Since $x_{2j-1} + ix_{2j}$ and $x_{2j-1} - ix_{2j}$ together generate x_{2j-1} and x_{2j} and since $\varphi(H)$ acts as 0 on x_{2n+1}^k, this equation tells us how to compute $\varphi(H)$ on any monomial, hence on any polynomial.

It is clear that the subspace of polynomials homogeneous of degree N is an invariant subspace under the representation. This invariant subspace is spanned by the weight vectors

$$(x_1 + ix_2)^{k_1}(x_1 - ix_2)^{l_1}(x_3 + ix_4)^{k_2} \cdots (x_{2n-1} - ix_{2n})^{l_n} x_{2n+1}^{k_0},$$

where $\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = N$. Hence the weights of the subspace are all expressions $\sum_{j=1}^n (l_j - k_j)e_j$ with $\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = N$.

2) Let $V = \bigwedge^l \mathbb{C}^{2n+1}$. The element H_1 of $\mathfrak{h}$ in the above example acts on $\varepsilon_1 + i\varepsilon_2$ by the scalar $+i$ and on $\varepsilon_1 - i\varepsilon_2$ by the scalar $-i$. Thus $\varepsilon_1 + i\varepsilon_2$ and $\varepsilon_1 - i\varepsilon_2$ are weight vectors in $\mathbb{C}^{2n+1}$ of respective weights $-e_1$ and $+e_1$. Also ε_{2n+1} has weight 0. Then the product rule for differentiation allows us to compute the weights in $\bigwedge^l \mathbb{C}^{2n+1}$ and find that they are all expressions

$$\pm e_{j_1} \pm \cdots \pm e_{j_r}$$

with

$$j_1 < \cdots < j_r \qquad \text{and} \qquad \begin{cases} r \le l & \text{if } l \le n \\ r \le 2n+1-l & \text{if } l > n. \end{cases}$$

Motivated by Proposition 4.59 for compact Lie groups, we say that a member λ of $\mathfrak{h}^*$ is **algebraically integral** if $2\langle \lambda, \alpha\rangle / |\alpha|^2$ is in $\mathbb{Z}$ for each $\alpha \in \Delta$.

Proposition 5.4. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the roots, and let $\mathfrak{h}_0 = \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$. If φ is a representation of $\mathfrak{g}$ on the finite-dimensional complex vector space V, then

(a) $\varphi(\mathfrak{h})$ acts diagonably on V, so that every generalized weight vector is a weight vector and V is the direct sum of all the weight spaces

(b) every weight is real-valued on $\mathfrak{h}_0$ and is algebraically integral

(c) roots and weights are related by $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.

PROOF.

(a, b) If α is a root and E_α and $E_{-\alpha}$ are nonzero root vectors for α and $-\alpha$, then $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ spans a subalgebra $\mathfrak{sl}_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, with $2|\alpha|^{-2}H_\alpha$ corresponding to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the restriction of φ to $\mathfrak{sl}_\alpha$ is a finite-dimensional representation of $\mathfrak{sl}_\alpha$, and Corollary 1.69

is a weight vector with weight $\mu = \lambda - q_1\beta_1 - \cdots - q_k\beta_k$, from which (c) follows. The number of expressions (5.12) leading to this μ is finite, and so $\dim M_\mu < \infty$. The number of expressions (5.12) leading to λ is 1, from v itself, and so $\dim M_\lambda = 1$.

Before defining Verma modules, we recall some facts about tensor products of associative algebras. (A special case has already been treated in §I.3.) Let M_1 and M_2 be complex vector spaces, and let A and B be complex associative algebras with identity. Suppose that M_1 is a right B module and M_2 is a left B module, and suppose that M_1 is also a left A module in such a way that $(am_1)b = a(m_1b)$. We define

$$M_1 \otimes_B M_2 = \frac{M_1 \otimes_{\mathbb{C}} M_2}{\text{subspace generated by all } m_1b \otimes m_2 - m_1 \otimes bm_2},$$

and we let A act on the quotient by $a(m_1 \otimes m_2) = (am_1) \otimes m_2$. Then $M_1 \otimes_B M_2$ is a left A module, and it has the following universal mapping property: Whenever $\psi : M_1 \times M_2 \to E$ is a bilinear map into a complex vector space E such that $\psi(m_1b, m_2) = \psi(m_1, bm_2)$, then there exists a unique linear map $\tilde{\psi} : M_1 \otimes_B M_2 \to E$ such that $\psi(m_1, m_2) = \tilde{\psi}(m_1 \otimes m_2)$.

Now let λ be in $\mathfrak{h}^*$, and make $\mathbb{C}$ into a left $U(\mathfrak{b})$ module $\mathbb{C}_{\lambda-\delta}$ by defining

$$\begin{aligned} Hz &= (\lambda - \delta)(H)z \qquad && \text{for } H \in \mathfrak{h},\ z \in \mathbb{C} \\ Xz &= 0 && \text{for } X \in \mathfrak{n}. \end{aligned} \tag{5.13}$$

(Equation (5.13) defines a 1-dimensional representation of $\mathfrak{b}$, and thus $\mathbb{C}_{\lambda-\delta}$ becomes a left $U(\mathfrak{b})$ module.) The algebra $U(\mathfrak{g})$ itself is a right $U(\mathfrak{b})$ module and a left $U(\mathfrak{g})$ module under multiplication, and we define the **Verma module** $V(\lambda)$ to be the left $U(\mathfrak{g})$ module

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\delta}.$$

Proposition 5.14. Let λ be in $\mathfrak{h}^*$.

(a) $V(\lambda)$ is a highest weight module under $U(\mathfrak{g})$ and is generated by $1 \otimes 1$ (the **canonical generator**), which is of weight $\lambda - \delta$.

(b) The map of $U(\mathfrak{n}^-)$ into $V(\lambda)$ given by $u \mapsto u(1 \otimes 1)$ is one-one onto.

(c) If M is any highest weight module under $U(\mathfrak{g})$ generated by a highest weight vector $v \neq 0$ of weight $\lambda - \delta$, then there exists one and only one $U(\mathfrak{g})$ homomorphism $\tilde{\psi}$ of $V(\lambda)$ into M such that $\tilde{\psi}(1 \otimes 1) = v$. The map $\tilde{\psi}$ is onto. Also $\tilde{\psi}$ is one-one if and only if $u \neq 0$ in $U(\mathfrak{n}^-)$ implies $u(v) \neq 0$ in M.

Proposition 5.15. Let λ be in $\mathfrak{h}^*$, and let $V(\lambda)_+ = \bigoplus_{\mu \neq \lambda - \delta} V(\lambda)_\mu$. Then every proper $U(\mathfrak{g})$ submodule of $V(\lambda)$ is contained in $V(\lambda)_+$. Consequently the sum S of all proper $U(\mathfrak{g})$ submodules is a proper $U(\mathfrak{g})$ submodule, and $L(\lambda) = V(\lambda)/S$ is an irreducible $U(\mathfrak{g})$ module. Moreover, $L(\lambda)$ is a highest weight module with highest weight $\lambda - \delta$.

PROOF. If N is a $U(\mathfrak{h})$ submodule, then $N = \bigoplus_\mu (N \cap V(\lambda)_\mu)$. Since $V(\lambda)_{\lambda-\delta}$ is 1-dimensional and generates $V(\lambda)$ (by Proposition 5.14a), the $\lambda - \delta$ term must be 0 in the sum for N if N is proper. Thus $N \subseteq V(\lambda)_+$. Hence S is proper, and $L(\lambda) = V(\lambda)/S$ is irreducible. The image of $1 \otimes 1$ in $L(\lambda)$ is not 0, is annihilated by $\mathfrak{n}$, and is acted upon by $\mathfrak{h}$ according to $\lambda - \delta$. Thus $L(\lambda)$ has all the required properties.

Theorem 5.16. Suppose that $\lambda \in \mathfrak{h}^*$ is real-valued on $\mathfrak{h}_0$ and is dominant and algebraically integral. Then the irreducible highest weight module $L(\lambda + \delta)$ is an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ.

REMARKS. Theorem 5.16 will complete the proof of the Theorem of the Highest Weight (Theorem 5.5). The proof of Theorem 5.16 will be preceded by two lemmas.

Lemma 5.17. In $U(\mathfrak{sl}(2, \mathbb{C}))$, $[e, f^n] = nf^{n-1}(h - (n-1))$.

PROOF. Let

$$\begin{aligned} Lf &= \text{left by } f \text{ in } U(\mathfrak{sl}(2,\mathbb{C})) \\ Rf &= \text{right by } f \\ \operatorname{ad} f &= Lf - Rf. \end{aligned}$$

Then $Rf = Lf - \operatorname{ad} f$, and the terms on the right commute. By the binomial theorem,

$$\begin{aligned} (Rf)^n e &= \sum_{j=0}^{n} \binom{n}{j} (Lf)^{n-j} (-\operatorname{ad} f)^j e \\ &= (Lf)^n e + n(Lf)^{n-1}(-\operatorname{ad} f)e + \frac{n(n-1)}{2} (Lf)^{n-2}(-\operatorname{ad} f)^2 e \end{aligned}$$

since $(\operatorname{ad} f)^3 e = 0$, and this expression is

$$\begin{aligned} &= (Lf)^n e + nf^{n-1}h + \frac{n(n-1)}{2} f^{n-2}(-2f) \\ &= (Lf)^n e + nf^{n-1}(h - (n-1)). \end{aligned}$$

Thus

$$[e, f^n] = (Rf)^n e - (Lf)^n e = nf^{n-1}(h - (n-1)).$$

In fact, if T is a finite-dimensional $U(\mathfrak{sl}_\alpha)$ submodule, then

$$\mathfrak{g}T = \left\{ \sum Xt \mid X \in \mathfrak{g} \text{ and } t \in T \right\}$$

is finite-dimensional and for $Y \in \mathfrak{sl}_\alpha$ and $X \in \mathfrak{g}$ we have

$$YXt = XYt + [Y, X]t = Xt' + [Y, X]t \in \mathfrak{g}T.$$

So $\mathfrak{g}T$ is $\mathfrak{sl}_\alpha$ stable, and the claim follows.

Since the sum of all finite-dimensional $U(\mathfrak{sl}_\alpha)$ submodules of $L(\lambda+\delta)$ is $\mathfrak{g}$ stable, the irreducibility of $L(\lambda + \delta)$ implies that this sum is all of $L(\lambda + \delta)$. By Corollary 1.70, $L(\lambda + \delta)$ is the direct sum of finite-dimensional irreducible $U(\mathfrak{sl}_\alpha)$ submodules.

Let μ be a weight, and let $t \neq 0$ be in V_μ. We have just shown that t lies in a finite direct sum of finite-dimensional irreducible $U(\mathfrak{sl}_\alpha)$ submodules. Let us write $t = \sum_{i \in I} t_i$ with t_i in a $U(\mathfrak{sl}_\alpha)$ submodule T_i and $t_i \neq 0$. Then

$$\sum H_\alpha t_i = H_\alpha t = \mu(H_\alpha)t = \sum \mu(H_\alpha)t_i,$$

and so
$$\frac{2H_\alpha}{|\alpha|^2}\, t_i = \frac{2\langle \mu, \alpha\rangle}{|\alpha|^2}\, t_i \qquad \text{for each } i \in I.$$

If $\langle \mu, \alpha \rangle > 0$, we know that $(E_{-\alpha})^{2\langle\mu,\alpha\rangle/|\alpha|^2} t_i \neq 0$ from Theorem 1.63. Hence $(E_{-\alpha})^{2\langle\mu,\alpha\rangle/|\alpha|^2} t \neq 0$, and we see that

$$\mu - \frac{2\langle \mu, \alpha\rangle}{|\alpha|^2}\, \alpha = s_\alpha \mu$$

is a weight. If $\langle \mu, \alpha \rangle < 0$ instead, we know that $(E_\alpha)^{-2\langle\mu,\alpha\rangle/|\alpha|^2} t_i \neq 0$ from Theorem 1.63. Hence $(E_\alpha)^{-2\langle\mu,\alpha\rangle/|\alpha|^2} t \neq 0$, and so

$$\mu - \frac{2\langle \mu, \alpha\rangle}{|\alpha|^2}\alpha = s_\alpha \mu$$

is a weight. If $\langle \mu, \alpha \rangle = 0$, then $s_\alpha \mu = \mu$. In any case $s_\alpha \mu$ is a weight. So the set of weights is stable under each reflection s_α for α simple, and Proposition 2.62 shows that the set of weights is stable under W.

Third we show: The set of weights of $L(\lambda + \delta)$ is finite, and $L(\lambda + \delta)$ is finite-dimensional. In fact, Corollary 2.68 shows that any linear functional on $\mathfrak{h}_0$ is W conjugate to a dominant one. Since the

Theorem 5.75 (Weyl Character Formula). Let V be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ. Then

$$\operatorname{char}(V) = d^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\delta)}.$$

REMARKS. We shall prove this theorem below after giving three lemmas. But first we deduce an alternative formulation of the theorem.

Corollary 5.76 (Weyl Denominator Formula).

$$e^{\delta} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} \varepsilon(w) e^{w\delta}.$$

PROOF. Take $\lambda = 0$ in Theorem 5.75. Then V is the 1-dimensional trivial representation, and $\operatorname{char}(V) = e^0 = 1$.

Theorem 5.77 (Weyl Character Formula, alternative formulation). Let V be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ. Then

$$\Big(\sum_{w \in W} \varepsilon(w) e^{w\delta}\Big) \operatorname{char}(V) = \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\delta)}.$$

PROOF. This follows by substituting the result of Corollary 5.76 into the formula of Theorem 5.75.

Lemma 5.78. If λ in $\mathfrak{h}^*$ is dominant, then no $w \neq 1$ in W fixes $\lambda + \delta$.

PROOF. If $w \neq 1$ fixes $\lambda + \delta$, then Chevalley's Lemma in the form of Corollary 2.73 shows that some root α has $\langle \lambda + \delta, \alpha \rangle = 0$. We may assume that α is positive. But then $\langle \lambda, \alpha \rangle \geq 0$ by dominance and $\langle \delta, \alpha \rangle > 0$ by Proposition 2.69, and we have a contradiction.

Lemma 5.79. The Verma module $V(\lambda)$ has a character belonging to $\mathbb{Z}\langle \mathfrak{h}^* \rangle$, and $\operatorname{char}(V(\lambda)) = d^{-1} e^{\lambda}$.

PROOF. Formula (5.67) shows that

$$\operatorname{char}(V(\lambda)) = e^{\lambda-\delta} \sum_{\gamma \in Q^+} \mathcal{P}(\gamma) e^{-\gamma} = K e^{-\delta} e^{\lambda},$$

and thus the result follows by substituting from Lemma 5.72.

Theorem 5.113 (Weyl Character Formula). Let G be a compact connected Lie group, let T be a maximal torus, let $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ be a positive system for the roots, and let $\lambda \in \mathfrak{t}^*$ be analytically integral and dominant. Then the character χ_{Φ_λ} of the irreducible finite-dimensional representation Φ_λ of G with highest weight λ is given by

$$\chi_{\Phi_\lambda}(t) = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)-\delta}(t)}{\prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t))}$$

at every $t \in T$ where no ξ_α takes the value 1 on t. If G is simply connected, then this formula can be rewritten as

$$\chi_{\Phi_\lambda}(t) = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)}(t)}{\xi_\delta(t) \prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t))} = \frac{\sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta)}(t)}{\sum_{w \in W} \varepsilon(w) \xi_{w\delta}(t)}.$$

REMARK. Theorem 4.36 says that every member of G is conjugate to a member of T. Since characters are constant on conjugacy classes, the above formulas determine the characters everywhere on G.

PROOF. Theorem 5.110 shows that Φ_λ exists when λ is analytically integral and dominant. We apply Theorem 5.75 in the form of (5.111). When we divide (5.111) by e^δ, Lemma 5.112 says that all the exponentials yield well defined functions on T. The first formula follows. If G is simply connected, then G is semisimple as a consequence of Proposition 1.99. The linear functional δ is algebraically integral by Proposition 2.69, hence analytically integral by Theorem 5.107. Thus we can regroup the formula as indicated. The version of the formula with an alternating sum in the denominator uses Theorem 5.77 in place of Theorem 5.75.

Finally we discuss how parabolic subalgebras play a role in the representation theory of compact Lie groups. With G and T given, fix a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ for the roots, define $\mathfrak{n}$ as in (5.8), and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Corollary 5.101 shows that $\mathfrak{l} = Z_{\mathfrak{g}}(H_{\delta(\mathfrak{u})})$, and we can equally well write $\mathfrak{l} = Z_{\mathfrak{g}}(iH_{\delta(\mathfrak{u})})$. Since $iH_{\delta(\mathfrak{u})}$ is in $\mathfrak{t}_0 \subseteq \mathfrak{g}_0$, $\mathfrak{l}$ is the complexification of the subalgebra

$$\mathfrak{l}_0 = Z_{\mathfrak{g}_0}(iH_{\delta(\mathfrak{u})})$$

of $\mathfrak{g}_0$. Define

$$L = Z_G(iH_{\delta(\mathfrak{u})}).$$

This is a compact subgroup of G containing T. Since the closure of $\exp i\mathbb{R}H_{\delta(\mathfrak{u})}$ is a torus in G, L is the centralizer of a torus in G and is connected by Corollary 4.51. Thus we have an inclusion of compact connected Lie groups $T \subseteq L \subseteq G$, and T is a maximal torus in both L and G. Hence analytic integrality is the same for L as for G. Combining Theorems 5.104 and 5.110, we obtain the following result.

Problems 36–41 concern fundamental representations. Let $\alpha_1, \dots, \alpha_l$ be the simple roots, and define $\varpi_1, \dots, \varpi_l$ by $2\langle \varpi_i, \alpha_j \rangle / |\alpha_j|^2 = \delta_{ij}$. The dominant algebraically integral linear functionals are then all expressions $\sum_i n_i \varpi_i$ with all n_i integers ≥ 0. We call ϖ_i the **fundamental weight** attached to the simple root α_i, and the corresponding irreducible representation is called the **fundamental representation** attached to that simple root.

36. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.
    (a) Verify that the fundamental weights are $\sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-1$.
    (b) Using Problem 7, verify that the fundamental representations are the usual alternating-tensor representations.

37. Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Let $\alpha_i = e_i - e_{i+1}$ for $i < n$, and let $\alpha_i = e_n$.
    (a) Verify that the fundamental weights are $\varpi_l = \sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-1$ and $\varpi_n = \frac{1}{2} \sum_{k=1}^{n} e_k$.
    (b) Using Problem 8, verify that the fundamental representations attached to simple roots other than the last one are alternating-tensor representations.
    (c) Using Problem 35, verify that the fundamental representation attached to the last simple root is the spin representation.

38. Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Let $\alpha_i = e_i - e_{i+1}$ for $i < n-1$, and let $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$.
    (a) Verify that the fundamental weights are $\varpi_l = \sum_{k=1}^{l} e_k$ for $1 \leq l \leq n-2$, $\varpi_{n-1} = \frac{1}{2} \sum_{k=1}^{n} e_k$, and $\varpi_n = \frac{1}{2} \big(\sum_{k=1}^{n-1} e_k - e_n \big)$.
    (b) Using Problem 9, verify that the fundamental representations attached to simple roots other than the last two are alternating-tensor representations.
    (c) Using Problem 34, verify that the fundamental representations attached to the last two simple roots are the spin representations.

39. Let λ and λ' be dominant algebraically integral, and suppose that $\lambda - \lambda'$ is dominant and nonzero. Prove that the dimension of an irreducible representation with highest weight λ is greater than the dimension of an irreducible representation with highest weight λ'.

40. Given $\mathfrak{g}$, prove for each integer N that there are only finitely many irreducible representations of $\mathfrak{g}$, up to equivalence, of dimension $\leq N$.

41. Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2.
    (a) Using Problem 42 in Chapter II, construct a 7-dimensional nonzero representation of $\mathfrak{g}$.
    (b) Let α_1 be the long simple root, and let α_2 be the short simple root. Verify that $\varpi_1 = 2\alpha_1 + 3\alpha_2$ and that $\varpi_2 = \alpha_1 + 2\alpha_2$.