of groups. His technique, now known as cohomological induction, has become a fundamental tool in representation theory.

Our goal is to start with elementary Lie theory and to arrive at the definition, elementary properties, and first applications of cohomological induction, all the while developing the computational techniques that are so important in handling Lie groups. A byproduct is that we are able to study homological algebra with a significant application in mind; we see as a consequence just what results are fundamental and what results are minor. A person who wants to study further may wish to read the general theory of roots and weights from any number of books (e.g., Knapp [1986], Chapters IV and V) and to study cohomological induction further from the book by Vogan [1981].

Prerequisites for reading the present book are a knowledge of metric spaces, an advanced course in linear algebra, and a passing acquaintance with topological groups. Also invariant integration for compact groups plays a role in Chapters III, VII, and VIII.

Chapter I is a quick introduction to Lie groups of matrices. The reader who already knows beginning Lie theory may skip this chapter except for the last section (representations). Representations of groups are the objects of interest in these notes, but the approach is by means of algebra. What we therefore study is representations of Lie algebras that are related to representations of Lie groups; the sense in which we can actually pass from the Lie algebras to the Lie groups is not addressed here very seriously. Since the group representations are the objects of interest, we constantly need examples of them in order to motivate what happens with Lie algebras.

The first half of Chapter II contains two kinds of topics alternately: a devlopment of multilinear algebra and the imposition of representations on various spaces of tensors. Most readers will be able to skip at least part of the material in §§1-4. The second half of Chapter II constructs some algebraic objects that will be used repeatedly throughout the later chapters——universal enveloping algebra, symmetrization, and tensor products over an algebra.

Chapter III treats the representation theory of compact groups, with emphasis on the unitary groups as examples. This material is included at this stage mostly for motivation and is not used until the latter part of Chapter VI.

Chapters IV and V, as well as the first half of Chapter VI, develop homological algebra as it applies to Lie algebras and their representations. The motivation that is provided comes largely from geometry and starts with the analytic Borel-Weil Theorem in §7 of Chapter III. A second line of motivation begins in §1 of Chapter IV. The two lines merge in §1 of Chapter VI and continue in §1 of Chapter VII.

Chapter VII makes a modification in the theory of the previous three chapters, so that the setting matches better what is needed for representation theory. The ingredients of cohomological induction are assembled in this chapter. Section 11 is the most important and illustrates how the theory can be applied in the ideal setting of a compact connected Lie group. This section contains the key computation (7.79) mentioned earlier.

Chapter VIII indicates the kinds of adjustments that need to be made when working with noncompact semisimple Lie groups, rather than compact groups. Specific illustrations are given for the unitary groups of indefinite quadratic forms.

At the end of the book is a chapter entitled "Notes," which gives historical comments, amplifies some notions, and points to the list of references at the end.

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A. W. K.
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