Technische Universität München
Zentrum Mathematik

Rational Parameter Rays of Multibrot Sets

Diplom-Arbeit
von
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Abgabetermin: 20.4.1999

Hiermit erkläre ich, diese Diplomarbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt zu haben.

München, den 20.4.1999,

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Version 1999/04/19-2, Printing: 22nd April 1999.

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1 Introduction

In this diploma thesis we will study the parameter space of the unicritical polynomials $z^d + c$ for an integer $d \geq 2$ and parameters $c \in \mathbb{C}$. In particular we are interested in Multibrot sets $\mathcal{M}_d$, i.e., the sets of parameters c for which $z^d + c$ has a connected Julia set. Multibrot sets are the immediate generalizations of the well known Mandelbrot set, which was first studied by Douady and Hubbard in [DH82] and the famous Orsay-Notes ([DH85]).

We have two main purposes: the first one is that we want to give a proof of the Structure Theorem for Multibrot sets, which gives us a combinatorial description of Multibrot sets. For the Mandelbrot set the Structure Theorem is well known and there exist several proofs: first the already mentioned proof in the Orsay-Notes by Douady and Hubbard. Moreover, in [S97] can be found a significantly simpler proof by Schleicher, which he gave first in his thesis ([S94]) and which will be published soon. Another proof was given in [M98] by Milnor. Each of these proofs would also prove with some modifications the Structure Theorem for Multibrot sets. However, the second main purpose is to combine parts of the proofs of Schleicher and Milnor with some new arguments and give by this a new proof of the Structure Theorem.

Next we state the Structure Theorem and describe below the organization of our proof.

Theorem 1.1. (Structure Theorem for Multibrot Sets)
For the Multibrot set $\mathcal{M}_d$ and the associated parameter rays the following statements hold:

(1) *Every periodic parameter ray lands at a parabolic parameter of $\mathcal{M}_d$.*

(2) *Every non-essential parabolic parameter of $\mathcal{M}_d$ is the landing point of exactly one periodic parameter ray.*

(3) *Every essential parabolic parameter of $\mathcal{M}_d$ is the landing point of exactly two periodic parameter rays.*

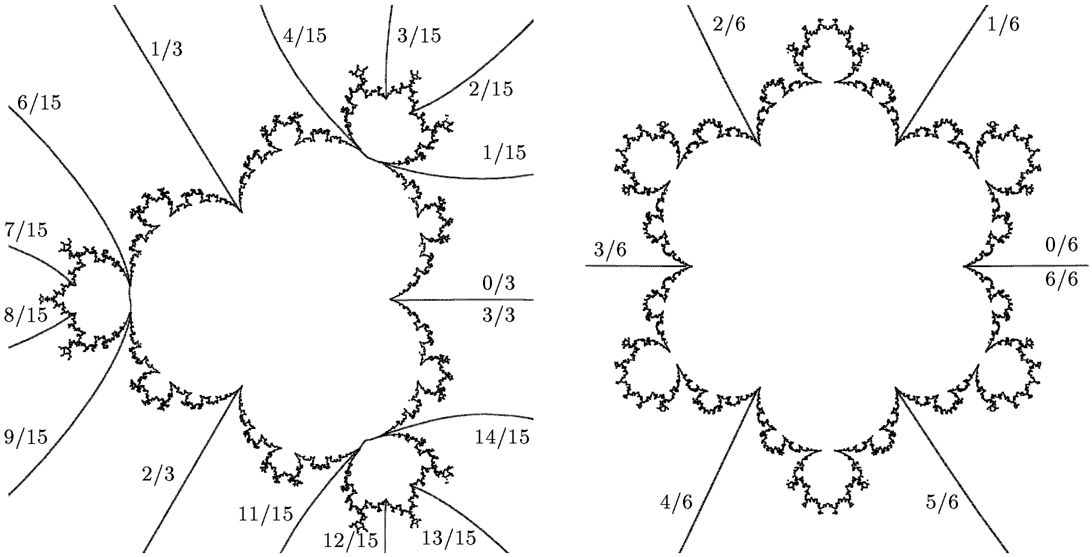
(4) *Every preperiodic parameter ray lands at a Misiurewicz point of $\mathcal{M}_d$.*

(5) *Every Misiurewicz point is the landing point of at least one preperiodic parameter ray.*

(6) *Every hyperbolic component of $\mathcal{M}_d$ has exactly one root and $d-2$ co-roots.*

For the definitions of the terms mentioned in the theorem see especially Subsections 2.3 and 3.1.

To get a first rough idea what Multibrot sets look like we show pictures of two of them. The picture on the left hand side is the $\mathcal{M}_4$ with its 1- and 2-periodic parameter rays. They are labeled by the corresponding angles. As stated in the Structure Theorem these parameter rays land and in particular some points, namely the essential parabolic parameters, are the landing points of precisely two parameter rays each. (The ray that is labeled by 0 and 1 has a special meaning and is counted twice.) The other landing points in the picture are non-essential parabolic parameters. On the right hand side is the $\mathcal{M}_7$ with the 1-periodic parameter rays. We should note that the bounded part belongs also the the Multibrot sets.



The crucial thing is to prove that exactly two parameter rays land at every essential parabolic parameter. As mentioned before we combine the proofs of Schleicher and Milnor. Therefore, a few words on their proofs of this statement: roughly speaking Schleicher shows that every parabolic parameter is the landing point of at most two parameter rays—in the quadratic case all parabolic parameters are essential—and combines this with a global counting argument, which implies that exactly two parameter rays land at each parabolic parameter. On the contrary Milnor shows that at least two parameter rays land at every parabolic parameter and uses then again a global counting argument, which shows that no parameter ray is left, i.e., they land all pairwise. Our aim is to show by Milnor's strategy that at least and by Schleicher's arguments that at most two parameter rays land at every essential parabolic parameter and to omit by this way the global counting arguments. Some time ago Milnor suggested this global strategy for the quadratic case. In more detail our organization is as follows: in Section 2 we restate some well known facts about complex dynamics, introduce Multibrot sets and show some of their basic properties.

Then in Section 3 we introduce following Milnor orbit portraits and show a few properties in the first subsection, which will be important for most of the further sections. In the second subsection we start the discussion of stability of portraits under perturbation of the parameter, which is the engine for several proofs. Moreover, as in the proofs for the quadratic case, we will use this concept to prove the first statement of the Structure Theorem (see Theorem 3.2.3). Then due to the fact that for $d > 2$ some parameter rays land in pairs and others alone, we have to start handling certain parameters different: in particular we show in Theorem 3.2.7 that at every non-essential parameter at least one ray lands and in Theorem 3.2.5 that some rays land pairwise.

Again as a tribute to the fact that in general not all parameter rays land pairwise but always a certain number of rays land at one hyperbolic component we have to introduce these objects in Section 4. In the proofs of Schleicher and Milnor the discussion of hyperbolic components starts after finishing the proof of the Structure Theorem.

In Section 5 we introduce the so-called Hubbard trees and prove with their help two Orbit Separation Lemmas. These will be a useful tool to see in more detail that if a parameter ray lands at a parameter, the corresponding dynamic ray must land at the so-called characteristic point of the parameter. Furthermore, this enables us to prove that every non-essential parabolic parameter is the landing point of exactly one periodic parameter ray (Corollary 5.3.2) and that at every essential parabolic parameter at least two parameter rays land (Corollary 5.3.3). Hence, in this subsection the second statement of the Structure Theorem will be proved.

For the proof of the third statement we have to show that at most two parameter rays land at any essential parabolic parameter. In Section 6 we have to see further properties of hyperbolic components, especially the number of roots and co-roots a hyperbolic component has. This will also prove the last statement of the Structure Theorem.

In Section 7 we can finish the proof of statement (*3*) by excluding for any essential parabolic parameter all rays, except for two, as candidates for landing at the parameter. The concept of kneading sequences is the main tool which we use for this.

By reducing the case of a preperiodic ray to the case of a periodic ray we can prove in Section 8 the fourth and fifth statement of the Structure Theorem as in the quadratic case.

Finally, I would like to thank Professor Königsberger for giving me the opportunity to do my diploma thesis at his chair. Furthermore, I thank Johannes Riedl for the discussions and especially for his support in preparing the pictures. But most grateful I am to my advisor, Dierk Schleicher. Certainly most of the new ideas in this paper are due to him. He introduced me to the world of complex dynamics and gave me several times the chance to join him at some of his interesting mathematical activities as for example conferences or seminars. In particular I want to thank him for his support to me during the work on this paper.

München — DE
April 1999

2 Elementary Definitions and Well Known Properties

In this section we want to fix our notation and repeat well known definitions and facts of complex analysis and holomorphic dynamics. We also refer to the "Introductory Lectures on Dynamics in One Complex Variable" of Milnor, [M90]. Moreover, we prove some basic properties of Multibrot sets in Subsection 2.3.

Now we set up our notation: by $\mathbb{R}$ we denote the field of *real numbers*, by $\mathbb{C}$ the one of the *complex numbers* and by $\mathbb{P}_1$ the *one dimensional projective space over* $\mathbb{C}$.

The *closed unit interval* we denote by $I := [\,0,1\,]$, the *open disk with radius* r *and center* a by $B_r(a) := \big\{\, z \in \mathbb{C} : |z-a| < r \,\big\}$ and in particular the *open unit disk* by $D := B_1(0)$. We denote the closure and the interior of a subset $A \subset \mathbb{C}$ with respect to the induced topology by $\overline{A}$ and A°, respectively. A bounded set $A \subset \mathbb{C}$ is called *full* if the complement $\mathbb{P}_1 - A$ is connected. By a *partition* of $\mathbb{C}$ we understand a countable family of open subsets of $\mathbb{C}$ such that their closure is equal to $\mathbb{C}$. The boundary of all these open sets is the *partition boundary* and often we identify the partition with its boundary.

Since we are mainly interested in unicritical polynomials, it is convenient to define $f_{c,d}(z) := z^d + c$, $c \in \mathbb{C}$, $d \geq 2$. In general we do not vary d and write therefore usually f_c for $f_{c,d}$. Sometimes it is convenient to identify f_c with the *parameter* c.

Let f and g be complex valued functions. Then we write as usual $f(z) = O\big(g(z)\big)$ for $z \in U \subset \mathbb{C}$ if a constant $C \in \mathbb{R}$ exists such that $|f(z)| \leq C \cdot |g(z)|$ holds for all $z \in U$.

In holomorphic dynamics it is common and convenient to measure angles in the fraction of a whole turn. Therefore, our angles are elements of $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. It is straightforward that we can identify every angle given in radians with a corresponding angle in $\mathbb{S}^1$. Moreover, $\mathbb{S}^1$ is isomorphic to $[\,0,1)$ and hence statements like $e^{2\pi i\vartheta}$ for an angle $\vartheta \in \mathbb{S}^1$ have a well-defined meaning. Since there is an equivalence between mapping a point in the dynamic plane by f_c and multiplying angles by d, it is convenient to denote the *d-tupling map* by $\sigma\colon \mathbb{S}^1 \to \mathbb{S}^1, \vartheta \mapsto d\vartheta$. Furthermore, we want to define intervals on $\mathbb{S}^1$: for two different angles $\vartheta_1, \vartheta_2 \in \mathbb{S}^1$ we define $(\vartheta_1, \vartheta_2)$ as the open connected component of $\mathbb{S}^1 - \{\vartheta_1, \vartheta_2\}$ that consists of the angles we reach if we go on $\mathbb{S}^1$ in positive direction from ϑ_1 to ϑ_2. We write $\vartheta_1 < \vartheta_2 < \ldots < \vartheta_s$ for at least three angles $\vartheta_1, \ldots, \vartheta_s \in \mathbb{S}^1$ if $\vartheta_{i+1} \in (\vartheta_i, \vartheta_{i+2})$, $i \in \{1, \ldots, s-2\}$. We like to note that ϑ_1 and ϑ_s are not required to be different if $s > 3$. We do not use the notation $\vartheta_1 < \vartheta_2$. Moreover, we denote the length of an interval $I_1 \subset \mathbb{S}^1$ by $\ell(I_1)$ such that $\ell(\mathbb{S}^1) = 1$.

2.1 Tools from Analysis

There are two concepts from analysis we would like to mention here. Namely, the term of a proper map and local connectivity of a set.

Definition. (Proper Map and Mapping Degree)
Let U be a region in $\mathbb{C}$ and $\varphi\colon U \to \mathbb{C}$ a holomorphic map. We call φ *proper* if for any sequence (c_n) in U with $c_n \to \partial U$ the image sequence $\big(\varphi(c_n)\big)$ leaves any compact set in $\varphi(U)$.

If the number of inverse images $\varphi^{-1}(z)$ is constant, say equal to d, for all $z \in \varphi(U)$ then φ has *mapping degree* d. ◊

Lemma 2.1.1. (Proper Maps Have a Mapping Degree)
Every proper holomorphic map has a well-defined mapping degree.

This is just a restatement of Lemma A.11 in [S98a]. There this lemma is proved.

Especially in Section 5 the following definition and topological consequences are essential. See Sections 15 and 16 in [M90] for more detailed information on these concepts.

Definition. (Arcs and Arcwise Connected Sets)
A topological embedding of I into $\mathbb{C}$ is an *arc* and a subset U of $\mathbb{C}$ is *arcwise connected* if any two points of U can be joined by an arc in U. ◊

If we say an arc $\gamma\colon [0,1] \to \mathbb{C}$ *connects two points* z_1 *and* z_2 then we mean that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Moreover, we define $\gamma(J) := \{\, \gamma(t) : t \in J \,\}$ for $J \subset I$ as usual.

Definition. (Locally Connected and Locally Arcwise Connected Sets)
A subset $U \subset \mathbb{C}$ is *locally (arcwise) connected* if every point $z \in U$ has the following property: for every neighborhood V of z there exists a neighborhood $V' \subset V$ of z such that $U \cap V'$ is (arcwise) connected. ◊

It is easy to see that every arcwise connected subset of the complex plane is connected and that the opposite direction is false in general. However, a subset of $\mathbb{C}$ is locally connected if and only if it is locally arcwise connected (see Lemma 16.4 in [M90]).

The proof of the following very important theorem on local connectivity can also be found in Section 16 in [M90].

Theorem 2.1.2. (Theorem of Carathéodory)
Let U be a simply connected region in $\mathbb{C}$ and $\varphi\colon D \to U$ a conformal map. Then φ extends to a continuous map from $\overline{D}$ onto $\overline{U}$ if and only if ∂U is locally connected.

2.2 The Dynamic Plane

In this section we restate some well known definitions and theorems with respect to the *dynamic plane*, i.e. the space where our functions $f_c(z) = z^d + c$ live in. They are the foundation on which our investigations on Multibrot sets base.

As usual we denote by $f^{\circ n}(z) := f\big(f^{\circ(n-1)}(z)\big)$ with $f^{\circ 0}(z) := \mathrm{Id}$ the *n-th iterate* of an entire function f. Furthermore, for a point $z \in \mathbb{C}$ the set $\{z, f(c), f^{\circ 2}(z), \dots\}$ is called the *orbit* of z with respect to f. Points for which an integer $k \geq 1$ exists such that $f^{\circ k}(z) = z$ are called *periodic* and the integer k is called *orbit period* of z with respect to f. The least orbit period of a point is *the exact orbit period* with respect to f. The orbit of a periodic point is called *periodic*, too. Evidently for a fixed f the exact orbit period of a point z divides every orbit period of z.

Among the points which are not periodic are some which jump at some time onto a periodic orbit. These are the preperiodic points. In more detail we say that a point $z \in \mathbb{C}$ is *preperiodic* if there exists an integer $l \geq 1$ such that $f^{\circ l}(z)$ is periodic. The least integer $l \geq 1$ with this property is called the *preperiod* of z with respect to f and the exact orbit period of $f^{\circ l}(z)$ is the *period* of z with respect to f. This means that periodic points are not preperiodic. Again the orbit of a preperiodic point is called *preperiodic*.

The *filled-in Julia set* $K(f)$ of a polynomial f is defined as the set of all points that have bounded orbit with respect to f. The boundary $\partial K(f)$ is the *Julia set* of f and the components of $\mathbb{C} - \partial K(f)$ are called the *Fatou components* of f. For the filled-in Julia set of $f_c(z) = z^d + c$ we write in general just $K_c := K(f_c)$.

Of particular interest is the derivative of the iterate fixing a periodic point z. Let f be a polynomial map and k be the exact orbit period of the point z with orbit $\mathcal{O} = \{z, f(z), f^{\circ 2}(z), \dots, f^{\circ (k-1)}\}$. Then we call $\lambda(f, \mathcal{O}) := \lambda(f, z) := \frac{d}{dz} f^{\circ k}(z)$ the *multiplier of* $\mathcal{O}$ with respect to f. In the case $f = f_c$ we write in general $\lambda(c, \mathcal{O})$ and $\lambda(c, z)$ instead of $\lambda(f_c, \mathcal{O})$ and $\lambda(f_c, z)$.

We call a periodic point $z \in \mathbb{C}$ and its orbit with respect to f_c, $c \in \mathbb{C}$, *repelling* if $|\lambda(c, z)| > 1$, *indifferent* if $|\lambda(c, z)| = 1$ and *attracting* if $|\lambda(c, z)| < 1$. If $\lambda(c, z) = 0$ they are *superattracting*. In general the case of an indifferent point z is more complicated and interesting. Therefore, we divide them again in subclasses: if z is rationally indifferent, i.e. $\lambda(c, z) = e^{2\pi i p/q}$ for a fraction $p/q \in \mathbb{Q}$, the point is called *parabolic*. Otherwise, z is irrationally indifferent and it is a *Cremer point* or a *Siegel point* according as $z \in \partial K_c$ or not. A parameter c for which f_c has a parabolic orbit is a *parabolic parameter*.

For parabolic parameters we should moreover note the term of a petal: let z_0 be a parabolic fixed point of f and U, U' neighborhoods of z_0 such that f maps U diffeomorphically onto U'. Then a connected open set U_0 is an *attracting petal for f at z_0* if $\overline{U}_0 \subset U_0 \cap U'$ and $f(\overline{U}_0) \subset U \cup \{z_0\}$ and $\bigcap f^{\circ n}(\overline{U}_0) = \{z_0\}$, $n \geq 0$. A set V_0 is a *repelling petal for f at z_0* if it is an attracting petal for f^{-1} at z_0. The Leau-Fatou Flower Theorem (see Theorem 7.2 in [M90]) says that every point z_0 of a parabolic orbit $\mathcal{O}$ of f_c with multiplier $\lambda(c, \mathcal{O}) = e^{2\pi i p/q}$ has q attracting and q repelling petals, which alternate each other, and form together with z_0 an open neighborhood of z_0.

For an attracting or parabolic orbit $\mathcal{O}$ the set of all points z with $f^{\circ n}(z) \to \mathcal{O}$ is called the *basin of attraction*. It is well known that this set is open.

Let Ω be the basin of attraction of an attracting orbit $\mathcal{O}$. Then Ω contains $\mathcal{O}$ and for a point $z' \in \mathcal{O}$ the connected component of Ω which contains z' is called the *immediate basin of attraction of z'*. The *immediate basin of attraction of* $\mathcal{O}$ is the union of the immediate basins of all points of $\mathcal{O}$.

In the case of a parabolic orbit $\mathcal{O}$ the *immediate basin of attraction of a point* $z' \in \mathcal{O}$ is the union of all Fatou components which contain an attracting petal for z'. As in the attracting case the *immediate basin of attraction of* $\mathcal{O}$ is the union of the immediate basins of all points of $\mathcal{O}$.

So far we have defined these terms only for periodic points and orbits. However, we use them also for a preperiodic point z and its orbit if they are applicable to the periodic iterates of z.

Now we can restate some theorems on Julia sets and periodic orbits. Most of them can be found in [M90].
The following theorem can be found in a similar form as Theorem 17.1 in [M90]:

Theorem 2.2.1.
Consider a polynomial map f. If the Julia set of f is connected, then the filled-in Julia set is full and every bounded Fatou component of f is simply connected.

Since f_c has the only critical point 0, i.e. the zero of $\frac{d}{dz}f_c(z)$, and the immediate basin of every attracting and parabolic orbit contains a critical point (see Corollary 7.10 and Lemma 10.2 in [M90]) and the basins do not have any points in common, the following lemma holds (see also Theorem 10.4 in [M90]):

Lemma 2.2.2. (At Most One Orbit is Not Repelling)
For a fixed parameter c all periodic orbits of f_c are repelling but possibly one. In particular a map f_c has at most one parabolic or attracting orbit in $\mathbb{C}$.

Thus every parabolic parameter has a well-defined parabolic orbit.

A bounded Fatou component which contains a parabolic periodic point on its boundary is called *parabolic Fatou component.*

Theorem 2.2.3. (At Most One Cycle of Bounded Fatou Components)
Every polynomial map f has at most one cycle of bounded Fatou components. That is: if there is a bounded Fatou component U of f then it is either periodic or preperiodic and every periodic Fatou component can be mapped onto any other periodic Fatou component by an iterate of f.

Furthermore, for a parabolic parameter c every periodic Fatou component is a parabolic Fatou component.

Remarks on the proof: The first statement is due to Sullivan and can be found as Theorem 13.5 in [M90]. Moreover, his results imply the second fact: by the classification of periodic Fatou components (see Section 13 in [M90]) we know that a periodic Fatou component is either the immediate attractive basin of an attracting periodic point, or contains an attracting petal of a parabolic periodic point, or a Siegel disk, or a Herman ring (that is a doubly connected Fatou component). By Lemma 2.2.2 we can exclude the first and the third possibility. Since every Fatou component of a polynomial is simply connected (Theorem 2.2.1), it can not be a Herman ring. □

We should note the following statement: let c be a parabolic parameter and U_0 the Fatou component containing the critical point. The other periodic Fatou components we denote by $U_1, \ldots, U_{n-1}, U_n = U_0$ with $U_l := f_c^{\circ l}(U_0)$. For $l \neq 0$ the restriction $f_c\colon \overline{U}_l \to \overline{U}_{l+1}$ is an one-to-one map and $f_c\colon \overline{U}_0 \to \overline{U}_1$ is a d-to-one map. Both maps are proper and holomorphic on the interior and continuous on the boundary.

For our next aim introducing the so-called dynamic rays we need some prerequisites. For the proofs of the statements which we will make in the following paragraph and for further information see Section 17 and 18 in [M90]. It is well known that for every parameter $c \in \mathbb{C}$ a neighborhood U of ∞ and a holomorphic function, the so-called *Böttcher Map* $\varphi_c\colon U \to U$, exists such that $\varphi_c \circ f_c \circ \varphi_c^{-1}(z) = z^d$ for $z \in U$ and $\varphi_c(\infty) = \infty$. Starting from this we define *Green's Function* g_c on U by $g_c(z) := \log|\varphi_c(z)|$ for $z \in U$ and note the functional equation $g_c(z) = g_c\big(f_c(z)\big)/d$. Now it follows easily that Green's Function can be extended continuously to $\mathbb{P}_1 - K_c$. It tends to zero as we reach K_c. Therefore, we define $g_c(z) := 0$ for $z \in K_c$. The value $g_c(z)$ is called *the potential of* z and for $t > 0$ the set $\big\{\, z \in \mathbb{C} : g_c(z) = t \,\big\}$ is the *equipotential curve of potential* t.

We should note that g_c has a critical point z whenever z is critical or precritical with respect to f_c, i.e, $f_c^{\circ l}(z) = c$ for an integer $l \geq 1$. If K_c is connected the critical point is inside of K_c and hence g_c has no critical point outside of K_c. However, if K_c is not connected g_c has infinitely many critical points.

For the Böttcher Map φ_c this means that we can extend φ_c holomorphically as long as $g_c(z) > g_c(0)$. In other words: if K_c is connected φ_c extends to a conformal map from $\mathbb{P}_1 - K_c$ onto $\mathbb{P}_1 - D$ and if K_c is not connected φ_c maps $\mathbb{P}_1 - \{ z \in \mathbb{C} : g_c(z) \le g_c(0) \}$ biholomorphically onto $\mathbb{P}_1 - \{ z \in \mathbb{C} : \log|z| \le g_c(0) \}$. In any case it has the following product expansion on its region of definition:

$$\varphi_c(z) = z \prod_{k=1}^{\infty} \left(1 + \frac{c}{\left(f_c^{\circ(k-1)}(z)\right)^d} \right)^{1/d^k} .$$

For each factor we choose the branch of the d^k-th root which maps 1 to 1. Since every z in the region of definition of φ_c is in the basin of infinity, there is a neighborhood N of 1 which does not contain 0 such that $1 + c/\left(f_c^{\circ(k-1)}(z)\right)^d \in U$ for almost every k. It follows now easily that the product expansion is well-defined (see also Theorem 3.1 in [S98a]).

Moreover, φ_c is tangent to the identity at infinity, i.e. $\varphi_c(z)/z \to 1$ as $z \to \infty$.

For a parameter c with connected Julia set we define the *dynamic ray* with angle ϑ as the set $R_\vartheta^c = \{ \varphi_c^{-1}(re^{2\pi i\vartheta}) : r > 1 \}$. If the limit $\lim \varphi_c^{-1}(re^{2\pi i\vartheta})$ exists for $r \searrow 1$ we say that R_ϑ^c *lands at* the limit point. Note that due to the conformality of the Böttcher map and the definition of dynamic rays, dynamic rays with different angles do not have any point in common. But certainly they may land at a common point.

If the Julia set of f_c is disconnected, we can define dynamic rays only at potentials which are greater than $g_c(0)$ as before. However, it is still possible to extend φ_c by the functional equation to the whole set $\mathbb{P}_1 - K_c$, if we do not require uniqueness. Using this extended φ_c we again obtain dynamic rays as inverse images of radial rays $\{ re^{2\pi i\vartheta} : r > 1 \}$. By construction of the extended φ_c these rays have branch points at the critical points of g_c. For a further discussion of this case see also Appendix A of [GM93].

The definition of periodic and preperiodic angles is completely analogous to the corresponding definitions with respect to orbits. Furthermore, we use these terms and the adjectives *rational* and *irrational* for rays if their angles satisfy these properties.

Moreover, we introduce rays inside a filled-in Julia set K_c if f_c has a superattracting orbit: let U be a Fatou component of K_c. Then it is well known that U is either preperiodic or periodic (see Theorem 2.2.3) and contains exactly one point, say z_U, which maps on the critical point 0 by some iterate of f_c. There is a Riemann map $\varphi\colon U \to D$ because U is simply connected by Theorem 2.2.1 and we may assume that it maps z_U to 0. Furthermore, it is well known that φ extends homeomorphically to $\overline{U}$. Therefore, we can define for any angle ϑ the set $R_\vartheta^U := \{ \varphi^{-1}(re^{2\pi i\vartheta}) : r \in I \}$ as the *internal dynamic ray of U at angle ϑ with respect to φ*. We note that for any rotation T around the origin it is obvious that $T \circ \varphi$ is again a Riemann map of U which maps z_U to 0.

There are a theorem and some lemmas we should mention before investigating elementary properties of Multibrot sets: the following theorem is due to Sullivan, Douady and Hubbard. For the periodic case including a proof see Theorem 18.1 in [M90]. In the preperiodic case it follows by reducing to the periodic one and taking backward images.

Theorem 2.2.4. (Every Periodic Dynamic Ray Lands)
Consider a parameter with connected Julia set. Then every periodic and preperiodic dynamic ray lands at a repelling or parabolic point which is periodic and preperiodic, respectively.

Theorem 2.2.5. (Every Repelling and Parabolic Point is Landing Point)
For a connected Julia set K_c every periodic and preperiodic repelling or parabolic point in ∂K_c is the landing point of at least one but only finitely many periodic or preperiodic dynamic rays, respectively.

Moreover, if K_c is in addition locally connected, the number of rays landing at $z \in \partial K_c$ is equal to the number of components of $K_c - \{z\}$.

Remarks on the proof: The first assertion can be found as Theorem 18.2 in [M90] and is due to Douady and Yoccoz. For a proof of the second statement see Lemma A.8 in [S98b]. It depends mainly on the Theorem of Carathéodory and a theorem of F. and M. Riesz. □

We should note the following, very useful lemma. For a connected K_c it can be found as Lemma 18.7 in [M90]. The proof there generalizes immediately for disconnected Julia sets.

Lemma 2.2.6. (Landing of the Image of A Dynamic Ray)
A dynamic ray R^c_ϑ lands at $z \in \partial K_c$ if and only if $R^c_{\sigma(\vartheta)}$ lands at $f_c(z)$.

The previous theorems and lemma show us that for repelling and parabolic periodic orbits beside the orbit period another kind of periodicity is involved: consider a repelling or parabolic periodic orbit $\mathcal{O}$ and let R^c_ϑ be a ray landing at some point of the orbit. It may happen that the period of the orbit and the period of the angle ϑ are different. Therefore, we define the *ray period of* $\mathcal{O}$ as the period of the angle ϑ. It is well known that the angles of all rays landing at a point of the orbit have the same period, i.e. the ray period is well-defined.

Finally, we should note the following correspondence between the multiplier of a parabolic orbit and its ray period:

Lemma 2.2.7. (Ray Period and Multiplier)
Let c be a parabolic parameter. Then the exact ray period of the parabolic orbit $\mathcal{O}$ is n if and only if for a point z of the orbit n is the least number with $\frac{d}{dz} f^{\circ n}(z) = 1$.

2.3 Definition and Some Properties of Multibrot Sets

As mentioned before we introduce in this subsection the Multibrot sets and show some of their basic properties. Multibrot sets are generalizations of the well known Mandelbrot set: they are the connectedness loci of the Julia sets of the polynomials $f_c(z) = z^d + c$. The term “Multibrot set” is due to Schleicher.

The definitions, properties and proofs in this subsection are well known for the Mandelbrot set and generalize easily.

In the following let $d \geq 2$ be an integer. We consider d fixed for the whole paper.

Definition. (Multibrot Set $\mathcal{M}_d$)
The *Multibrot set* of degree d is the set $\mathcal{M}_d := \{ c \in \mathbb{C} : K_c \text{ is connected.} \}$. ◊

Certainly the Multibrot set $\mathcal{M}_2$ is just the Mandelbrot set. A very essential fact is that the polynomials f_c have only one critical point. It is easy to see that every quadratic polynomial can be written by conformal conjugation as $z^2 + c$. However, in the case $d > 2$ this is not true in general: exactly the polynomials with more than one critical point can not be written as $z^d + c$. There are equivalent definitions of $\mathcal{M}_d$ as the following theorem shows:

Theorem 2.3.1. (Equivalent Definitions of $\mathcal{M}_d$)
The following statements are equivalent:

(1) $c \in \mathcal{M}_d$.

(2) The critical orbit with respect to f_c is bounded.

(3) $|f_c^{\circ n}(0)| \leq 2$ *for every integer* $n \geq 1$.

Proof: By Theorem 3.5 in [S98c] the Julia set of a polynomial is connected if and only if all critical orbits, i.e. the orbits of the critical points, are bounded. Thus, the first two statements are equivalent.

Now we show that the negation of *(3)* implies the negation of *(2)*: if $|c| > 2$ then we obtain by induction

$$|f_c^{\circ(n+1)}(c)| \geq |f_c^{\circ n}(c)|^d - |c| \geq |c|^d(|c|-1)^{d^{n-1}\cdot d} - |c| \geq |c|(|c|-1)^{d^n}$$

and this means that the critical orbit is unbounded.

Now we consider a $c \in \mathbb{C}$ and an integer $n \geq 1$ such that $|c| \leq 2$ and $|f_c^{\circ n}(0)| > 2 + \epsilon$ for an $\epsilon > 0$. Thus, by induction

$$|f_c^{\circ(n+k+1)}(0)| \geq (2 + 2^k\epsilon)^d - 2 \geq 2 + 2^{k+1}\epsilon,$$

i.e., the critical orbit escapes. The other direction, *(3)* $\Longrightarrow$ *(2)*, is trivial. $\square$

As in the dynamic plane we want to define rays in the parameter plane, i.e., the complex plane regarded as the set of parameters c of f_c. For this purpose we need an analogue to the Böttcher map in the quadratic case. The following theorem guarantees us the existence of such a map.

Theorem 2.3.2. (Properties of Multibrot Sets)
The Multibrot set $\mathcal{M}_d$ has the following properties:

(1) $\mathcal{M}_d$ is compact and full.

(2) $\mathbb{P}_1 - \mathcal{M}_d$ is simply connected.

(3) $\mathcal{M}_d$ is connected.

Proof: In the proof of Theorem 2.3.1 we already verified that $\mathcal{M}_d$ is contained in the disk $B_2(0)$. Since the nested intersection of countably many compact sets is compact, the characterization of $\mathcal{M}_d$ by Statement *(3)* in Theorem 2.3.1 shows that $\mathcal{M}_d$ is compact.

The third assertion follows by a classical theorem of Alexandroff from *(2)*.

In order to prove the second statement we show that there is a biholomorphic map $\Phi(c)$ from $\mathbb{P}_1 - \mathcal{M}_d$ onto $\mathbb{P}_1 - D$. This implies that $\mathbb{P}_1 - \mathcal{M}_d$ is simply connected.

For $c \in \mathbb{P}_1 - \mathcal{M}_d$ we define $\Phi(c) := \varphi_c(c)$. The map $\Phi(c)$ is well-defined, because $g_c(c) > g_c(0)$ and, if we choose the branch of the d^l-th root as in Subsection 2.2, Φ has the well-defined product expansion

$$\Phi(c) = c \prod_{l=1}^{\infty} \left(1 + \frac{c}{\left(f_c^{\circ(l-1)}(c)\right)^d} \right)^{1/d^l}.$$

It is quite easy to see that $\Phi(c)$ converges locally uniform on $\mathbb{P}_1 - \mathcal{M}_d$ and hence $\Phi(c)$ is there holomorphic. Moreover, we see that $\Phi(c)/c \to 1$ as $c \to \infty$. Since we want to show that Φ maps $\mathbb{P}_1 - \mathcal{M}_d$ onto $\mathbb{P}_1 - D$, we should verify $|\Phi(c)| \to 1$ as $c \to \partial\mathcal{M}_d$. For this purpose let $R > 2$ and define for a $c \in \overline{B}_R(0) - \mathcal{M}_d$ the sequence $c_l := f_c^{\circ l}(c)$, $l \geq 0$. Evidently (c_l) will leave $\overline{B}_R(0)$ and hence there is a well-defined index $N(c) := \min\{\, l \in \mathbb{N} : c_l \notin \overline{B}_R(0) \,\}$. Now we obtain for every $c \in \overline{B}_R(0)$ the inequalities $|c_{N(c)}| \leq R^d + R$ and $|c_k|^d > 2|c|$ for almost all k (see the proof of Theorem 2.3.1). By combining this with the functional equation of the Böttcher map of c we get for all $c \in \overline{B}_R(0) - \mathcal{M}_d$ and some $S > 1$

$$|\Phi(c)| = |\varphi_c(c_{N(c)})|^{1/d^{N(c)}} = \left| c_{N(c)} \prod_{l=1}^{\infty} \left(1 + \frac{c}{c_{N(c)+l-1}^d} \right)^{1/d^l} \right|^{1/d^{N(c)}} \leq S^{1/d^{N(c)}}.$$

Together with $N(c) \to \infty$ as $c \to \partial\mathcal{M}_d$ this implies $|\Phi(c)| \to 1$ as $c \to \partial\mathcal{M}_d$. Hence Φ is a proper map from $\mathbb{P}_1 - \mathcal{M}_d$ onto $\mathbb{P}_1 - D$ and has therefore by Lemma 2.1.1 a well-defined mapping degree. It is 1 because Φ is tangent to the identity near ∞. This means that Φ is a conformal isomorphism from $\mathbb{P}_1 - \mathcal{M}_d$ onto $\mathbb{P}_1 - D$. □

Now we can define the analogue for the dynamic rays in the parameter plane: let Φ be the biholomorphic map from $\mathbb{P}_1 - \mathcal{M}_d$ onto $\mathbb{P}_1 - D$ as in the proof just before. The *parameter ray with angle* ϑ is defined as the set

$$R_\vartheta^{\mathcal{M}} := \{\, \Phi^{-1}(re^{2\pi i t}) : r > 1 \,\}.$$

If $\lim \Phi^{-1}(re^{2\pi i t})$ for $r \downarrow 1$ exists we say that $R_\vartheta^{\mathcal{M}}$ *lands at* the limit point.

If a parameter $c \notin \mathcal{M}_d$ lies on a parameter ray with angle ϑ we call ϑ the *external angle* of c. Evidently the external angle is well-defined for every parameter in $\mathbb{C} - \mathcal{M}_d$.

Although there is no dynamics in the parameter plane it is convenient to use the adjectives *periodic, preperiodic, rational* and *irrational* as in the case of dynamic rays for parameter rays if their angles have these properties.

The following lemma shows how dynamic rays depend on the corresponding parameter.

Lemma 2.3.3. (When Rational Dynamic Rays Land)
Let c be a parameter with disconnected Julia set. Then the dynamic ray R_ϑ^c lands if and only if $\vartheta_0 \neq \sigma^{\circ n}(\vartheta)$ for every integer $n \geq 1$. Moreover, the parameter c lies on the parameter ray $R_{\vartheta_0}^c$.

Proof: First we show that for a parameter $c \notin \mathcal{M}_d$ with external angle ϑ_0 the critical value lies on $R_{\vartheta_0}^c$. Since $g_c(c) > g_c(0)$ the Böttcher Map φ_c is well-defined at c and hence $e^{2\pi i \vartheta_0} = \Phi(c)/|\Phi(c)| = \varphi_c(c)/|\varphi_c(c)|$.

The dynamic ray R_ϑ^c is well-defined at potential t if and only if the dynamic ray $R_{\sigma(\vartheta)}^c$ is well-defined at potential $d \cdot t$, i.e. does not contain the critical value.

Therefore, $R^c_\vartheta = \left\{ \varphi_c^{-1}(re^{2\pi i\vartheta}) \,:\, r > 1 \right\}$ is well-defined if and only if $\vartheta_0 \neq \sigma^{\circ n}(\vartheta)$ for all integers $n \geq 1$. Since the limit set of a dynamic ray is a compact subset of the Julia set (see for example the remark after Definition 2.4 in [S98b]) and K_c is completely disconnected, a well-defined dynamic ray R^c_ϑ lands at a single point.

□

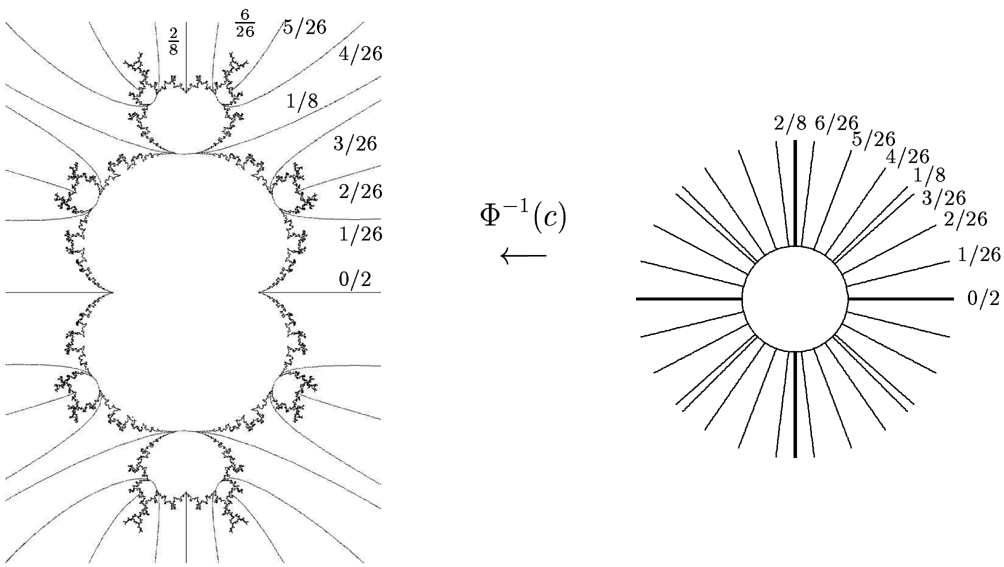


Figure 1: On the left hand side we can see the Multibrot set $\mathcal{M}_3$ with the periodic parameter rays of period 3 and below. The corresponding image rays under the map Φ are shown on the other side.

When studying the landing properties of parameter rays—we will see that they can only land at parabolic parameters—it is important to know that there are not too many parabolic parameters:

Lemma 2.3.4. (The Number of Parabolic Parameters is Countable)
The number of parameters which have a parabolic orbit of a given period is finite. In particular the number of all parabolic parameters is countable.

Proof: Let $k \geq 1$ be a fixed integer and define $Q(c,z) := f_c^{\circ k}(z) - z$. Then the number of parameters which have a parabolic orbit with ray period k is less than the number of points $(c,z) \in \mathbb{C}^2$ with $Q(c,z) = 0 = \frac{d}{dz}Q(c,z)$. Since any complex algebraic curve can be regarded as a projective curve by adding points at infinity, we obtain that $Q(c,z) = 0 = \frac{d}{dz}Q(c,z)$ holds only for finitely many pairs (c,z) by Bezout's theorem (see for example Section 3.1 in [K92]). This proves the lemma.

□

3 Orbit Portraits

In this section we introduce *orbit portraits* following Milnor in [M98], i.e., the landing patterns of dynamic rays landing at a periodic orbit, and show some properties of them. They help us to give a combinatorial description of Julia sets and are a tool to investigate the landing properties of parameter rays. In particular we use them in this section to show that every periodic parameter ray lands at a parabolic parameter (see Theorem 3.2.3) and that some of them land in pairs (see Theorem 3.2.5). Moreover, we prove that at each so-called primitive parabolic parameter at least one parameter ray lands (see Theorem 3.2.7). However, we need further techniques, which we will introduce in the following sections, to finish the proof of Theorem 1.1. The concept of orbit portraits is essential in Milnor's proof of the Structure Theorem in the quadratic case in [M98].

3.1 Definitions and Elementary Properties

The definitions and properties which we will give in this subsection are mostly well known, at least for the case $d = 2$. However, for the sake of completeness we recall them. The proofs of the quadratic case generalize easily to the case $d \geq 2$.

Definition. (Orbit Portraits)
Consider a parameter $c \in \mathbb{C}$ and let A_i be the set of angles for a periodic orbit $\mathcal{O} = \{z_0, \dots, z_{k-1}\}$ with respect to f_c for which the dynamic rays land at z_i. We call the set $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ *the orbit portrait* of $\mathcal{O}$ with respect to f_c. We denote the set of all angles that land at the orbit $\mathcal{O}$ by $A_{\mathcal{P}} = A_0 \cup \ldots \cup A_{k-1}$.

A portrait $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ is called *essential* if each A_i contains at least two angles and otherwise *non-essential*. However, there is one exceptional case: the portrait $\mathcal{P} = \{\{0\}\}$ is also essential.

It is convenient to call a parameter c with essential parabolic portrait an *essential parabolic parameter*. The definition of a *non-essential parabolic parameter* is analogous.

If the period of all angles in $A_0 \cup \ldots \cup A_{k-1}$ is equal to k, i.e., ray and orbit period are equal, then the portrait is called *primitive* and otherwise *non-primitive*.

Again it is convenient to refer to a parabolic parameter with primitive parabolic portrait as *primitive parabolic parameter* and to define a *non-primitive parabolic parameter* analogous.

We say that an orbit portrait $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ is *pairwise unlinked* if for every $i \neq j$ the set A_j is contained in a connected component of $\mathbb{S}^1 - A_i$.

For an element $A = \{\vartheta_0, \dots, \vartheta_{s-1}\} \in \mathcal{P}$, $s \geq 2$ with $\vartheta_0 < \vartheta_1 < \ldots < \vartheta_{s-1} < \vartheta_0$ (see the beginning of Section 2 for the definition of this notation) we call the intervals $(\vartheta_0, \vartheta_1), \dots, (\vartheta_{s-1}, \vartheta_0)$ the *complementary intervals* of A. $\Diamond$

We should note one major difference between $d = 2$ and $d > 2$: in the quadratic case all parabolic parameters have a parabolic orbit with essential portrait (see Corollary 4.8 in [M98]). But for $d > 2$ this is not true.

Next we state some basic properties for portraits. Actually they are characteristic for portraits and below we will use them to define so-called formal orbit portraits.

Lemma 3.1.1. (Elementary Properties of Orbit Portraits)
Let $\mathcal{P} = \{A_0, \ldots, A_{k-1}\}$, $A_k := A_0$, be the portrait of the orbit of a periodic point z with respect to f_c, $c \in \mathbb{C}$. Then:

(*1*) *Every A_i is mapped by the d-tupling map σ bijectively onto A_{i+1} the cyclic order preserving.*

(*2*) *All angles in $A_{\mathcal{P}}$ have the same exact period.*

(*3*) *The A_i are pairwise unlinked.*

(*4*) *If the portrait is essential we have: for any A_i every complementary interval but one, say I_0, is mapped by σ homeomorphically onto a complementary interval of A_{i+1}.*

This implies that for I_0 exists a complementary interval I'_0 of A_{i+1} such that $\sigma(I_0)$ covers I'_0 exactly d times, i.e., for every $\vartheta \in I'_0$ are d different angles in I_0 which map to ϑ by σ. Moreover, all complementary intervals of A_i except for I_0 together have length less than $1/d$ and the length of I_0 is greater than $1 - 1/d$.

Remark: Note that Properties (*1*) and (*2*) imply that every A_i is finite and all A_i contain the same number of angles. Moreover, it is easy to show by the second property that every angle in $A_{\mathcal{P}}$ is rational, i.e. is in $\mathbb{Q}/\mathbb{Z}$: since any angle $\vartheta \in A_{\mathcal{P}}$ is periodic, say with period n, $\vartheta = \sigma^{\circ n}(\vartheta) = d^n\vartheta$ holds and hence $\vartheta = a/(d^n - 1) + \mathbb{Z}$ for an integer a.
Proof: The proofs of Properties (*1*) to (*3*) are well known, easily supplied and exactly the same as in the quadratic case. Therefore, we omit them here and refer to Lemma 2.3 in [M98].

In the case $d = 2$ Property (*4*) follows from the previous properties, since there is no quadratic polynomial with more than one critical point. However, we give the proof of Property (*4*) for $d \geq 2$: let z_i be the landing point of the dynamic rays at angles in A_i. For a fixed m the partition $z_m \cup \bigcup R^c_\vartheta$, $\vartheta \in A_m$ has exactly one open component which contains the only critical point 0. Thus, every component, except for the component containing the critical point, is mapped homeomorphically by f_c. This means that all the complementary intervals of A_m, except for one, are also mapped homeomorphically by σ. The complementary intervals which are mapped homeomorphically have necessarily length less than $1/d$. Using Property (*1*) it follows that the images are complementary intervals of A_{m+1}.

Now we show how this implies the rest: let $I_1, \ldots, I_{s-1}$ be the complementary intervals of an A_l which are mapped homeomorphically onto a complementary interval of A_{l+1} and have therefore each length $\ell\bigl(\sigma(I_i)\bigr) = d\ell(I_i) < 1$. Since for any two different I_i, I_j with $i, j \in \{1, \ldots, s-1\}$ the images are disjoint by the previous properties, it follows that $\ell\bigl(\sigma(I_1)\bigr) + \cdots + \ell\bigl(\sigma(I_{s-1})\bigr) < 1$, i.e. $\ell(I_1) + \cdots + \ell(I_{s-1}) < 1/d$. However, the length of the images of all complementary intervals is d. This means that the only complementary interval of A_l, say I_0, which is not mapped homeomorphically must have length greater than $1 - 1/d$. It follows that $\ell\bigl(\sigma(I_0)\bigr) > d - 1$ and therefore a complementary interval of A_{l+1} exists such that every point of this interval has d preimages in I_0 and all other complementary intervals of A_{i+1} are covered $d - 1$ times by I_0. □

For an essential orbit portrait $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ we call the shortest of all complementary intervals of all A_i, say $(\vartheta_-, \vartheta_+)$, *characteristic interval of* $\mathcal{P}$ and the angles ϑ_-, ϑ_+ are the characteristic angles of $\mathcal{P}$. If $\mathcal{P} = \{\{0\}\}$ the characteristic interval is the open component of $\mathbb{S}^1 - \{0\}$ and the characteristic angles are 0 and 1. In this special case we distinct between 0 and 1 although they are equal as elements in $\mathbb{S}^1$. We consider also for such portraits the dynamic and parameter rays at angles 0 and 1 as two different angles.

There are two ways to define portraits: the first one, introduced just before, starts from an orbit and the set of angles landing at this orbit. The other one starts from a set $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ satisfying the properties of Lemma 3.1.1. This leads to the following definition:

Definition. (Formal Orbit Portrait)
A set $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ of subsets $A_i \subset \mathbb{S}^1$ is called a *formal orbit portrait* if $\mathcal{P}$ satisfies the properties of Lemma 3.1.1.

The terms *essential*, *primitive*, *non-essential*, *non-primitive*, *characteristic interval*, *characteristic angles* and *complementary interval* are defined for formal orbit portraits analogous to the above definitions. $\Diamond$

Non-essential formal orbit portraits are trivially realized by orbits of $f_0(z) = z^d$ and by Theorem 3.1.4 we will see that for every essential formal orbit portrait $\mathcal{P}$ there exists a parameter c with orbit which realizes $\mathcal{P}$, i.e. has portrait $\mathcal{P}$. The other direction, i.e., that every orbit portrait is a formal orbit portrait, is certainly obvious.

Lemma 3.1.2. (The Characteristic Interval of a Portrait is Well-Defined)
Let $\mathcal{P} = \{A_0, \dots, A_{k-1}\}$ be an essential formal orbit portrait. Then among the union of complementary intervals of each A_i for $0 \le i \le k-1$ there is exactly one with shortest length, i.e., the characteristic interval of a portrait is well-defined.

Proof: First we note that the set of lengths of the complementary intervals of sets in $\mathcal{P}$ has a minimum, say ℓ. Hence, we assume without loss of generality that a complementary interval of some A_{l+1}, say $I_{\mathcal{P}}$, has length ℓ. Since ℓ is minimal and all intervals with length less than $1/d$ are mapped homeomorphically by σ and become d-times larger, there is no complementary interval which is mapped homeomorphically onto $I_{\mathcal{P}}$, i.e., $I_{\mathcal{P}}$ must be the interval which is covered d times by a complementary interval of A_i, say I_c, with length greater than $1 - 1/d$. Denote the d preimages of $I_{\mathcal{P}}$ by $I_c^1, \dots, I_c^d \subset I_c$. Certainly the length of each of these intervals is equal to ℓ/d and they are not complementary intervals of A_l. Due to the minimality of $I_{\mathcal{P}}$ and the unlinking property $I_{\mathcal{P}}$ and hence also $I_c^1, \dots, I_c^d$ do not contain any point of $A_{\mathcal{P}}$. The unlinking property implies that any A_i must be completely contained in $\mathbb{S}^1 - I_c$ or in $I_c - (\bigcup I_c^m)$ for $m = 1, \dots, d$. In both cases the union of the images of all complementary intervals of any A_i with length less than $1/d$ is contained in $\mathbb{S}^1 - I_{\mathcal{P}}$. Now we consider an arbitrary $j \in \{0, \dots, k-1\}$ and denote by $I'_{\mathcal{P}}$ the complementary interval of A_{j+1} which is covered d-times by the complementary interval of A_j with length greater than $1 - 1/d$: it follows that $I'_{\mathcal{P}}$ is not contained in $S^1 - I_{\mathcal{P}}$, i.e. $I'_{\mathcal{P}} \supset I_{\mathcal{P}}$. If $\ell(I'_{\mathcal{P}}) = \ell$ then we obtain by applying the above argument to $I'_{\mathcal{P}}$ that $I_{\mathcal{P}} \supset I'_{\mathcal{P}}$, i.e. $I_{\mathcal{P}} = I'_{\mathcal{P}}$, and therefore $I_{\mathcal{P}}$ is uniquely determined. $\square$

Now we introduce some more objects which help us to describe parabolic parameters. Especially in Sections 5, 6 and 7 we will use them.

Definition. (Characteristics of Parabolic Parameters)
Consider a parameter c with parabolic orbit of portrait $\mathcal{P}$. The Fatou component U_1 containing the critical value c is called *characteristic Fatou component* of c. It is well known that there is exactly one parabolic point on the boundary of the characteristic Fatou component. It is called the *characteristic point* of the parabolic orbit of c. $\Diamond$

Let $\mathcal{P}$ be a portrait. Then we call the minimum number of angles $\vartheta_0, \ldots, \vartheta_{s-1} \in A_{\mathcal{P}}$ which we need to represent any other angle $\vartheta \in A_{\mathcal{P}}$ by $\vartheta = \sigma^{\circ l}(\vartheta_i)$ the *number of cycles* of $A_{\mathcal{P}}$ or $\mathcal{P}$.

An important difference between primitive and non-primitive parabolic portraits shows the following lemma. Due to this fact the results and proofs for both kinds of portraits and corresponding parameters are often quite different.

Lemma 3.1.3. (Primitive and Non-Primitive Portraits)
Let $\mathcal{P} = \{A_0, \ldots, A_{k-1}\}$ be a formal orbit portrait. If $\mathcal{P}$ is primitive then each A_i contains at most two angles and $\mathcal{P}$ has one cycle in the non-essential case and two in the essential.

Otherwise, if $\mathcal{P}$ is non-primitive, each A_i contains at least two angles and $\mathcal{P}$ has exactly one cycle.

Proof: Obviously non-essential portraits are primitive and have exactly one cycle. Hence, we consider for the rest of the proof only essential portraits. Let n be the ray period of the angles in $A_{\mathcal{P}}$ and v the number of angles that each A_i contains. Then the number of angles in $A_{\mathcal{P}}$ is $k \cdot v$ and the integer $k \cdot v/n$ is the number of different cycles in $A_{\mathcal{P}}$.

Now we make two assumptions: the first one is that $v \geq 3$ and the second one is that there are at least two distinct cycles. As before we denote the characteristic interval of $\mathcal{P}$ by $I_{\mathcal{P}} = (\vartheta_-, \vartheta_+)$ and assume without loss of generality that it is a complementary interval of A_0. Since $v \geq 3$ there are two not necessarily distinct angles $\vartheta_1, \vartheta_2 \in A_0$ such that $\vartheta_1 < \vartheta_- < \vartheta_+ < \vartheta_2 \leq \vartheta_1$ and we denote the corresponding intervals by $I_- = (\vartheta_1, \vartheta_-)$ and $I_+ = (\vartheta_+, \vartheta_2)$. Without restriction we assume that $\ell(I_-) \leq \ell(I_+)$. Since I_- has not minimal length we obtain as in the proof of Lemma 3.1.3, that there is a complementary interval $I_1 \supset I_{\mathcal{P}}$ of some A_j which maps homeomorphically onto I_- by $\sigma^{\circ m}$ for an integer m. This implies that $\ell(I_1) < \ell(I_-)$. Using the second assumption, i.e. that there are two different cycles, it follows that ϑ_- and ϑ_+ belong to different cycles of $A_{\mathcal{P}}$. Otherwise, there would be an integer $l \geq 1$ such that $\sigma^{\circ l}(\vartheta_-) = \vartheta_+$. Every interval $(\sigma^{\circ lj}(\vartheta_-), \sigma^{\circ lj}(\vartheta_+)) = (\sigma^{\circ lj}(\vartheta_-), \sigma^{\circ l(j+1)}(\vartheta_-))$ would be a complementary interval of A_0. Since $A_{\mathcal{P}} \cap I_{\mathcal{P}} = \emptyset$ and σ is order preserving, this would imply that there is only one cycle. Therefore, $\sigma^{\circ l}(\vartheta_+) \neq \vartheta_-$ for all integer $l \geq 1$, i.e. $I_{\mathcal{P}}$ can not map on I_-. This shows that $I_1 \supsetneq I_{\mathcal{P}}$ and that I_1 is not a complementary interval of A_0. We write $I_1 = (\vartheta'_-, \vartheta'_+)$ and it follows that $\vartheta'_- \in I_-$ because $\ell(I_1) < \ell(I_-)$. We obtain by the unlinking property that I_1 must be in a connected component of $\mathbb{S}^1 - \{\vartheta_1, \vartheta_-\}$ and hence, $\vartheta'_+ \in I_-$, too. This implies that $I_+ \subset I_1$ which is a contradiction to $\ell(I_1) < \ell(I_-) \leq \ell(I_+)$. Thus, we have proved the following: if $v \geq 3$ then there is only one cycle in $A_{\mathcal{P}}$ and if there are two cycles in $A_{\mathcal{P}}$ then $v \leq 2$. This finishes the proof of the lemma. $\square$

Of particular interest is the following theorem: it shows that for every essential formal orbit portrait a parameter c exists which has an orbit realizing the portrait. The proof generalizes easily from the quadratic case to $d \geq 2$ (see Lemmas 2.9 and 2.8 in [M98]). This statement will be important for the proof of Theorem 3.2.5.

Theorem 3.1.4. (Orbits Realizing a Given Portrait)
Let $\mathcal{P}$ be an essential formal orbit portrait with characteristic interval $(\vartheta_-, \vartheta_+)$. For a parameter $c \in \mathbb{C} - \mathcal{M}_d$ the map f_c has an orbit with portrait $\mathcal{P}$ if and only if the external angle of c is in $(\vartheta_-, \vartheta_+)$.

Proof: Let ϑ_c be the external angle of c. First we should note a criterion for the landing of dynamic rays at a common point: since c lies on the dynamic ray at angle ϑ_c the d dynamic rays at the angles mapping to ϑ_c by σ, say $\vartheta_c^1, \ldots, \vartheta_c^d$, meet at the critical point 0 and we obtain a partition of the dynamic plane with d open components. We label these components such that every component has a unique label and with every angle ϑ we associate the sequence of labels of $\vartheta, \sigma(\vartheta), \sigma^{\circ 2}(\vartheta), \ldots$ It is well known that two dynamic rays land at a common point of ∂K_c if and only if their sequences of labels are equal. (In fact the sequence of labels is just the ϑ_c-itinerary of ϑ.)

If $\vartheta_c \in (\vartheta_-, \vartheta_+)$ then the angles $\vartheta_c^1, \ldots, \vartheta_c^d$ lie in the components $I_c^1, \ldots, I_c^d$ of the preimage of $(\vartheta_-, \vartheta_+)$. By the unlinking property this means for $\mathcal{P} = \{A_0, \ldots, A_{k-1}\}$ that every A_j lies in a connected component of $\mathbb{S}^1 - \bigcup_i I_c^i$. Now using the above criterion it follows for every fixed j that the dynamic rays at the angles in A_j land at a common point. Denoting by $\mathcal{P}' = \{A_0', \ldots, A_{k'-1}'\}$ the portrait that the angles form this means that $k' \leq k$ and we may assume without loss of generality that $A_i \subset A_i'$ for $0 \leq i < k'$. Let n be the common ray period of the angles in $A_{\mathcal{P}}$. We assume that $k > k'$. If $\mathcal{P}$ is a primitive portrait it follows by Lemma 3.1.3 that $\mathcal{P}$ has two different cycles because it is essential. However, since $k' < k$ every $A_i' \in \mathcal{P}'$ consists of at least three angles and has therefore again by Lemma 3.1.3 only one cycle. This is a contradiction to $A_{\mathcal{P}} = A_{\mathcal{P}'}$. If $\mathcal{P}$ is non-primitive, $k > k'$ implies that $A_{k'} \subset A_0'$. Since A_0 and $A_{k'}$ are unlinked and σ is order preserving, this would mean that $\sigma^{\circ k}$ fixes each ray in $A_0 \cup A_{k'}$, i.e. $k = n$. This is a contradiction to the assumption that $\mathcal{P}$ is non-primitive.

It remains to show the other direction of the theorem: if $\vartheta_c = \vartheta_-$ or $\vartheta_c = \vartheta_+$ then the dynamic rays at the characteristic angles pass a precritical point and do not land therefore. Finally, if ϑ_c is not one of the characteristic angles and $\vartheta_c \notin (\vartheta_-, \vartheta_+)$ then it is easy to see that $\vartheta_c^i \in (\vartheta_-, \vartheta_+)$, i.e. both characteristic angles have different sequences of labels and do not land in a common point. □

Definition. (Combinatorial Rotation Number)
Let $\mathcal{P} = \{A_0, \ldots, A_{k-1}\}$ be a portrait of an orbit with s rays landing at each point of the orbit. Moreover, let some $A_i = \{\vartheta_0, \ldots, \vartheta_{s-1}\} \in \mathcal{P}$ with $\vartheta_0 < \ldots < \vartheta_{s-1} < \vartheta_0$ and let r be the integer in $\{0, \ldots, s-1\}$ such that $\sigma^{\circ k}(\vartheta_i) = \vartheta_{r+i}$. (We read subscripts of angles modulo s.) Then we call the number r/s *combinatorial rotation number* of $\mathcal{P}$. ◊

It is well known and easy to see that the combinatorial rotation number of a portrait $\mathcal{P}$ is independent of the subset $A_i \subset \mathcal{P}$ used to define it.

We should note that if $e^{2\pi i p/q}$ is the multiplier of a parabolic parameter then the combinatorial rotation number of the associated parabolic orbit portrait is p/q.

3.2 Stability of Orbit Portraits

Now we can use the just before defined objects. In particular we will obtain some results about the perturbation of parameters and the consequences for certain orbits and their portraits. These results will then lead to the theorems on parameter rays which we mentioned at the beginning of this section.

We start with a statement about the stability of periodic points under perturbation of the parameter. It will be of interest in later sections, too. Obviously the proof does not depend on the degree d.

Lemma 3.2.1. (Continuous Dependence of Periodic Points on the Parameter)
Let c_0 be a parameter and z_0 a periodic point of f_{c_0} with exact period k and multiplier $\lambda(c_0, z_0) \neq 1$. Then there exists a neighborhood U of c_0 and a holomorphic function $z\colon U \to \mathbb{C}$ such that $z(c)$ is a periodic point of f_c with exact period k for every $c \in U$ and $z(c_0) = z_0$.

Proof: The assertion follows easily by the Implicit Function Theorem: let $Q(c,z) := f_c^{\circ k}(z) - z$. This function is clearly holomorphic in c and z, has the zero (c_0, z_0) and $\frac{d}{dz}Q(c_0, z_0) \neq 0$ since $\lambda(c_0, z_0) \neq 1$. Hence, we get by the Implicit Function Theorem that there are neighborhoods U and V of c_0 and z_0, respectively, and a holomorphic function $z\colon U \to V$ such that $Q\big(c, z(c)\big) = 0$ or in other words $z(c)$ is a point of exact period k for every $c \in U$. $\square$

The following theorem is a slight generalization of Lemma B.1 in [GM93] and guarantees local stability of the portrait of a repelling orbit under perturbation of the parameter.

Theorem 3.2.2. (Stability of Portraits)
Let c_0 be a parameter with repelling periodic point z_0 and portrait $\mathcal{P}$. Then there exists a neighborhood U of c_0 such that there is a holomorphic function $z\colon U \to \mathbb{C}$ with $z(c_0) = z_0$ and $z(c)$ is a repelling periodic point with associated portrait $\mathcal{P}$ for every $c \in U$.

Proof: Let k be the orbit and n be the ray period of the parabolic orbit of c_0. By Lemma 3.2.1 the existence of a neighborhood U of c_0 and a holomorphic function z on U with the asserted properties is guaranteed except for the statements that $z(c)$ is repelling and the portrait remains stable. We will show that a restriction of z satisfies the remaining properties. We state four requirements which U must satisfy: by possibly shrinking U we may assume that $z(c)$ is repelling for all $c \in U$, since the multiplier $\lambda\big(c, z(c)\big)$ depends holomorphically on c. Moreover, by the Kœnigs Linearization Theorem (see Theorem 6.1 and Remark 6.2 in [M90]) there exists again for a possibly shrunken U a holomorphic map $\Phi_c\colon z(U) \to \mathbb{C}$ depending also holomorphically on $c \in U$ such that $\Phi_c(z_0) = 0$ and $\Phi_c \circ f_c^{\circ k}\big(z(c)\big) = \lambda\big(c, z(c)\big) \cdot \Phi_c \circ z(c)$ for all $c \in U$. Now let $R_\vartheta^{c_0}$ be a dynamic ray that lands at z_0. Again by shrinking U, if necessary, we can assume that a point of R_ϑ^c with some sufficiently small potential $t > 0$ is contained in $z(U)$ for every $c \in U$. The last requirement on U is according to whether $c_0 \in \mathcal{M}_d$ or not: if $c_0 \notin \mathcal{M}_d$ we know by Lemma 2.3.3 that the external angle of c_0 is different from the finitely many angles $\sigma(\vartheta), \sigma^{\circ 2}(\vartheta), \ldots$ and we can hence assume that the external angles of the parameters in U are different from $\sigma(\vartheta), \sigma^{\circ 2}(\vartheta), \ldots$ In the case $c_0 \in \mathcal{M}_d$ we assume that U is small enough such that all points in $z(U)$ have a potential less than $t/2$.

Now the neighborhood U of c_0 has the properties we need to finish the proof: by construction of U, by Theorem 2.2.4 and Lemma 2.3.3 the dynamic rays R^c_ϑ land for all $c \in U$. For every $c \in U$ a branch of the inverse map of $f_c^{\circ k}$ exists. Let q_c be the branch that fixes $z(c)$ and the dynamic ray R^c_ϑ. Since $\left|\lambda\big(f_c, z(c)\big)\right| > 1$ every point in $z(U)$ converges to $z(c)$ by iterating q_c, especially the point on R^c_ϑ with potential t. Hence, R^c_ϑ lands at $z(c)$ for every $c \in U$. This shows that the portrait remains stable in U. □

Now we are ready to make our first statement about the landing properties of periodic parameter rays. It shows that periodic parameter rays land. Hence, it proves the first assertion of the Structure Theorem 1.1 and tells us that it could make sense to study the landing properties of parameter rays further. The proof is due to Goldberg, Milnor, Douady and Hubbard (see Theorem C.7 in [GM93]). Although they state the theorem only for the quadratic case, their proof also holds for $d \geq 2$. Milnor and Schleicher use this theorem also for their proofs of the Structure Theorem in the quadratic case (see the proof of Theorem 3.1 in [M98] and Proposition 3.1 in [S97]).

Theorem 3.2.3. (Periodic Parameter Rays Land)
Let ϑ be an angle of exact period n. Then the parameter ray $R^{\mathcal{M}}_\vartheta$ lands at a parabolic parameter c_0 with parabolic orbit of exact ray period n and the dynamic ray $R^{c_0}_\vartheta$ lands at a point of the parabolic orbit.

Remark: In fact we will see in Theorem 5.3.1 that the dynamic ray $R^{c_0}_\vartheta$ lands at a special point of the parabolic orbit, the so-called characteristic point.
Proof: Let c_0 be an accumulation point of the parameter ray $R^{\mathcal{M}}_\vartheta$. Then the dynamic ray $R^{c_0}_\vartheta$ lands at a parabolic or repelling point z of period n by Theorem 2.2.4. Assume that z is repelling with respect to c_0. Then by Theorem 3.2.2 there is a neighborhood U of c_0 such that for every $c \in U$ the dynamic ray R^c_ϑ lands at a repelling point. But by Lemma 2.3.3 we see that every neighborhood of c_0 contains parameters c such that R^c_ϑ does not land and hence c_0 is parabolic. Since the limit set of the ray is connected (see for example the remark after Definition 2.4 in [S98b]) and by Lemma 2.3.4 there are only finitely many parameters having parabolic orbits of period n, c_0 is the unique limit point of the parameter ray. □

By analytic continuation of the function $z(c)$ in Theorem 3.2.2 we obtain now a global version as the following corollary:

Corollary 3.2.4. (Stability of Portraits)
Let c_0 be a parameter with repelling periodic point z_0 and associated portrait $\mathcal{P}$ of ray period n. Furthermore, let U be a simply connected neighborhood of c_0 such that every $c \in U$ satisfies the following two properties:

(1) c has no parabolic orbit of ray period n and

(2) c does not lie on any parameter ray of period n.

Then there exists a holomorphic function $z\colon U \to \mathbb{C}$ such that $z(c_0) = z_0$ and $z(c)$ is a repelling periodic point with associated portrait $\mathcal{P}$ for every $c \in U$.

Remark: Hence, the portrait of a repelling orbit can be destroyed only at parabolic points or at a parameter ray of equal period. Indeed, we will see in Theorem 4.2 that the portrait of a repelling orbit is destroyed in some cases while perturbing into a parabolic parameter. If an angle ϑ is contained in a subset of the portrait of a repelling orbit with respect to a parameter c_0, the portrait will be destroyed by Lemma 2.3.3 if we perturb into a parameter ray with angle which is equal to a forward image of ϑ.

Now, having the global version of the statement on stability of repelling orbit portraits, we can give a slightly weaker version of Milnor's Theorem 3.1 in [M98] following his proof, which does not really depend on the degree. In this paper we use it to show that at every essential parabolic parameter at least two parameter rays land. In the quadratic case this shows that all parameter rays land in pairs because all parabolic orbit portraits are essential. However, in the case $d > 2$ this is not true.

Theorem 3.2.5. (Parameter Rays Landing at a Common Point)
Let $\mathcal{P}$ be an essential portrait with characteristic angles ϑ_- and ϑ_+. Then the parameter rays $R^{\mathcal{M}}_{\vartheta_-}$ and $R^{\mathcal{M}}_{\vartheta_+}$ land at a common parabolic parameter c_0. Moreover, every parameter which is in the component of $\mathbb{C} - (R^{\mathcal{M}}_{\vartheta_-} \cup R^{\mathcal{M}}_{\vartheta_-} \cup \{z_0\})$ that does not contain 0 has an orbit with portrait $\mathcal{P}$.

Proof: We denote the common period of the angles in $A_{\mathcal{P}}$ by n and let F_n be the set of all parabolic parameters with parabolic orbit of ray period n. Now consider the connected components U_i of the partition $F_n \cup \bigcup_{\vartheta \in A_{\mathcal{P}}} R^{\mathcal{M}}_{\vartheta}$. Since $A_{\mathcal{P}}$ and F_n are finite (Lemmas 3.1.1 and 2.3.4) there are only finitely many components and by Theorem 3.2.3 they are open. Furthermore, using Corollary 3.2.4 we obtain that the portrait the angles $A_{\mathcal{P}}$ form remains stable in every U_i, i.e. given any U_i the portrait of the angles $A_{\mathcal{P}}$ is the same for all parameters in U_i. Since by Lemma 3.1.2 there is no angle in $A_{\mathcal{P}}$ within the interval $(\vartheta_-, \vartheta_+)$ and parameter rays at different angles have no common point in $\mathbb{C} - \mathcal{M}_d$, there is exactly one component, say U_0, containing parameters in $\mathbb{C} - \mathcal{M}_d$ having external angles in $(\vartheta_-, \vartheta_+)$. By Theorem 3.1.4 these are the only parameters in $\mathbb{C} - \mathcal{M}_d$ having an orbit with portrait $\mathcal{P}$. Combining this with the just described result of Corollary 3.2.4 we see that U_0 can not contain any other parameters in $\mathbb{C} - \mathcal{M}_d$ and this means that U_0 is bounded by $R^{\mathcal{M}}_{\vartheta_-}$, $R^{\mathcal{M}}_{\vartheta_+}$ and exactly one point of F_n, say c_0. This proves the first assertion. Now for a parameter $c \notin \mathcal{M}_d$ the second assertion follows by Theorem 3.1.4. Combining this with Corollary 3.2.4 we see that the assertion holds also for $c \in \mathcal{M}_d$. Since all periodic orbits of $f_0(z) = z^d$ have a non-essential portrait, it is evident that $0 \notin U_0$. □

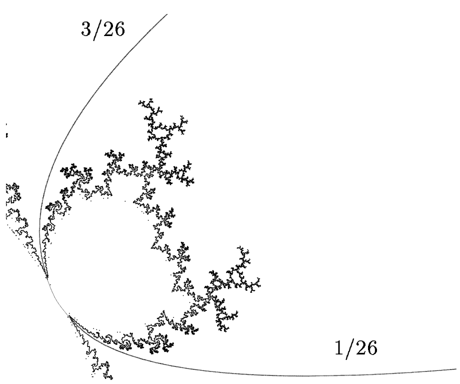


Figure 2: The $\mathcal{P}$-wake of $\mathcal{M}_3$ for the portrait $\mathcal{P}$ with the characteristic interval $(1/26, 3/26)$.

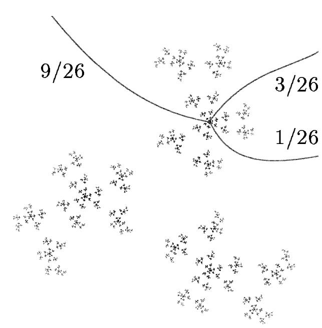


Figure 3: The Julia set of $z \mapsto z^3 + c$ for a parameter c inside the $\mathcal{P}$-wake but outside of $\mathcal{M}_3$ (c is near $0.65 + 0.59\,i$).

For two parameter rays $R_{\vartheta_-}, R_{\vartheta_+}$ which land at a common point z_0 the component of $\mathbb{C} - (R^{\mathcal{M}}_{\vartheta_-} \cup R^{\mathcal{M}}_{\vartheta_-} \cup \{z_0\})$ which does not contain 0 is called the *$\mathcal{P}$-wake of $\mathcal{M}_d$*. See also Section 3 of [M98].

We will combine the following lemma in Theorem 3.2.7 with a new argument, which is due to Schleicher, in order to prove that every primitive parameter is the landing point of at least two or one parameter rays according to whether it is essential or not.

Lemma 3.2.6. (In the Neighborhood of a Primitive Parabolic Point) *Let c_0 be a primitive parabolic parameter and z_0 a point of the parabolic orbit with exact period k. Then there are two possibilities: either there are neighborhoods U and V of c_0 and z_0, respectively, and holomorphic functions $z_1, z_2\colon U \to V$ such that $z_1(c)$ and $z_2(c)$ are points of exact period k and $z_1(c_0) = z_2(c_0) = z_0$.*

Or there is a two-sheeted cover $\pi\colon U' \to U$ of a neighborhood U of c_0 with the only ramification point $\pi(c_0') = c_0$, a neighborhood V of z_0 and a holomorphic function $z\colon U' \to V$ such that $z(c')$ is a point of exact period k for $c = \pi(c')$ and $z(c_0') = z_0$.

Remark: We can say that the parabolic orbit of c_0 splits into two orbits with exact period k. Later we will be able to see that in the primitive case always the second possibility occurs. This means that the two orbits the parabolic orbit of c_0 splits into will be interchanged if we go along a sufficiently small closed arc around c_0.

Proof: Let $Q(c,z) := f_c^{\circ k}(z) - z$. We first prove that there are neighborhoods U and V of c_0 and z_0, respectively, such that $Q(c,z)$ has for a fixed $c \in U$ exactly two zeroes in V. Let U be a neighborhood of c_0 such that no other parameters with k-periodic parabolic orbit are in U. Since the multiplier is $\lambda(c_0, z_0) = 1$ by Lemma 2.2.7, the Taylor expansion for a fixed $c \in U$ near z_0 with respect to z is $Q(c,z) = a(z-z_0)^{q+1} + O\big((z-z_0)^{q+2}\big)$. Here are $q \geq 1$, $a \in \mathbb{C}$. By the Leau-Fatou Flower Theorem (see the beginning of Subsection 2.2 and Theorem 7.2 in [M90]) we know that q is the exact number of attracting petals. Since f_c has an unique critical point and by Corollary 7.10 in [M90] the immediate basin of each attracting petal contains at least one critical point, $f_c^{\circ k}$ has only one critical point in the immediate basin of the orbit of z_0 and hence there is exactly one attracting petal with respect to $f_{c_0}^{\circ k}$, i.e., $q = 1$. This means that z_0 is a double zero of $z \mapsto Q(c_0, z)$ and using the Argument Principle it follows easily for a possibly shrunken U that for every $c \in U$ there exists an $\epsilon > 0$ such that there are exactly two simple zeros $z_1(c), z_2(c) \in V := B_\epsilon(z_0)$. Now we will see that which possibility takes place depends on whether the two orbits are interchanged or not while the parameter goes around c_0. Since U does not contain any parameter with k-periodic parabolic orbit by assumption, the multiplier $\lambda(c,z)$ is for every $c \in U - \{c_0\}$ and periodic point z in V of exact period k always different from 1. This means, if we choose a parameter $c^* \in U - \{c_0\}$ with zeroes $z_1^*, z_2^* \in V$ of $Q(c^*, z)$ then there are by Lemma 3.2.1 analytic germs $\mathbf{z_1}$ and $\mathbf{z_2}$ in c^* with $\mathbf{z_1}(c^*) = z_1^*$ and $\mathbf{z_2}(c^*) = z_2^*$ and we can continue them analytically along any arc in $U - \{c_0\}$. If we consider an arc $\gamma\colon I \to U - \{c_0\}$ around c_0 with $\gamma(0) = \gamma(1) = c^*$ then there are two situations for the analytic continuations of $\mathbf{z_1}$ and $\mathbf{z_2}$ along γ: either the continuation of $\mathbf{z_1}$ leads again to z_1^* and hence the one of $\mathbf{z_2}$ to z_2^* or they are interchanged.

Using the Monodromy Theorem we obtain in the first case that we can continue the germs $\mathbf{z_1}, \mathbf{z_2}$ to bounded holomorphic functions on $U - \{c_0\}$ and then continue further to holomorphic functions $z_1, z_2\colon U \to V$ with $z_1(c_0) = z_2(c_0) = z_0$ such that $z_1(c)$ and $z_2(c)$ are for every $c \in U$ both points with exact period k with respect to f_c.

In the second case we have just seen that there is no holomorphic function with the required properties on U. However, on a two-sheeted cover $\pi\colon U' \to V$ with the only ramification point c_0 the $\mathbf{z_1}, \mathbf{z_2}$ lead to a holomorphic function $z\colon U' \to V$ with $z(c_0') = z_0$, $\pi^{-1}(c_0) = \{c_0'\}$, such that $z(c')$ is a point of period k for every $c' \in U'$ with respect to $f_{\pi(c')}$. □

As announced before we can now show the following theorem on primitive parabolic parameters:

Theorem 3.2.7. (Primitive Parabolic Parameters)
Let c_0 be a primitive parabolic parameter. Then at least two parameter rays land at c_0 if c_0 is essential and at least one parameter ray lands at c_0 if it is non-essential.

The idea of the following proof is: if we wind once around a primitive parabolic parameter, then there is a necessarily repelling orbit portrait which must be destroyed, since the repelling orbit becomes attracting at some time while winding around the primitive parameter. If the curve along we go is small enough this can only happen at a parameter ray. Hence, a parameter ray lands at the primitive parabolic parameter.
Proof: Let k be the exact orbit period of the parabolic orbit. Again we have to discuss the two possibilities of the previous lemma: if the first possibility occurs, then there is a neighborhood U of c_0 such that no other parabolic points of period k are in U and the multipliers $\lambda_1(c) := \lambda\big(c, z_1(c)\big)$, $\lambda_2(c) := \lambda_2\big(c, z_2(c)\big)$ are holomorphic on U and hence by the Open Mapping Principle they map any neighborhood in U of c_0 onto a neighborhood of 1. Let γ be a closed arc in U winding around c_0 with $\gamma(0) = c^* = \gamma(1)$. Then there is a $t_0 \in I$ such that $\big|\lambda_1\big(\gamma_\epsilon(t_0)\big)\big| > 1$ and $\big|\lambda_2\big(\gamma_\epsilon(t_0)\big)\big| < 1$. By Theorem 2.2.5 the point $z_1\big(\gamma_\epsilon(t_0)\big)$ is the landing point of a dynamic ray and its orbit has therefore a portrait $\mathcal{P} \neq \emptyset$. Now we assume without loss of generality that there is a t_1 with $t_0 < t_1$ such that $\big|\lambda_1\big(\gamma_\epsilon(t_1)\big)\big| < 1$. Since an attracting point is never the landing point of a dynamic ray and the portrait of repelling points is stable under perturbation (see Corollary 3.2.4), there must be a $t^* \in (t_0, t_1)$ such that $\gamma_\epsilon(t^*)$ is on a parameter ray with angle of period k. This parameter ray lands at c_0 by Theorem 3.2.3, since by assumption no other parameters with parabolic orbit of period k are in U. The angle of this parameter ray landing at c_0 must be equal to the angle of a dynamic ray landing at the parabolic orbit of c_0. Since in the essential case the parabolic orbit portrait has two cycles, we can apply the above discussion for both cycles and obtain that at least two parameter rays land at c_0.

In the second case there is a two-sheeted cover $\pi\colon U' \to U$ of a neighborhood U of c_0 with the only ramification point $c_0 = \pi(c_0')$ and a holomorphic function $z\colon U' \to \mathbb{C}$ such that $z(c')$ is a point of period k and $z(c_0')$ is a parabolic point. Hence, the multiplier $\lambda(c') := \lambda\big(\pi(c'), z(c')\big)$ is holomorphic and $\lambda(c_0') = 1$. Now, let $c_1', c_2' \in U'$ such that $\pi(c_1') = \pi(c_2')$ but $c_1' \neq c_2'$ and consider an arc γ in U' such that $\gamma(0) = c_1', \gamma(1) = c_2'$ and $\pi\big(\gamma(I)\big)$ is a closed arc in U. We may assume by the Open Mapping Principle that $|\lambda(c_1')| < 1$ and this implies that $|\lambda(c_2')| > 1$. Hence, there is a $t^* \in I^\circ$ such that $z\big(\gamma(t^*)\big)$ lies on a parameter ray of period k. This ray can only land at c_0. As in the first case it follows that for an essential portrait a second parameter ray lands at c_0 and the proof is finished. □

4 Hyperbolic Components

In this section we introduce the so-called hyperbolic components of $\mathcal{M}_d$. These are the connected components of parameters which have attracting orbits.

Comparing this proof of the Structure Theorem with the proofs of Milnor and Schleicher in [M98] and [S97] we notice that they do not use hyperbolic components for their proofs. In contrary their discussion of hyperbolic components bases on the Structure Theorem. This has two reasons: the first one is that we develop in Section 5 the tool of orbit separation starting from parameters with superattracting orbits which we have to perturb through hyperbolic components to parabolic parameters. The second and more important reason is that we have to see for our proof that periodic parameter rays land in groups. In the quadratic case this is easier because there all periodic parameter rays land pairwise. However, for $d > 2$ this is not true and they land in groups of d rays at hyperbolic components.

We start with the definition of hyperbolic components and related objects. Moreover, we will associate these objects in Theorems 4.2 and 6.1 with parabolic parameters and stability of the landing points of dynamic rays under perturbation of the parameter.

Definition. (Hyperbolicity)
For a parameter $c \in \mathbb{C}$ we call f_c a *hyperbolic map* if it has an attracting orbit. A parameter c is *hyperbolic* if f_c is a hyperbolic map. $\Diamond$

These terms are due to Douady and Hubbard (see [DH85]). However, we should note that the definition of hyperbolicity given by them is more general (but equivalent for our maps) than the one we use here. See also Section 14 in [M90] for the discussion of hyperbolic maps.

Definition. (Hyperbolic Component)
A *hyperbolic component with period* n of $\mathcal{M}_d$ is a connected component of the set of parameters which have an attracting orbit with exact period n. $\Diamond$

Lemma 4.1. (Elementary Properties of Hyperbolic Components)
Let $\mathcal{H}$ be a hyperbolic component with period n. Then $\mathcal{H}$ is an open subset of $\mathcal{M}_d$ and there is a holomorphic map $z\colon \mathcal{H} \to \mathbb{C}$ such that $z(c)$ is attracting for all $c \in \mathcal{H}$ and has exact orbit period n.

Proof: Since $\mathcal{H}$ has period n there is per definitionem a $c_0 \in \mathcal{H}$ with attracting orbit of exact period n. By Lemma 2.2.2 f_{c_0} has a well-defined attracting orbit. Attracting means that the absolute value of the corresponding multiplier is less than 1 and hence by Lemma 3.2.1 there is a map $z\colon \mathcal{H} \to D$ with the required properties. By the same lemma it follows that every $c' \in \mathcal{H}$ has a neighborhood U such that all $c \in U$ are hyperbolic, i.e. $\mathcal{H}$ is open. Moreover, if f_c has an attracting orbit, then the critical point is in the immediate basin of this orbit (see the remark before Lemma 2.2.2) and thus the critical orbit is bounded, i.e. $c \in \mathcal{M}_d$ and this means $\mathcal{H} \subset \mathcal{M}_d$. $\Box$

Definition. (Roots, Co-Roots and Centers)
Let $\mathcal{H}$ be a hyperbolic component of period n. Then a parameter on $\partial\mathcal{H}$ with essential parabolic orbit of exact ray period n is called a *root* of $\mathcal{H}$. Similarly a parameter on $\partial\mathcal{H}$ with non-essential parabolic orbit of exact ray period n is called a *co-root* of $\mathcal{H}$. Moreover, a parameter in $\mathcal{H}$ which has a superattracting orbit of exact period n is called a *center* of $\mathcal{H}$. $\Diamond$

Theorems 6.5 and 6.4 will show that every hyperbolic component has exactly one root and one center. Moreover, we will see in Corollary 6.6 that the number of co-roots of a hyperbolic component is exactly $d-2$. In particular in the quadratic case there are no co-roots.

The following theorem gives us again information about parabolic parameters. In particular it shows that the parabolic orbit splits into several orbits if we perturb the parameter. The proof of this theorem in the quadratic case generalizes immediately to $d \geq 2$ (see Lemmas 6.1 and 6.2 in [M98] and Lemma 5.1 in [S97]).

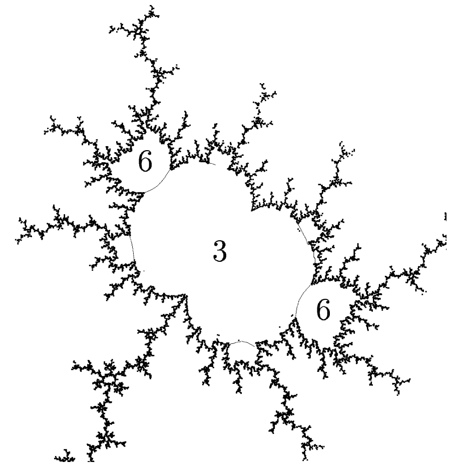


Figure 4: Here we can see hyperbolic components of $\mathcal{M}_3$. Among them is one with period 3, which has two neighboring components of period 6.

Theorem 4.2. (In the Neighborhood of Parabolic Parameters)
Let c_0 be a parabolic parameter with exact parabolic orbit period k, exact ray period n and z_0 a point of the parabolic orbit of c_0. Then the following holds:

(*1*) *If the parabolic orbit portrait is non-primitive then c_0 lies on the boundary of hyperbolic components with period k and n. Moreover, a neighborhood U of c_0 and a holomorphic function $z_1 \colon U \to \mathbb{C}$ exist such that $z_1(c)$ is a point of exact period k and $z_1(c_0) = z_0$. Furthermore, there exists for every $c \in U - \{c_0\}$ an orbit $\mathcal{O}(c)$ with exact period n that merges into the parabolic orbit $\mathcal{O}(c_0)$ as $c \to c_0$ and for that $c \mapsto \lambda\big(c, \mathcal{O}(c)\big)$ is holomorphic on U.*

(*2*) *If the parabolic orbit portrait is primitive then c_0 is a root or co-root of a hyperbolic component with period n. Furthermore, a two-sheeted cover $\pi \colon U' \to U$ of a neighborhood U of c_0 with the only ramification point $\pi(c_0') = c_0$ and a holomorphic function $z \colon U' \to \mathbb{C}$ exist such that $z(c)$ is a point of exact period n and $z(c_0') = z_0$.*

Proof: First we consider the case of a non-primitive parabolic orbit portrait. It follows by Lemma 2.2.7 and Lemma 3.2.1 that there are a neighborhood U of c_0 and a holomorphic function $z_1 \colon U \to \mathbb{C}$ such that $z_1(c)$ is a point with exact orbit period k and $z_1(c_0) = z_0$. Moreover, the multiplier $\lambda\big(c, z_1(c)\big)$ is a holomorphic function in c on U because $z_1(c)$ is. By the Open Mapping Principle and $\big|\lambda\big(c_0, z_1(c_0)\big)\big| = 1$ it follows that every neighborhood of c_0 contains parameters c such that $z_1(c)$ is attracting. Thus, c_0 lies on the boundary of a hyperbolic component with period k.

In order to show that it lies also on the boundary of a hyperbolic component with period n, we have to prove that an attracting orbit with exact period n exists. Since $\lambda(f_{c_0}^{\circ n}, z_0) = 1$, we obtain $f_{c_0}^{\circ n}(z) = z + a(z-z_0)^{q+1} + O\big((z-z_0)^{q+2}\big)$ for an integer $q \geq 1$, $a \in \mathbb{C}$ as the Taylor expansion of $f_{c_0}^{\circ n}$ near z_0. This means by the Leau-Fatou Flower Theorem that z_0 has q attracting petals and that $f^{\circ n}$ is the first iterate of f which fixes them and the q dynamic rays landing at z_0. These rays are permuted transitively by the first return map $f^{\circ k}$ because f_c has only one critical point. Moreover, $\lambda(f_{c_0}, z_0) = 1$ implies by Lemma 2.2.7 that $\lambda(f_{c_0}^{\circ k}, z_0)$ is an exact n/k-th root of 1. Finally this shows $q = n/k$. As in the proof of Lemma 3.2.6 it follows by the Argument Principle that there are neighborhoods U of c_0 and V of z_0 such that $f_c^{\circ n}(z)$ has for every $c \in U$ exactly $n/k + 1$ fixed points in V, counted with multiplicities. We are interested in the exact periods of these points with respect to f_c for $c \in U - \{c_0\}$. By the above discussion exactly one of them has exact period k and no one has a lower period. Since $\lambda(f_{c_0}^{\circ l \cdot k}, z_0) \neq 1$ for $l = 2, \ldots, n/k - 1$, we get that the iterates $f_c^{\circ l \cdot k}$ have for $c \in U$ exactly one fixed point in V or in other words: for every $c \in U$ and $l \in \{2, \ldots, n/k - 1\}$ the map f_c has exactly one point of period $l \cdot k$ in V. But we already know that there is a point with exact period k with respect to f_c in V. Hence, all the $l \cdot k$-periodic points are just the point with exact period k in V. This shows that n/k points in V have exact period n. Therefore, we have for every $c \in U - \{c_0\}$ an orbit $\mathcal{O}(c)$ with exact period n and well-defined multiplier such that n/k points of $\mathcal{O}(c)$ each coalesce at a point of the parabolic orbit $\mathcal{O}(c_0)$ if $c \to c_0$. The multiplier $c \mapsto \lambda\big(c, \mathcal{O}(c)\big)$ defines a holomorphic function on every simply connected region in $U - \{c_0\}$. Since the multiplier is bounded on U, we can continue it to a holomorphic function on U. Again by the Open Mapping Principle it follows with $|\lambda\big(c_0, \mathcal{O}(c_0)\big)| = 1$ that every neighborhood of c_0 contains parameters c such that $\mathcal{O}(c)$ is an attracting orbit, i.e. c_0 lies on the boundary of a hyperbolic component with period n. This proves the theorem in the non-primitive case.

In the primitive case the second assertion, i.e. the existence of the holomorphic function, follows directly by Lemma 3.2.6. This implies again by using the Open Mapping Principle the first assertion. □

As a consequence we should note the correspondence between parabolic parameters and roots. For a further discussion of hyperbolic components the so-called multiplier map of a hyperbolic component is very important.

Definition. (Multiplier Map of a Hyperbolic Component)
Let $\mathcal{H}$ be a hyperbolic component and for $c \in \mathcal{H}$ let $\lambda(c, \mathcal{O})$ be the multiplier of the attracting orbit of f_c. Then we call $\lambda_{\mathcal{H}}\colon \mathcal{H} \to D, c \mapsto \lambda(c, \mathcal{O})$ the *multiplier map of* $\mathcal{H}$. ◊

It is immediate that $\lambda_{\mathcal{H}}$ is well-defined: for hyperbolic parameters there is an unique attracting orbit and the absolute value of the multiplier of an attracting orbit is less than one. Next we want to show some interesting properties of the multiplier map $\lambda_{\mathcal{H}}$:

Lemma 4.3. (Properties of the Multiplier Map of a Hyperbolic Component)
The multiplier map $\lambda_{\mathcal{H}}$ of a hyperbolic component $\mathcal{H}$ is a proper holomorphic map and has a continuous extension $\lambda_{\overline{\mathcal{H}}}$ from $\overline{\mathcal{H}}$ onto $\overline{D}$.

Proof: Let k be the period of $\mathcal{H}$. By Lemma 4.1 there is a holomorphic map z on $\mathcal{H}$ such that $z(c)$ is an attracting point with exact period k. This implies that $\lambda_{\mathcal{H}}$ is holomorphic on $\mathcal{H}$. For parameters $c \in \partial\mathcal{H}$ which do not have a parabolic point with ray period k we can continue $z(c)$ analytically in a neighborhood of c by Lemma 3.2.1. In the other parameters on $\partial\mathcal{H}$ we can still continue $z(c)$ by Theorem 4.2. Moreover, for two sequences (c_i) and (c_j) in $\mathcal{H}$ which converge to a parameter $c \in \partial\mathcal{H}$ the limits $\lim \lambda_{\mathcal{H}}(c_i)$ and $\lim \lambda_{\mathcal{H}}(c_j)$ are equal, because every $c \in \partial H$ has an unique indifferent orbit. This shows that $\lambda_{\mathcal{H}}$ has a continuous extension $\lambda_{\overline{\mathcal{H}}}\colon \overline{\mathcal{H}} \to \overline{D}$.

Now consider a sequence $(c_n)_{\mathbb{N}}$ in $\mathcal{H}$ with $c_n \to c \in \partial\mathcal{H}$ and assume that there exists a compact set $K \subset D$ such that $\lambda_{\mathcal{H}}(c_n) \in K$ for all $n \in \mathbb{N}$. Then there is a $\lambda < 1$ with $|\lambda_{\mathcal{H}}(c_n)| < \lambda < 1$ for all $n \in \mathbb{N}$, i.e., all accumulation points of $\big(\lambda_{\mathcal{H}}(c_n)\big)$ have absolute value less than one. But this is a contradiction to $\lim |\lambda_{\mathcal{H}}(c_n)| = |\lambda_{\overline{\mathcal{H}}}(c)| = 1$ by continuity of $|\lambda_{\overline{\mathcal{H}}}|$. $\square$

5 Orbit Separation

Our aim is now to get more information on the landing properties of periodic parameter rays. In particular we want to show that if a periodic parameter ray lands at a parameter then the dynamic ray at the same angle lands at the characteristic point of the parabolic orbit. The main tool we use for this is the concept of orbit separation, which is due to Schleicher. We will prove certain orbit separation lemmas (or orbit separation properties). Per se they only concern the dynamic plane and guarantee the existence of dynamic rays separating certain points. Two points z and z' are *separated* by dynamic rays in K_c if angles ϑ and ϑ' exist such that $R^c_\vartheta, R^c_{\vartheta'}$ land at a common point, say z_0, and z and z' are in different components of $\mathbb{C} - (R^c_\vartheta \cup R^c_{\vartheta'} \cup z_0)$. Following Schleicher we call two such rays together with their landing point a *ray pair at angles* (ϑ, ϑ'). In the concluding Subsection 5.3 these techniques lead to the statement mentioned just before. Furthermore, combining this with the results of Section 3 we are able to show that at every primitive parameter exactly two or one parameter rays land according to whether it is essential or not (Corollary 5.3.2). As an analogue to Theorem 3.2.7 for non-primitive parameters we show that every essential parabolic parameter is the landing point of the parameter rays at the characteristic angles of its parabolic portrait (Corollary 5.3.3).

For a discussion of orbit separation in the quadratic case see Sections 3 and 5 in [S97]. The following approach is based on this and the ideas are the same.

5.1 Hubbard Trees

In order to prove the orbit separation properties we start from parameters with superattracting orbits and so-called Hubbard trees connecting them. This is possible since Julia sets of hyperbolic maps are locally connected and hence arcwise connected. We will show that in the superattracting case a ray pair exists which separates the critical value from any other point of the critical orbit. Then we verify that this separation remains stable under perturbation of the parameter to a parabolic parameter through a hyperbolic component. This will show us that any two points of the parabolic orbit can be separated.

As suggested by Schleicher we use now standard Hubbard trees and not parabolic Hubbard trees as in [S97] because some arguments are easier to see for standard Hubbard trees and the local connectivity of hyperbolic Julia sets is much easier to prove than the one of parabolic ones.

During the whole subsection we assume that all superattracting orbits have at least exact period 2 (for the case of a superattracting fixed point we do not need Hubbard trees). First we give the definition of a tree and some properties:

Definition. (Hubbard Tree)
Let c be a parameter with superattracting orbit $\mathcal{O}$. Then a compact and connected subset Γ_c of K_c is called *Hubbard tree for* c if

(1) every end point of Γ_c is a point of the superattracting orbit and

(2) for every Fatou component U of K_c the intersection $\Gamma_c \cap \overline{U}$ is either empty or is the union of finitely many internal rays of U.

$\Diamond$

For the definition of an internal ray see Subsection 2.2. In particular note that the "landing point", i.e. the point in ∂K_c, is part of the ray.

An *end point* of a tree Γ is a point $z \in \Gamma$ for which $\Gamma - \{z\}$ is connected.

By an arc connecting a point z with some arcwise connected set $S \neq \emptyset$ in K_c we mean an arc γ such that γ connects z with a point of S and $\gamma(t) \notin S$ for all $t \in [0, 1)$. Evidently such an arc always exists.

This definition raises some questions. In particular we should say a few words about existence and uniqueness. Before that we note that the superattracting orbit of a map f_c is precisely the critical orbit. We have the following lemma:

Lemma 5.1.1. (Existence and Uniqueness of Hubbard Trees)
For every parameter c with superattracting orbit $\mathcal{O}$ there exists a Hubbard tree and it is unique, i.e. if Γ_1 and Γ_2 are Hubbard trees for c then $\Gamma_1 = \Gamma_2$.

Proof: We give a strategy how to construct a Hubbard tree for c: since hyperbolic filled-in Julia sets are compact, connected and locally connected (see Theorem 17.5 in [M90]), they are by Lemma 16.4 and Lemma 16.3 in [M90] arcwise connected. Therefore, we can start the construction of a Hubbard tree by connecting the critical point with another point of the critical orbit by an arc in K_c. After that we iteratively connect any point of $\mathcal{O}$ (which is not already part of the tree) by an arc in K_c with the tree constructed so far. Since $\mathcal{O}$ is finite this process terminates after finitely many steps. During the whole process we require that for every Fatou component U of K_c the intersection of $\overline{U}$ and the so far constructed tree is either empty or is the union of finitely many internal rays of U. This is possible because for every Fatou component U the Riemann map $\varphi\colon U \to D$ extends to a homeomorphism on $\overline{U}$. By construction the tree is compact, connected (even arcwise connected) and every end point is a point of $\mathcal{O}$, i.e. the tree is a Hubbard tree for c.

We prove the second assertion by contradiction: we assume that there are Hubbard trees Γ_1 and Γ_2 for c such that there is a point $z^* \in \Gamma_1$ but $z^* \notin \Gamma_2$. Let $z^* \notin \partial K_c$. Then it lies by the definition of a Hubbard tree on an internal ray, say R^U_ϑ, of a Fatou component U. Since any two internal rays of a Fatou component have a common point in ∂K_c if and only if they are identically, we may assume that $z^* \in \partial K_c$. Then z^* is not an end point of Γ_1 and hence there are points $z, z' \in \mathcal{O}$ and an arc γ_1 connecting z with z' in Γ_1 such that $z^* \in \gamma_1(I^\circ)$. Moreover, there is an arc γ_2 connecting z and z' in Γ_2. Since $z^* \in K_c$ this means that $\gamma_1(I)$ and $\gamma_2(I)$ bound a subset of ∂K_c in contradiction to the fact that K_c is full. $\square$

Moreover, we should note that the definition using internal rays implies that the Hubbard tree for a parameter c is invariant under forward mapping by f_c.

The next lemma shows us in particular that the critical value is an end point of every Hubbard tree. This will be important in the proof of the Orbit Separation Lemmas.

Lemma 5.1.2. (Intersection Properties of Hubbard Trees)
Let c be a parameter with superattracting orbit $\mathcal{O}$ and Γ a Hubbard tree of c. Then the intersection of Γ and the boundary of the Fatou component containing the critical value consists of exactly one periodic point. However, the intersection of Γ and the boundary of any other bounded Fatou component consists of at most d points which are periodic or preperiodic points of $\mathcal{O}$.

Proof: Let U_0 be the Fatou component containing the critical point and $U_1, \dots, U_{n-1}$ the other bounded periodic components with $U_l := f_c^{\circ l}(U_0)$. It is convenient to define $U_n := U_0$. Moreover, we denote by a_l the number of points that the intersection of Γ with ∂U_l has. Now consider two different points $z, z' \in \mathcal{O}$ and an arc γ connecting these two points in Γ and let $z^* \in \partial U_l \cap \gamma(I)$ for an l. Since Γ is unique and invariant, we see that $f_c(z^*) \in \partial U_{l+1} \cap \Gamma$. This means that intersection points of Γ with the boundary of periodic Fatou components are mapped onto such intersection points. Since $f_c\colon \overline{U}_l \to \overline{U}_{l+1}$ is an one-to-one map for $l \in \{1, 2, \ldots, n-1\}$ and a d-to-one map for $l = 0$, we have $a_0/d \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_0$.

We want to show that there is an $l_1 \in \{0, 1, 2, \ldots, n-1\}$ with $a_{l_1} = 1$. Note that if $\overline{U}_l$ does not disconnect the tree, i.e., $\Gamma - \overline{U}_l$ is connected, then $a_l = 1$ because only points of the critical orbit—which are inside a bounded and periodic Fatou component—can be end points of the tree. We assume that every $\overline{U}_l$ disconnects Γ and consider one of the components of $K_c - \overline{U_0}$ and denote it by $K_c^{(0)}$. Moreover, we denote the part of Γ which lies in $K_c^{(0)}$ by $\Gamma^{(0)}$. Due to the fact that Γ connects the critical orbit $\Gamma^{(0)}$ contains at least one point of the critical orbit lying in a periodic Fatou component. Let U_{l_1} be one of those Fatou components such that no point of the critical orbit lies on the part of the tree connecting the point of the critical orbit in U_0 and U_{l_1}. Since $\overline{U}_{l_1}$ disconnects Γ, there is again at least one component of $K_c^{(0)} - \overline{U}_{l_1}$ such that it contains a point of the critical orbit. By iterating we get a contradiction because the critical orbit is finite. Hence, there is an $l_1 \in \{0, 1, 2, \ldots, n-1\}$ with $a_{l_1} = 1$ and $\Gamma - \overline{U}_{l_1}$ is connected. Using this together with the above inequality we obtain $a_0/d \leq a_1 \leq a_{l_1} = 1$ and $a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_0 \leq d$. This proves the lemma. $\square$

It is well known (Theorem 2.2.5) that a point z in a locally connected filled-in Julia set K which disconnects K is the landing point of as many rays as $K - \{z\}$ has components. Necessarily z is contained in ∂K. As mentioned before we want to prove the existence of dynamic ray pairs separating certain points. This is the reason for our interest in branch points of a Hubbard tree:

Definition. (Branch and Branch Point)
Let c be a parameter with superattracting orbit and let Γ be the Hubbard tree of c. Then for $z \in \Gamma$ the components of $\Gamma - \{z\}$ are called *branches of* Γ *at* z.

If the number of branches with respect to z is greater or equal to three then z is called *branch point* of Γ. $\Diamond$

Lemma 5.1.3. (Properties of Branch Points)
Let c be a parameter with superattracting orbit and Γ the Hubbard tree for c. Consider a $z \in \Gamma$ such that Γ has m branches at z. Then:

(1) *If z does not lie in the closure of the critical Fatou component, i.e. the Fatou component that contains the critical point, then the image $f_c(z)$ has at least m branches.*

(2) *However, if z does lie in the closure of the critical Fatou component then the image $f_c(z)$ has at least $m-1$ branches.*

(3) *If z is a branch point then it is periodic or preperiodic and lies on a repelling or the superattracting orbit.*

Proof: The first statement can be verified by Lemma 5.1.1: since z has m branches, there is a subset $\Gamma_z \subset \Gamma$ connecting a subset M of the critical orbit with m points such that $\Gamma_z - \{z\}$ has m components. Then for any two points $z', z'' \in M$ there is an arc γ connecting z' with z'' in Γ_z and hence $f_c \circ \gamma$ connects $f_c(z')$ with $f_c(z'')$ which lies on the critical orbit of f_c. Since the Hubbard tree is invariant and z is not on the boundary of the critical Fatou component, the restriction of f_c to a neighborhood of z is one-to-one and the assertion follows.

If z lies on the critical Fatou component all branches of z but possibly the branches which are in the critical Fatou component are mapped homeomorphically by f_c. Therefore, we can at most lose the branch of z which is in the critical Fatou component, i.e., the number of branches of $f_c(z)$ is possibly by one smaller than the one of z. This shows (*2*).

Now Statement (*3*) is obvious because the Hubbard tree has only finitely many branches and by Lemma 2.2.2 all periodic orbits, except for the superattracting orbit, are repelling. □

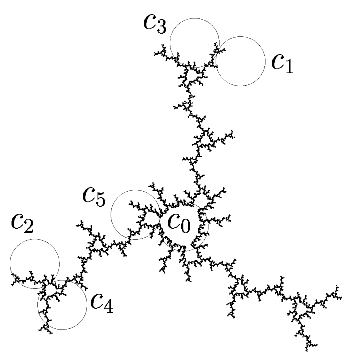


Figure 5: The Julia set of a map $z \mapsto z^3 + c$ with 6-periodic superattracting orbit (c is near $0.279484 + 1.25140\,i$). The Fatou components which contain the points $c_l = f_c^{\circ l}(0)$ of the critical orbit are inside the corresponding disks.

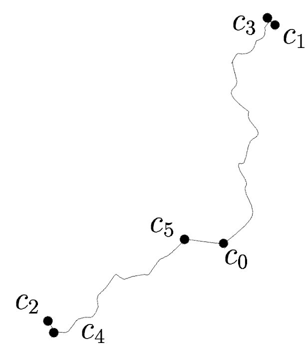


Figure 6: The Hubbard tree for the Julia set on the left hand side. As we can see it is unbranched in this case. Note that $c_1 = c$ is an end point of the tree.

5.2 Two Orbit Separation Lemmas

Now we have sufficient techniques to prove the following two orbit separation lemmas. The first one shows that parabolic points can be separated from each other (see also Lemma 3.7 in [S97]). Similarly the second one shows that parabolic and repelling periodic points can be separated in the case of a primitive parabolic portrait. In Section 5 in [S97] it is called Orbit Separation Property.

Lemma 5.2.1. (Orbit Separation Lemma For Two Parabolic Points) *Let c_0 be a parabolic parameter and z, z' associated different parabolic points. Then there is a ray pair separating z from z'.*

Proof: Let c_1 be a center of a hyperbolic component $\mathcal{H}$ with period n for which c_0 is a root or co-root and let Γ be the Hubbard tree of c_1. (By Theorem 4.2 we know that c_0 is a root or co-root of a hyperbolic component.) First we show that for every point z_1 of the critical orbit of f_{c_1}, except for c_1 itself, there is a ray pair separating z_1 from c_1. Let γ be an arc connecting z_1 with c_1 in Γ. Without loss of generality we assume that no point of the critical orbit lies on $\gamma(I^\circ)$. If there is a branch point of Γ on $\gamma(I^\circ)$, there is a ray pair separating z_1 from c_1 by Theorem 2.2.5. Otherwise let $m \geq 1$ be the least integer such that $f_{c_1}^{\circ m}(z_1) = c_1$. Then $f_{c_1}^{\circ m} \circ \gamma$ connects c_1 with $f_{c_1}^{\circ m}(c_1)$ and by the uniqueness of the Hubbard tree $f_{c_1}^{\circ m} \circ \gamma(I) \subset \Gamma$. Since no point of $\gamma(I^\circ)$ is a branch point or a point of the critical orbit and c_1 is by Lemma 5.1.2 an end point, there is a $t^* \in I^\circ$ such that $\gamma(t^*) = f_{c_1}^{\circ m} \circ \gamma(t^*)$. This point $z^* := \gamma(t^*)$ must be repelling, since it is periodic and not a point of the critical orbit, which is the only non-repelling periodic orbit (see Lemma 2.2.2). It follows that $z^* \in \partial K_c$ and thus that $\partial K_c - \{z^*\}$ is disconnected. Therefore, it is by Theorem 2.2.5 the landing point of a dynamic ray pair, say at angles ϑ and ϑ'. These rays separate c_1 from z_1 because Γ has at z^* only two branches and c_1 and z_1 are on different branches.

Next we want to show that we can perturb c_1 to c_0 through the hyperbolic component $\mathcal{H}$ such that c_1 and z_1 become the characteristic point and some other point of the parabolic orbit of c_0, respectively, and the ray pair at angles ϑ and ϑ' separates the two points. Using Lemma 4.1 and Corollary 3.2.4 it is easy to see that there are continuous continuations z_{z_1}, z_{c_1} and z_{z^*} of z_1, c_1 and z^*, respectively, to $\mathcal{H} \cup \{c_0\}$ such that $z_{c_1}(c_0)$ is the characteristic point and $z_{z_1}(c_0)$ another point of the parabolic orbit of c_0 and $z_{z^*}(c)$ is the landing point of the ray pair at angles ϑ and ϑ' for all $c \in \mathcal{H} \cup \{c_0\}$. Hence, this ray pair separates the characteristic point from $z_{z_1}(c_0)$. Every point of the critical orbit has such a continuation to $\mathcal{H} \cup \{c_0\}$, which leads to a point of the parabolic orbit of c_0. Moreover, every point of the parabolic orbit of c_0 splits into an attracting and repelling point if the parameter is perturbed into $\mathcal{H}$ by Theorem 4.2. Therefore, every parabolic point of c_0 can be separated from the characteristic point.

If z and z' are parabolic points different from the characteristic point z_1 let m be the least natural number with $z = f_{c_0}^{\circ m}(z_1)$. Without loss of generality we assume that $f_{c_0}^{\circ m'}(z_1) \neq z'$ for $0 < m' < m$. Using the above result we obtain that there are two rays landing at a common periodic or preperiodic point z^* of a repelling orbit and separating z_1 from the periodic point $f_{c_0}^{\circ(-m)}(z')$, defined by the pull-back along the parabolic orbit. Hence, the m-th forward image of z^* together with the rays landing there separate z and z'. $\square$

Lemma 5.2.2. (Orbit Separation Lemma for One Parabolic and One Repelling Point)
Let c_0 be a primitive parabolic parameter and k the exact period of the parabolic orbit with the characteristic point z_1. Moreover, let z' be any repelling point with orbit period k which does not lie on the boundary of the characteristic Fatou component. Then there is a ray pair separating z_1 from z'.

Proof: As in the proof before let c_1 be a center of a hyperbolic component $\mathcal{H}$ for which c_0 is a root or co-root. (Theorem 4.2 shows us that c_0 is a root or co-root of a hyperbolic component.) By Theorem 4.2 and Corollary 3.2.4 there are continuous functions $z_1(c)$ and $z'(c)$ on $\mathcal{H} \cup \{c_0\}$ such that $z_1(c_0) = z_1$ and $z'(c_0) = z'$ and $z_1(c_1) = c_1$ and $z'(c)$ is for all $c \in \mathcal{H} \cup \{c_0\}$ a repelling point of the same period as z'. Since z' does not lie on the boundary of the characteristic Fatou component, $z'(c_1)$ does not lie in the closure of the Fatou component which contains the critical value. Let Γ be the Hubbard tree for c_1 and γ be an arc in Γ which connects c_1 with $z'(c_1)$. Since every bounded Fatou component is periodic or preperiodic, the number

$$m := \min\{\, n \in \mathbb{N}_0 : f_{c_1}^{\circ n} \circ \gamma(I^\circ) \cap U \neq \emptyset \text{ for a periodic Fatou component } U \,\}$$

is well-defined. By construction $f_{c_1}^{\circ m} \circ \gamma$ connects a boundary point of a periodic Fatou component containing a point of the critical orbit, say c^*, with $f_{c_1}^{\circ m}(c_1)$. By construction any ray pair separating c^* and $f_{c_1}^{\circ m}(c_1)$ separates also $z'(c_1)$ and c_1. Such a ray pair exists by the previous Orbit Separation Lemma because two periodic Fatou components do not have any common boundary point in the primitive case. As before we can see that the landing point of the ray pair remains repelling if we perturb from c_1 to c_0. □

5.3 Parameter Space and Orbit Separation

In this subsection we use the just developed tools to obtain more information about the landing properties of periodic parameter rays. If we consider a periodic parameter ray at a given angle, we are now able to say more about the landing point of the dynamic rays at the same angle. This enables us to show that at least certain rays land pairwise and give even a complete description of the periodic rays landing at primitive parabolic parameters.

The following theorem can be found as Proposition 3.2 in [S97] for the quadratic case and the proof is the same for $d \geq 2$.

Theorem 5.3.1. (A Necessary Condition)
If a periodic parameter ray $R_\vartheta^{\mathcal{M}}$ lands at a parameter c_0 then the dynamic ray $R_\vartheta^{c_0}$ lands at the characteristic point of the parabolic orbit of c_0.

Proof: Consider a periodic parameter ray $R_\vartheta^{\mathcal{M}}$ landing at a parameter c_0, which is necessarily parabolic by Theorem 3.2.3. Then again by Theorem 3.2.3 the dynamic ray $R_\vartheta^{c_0}$ lands at a point of the parabolic orbit of c_0. For a parabolic fixed point this proves the theorem. Therefore, we assume now that the exact parabolic orbit period is at least 2. The Orbit Separation Lemma 5.2.1 tells us that for the characteristic point z_1 and any other point z_i of the parabolic orbit there is a ray pair at angles $\vartheta_i, \vartheta_i'$ separating both parabolic points. Hence, there is a finite number of ray pairs landing at repelling points and dividing the complex plane in several components such that in the component containing the characteristic point z_1 is no other point of the parabolic orbit. By Theorem 3.2.2 there is a neighborhood U of c_0 such that the landing points of the rays at angles $\vartheta_i, \vartheta_i'$ depend holomorphically on c for all $c \in U$, i.e., the continuation of the characteristic point is still separated from the continuations of the other parabolic points. Combining this with the fact that for all parameters c on the parameter ray $R_\vartheta^{\mathcal{M}}$ the critical value c lies on the dynamic ray R_ϑ^c (Lemma 2.3.3) we obtain that R_ϑ^c must land in the partition containing the critical value. Therefore, again by Theorem 3.2.3 it lands at the characteristic point of the parabolic orbit of c_0. □

There are some immediate consequences. We should note that the following corollary is only of partial interest in the quadratic case, because there are no non-essential parameter rays, i.e. no parameter ray lands for $d = 2$ alone. However, it gives us a complete description of the periodic parameter rays landing at primitive parameters.

Corollary 5.3.2. (Parameter Rays Landing at Primitive Parameters) *Every primitive parabolic parameter c is the landing point of exactly two periodic parameter rays if c is essential and exactly one periodic parameter ray lands at c if it is non-essential. Moreover, the dynamic rays at the same angles lands in the dynamic plane of c at the characteristic point of the parabolic orbit of c.*

Proof: Choose an arbitrary primitive parabolic parameter c. We obtain by Theorem 3.2.7 that at least two periodic parameter rays land at c if c is essential and at least one periodic parameter ray lands at c if it is non-essential.

On the other hand, because c has a primitive parabolic portrait, we see by Theorem 5.3.1 immediately that at most two or one parameter rays can land at c according to whether c is essential or not. This finishes the proof. $\square$

This proves Statement (*2*) of the Structure Theorem 1.1. However, the proof of a corresponding statement for non-primitive parameters is not as easy. We will work on this during the following two sections.

There is another corollary, which uses Milnor's Theorem 3.2.5 and Schleicher's orbit separation. Somehow it is an analogue in the non-primitive case to Theorem 3.2.7 and a sharpened form of Theorem 3.2.5. It is the same statement as Corollary 4.3 in [M98]. However, Milnor does the work which we do by orbit separation by a "deformation preserving the portrait" using certain "convenient coordinates". See Section 4 in [M98] for his discussion on this topic.

Corollary 5.3.3. (At Least the Characteristic Rays Land at a Parameter) *At every essential parabolic parameter land the parameter rays with the characteristic angles of the parabolic orbit portrait.*

Proof: In the primitive case we have already a stronger statement by Corollary 5.3.2. Therefore let c be a non-primitive parabolic parameter and denote the characteristic angles of the corresponding parabolic portrait $\mathcal{P}$ by ϑ_- and ϑ_+. Then, by Theorem 3.2.5 the rays $R^{\mathcal{M}}_{\vartheta_-}$ and $R^{\mathcal{M}}_{\vartheta_+}$ land at a common parabolic parameter c'. Using Theorem 5.3.1 we see that the dynamic rays $R^{c'}_{\vartheta_-}$ and $R^{c'}_{\vartheta_+}$ land at the characteristic point of $f_{c'}$. This means that the angles ϑ_- and ϑ_+ are contained in the same element of the parabolic orbit portrait $\mathcal{P}'$ of c'. The sets $A_{\mathcal{P}}$ and $A_{\mathcal{P}'}$ are identical because $\vartheta_-, \vartheta_+ \in A_{\mathcal{P}'}$ and $\mathcal{P}'$ is non-primitive and has hence only one cycle (Lemma 3.1.3). Since ϑ_- and ϑ_+ are the characteristic angles of $\mathcal{P}$, there is by Lemma 3.1.2 no angle within $(\vartheta_-, \vartheta_+)$ in $A_{\mathcal{P}} = A_{\mathcal{P}'}$. This implies that the parabolic orbit period of c' is less or equal than the one of c. As in the proof of Theorem 3.1.4 it follows that they are equal and therefore $(\vartheta_-, \vartheta_+)$ is the characteristic interval of $\mathcal{P}'$, too. This finishes the proof. $\square$

6 Hyperbolic Components and Portraits

In this section we will continue our investigations on hyperbolic components, which we introduced in Section 4. In particular we will show that every hyperbolic component has exactly one center, exactly one root and exactly $d-2$ co-roots. To do the proofs we need more information on the stability of portraits in the closure of a hyperbolic component. Obviously a necessary condition for the stability of a portrait is that the landing point of a dynamic ray at some angle in the portrait depends continuously on the parameter. This gives us the motivation to prove the following theorem. It is due to Schleicher (see Proposition 5.2 in [S97]) and his proof of the quadratic case generalizes easily to the case $d \geq 2$. It bases on the Orbit Separation Lemmas of the previous section.

Theorem 6.1. (Continuous Dependence of Landing Points on Parameters) *Let z_0 be a repelling or parabolic periodic point of f_{c_0}. For a dynamic ray $R_\vartheta^{c_0}$ landing at z_0 we define the set $\Omega(\vartheta) := \{ c \in \mathbb{C} : R_\vartheta^c \text{ lands} \}$. Then there is a continuous function $z\colon \Omega(\vartheta) \to \mathbb{C}$ such that $z(c)$ is the landing point of R_ϑ^c.*

Remark: In the non-primitive case two rays R_ϑ^c and $R_{\vartheta'}^c$ land at the same k-periodic point z if and only if $\vartheta = \sigma^{\circ l \cdot k}(\vartheta')$ for an $l \in \mathbb{N}_0$ (see Lemma 3.1.3). This means that the definition of $\Omega(\vartheta)$ does not depend on a particular ray in this case. However, in the primitive essential case there are two different cycles of rays and $\Omega(\vartheta)$ may depend on the angle for which we defined it.

Although the landing point of the dynamic ray depends continuously on the parameter c (if the ray lands), the portrait may be destroyed. This is certainly always the case whenever the orbits that we consider have different periods. For example it may occur that $z(c)$ splits while perturbing away from a parabolic parameter into several periodic points, among which the rays of the parabolic point are distributed. Then the portrait of $z(c)$ is disturbed.

Proof: For the subset of $\Omega(\vartheta)$ which contains parameters such that R_ϑ^c lands at a repelling point we have proved an even stronger statement in Corollary 3.2.4. Hence, we assume that c_0 is a parabolic parameter and z_0 is a point of the parabolic orbit $\mathcal{O}_p$ of c_0 and denote the exact parabolic orbit period by k and the corresponding exact ray period by n. We discuss two cases: the first one is that the portrait of the parabolic orbit is non-primitive, i.e. $k < n$. Now we consider a neighborhood U of c_0 and a holomorphic function $z_1\colon U \to \mathbb{C}$ as in Theorem 4.2 with $z_1(c_0) = z_0$. The multiplier $\lambda_1(c) := \lambda\big(c, z_1(c)\big)$ of the k-periodic point $z_1(c)$ is holomorphic on U because $z_1(c)$ is. If $\mathcal{O}(c)$ is the n-periodic orbit that merges into the parabolic orbit then the multiplier $\lambda_2(c) := \lambda\big(c, \mathcal{O}(c)\big)$ is holomorphic on U, too (see Theorem 4.2). For a dynamic ray at some angle which lands at z_0 we have to show that the dynamic ray at the same angle lands at $z_1(c)$ or at a point of $\mathcal{O}(c)$, if we go away from c_0 along a small arc in U on which c lies. Therefore, we consider an arc $\gamma\colon I \to U$ ending in c_0, i.e. $\gamma(1) = c_0$, such that $z_1\big(\gamma(t)\big)$ is repelling for all $\gamma(t)$, $t \in [0,1)$. By Theorem 2.2.5 at least one dynamic ray, say $R_{\vartheta'}^{\gamma(0)}$, lands at $z_1\big(\gamma(0)\big)$ and because the $z_1\big(\gamma(t)\big)$ are repelling for $t \in [0,1)$ we know by Corollary 3.2.4 that $R_{\vartheta'}^{\gamma(t)}$ lands at $z_1\big(\gamma(t)\big)$ for all $t \in [0,1)$. Since the parameter $\gamma(1) = c_0$ is in $\mathcal{M}_d$ and is parabolic, the dynamic ray $R_{\vartheta'}^{\gamma(1)}$ will still land at a n-periodic parabolic or repelling point of K_{c_0}. We can exclude ad hoc that the landing point is repelling because otherwise the ray would land at a point different from $z_1\big(\gamma(t)\big)$ in the dynamic plane of $\gamma(t)$, $t \in [0,1)$. To prove that this dynamic ray $R^{c_0}_{\vartheta'}$ will land at z_0, we use orbit separation: by the Orbit Separation Lemma 5.2.1 for any $z \in \mathcal{O}_p - \{z_0\}$ exists a ray pair in the dynamic plane of c_0 separating z from z_0. We denote this ray pair by $S(z)$ and let $P(c_0) := \bigcup S(z)$ for all $z \in \mathcal{O}_p - \{z_0\}$. By construction of $P(c_0)$ there is a component $V(c_0)$ such that z_0 is the only parabolic point of c_0 in $V(c_0)$. Since the landing points of the ray pairs are repelling, we can continue them by Theorem 3.2.2 in the, if necessary shrunken, neighborhood U of c_0 and get for every $c \in U$ a new partition $P(c)$. This partition $P(c)$ consists of the dynamic ray pairs at the same angles as the one of $P(c_0)$. By construction a dynamic ray is contained in a component of $P(c)$ if and only if it is contained in the component of $P(c_0)$ bounded by the ray pair at the same angles. Let $V(c)$ be the component of $P(c)$ which is bounded by the dynamic ray pair at the same angles as $V(c_0)$. Since $z_1\big(\gamma(t)\big) \in V\big(\gamma(t)\big)$ for $t \in I$, the dynamic rays $R^{\gamma(t)}_{\vartheta'}$ are also subsets of $V\big(\gamma(t)\big)$ for all $t \in I$. Since z_0 is the only parabolic point in $V(c_0)$, the dynamic ray $R^{c_0}_{\vartheta'}$ lands at z_0. Moreover, if we consider any other dynamic ray, say $R^{c_0}_{\vartheta}$, landing at z_0 then the dynamic ray $R^{\gamma(t)}_{\vartheta}$ will land at $z_1\big(\gamma(t)\big)$ for all $t \in I$: since c_0 is a non-primitive parabolic parameter, all dynamic rays landing at z_0 are in the same cycle (see Lemma 3.1.3). This means that for some integer $m \geq 1$ we have $\sigma^{\circ mk}(\vartheta') = \vartheta$ and hence $R^{\gamma(t)}_{\vartheta}$ lands at $z_1\big(\gamma(t)\big)$, too. If we consider an arc $\gamma\colon I \to U$ ending in c_0 such that $\big|\lambda_1\big(\gamma(t)\big)\big| < 1$, i.e. $\big|\lambda_2\big(\gamma(t)\big)\big| > 1$, then we can apply for parameters $c \in U$, U possibly shrunken, the same arguments as above to a point $z_2(c)$ that merges into the characteristic point of the parabolic orbit as $c \to c_0$. We obtain that if a dynamic ray, say $R^{\gamma(t)}_{\vartheta}$, lands at $z_2(c)$ then $R^{c_0}_{\vartheta}$ lands at the characteristic point of the parabolic orbit. The other dynamic rays which land at the parabolic orbit are distributed among the other points of the orbit of $z_2(c)$. This finishes the proof in the non-primitive case.

In the primitive case we could prove the assertion analogous for one ray landing at z_0. However, by this way we would not be able to handle in the essential case the other ray landing at z_0, since it is not in the same cycle. We can overcome this difficulty by using also the Orbit Separation Lemma 5.2.2 for a parabolic and repelling point, which holds only in the primitive case. We denote by $\mathcal{O}_r$ the set of all repelling points of f_{c_0} with orbit period n which do not lie on the boundary of the characteristic Fatou component. Now we start separating: by the Orbit Separation Lemmas 5.2.1 and 5.2.2 we know that for any $z \in \mathcal{O}_p \cup \mathcal{O}_r - \{z_0\}$ there is a ray pair, say $S(z)$, separating z from z_0. Therefore, the partition $P(c_0) := \bigcup S(z)$ for all $z \in \mathcal{O}_p \cup \mathcal{O}_r - \{z_0\}$ of $\mathbb{C}$ has the property that the points on the boundary of the characteristic Fatou component are the only parabolic and repelling points of period n in the component $V(c_0)$ containing z_0. Let $R^{c_0}_{\vartheta}$ be a dynamic ray landing at z_0 and if c_0 is essential let $R^{c_0}_{\vartheta'}$ be the second dynamic ray landing at z_0. By construction of $P(c_0)$ these rays are completely contained in $V(c_0)$. Since $P(c_0)$ consists of dynamic ray pairs landing at repelling points, there is by Theorem 3.2.2 a neighborhood U of c_0 such that for every $z \in \mathcal{O}_p \cup \mathcal{O}_r - \{z_0\}$ the ray pair separating z from z_0 in the dynamic plane of c_0 still has a repelling landing point for $c \in U$. This separation in the dynamic plane of c defines a partition $P(c)$ with component $V(c)$ which is bounded by the ray pair at the same angles as $V(c_0)$. Moreover, for all $c \in U - \{c_0\}$ the component $V(c)$ does not contain the continuations of the n-periodic repelling and parabolic points of c_0 other than z_0 and the repelling points on the boundary of the characteristic Fatou component. As before we use the results of Theorem 4.2: there is a two-sheeted cover $\pi\colon U' \to U$ with the only ramification point $\pi(c_0') = c_0$ of the, if necessary shrunken, neighborhood U of c_0 and a holomorphic function $z\colon U' \to \mathbb{C}$ such that $z(c')$ is a point of period n with respect to $f_{\pi(c')}$ and $z(c_0') = z_0$. It follows again that the multiplier $\lambda(c') := \lambda\big(\pi(c'), z(c')\big)$ is holomorphic on U', since z is, and hence by the Open Mapping Principle λ maps a neighborhood of c_0' onto a neighborhood of 1. This implies that if $\gamma\colon I \to U'$ is an arc ending in c_0', i.e. $\gamma(1) = c_0'$, and which is not closed, then there are branches $z_1, z_2\colon \pi \circ \gamma(I) \to \mathbb{C}$ of z. In particular for every $t \in I$ the only repelling or parabolic points in $V\big(\pi \circ \gamma(t)\big)$ with period n are $z_1\big(\pi \circ \gamma(t)\big)$ and $z_2\big(\pi \circ \gamma(t)\big)$ and the continuations of the n-periodic points on the boundary of the characteristic Fatou component. We have to show that $R_\vartheta^{\pi\circ\gamma(t)}$ and if c_0 is essential $R_{\vartheta'}^{\pi\circ\gamma(t)}$ do not land at the continuations of the n-periodic repelling points on the boundary of the Fatou component containing the critical value: since these rays land at repelling or parabolic periodic points and are contained in $V\big(\pi \circ \gamma(t)\big)$, it follows then that they land at $z_1\big(\pi\circ\gamma(t)\big)$ or $z_2\big(\pi\circ\gamma(t)\big)$ and finishes the proof. In the non-essential case we can see as in the proof of the non-primitive case that $R_\vartheta^{\pi\circ\gamma(t)}$ can not land at a point that is different from $z_1\big(\pi\circ\gamma(t)\big)$ and $z_2\big(\pi\circ\gamma(t)\big)$. However, in the essential case we know by Theorem 3.2.5 that $R_\vartheta^{\pi\circ\gamma(t)}$ and $R_{\vartheta'}^{\pi\circ\gamma(t)}$ land at a common point for $t \in [0, 1)$. Since one of them must land at $z_1\big(\pi \circ \gamma(t)\big)$ or $z_2\big(\pi \circ \gamma(t)\big)$, the theorem follows. $\square$

As mentioned before the proof of the continuous dependence of the landing points does not necessarily imply the stability of portraits. However, we have the following corollary:

Corollary 6.2. (Stability of Portraits in a Hyperbolic Component and its Roots and Co-Roots)
Let $\mathcal{H}$ be a hyperbolic component and E be the set of all roots and co-roots of $\mathcal{H}$. Then for every parameter $c_0 \in \mathcal{H} \cup E$ and associated repelling or parabolic periodic point z_0 there is a continuous map $z\colon \mathcal{H} \cup E \to \mathbb{C}$ such that $z(c_0) = z_0$ and the portrait of the orbit of $z(c)$ is the same for all $c \in \mathcal{H} \cup E$.

Proof: First we note, that for all $c \in \mathcal{H} \cup E$ all periodic dynamic rays land in the dynamic plane of c. Hence, their landing points $z(c)$ depend continuously on the parameter by Theorem 6.1. It remains to show that the essential portraits are preserved: if for $c_0 \in \mathcal{H} \cup E$ the dynamic rays, say $R_\vartheta^{c_0}$ and $R_{\vartheta'}^{c_0}$, land at z_0 then $z(c)$ is the landing point of R_ϑ^c and $R_{\vartheta'}^c$ for all $c \in \mathcal{H} \cup E$. Let k be the orbit and n be the ray period of z_0. If z_0 and $z(c)$ are repelling the statement follows by Corollary 3.2.4. However, if z_0 is parabolic then by Theorem 4.2 at z_0 coalesce points of a k-periodic and an n-periodic orbit. Since one of them is attracting, namely the n-periodic orbit in the non-primitive case, $z(c)$ has period k. It follows by the previous theorem that $z(c)$ is the landing point of the dynamic rays at angles ϑ and ϑ' for all $c \in \mathcal{H} \cup E$. $\square$

The next lemma and theorems lead us to statements on roots, co-roots and centers of hyperbolic components:

Lemma 6.3. (On the Boundary of the Characteristic Fatou Component)
Let c_0 be a parameter with superattracting orbit of exact period n and $\mathcal{H}$ be the hyperbolic component for which c_0 is a center. Then the Fatou component of f_{c_0} containing the critical value c_0 has exactly $d-1$ points of period n on its boundary, say $z_0^{(1)}, \dots, z_0^{(d-1)}$. At one of them, say $z_0^{(1)}$, land two dynamic rays.

Moreover, let E be the set of roots and co-roots of $\mathcal{H}$. Then there are continuous functions $z^{(1)}, \dots, z^{(d-1)}, z^*$ on $\mathcal{H} \cup E$ such that $z^{(i)}(c_0) = z_0^{(i)}$ for $1 \le i \le d-1$ and $z^*(c_0) = c_0$. Furthermore, for every $c \in E$, $z^*(c)$ and one of the $z^{(i)}(c)$ are equal to the characteristic point of the parabolic orbit of c.

Proof: First we prove that the Fatou component U_1 of f_{c_0} containing the critical value has precisely $d-1$ points of exact period n on its boundary: since c_0 is a point of the superattracting orbit, there is a Böttcher map φ on $\overline{U}_1$ such that $\varphi^{-1} \circ f_{c_0}^{\circ n} \circ \varphi(z) = z^d$. It follows that on ∂U_1 are $d-1$ fixed points of $f_{c_0}^{\circ n}$. Since they are not on the only non-repelling periodic orbit of c_0, they are repelling.

Moreover, as in the proof of Lemma 5.2.1 we see if $n \neq 1$ that one of these points, say $z_0^{(1)}$, lies on the Hubbard tree Γ of c_0 and it is the only point in $\Gamma \cap \overline{U}_1$ by Lemma 5.1.2. If $n = 1$ then there is clearly only one parameter with superattracting orbit and the dynamic rays at angles 0 and 1, which we consider as two different rays in this case, land trivially at a common point.

Therefore, $z_0^{(1)}$ disconnects ∂K_c and this means by Theorem 2.2.5 that at least two dynamic rays land at $z_0^{(1)}$. By Lemma 4.1 there is a continuous function $z^* \colon \mathcal{H} \to \mathbb{C}$ such that $z^*(c_0) = c_0$ and as in the proof of Lemma 4.3 it can be extended to a continuous function on $\mathcal{H} \cup E$. Since the multiplier map $\lambda_{\overline{\mathcal{H}}}$ is continuous on $\overline{\mathcal{H}}$, $z^*(c)$ is the characteristic point of the parabolic orbit for every $c \in E$. Since the $z_0^{(i)}$ are repelling, there are by Corollary 3.2.4 continuous functions $z^{(i)}$ on $\mathcal{H}$ such that $z^{(i)}(c_0) = z_0^{(i)}$ for $1 \le i \le d-1$ and all $z^{(i)}(c)$ are repelling for $c \in \mathcal{H}$. Again the functions $z^{(i)}$ can be extended continuously to $\mathcal{H} \cup E$. Moreover, since $z^*(c)$ is the characteristic point of the parabolic orbit of c for $c \in E$ and lies therefore on the boundary of the characteristic Fatou component and has exact period n, it must be equal to one of the points $z^{(i)}(c)$. □

Next we will show that the mapping degree of $\lambda_{\overline{\mathcal{H}}}$ is $d-1$ and by doing this we will see that every hyperbolic component has an unique center and at least one root. We note that it is not possible to apply the same proof in the case $d \geq 2$ as it is given for Corollary 5.4 in [S97] for $d = 2$. The reason is: in the quadratic case it is sufficient to see that there is at least one root, which is clear because there are no co-roots and $\lambda_{\mathcal{H}}$ is a proper holomorphic map. Then it is possible to show that the root is unique and it follows that $\lambda_{\mathcal{H}}$ has mapping degree 1 and moreover that it is a conformal isomorphism from $\mathcal{H}$ onto D. Milnor shows in [M98] also that $\lambda_{\mathcal{H}}$ is a conformal isomorphism by using some global counting arguments. However, there are other possibilities and we follow a suggestion of Schleicher.

Theorem 6.4. (Mapping Degree of $\lambda_{\mathcal{H}}$)
The multiplier map of a hyperbolic component has mapping degree $d-1$. Moreover, every hyperbolic component has an unique center and at least one root.

Proof: Let $\mathcal{H}$ be a hyperbolic component with period n. In order to show that the mapping degree of $\lambda_{\overline{\mathcal{H}}}$ is at least $d-1$ we consider a parameter $c_0 \in \lambda_{\overline{\mathcal{H}}}^{-1}(0)$. The critical point is on the superattracting orbit and its exact period is n. By Lemma 4.1 there is a holomorphic function $z(c)$ such that $z(c_0) = c_0$. Since we can locally write $\lambda_{\overline{\mathcal{H}}}(c) = d^n f_c^{\circ(n-1)}\big(z(c)\big)^{d-1} \dots f_c\big(z(c)\big)^{d-1} z(c)^{d-1}$ and $z(c)$ has the only zero c_0, it follows that $\lambda_{\overline{\mathcal{H}}}(c) = d^n (c-c_0)^{d-1} g(c)^{d-1}$ for some holomorphic function g near c_0. Hence, $\lambda_{\overline{\mathcal{H}}}$ has at least mapping degree $d-1$. Moreover, it follows that c_0 is the only parameter with $\lambda_{\overline{\mathcal{H}}}(c_0) = 0$, i.e. the center is unique. If c_1 would be an another center of $\mathcal{H}$, one of the points $z(c_1), \ldots, f_{c_1}^{\circ n-1}\big(z(c_1)\big)$ would be 0 in contradiction to the fact that the exact orbit period is n and $z(c) = 0$ if and only if $c = c_0$.

Now we show that the mapping degree is at most $d-1$: by Theorem 5.3.1 we know that if a parameter ray lands at a parameter c, the dynamic ray at the same angle lands at the characteristic point of the parabolic orbit of c. But by the previous Lemma 6.3 we know that there are only $d-1$ candidates for characteristic points of parabolic parameters with ray period n, i.e., there are at most $d-1$ roots and co-roots of $\mathcal{H}$. This means that the mapping degree of $\lambda_{\overline{\mathcal{H}}}$ is at most $d-1$ and hence precisely $d-1$.

Moreover, this shows that all candidates for characteristic points are realized. Since portraits are stable for all parameters in $\mathcal{H}$ and all of its roots and co-roots by Corollary 6.2 we obtain by Lemma 6.3 that at least one parameter has a parabolic orbit with characteristic point at which at least two rays land. Hence, there is at least one root. □

The uniqueness proof for roots of hyperbolic components can be done as in the quadratic case (see again Corollary 5.4 in [S97]).

Theorem 6.5. (Roots Exist and Are Unique)
Every hyperbolic component has exactly one root.

Proof: Let $\mathcal{H}$ be a hyperbolic component of period n. By the previous theorem there is at least one root. Now assume that there are two roots c_0 and c_1 of $\mathcal{H}$. Then the sets of the portraits of all repelling and parabolic periodic orbits are equal for c_0 and c_1 by Corollary 6.2. This means that for any orbit of c_0 with some portrait $\mathcal{P}$ an orbit of c_1 exists with portrait $\mathcal{P}$ and vice versa. Hence, they have the same angles, say ϑ_- and ϑ_+, with the property that the critical point and the critical value are separated by the ray pair at these angles and that all other ray pairs separating 0 and the critical value lie in the component containing 0. It follows that the characteristic angles of the parabolic orbit portraits are equal.

Since by Corollary 5.3.3 every essential parabolic parameter is the landing point of the parameter rays at the characteristic angles of the parabolic orbit portraits, this proves the theorem. □

Now it is quite easy to determine the number of co-roots of a hyperbolic component:

Corollary 6.6. (Number of Co-Roots)
Every hyperbolic component has exactly $d-2$ co-roots.

Proof: Let $\mathcal{H}$ be a hyperbolic component with period n. It follows from the definition of $\lambda_{\overline{\mathcal{H}}}$ and Theorem 4.2 that precisely the parameters which have a parabolic orbit with exact ray period n are mapped on 1 by $\lambda_{\overline{\mathcal{H}}}$. The number of these parameters is exactly $d-1$, because by Theorem 6.4 the mapping degree of $\lambda_{\overline{\mathcal{H}}}$ is $d-1$. By Theorem 6.5 every hyperbolic component has exactly one root, so the other $d-2$ parameters must be co-roots. □

The previous two statements, Theorem 6.5 and Corollary 6.6, are restatements of the last assertion of the Structure Theorem.

So far we know that at every hyperbolic component at least d parameter rays land. Our next aim is to show that at most d parameter rays land at every hyperbolic component. Then the proof of the Structure Theorem for periodic rays is finished. In the following section we will use the so-called internal rays of a hyperbolic component to join the landing points of the parameter rays landing there. Here is the definition:

Definition. (Internal Rays and Angles of a Hyperbolic Component) Let $\mathcal{H}$ be a hyperbolic component and $\gamma\colon I \to \overline{\mathcal{H}}$ an arc starting at the center of $\mathcal{H}$ such that there is an angle ϑ with $\lambda_{\overline{\mathcal{H}}}\big(\gamma(t)\big) = t \cdot e^{2\pi i\vartheta}$ for all $t \in I$. Then we call $\gamma(I)$ an *internal ray* of $\mathcal{H}$ with angle ϑ and write $R^{\mathcal{H}}_{\vartheta}$ for $\gamma(I)$.

For a parameter $c \in \mathcal{H}$ different from the center of $\mathcal{H}$ which lies on an internal ray with angle ϑ of $\mathcal{H}$ we call ϑ the *internal angle* of c with respect to $\mathcal{H}$. ◊

Remark: Since $\lambda_{\mathcal{H}}$ is a $(d-1)$-to-one map an internal ray of $\mathcal{H}$ with a given angle is not uniquely defined. On the contrary for every angle ϑ a hyperbolic component has $d-1$ internal rays with this angle ϑ.

7 Kneading Sequences

In this section we will finish as mentioned before the proof of the Structure Theorem by showing that at every hyperbolic component at most d parameter rays land. We will do this by proving a necessary condition for the landing of parameter rays at a common point (see Theorem 7.2 and 7.4). For this purpose we introduce kneading sequences of angles. The proofs in this section are more or less the same as the one of Schleicher in Section 3 in [S97] (see in particular Lemmas 3.9 and 3.10). However, to use these methods, namely the partition in Theorem 7.2, for $d \geq 2$ we need some knowledge on hyperbolic components. We accumulated this in the previous sections and can use it now.

Definition. (Itineries and Kneading Sequences)
For an angle $\vartheta \in \mathbb{S}^1$ we divide $\mathbb{S}^1$ by the inverse of the d-tupling map σ and label the components in the following manner:

$$l_\vartheta(\eta) := \begin{cases} m & \text{if } \eta \in \left(\frac{\vartheta+(m-1)}{d}, \frac{\vartheta+m}{d}\right) \\ \overset{m_2}{m_1} & \text{if } \eta = \frac{\vartheta+(m_2-1)}{d} = \frac{\vartheta+m_1}{d} \end{cases}$$

The infinite sequence $I_\vartheta(\eta) := l_\vartheta(\eta), l_\vartheta(d\eta), l_\vartheta(d^2\eta), \ldots$ is called the *ϑ-itinerary of η* with respect to the d-tupling map. For the special itinerary $I_\vartheta(\vartheta)$ we write $K(\vartheta) := I_\vartheta(\vartheta) = l_\vartheta(\vartheta), l_\vartheta(d\vartheta), l_\vartheta(d^2\vartheta), \ldots$ and call $K(\vartheta)$ the *kneading sequence of ϑ* with respect to the d-tupling map.

The symbols $\overset{1}{0}, \overset{2}{1}, \ldots, \overset{d-1}{d-2}, \overset{0}{d-1}$ are named *boundary symbols* and if it does not matter which of the boundary symbols we mean, we replace them sometimes by an asterisk $(*)$. $\Diamond$

Remark: We can consider the kneading sequence as a mapping $K \colon \mathbb{S}^1 \to KS$ with

$$KS := \left\{ (a_n)_{\mathbb{N}} : a_n \text{ is one of the symbols } 0, 1, \ldots, d-1, \overset{1}{0}, \overset{2}{1}, \ldots, \overset{d-1}{d-2}, \overset{0}{d-1}. \right\}.$$

It is convenient to write $K(\vartheta_1) = K(\vartheta_2)$ for angles ϑ_1, ϑ_2 if both angles have boundary symbols at the same entries and all other symbols coincide.

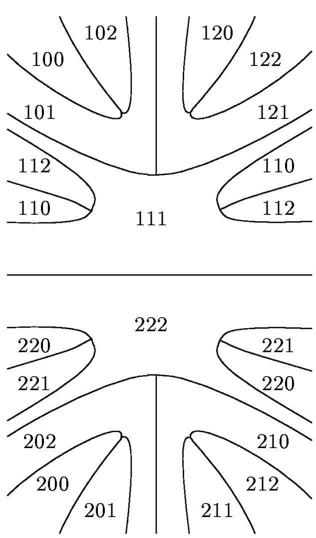


Figure 7: On the left hand side is the partition P_3 of initial kneading sequences for $\mathcal{M}_3$ shown, which we use in the proof of the following theorem. It consists of parameter rays at angles which have period 3 or below, their landing points and the internal rays at angle 0 of the corresponding hyperbolic components (to connect the various landing points). The first three entries of the kneading sequence of the angle of any parameter ray which is contained in one of these components are as indicated in the figure.

We should note the following statement:

Lemma 7.1. (Changing of the Entries of a Kneading Sequence)
Consider the kneading sequence as mapping $K\colon \mathbb{S}^1 \to KS$. Then the k-th entry of $K(\vartheta)$ changes if and only if $l_\vartheta(d^{k-1}\vartheta)$ is a boundary symbol. In more detail we can say: if ϑ goes in positive direction, it is incremented as indicated by the boundary symbol ${}^{m+1}_{\ m}$, i.e., it changes from m to $m+1$ (again $d \equiv 0$).

Proof: The assertion follows immediately by the definition: the k-th entry of $K(\vartheta)$ is defined as $l_\vartheta(d^{k-1}\vartheta)$. □

In the following theorem we define a partition of initial kneading sequences and do most of the proof of the induction step of Theorem 7.4.

Theorem 7.2. (A Partition of Initial Kneading Sequences)
Let $n \geq 2$ be an integer. If at the root of every hyperbolic component with period $n-1$ or lower exactly two periodic parameter rays land then any two parameter rays with angles ϑ_1, ϑ_2 of exact ray period n can land at the same parameter only if $K(\vartheta_1) = K(\vartheta_2)$.

Proof: To verify the statement we construct a partition P_{n-1} of $\mathbb{C}$ such that every parameter ray with exact ray period n together with its landing point is completely contained in an open component of P_{n-1}. Furthermore, we require that for any two parameter rays $R^{\mathcal{M}}_{\vartheta_1}, R^{\mathcal{M}}_{\vartheta_2}$ which are both in the same open component of P_{n-1} the kneading sequences of ϑ_1 and ϑ_2 coincide in the first $n-1$ entries. This proves the theorem, because any parameter ray with angle of exact period n has a kneading sequence of period n and the n-th entry of the kneading sequence is $(*)$. It remains to prove that there is such a partition.

Let Θ_k be the set of all angles with exact period k and Λ_k the set of multiplier maps of the k-periodic hyperbolic components. We define

$$P_{n-1} := \bigcup_{k=1}^{n-1}\Bigg(\bigcup_{\vartheta\in\Theta_k} R^{\mathcal{M}}_\vartheta \cup \bigcup_{\lambda_{\mathcal{H}}\in\Lambda_k} \lambda^{-1}_{\mathcal{H}}(I)\Bigg)$$

and assert that P_{n-1} is a partition with the required properties. By construction P_{n-1} is a partition of $\mathbb{C}$. Due to Theorem 3.2.3 all rays with exact ray period k land at a parameter which has a parabolic orbit with exact ray period k, too. Moreover, for a hyperbolic component $\mathcal{H}$ the inverse image $\lambda^{-1}_{\mathcal{H}}(I)$ is exactly the set of all internal rays with angle 0. Each of these $d-1$ internal rays lands at a root or co-root of $\mathcal{H}$ and conversely the root and every co-root of $\mathcal{H}$ is a landing point of one of these internal rays. Since parameter rays do not cross we get by using these considerations that every parameter ray of period n together with its landing point is completely contained in one of the open components of P_{n-1}.

Now assume that two parameter rays $R^{\mathcal{M}}_{\vartheta_1}, R^{\mathcal{M}}_{\vartheta_2}$ are both contained in the same open component of P_{n-1}. Since every hyperbolic component has $d-2$ co-roots and exactly one root (Corollary 6.6 and Theorem 6.5) and by assumption at every root exactly two parameter rays land, we see that at the boundary of every hyperbolic component of period k exactly d parameter rays of period k land for $k \in \{1, \ldots, n-1\}$. Thus, for every $k \in \{1, \ldots, n-1\}$ the number of angles which are in Θ_k and in $(\vartheta_1, \vartheta_2)$ is $m \cdot d$ for an $m \in \mathbb{N}_0$. Using Lemma 7.1 this yields that, again for every $k \in \{1, \ldots, n-1\}$, the k-th entry of $K(\vartheta)$ is incremented $m \cdot d$ times as ϑ goes from ϑ_1 to ϑ_2, i.e., it is the same for ϑ_1 and ϑ_2. □

The next step is to see that given a root the kneading sequences of all angles, except for possibly the characteristic angles, of the dynamic rays landing at the characteristic point of the parabolic orbit of the root are different.

Theorem 7.3. (Different Kneading Sequences)
Let c_0 be a root. Then the angles of the dynamic rays landing at the associated characteristic point, except for possibly the two characteristic angles, have pairwise different kneading sequences.

Proof: First we introduce some notation: let z_1 be the characteristic point of the parabolic orbit and $R^{c_0}_{\vartheta_1}, \dots, R^{c_0}_{\vartheta_s}$ all the dynamic rays landing at z_1. For $s = 2$ there is nothing to prove, so we assume $s \geq 3$. If the exact period of the angles $\vartheta_1, \dots, \vartheta_s$ is n, then by Lemma 3.1.1 the orbit period of the parabolic orbit is $k := n/s$. We assume that $n \geq 2$ because the statement is trivial for $n = 1$. For $i \in \{1, \dots, s\}$ we denote the d inverse images of ϑ_i with respect to the d-tupling map by $\vartheta_i^{(l)} := (\vartheta_i + l)/d \in \mathbb{S}^1$ $(l \in \{0, \dots, d-1\})$ and the landing point of $\vartheta_i^{(l)}$ by $z_0^{(l)}$ for $l \in \{0, \dots, d-1\}$. Obviously $z_0^{(l)}$ is the landing point of $\vartheta_i^{(l)}$ if and only if it is the one of a $\vartheta_j^{(l)}$, i.e., the $z_0^{(0)}, \dots, z_0^{(d-1)}$ do not depend on the choice of a specific angle ϑ_i $(i, j \in \{1, \dots, s\})$. Let $\mathcal{H}$ be the hyperbolic component for which c_0 is the root and let c_1 be the corresponding center. By Corollary 6.2 and Lemma 6.3 we see that there are continuous functions $z^{(0)}, \dots, z^{(d-1)}$ on $\mathcal{H} \cup \{c_0\}$ such that $z^{(i)}(c_0) = z_0^{(i)}$ for all i and at $z^{(i)}(c)$ land the dynamic rays at the same angles as at $z_0^{(i)}$ for all i and $c \in \mathcal{H} \cup \{c_0\}$. The points $z^{(i)}(c_1)$ lie on the boundary of the Fatou component U_0 of f_{c_1} containing the critical point. Let Γ be the Hubbard tree of c_1, U_1 the characteristic Fatou component and $\gamma\colon I \to \Gamma$ be the arc which connects the unique intersection point $\Gamma \cap \overline{U}_1$ with the critical value c_1. Then $\gamma(I)$ has d inverse images. Each of them lies in $\overline{U}_0$ and connects the critical point with one of the points $z^{(i)}(c_1)$. Therefore, the partition $P_{\vartheta_i} := f_{c_1}^{-1}\big(\gamma(I)\big) \cup \bigcup R^{c_1}_{\vartheta_i^{(l)}}$ has precisely d open components. Now we label the boundary P_{ϑ_i} by $(*)$ and the component which contains the critical value c_1 by 1. We label the subsequent components by subsequent numbers in positive direction. By construction the branch of $\Gamma - \overline{U}_0$ on which the critical value lies has always label 1. Since f_{c_1} is orientation preserving this implies that every branch of $\Gamma - \overline{U}_0$ has the same label with respect to every partition P_{ϑ_i}. Since the Hubbard tree connects the critical orbit and every $z^{(i)}(c_1)$ lies on the boundary of a component which contains a point of the critical orbit, the dynamic rays landing at a $z^{(i)}(c_1)$ have the same label for all partitions. Hence, the ϑ_{i_1}-itinerary of ϑ_{j_1} is equal to the ϑ_{i_2} itinerary of ϑ_{j_2} for all $i_1, i_2, j_1, j_2 \in \{0, \dots, d-1\}$ except for possibly the positions $mk - 1$, $m \in \mathbb{N}$. This means that the kneading sequences of all ϑ_i can only differ in the position $mk - 1$, $m \in \mathbb{N}$. Next we verify that the kneading sequences of all ϑ_i except for two angles are pairwise different at an $(mk-1)$-th position. The $(mk-1)$-th entry of the kneading sequence of ϑ_i is just the label of $\sigma^{\circ(mk-1)}(\vartheta_i)$ with respect to P_{ϑ_i}. Thus, two angles ϑ_i and ϑ_j can have the same kneading sequence only if the number of dynamic rays among the $R^{c_1}_{\vartheta_0^{(l)}}$ which have a certain label is equal with respect to P_{ϑ_i} and P_{ϑ_j}. However, if at least two of the rays $R^{c_1}_{\vartheta_0^{(l)}}$ have a different label with respect to P_{ϑ_i}, the number of rays which have the smaller label is different with respect to P_{ϑ_i} and P_{ϑ_j} for $\vartheta_i \neq \vartheta_j$. Therefore, all these dynamic rays must have the same label. This is only possibly if none of them lies in the component of $R_{\vartheta_0^{(i)}} \cup R_{\vartheta_0^{(j)}} \cup z^{(0)}(c_1)$ which contains U_0, i.e. if ϑ_i and ϑ_j are the characteristic angles. $\square$

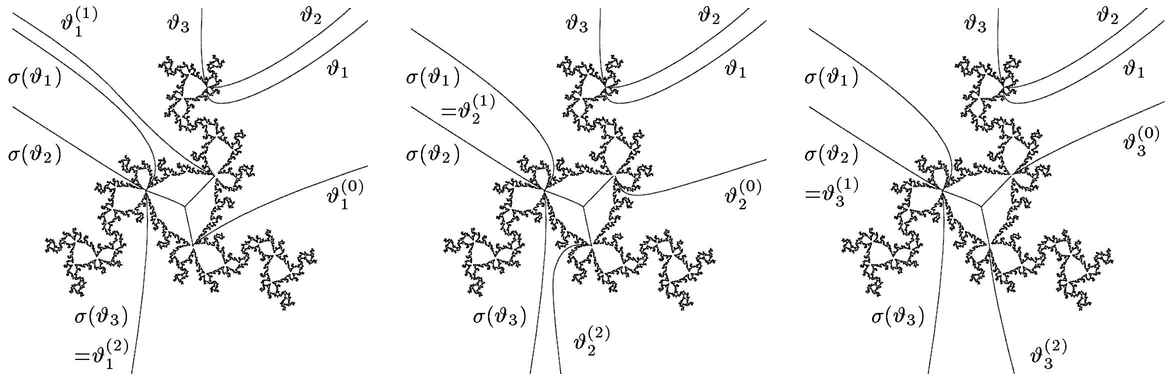


Figure 8: The Julia set of a polynomial $z \mapsto z^3 + c$ which has a 6-periodic superattracting orbit (c is near $0.225345 + 0.941406\,i$). With the notation of the previous proof the dynamic rays landing at z_1 have the angles $\vartheta_1 = 92/728$, $\vartheta_2 = 100/728$ and $\vartheta_3 = 172/728$. The orbit of ϑ_1 is $\vartheta_1 \mapsto \sigma(\vartheta_1) \mapsto \vartheta_2 \mapsto \sigma(\vartheta_2) \mapsto \vartheta_3 \mapsto \sigma(\vartheta_3) \mapsto \vartheta_1$.

Finally we combine the previous results and finish the proof of the third assertion of the Structure Theorem by showing the following statement.

Theorem 7.4. (Every Root is the Landing Point of Precisely Two Parameter Rays)
Every root c_0 is the landing point of exactly two parameter rays. Moreover, the angles of the parameter rays landing at c_0 are the characteristic angles of the parabolic orbit portrait of c_0.

Proof: We prove the theorem by induction: let n be the ray period of c_0. For $n = 1$ the only root is the landing point of the parameter rays at angles 0 and 1 that we consider as two rays in this case. We assume that the roots of all hyperbolic components with period $n-1$ or lower are the landing points of exactly two parameter rays. Then we obtain by Theorem 7.2 that at the root c_0 of any hyperbolic component with period n only parameter rays with n-periodic angles which have the same kneading sequences can land. Note that a parameter ray with a given angle can only land at c_0 if the dynamic ray with the same angle lands at the characteristic point z_0 of the parabolic orbit of c_0 (Theorem 5.3.1) and that all angles of the dynamic ray landing at z_0, except for possibly the characteristic angles ϑ_-, ϑ_+, have different kneading sequences. This shows us that at most the parameter rays with the angles ϑ_-, ϑ_+ can land at c_0. By Corollary 5.3.3 we know that they land indeed. This finishes the induction. □

8 Preperiodic Parameter Rays

To complete the discussion of rational parameter rays we have to study the landing points of preperiodic parameter rays. Since any preperiodic angle is mapped by some iterate onto a periodic orbit we can use our results for periodic parameter rays. The ideas are the same as in the quadratic case. We follow Schleicher's proofs of Section 4 in [S97].

Definition. (Misiurewicz Point)
A parameter c for which the critical orbit is preperiodic but not periodic is called *Misiurewicz point.* $\diamondsuit$

Theorem 8.1. (Preperiodic Parameter Rays Land)
Every parameter ray at preperiodic angle ϑ lands at a Misiurewicz point c_0. The dynamic ray $R^{c_0}_\vartheta$ lands at the critical value c_0.

Before the proof we should note that by taking backward images we obtain by Theorem 3.2.2 that also for every parameter c with preperiodic repelling point z_0 which is the landing point of a dynamic ray $R^{c_0}_\vartheta$ a neighborhood U of c and a holomorphic function $z\colon U \to \mathbb{C}$ exist such that $z(c_0) = z_0$ and every $z(c)$ is the landing point of R^c_ϑ.

Proof: Let $c_0 \in \mathcal{M}_d$ be a parameter in the limit set of $R^{\mathcal{M}}_\vartheta$. Using the results about kneading sequences it is easy to see that c_0 can not be a parabolic parameter: by Theorems 7.4 and 7.2 we know that two parameter rays can land at the same parameter only if the kneading sequences of their angles are equal. Since every parabolic parameter is the landing point of at least one periodic parameter ray and the kneading sequence of periodic angles is periodic again, only parameter rays at angles with a periodic kneading sequence can land at a parabolic parameter. But the kneading sequence of a preperiodic angle does not contain any boundary symbol and hence c_0 can not be a parabolic parameter.

We want to show that the dynamic ray $R^{c_0}_\vartheta$ lands at the critical value c_0. Then it follows by Lemma 2.2.6 that the critical orbit is preperiodic and hence, c_0 is a Misiurewicz point. Since limit sets are connected (see for example the remark after Definition 2.4 in [S98b]) and the set of Misiurewicz points of $\mathcal{M}_d$ is countable (note, that every Misiurewicz point c must satisfy the equation $f_c^{\circ p}(c) = f_c^{\circ(p+k)}(c)$ for integers $p, k \geq 1$), the theorem then follows.

Now to the proof that $R^{c_0}_\vartheta$ lands at c_0: since c_0 is not a parabolic parameter, $R^{c_0}_\vartheta$ lands at a preperiodic repelling point z_0 by Theorem 2.2.4. If z_0 is not mapped by any iterate of f_{c_0} to the critical point 0, then there is a neighborhood U of c_0 and a holomorphic function z on U such that $z(c_0) = z_0$ and $z(c)$ is the landing point of R^c_ϑ by Theorem 3.2.2 for all $c \in U$. Furthermore, by Lemma 2.3.3 for every parameter $c \in U \cap R^{\mathcal{M}}_\vartheta$ the critical value c lies on the dynamic ray R^c_ϑ. Since $R^{c_0}_\vartheta$ lands and z is continuous this implies $z(c_0) = c_0$.

However, if z_0 lies on the backward orbit of the critical value c_0 then there is an integer $l \geq 1$ such that $f^{\circ l}(z_0) = c_0$ and again by Theorem 3.2.2 there is a neighborhood U of c_0 and a holomorphic function z on U such that $z(c_0) = c_0$ and $z(c)$ is the landing point of $R^c_{\sigma^{\circ l}(\vartheta)}$. Now we can not go back uniquely because the critical point lies on the orbit. But the d branches of $f_c^{\circ(-l)} \circ z(c)$ are then the landing points of the branches of R^c_ϑ in which $R^c_{\sigma^{\circ l}(\vartheta)}$ splits into. These landing points again depend holomorphically on c and as above it follows that a branch of $R_\vartheta^{c_0}$ lands at c_0. $\square$

We should note that the last case in the previous proof can never occur: it would follow that c_0 is periodic and this would be a contradiction to the preperiodiness of ϑ and to the assumption that $R_\vartheta^{c_0}$ lands at the backward orbit of c_0.

Now we have also finished the proof of the fourth assertion of the Structure Theorem and will now show the fifth:

Theorem 8.2. (Every Misiurewicz Point is Landing Point)
At every Misiurewicz point a preperiodic parameter ray lands.

Proof: Let c_0 be a Misiurewicz point. In the proof of the previous theorem we have already seen that c_0 is not parabolic. By Theorem 2.2.5 the critical value c_0 is the landing point of a dynamic ray, say $R_\vartheta^{c_0}$. Moreover, by Theorem 3.2.2 there is a neighborhood U of c_0 and a holomorphic function z on U such that $z(c_0) = c_0$ and $z(c)$ is the landing point of R_ϑ^c for all $c \in U$. Since the number of Misiurewicz points with a given preperiod and period is finite, we may assume that U does not contain any other Misiurewicz point. For $c \in U$ let now φ_c be the Böttcher map which maps the complement of the filled-in Julia set K_c onto the complement of D and let $z(c,t) := \varphi_c^{-1}(te^{2\pi i\vartheta})$. Since R_ϑ^c lands, $z(c,t)$ is well-defined for all $t \in [1, \infty]$. We define the winding number of R_ϑ^c around c as the total change of $\arg\bigl(z(c,t) - c\bigr)/(2\pi)$ while decreasing t from ∞ to 1. If R_ϑ^c does not contain the critical value c and does not land there, the winding number is well-defined, finite and depends continuously on the parameter. Under this assumption the change of the winding number if we go along any small closed arc around c_0, is by the Argument Principle the multiplicity of the zero c_0 with respect to $z(c) - c$. But the value of the winding number is the same at the start and end point of the closed arc. Hence, there is a discontinuity on any closed arc around c_0 and there are parameters c for which the critical value lies on R_ϑ^c. This implies that c_0 is a limit point of $R_\vartheta^{\mathcal{M}}$ and therefore by the previous theorem $R_\vartheta^{\mathcal{M}}$ lands at c_0. $\square$

Different to the periodic case we are in general not able to say how many parameter rays land at a given Misiurewicz point. However, we can still make some statement in Theorem 8.4. Its proof depends on the following lemma:

Lemma 8.3. (The Kneading Sequence of Preperiodic Angles)
For a parameter c_0 let $R_\vartheta^{c_0}$ be a preperiodic ray landing at z_0 with preperiod l and period n. Then $K(\vartheta)$ has preperiod l, too, and its period is equal to the orbit period of z_0.

Proof: Clearly the preperiod of $K(\vartheta)$ can not be larger than l. Similarly to the proof of Theorem 7.3 we construct a partition to handle the kneading sequence of a given angle. By Theorem 8.1 the dynamic ray $R_\vartheta^{c_0}$ lands at the critical value c_0 and hence the d dynamic rays with the preimages of ϑ as angles land at the critical point 0. If we label the components of this partition, consisting of the critical value and this dynamic rays landing there, the labels of $R_\vartheta^{c_0}, R_{\sigma(\vartheta)}^{c_0}, \ldots$ reflect again the kneading sequence of ϑ. If the preperiod of $K(\vartheta)$ would be smaller than l, then the dynamic rays at angles $\sigma^{\circ(l-1)}(\vartheta)$ and $\sigma^{\circ(l-1+n)}$ would have the same label. But this can not be the case because their landing points $f_{c_0}^{\circ(l-1)}(c_0)$ and $f_{c_0}^{\circ(l-1+n)}(c_0)$ are in different components of the partition.

Next we show that the orbit period of $f_{c_0}^{\circ l}(c)$ is exactly the period of the kneading sequence of $\sigma^{\circ l}(\vartheta)$: clearly the period of the kneading sequence divides the orbit period. If the labels of $\sigma^{\circ l}(\vartheta)$ and $\sigma^{\circ(l+k')}(\vartheta)$ are always equal, i.e., k' is the period of the kneading sequence, the dynamic rays at these angles land always in the same partition. We can then connect their landing points by an arc within every component. By iteratively taking inverse images of these dynamic rays, the landing points and the arc we see that the landing points converge to a single point, i.e., the dynamic rays at angles $\sigma^{\circ l}(\vartheta)$ and $\sigma^{\circ(l+k')}(\vartheta)$ land at the same points. □

Theorem 8.4. (Number of Rays at a Misiurewicz Point)
Let ϑ be a preperiodic angle with preperiod l and period n. Furthermore, let k be the period of $K(\vartheta)$ and denote the Misiurewicz point at which the parameter ray at angle ϑ lands by c_0. If $n/k > 1$ then exactly n/k parameter rays land at c_0 and if $n/k = 1$ then 1 or 2 parameter rays land at c_0.

Proof: Using Lemmas 8.3 and 3.1.3 we already know that n/k dynamic rays land at each point of the orbit of $f_{c_0}^{\circ l}(c_0)$ if $n/k > 1$ and at most two dynamic rays land at each point if $n/k = 1$. Moreover, since f_{c_0} is a local homeomorphism in a neighborhood of $c_0, f_{c_0}(c_0), \dots, f_{c_0}^{\circ(l-1)}(c_0)$ at these points the same number of dynamic rays land as at the periodic orbit. By Theorems 8.1 and 8.2 the number of parameter rays landing at c_0 is precisely the number of dynamic rays landing at each point of the orbit of c_0. This finishes the proof. □

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