CHAPTER VI

Structure Theory of Semisimple Groups

Abstract. Every complex semisimple Lie algebra has a compact real form, as a consequence of a particular normalization of root vectors whose construction uses the Isomorphism Theorem of Chapter II. If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then the use of a compact real form of $(\mathfrak{g}_0)^{\mathbb{C}}$ leads to the construction of a "Cartan involution" θ of $\mathfrak{g}_0$. This involution has the property that if $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the corresponding eigenspace decomposition or "Cartan decomposition," then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $(\mathfrak{g}_0)^{\mathbb{C}}$. Any two Cartan involutions of $\mathfrak{g}_0$ are conjugate by an inner automorphism. The Cartan decomposition generalizes the decomposition of a classical matrix Lie algebra into its skew-Hermitian and Hermitian parts.

If G is a semisimple Lie group, then a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of its Lie algebra leads to a global decomposition $G = K \exp \mathfrak{p}_0$, where K is the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. This global decomposition generalizes the polar decomposition of matrices. The group K contains the center of G and, if the center of G is finite, is a maximal compact subgroup of G.

The Iwasawa decomposition $G = KAN$ exhibits closed subgroups A and N of G such that A is simply connected abelian, N is simply connected nilpotent, A normalizes N, and multiplication from $K \times A \times N$ to G is a diffeomorphism onto. This decomposition generalizes the Gram-Schmidt orthogonalization process. Any two Iwasawa decompositions of G are conjugate. The Lie algebra $\mathfrak{a}_0$ of A may be taken to be any maximal abelian subspace of $\mathfrak{p}_0$, and the Lie algebra of N is defined from a kind of root-space decomposition of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$. The simultaneous eigenspaces are called "restricted roots," and the restricted roots form an abstract root system. The Weyl group of this system coincides with the quotient of normalizer by centralizer of $\mathfrak{a}_0$ in K.

A Cartan subalgebra of $\mathfrak{g}_0$ is a subalgebra whose complexification is a Cartan subalgebra of $(\mathfrak{g}_0)^{\mathbb{C}}$. One Cartan subalgebra of $\mathfrak{g}_0$ is obtained by adjoining to the above $\mathfrak{a}_0$ a maximal abelian subspace of the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$. This Cartan subalgebra is θ stable. Any Cartan subalgebra of $\mathfrak{g}_0$ is conjugate by an inner automorphism to a θ stable one, and the subalgebra built from $\mathfrak{a}_0$ as above is maximally noncompact among all θ stable Cartan subalgebras. Any two maximally noncompact Cartan subalgebras are conjugate, and so are any two maximally compact ones. Cayley transforms allow one to pass between any two θ stable Cartan subalgebras, up to conjugacy.

A Vogan diagram of $\mathfrak{g}_0$ superimposes certain information about the real form $\mathfrak{g}_0$ on the Dynkin diagram of $(\mathfrak{g}_0)^{\mathbb{C}}$. The extra information involves a maximally compact θ stable Cartan subalgebra and an allowable choice of a positive system of roots. The effect of θ on simple roots is labeled, and imaginary simple roots are painted if they are "noncompact," left unpainted if they are "compact." Such a diagram is not unique for

according to Proposition 1.96, and this property is not always shared by other forms. To take advantage of this property, we shall insist that B is the Killing form in §§1–3. After that, we shall allow more general forms in place of B.)

For each pair $\{\alpha, -\alpha\}$ in Δ, we fix $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $B(E_\alpha, E_{-\alpha}) = 1$. Then $[E_\alpha, E_{-\alpha}] = H_\alpha$ by Lemma 2.18a. Let α and β be roots. If $\alpha + \beta$ is in Δ, define $C_{\alpha,\beta}$ by

$$[E_\alpha, E_\beta] = C_{\alpha,\beta} E_{\alpha+\beta}.$$

If $\alpha + \beta$ is not in Δ, put $C_{\alpha,\beta} = 0$.

Lemma 6.2. $C_{\alpha,\beta} = -C_{\beta,\alpha}$.

PROOF. This follows from the skew symmetry of the bracket.

Lemma 6.3. If α, β, and γ are in Δ and $\alpha + \beta + \gamma = 0$, then

$$C_{\alpha,\beta} = C_{\beta,\gamma} = C_{\gamma,\alpha}.$$

PROOF. By the Jacobi identity,

$$[[E_\alpha, E_\beta], E_\gamma] + [[E_\beta, E_\gamma], E_\alpha] + [[E_\gamma, E_\alpha], E_\beta] = 0.$$

Thus $$C_{\alpha,\beta}[E_{-\gamma}, E_\gamma] + C_{\beta,\gamma}[E_{-\alpha}, E_\alpha] + C_{\gamma,\alpha}[E_{-\beta}, E_\beta] = 0$$

and $$C_{\alpha,\beta} H_\gamma + C_{\beta,\gamma} H_\alpha + C_{\gamma,\alpha} H_\beta = 0.$$

Substituting $H_\gamma = -H_\alpha - H_\beta$ and using the linear independence of $\{H_\alpha, H_\beta\}$, we obtain the result.

Lemma 6.4. Let α, β, and $\alpha + \beta$ be in Δ, and let $\beta + n\alpha$, with $-p \le n \le q$, be the α string containing β. Then

$$C_{\alpha,\beta}, C_{-\alpha,-\beta} = -\tfrac{1}{2} q(1+p)|\alpha|^2.$$

PROOF. By Corollary 2.37,

$$[E_{-\alpha}, [E_\alpha, E_\beta]] = \tfrac{1}{2} q(1+p)|\alpha|^2 B(E_\alpha, E_{-\alpha}) E_\beta.$$

The left side is $C_{-\alpha,\alpha+\beta} C_{\alpha,\beta} E_\beta$, and $B(E_\alpha, E_{-\alpha}) = 1$ on the right side. Therefore

$$C_{-\alpha,\alpha+\beta} C_{\alpha,\beta} = \tfrac{1}{2} q(1+p)|\alpha|^2. \tag{6.5}$$

Since $(-\alpha) + (\alpha + \beta) + (-\beta) = 0$, Lemmas 6.3 and 6.2 give

$$C_{-\alpha,\alpha+\beta} = C_{-\beta,-\alpha} = -C_{-\alpha,-\beta},$$

and the result follows by substituting this formula into (6.5).

Finally

$$[(X_\alpha - X_{-\alpha}), i(X_\alpha + X_{-\alpha})] = 2iH_\alpha,$$

and therefore $\mathfrak{u}_0$ is closed under brackets. Consequently $\mathfrak{u}_0$ is a real form.

To show that $\mathfrak{u}_0$ is a compact Lie algebra, it is enough, by Proposition 4.27, to show that the Killing form of $\mathfrak{u}_0$ is negative definite. The Killing forms $B_{\mathfrak{u}_0}$ of $\mathfrak{u}_0$ and B of $\mathfrak{g}$ are related by $B_{\mathfrak{u}_0} = B|_{\mathfrak{u}_0 \times \mathfrak{u}_0}$, according to (1.20). The first term on the right side of (6.12) is orthogonal to the other two terms by Proposition 2.17a, and B is positive on $\sum \mathbb{R}H_\alpha$ by Corollary 2.38. Hence B is negative on $\sum \mathbb{R}iH_\alpha$. Next we use Proposition 2.17a to observe for $\beta \neq \pm\alpha$ that

$$\begin{aligned} B((X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta})) &= 0 \\ B((X_\alpha - X_{-\alpha}), i(X_\beta + X_{-\beta})) &= 0 \\ B(i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})) &= 0. \end{aligned}$$

Finally we have

$$\begin{aligned} B((X_\alpha - X_{-\alpha}), (X_\alpha - X_{-\alpha})) &= -2B(X_\alpha, X_{-\alpha}) = -2 \\ B(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) &= -2B(X_\alpha, X_{-\alpha}) = -2, \end{aligned}$$

and therefore $B|_{\mathfrak{u}_0 \times \mathfrak{u}_0}$ is negative definite.

2. Cartan Decomposition on the Lie Algebra Level

To detect semisimplicity of some specific Lie algebras of matrices in §I.8, we made critical use of the conjugate transpose mapping $X \mapsto X^*$. Slightly better is the map $\theta(X) = -X^*$, which is actually an **involution**, i.e., an automorphism of the Lie algebra with square equal to the identity. To see that θ respects brackets, we just write

$$\theta[X, Y] = -[X, Y]^* = -[Y^*, X^*] = [-X^*, -Y^*] = [\theta(X), \theta(Y)].$$

Let B be the Killing form. The involution θ has the property that $B_\theta(X, Y) = -B(X, \theta Y)$ is symmetric and positive definite because Proposition 1.96 gives

$$\begin{aligned} B_\theta(X, Y) &= -B(X, \theta Y) = -B(\theta X, \theta^2 Y) \\ &= -B(\theta X, Y) = -B(Y, \theta X) = B_\theta(Y, X) \end{aligned}$$

and (6.1) gives

$$\begin{aligned} B_\theta(X, X) &= -B(X, \theta X) = -\mathrm{Tr}((\mathrm{ad}\, X)(\mathrm{ad}\, \theta X)) \\ &= \mathrm{Tr}((\mathrm{ad}\, X)(\mathrm{ad}\, X^*)) = \mathrm{Tr}((\mathrm{ad}\, X)(\mathrm{ad}\, X)^*) \geq 0. \end{aligned}$$

Lemma 6.15. Let $\mathfrak{g}_0$ be a real finite-dimensional Lie algebra, and let ρ be an automorphism of $\mathfrak{g}_0$ that is diagonable with positive eigenvalues $d_1, \ldots, d_m$ and corresponding eigenspaces $(\mathfrak{g}_0)_{d_j}$. For $-\infty < r < \infty$, define ρ^r to be the linear transformation on $\mathfrak{g}_0$ that is d_j^r on $(\mathfrak{g}_0)_{d_j}$. Then $\{\rho^r\}$ is a one-parameter group in $\operatorname{Aut}\mathfrak{g}_0$. If $\mathfrak{g}_0$ is semisimple, then ρ^r lies in $\operatorname{Int}\mathfrak{g}_0$.

PROOF. If X is in $(\mathfrak{g}_0)_{d_i}$ and Y is in $(\mathfrak{g}_0)_{d_j}$, then

$$\rho[X, Y] = [\rho X, \rho Y] = d_i d_j [X, Y]$$

since ρ is an automorphism. Hence $[X, Y]$ is in $(\mathfrak{g}_0)_{d_i d_j}$, and we obtain

$$\rho^r[X, Y] = (d_i d_j)^r [X, Y] = [d_i^r X, d_j^r Y] = [\rho^r X, \rho^r Y].$$

Consequently ρ^r is an automorphism. Therefore $\{\rho^r\}$ is a one-parameter group in $\operatorname{Aut}\mathfrak{g}_0$, hence in the identity component $(\operatorname{Aut}\mathfrak{g}_0)_0$. If $\mathfrak{g}_0$ is semisimple, then Propositions 1.97 and 1.98 show that $(\operatorname{Aut}\mathfrak{g}_0)_0 = \operatorname{Int}\mathfrak{g}_0$, and the lemma follows.

Theorem 6.16. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution, and let σ be any involution. Then there exists $\varphi \in \operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.

PROOF. Since θ is given as a Cartan involution, B_θ is an inner product for $\mathfrak{g}_0$. Put $\omega = \sigma\theta$. This is an automorphism of $\mathfrak{g}_0$, and Proposition 1.96 shows that it leaves B invariant. From $\sigma^2 = \theta^2 = 1$, we therefore have

$$B(\omega X, \theta Y) = B(X, \omega^{-1}\theta Y) = B(X, \theta\omega Y)$$

and hence $$B_\theta(\omega X, Y) = B_\theta(X, \omega Y).$$

Thus ω is symmetric, and its square $\rho = \omega^2$ is positive definite. Write ρ^r for the positive-definite r^{th} power of ρ, $-\infty < r < \infty$. Lemma 6.15 shows that ρ^r is a one-parameter group in $\operatorname{Int}\mathfrak{g}_0$. Consideration of ω as a diagonal matrix shows that ρ^r commutes with ω. Now

$$\rho\theta = \omega^2\theta = \sigma\theta\sigma\theta\theta = \sigma\theta\sigma = \theta\theta\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}.$$

In terms of a basis of $\mathfrak{g}_0$ that diagonalizes ρ, the matrix form of this equation is

$$\rho_{ii}\theta_{ij} = \theta_{ij}\rho_{jj}^{-1} \qquad \text{for all } i \text{ and } j.$$

Considering separately the cases $\theta_{ij} = 0$ and $\theta_{ij} \neq 0$, we see that

$$\rho_{ii}^r\theta_{ij} = \theta_{ij}\rho_{jj}^{-r}$$

PROOF. Let θ and θ' be two Cartan involutions. Taking $\sigma = \theta'$ in Theorem 6.16, we can find $\varphi \in \operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with θ'. Here $\varphi\theta\varphi^{-1}$ is another Cartan involution of $\mathfrak{g}_0$. So we may as well assume that θ and θ' commute from the outset. We shall prove that $\theta = \theta'$.

Since θ and θ' commute, they have compatible eigenspace decompositions into $+1$ and -1 eigenspaces. By symmetry it is enough to show that no nonzero $X \in \mathfrak{g}_0$ is in the $+1$ eigenspace for θ and the -1 eigenspace for θ'. Assuming the contrary, suppose that $\theta X = X$ and $\theta' X = -X$. Then we have

$$0 < B_\theta(X, X) = -B(X, \theta X) = -B(X, X)$$
$$0 < B_{\theta'}(X, X) = -B(X, \theta' X) = +B(X, X),$$

contradiction. We conclude that $\theta = \theta'$, and the proof is complete.

Corollary 6.20. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two compact real forms of $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$.

PROOF. Each compact real form has an associated conjugation of $\mathfrak{g}$ that determines it, and this conjugation is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, by Proposition 6.14. Applying Corollary 6.19 to $\mathfrak{g}^{\mathbb{R}}$, we see that the two conjugations are conjugate by a member of $\operatorname{Int}(\mathfrak{g}^{\mathbb{R}})$. Since $\operatorname{Int}(\mathfrak{g}^{\mathbb{R}}) = \operatorname{Int}\mathfrak{g}$, the corollary follows.

Corollary 6.21. If $A = (A_{ij})_{i,j=1}^{l}$ is an abstract Cartan matrix, then there exists, up to isomorphism, one and only one compact semisimple Lie algebra $\mathfrak{g}_0$ whose complexification $\mathfrak{g}$ has a root system with A as Cartan matrix.

PROOF. Existence of $\mathfrak{g}$ is given in Theorem 2.111, and uniqueness of $\mathfrak{g}$ is given in Example 1 of §II.10. The passage from $\mathfrak{g}$ to $\mathfrak{g}_0$ is accomplished by Theorem 6.11 and Corollary 6.20.

Corollary 6.22. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are the conjugations with respect to the compact real forms of $\mathfrak{g}$.

PROOF. Theorem 6.11 and Proposition 6.14 produce a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ that is conjugation with respect to some compact real form of $\mathfrak{g}$. Any other Cartan involution is conjugate to this one, according to Corollary 6.19, and hence is also the conjugation with respect to a compact real form of $\mathfrak{g}$.

A Cartan involution θ of $\mathfrak{g}_0$ yields an eigenspace decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \tag{6.23}$$

of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces, and these must bracket according to the rules

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0 \tag{6.24}$$

since θ is an involution. From (6.23) and (6.24) it follows that

$$\mathfrak{k}_0 \text{ and } \mathfrak{p}_0 \text{ are orthogonal under } B_{\mathfrak{g}_0} \text{ and under } B_\theta \tag{6.25}$$

In fact, if X is in $\mathfrak{k}_0$ and Y is in $\mathfrak{p}_0$, then $\operatorname{ad} X \operatorname{ad} Y$ carries $\mathfrak{k}_0$ to $\mathfrak{p}_0$ and $\mathfrak{p}_0$ to $\mathfrak{k}_0$. Thus it has trace 0, and $B_{\mathfrak{g}_0}(X, Y) = 0$; since $\theta Y = -Y$, $B_\theta(X, Y) = 0$ also.

Since B_θ is positive definite, the eigenspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ in (6.23) have the property that

$$B_{\mathfrak{g}_0} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0. \end{cases} \tag{6.26}$$

A decomposition (6.23) of $\mathfrak{g}_0$ that satisfies (6.24) and (6.26) is called a **Cartan decomposition** of $\mathfrak{g}_0$.

Conversely a Cartan decomposition determines a Cartan involution θ by the formula

$$\theta = \begin{cases} +1 & \text{on } \mathfrak{k}_0 \\ -1 & \text{on } \mathfrak{p}_0. \end{cases}$$

Here (6.24) shows that θ respects brackets, and (6.25) and (6.26) show that B_θ is positive definite. (B_θ is symmetric by Proposition 1.96 since θ has order 2.)

If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Conversely if $\mathfrak{h}_0$ and $\mathfrak{q}_0$ are the $+1$ and -1 eigenspaces of an involution σ, then σ is a Cartan involution only if the real form $\mathfrak{h}_0 \oplus i\mathfrak{q}_0$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is compact.

If $\mathfrak{g}$ is a complex semisimple Lie algebra, then it follows from Corollary 6.22 that the most general Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ is $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$, where $\mathfrak{u}_0$ is a compact real form of $\mathfrak{g}$.

Corollaries 6.18 and 6.19 have shown for an arbitrary real semisimple Lie algebra $\mathfrak{g}_0$ that Cartan decompositions exist and are unique up to conjugacy by $\operatorname{Int} \mathfrak{g}_0$. Let us see as a consequence that every real semsimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose.

and 1.98 show that $\operatorname{Int}\mathfrak{g}_0 = (\operatorname{Aut}\mathfrak{g}_0)_0$, $\bar{K}^\#$ is closed in $GL(\mathfrak{g}_0)$. Since $\bar{K}^\#$ is contained in the orthogonal group, $\bar{K}^\#$ is compact. The Lie algebra of $\bar{K}^\#$ is the subalgebra of all $\bar{T} \in \operatorname{ad}\mathfrak{g}_0$ where $\bar{\theta}(\bar{T}) = \bar{T}$, and this is just $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$.

Consider the smooth mapping $\varphi_{\bar{G}} : \bar{K}^\# \times \operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0) \to \bar{G}$ given by $\varphi_{\bar{G}}(\bar{k}, \bar{S}) = \bar{k}\exp\bar{S}$. Let us prove that $\varphi_{\bar{G}}$ maps onto $\bar{G}$. Given $\bar{x} \in \bar{G}$, define $\bar{X} \in \operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0)$ by (6.35), and put $\bar{p} = \exp\frac{1}{2}\bar{X}$. The element $\bar{p}$ is in $\operatorname{Ad}(G)$, and $\bar{p}^* = \bar{p}$. Put $\bar{k} = \bar{x}\bar{p}^{-1}$, so that $\bar{x} = \bar{k}\bar{p}$. Then $\bar{k}^*\bar{k} = (\bar{p}^{-1})^*\bar{x}^*\bar{x}\bar{p}^{-1} = (\exp -\frac{1}{2}\bar{X})(\exp\bar{X})(\exp -\frac{1}{2}\bar{X}) = 1$, and hence $\bar{k}^* = \bar{k}^{-1}$. Consequently $\bar{\Theta}(\bar{k}) = (\bar{k}^*)^{-1} = \bar{k}$, and we conclude that $\varphi_{\bar{G}}$ is onto.

Let us see that $\varphi_{\bar{G}}$ is one-one. If $\bar{x} = \bar{k}\exp\bar{X}$, then $\bar{x}^* = (\exp\bar{X}^*)\bar{k}^* = (\exp\bar{X})\bar{k}^* = (\exp\bar{X})\bar{k}^{-1}$. Hence $\bar{x}^*\bar{x} = \exp 2\bar{X}$. The two sides of this equation are equal positive definite linear transformations. Their positive definite r^{th} powers must be equal for all real r, necessarily to $\exp 2r\bar{X}$. Differentiating $(\bar{x}^*\bar{x})^r = \exp 2r\bar{X}$ with respect to r and putting $r = 0$, we see that $\bar{x}$ determines $\bar{X}$. Hence $\bar{x}$ determines also $\bar{k}$, and $\varphi_{\bar{G}}$ is one-one.

To complete the proof of (c) (but with K replaced by $\bar{K}^\#$), we are to show that the inverse map is smooth. It is enough to prove that the corresponding inverse map in the case of all n-by-n real nonsingular matrices is smooth, where $n = \dim\mathfrak{g}_0$. In fact, the given inverse map is a restriction of the inverse map for all matrices, and we recall from §I.10 that if M is an analytic subgroup of a Lie group M', then a smooth map into M' with image in M is smooth into M.

Thus we are to prove smoothness of the inverse for the case of matrices. The forward map is $O(n) \times \mathfrak{p}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ with $(k, X) \mapsto ke^X$, where $\mathfrak{p}(n, \mathbb{R})$ denotes the vector space of real symmetric matrices. It is enough to prove local invertibility of this mapping near $(1, X_0)$. Thus we examine the differential at $k = 1$ and $X = X_0$ of $(k, X) \mapsto ke^Xe^{-X_0}$, identifying tangent spaces as follows: At $k = 1$, we use the linear Lie algebra of $O(n)$, which is the space $\mathfrak{so}(n)$ of skew-symmetric real matrices. Near $X = X_0$, write $X = X_0 + S$, and use $\{S\} = \mathfrak{p}(n, \mathbb{R})$ as tangent space. In $GL(n, \mathbb{R})$, we use the linear Lie algebra, which consists of all real matrices.

To compute the differential, we consider restrictions of the forward map with each coordinate fixed in turn. The differential of $(k, X_0) \mapsto k$ is $(T, 0) \mapsto T$ for $T \in \mathfrak{so}(n)$. The map $(1, X) \mapsto e^Xe^{-X_0}$ has derivative at $t = 0$ along the curve $X = X_0 + tS$ equal to

$$\frac{d}{dt}e^{X_0+tS}e^{-X_0}\big|_{t=0}.$$

Thus we ask whether it is possible to have and therefore the diagram

$$\begin{array}{ccc} K \times \mathfrak{p}_0 & \xrightarrow{\varphi_G} & G \\ {\scriptstyle e|_K \times \mathrm{ad}_{\mathfrak{g}_0}}\downarrow & & \downarrow{\scriptstyle e} \\ \bar{K} \times \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0) & \xrightarrow{\varphi_{\bar{G}}} & \bar{G} \end{array}$$

commutes. The maps on the sides are covering maps since K is connected, and $\varphi_{\bar{G}}$ is a diffeomorphism by (c) for $\bar{G}$. If we show that φ_G is one-one onto, then it follows that φ_G is a diffeomorphism, and (c) is proved for G.

First let us check that φ_G is one-one. Suppose $k \exp_G X = k' \exp_G X'$. Applying e, we have $e(k) \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X)) = e(k') \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X'))$. Then $X = X'$ from (c) for $\bar{G}$, and consequently $k = k'$.

Second let us check that φ_G is onto. Let $x \in G$ be given. Write $e(x) = \bar{k} \exp_{\bar{G}}(\mathrm{ad}_{\mathfrak{g}_0}(X))$ by (c) for $\bar{G}$, and let k be any member of $e^{-1}(\bar{k})$. Then $e(x) = e(k \exp_G X)$, and we see that $x = zk \exp_G X$ for some $z \in Z$. Since $Z \subseteq K$, $x = (zk) \exp_G X$ is the required decomposition. This completes the proof of (c) for G.

The next step is to construct Θ. Let $\tilde{G}$ be a simply connected covering group of G, let $\tilde{K}$ be the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}_0$, let $\tilde{Z}$ be the center of $\tilde{G}$, and let $\tilde{e} : \tilde{G} \to G$ be the covering homomorphism. Since $\tilde{G}$ is simply connected, there exists a unique involution $\tilde{\Theta}$ of $\tilde{G}$ with differential θ. Since θ is 1 on $\mathfrak{k}_0$, $\tilde{\Theta}$ is 1 on $\tilde{K}$. By (e) for $\tilde{G}$, $\tilde{Z} \subseteq \tilde{K}$. Therefore $\ker \tilde{e} \subseteq \tilde{K}$, and $\tilde{\Theta}$ descends to an involution Θ of G with differential θ. This proves (a) for G.

Suppose that x is a member of G with $\Theta(x) = x$. Using (c), we can write $x = k \exp_G X$ and see that

$$k(\exp_G X)^{-1} = k \exp_G \theta X = k\Theta(\exp_G X) = \Theta(x) = x = k \exp_G X.$$

Then $\exp_G 2X = 1$, and it follows from (c) that $X = 0$. Thus x is in K, and (b) is proved for G.

Finally we are to prove (g) for G. Suppose that K is compact and that $K \subseteq K_1$ with K_1 compact. Applying e, we obtain a compact subgroup $e(K_1)$ of $\bar{G}$ that contains $\bar{K}$. By (g) for $\bar{G}$, $e(K_1) = e(K)$. Therefore $K_1 \subseteq ZK = K$, and we must have $K_1 = K$. This completes the proof of the theorem.

The Cartan decomposition on the Lie algebra level led in Proposition 6.28 to the conclusion that any real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose. There of $SL(m, \mathbb{C})$ amounts to the Gram-Schmidt orthogonalization process, let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{C}^m$, let $g \in G$ be given, and form the basis $\{ge_1, \ldots, ge_m\}$. The Gram-Schmidt process yields an orthonormal basis $v_1, \ldots, v_m$ such that

$$\operatorname{span}\{ge_1, \ldots, ge_j\} = \operatorname{span}\{v_1, \ldots, v_j\}$$
$$v_j \in \mathbb{R}^+(ge_j) + \operatorname{span}\{v_1, \ldots, v_{j-1}\}$$

for $1 \le j \le m$. Define a matrix $k \in U(m)$ by $k^{-1}v_j = e_j$. Then $k^{-1}g$ is upper triangular with positive diagonal entries. Since g has determinant 1 and k has determinant of modulus 1, k must have determinant 1. Then k is in $K = SU(m)$, $k^{-1}g$ is in AN, and $g = k(k^{-1}g)$ exhibits g as in $K(AN)$. This proves that $K \times A \times N \to G$ is onto. It is one-one since $K \cap AN = \{1\}$, and the inverse is smooth because of the explicit formulas for the Gram-Schmidt process.

The decomposition in the example extends to all semisimple Lie groups. To prove such a theorem, we first obtain a Lie algebra decomposition, and then we lift the result to the Lie group.

Throughout this section, G will denote a semisimple Lie group. Changing notation from earlier sections of this chapter, we write $\mathfrak{g}$ for the Lie algebra of G. (We shall have relatively little use for the complexification of the Lie algebra in this section and write $\mathfrak{g}$ in place of $\mathfrak{g}_0$ to make the notation less cumbersome.) Let θ be a Cartan involution of $\mathfrak{g}$ (Corollary 6.18), let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition (6.23), and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$.

Insistence on using the Killing form as our nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ will turn out to be inconvenient later when we want to compare the form on $\mathfrak{g}$ with a corresponding form on a semisimple subalgebra of $\mathfrak{g}$. Thus we shall allow some flexibility in choosing a form B. For now it will be enough to let B be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that $B(\theta X, \theta Y) = B(X, Y)$ for all X and Y in $\mathfrak{g}$ and such that the form B_θ defined in terms of B by (6.13) is positive definite. Then it follows that B is negative definite on the compact real form $\mathfrak{k} \oplus i\mathfrak{p}$. Therefore B is negative definite on a maximal abelian subspace of $\mathfrak{k} \oplus i\mathfrak{p}$, and we conclude as in the remarks with Corollary 2.38 that, for any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, B is positive definite on the real subspace where all the roots are real-valued.

The Killing form is one possible choice for B, but there are others. In any event, B_θ is an inner product on $\mathfrak{g}$, and we use it to define orthogonality and adjoints.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. This exists by finite-dimensionality. Since $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ by Lemma 6.27, the set consists of all skew-Hermitian diagonal matrices in $\mathfrak{g}$. For $\mathbb{K}=\mathbb{R}$ this is 0, and for $\mathbb{K}=\mathbb{C}$ it is all purely imaginary matrices of trace 0 and has dimension $n-1$. For $\mathbb{K}=\mathbb{H}$, $\mathfrak{m}$ consists of all diagonal matrices whose diagonal entries x_j have $\bar{x}_j=-x_j$ and is isomorphic to the direct sum of n copies of $\mathfrak{su}(2)$; its dimension is $3n$.

2) Let $G=SU(p,q)$ with $p\geq q$. We can write the Lie algebra in block form as

$$\text{(6.41)}\qquad \mathfrak{g}=\begin{array}{c} \begin{matrix} p & q \end{matrix} \\ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \end{array} \begin{matrix} \\ p \\ q \end{matrix}$$

with all entries complex, with a and d skew Hermitian, and with $\operatorname{Tr} a+\operatorname{Tr} d=0$. We take $\mathfrak{k}$ to be all matrices in $\mathfrak{g}$ with $b=0$, and we take $\mathfrak{p}$ to be all matrices in $\mathfrak{g}$ with $a=0$ and $d=0$. One way of forming a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is to allow b to have nonzero real entries only in the lower-left entry and the entries extending diagonally up from that one:

$$\text{(6.42)}\qquad b=\begin{pmatrix} \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_q \\ & ⋰ & \\ a_1 & \cdots & 0 \end{pmatrix},$$

with $p-q$ rows of 0's at the top. Let f_i be the member of $\mathfrak{a}^*$ whose value on the $\mathfrak{a}$ matrix indicated in (6.42) is a_i. Then the restricted roots include all linear functionals $\pm f_i\pm f_j$ with $i\neq j$ and $\pm 2f_i$ for all i. Also the $\pm f_i$ are restricted roots if $p\neq q$. The restricted-root spaces are described as follows: Let $i<j$, and let $J(z)$, $I_+(z)$, and $I_-(z)$ be the 2-by-2 matrices

$$J(z)=\begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix},\qquad I_+(z)=\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix},\qquad I_-(z)=\begin{pmatrix} z & 0 \\ 0 & -\bar{z} \end{pmatrix}.$$

Here z is any complex number. The restricted-root spaces for $\pm f_i\pm f_j$ are 2-dimensional and are nonzero only in the 16 entries corresponding to row and column indices $p-j+1$, $p-i+1$, $p+i$, $p+j$, where they are

$$\mathfrak{g}_{f_i-f_j}=\left\{\begin{pmatrix} J(z) & -I_+(z) \\ -I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix}\right\},\qquad \mathfrak{g}_{-f_i+f_j}=\left\{\begin{pmatrix} J(z) & I_+(z) \\ I_+(\bar{z}) & -J(\bar{z}) \end{pmatrix}\right\},$$

$$\mathfrak{g}_{f_i+f_j}=\left\{\begin{pmatrix} J(z) & -I_-(z) \\ -I_-(\bar{z}) & J(\bar{z}) \end{pmatrix}\right\},\qquad \mathfrak{g}_{-f_i-f_j}=\left\{\begin{pmatrix} J(z) & I_-(z) \\ I_-(\bar{z}) & J(\bar{z}) \end{pmatrix}\right\}.$$

The restricted-root spaces for $\pm 2f_i$ have dimension 1 and are nonzero only in the 4 entries corresponding to row and column indices $p-i+1$ and $p+i$, where they are

$$\mathfrak{g}_{2f_i} = i\mathbb{R}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{-2f_i} = i\mathbb{R}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The restricted-root spaces for $\pm f_i$ have dimension $2(p-q)$ and are nonzero only in the entries corresponding to row and column indices 1 to $p-q$, $p-i+1$, and $p+i$, where they are

$$\mathfrak{g}_{f_i} = \left\{ \begin{pmatrix} 0 & v & -v \\ -v^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_{-f_i} = \left\{ \begin{pmatrix} 0 & v & v \\ -v^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \right\}.$$

Here v is any member of $\mathbb{C}^{p-q}$. The subalgebra $\mathfrak{m}$ of Proposition 6.40d consists of all skew-Hermitian matrices of trace 0 that are arbitrary in the upper left block of size $p-q$, are otherwise diagonal, and have the $(p-i+1)^{\text{st}}$ diagonal entry equal to the $(p+i)^{\text{th}}$ diagonal entry for $1 \le i \le q$; thus $\mathfrak{m} \cong \mathfrak{su}(p-q) \oplus \mathbb{R}^q$. In the next section we shall see that Σ is an abstract root system; this example shows that this root system need not be reduced.

3) Let $G = SO(p,q)_0$ with $p \ge q$. We can write the Lie algebra in block form as in (6.41) but with all entries real and with a and d skew symmetric. As in Example 2, we take $\mathfrak{k}$ to be all matrices in $\mathfrak{g}$ with $b = 0$, and we take $\mathfrak{p}$ to be all matrices in $\mathfrak{g}$ with $a = 0$ and $d = 0$. We again choose $\mathfrak{a}$ as in (6.42). Let f_i be the member whose value on the matrix in (6.42) is a_i. Then the restricted roots include all linear functionals $\pm f_i \pm f_j$ with $i \ne j$. Also the $\pm f_i$ are restricted roots if $p \ne q$. The restricted-root spaces are the intersections with $\mathfrak{so}(p,q)$ of the restricted-root spaces in Example 2. Then the restricted-root spaces for $\pm f_i \pm f_j$ are 1-dimensional, and the restricted-root spaces for $\pm f_i$ have dimension $p-q$. The linear functionals $\pm 2f_i$ are no longer restricted roots. The subalgebra $\mathfrak{m}$ of Proposition 6.40d consists of all skew-symmetric matrices that are nonzero only in the upper left block of size $p-q$; thus $\mathfrak{m} \cong \mathfrak{so}(p-q)$.

Choose a notion of positivity for $\mathfrak{a}^*$ in the manner of §II.5, as for example by using a lexicographic ordering. Let Σ^+ be the set of positive roots, and define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. By Proposition 6.40b, $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ and is nilpotent.

Proposition 6.43 (Iwasawa decomposition of Lie algebra). With notation as above, $\mathfrak{g}$ is a vector-space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Lemma 6.45. There exists a basis $\{X_i\}$ of $\mathfrak{g}$ such that the matrices representing $\operatorname{ad}\mathfrak{g}$ have the following properties:

(a) the matrices of $\operatorname{ad}\mathfrak{k}$ are skew symmetric
(b) the matrices of $\operatorname{ad}\mathfrak{a}$ are diagonal with real entries
(c) the matrices of $\operatorname{ad}\mathfrak{n}$ are upper triangular with 0's on the diagonal.

PROOF. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ compatible with the orthogonal decomposition of $\mathfrak{g}$ in Proposition 6.40a and having the property that $X_i \in \mathfrak{g}_{\lambda_i}$ and $X_j \in \mathfrak{g}_{\lambda_j}$ with $i < j$ implies $\lambda_i \geq \lambda_j$. For $X \in \mathfrak{k}$, we have $(\operatorname{ad} X)^* = -\operatorname{ad}\theta X = -\operatorname{ad} X$ from Lemma 6.27, and this proves (a). Since each X_i is a restricted-root vector or is in $\mathfrak{g}_0$, the matrices of $\operatorname{ad}\mathfrak{a}$ are diagonal, necessarily with real entries. This proves (b). Conclusion (c) follows from Proposition 6.40b.

Theorem 6.46 (Iwasawa decomposition). Let G be a semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G, and let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto. The groups A and N are simply connected.

PROOF. Let $\bar{G} = \operatorname{Ad}(G)$, regarded as the closed subgroup $(\operatorname{Aut}\mathfrak{g})_0$ of $GL(\mathfrak{g})$ (Propositions 1.97 and 1.98). We shall prove the theorem for $\bar{G}$ and then lift the result to G.

We impose the inner product B_θ on $\mathfrak{g}$ and write matrices for elements of $\bar{G}$ and $\operatorname{ad}\mathfrak{g}$ relative to the basis in Lemma 6.45. Let $\bar{K} = \operatorname{Ad}_{\mathfrak{g}}(K)$, $\bar{A} = \operatorname{Ad}_{\mathfrak{g}}(A)$, and $\bar{N} = \operatorname{Ad}_{\mathfrak{g}}(N)$. Lemma 6.45 shows that the matrices of $\bar{K}$ are rotation matrices, those for $\bar{A}$ are diagonal with positive entries on the diagonal, and those for $\bar{N}$ are upper triangular with 1's on the diagonal. We know that $\bar{K}$ is compact (Proposition 6.30 and Theorem 6.31f). The diagonal subgroup of $GL(\mathfrak{g})$ with positive diagonal entries is simply connected abelian, and $\bar{A}$ is an analytic subgroup of it. By Corollary 1.111, $\bar{A}$ is closed in $GL(\mathfrak{g})$ and hence closed in $\bar{G}$. Similarly the upper-triangular subgroup of $GL(\mathfrak{g})$ with 1's on the diagonal is simply connected nilpotent, and $\bar{N}$ is an analytic subgroup of it. By Corollary 1.111, $\bar{N}$ is closed in $GL(\mathfrak{g})$ and hence closed in $\bar{G}$.

The map $\bar{A} \times \bar{N}$ into $GL(\mathfrak{g})$ given by $(\bar{a}, \bar{n}) \mapsto \bar{a}\bar{n}$ is one-one since we can recover $\bar{a}$ from the diagonal entries, and it is onto a subgroup $\bar{A}\bar{N}$ since $\bar{a}_1\bar{n}_1\bar{a}_2\bar{n}_2 = \bar{a}_1\bar{a}_2(\bar{a}_2^{-1}\bar{n}_1\bar{a}_2)\bar{n}_2$ and $(\bar{a}\bar{n})^{-1} = \bar{n}^{-1}\bar{a}^{-1} = \bar{a}^{-1}(\bar{a}\bar{n}\bar{a}^{-1})$. This subgroup is closed. In fact, if $\lim \bar{a}_m\bar{n}_m = x$, let $\bar{a}$ be the diagonal matrix with the same diagonal entries as x. Then $\lim \bar{a}_m = \bar{a}$, and $\bar{a}$ must be in $\bar{A}$ since $\bar{A}$ is closed in $GL(\mathfrak{g})$. Also $\bar{n}_m = \bar{a}_m^{-1}(\bar{a}_m\bar{n}_m)$ has limit $\bar{a}^{-1}x$, which has to be in $\bar{N}$ since $\bar{N}$ is closed in $\bar{G}$. Thus $\lim \bar{a}_m\bar{n}_m$ is in $\bar{A}\bar{N}$, and $\bar{A}\bar{N}$ is closed.

Clearly the closed subgroup $\bar{A}\bar{N}$ has Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. By Lemma 6.44, $\bar{A} \times \bar{N} \to \bar{A}\bar{N}$ is a diffeomorphism.

The subgroup $\bar{K}$ is compact, and thus the image of $\bar{K} \times \bar{A} \times \bar{N} \to \bar{K} \times \bar{A}\bar{N} \to \bar{G}$ is the product of a compact set and a closed set and is closed. Also the image is open since the map is everywhere regular (Lemma 6.44) and since the equality $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ shows that the dimensions add properly. Since the image of $\bar{K} \times \bar{A} \times \bar{N}$ is open and closed and since $\bar{G}$ is connected, the image is all of $\bar{G}$.

Thus the multiplication map is smooth, regular, and onto. Finally $\bar{K} \cap \bar{A}\bar{N} = \{1\}$ since a rotation matrix with positive eigenvalues is 1. Since $\bar{A} \times \bar{N} \to \bar{A}\bar{N}$ is one-one, it follows that $\bar{K} \times \bar{A} \times \bar{N} \to \bar{G}$ is one-one. This completes the proof for the adjoint group $\bar{G}$.

We now lift the above result to G. Let $e : G \to \bar{G} = \mathrm{Ad}(G)$ be the covering homomorphism. Using a locally defined inverse of e, we can write the map $(k, a, n) \mapsto kan$ locally as

$$(k, a, n) \mapsto (e(k), e(a), e(n)) \mapsto e(k)e(a)e(n) = e(kan) \mapsto kan,$$

and therefore the multiplication map is smooth and everywhere regular. Since A and N are connected, $e|_A$ and $e|_N$ are covering maps to $\bar{A}$ and $\bar{N}$, respectively. Since $\bar{A}$ and $\bar{N}$ are simply connected, it follows that e is one-one on A and on N and that A and N are simply connected.

Let us prove that the multiplication map is onto G. If $g \in G$ is given, write $e(g) = \bar{k}\bar{a}\bar{n}$. Put $a = (e|_A)^{-1}(\bar{a}) \in A$ and $n = (e|_N)^{-1}(\bar{N}) \in N$. Let k be in $e^{-1}(\bar{k})$. Then $e(kan) = \bar{k}\bar{a}\bar{n}$, so that $e(g(kan)^{-1}) = 1$. Thus $g(kan)^{-1} = z$ is in the center of G. By Theorem 6.31e, z is in K. Therefore $g = (zk)an$ exhibits g as in the image of the multiplication map.

Finally we show that the multiplication map is one-one. Since $\bar{A} \times \bar{N} \to \bar{A}\bar{N}$ is one-one, so is $A \times N \to AN$. The set of products AN is a group, just as in the adjoint case, and therefore it is enough to prove that $K \cap AN = \{1\}$. If x is in $K \cap AN$, then $e(x)$ is in $\bar{K} \cap \bar{A}\bar{N} = \{1\}$. Hence $e(x) = 1$. Write $x = an \in AN$. Then $1 = e(x) = e(an) = e(a)e(n)$, and the result for the adjoint case implies that $e(a) = e(n) = 1$. Since e is one-one on A and on N, $a = n = 1$. Thus $x = 1$. This completes the proof.

Recall from §IV.5 that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The **rank** of $\mathfrak{g}$ is the dimension of any Cartan subalgebra; this is well defined since Proposition 2.15 shows that any two Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate via $\mathrm{Int}\,\mathfrak{g}^{\mathbb{C}}$.

Proposition 6.47. If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

PROOF. By Corollary 2.13 it is enough to show that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}^{\mathbb{C}}$ and that $\operatorname{ad}_{\mathfrak{g}^{\mathbb{C}}} \mathfrak{h}^{\mathbb{C}}$ is simultaneously diagonable.

Certainly $\mathfrak{h}^{\mathbb{C}}$ is abelian. Let us see that it is maximal abelian. If $Z = X + iY$ commutes with $\mathfrak{h}^{\mathbb{C}}$, then so do X and Y. Thus there is no loss in generality in considering only X. The element X commutes with $\mathfrak{h}^{\mathbb{C}}$, hence commutes with $\mathfrak{a}$, and hence is in $\mathfrak{a} \oplus \mathfrak{m}$. The same thing is true of θX. Then $X + \theta X$, being in $\mathfrak{k}$, is in $\mathfrak{m}$ and commutes with $\mathfrak{t}$, hence is in $\mathfrak{t}$, while $X - \theta X$ is in $\mathfrak{a}$. Thus X is in $\mathfrak{a} \oplus \mathfrak{t}$, and we conclude that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian.

In the basis of Lemma 6.45, the matrices representing $\operatorname{ad} \mathfrak{t}$ are skew symmetric and hence are diagonable over $\mathbb{C}$, while the matrices representing $\operatorname{ad} \mathfrak{a}$ are already diagonal. Since all the matrices in question form a commuting family, the members of $\operatorname{ad} \mathfrak{h}^{\mathbb{C}}$ are diagonable.

With notation as in Proposition 6.47, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, and it is meaningful to speak of the set $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. We can write the corresponding root-space decomposition as

(6.48a)
$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}^{\mathbb{C}})_{\alpha}.$$

Then it is clear that

(6.48b)
$$\mathfrak{g}_{\lambda} = \mathfrak{g} \cap \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha|_{\mathfrak{a}} = \lambda}} (\mathfrak{g}^{\mathbb{C}})_{\alpha}$$

and

(6.48c)
$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha \in \Delta, \\ \alpha|_{\mathfrak{a}} = 0}} (\mathfrak{g}^{\mathbb{C}})_{\alpha}.$$

That is, the restricted roots are the nonzero restrictions to $\mathfrak{a}$ of the roots, and $\mathfrak{m}$ arises from $\mathfrak{t}$ and the roots that restrict to 0 on $\mathfrak{a}$.

Corollary 6.49. If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then the Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ of $\mathfrak{g}$ has the property that all of the roots are real on $\mathfrak{a} \oplus i\mathfrak{t}$. If $\mathfrak{m} = 0$, then $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$.

PROOF. In view of (6.48) the values of the roots on a member H of $\mathfrak{h}$ are the eigenvalues of $\operatorname{ad} H$. For $H \in \mathfrak{a}$, these are real since $\operatorname{ad} H$ is self adjoint. For $H \in \mathfrak{t}$, they are purely imaginary since $\operatorname{ad} H$ is skew adjoint. The first assertion follows.

If $\mathfrak{m} = 0$, then $\mathfrak{t} = 0$. So the roots are real on $\mathfrak{h} = \mathfrak{a}$. Thus $\mathfrak{g}$ contains the real subspace of a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ where all the roots are real, and $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$.

By nondegeneracy of B on $\mathfrak{a}$, $[E_\lambda, \theta E_\lambda] = B(E_\lambda, \theta E_\lambda) H_\lambda$. Finally $B(E_\lambda, \theta E_\lambda) = -B_\theta(E_\lambda, E_\lambda) < 0$ since B_θ is positive definite.

(b) Put

$$H'_\lambda = \frac{2}{|\lambda|^2} H_\lambda, \quad E'_\lambda = \frac{2}{|\lambda|^2 B(E_\lambda, \theta E_\lambda)} E_\lambda, \quad E'_{-\lambda} = \theta E_\lambda.$$

Then (a) shows that

$$[H'_\lambda, E'_\lambda] = 2E'_\lambda, \quad [H'_\lambda, E'_{-\lambda}] = -2E'_{-\lambda}, \quad [E'_\lambda, E'_{-\lambda}] = H'_\lambda,$$

and (b) follows.

(c) Note from (a) that the normalization $B(E_\lambda, \theta E_\lambda) = -2/|\lambda|^2$ is allowable. If $\lambda(H) = 0$, then

$$\begin{aligned} \operatorname{Ad}(k)H &= \operatorname{Ad}(\exp \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= (\exp \operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^n H \\ &= H. \end{aligned}$$

On the other hand, for the element H'_λ, we first calculate that

$$(\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H'_\lambda = \pi(\theta E_\lambda - E_\lambda)$$

and

$$(\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2 H'_\lambda = -\pi^2 H'_\lambda.$$

Therefore

$$\begin{aligned} \operatorname{Ad}(k)H'_\lambda &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^n H'_\lambda \\ &= \sum_{m=0}^{\infty} \tfrac{1}{(2m)!}((\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2)^m H'_\lambda \\ &\quad + \sum_{m=0}^{\infty} \tfrac{1}{(2m+1)!}(\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))((\operatorname{ad} \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))^2)^m H'_\lambda \\ &= \sum_{m=0}^{\infty} \tfrac{1}{(2m)!}(-\pi^2)^m H'_\lambda + \sum_{m=0}^{\infty} \tfrac{1}{(2m+1)!}(-\pi^2)^m \pi(\theta E_\lambda - E_\lambda) \\ &= (\cos\pi)H'_\lambda + (\sin\pi)(\theta E_\lambda - E_\lambda) \\ &= -H'_\lambda, \end{aligned}$$

and (c) follows.

This is a group of linear transformations of $\mathfrak{a}$, telling all possible ways that members of K can act on $\mathfrak{a}$ by Ad. We have already seen that $W(\Sigma) \subseteq W(G, A)$, and we are going to prove that $W(\Sigma) = W(G, A)$.

We write M for the group $Z_K(\mathfrak{a})$. Modulo the center of G, M is a compact group (being a closed subgroup of K) with Lie algebra $Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{m}$. After Proposition 6.40 we saw examples of restricted-root space decompositions and the associated Lie algebras $\mathfrak{m}$. The following examples continue that discussion.

EXAMPLES.

1) Let $G = SL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The subgroup M consists of all diagonal members of K. When $\mathbb{K} = \mathbb{R}$, the diagonal entries are ± 1, but there are only $n - 1$ independent signs since the determinant is 1. Thus M is finite abelian and is the product of $n - 1$ groups of order 2. When $\mathbb{K} = \mathbb{C}$, the diagonal entries are complex numbers of modulus 1, and again the determinant is 1. Thus M is a torus of dimension $n - 1$. When $\mathbb{K} = \mathbb{H}$, the diagonal entries are quaternions of absolute value 1, and there is no restriction on the determinant. Thus M is the product of n copies of $SU(2)$.

2) Let $G = SU(p, q)$ with $p \geq q$. The group M consists of all unitary matrices of determinant 1 that are arbitrary in the upper left block of size $p - q$, are otherwise diagonal, and have the $(p - i + 1)^{\text{st}}$ diagonal entry equal to the $(p + i)^{\text{th}}$ diagonal entry for $1 \leq i \leq q$. Let us abbreviate such a matrix as

$$m = \operatorname{diag}(\omega, e^{i\theta_q}, \dots, e^{i\theta_1}, e^{i\theta_1}, \dots, e^{i\theta_q}),$$

where ω is the upper left block of size $p - q$. When $p = q$, the condition that the determinant be 1 says that $\sum_{j=1}^{q} \theta_j \in \pi\mathbb{Z}$. Thus we can take $\theta_1, \dots, \theta_{q-1}$ to be arbitrary and use $e^{i\theta_q} = \pm e^{-i(\theta_1 + \cdots + \theta_{q-1})}$. Consequently M is the product of a torus of dimension $q - 1$ and a 2-element group. When $p > q$, M is connected. In fact, the homomorphism that maps the above matrix m to the $2q$-by-$2q$ diagonal matrix

$$\operatorname{diag}(e^{i\theta_q}, \dots, e^{i\theta_1}, e^{i\theta_1}, \dots, e^{i\theta_q})$$

has a (connected) q-dimensional torus as image, and the kernel is isomorphic to the connected group $SU(p - q)$; thus M itself is connected.

3) Let $G = SO(p, q)_0$ with $p \geq q$. The subgroup M for this example is the intersection of $SO(p) \times SO(q)$ with the M of the previous example. Thus M here consists of matrices that are orthogonal matrices of total determinant 1, are arbitrary in the upper left block of size $p - q$, are

otherwise diagonal, have q diagonal entries ± 1 after the upper left block, and then have those q diagonal entries ± 1 repeated in reverse order. For the lower right q entries to yield a matrix in $SO(q)$, the product of the q entries ± 1 must be 1. For the upper left p entries to yield a matrix in $SO(p)$, the orthogonal matrix in the upper left block of size $p-q$ must have determinant 1. Therefore M is isomorphic to the product of $SO(p-q)$ and the product of $q-1$ groups of order 2.

Lemma 6.56. The Lie algebra of $N_K(\mathfrak{a})$ is $\mathfrak{m}$. Therefore $W(G,A)$ is a finite group.

PROOF. The second conclusion follows from the first, since the first conclusion implies that $W(G,A)$ is 0-dimensional and compact, hence finite. For the first conclusion, the Lie algebra in question is $N_{\mathfrak{k}}(\mathfrak{a})$. Let $X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$ be a member of $N_{\mathfrak{k}}(\mathfrak{a})$, with $H_0 \in \mathfrak{a}$, $X_0 \in \mathfrak{m}$, and $X_\lambda \in \mathfrak{g}_\lambda$. Since X is to be in $\mathfrak{k}$, θ must fix X, and we see that X may be rewritten as $X = X_0 + \sum_{\lambda\in\Sigma^+}(X_\lambda + \theta X_\lambda)$. When we apply $\operatorname{ad} H$ for $H \in \mathfrak{a}$, we obtain $[H,X] = \sum_{\lambda\in\Sigma^+}\lambda(H)(X_\lambda - \theta X_\lambda)$. This element is supposed to be in $\mathfrak{a}$, since we started with X in the normalizer of $\mathfrak{a}$, and that means $[H,X]$ is 0. But then $X_\lambda = 0$ for all λ, and X reduces to the member X_0 of $\mathfrak{m}$.

Theorem 6.57. The group $W(G,A)$ coincides with $W(\Sigma)$.

REMARK. This theorem should be compared with Theorem 4.54.

PROOF. Let us observe that $W(G,A)$ permutes the restricted roots. In fact, let k be in $N_K(\mathfrak{a})$, let λ be in Σ, and let E_λ be in $\mathfrak{g}_\lambda$. Then

$$\begin{aligned}[H, \operatorname{Ad}(k)E_\lambda] &= \operatorname{Ad}(k)[\operatorname{Ad}(k)^{-1}H, E_\lambda] = \operatorname{Ad}(k)(\lambda(\operatorname{Ad}(k)^{-1}H)E_\lambda)\\ &= \lambda(\operatorname{Ad}(k)^{-1}H)\operatorname{Ad}(k)E_\lambda = (k\lambda)(H)\operatorname{Ad}(k)E_\lambda\end{aligned}$$

shows that $k\lambda$ is in Σ and that $\operatorname{Ad}(k)E_\lambda$ is a restricted-root vector for $k\lambda$. Thus $W(G,A)$ permutes the restricted roots.

We have seen that $W(\Sigma) \subseteq W(G,A)$. Fix a simple system Σ^+ for Σ. In view of Theorem 2.63, it suffices to show that if $k \in N_K(\mathfrak{a})$ has $\operatorname{Ad}(k)\Sigma^+ = \Sigma^+$, then k is in $Z_K(\mathfrak{a})$.

The element $\operatorname{Ad}(k) = w$ acts as a permutation of Σ^+. Let 2δ denote the sum of the reduced members of Σ^+, so that w fixes δ. If λ_i is a simple restricted root, then Lemma 2.91 and Proposition 2.69 show that $2\langle\delta,\lambda_i\rangle/|\lambda_i|^2 = 1$. Therefore $\langle\delta,\lambda\rangle > 0$ for all $\lambda \in \Sigma^+$.

Let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ be the compact real form of $\mathfrak{g}^{\mathbb{C}}$ associated to θ, and let U be the adjoint group of $\mathfrak{u}$. Then $\operatorname{Ad}_{\mathfrak{g}^{\mathbb{C}}}(K) \subseteq U$, and in particular $\operatorname{Ad}(k)$ is a member of U. Form $S = \overline{\{\exp ir \operatorname{ad} H_\delta\}} \subseteq U$. Here S is a torus in U, and we let $\mathfrak{s}$ be the Lie algebra of S. The element $\mathrm{Ad}(k)$ is in $Z_U(S)$, and the claim is that every member of $Z_U(S)$ centralizes $\mathfrak{a}$. If so, then $\mathrm{Ad}(k)$ is 1 on $\mathfrak{a}$, and k is in $Z_K(\mathfrak{a})$, as required.

By Corollary 4.51 we can verify that $Z_U(S)$ centralizes $\mathfrak{a}$ by showing that $Z_{\mathfrak{u}}(\mathfrak{s})$ centralizes $\mathfrak{a}$. Here

$$Z_{\mathfrak{u}}(\mathfrak{s}) = \mathfrak{u} \cap Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{u} \cap Z_{\mathfrak{g}^{\mathbb{C}}}(H_\delta).$$

To evaluate the right side, we complexify the statement of Lemma 6.50. Since $\langle \lambda, \delta \rangle \neq 0$, the centralizer $Z_{\mathfrak{g}^{\mathbb{C}}}(H_\delta)$ is just $\mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. Therefore

$$Z_{\mathfrak{u}}(\mathfrak{s}) = \mathfrak{u} \cap (\mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}) = i\mathfrak{a} \oplus \mathfrak{m}.$$

Every member of the right side centralizes $\mathfrak{a}$, and the proof is complete.

6. Cartan Subalgebras

Proposition 6.47 showed that every real semisimple Lie algebra has a Cartan subalgebra. But as we shall see shortly, not all Cartan subalgebras are conjugate. In this section and the next we investigate the conjugacy classes of Cartan subalgebras and some of their relationships to each other.

We revert to the use of subscripted Gothic letters for real Lie algebras and to unsubscripted letters for complexifications. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexification of the Cartan decomposition. Let B be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_0$ such that $B(\theta X, \theta Y) = B(X, Y)$ and such that B_θ, defined by (6.13), is positive definite.

All Cartan subalgebras of $\mathfrak{g}_0$ have the same dimension, since their complexifications are Cartan subalgebras of $\mathfrak{g}$ and are conjugate via $\mathrm{Int}\,\mathfrak{g}$, according to Theorem 2.15.

Let $K = \mathrm{Int}_{\mathfrak{g}_0}(\mathfrak{k}_0)$. This subgroup of $\mathrm{Int}\,\mathfrak{g}_0$ is compact.

EXAMPLE. Let $G = SL(2, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$. A Cartan subalgebra $\mathfrak{h}_0$ complexifies to a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ and therefore has dimension 1. Therefore let us consider which 1-dimensional subspaces $\mathbb{R}X$ of $\mathfrak{sl}(2, \mathbb{R})$ are Cartan subalgebras. The matrix X has trace 0, and we divide matters into cases according to the sign of $\det X$. If $\det X < 0$, then X has real eigenvalues μ and $-\mu$, and X is conjugate via $SL(2, \mathbb{R})$ to a diagonal matrix. Thus, for some $g \in SL(2, \mathbb{R})$,

$$\mathbb{R}X = \{\mathrm{Ad}(g)\mathbb{R}h\}.$$

noncompact Cartan subalgebra. To correlate this information, we need to be able to track down the conjugacy via $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ of a maximally compact Cartan subalgebra and a maximally noncompact one.

Cayley transforms are one-step conjugacies of θ stable Cartan subalgebras whose iterates explicitly relate any θ stable Cartan subalgebra with any other. We develop Cayley transforms in this section and show that in favorable circumstances we can see past the step-by-past process to understand the composite conjugation all at once.

There are two kinds of Cayley transforms, essentially inverse to each other. They are modeled on what happens in $\mathfrak{sl}(2, \mathbb{R})$. In the case of $\mathfrak{sl}(2, \mathbb{R})$, we start with the standard basis h, e, f for $\mathfrak{sl}(2, \mathbb{C})$ as in (1.5), as well as the members h_B, e_B, f_B of $\mathfrak{sl}(2, \mathbb{C})$ defined in (6.58). The latter elements satisfy the familiar bracket relations

$$[h_B, e_B] = 2e_B, \quad [h_B, f_B] = -2f_B, \quad [e_B, f_B] = h_B.$$

The definitions of e_B and f_B make $e_B + f_B$ and $i(e_B - f_B)$ be in $\mathfrak{sl}(2, \mathbb{R})$, while $i(e_B + f_B)$ and $e_B - f_B$ are in $\mathfrak{su}(2)$. The first kind of Cayley transform within $\mathfrak{sl}(2, \mathbb{C})$ is the mapping

$$\operatorname{Ad}\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right) = \operatorname{Ad}(\exp \tfrac{\pi}{4}(f_B - e_B)),$$

which carries h_B, e_B, f_B to complex multiples of h, e, f and carries the Cartan subalgebra $\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $i\mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. When generalized below, this Cayley transform will be called $\mathbf{c}_\beta$.

The second kind of Cayley transform within $\mathfrak{sl}(2, \mathbb{C})$ is the mapping

$$\operatorname{Ad}\left(\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\right) = \operatorname{Ad}(\exp i\tfrac{\pi}{4}(-f - e)),$$

which carries h, e, f to complex mutliples of h_B, e_B, f_B and carries the Cartan subalgebra $\mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $i\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In view of the explicit formula for the matrices of the Cayley transforms, the two transforms are inverse to one another. When generalized below, this second Cayley transform will be called $\mathbf{d}_\alpha$.

The idea is to embed each of these constructions into constructions in the complexification of our underlying semisimple algebra that depend upon a single root of a special kind, leaving fixed the part of the Cartan subalgebra that is orthogonal to the embedded copy of $\mathfrak{sl}(2, \mathbb{C})$.

In terms of our discussion above of $\mathfrak{sl}(2,\mathbb{C})$, the correspondence is

$$H'_\alpha \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_\alpha \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\theta E_\alpha \leftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$i(\theta E_\alpha - E_\alpha) \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Define

$$\mathbf{d}_\alpha = \mathrm{Ad}(\exp i\,\tfrac{\pi}{4}(\theta E_\alpha - E_\alpha)) \tag{6.67a}$$

and

$$\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathbf{d}_\alpha(\mathfrak{h}') = \ker(\alpha|_{\mathfrak{h}'_0}) \oplus \mathbb{R}(E_\alpha + \theta E_\alpha). \tag{6.67b}$$

To see that (6.67b) is valid, we can use infinite series to calculate that

$$\mathbf{d}_\alpha(H'_\alpha) = i(E_\alpha + \theta E_\alpha) \tag{6.68a}$$

$$\mathbf{d}_\alpha(E_\alpha - \theta E_\alpha) = E_\alpha - \theta E_\alpha \tag{6.68b}$$

$$\mathbf{d}_\alpha(E_\alpha + \theta E_\alpha) = iH'_\alpha. \tag{6.68c}$$

Then (6.68a) implies (6.67b).

Proposition 6.69. The two kinds of Cayley transforms are essentially inverse to each other in the following senses:

(a) If β is a noncompact imaginary root, then in the computation of $\mathbf{d}_{\mathbf{c}_\beta(\beta)} \circ \mathbf{c}_\beta$ the root vector $E_{\mathbf{c}_\beta(\beta)}$ can be taken to be $i\mathbf{c}_\beta(E_\beta)$ and this choice makes the composition the identity.

(b) If α is a real root, then in the the computation of $\mathbf{c}_{\mathbf{d}_\alpha(\alpha)} \circ \mathbf{d}_\alpha$ the root vector $E_{\mathbf{d}_\alpha(\alpha)}$ can be taken to be $-i\mathbf{d}_\alpha(E_\alpha)$ and this choice makes the composition the identity.

Proof.

(a) By (6.66),

$$\mathbf{c}_\beta(E_\beta) = \tfrac{1}{2}\mathbf{c}_\beta(E_\beta + \overline{E_\beta}) + \tfrac{1}{2}\mathbf{c}_\beta(E_\beta - \overline{E_\beta}) = -\tfrac{1}{2}H'_\beta + \tfrac{1}{2}(E_\beta - \overline{E_\beta}).$$

Both terms on the right side are in $i\mathfrak{g}_0$, and hence $i\mathbf{c}_\beta(E_\beta)$ is in $\mathfrak{g}_0$. Since H'_β is in $\mathfrak{k}$ while E_β and $\overline{E_\beta}$ are in $\mathfrak{p}$,

$$\theta\mathbf{c}_\beta(E_\beta) = -\tfrac{1}{2}H'_\beta - \tfrac{1}{2}(E_\beta - \overline{E_\beta}).$$

Put $E_{\mathbf{c}_\beta(\beta)} = i\mathbf{c}_\beta(E_\beta)$. From $B(E_\beta, \overline{E_\beta}) = 2/|\beta|^2$, we obtain

$$B(E_{\mathbf{c}_\beta(\beta)}, \theta E_{\mathbf{c}_\beta(\beta)}) = -2/|\beta|^2 = -2/|\mathbf{c}_\beta(\beta)|^2.$$

Thus $E_{\mathbf{c}_\beta(\beta)}$ is properly normalized. Then $\mathbf{d}_{\mathbf{c}_\beta(\beta)}$ becomes

$$\begin{aligned}\mathbf{d}_{\mathbf{c}_\beta(\beta)} &= \operatorname{Ad}(\exp i\tfrac{\pi}{4}(\theta E_{\mathbf{c}_\beta(\beta)} - E_{\mathbf{c}_\beta(\beta)}))\\ &= \operatorname{Ad}(\exp \tfrac{\pi}{4}(\mathbf{c}_\beta(E_\beta) - \theta\mathbf{c}_\beta(E_\beta)))\\ &= \operatorname{Ad}(\exp \tfrac{\pi}{4}(E_\beta - \overline{E_\beta})),\end{aligned}$$

and this is the inverse of

$$\mathbf{c}_\beta = \operatorname{Ad}(\exp \tfrac{\pi}{4}(\overline{E_\beta} - E_\beta)).$$

(b) By (6.68),

$$\mathbf{d}_\alpha(E_\alpha) = \tfrac{1}{2}\mathbf{d}_\alpha(E_\alpha + \theta E_\alpha) + \tfrac{1}{2}\mathbf{d}_\alpha(E_\alpha - \theta E_\alpha) = \tfrac{1}{2}iH'_\alpha + \tfrac{1}{2}(E_\alpha - \theta E_\alpha).$$

Since H'_α, E_α, and θE_α are in $\mathfrak{g}_0$,

$$\overline{\mathbf{d}_\alpha(E_\alpha)} = -\tfrac{1}{2}iH'_\alpha + \tfrac{1}{2}(E_\alpha - \theta E_\alpha).$$

Put $E_{\mathbf{d}_\alpha(\alpha)} = -i\mathbf{d}_\alpha(E_\alpha)$. From $B(E_\alpha, \theta E_\alpha) = -2/|\alpha|^2$, we obtain

$$B(E_{\mathbf{d}_\alpha(\alpha)}, \overline{E_{\mathbf{d}_\alpha(\alpha)}}) = 2/|\alpha|^2 = 2/|\mathbf{d}_\alpha(\alpha)|^2.$$

Thus $E_{\mathbf{d}_\alpha(\alpha)}$ is properly normalized. Then $\mathbf{c}_{\mathbf{d}_\alpha(\alpha)}$ becomes

$$\begin{aligned}\mathbf{c}_{\mathbf{d}_\alpha(\alpha)} &= \operatorname{Ad}(\exp \tfrac{\pi}{4}(\overline{E_{\mathbf{d}_\alpha(\alpha)}} - E_{\mathbf{d}_\alpha(\alpha)}))\\ &= \operatorname{Ad}(\exp i\tfrac{\pi}{4}(\mathbf{d}_\alpha(E_\alpha) + \overline{\mathbf{d}_\alpha(E_\alpha)}))\\ &= \operatorname{Ad}(\exp i\tfrac{\pi}{4}(E_\alpha - \theta E_\alpha)),\end{aligned}$$

and this is the inverse of

$$\mathbf{d}_\alpha = \operatorname{Ad}(\exp i\tfrac{\pi}{4}(\theta E_\alpha - E_\alpha)).$$

Proposition 6.70. Let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Then there are no noncompact imaginary roots if and only if $\mathfrak{h}_0$ is maximally noncompact, and there are no real roots if and only if $\mathfrak{h}_0$ is maximally compact.

Computation shows that

$$\Delta = \{\pm e_j \pm e_k \pm (f_j - f_k) \mid j \neq k\} \cup \{\pm 2e_l \mid 1 \leq l \leq n\}.$$

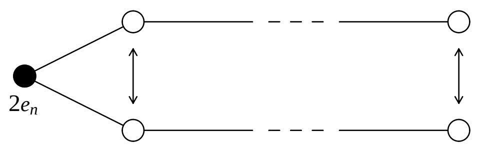
Roots that involve only e_j's are imaginary, those that involve only f_j's are real, and the remainder are complex. It is apparent that there are no real roots, and therefore $\mathfrak{h}_0$ is maximally compact. The involution θ acts as $+1$ on the e_j's and -1 on the f_j's. We define a lexicographic ordering by using the spanning set

$$e_1, \ldots, e_n, f_1, \ldots, f_n,$$

and we obtain

$$\Delta^+ = \begin{cases} e_j + e_k \pm (f_j - f_k), & \text{all } j \neq k \\ e_j - e_k \pm (f_j - f_k), & j < k \\ 2e_l, & 1 \leq l \leq n. \end{cases}$$

The Vogan diagram is



Theorem 6.74. Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be real semisimple Lie algebras. With notation as above, if two triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', (\Delta')^+)$ have the same Vogan diagram, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.

REMARK. This theorem is an analog for real semisimple Lie algebras of the Isomorphism Theorem (Theorem 2.108) for complex semisimple Lie algebras.

PROOF. Since the Dynkin diagrams are the same, the Isomorphism Theorem (Theorem 2.108) shows that there is no loss of generality in assuming that $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ have the same complexification $\mathfrak{g}$. Let $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ and $\mathfrak{u}_0' = \mathfrak{k}_0' \oplus i\mathfrak{p}_0'$ be the associated compact real forms of $\mathfrak{g}$. By Corollary 6.20, there exists $x \in \operatorname{Int} \mathfrak{g}$ such that $x\mathfrak{u}_0' = \mathfrak{u}_0$. The real form $x\mathfrak{g}_0'$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_0'$ and has Cartan decomposition $x\mathfrak{g}_0' = x\mathfrak{k}_0' \oplus x\mathfrak{p}_0'$. Since $x\mathfrak{k}_0' \oplus ix\mathfrak{p}_0' = x\mathfrak{u}_0' = \mathfrak{u}_0$, there is no loss of generality in assuming that $\mathfrak{u}_0' = \mathfrak{u}_0$ from the outset. Then

$$\theta(\mathfrak{u}_0) = \mathfrak{u}_0 \quad \text{and} \quad \theta'(\mathfrak{u}_0) = \mathfrak{u}_0. \tag{6.75}$$

Let us write the effect of the Cartan decompositions on the Cartan subalgebras as $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and $\mathfrak{h}'_0 = \mathfrak{t}'_0 \oplus \mathfrak{a}'_0$. Then $\mathfrak{t}_0 \oplus i\mathfrak{a}_0$ and $\mathfrak{t}'_0 \oplus i\mathfrak{a}'_0$ are maximal abelian subspaces of $\mathfrak{u}_0$. By Theorem 4.34 there exists $k \in \operatorname{Int}\mathfrak{u}_0$ with $k(\mathfrak{t}'_0 \oplus i\mathfrak{a}'_0) = \mathfrak{t}_0 \oplus i\mathfrak{a}_0$. Replacing $\mathfrak{g}'_0$ by $k\mathfrak{g}'_0$ and arguing as above, we may assume that $\mathfrak{t}'_0 \oplus i\mathfrak{a}'_0 = \mathfrak{t}_0 \oplus i\mathfrak{a}_0$ from the outset. Therefore $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ have the same complexification, which we denote $\mathfrak{h}$. The space

$$\mathfrak{u}_0 \cap \mathfrak{h} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0 = \mathfrak{t}'_0 \oplus i\mathfrak{a}'_0$$

is a maximal abelian subspace of $\mathfrak{u}_0$.

Now that the complexifications $\mathfrak{g}$ and $\mathfrak{h}$ have been aligned, the root systems are the same. Let the positive systems given in the respective triples be Δ^+ and $\Delta^{+\prime}$. By Theorems 4.54 and 2.63 there exists $k' \in \operatorname{Int}\mathfrak{u}_0$ normalizing $\mathfrak{u}_0 \cap \mathfrak{h}$ with $k'\Delta^{+\prime} = \Delta^+$. Replacing $\mathfrak{g}'_0$ by $k'\mathfrak{g}'_0$ and arguing as above, we may assume that $\Delta^{+\prime} = \Delta^+$ from the outset.

The next step is to choose normalizations of root vectors relative to $\mathfrak{h}$. For this proof let B be the Killing form of $\mathfrak{g}$. We start with root vectors X_α produced from $\mathfrak{h}$ as in Theorem 6.6. Using (6.12), we construct a compact real form $\tilde{\mathfrak{u}}_0$ of $\mathfrak{g}$. The subalgebra $\tilde{\mathfrak{u}}_0$ contains the real subspace of $\mathfrak{h}$ where the roots are imaginary, which is just $\mathfrak{u}_0 \cap \mathfrak{h}$. By Corollary 6.20, there exists $g \in \operatorname{Int}\mathfrak{g}$ such that $g\tilde{\mathfrak{u}}_0 = \mathfrak{u}_0$. Then $g\tilde{\mathfrak{u}}_0 = \mathfrak{u}_0$ is built by (6.12) from $g(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors gX_α. Since $\mathfrak{u}_0 \cap \mathfrak{h}$ and $g(\mathfrak{u}_0 \cap \mathfrak{h})$ are maximal abelian in $\mathfrak{u}_0$, Theorem 4.34 produces $u \in \operatorname{Int}\mathfrak{u}_0$ with $ug(\mathfrak{u}_0 \cap \mathfrak{h}) = \mathfrak{u}_0 \cap \mathfrak{h}$. Then $\mathfrak{u}_0$ is built by (6.12) from $ug(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors ugX_α. For $\alpha \in \Delta$, put $Y_\alpha = ugX_\alpha$. Then we have established that

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(Y_\alpha - Y_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(Y_\alpha + Y_{-\alpha}). \tag{6.76}$$

We have not yet used the information that is superimposed on the Dynkin diagram of Δ^+. Since the automorphisms of Δ^+ defined by θ and θ' are the same, θ and θ' have the same effect on $\mathfrak{h}^*$. Thus

$$\theta(H) = \theta'(H) \qquad \text{for all } H \in \mathfrak{h}. \tag{6.77}$$

If α is an imaginary simple root, then

$$\theta(Y_\alpha) = Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is unpainted}, \tag{6.78a}$$

$$\theta(Y_\alpha) = -Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is painted}. \tag{6.78b}$$

We still have to deal with the complex simple roots. For $\alpha \in \Delta$, write $\theta Y_\alpha = a_\alpha Y_{\theta\alpha}$. From (6.75) we know that

$$\theta(\mathfrak{u}_0 \cap \operatorname{span}\{Y_\alpha, Y_{-\alpha}\}) \subseteq \mathfrak{u}_0 \cap \operatorname{span}\{Y_{\theta\alpha}, Y_{-\theta\alpha}\}.$$

In view of (6.76) this inclusion means that

$$\theta(\mathbb{R}(Y_\alpha - Y_{-\alpha}) + \mathbb{R}i(Y_\alpha + Y_{-\alpha})) \subseteq \mathbb{R}(Y_{\theta\alpha} - Y_{-\theta\alpha}) + \mathbb{R}i(Y_{\theta\alpha} + Y_{-\theta\alpha}).$$

If x and y are real and if $z = x + yi$, then we have

$$x(Y_\alpha - Y_{-\alpha}) + yi(Y_\alpha + Y_{-\alpha}) = zY_\alpha - \bar{z}Y_{-\alpha}.$$

Thus the expression $\theta(zY_\alpha - \bar{z}Y_{-\alpha}) = za_\alpha Y_{\theta\alpha} - \bar{z}a_{-\alpha}Y_{-\theta\alpha}$ must be of the form $wY_{\theta\alpha} - \bar{w}Y_{-\theta\alpha}$, and we conclude that

(6.79) $$a_{-\alpha} = \overline{a_\alpha}.$$

Meanwhile $a_\alpha a_{-\alpha} = B(a_\alpha Y_{\theta\alpha}, a_{-\alpha}Y_{-\theta\alpha}) = B(\theta Y_\alpha, \theta Y_{-\alpha}) = B(Y_\alpha, Y_{-\alpha}) = 1$ shows that

(6.80) $$a_\alpha a_{-\alpha} = 1.$$

Combining (6.79) and (6.80), we see that

(6.81) $$|a_\alpha| = 1.$$

Next we observe that

(6.82) $$a_\alpha a_{\theta\alpha} = 1$$

since $Y_\alpha = \theta^2 Y_\alpha = \theta(a_\alpha Y_{\theta\alpha}) = a_\alpha a_{\theta\alpha} Y_\alpha$.

For each pair of complex simple roots α and $\theta\alpha$, choose square roots $a_\alpha^{1/2}$ and $a_{\theta\alpha}^{1/2}$ so that

(6.83) $$a_\alpha^{1/2} a_{\theta\alpha}^{1/2} = 1.$$

This is possible by (6.82).

Similarly write $\theta' Y_\alpha = b_\alpha Y_{\theta\alpha}$ with

(6.84) $$|b_\alpha| = 1,$$

and define $b_\alpha^{1/2}$ and $b_{\theta\alpha}^{1/2}$ for α and $\theta\alpha$ simple so that

(6.85) $$b_\alpha^{1/2} b_{\theta\alpha}^{1/2} = 1.$$

By (6.81) and (6.84), we can define H and H' in $\mathfrak{u}_0 \cap \mathfrak{h}$ by the conditions that $\alpha(H) = \alpha(H') = 0$ for α imaginary simple and

$$\exp(\tfrac{1}{2}\alpha(H)) = a_\alpha^{1/2}, \quad \exp(\tfrac{1}{2}\theta\alpha(H)) = a_{\theta\alpha}^{1/2},$$
$$\exp(\tfrac{1}{2}\alpha(H')) = b_\alpha^{1/2}, \quad \exp(\tfrac{1}{2}\theta\alpha(H')) = b_{\theta\alpha}^{1/2}$$

for α and $\theta\alpha$ complex simple.

We shall show that

$$\theta' \circ \operatorname{Ad}(\exp \tfrac{1}{2}(H - H')) = \operatorname{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \theta. \tag{6.86}$$

In fact, the two sides of (6.86) are equal on $\mathfrak{h}$ and also on each X_α for α imaginary simple, by (6.77) and (6.78), since the Ad factor drops out from each side. If α is complex simple, then

$$\begin{aligned}
\theta' \circ \operatorname{Ad}(\exp \tfrac{1}{2}(H - H'))Y_\alpha &= \theta'(e^{\frac{1}{2}\alpha(H-H')}Y_\alpha) \\
&= b_\alpha a_\alpha^{1/2} b_\alpha^{-1/2} Y_{\theta\alpha} \\
&= b_\alpha^{1/2} a_\alpha^{-1/2} \theta Y_\alpha \\
&= b_{\theta\alpha}^{-1/2} a_{\theta\alpha}^{1/2} \theta Y_\alpha \quad \text{by (6.83) and (6.85)} \\
&= \operatorname{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \theta Y_\alpha .
\end{aligned}$$

This proves (6.86).

Applying (6.86) to $\mathfrak{k}$ and then to $\mathfrak{p}$, we see that

$$\begin{aligned}
\operatorname{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{k}) &\subseteq \mathfrak{k}' \\
\operatorname{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{p}) &\subseteq \mathfrak{p}',
\end{aligned} \tag{6.87}$$

and then equality must hold in each line of (6.87). Since the element $\operatorname{Ad}(\exp \frac{1}{2}(H-H'))$ carries $\mathfrak{u}_0$ to itself, it must carry $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ to $\mathfrak{k}_0' = \mathfrak{u}_0 \cap \mathfrak{k}'$ and $\mathfrak{p}_0 = \mathfrak{u}_0 \cap \mathfrak{p}$ to $\mathfrak{p}_0' = \mathfrak{u}_0 \cap \mathfrak{p}'$. Hence it must carry $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ to $\mathfrak{g}_0' = \mathfrak{k}_0' \oplus \mathfrak{p}_0'$. This completes the proof.

Now let us address the question of existence. We define an **abstract Vogan diagram** to be an abstract Dynkin diagram with two pieces of additional structure indicated: One is an automorphism of order 1 or 2 of the diagram, which is to be indicated by labeling the 2-element orbits. The other is a subset of the 1-element orbits, which is to be indicated by painting the vertices corresponding to the members of the subset. Every Vogan diagram is of course an abstract Vogan diagram.

Theorem 6.88. If an abstract Vogan diagram is given, then there exist a real semisimple Lie algebra $\mathfrak{g}_0$, a Cartan involution θ, a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, and a positive system Δ^+ for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$.

REMARK. Briefly the theorem says that any abstract Vogan diagram comes from some $\mathfrak{g}_0$. Thus the theorem is an analog for real semisimple Lie algebras of the Existence Theorem (Theorem 2.111) for complex semisimple Lie algebras.

positive roots α and β and if $\alpha+\beta$ is a root, then it holds for $\alpha+\beta$. In the notation of Theorem 6.6, we have

$$\begin{aligned}\theta X_{\alpha+\beta} &= N_{\alpha,\beta}^{-1}\theta[X_\alpha, X_\beta] = N_{\alpha,\beta}^{-1}[\theta X_\alpha, \theta X_\beta]\\ &= N_{\alpha,\beta}^{-1}a_\alpha a_\beta[X_{\theta\alpha}, X_{\theta\beta}] = N_{\alpha,\beta}^{-1}N_{\theta\alpha,\theta\beta}a_\alpha a_\beta X_{\theta\alpha+\theta\beta}.\end{aligned}$$

Therefore

$$a_{\alpha+\beta} = N_{\alpha,\beta}^{-1}N_{\theta\alpha,\theta\beta}a_\alpha a_\beta.$$

Here $a_\alpha a_\beta = \pm 1$ by assumption, while Theorem 6.6 and the fact that θ is an automorphism of Δ say that $N_{\alpha,\beta}$ and $N_{\theta\alpha,\theta\beta}$ are real with

$$N_{\alpha,\beta}^2 = \tfrac{1}{2}q(1+p)|\alpha|^2 = \tfrac{1}{2}q(1+p)|\theta\alpha|^2 = N_{\theta\alpha,\theta\beta}^2.$$

Hence $a_{\alpha+\beta} = \pm 1$, and (6.91) is proved.

Let us see that

(6.92)
$$\theta(\mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}i(X_\alpha + X_{-\alpha})) \subseteq \mathbb{R}(X_{\theta\alpha} - X_{-\theta\alpha}) + \mathbb{R}i(X_{\theta\alpha} + X_{-\theta\alpha}).$$

If x and y are real and if $z = x + yi$, then we have

$$x(X_\alpha - X_{-\alpha}) + yi(X_\alpha + X_{-\alpha}) = zX_\alpha - \bar{z}X_{-\alpha}.$$

Thus (6.92) amounts to the assertion that the expression

$$\theta(zX_\alpha - \bar{z}X_{-\alpha}) = za_\alpha X_{\theta\alpha} - \bar{z}a_{-\alpha}X_{-\theta\alpha}$$

is of the form $wX_{\theta\alpha} - \bar{w}X_{-\theta\alpha}$, and this follows from (6.91) and (6.90). Since θ carries roots to roots,

$$\theta\Big(\sum_{\alpha\in\Delta}\mathbb{R}(iH_\alpha)\Big) = \sum_{\alpha\in\Delta}\mathbb{R}(iH_\alpha). \tag{6.93}$$

Combining (6.92) and (6.93) with (6.89), we see that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$.

Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces for θ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$, we have

$$\mathfrak{u}_0 = (\mathfrak{u}_0 \cap \mathfrak{k}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p}).$$

Define $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 = i(\mathfrak{u}_0 \cap \mathfrak{p})$, so that

$$\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0.$$

Lemma 6.98. Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra, and let the Vogan diagram of $\mathfrak{g}_0$ be given that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Write $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ as usual. Let V be the span of the simple roots that are imaginary, let Δ_0 be the root system $\Delta \cap V$, let $\mathcal{H}$ be the subset of $i\mathfrak{t}_0$ paired with V, and let Λ be the subset of $\mathcal{H}$ where all roots of Δ_0 take integer values and all noncompact roots of Δ_0 take odd-integer values. Then Λ is nonempty. In fact, if $\alpha_1, \ldots, \alpha_m$ is any simple system for Δ_0 and if $\omega_1, \ldots, \omega_m$ in V are defined by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$, then the element

$$\omega = \sum_{\substack{i \text{ with } \alpha_i \\ \text{noncompact}}} \omega_i .$$

is in Λ.

PROOF. Fix a simple system $\alpha_1, \ldots, \alpha_m$ for Δ_0, and let Δ_0^+ be the set of positive roots of Δ_0. Define $\omega_1, \ldots, \omega_m$ by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. If $\alpha = \sum_{i=1}^m n_i \alpha_i$ is a positive root of Δ_0, then $\langle \omega, \alpha \rangle$ is the sum of the n_i for which α_i is noncompact. This is certainly an integer.

We shall prove by induction on the level $\sum_{i=1}^m n_i$ that $\langle \omega, \alpha \rangle$ is even if α is compact, odd if α is noncompact. When the level is 1, this assertion is true by definition. In the general case, let α and β be in Δ_0^+ with $\alpha + \beta$ in Δ, and suppose that the assertion is true for α and β. Since the sum of the n_i for which α_i is noncompact is additive, we are to prove that imaginary roots satisfy

$$\begin{aligned} \text{compact} + \text{compact} &= \text{compact} \\ \text{compact} + \text{noncompact} &= \text{noncompact} \\ \text{noncompact} + \text{noncompact} &= \text{compact}. \end{aligned} \tag{6.99}$$

But this is immediate from Corollary 2.35 and the bracket relations (6.24).

PROOF OF THEOREM 6.96. Define V, Δ_0, and Λ as in Lemma 6.98. Before we use Lemma 6.97, it is necessary to observe that the Dynkin diagram of Δ_0 is connected, i.e., that the roots in the Dynkin diagram of Δ fixed by the given automorphism form a connected set. There is no problem when the automorphism is the identity, and we observe the connectedness in the other cases one at a time by inspection.

Let $\Delta_0^+ = \Delta^+ \cap V$. The set Λ is discrete, being a subset of a lattice, and Lemma 6.98 has just shown that it is nonempty. Let H_0 be a member of Λ with norm as small as possible. By Proposition 2.67 we can choose a new positive system $\Delta_0^{+\prime}$ for Δ_0 that makes H_0 dominant. The main step is to show that

(6.100) at most one simple root of $\Delta_0^{+\prime}$ is painted.

Let us summarize our results.

Theorem 6.105 (classification). Up to isomorphism every simple real Lie algebra is in the following list, and everything in the list is a simple real Lie algebra:

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is complex simple of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, or G_2

(b) the compact real form of any $\mathfrak{g}$ as in (a)

(c) the classical matrix algebras

$\mathfrak{su}(p,q)$	with	$p \geq q > 0,\ p+q \geq 2$
$\mathfrak{so}(p,q)$	with	$p > q > 0,\ p+q$ odd, $p+q \geq 5$
	or with	$p \geq q > 0,\ p+q$ even, $p+q \geq 8$
$\mathfrak{sp}(p,q)$	with	$p \geq q > 0,\ p+q \geq 3$
$\mathfrak{sp}(n,\mathbb{R})$	with	$n \geq 3$
$\mathfrak{so}^*(2n)$	with	$n \geq 4$
$\mathfrak{sl}(n,\mathbb{R})$	with	$n \geq 3$
$\mathfrak{sl}(n,\mathbb{H})$	with	$n \geq 2$

(d) the 12 exceptional noncomplex noncompact simple Lie algebras given in Figures 6.2 and 6.3.

The only isomorphism among Lie algebras in the above list is $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.

REMARKS. The restrictions on rank in (a) prevent coincidences in Dynkin diagrams. These restrictions are maintained in (b) and (c) for the same reason. Note for $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{sl}(n,\mathbb{H})$ that the restrictions on n force the automorphism to be nontrivial. In (c) there are no isomorphisms within a series because the $\mathfrak{k}_0$'s are different. To have an isomorphism between members of two series, we need at least two series with the same Dynkin diagram and automorphism. Then we examine the possibilities and are led to compare $\mathfrak{so}^*(8)$ with $\mathfrak{so}(6,2)$. The standard Vogan diagrams for these two Lie algebras are isomorphic, and hence the Lie algebras are isomorphic by Theorem 6.74.

11. Restricted Roots in the Classification

Additional information about the simple real Lie algebras of §10 comes by switching from a maximally compact Cartan subalgebra to a maximally noncompact Cartan subalgebra. The switch exposes the system of restricted roots, which governs the Iwasawa decomposition and some further structure theory that will be developed in Chapter VII.

According to §7 the switch in Cartan subalgebra is best carried out when we can find a maximal strongly orthogonal sequence of noncompact imaginary roots such that, after application of the Cayley transforms, no noncompact imaginary roots remain. If $\mathfrak{g}_0$ is a noncomplex simple real Lie algebra and if we have a Vogan diagram for $\mathfrak{g}_0$ as in Theorem 6.96, such a sequence is readily at hand by an inductive construction. We start with a noncompact imaginary simple root, form the set of roots orthogonal to it, label their compactness or noncompactness by means of Proposition 6.72, and iterate the process.

EXAMPLE. Let $\mathfrak{g}_0 = \mathfrak{su}(p, n-p)$ with $p \le n-p$. The distinguished Vogan diagram is of type A_{n-1} with $e_p - e_{p+1}$ as the unique noncompact imaginary simple root. Since the Dynkin diagram does not have a double line, orthogonality implies strong orthogonality. The above process yields the sequence of noncompact imaginary roots

$$\begin{aligned} 2f_p &= e_p - e_{p+1} \\ 2f_{p-1} &= e_{p-1} - e_{p+2} \\ &\vdots \\ 2f_1 &= e_1 - e_{2p}. \end{aligned} \tag{6.106}$$

We do a Cayley transform with respect to each of these. The order is irrelevant; since the roots are strongly orthogonal, the individual Cayley transforms commute. It is helpful to use the same names for roots before and after Cayley transform but always to remember what Cartan subalgebra is being used. After Cayley transform the remaining imaginary roots are those roots involving only indices $2p+1, \dots, n$, and such roots are compact. Thus a maximally noncompact Cartan subalgebra has noncompact dimension p. The restricted roots are obtained by projecting all $e_k - e_l$ on the linear span of (6.106). If $1 \le k < l \le p$, we have

$$\begin{aligned} e_k - e_l &= \tfrac{1}{2}(e_k - e_{2p+1-k}) - \tfrac{1}{2}(e_l - e_{2p+1-l}) + (\text{orthogonal to (6.106)}) \\ &= (f_k - f_l) + (\text{orthogonal to (6.106)}). \end{aligned}$$

Thus $f_k - f_l$ is a restricted root. For the same k and l, $e_k - e_{2p+1-l}$ restricts to $f_k + f_l$. In addition, if $k + l = 2p + 1$, then $e_k - e_l$ restricts to $2f_k$, while if $k \le p$ and $l > 2p$, then $e_k - e_l$ restricts to f_k. Consequently the set of restricted roots is

$$\Sigma = \begin{cases} \{\pm f_k \pm f_l\} \cup \{\pm 2f_k\} \cup \{\pm f_k\} & \text{if } 2p < n \\ \{\pm f_k \pm f_l\} \cup \{\pm 2f_k\} & \text{if } 2p = n. \end{cases}$$

Thus Σ is of type $(BC)_p$ if $2p < n$ and of type C_p if $2p = n$.