Corrections to
***Lie Groups Beyond an Introduction*, *Second Edition*:**
Short Corrections, An Addition, A Long Correction, and Some Remarks

Short Corrections

Page 6, line -2. Change " $\sum_{n=0}^{\infty}$ " to " $\sum_{N=0}^{\infty}$ ".

Page 7, line -1. Change " $\sum_{N=0}^{\infty}$ " to " $\sum_{N=1}^{\infty}$ " in two places.

Page 42, line 13. Change "Proposition 1" to "Proposition 1.10".

Page 56, line 12. Change "of the maximum possible dimension" to "with the maximum possible dimension".

Page 64, line 2. Change "$\pi(\mathfrak{sl}(2,\mathbb{C})$" to "$\pi(\mathfrak{sl}(2,\mathbb{C}))$".

Page 72, line -7. Change "the image of Φ" to "the image of the identity component of G under Φ".

Page 90, last line of statement of Proposition 1.101. Change "D of G" to "D of $\widetilde{G}$".

Page 150, table (2.43). With A_n, change the condition "$\sum a_i e_i = 0$" to "$\sum a_i = 0$".

Page 153, line -4. Change "strict equality" to "strict inequality".

Page 172, line 15. Change subscript "α_{i+1}" to subscript "α_j".

Page 232, line 13. Change "$H(V)$ of" to "$H(V)$ on".

Page 237, line 10. Change "which another element" to "which is another element".

Page 241, line 10. Change "V" to "V'" at the end of the line.

Page 248, line 13. Conrado Lacerda has pointed out that the words "It follows from Theorem 4.20 that" need some elaboration. Thus change "Theorem 4.20" on line 13 to "Corollary 4.21a", and insert the statement and proof of Corollary 4.21a, which are given in the section "An Addition" later in these notes, between lines 3 and 4 on page 248.

Page 259, line -5. Change "of A_0, some" to "of A, some".

Page 267. Replace the proof of Proposition 4.67 by the following:

"PROOF. Let $\varphi : \widetilde{G} \to G$ be the quotient homomorphism, let Z be the kernel, let $\widetilde{T}$ be a maximal torus of $\widetilde{G}$, and let $T = \varphi(\widetilde{T})$. Corollary 4.47 shows that $\varphi\big|_{\widetilde{T}}$ has kernel Z. Consequently the mapping φ^* of the group $\widehat{T}$ of multiplicative characters of T into the group $\widehat{\widetilde{T}}$ given by $\varphi^*(\chi) = \chi \circ \varphi$ is a one-one homomorphism such that the index of $\varphi^*(T)$ in $\widehat{\widetilde{T}}$ is at most the order $|Z|$ of Z. On the other hand, if σ is any member of the group $\widehat{Z}$ of multiplicative characters of Z, then some multiplicative character τ of $\widetilde{T}$ has $\tau\big|_Z = \sigma$. (This can be seen as follows: The set of restrictions

$\tau\big|_Z$ is a subgroup $\widehat{Z}_1$ of $\widehat{Z}$. If $\widehat{Z}_1$ is a proper subgroup, then its linear span is a set of functions on Z of dimension $< |Z|$. However, the members of $\widehat{\widetilde{T}}$ separate point of $\widetilde{T}$, and the Stone–Weierstrass Theorem implies that their linear span, when restricted to any finite subset of $\widetilde{T}$, yields all functions on that set.) Consequently the index of $\varphi^*(\widehat{T})$ in $\widehat{\widetilde{T}}$ is at least $|Z|$. Therefore it equals $|Z|$. Application of Proposition 4.58 translates this conclusion into the desired conclusion about analytically integral forms."

Page 278, line 5. Change "$(x_{2j-1} \pm x_{2j})$" to "$(x_{2j-1} \pm ix_{2j})$".

Page 283, line 5. Change "$\varphi(U(\mathfrak{g}))$" to "$(\varphi \oplus \varphi')(U(\mathfrak{g}))$".

Page 283, line 6. Change "φ" to "$\varphi \oplus \varphi'$".

Page 292, proof of Proposition 5.21. At the end of the second display, change the period to a comma. Change "Then (a) follows from Proposition 1.91, and (b) follows from Corollary 1.85" to "the second inequality following from Proposition 1.91. This proves (a), and (b) follows from Corollary 1.85".

Page 295, line -5. Change "(Proposition 5.1)" to "(in the formulation of Corollary 5.2)".

Page 300, line -5. Change "$\lambda^w(H) = \lambda(H^{w^{-1}}))$" to "$(w\lambda)(H) = \lambda(w^{-1}H))$".

Page 305, line 6. Change "is related in" to "is related to".

Page 306, line 2. Change the displayed line from "$H_\delta^m E_{\beta_1}^{r_1} \cdots E_{\beta^k}^{r_k} \mod U^{m+\sum r_j - 1}(\mathfrak{g})$" to "$H_\delta^m E_{\beta_1}^{r_1} \cdots E_{\beta_k}^{r_k} \mod U_{m+\sum r_j - 1}(\mathfrak{g})$".

Page 306, line -4. Change "$-\beta_n$" to "$-\beta_k$".

Page 311, line -4. Change "$nH_\nu^{n-1}H_{\nu'}$ to "$nH_\nu^{n-1}H_{\nu'} + CH_{\nu'}^n$,", and insert on the next line at the left margin the line "where C is the constant $\sum_{j=0}^n c_j j^n$".

Page 312, line -1. Change "$\mathcal{H}^W$" to "$Z(\mathfrak{g})$".

Page 313, line -6. Change "$|\lambda - \delta|^2 - |\delta|^2$" to "$|\lambda + \delta|^2 - |\delta|^2$".

Page 314, line -10. Change "1.65" to "1.66".

Page 316, line 8. Change "$\nu - \lambda_0 - \mu_0$" to "$\lambda_0 + \mu_0 - \nu$".

Page 316, line 10. Change "$\mathcal{P}(\nu - \lambda_0 - \mu_0)$" to "$\mathcal{P}(\lambda_0 + \mu_0 - \nu)$".

Page 318, display (5.70). Change "$(V_1 \otimes V_2)$" to "$\mathrm{char}(V_1 \otimes V_2)$".

Page 321, line 11. Change "image φ" to "$\varphi(V(\mu)^m)_{\mu-\delta}$".

Page 323, line 5. Change "For $H \in \mathfrak{h}^*$" to "For $H \in \mathfrak{h}$".

Page 336, paragraph 5, line 1. Change "Let $\widetilde{G}$ be the universal covering group of G" to Let $\widetilde{G}$ be the universal covering group of G, and identify the Lie algebra of $\widetilde{G}$ with the Lie algebra $\mathfrak{g}_0$ of G via the differential of the covering map."

Page 355, line 4. Change "Let B be" to "Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and let B be".

Page 355, line 11. Change period to comma at the end of the display, and add afterward the text "the inequality being strict if $X \neq 0$."

Page 366, line 2. Change "Because of (6.37)" to "Because of (6.38)".

Page 379, between the statement of Proposition 6.52 and the proof. Insert the following:

"REMARK. In (b) the existence of a restricted root is actually equivalent with the existence of a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Indeed, if there is no restricted root, then $\mathfrak{a}=0$. Thus $\mathfrak{p}=0$ and $\mathfrak{g}=\mathfrak{k}$. By Proposition 6.28, $\mathfrak{g}$ is isomorphic to a Lie subalgebra of some $\mathfrak{so}(n)$. An analytic subgroup of $SO(n)$ whose Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ would have to be a closed subgroup of the compact group $SO(n)$ by Proposition 7.9 in the next chapter, and there is no such subgroup."

Page 455, line 4 of statement of Proposition 7.29. Change "$k \in K_{ss}$" to "$k \in (K \cap G_{ss})$".

Page 488, proof of Proposition 7.90a. Change this so as to read:
"(a) If $\mathfrak{h}_0$ is maximally noncompact, then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$, and $\mathfrak{h}_0=\mathfrak{a}_0\oplus\mathfrak{t}_0$, where $\mathfrak{t}_0=Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. If $M=Z_K(\mathfrak{a}_0)$ as in Section 5, then Proposition 7.33 gives $G=MG_0$, and Proposition 7.49 gives $M=Z_M(\mathfrak{t}_0)M_0$. The Cartan subgroup H is reductive and thus has the form $H=Z_G(\mathfrak{a}_0)\cap Z_G(\mathfrak{t}_0)=MA\cap Z_G(\mathfrak{t}_0)$. Intersecting both sides with K gives $H\cap K=M\cap Z_K(\mathfrak{t}_0)=Z_M(\mathfrak{t}_0)$. Substituting for $Z_M(\mathfrak{t}_0)$ into the formula for M and using the result in the formula for G gives $G=MG_0=Z_M(\mathfrak{t}_0)M_0G_0=(H\cap K)G_0$, and (a) follows."

Page 495, last paragraph. Replace this with:
"We are left with proving that any regular element X_0 of $\mathfrak{h}$ has $Z_{G_c}(X_0)=H_c$. Let $x\in G_c$ satisfy $\mathrm{Ad}(x)X_0=X_0$. The Bruhat decomposition of G_c given in Theorem 7.40 shows that there exists an element s in $N_K(\mathfrak{a})$ with x in the MAN double coset $MANsMAN$ within G. Write $x=(m_1a_1n_1)s(n_2a_2m_2)$. Then $\mathrm{Ad}(m_1a_1n_1)\mathrm{Ad}(s)\mathrm{Ad}(n_2a_2m_2)X_0=X_0$, and $\mathrm{Ad}(s)\mathrm{Ad}(n_2a_2m_2)X_0=\mathrm{Ad}(m_1a_1n_1)^{-1}X_0$. Since G_c is complex, M and A fix X_0, and thus $\mathrm{Ad}(n_1^{-1})X_0=\mathrm{Ad}(s)\mathrm{Ad}(n_2)X_0$. Theorem 1.127 shows that exp carries $\mathfrak{n}_0$ onto N, and hence $\mathrm{Ad}(n_1)^{-1}X_0$ is a member of $X_0+\mathfrak{n}_0$. Similarly $\mathrm{Ad}(s)\mathrm{Ad}(n_2)X_0$ is a member of $\mathrm{Ad}(s)X_0+\mathrm{Ad}(s)\mathfrak{n}_0$. Equating the $\mathfrak{h}$ components of these two expressions gives $\mathrm{Ad}(s)X_0=X_0$. The regularity of X_0 implies that no root vanishes on X_0, and it follows that $\mathrm{Ad}(s)$ acts as the identity on X_0. In other words, x is in MAN. Say that $x=n_0a_0m_0$. From $\mathrm{Ad}(x)X_0=X_0$, we obtain $\mathrm{Ad}(n)X_0=X_0$. On the left side we write n as an exponential and expand $\mathrm{Ad}(n)$ in series. Every root is nonzero on X_0 by regularity, and thus the exponential series collapses to its constant term. In other words, $n=1$, and x is in the subgroup $MA=H$, as required."

Page 526, line 16. Change "$\int_M f(x)\,du_\omega(x)$" to "$\int_M f(x)\,d\mu_\omega(x)$".

Page 615, lines 2–3. Change "finite-dimensional vector V" to "finite-dimensional vector space V".

Page 641, line 11. Change "$\mathrm{Hom}(\Bbbk,F)$" to "$\mathrm{Hom}_{\Bbbk}(\Bbbk,F)$".

Page 641, line 13. Change "spaces, Suppose" to "spaces. Suppose".

Page 703, formula for Σ. Change "B_p" to "B_{2p+1}", and change "D_p" to "D_{2p+1}".

Page 704, formula for Σ. Change "B_p" to "B_{2p}", and change "D_p" to "D_{2p}".

Page 763, line 4–6. Change the sentence "Goto [1948] proved that a semisimple matrix group is a closed subgroup of matrices, and the proof of Theorem 4.29 makes use of some of Goto's ideas" to
"Goto [1948] proved that a semisimple matrix group is a closed subgroup of matrices, and the proof of Theorem 4.29 makes use of some of Goto's ideas; this theorem had been proved earlier in a slightly different way by Yosida [1938]".

Page 767, lines 13–14. Change "Helgason [1978] gives a proof of the classification that is based on classifying automorphisms in a different way" to
"Helgason [1978] gives a proof of the classification of real semisimple Lie algebras that establishes and applies the classification of automorphisms of finite order for complex semisimple Lie algebras as given by Kac [1969]".

Pages 769–770. A long correction to the Historical Notes appears below in the section "A Long Correction."

Add the following two items to the section of References:

Kac (Kats), V. G., Automorphisms of finite order of semisimple Lie algebras, *Funktsional'nyi Analiz i Ego Prilozheniya* 3 (1969), No. 3, 94–96 (Russian). English translation: *Functional Anal. and Its Appl.* 3 (1969), 252–254.

Yosida, K., A theorem concerning the semi-simple Lie groups, *Tohoku Math. J.* 44 (1938), 81–84.

An Addition

On page 248, between lines 3 and 4, insert the following corollary, remarks, and proof.

Corollary 4.21a (Approximation Theorem). If G is a compact group, then the linear span of all matrix coefficients for all finite-dimensional irreducible representations of G is uniformly dense in the set $C(G)$ of continuous complex-valued functions on G.

Remarks. In the set $C(G)$, let us write $\|h\|_{\sup}$ for the maximum value of $|h(x)|$ for $x \in G$. The set $C(G)$ becomes a metric space if we define the distance between two continuous functions h_1 and h_2 to be $\|h_1 - h_2\|_{\sup}$. Convergence of a sequence in $C(G)$ is uniform convergence of the sequence of functions. The uniform continuity of a member h of $C(G)$ amounts to the fact that the function $y \mapsto h(y^{-1}x)$ of G into $C(G)$ is continuous.

Proof. If h is in $C(G)$ and f is in $L^1(G)$, then the function

$$F(x) = \int_G h(xy^{-1}) f(y)\, dy$$

is continuous as a consequence of the estimate

$$|F(x_1) - F(x_2)| \le \sup_y |h(x_1 y^{-1}) - h(x_2 y^{-1})|$$

and the uniform continuity of h. It is called the **convolution** of h and f, and we write $h * f$ for it.

Let $\epsilon > 0$ and h continuous be given. For each neighborhood N of the identity, let f_N be the characteristic function of N divided by the measure $|N|$ of N. Since f_N is nonnegative and has integral 1, $|(h * f_N)(x) - h(x)|$ is

$$\begin{aligned} &= \big| |N|^{-1} \textstyle\int_N h(xy^{-1})\, dy - h(x) \big| = |N|^{-1} \big| \textstyle\int_N (h(xy^{-1}) - h(x))\, dx \big| \\ &\le |N|^{-1} \textstyle\int_N |h(xy^{-1}) - h(x)|\, dx \le \sup_{y \in N} |h(xy^{-1}) - h(x)|. \end{aligned}$$

The uniform continuity of h implies that the right side can be made small for all x by choosing N large enough. We can thus choose N such that $\|h * f_N - h\|_{\sup} \le \epsilon$.

With N fixed and satisfying this condition, choose by the Peter-Weyl Theorem a finite linear combination m of matrix coefficients such that $\|m - f_N\|_2 \le \epsilon / \|h\|_2$. Then

$$\begin{aligned} \|h * m - h\|_{\sup} &\le \|h * (m - f_N)\|_{\sup} + \|h * f_N - h\|_{\sup} \\ &\le \|h\|_2 \|m - f_N\|_2 + \epsilon \le 2\epsilon, \end{aligned}$$

the next-to-last inequality following from the Schwarz inequality.

Going over the proofs of Lemmas 4.18 and 4.19 and replacing $\|\cdot\|_2$ everywhere by $\|\cdot\|_{\sup}$, we see that if the given L^2 function in the lemmas is continuous, then the lemmas remain valid with uniform convergence in place of L^2 convergence.

The left translates of m all lie within a finite-dimensional vector subspace V of $C(G)$, and the modified Lemma 4.19 says that $h * m$ is the uniform limit of a sequence of functions in V. Since V is finite-dimensional, this limit is in V. Thus $h * m$ is a finite linear combination of matrix coefficients that is uniformly within 2ϵ of h, and Corollary 4.21a is proved.

A Long Correction

Page 769, last two lines, and page 770, lines 1–18. Change

"Theorem 8.49, called Helgason's Theorem in the text, is from Helgason [1970], §III.3. Warner [1972a], p. 210, calls the result the "Cartan–Helgason Theorem." In fact at least four people were involved in the evolution of the theorem as it is stated in the text. Cartan [1929b], §§23–32, raised the question of characterizing the irreducible representations of G with a nonzero K fixed vector, G being a compact semisimple Lie group and K being the fixed subgroup under an involution. His answer went in the direction of the equivalence of (a) and (c) but was incomplete. In addition the proof contained errors, as is acknowledged by the presence of corrections in the version of the paper in his *Œuvres Complètes*. Cartan's work was redone by Harish-Chandra and Sugiura. Harish-Chandra [1958], §2, worked in a dual setting, dealing with a noncompact semisimple group G with finite center and a maximal compact subgroup K. He proved that if ν is the highest restricted weight of an irreducible finite-dimensional representation of G with a K fixed vector, then $\langle \nu, \beta\rangle/|\beta|^2$ is an integer ≥ 0 for every positive restricted root. Sugiura [1962] proved conversely that any ν such that $\langle \nu, \beta\rangle/|\beta|^2$ is an integer ≥ 0 for every positive restricted root is the highest restricted weight of some irreducible finite-dimensional representation of G with a K fixed vector. Thus Harish-Chandra and Sugiura together completed the proof of the equivalence of (a) and (c). Helgason added the equivalence of (b) with (a) and (c), and he provided a geometric interpretation of the theorem."

to

"Theorem 8.49, called Helgason's Theorem in the text, is from Helgason [1970], §III.3, and the proof in the text is substantially unchanged from Helgason's. Inspection of the proof shows that a version of the theorem remains valid for the compact form U of G relative to $G^{\mathbb{C}}$, as described in Proposition 7.15: if a finite-dimensional representation of U is given, then the equivalence of (a), (b), and (c) in Theorem 8.49 is still valid; however, the converse assertion that produces a representation requires a further hypothesis, such as simple connectivity of U, as examples with $U = \mathrm{Ad}_{\mathfrak{su}(3)}(SU(3))$ and $K = \mathrm{Ad}_{\mathfrak{su}(3)}(SO(3))$ show. As a result of the attribution of Warner [1972a], p. 210, the direct part of the theorem, i.e., the equivalence of (a), (b), (c) when a representation is given, is sometimes called the "Cartan–Helgason Theorem." The inclusion of Cartan's name is based on work in Cartan [1929b], §§23–32, which raised the question of characterizing the irreducible representations of U with a nonzero K fixed vector, U being a compact semisimple Lie group and K being the fixed subgroup under an involution. Cartan's answer went in the direction of the equivalence of (a) and (c) but was incomplete. In addition, the proof contained errors, as is acknowledged by the presence of corrections in the version of the paper in Cartan's *Œuvres Complètes*. Cartan's work was addressed anew by Harish-Chandra and Sugiura. Harish-Chandra [1958], Lemma 1, worked with a noncompact semisimple group G with finite center and a maximal compact subgroup K. He proved that the highest weight of an irreducible finite-dimensional representation of G with a K fixed vector vanishes on $\mathfrak{t}_{\mathfrak{p}}$. Sugiura [1962] worked with a simply connected compact semisimple group U and the fixed subgroup K under an involution. He announced for that setting, on the basis of what he later acknowledged to be an incomplete case-by-case analysis, the equivalence of (a) and (c) for the highest weight of an irreducible finite-dimensional representation of U with a K fixed vector. Thus Helgason's contribution was to introduce the equivalence of (b) with (a) and (c), supply proofs for all the equivalences, and add the converse result; in addition, Helgason provided a geometric interpretation of the theorem".

Remarks

Page 50, Proposition 1.43. Whether or not C is nondegenerate, it is still true that

$$\dim U + \dim U^\perp \geq \dim V.$$

In fact, going over the proof of Proposition 1.43 shows that the equality $\ker \psi = U^\perp$ is still valid. Hence

$$\begin{aligned}\dim V = \dim(\mathrm{domain}(\psi)) &= \dim(\ker(\psi)) + \dim(\mathrm{image}(\psi)) \\ &\leq \dim U^\perp + \dim U^* = \dim U^\perp + \dim U,\end{aligned}$$

and the inequality follows.

Page 50, Corollary 1.44. Whether or not C is nondegenerate, it is still true that $V = U \oplus U^\perp$ if and only if $C|_{U\times U}$ is nondegenerate. In fact, if $V = U \oplus U^\perp$, then $U \cap U^\perp = 0$ and the equality $U \cap U^\perp = \mathrm{rad}(C|_{U\times U})$ of (1.42) shows that $C|_{U\times U}$ is nondegenerate. Conversely if $C|_{U\times U}$ is nondegenerate, then $U \cap U^\perp = 0$ by (1.42). From the previous remark we see that

$$\dim(U + U^\perp) = \dim U + \dim U^\perp - \dim(U^\perp \cap U) \geq \dim V - 0 = \dim V,$$

and thus $U + U^\perp = V$. Hence $V = U \oplus U^\perp$.

Pages 108, line -2, to page 110, statement of Corollary 1.134. This material proves that the exponential map is everywhere regular when the Lie algebra is nilpotent. An alternative approach to this question is to establish the following general formula for the differential of the exponential map:

$$(d\exp)_X = d(L_{\exp X})_1 \circ \frac{1 - e^{-\mathrm{ad}\, X}}{\mathrm{ad}\, X}.$$

When the Lie algebra is nilpotent, each $\mathrm{ad}\, X$ is nilpotent. Consequently $\frac{1-e^{-\mathrm{ad}\, X}}{\mathrm{ad}\, X}$ is everywhere nonsingular, and the differential is everywhere one-one onto.

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