Anthony W. Knapp

Lie Groups Beyond an Introduction

Second Edition

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To Susan

and

Sarah and Dave and Will

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PREFACE TO THE SECOND EDITION

The publication of a second edition is an opportunity to underscore that the subject of Lie groups is important both for its general theory and for its examples. To this end I have added material at both the beginning and the end of the first edition.

At the beginning is now an Introduction, directly developing some of the elementary theory just for matrix groups, so that the reader at once has a large stock of concrete and useful examples. In addition, the part of Chapter I summarizing the full elementary theory of Lie groups has been expanded to provide greater flexibility where one begins to study the subject. The goal has been to include a long enough summary of the elementary theory so that a reader can proceed in the subject confidently with or without prior knowledge of the detailed foundations of Lie theory.

At the end are two new chapters, IX and X. Partly these explore specific examples and carry the theory closer to some of its applications, especially infinite-dimensional representation theory. Chapter IX is largely about branching theorems, which have applications also to mathematical physics and which relate compact groups to the structure theory of noncompact groups. Chapter X is largely about actions of compact Lie groups on polynomial algebras. It points toward invariant theory and some routes to infinite-dimensional representation theory.

The reader's attention is drawn to the Historical Notes near the end of the book. These notes often put the content of the text in a wider perspective, they supply certain details that have been omitted in the text, and they try to anticipate and answer questions that the reader might ask.

Here is more detail about how the second edition differs from the first, apart from minor changes: The Introduction is all new, expanding upon two pages from §I.10 of the first edition. The main change within Chapter I is that the discussion of the elementary theory of Lie groups in §10 has been expanded into four sections, providing more detail about the theory itself and adding material about covering groups, complex structures and complex Lie groups, and the real analytic structure of Lie groups. Results about the largest nilpotent ideal in a Lie algebra have been added to §6, and the section on classical semisimple Lie groups has been adjusted to be compatible with the new Introduction. In addition, some of the problems at the end of the chapter have been replaced or adjusted.

In Chapters II through VIII, the text contains only a few significant additions. A new Proposition 2.13 improves on the first edition's Corollary 2.13 by enabling one to recognize subalgebras of a complex semisimple Lie algebra that can be extended to Cartan subalgebras. To §III.3 has been added the left Noetherian property of universal enveloping algebras. A paragraph has been added at the beginning of §IV.3 to smooth the transition to the new Chapter IX. In Chapter VII, Sections 1 and 9 make use of new material from Chapter I concerning complex structures. In Chapter VIII, a misleading and incorrect example in §5 has been excised, and Lemma 8.57 represents a tightening of the proof of Theorem 8.60. Most chapters from II through VIII contain additional problems, either just before the blocks of related problems begin or at the very end. One block of problems in Chapter V has been postponed to Chapter IX.

Chapters IX and X are new to the second edition. Appendix A contains a new section on left Noetherian rings, and Appendix B contains new sections that state and prove Ado's Theorem and the Campbell–Baker–Hausdorff Formula. The Historical Notes and the References have been expanded to take the new material into account.

The only chapter in which sections have been renumbered is Chapter I, and the only places in which results have been renumbered are in Chapters I and III and in Appendix A.

In writing the second edition, I was greatly assisted by Paul Friedman, who read and criticized several drafts, spending a great deal of time helping to get the exposition right. I could not have finished the project successfully without him and am extremely grateful for his assistance. I was helped also by P. Batra, R. Donley, and D. Vogan, who told me of the errors that they had found in the first edition, and I thank them for their efforts.

Much of the second edition was prepared while I was a visitor at the Institute for Advanced Study, and I appreciate the Institute's hospitality. As was the case with the first edition, the typesetting was by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX, and the figures were drawn with Mathematica®.

June 2002

PREFACE TO THE FIRST EDITION

Fifty years ago Claude Chevalley revolutionized Lie theory by publishing his classic *Theory of Lie Groups I*. Before his book Lie theory was a mixture of local and global results. As Chevalley put it, "This limitation was probably necessary as long as general topology was not yet sufficiently well elaborated to provide a solid base for a theory in the large. These days are now passed."

Indeed, they are passed because Chevalley's book changed matters. Chevalley made global Lie groups into the primary objects of study. In his third and fourth chapters he introduced the global notion of analytic subgroup, so that Lie subalgebras corresponded exactly to analytic subgroups. This correspondence is now taken as absolutely standard, and any introduction to general Lie groups has to have it at its core. Nowadays "local Lie groups" are a thing of the past; they arise only at one point in the development, and only until Chevalley's results have been stated and have eliminated the need for the local theory.

But where does the theory go from this point? Fifty years after Chevalley's book, there are clear topics: É. Cartan's completion of W. Killing's work on classifying complex semisimple Lie algebras, the treatment of finite-dimensional representations of complex semisimple Lie algebras and compact Lie groups by Cartan and H. Weyl, the structure theory begun by Cartan for real semisimple Lie algebras and Lie groups, and harmonic analysis in the setting of semisimple groups as begun by Cartan and Weyl.

Since the development of these topics, an infinite-dimensional representation theory that began with the work of Weyl, von Neumann, and Wigner has grown tremendously from contributions by Gelfand, Harish-Chandra, and many other people. In addition, the theory of Lie algebras has gone in new directions, and an extensive theory of algebraic groups has developed. All of these later advances build on the structure theory, representation theory, and analysis begun by Cartan and Weyl.

With one exception all books before this one that go beyond the level of an introduction to Lie theory stick to Lie algebras, or else go in the direction of algebraic groups, or else begin beyond the fundamental "Cartan decomposition" of real semisimple Lie algebras. The one exception is the book Helgason [1962],* with its later edition Helgason [1978]. Helgason's books follow Cartan's differential-geometry approach, developing geometry and Lie groups at the same time by geometric methods.

The present book uses Lie-theoretic methods to continue Lie theory beyond the introductory level, bridging the gap between the theory of complex semisimple Lie algebras and the theory of global real semisimple Lie groups and providing a solid foundation for representation theory. The goal is to understand Lie groups, and Lie algebras are regarded throughout as a tool for understanding Lie groups.

The flavor of the book is both algebraic and analytic. As I said in a preface written in 1984, "Beginning with Cartan and Weyl and lasting even beyond 1960, there was a continual argument among experts about whether the subject should be approached through analysis or through algebra. Some today still take one side or the other. It is clear from history, though, that it is best to use both analysis and algebra; insight comes from each." That statement remains true.

Examples play a key role in this subject. Experts tend to think extensively in terms of examples, using them as a guide to seeing where the theory is headed and to finding theorems. Thus examples properly play a key role in this book. A feature of the presentation is that the point of view—about examples and about the theory—has to evolve as the theory develops. At the beginning one may think about a Lie group of matrices and its Lie algebra in terms of matrix entries, or in terms of conditions on matrices. But soon it should no longer be necessary to work with the actual matrices. By the time one gets to Chapters VII and VIII, the point of view is completely different. One has a large stock of examples, but particular features of them are what stand out. These features may be properties of an underlying root system, or relationships among subgroups, or patterns among different groups, but they are far from properties of concrete matrices.

A reader who wants only a limited understanding of the examples and the evolving point of view can just read the text. But a better understanding comes from doing problems, and each chapter contains some in its last section. Some of these are really theorems, some are examples that show the degree to which hypotheses can be stretched, and some are exercises. Hints for solutions, and in many cases complete solutions, appear in a section near the end of the book. The theory in the text never relies on

*A name followed by a bracketed year points to the list of references at the end of the book.

a problem from an earlier chapter, and proofs of theorems in the text are never left as problems at the end of the current chapter.

A section called Historical Notes near the end of the book provides historical commentary, gives bibliographical citations, tells about additional results, and serves as a guide to further reading.

The main prerequisite for reading this book is a familiarity with elementary Lie theory, as in Chapter IV of Chevalley [1946] or other sources listed at the end of the Notes for Chapter I. This theory itself requires a modest amount of linear algebra and group theory, some point-set topology, the theory of covering spaces, the theory of smooth manifolds, and some easy facts about topological groups. Except in the case of the theory of involutive distributions, the treatments of this other material in many recent books are more consistent with the present book than is Chevalley's treatment. A little Lebesgue integration plays a role in Chapter IV. In addition, existence and uniqueness of Haar measure on compact Lie groups are needed for Chapter IV; one can take these results on faith or one can know them from differential geometry or from integration theory. Differential forms and more extensive integration theory are used in Chapter VIII. Occasionally some other isolated result from algebra or analysis is needed; references are given in such cases.

Individual chapters in the book usually depend on only some of the earlier chapters. Details of this dependence are given on page xvii.

My own introduction to this subject came from courses by B. Kostant and S. Helgason at M.I.T. in 1965–67, and parts of those courses have heavily influenced parts of the book. Most of the book is based on various courses I taught at Cornell University or SUNY Stony Brook between 1971 and 1995. I am indebted to R. Donley, J. J. Duistermaat, S. Greenleaf, S. Helgason, D. Vogan, and A. Weinstein for help with various aspects of the book and to the Institut Mittag-Leffler for its hospitality during the last period in which the book was written. The typesetting was by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX, and the figures were drawn with Mathematica®

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PREREQUISITES BY CHAPTER

This book assumes knowledge of a modest amount of linear algebra and group theory, some point-set topology, the theory of covering spaces, the theory of smooth manifolds, and some easy facts about topological groups. The main prerequisite is some degree of familiarity with elementary Lie theory, as in Chapter IV of Chevalley [1946]. The dependences of chapters on earlier chapters, as well as additional prerequisites for particular chapters, are listed here.

INTRODUCTION. No additional prerequisites.

CHAPTER I. Tensor products of vector spaces (cf. §1 of Appendix A). In §12 Proposition 1.110 makes use of Ado's Theorem from Appendix B; however, the proposition is used in this book only for matrix groups, and for matrix groups Ado's Theorem is not needed in the proof of Proposition 1.110. The material in §13 is an aside and makes use of Ado's Theorem.

CHAPTER II. Chapter I. Starting in §9: The proof of Proposition 2.96 is deferred to Chapter III, where the result is restated and proved as Proposition 3.31. Starting in §11: Tensor algebra as in §1 of Appendix A.

CHAPTER III. Chapter I, all of Appendix A.

CHAPTER IV. Chapter I, tensor and exterior algebras as in §§1–3 of Appendix A, a small amount of Lebesgue integration, existence of Haar measure for compact groups. The proof of Theorem 4.20 uses the Hilbert–Schmidt Theorem from functional analysis. Starting in §5: Chapter II.

CHAPTER V. Chapters II, III, and IV. The proof of Theorem 5.62 uses the Hilbert Nullstellensatz.

CHAPTER VI. Chapters II and IV. Problems 28–35 use §V.7.

CHAPTER VII. Chapter VI. Starting in §5: Chapter V.

CHAPTER VIII. Chapter VII, differential forms, additional Lebesgue integration.

CHAPTER IX. Chapters IV and V. Starting in §4: Chapters VI and VII. Starting in §6: Chapter VIII.

CHAPTER X. Chapter IX.

APPENDIX B. Chapter I and Theorem 5.29. Starting in §3: Chapter III.

STANDARD NOTATION

Item	Meaning
$\#S$ or $\|S\|$	number of elements in S
$\emptyset$	empty set
E^c	complement of set, contragredient module
δ_{ij}	1 if $i = j$, 0 if $i \neq j$
n positive	$n > 0$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integers, rationals, reals, complex numbers
$[x]$	greatest integer $\leq x$ if x is real
$\operatorname{Re} z, \operatorname{Im} z$	real and imaginary parts of z
$\bar{z}$	complex conjugate of z
1	multiplicative identity
1 or I	identity matrix or operator
$\dim V$	dimension of vector space
V^*	dual of vector space
$\mathbb{R}^n, \mathbb{C}^n$	spaces of column vectors
$\operatorname{Tr} A$	trace of A
$\det A$	determinant of A
A^t	transpose of A
A^*	conjugate transpose of A
A diagonable	A has a basis of eigenvectors with eigenvalues in the given field
$\operatorname{diag}(a_1, \ldots, a_n)$	diagonal matrix
$\operatorname{End} V$	linear maps of V into itself
$GL(V)$	invertible linear maps of V into itself
$[A : B]$	index or multiplicity of B in A
$\bigoplus V_i$	direct sum of the V_i
$\operatorname{span}(S)$	linear span of S
$\cong$	is isomorphic to, is equivalent with
G_0	identity component of group G
$Z_A(B)$	centralizer of B in A
$N_A(B)$	normalizer of B in A
C^∞	infinitely differentiable

Notation introduced in Appendix A and used throughout the book is generally defined at its first occurrence and appears in the Index of Notation at the end of the book.