INTRODUCTION

Closed Linear Groups

Abstract. A closed linear group G is a group of real or complex matrices that is topologically closed in a complex general linear group. Rotation groups, unitary groups, and special linear groups provide familiar examples. The linear Lie algebra $\mathfrak{g}$ of G is the set of derivatives at 0 of all smooth curves $c(t)$ of matrices that lie in G for all t and are equal to the identity at $t = 0$. The set of matrices $\mathfrak{g}$ is indeed a Lie algebra over $\mathbb{R}$.

The exponential of a square matrix is defined by the familiar power series of the exponential function. The exponential map enables one to compute explicitly the linear Lie algebra of each of the familiar examples. It turns out that the exponential map carries $\mathfrak{g}$ into G. From this fact one deduces the main result of the Introduction, that any closed linear group has a natural structure as a smooth manifold that makes the group into a Lie group.

A homomorphism π between two closed linear groups G and H carries smooth curves through the identity in G to smooth curves through the identity in H. The map on the derivatives at 0 of such curves is well defined as a Lie algebra homomorphism $d\pi$ between the linear Lie algebras of G and H. The homomorphisms π and $d\pi$ are related by the important identity $\pi \circ \exp = \exp \circ d\pi$. This identity is a quantitative version of the statement that the infinitesimal behavior of a homomorphism at the identity determines the homomorphism in a neighborhood of the identity.

1. Linear Lie Algebra of a Closed Linear Group

Many readers of this book will already know a certain amount of the elementary theory of Lie groups. Yet it may be helpful to review some of that theory in a special case, namely for groups of matrices that are topologically closed. The reason is that the techniques that come into play for these groups are more like the techniques used for all Lie groups in the more advanced parts of Lie theory. Thus the review in this introductory chapter is intended to establish the spirit in which many results and examples will be approached later in the book.

We denote by $GL(n, \mathbb{R})$ the real **general linear group** consisting of all nonsingular n-by-n real matrices with matrix multiplication as group operation. Similarly $GL(n, \mathbb{C})$ denotes the group of nonsingular n-by-n complex matrices. The groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ have topologies