A section called Notes near the end of the book provides historical commentary, gives bibliographical citations, tells about additional results, and serves as a guide to further reading.

The main prerequisite for reading this book is a familiarity with elementary Lie theory, as in Chapter IV of Chevalley [1946] or other sources listed at the end of the Notes for Chapter I. This theory itself requires a modest amount of linear algebra and group theory, some point-set topology, the theory of covering spaces, the theory of smooth manifolds, and some easy facts about topological groups. Except in the case of the theory of involutive distributions, the treatments of this other material in many recent books are more consistent with the present book than is Chevalley's treatment. A little Lebesgue integration plays a role in Chapter IV. In addition, existence and uniqueness of Haar measure on compact Lie groups are needed for Chapter IV; one can take these results on faith or one can know them from differential geometry or from integration theory. Differential forms and more extensive integration theory are used in Chapter VIII. Occasionally some other isolated result from algebra or analysis is needed; references are given in such cases.

Individual chapters in the book usually depend on only some of the earlier chapters. Details of this dependence are given on page xiv.

My own introduction to this subject came from courses by B. Kostant and S. Helgason at M.I.T. in 1965–67, and parts of those courses have heavily influenced parts of the book. Most of the book is based on various courses I taught at Cornell University or SUNY Stony Brook between 1971 and 1995. I am indebted to R. Donley, J. J. Duistermaat, S. Greenleaf, S. Helgason, D. Vogan, and A. Weinstein for help with various aspects of the book and to the Institut Mittag-Leffler for its hospitality during the last period in which the book was written. The typesetting was by AMS-TEX, and the figures were drawn with Mathematica®.

May 1996

In the reprinted version a number of errors have been corrected. All but one of these are fairly minor. The one serious error was an incorrect and misleading example in §VIII.5. I am indebted to P. Batra, R. Donley, P. Friedman, and D. Vogan for telling me of the errors that they had found in the original version.

September 2000

PREREQUISITES BY CHAPTER

This book assumes knowledge of a modest amount of linear algebra and group theory, some point-set topology, the theory of covering spaces, the theory of smooth manifolds, and some easy facts about topological groups. The main prerequisite is a familiarity with elementary Lie theory, as in Chapter IV of Chevalley [1946]. The dependence of chapters on earlier chapters, as well as additional prerequisites for particular chapters, are listed here.

CHAPTER I. Tensor products of vector spaces (cf. §1 of Appendix A).

CHAPTER II. Chapter I. Starting in §9: The proof of Proposition 2.96 is deferred to Chapter III, where the result is restated and proved as Proposition 3.29. Starting in §11: Tensor algebra as in §1 of Appendix A.

CHAPTER III. Chapter I, all of Appendix A.

CHAPTER IV. Chapter I, tensor and exterior algebras as in §§1-3 of Appendix A, a small amount of Lebesgue integration, existence of Haar measure for compact groups. The proof of Theorem 4.20 uses the Hilbert-Schmidt Theorem from functional analysis. Starting in §5: Chapter II.

CHAPTER V. Chapters II, III, and IV. The proof of Theorem 5.62 uses the Hilbert Nullstellensatz.

CHAPTER VI. Chapters II and IV.

CHAPTER VII. Chapter VI. Starting in §5: Chapter V. Starting in §8: complex manifolds (apart from complex Lie groups).

CHAPTER VIII. Chapter VII, differential forms, more Lebesgue integration.

APPENDIX B. Chapters I and V.

for all X, Y, and Z in $\mathfrak{g}$. An alternative way of writing (1.19a) is

$$B([X, Y], Z) = B(X, [Y, Z]). \tag{1.19b}$$

Equation (1.19) is straightforward to verify; we simply expand both sides and use the fact that $\operatorname{Tr}(LM) = \operatorname{Tr}(ML)$.

EXAMPLES.

1) Let $\mathfrak{g}$ be 2-dimensional nonabelian as in Example 11b of §1. Then $\mathfrak{g}$ has a basis $\{X, Y\}$ with $[X, Y] = Y$. To understand the Killing form B, it is enough to know what B is on every pair of basis vectors. Thus we have to compute the traces of ad X ad X, ad X ad Y, and ad Y ad Y. The matrix of ad X ad X in the basis $\{X, Y\}$ is

$$\operatorname{ad} X \operatorname{ad} X = \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ X \\ Y \end{array}$$

and hence $B(X, X) = 1$. Calculating $B(X, Y)$ and $B(Y, Y)$ similarly, we see that B is given by the matrix

$$B = \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ X \\ Y \end{array} .$$

2) Let $\mathfrak{g} = \mathfrak{sl}(2, \Bbbk)$ with basis $\{h, e, f\}$ as in (1.5) and bracket relations as in (1.6). Computing as in the previous example, we see that the matrix of B in this basis is

$$B = \begin{array}{c} \begin{array}{ccc} h & e & f \end{array} \\ \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ h \\ e \\ f \end{array} .$$

Returning to a general finite-dimensional Lie algebra $\mathfrak{g}$ over $\Bbbk$, let us extend the scalars from $\Bbbk$ to $\mathbb{K}$, forming the Lie algebra $\mathfrak{g}^{\mathbb{K}}$. Let $B^{\mathbb{K}}$ be the Killing form of $\mathfrak{g}^{\mathbb{K}}$. If we fix a basis of $\mathfrak{g}$ over $\Bbbk$, then that same set is a basis of $\mathfrak{g}^{\mathbb{K}}$ over $\mathbb{K}$. Consequently if X and Y are in $\mathfrak{g}$, the matrix of ad X ad Y is the same for $\mathfrak{g}$ as it is for $\mathfrak{g}^{\mathbb{K}}$, and it follows that

$$B^{\mathbb{K}}|_{\mathfrak{g}\times\mathfrak{g}} = B. \tag{1.20}$$

When we shrink the scalars from $\mathbb{K}$ to $\Bbbk$, passing from a Lie algebra $\mathfrak{h}$ over $\mathbb{K}$ to the Lie algebra $\mathfrak{h}^{\Bbbk}$, the dimension is not preserved. In fact, the $\Bbbk$ dimension of $\mathfrak{h}^{\Bbbk}$ is the product of the degree of $\mathbb{K}$ over $\Bbbk$ and the $\mathbb{K}$ dimension of $\mathfrak{h}$. Thus the Killing forms of $\mathfrak{h}$ and $\mathfrak{h}^{\Bbbk}$ are not related so simply. We shall be interested in this relationship only in the special case that $\Bbbk = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, and we return to it in §8.

Before carrying out the induction indicated in Remark 2, we observe something about eigenvalues in connection with representations. Let π be a representation of $\mathfrak{g}$ on a finite-dimensional V, and let $U \subseteq V$ be an **invariant subspace**: $\pi(\mathfrak{g})U \subseteq U$. Then $\pi(X)(v+U) = \pi(X)v + U$ defines a **quotient representation** of $\mathfrak{g}$ on V/U. The characteristic polynomial of $\pi(X)$ on V is the product of the characteristic polynomial on U and that on V/U, and hence the eigenvalues for V/U are a subset of those for V.

Corollary 1.29 (Lie's Theorem). Under the assumptions on $\mathfrak{g}$, V, π, and $\mathbb{K}$ as in Theorem 1.25, there exists a sequence of subspaces

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_m = 0$$

such that each V_i is stable under $\pi(\mathfrak{g})$ and $\dim V_i/V_{i+1} = 1$. Consequently V has a basis with respect to which all the matrices of $\pi(\mathfrak{g})$ are upper triangular.

REMARK. The sequence of subspaces in the corollary is called an **invariant flag** of V.

PROOF. We induct on $\dim V$, the case $\dim V = 1$ being trivial. If V is given, find by Theorem 1.25 an eigenvector $v \neq 0$ for $\pi(\mathfrak{g})$, and put $U = \mathbb{k}v$. Then U is an invariant subspace, and π provides a quotient representation on V/U, where $\dim(V/U) < \dim V$. Find by inductive hypothesis an invariant flag for V/U, say

$$V/U = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_{m-1} = 0,$$

and put $V_i = \sigma^{-1}(W_i)$, where $\sigma : V \to V/U$ is the quotient map (which commutes with all $\pi(X)$ by definition). Taking $V_m = 0$, we have

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{m-1} \supseteq V_m = 0$$

as the required sequence.

A solvable Lie algebra $\mathfrak{g}$ is said to be **split-solvable** if there is an elementary sequence

$$\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n = 0$$

in which each $\mathfrak{a}_i$ is an ideal in $\mathfrak{g}$ (rather than just in $\mathfrak{a}_{i-1}$). Notice that a subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal if and only if $\mathfrak{a}$ is stable under $\operatorname{ad}\mathfrak{g}$. Thus in the terminology above, $\mathfrak{g}$ is split-solvable if and only if there is an invariant flag for the adjoint representation.

Let $\mathfrak{s} = \mathfrak{h} + \Bbbk X_0$. Then (1.37) shows that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{h}$, and hence $\mathfrak{s} = \mathfrak{g}$ by maximality of $\dim \mathfrak{h}$. Consequently $\mathfrak{h}$ has codimension 1 in $\mathfrak{g}$. Also (1.37) shows that $\mathfrak{h}$ is an ideal.

To complete the proof, let $V_0 = \{v \in V \mid Hv = 0 \text{ for all } H \in \mathfrak{h}\}$. Since $\mathfrak{h}$ acts as nilpotent endomorphisms, the inductive hypothesis shows that V_0 is not 0. If v is in V_0, then

$$HX_0v = [H, X_0]v + X_0Hv = 0 + 0 = 0$$

since $\mathfrak{h}$ is an ideal. Thus $X_0(V_0) \subseteq V_0$. By assumption X_0 is nilpotent, and thus 0 is its only eigenvalue. Hence X_0 has an eigenvector v_0 in V_0. Then $X_0(v_0) = 0$ and $\mathfrak{h}(v_0) = 0$, so that $\mathfrak{g}(v_0) = 0$. Consequently (b) holds for $\mathfrak{g}$, and the induction is complete.

Corollary 1.38. If $\mathfrak{g}$ is a Lie algebra such that each $\operatorname{ad} X$ for $X \in \mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is a nilpotent Lie algebra.

PROOF. Theorem 1.35 shows that $\operatorname{ad} \mathfrak{g}$ is nilpotent, and Proposition 1.32 allows us to conclude that $\mathfrak{g}$ is nilpotent.

7. Cartan's Criterion for Semisimplicity

In this section, $\Bbbk$ denotes a subfield of $\mathbb{C}$, and $\mathfrak{g}$ denotes a finite-dimensional Lie algebra over $\Bbbk$. We shall relate semisimplicity of $\mathfrak{g}$ to a nondegeneracy property of the Killing form of $\mathfrak{g}$, the Killing form having been defined in (1.18).

First we make some general remarks about bilinear forms. Let V be a finite-dimensional vector space, and let $C(\,\cdot\,,\,\cdot\,)$ be a bilinear form on $V \times V$. Define

$$\operatorname{rad} C = \{v \in V \mid C(v, u) = 0 \text{ for all } u \in V\}.$$

Writing $\langle\,\cdot\,,\,\cdot\,\rangle$ for the pairing of the dual V^* with V, define $\varphi : V \to V^*$ by $\langle \varphi(v), u\rangle = C(v, u)$. Then $\ker \varphi = \operatorname{rad} C$, and so φ is an isomorphism (onto) if and only if C is **nondegenerate** (i.e., $\operatorname{rad} C = 0$).

If U is a subspace of V, let

$$U^\perp = \{v \in V \mid C(v, u) = 0 \text{ for all } u \in U\}.$$

Then

$$U \cap U^\perp = \operatorname{rad}(C|_{U\times U}). \tag{1.39}$$

Even if C is nondegenerate, we may have $U \cap U^\perp \neq 0$. For example, take $\Bbbk = \mathbb{R}$, $V = \mathbb{R}^2$, $C(x, y) = x_1y_1 - x_2y_2$, and $U = \{(x_1, x_1)\}$; then C is nondegenerate, but $U = U^\perp \neq 0$. However, we can make the positive statement given in the following proposition.

First we define automorphisms. An **automorphism** of a Lie algebra is an invertible linear map L that preserves brackets: $[L(X), L(Y)] = L[X, Y]$. For example if $\mathfrak{g}$ is the (real) Lie algebra of a Lie group G and if g is in G, then $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$.

If $\mathfrak{g}$ is real, let $\mathrm{Aut}_{\mathbb{R}}\,\mathfrak{g} \subseteq GL_{\mathbb{R}}(\mathfrak{g})$ be the subgroup of $\mathbb{R}$ linear automorphisms of $\mathfrak{g}$. This is a closed subgroup of a general linear group, hence a Lie group. If $\mathfrak{g}$ is complex, we can regard

$$\mathrm{Aut}_{\mathbb{C}}\,\mathfrak{g} \subseteq GL_{\mathbb{C}}(\mathfrak{g}) \subseteq GL_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}}),$$

the subscript $\mathbb{C}$ referring to complex-linearity and $\mathfrak{g}^{\mathbb{R}}$ denoting the underlying real Lie algebra of $\mathfrak{g}$ as in §3. But also we have the option of regarding $\mathfrak{g}$ as the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ directly. Then we have

$$\mathrm{Aut}_{\mathbb{C}}\,\mathfrak{g} \subseteq \mathrm{Aut}_{\mathbb{R}}\,\mathfrak{g}^{\mathbb{R}} \subseteq GL_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}}).$$

Lemma 1.95. If a is an automorphism of $\mathfrak{g}$ and if X is in $\mathfrak{g}$, then $\mathrm{ad}(aX) = a(\mathrm{ad}\,X)a^{-1}$.

PROOF. We have $\mathrm{ad}(aX)Y = [aX, Y] = a[X, a^{-1}Y] = (a(\mathrm{ad}\,X)a^{-1})Y$.

Proposition 1.96. If B is the Killing form of $\mathfrak{g}$ and if a is an automorphism of $\mathfrak{g}$, then $B(aX, aY) = B(X, Y)$ for all X and Y in $\mathfrak{g}$.

PROOF. By Lemma 1.95 we have

$$\begin{aligned} B(aX, aY) &= \mathrm{Tr}(\mathrm{ad}(aX)\mathrm{ad}(aY)) \\ &= \mathrm{Tr}(a(\mathrm{ad}\,X)a^{-1}a(\mathrm{ad}\,Y)a^{-1}) \\ &= \mathrm{Tr}((\mathrm{ad}\,X)(\mathrm{ad}\,Y)) \\ &= B(X, Y), \end{aligned}$$

as required.

Next we recall that derivations of the Lie algebra $\mathfrak{g}$ were defined in (1.2). In §4 we introduced $\mathrm{Der}\,\mathfrak{g}$ as the Lie algebra of all derivations of $\mathfrak{g}$. If $\mathfrak{g}$ is real, then $\mathrm{Der}\,\mathfrak{g}$ has just one interpretation, namely the Lie subalgebra $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{g} \subseteq \mathrm{End}_{\mathbb{R}}\,\mathfrak{g}$. If $\mathfrak{g}$ is complex, then two interpretations are possible, namely as $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{g}^{\mathbb{R}} \subseteq \mathrm{End}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}})$ or as $\mathrm{Der}_{\mathbb{C}}\,\mathfrak{g} \subseteq \mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \subseteq \mathrm{End}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}})$.

Proposition 1.97. If $\mathfrak{g}$ is real, the Lie algebra of $\mathrm{Aut}_{\mathbb{R}}\,\mathfrak{g}$ is $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{g}$. If $\mathfrak{g}$ is complex, the Lie algebra of $\mathrm{Aut}_{\mathbb{C}}\,\mathfrak{g}$ is $\mathrm{Der}_{\mathbb{C}}\,\mathfrak{g}$. In either case the Lie algebra contains $\mathrm{ad}\,\mathfrak{g}$.

PROOF. For the direct part, H_1 and H_2 are closed and normal. Hence they are Lie subgroups, and their Lie algebras are ideals in $\mathfrak{g}$. The vector space direct sum relationship depends only on the product structure of the manifold G.

For the converse the inclusions of H_1 and H_2 into G give us a smooth homomorphism $H_1 \oplus H_2 \to G$. On the other hand, the isomorphism of $\mathfrak{g}$ with $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, in combination with the fact that G is connected and simply connected, gives us a homomorphism $G \to H_1 \oplus H_2$. The composition of the two group homomorphisms in either order has differential the identity and is therefore the identity homomorphism.

As in §4 the next step is to expand the theory of direct sums to a theory of semidirect products. Let G and H be Lie groups. We say that G **acts on** H **by automorphisms** if a smooth map $\tau : G \times H \to H$ is specified such that $g \mapsto \tau(g, \cdot)$ is a homomorphism of G into the abstract group of automorphisms of H. In this case the **semidirect product** $G \times_\tau H$ is the Lie group with $G \times H$ as its underlying manifold and with multiplication and inversion given by

$$\begin{aligned} (g_1, h_1)(g_2, h_2) &= (g_1 g_2, \tau(g_2^{-1}, h_1)h_2) \\ (g, h)^{-1} &= (g^{-1}, \tau(g, h^{-1})). \end{aligned} \tag{1.100}$$

(To understand the definition of multiplication, think of the formula as if it were written $g_1 h_1 g_2 h_2 = g_1 g_2 (g_2^{-1} h_1 g_2) h_2$.) A little checking shows that this multiplication is associative. Then $G \times_\tau H$ is a Lie group, G and H are closed subgroups, and H is normal.

EXAMPLE. Let $G = SO(n)$, $H = \mathbb{R}^n$, and $\tau(r, x) = r(x)$. Then $G \times_\tau H$ is the group of translations and rotations (with arbitrary center) in $\mathbb{R}^n$.

Let us compute the Lie algebra of a semidirect product $G \times_\tau H$. We consider the differential $\bar\tau(g)$ of $\tau(g, \cdot)$ at the identity of H. Then $\bar\tau(g)$ is a Lie algebra isomorphism of $\mathfrak{h}$. As with Ad in §10, we find that

$$\begin{gathered} \bar\tau \text{ is smooth into } GL(\mathfrak{h}) \\ \bar\tau(g_1 g_2) = \bar\tau(g_1)\bar\tau(g_2). \end{gathered}$$

Thus $\bar\tau$ is a smooth homomorphism of G into $\mathrm{Aut}_{\mathbb{R}}\mathfrak{h}$. Its differential $d\bar\tau$ is a homomorphism of $\mathfrak{g}$ into $\mathrm{Der}_{\mathbb{R}}\,\mathfrak{h}$, by Proposition 1.97, and Proposition 1.22 allows us to form the semidirect product of Lie algebras $\mathfrak{g} \oplus_{d\bar\tau} \mathfrak{h}$.

is the group of real matrices of size $m+n$ preserving the symmetric bilinear form on $\mathbb{R}^{m+n}\times\mathbb{R}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1y_1+\cdots+x_my_m-x_{m+1}y_{m+1}-\cdots-x_{m+n}y_{m+n},$$

and $SO(m,n)$ is the subgroup of members of $O(m,n)$ of determinant 1. The group

$$U(m,n)=\{g\in GL(m+n,\mathbb{C})\mid g^*I_{m,n}g=I_{m,n}\}$$

is the group of complex matrices of size $m+n$ preserving the Hermitian form on $\mathbb{C}^{m+n}\times\mathbb{C}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1\overline{y_1}+\cdots+x_m\overline{y_m}-x_{m+1}\overline{y_{m+1}}-\cdots-x_{m+n}\overline{y_{m+n}},$$

and $SU(m,n)$ is the subgroup of members of $U(m,n)$ of determinant 1. The group $Sp(m,n)$ is the group of quaternion matrices of size $m+n$ preserving the Hermitian form on $\mathbb{H}^{m+n}\times\mathbb{H}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1\overline{y_1}+\cdots+x_m\overline{y_m}-x_{m+1}\overline{y_{m+1}}-\cdots-x_{m+n}\overline{y_{m+n}},$$

with no condition needed on the determinant.

The linear Lie algebras of the closed linear groups in (1.120) are given in a table in Example 3 of §8, and the table in §8 tells which values of m and n lead to semisimple Lie algebras. It will be a consequence of results below that all the closed linear groups in (1.120) are topologically connected except for $SO(m,n)$. In the case of $SO(m,n)$, one often works with the identity component $SO(m,n)_0$ in order to have access to the full set of results about semisimple groups in later chapters.

Let us now address the subject of connectedness in detail. We shall work with a closed linear group of complex matrices that is closed under adjoint and is defined by polynomial equations. We begin with a lemma.

Lemma 1.121. Let $P:\mathbb{R}^n\to\mathbb{R}$ be a polynomial, and suppose $(a_1,\dots,a_n)$ has the property that $P(e^{ka_1},\dots,e^{ka_n})=0$ for all integers $k\geq 0$. Then $P(e^{ta_1},\dots,e^{ta_n})=0$ for all real t.

9. Show that the solvable Lie algebra $\mathfrak{g} = \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$ over $\mathbb{R}$ is not split solvable

    (a) by showing that $\mathfrak{g}$ has no 1-dimensional ideal.

    (b) by producing nonreal eigenvalues for some ad X with $X \in \mathfrak{g}$.

    Show also that $\mathfrak{g}^{\mathbb{C}}$ can be regarded as all complex matrices of the form $\mathfrak{g} = \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$, and exhibit a 1-dimensional ideal in $\mathfrak{g}^{\mathbb{C}}$ (which exists since $\mathfrak{g}^{\mathbb{C}}$ has to be split solvable over $\mathbb{C}$).

10. Prove for a finite-dimensional solvable Lie algebra over $\mathbb{R}$ that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

11. Prove that if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over $\mathbb{R}$, then the Killing form of $\mathfrak{g}$ is identically 0.

12. Let $\mathfrak{g}$ be a complex Lie algebra of complex matrices, and suppose that $\mathfrak{g}$ is simple over $\mathbb{C}$. Let $C(X, Y) = \operatorname{Tr}(XY)$ for X and Y in $\mathfrak{g}$. Prove that C is a multiple of the Killing form.

13. For $k = \mathbb{R}$, prove that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ are not isomorphic.

14. (a) Show that $\mathfrak{so}(3)$ is isomorphic with $\mathfrak{su}(2)$.

    (b) Prove that $\mathfrak{su}(2)$ is simple.

    (c) Prove that there exists a covering homomorphism of $SU(2)$ onto $SO(3)$ with 2-element kernel.

15. Prove that $\mathfrak{so}(2, 1)$ is isomorphic with $\mathfrak{sl}(2, \mathbb{R})$.

16. For $\mathfrak{u}(n)$, we have an isomorphism $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathbb{R}$, where $\mathbb{R}$ is the center. Let Z be the analytic subgroup of $U(n)$ with Lie algebra the center. Is $U(n)$ isomorphic with the direct sum of $SU(n)$ and Z? Why or why not?

17. Let V_n be the complex vector space of all polynomials in two complex variables z_1 and z_2 homogeneous of degree n. Define a representation of $SL(2, \mathbb{C})$ by

$$\Phi_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right).$$

    Then $\dim V_n = n + 1$, Φ is a homomorphism, and Φ is holomorphic. Let φ be the differential of Φ at 1. Prove that φ is isomorphic with the irreducible complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $n + 1$ given in Theorem 1.63.