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# *Representation Theory of Semisimple Groups*

## AN OVERVIEW BASED ON EXAMPLES

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*To Susan*

*and*

*Sarah and William*

*for their patience*

# Contents

# Preface

The intention with this book is to give a survey of the representation theory of semisimple Lie groups, including results and techniques, in a way that reflects the spirit of the subject, corresponds more to a person's natural learning process, and stops at the end of a single volume.

Our approach is based on examples and has unusual ground rules. Although we insist (at least ultimately) on precisely stated theorems, we allow proofs that handle only an example. This is especially so when the example captures the idea for the general case. In fact, we prefer such a proof when the difference between the special case and the general case is merely a matter of technique and a presentation of the technique would not contribute to the goals of the book. The reader will be confronted with a first instance of this style of proof with Proposition 1.2. In some cases later on, when the style of a proof is atypical of the subject matter of the book, we omit the proof altogether.

Another aspect of the ground rules is that we feel no compulsion to state results in maximum generality. Even when the effect is to break with tradition, we are willing to define a concept narrowly. This is especially so with concepts for which one traditionally makes a wider definition and then proves as a theorem that the narrower definition gives all examples. Thus, for instance, a semisimple Lie group for us has a built-in Cartan involution, whereas traditionally one proves the existence of a Cartan involution as a theorem; since the involution is apparent in examples, we take it as part of the definition.

An essential companion to this style of writing is a careful guide to further reading for people who are interested. The section of Notes and its accompanying References are for just this purpose—so that a reader can selectively go more deeply into an aspect of the subject at will.

Twice we depart somewhat from our ground rules and proceed in a more thorough fashion. The first time is in Chapter IV with the Cartan-Weyl theory for compact Lie groups. The theory is applied often, and its general techniques are used frequently. The second time is in Chapter VIII and Appendix B with admissible representations. The heart of this theory consists of two brilliant papers by Harish-Chandra [1960] on the role of differential equations, a fundamental contribution by Langlands

[1973] on the classification of irreducible admissible representations, and a striking application of the theory by Casselman [1975]. The original papers are unpublished manuscripts, although Harish-Chandra's have been included in his collected works and parts of all the papers have been incorporated into the books by Warner [1972b] and by Borel and Wallach [1980] and into the paper by Casselman and Miličić [1982]. Since the original papers are not otherwise widely accessible, since they have been simplified somewhat by several people, and since their content is so important, we have chosen to go into some detail about them.

The finite-dimensional representation theory of semisimple groups is due chiefly to E. Cartan and H. Weyl. The infinite-dimensional theory began with Bargmann's treatment of $SL(2, \mathbb{R})$ in 1947 and then was dominated for many years by Harish-Chandra in the United States and by Gelfand and Naimark in the Soviet Union. Although functional analysts such as Godement, Mackey, Mautner, and Segal made early contributions to the foundations of the subject, it was Harish-Chandra, Gelfand, and Naimark who set the tone for research by using deeper structural properties of the groups to get at explicit results in representation theory. The early work by these three leaders established the explicit determination of the Plancherel formula and the explicit description of the unitary dual as important initial goals. This attitude of requiring explicit results ultimately forced a more concrete approach to the subject than was possible with abstract functional analysis, and the same attitude continues today. More recently this attitude has been refined to insist that significant results not only be explicit but also be applicable to all semisimple groups. A group-by-group analysis is rarely sufficient now: It usually does not give the required amount of insight into the subject. To be true to the field, this book attempts to communicate such attitudes and approaches, along with the results.

Bruhat's 1956 thesis was the first major advance in the field by another author that was consistent with the attitudes and approaches of the three leaders. Toward 1960 other mathematicians began to make significant contributions to parts of the theory beyond the foundations, but the goals and attitudes remained.

Beginning with Cartan and Weyl and lasting even beyond 1960, there was a continual argument among experts about whether the subject should be approached through analysis or through algebra. Some today still take one side or the other. It is clear from history, though, that it is best to use both analysis and algebra; insight comes from each. This book reflects that philosophy. To present both viewpoints for compact groups, for example, we begin with Cartan's algebraic approach and switch abruptly to Weyl's analytic approach in the middle. The reader will notice other instances of this philosophy in later chapters.

The author's introduction to this subject came from a course taught by S. Helgason at M.I.T. in 1967, a seminar run with C. J. Earle, W.H.J. Fuchs, S. Halperin, O. S. Rothaus, and H.-C. Wang in 1968, a course from Harish-Chandra in the fall of 1968, and conversations with E. M. Stein beginning in 1968. Some of these first insights are reproduced in this book. More of the book comes from lectures and courses given by the author over a period of fifteen years. There are a few new theorems and many new proofs.

All of this material came together for a course at Université Paris VII in Spring 1982, and the notes given for that course constituted a preliminary edition of the present book.

Prerequisites for the book are a one-semester course in Lie groups, some measure theory, some knowledge of one complex variable, and a few things about Hilbert spaces. For the one-semester course in Lie groups, knowledge of the first four chapters of the book by Chevalley [1947] and some supplementary material on Lie algebras are appropriate; a summary of this material constitutes Appendix A. In addition to these prerequisites, existence and uniqueness of Haar measure are assumed, as is the definition of a complex manifold; references are provided for this material. Other theorems are sometimes cited in the text; they are not intended as part of the prerequisites, and references are given.

Beginning at a certain point in one's mathematical career—corresponding roughly to the second or third year of graduate school in the United States and to the troisième cycle in France—one rarely learns a field of mathematics by studying it from start to finish. Later courses may be given as logical progressions through a subject, but the alert instructor recognizes that the students who master the mathematics do not do so by mastering the logical progressions. Instead the mastery comes through studying examples, through grasping patterns, through getting a feeling for how to approach aspects of the subject, and through other intangibles. Yet our advanced mathematics books seldom reflect this reality.

The subject of semisimple Lie groups is especially troublesome in this respect. It has a reputation for being both beautiful and difficult, and many mathematicians seem to want to know something about it. But it seems impossible to penetrate. A thorough logical-progression approach might require ten thousand pages.

Thus the need and the opportunity are present to try a different approach. The intention is that an approach to representation theory through examples be a response to that need and opportunity.

A.W.K.
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