Dynamics of complex unicritical polynomials

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Abstract of the Dissertation

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In this dissertation we study two relatively independent problems in one dimensional complex dynamics. One on the parameter space of unicritical polynomials and the other one on measurable dynamics of certain quadratic polynomials with positive area Julia sets.

It has been conjectured that combinatorially equivalent non-hyperbolic unicritical polynomials are conformally conjugate. The conjecture has been already established for finitely renormalizable unicritical polynomials. In the first part of this work we prove that a type of compactness, called a *priori* bounds, on the renormalization levels of a unicritical polynomial, under a certain combinatorial condition, implies this conjecture.

It has been recently shown that there are quadratic polynomials with positive area Julia set. The second part of this work investigates typical trajectories of quadratic polynomials with a non-linearizable fixed point of high return

type. This class contains the examples with positive area Julia set mentioned above. In particular, we prove that Lebesgue almost every point in the Julia set of these maps accumulates on the non-linearizable fixed point. We show that the post-critical sets of these maps, which are the measure theoretic attractors, have measure zero and are connected subsets of the plane. This has some interesting corollaries such as, almost every point in the Julia set of such maps is non-recurrent, or, there is no finite absolutely continuous invariant measure supported on the Julia set of these maps.

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Chapter 1

Introduction

In this work we study two different problems in one dimensional holomorphic dynamics. The first one is on *combinatorial rigidity* of a class of unicritical maps. The second one describes typical (with respect to the Lebesgue measure) trajectories of certain quadratic polynomials with positive area Julia set. Below we describe the content of these two studies.

Part 1. Dynamics of a rational map on the Riemann sphere breaks into two pieces; dynamics on the *Fatou set* and dynamics on the *Julia set*. Fatou set of a map is "nicely behaved" part of the dynamics, and it is completely understood through works of Fatou [Fat19], [Fat20a], [Fat20b], Julia [Jul18] and Sullivan [Su85]. The Julia set of a rational map is the "chaotic" part of the dynamics. Substantial work by many people has been devoted to understanding these two sets for different families of rational maps. Basic properties of these sets has been collected in several books and surveys: See for example Beardon [Be91], Milnor [M06], Blanchard [Bl84], and Lyubich [Ly86].

Hyperbolic maps form a class of thoroughly understood and well behaved maps. These are rational maps for which the orbits of all critical points tend to *attracting* periodic points. For every map in this class, there exists a finite subset of the Riemann sphere such that a full measure set of points on the sphere is attracted to this set under dynamics of the map. One of the main conjectures in holomorphic dynamics, which goes back to Fatou, states that the hyperbolic maps are dense in the space of rational maps of degree d (also in the space of polynomials of degree d). By an approach developed through the work of R. Mãne, P. Sad and D. Sullivan [MSS83] for rational maps to tackle this problem, "studying families of rational maps has been reduced to studying of dynamics of individual maps on their Julia set".

Every polynomial with one critical point, which is referred as *unicritical polynomial* through this thesis, can be written (up to conformal conjugacy) in the form $z \mapsto z^d + c$, for some complex number c. If the critical point 0 tends to infinity under iteration of such a map (attracted to the super-attracting fixed point at infinity), the map is hyperbolic. The Julia set is a Cantor set in this case. Therefore, attention reduces to parameters for which orbit of 0 stays bounded under iteration. The set of such parameters is called the *connectedness locus* or the *Mulribrot* set, (the well-known *Mandelbrot* set corresponds to $d = 2$) see Figures 2.1 and 2.2.

There is a way of defining graded partitions of the Multibrot set into pieces such that dynamics of maps in each piece have some special combinatorial property. All maps in a given piece of partition of a certain level will be called *combinatorially equivalent* up to that level. Conjecturally, combinatorially equivalent (up to all levels) *non-hyperbolic* maps in the uniciritcal family are *conformally conjugate*. As stated in [DH82] for $d = 2$, this *Rigidity Conjecture* is equivalent to the local connectivity of the Mandelbrot set (MLC for short) and naturally extends to degree d unicritical polynomials. In [DH82], they prove that MLC implies Density of hyperbolic quadratic polynomials among quadratic polynomials. These discussions have been extended to degree d unicritical polynomials by D. Schleicher in [Sch04].

Roughly speaking, the rigidity conjecture says that combinatorics of a map in this class uniquely determines (fine scale) geometry of the chaotic part of the dynamics of that map. In particular, combinatorics of such map determine the parameter c up to a finite number of values.

A unicritical polynomial f is called Douady-Hubbard *renormalizable*, or renormalizable for short, if there exist a neighborhood U of the critical point and an integer $n > 1$, such that $f^{\circ n} : U \to V$ is a degree d proper branched covering with U compactly contained in V, and $f^{\circ nk}(0) \in U$, for $k = 0, 1, 2, \ldots$. By [DH85], such a map $f^{\circ n} : U \to V$, denoted by $\mathcal{R}^1(f)$, is conjugate to a unicritical polynomial, which we denote it by $\mathcal{SR}^1(f)$, of the same degree as the one of f. Moreover, this renormalization, under a certain combinatorial condition which will be clear later, provides a homeomorphism from a subset of the connectedness locus onto all of the connectedness locus. Such homeomorphim is determined up finite number of rotations, however, one can make it unique by marking a particular point on the connectedness locus (which will be made clear later). We will denote the corresponding maximal homeomorphic copy of the connectedness locus (domain of this homeomorphism) within the actual connectedness locus by $\tau(f)$. By maximal homeomorphic copy we mean that it is not strictly contained in any other homeomorphic copy except the actual connectedness locus.

Given a renormalizable unicritical polynomial f, if $\mathcal{R}(f)$ is also renormalizable we say that f is twice renormalizable. So, a unicritical polynomial is called *infinitely renormalizable* if the renormalization process defined above can be carried out infinite number of times, that is, the sequence $\mathcal{R}(f)$, $\mathcal{R}(\mathcal{R}(f))$,

$\mathcal{R}(\mathcal{R}(\mathcal{R}(f))), \ldots$ is well defined. *Combinatorics* of an infinitely renormalizable unicritical polynomial is defined as the sequence of maximal homeomorphic copies $\langle \tau(f), \tau(\mathcal{SR}^1(f)), \tau((\mathcal{SR})^2(f)), \ldots \rangle$ of the connectedness locus.

In 1990's, J. C. Yoccoz [H93] proved MLC conjecture at all non-hyperbolic parameter values which are at most finitely *renormalizable*. He also proved local connectivity of the Julia set for such maps with all periodic points repelling. Degree 2 assumption was essential in his argument. Local connectivity of Julia sets for unicritical degree d polynomials which are at most finitely renormalizable, and with all their periodic points repelling has been established by J. Kahn and M. Lyubich [KL05] in 2005. Their proof is based on "controlling" geometry of a *Modified principal nest*. The same controlling technique is used to settle the rigidity problem for these parameters in [AKLS05].

The rigidity conjecture for quadratic maps $z^2 + c$ with c real is proved independently by M. Lyubich [Ly97] and J. Graczyk, G. Świątek [GS98]. Recently, the conjecture has been proved for polynomials with all their critical points real and non-degenerate by O. Kozlovski, W. Shen and S. Van Strien in [KSvS07].

An infinitely renormalizable unicritical polynomial satisfies *a priori bounds* condition, a notion introduced by D. Sullivan [S92], if infimum of moduli of annuli $V_n \setminus U_n$ is non-zero, where $(\mathcal{R}^n(f))^{t_n} : U_n \to V_n$ is a renormalization of $\mathcal{R}^n(f)$. Here we prove that this property, under certain combinatorial assumption, implies the rigidity conjecture. We say that an infinitely renormalizable map satisfies the *secondary limbs condition*, denoted by $\mathcal{SL}$ for short, if the maximal connectedness loci copies $\tau(f), \tau(\mathcal{R}^1(f)), \ldots$ in the definition of the combinatorics of f belong to finite number of *secondary limbs* of the connectedness locus.

Theorem 1.1 (Rigidity)**.** *If f_c is an infinitely renormalizable degree d uni-*

critical polynomial, with a priori bounds, and satisfies $\mathcal{SL}$ condition, then it is combinatorially rigid.

This result was proved in part II of [Ly97] for degree 2 polynomials. The main difference between our proof and the one in [Ly97] is in the construction of the *Thurston conjugacy*. In [Ly97] such a conjugacy is built along the whole principal nest and uses linear growth of *moduli* along this nest. However, such a growth is not known for arbitrary degree unicritical polynomials. Certain annuli in the modified principal nest introduced in [KL05] have definite module and this helps us to "pass" over principal nest much easier. This makes the whole construction simpler and more general to include arbitrary degree d.

To prove this statement, we first construct a quasi-conformal map of the plane which conjugates the two combinatorially equivalent maps on their post-critical sets. It does not respect the dynamics outside of the post-critical sets, but it is in a "right" homotopy class of homeomorphisms of the complex plane minus the post-critical sets. Verifying such a topological property with the rich possibilities of combinatorics of these maps is the source of difficulty in the proof. Being in such a homotopy class enables one to lift this quasi-conformal map infinite number of times (via the two polynomials) and obtain an actual quasi-conformal conjugacy in the limit. Finally, this quasi-conformal conjugacy can be promoted to a conformal conjugacy by an open-closed argument.

Different classes of maps are known to enjoy *a priori bounds* property; real infinitely renormalizable unicritical polynomials (see [LvS98] and [LY97]), *primitive* infinitely renormalizable maps of bounded type in [K06], parameters under a *Decorations* condition in [KL06], and in [KL07] under a *Molecule* condition. The class of maps satisfying the molecule condition contains the primitive infinitely renormalizable parameters of bounded type and the ones satisfying decoration condition. We will see that the combinatorial class $\mathcal{SL}$ contains the combinatorial class of all these parameters. Therefore, combining the above theorem with [KL07] we obtain the following:

Corollary 1.2. *Infinitely renormalizable, combinatorially equivalent, quadratic polynomials satisfying the molecule condition are conformally equivalent.*

Part 2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of the Riemann sphere. A central problem in dynamics of a system is to understand behaviour of orbit of a typical point; $z, f(z), f^2(z), \dots, f^n(z), \dots$. Indeed, this is a rather old subject in holomorphic dynamics starting with contributions of Koenigs [Ko84], Schröder [Sch71], Böttcher [Bö04] in the late 19th century on the local dynamics of holomorphic maps. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic map defined on some neighborhood of z_0 with $f(z_0) = z_0$ and *multiplier* $f'(z_0) = \lambda$. If $|\lambda| \neq 0, 1$, by [Ko84], it is known that there exists a local conformal change of coordinate near z_0, with $\varphi(z_0) = 0$, and $\varphi \circ f \circ \varphi^{-1}(z) = z \mapsto \lambda z$ near zero. There is a similar normal form by [Le97] and [Bö04] when $\lambda = e^{2\pi \frac{p}{q} i}$, and $\lambda = 0$, respectively. See [M06] for these results and more. This describes orbits near such fixed points.

If the multiplier at a fixed point is $\lambda = e^{2\pi\alpha\mathbf{i}}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, f is called *linearizable* at that fixed point, if there exists a conformal change of coordinate on a neighborhood of the fixed point so that in the new system of coordinate the map becomes the linear map $z \mapsto e^{2\pi\alpha\mathbf{i}} z$. If there is such a domain of linearization, the maximal domain of linearizability is called *Siegel disk*, otherwise the fixed point is called *Cremer fixed point.* G. Pfeiffer [Pf17] found first examples of rational maps with a non linearizable fixed point, and H. Cremer [Cre38] proved that for a Baire generic choice of α in $(0, 1)$ every rational map with a fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ is non linearizable. C. L. Siegel [Sie42] showed that for Lebesgue almost every α in $(0,1)$ every germ with a fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ is linearizable. However, these results did not cover all rotations.

It was apparent from Cremer and Siegel's proofs that the answer to this problem depends on the arithmetic nature of α. Let

$$\alpha = [a_1, a_2, a_3, \dots] := \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

denote the continued fraction expansion of α, and $\frac{p_n}{q_n} = [a_1, a_2, \cdots, a_n]$ denote its rational approximants. Following Siegel's ideas, Brjuno [Brj71] proved that given approximants $\frac{p_n}{q_n}$ of α, under the weaker condition

$$\sum_{n=1}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty,$$

every germ with a fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ is locally linearizable. We say that a real number α is *Brjuno*, and write $\alpha \in \mathscr{B}$, if the above series is finite. Yoccoz [Yoc95] showed that the Brjuno condition is sharp for the quadratic maps. In other words, if $\alpha \notin \mathscr{B}$, $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2$ is not linearizable at 0.

Now assume that $f : \mathbb{C} \to \mathbb{C}$ is a quadratic polynomial. As the dynamics on the Fatou set is very well understood, we have a complete understanding of typical orbits of a map, if Julia set of that map has zero area. Indeed, this is known in several cases including

- if f is hyperbolic,
- if f has a *parabolic* cycle, see [DH82] or [Ly83b],
- if f is at most finitely renormalizable with all periodic points repelling, see [Ly91] or [Sh91],
- if f has a Siegel disk with rotation number satisfying $\log a_n = \mathcal{O}(\sqrt{n})$, see [PZ04],

In particular, siegel disks of bounded type have been studied in greater detail. In [Mc98] and [Ya08] two different types of renormalization technique have been used to describe geometry of the coresponding Julia sets and the dynamics of the map on them.

However, X. Buff and A. Chéritat [BC05] in 2005, by completing a program initiated by A. Douady, proved that there are quadratic polynomials with a Cremer fixed point and positive area Julia set. This motivates the problem of describing typical orbits of points in the Julia set of such quadratic polynomials. In the presence of a Cremer fixed point, there is only one Fatou component, the *basin of infinity*, which is the set of points that tend to the attracting fixed point at infinity under iteration. The Julia set is known to have a complicated topology. For example, it is a non locally connected subset of the plane. Furthermore, by Mañé [Ma83], the finite critical point is recurrent and its orbit accumulates on the Cremer fixed point.

Given $N > 0$, let Irr_N denote the set of irrational numbers $\alpha = [a_1, a_2, \dots]$, with $a_i \geq N$ for $i = 1, 2, \dots$. Our first result in this direction is that

Theorem 1.3. *There exists a constant N such that if $\alpha \in Irr_N$ is a non-Brjuno number, then orbit of almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha i}z + z^2$ accumulates on the 0 fixed point.*

The statement of the above theorem is trivial if the Julia set has zero area. However, the positive area Julia sets constructed by Buff and Chéritat fall into class of maps satisfying our conditions in the above theorem. Thus, above theorem applies to these examples.

Proof of this theorem uses a Douady-Ghys type *renormalization*; return map to a certain domain. In a fruitful work by H. Inou and M. Shishikura [IS06] in 2005 this renormalization was developed to study large iterate of maps near fixed points of high return times. They have introduced a compact (in compact-open topology) class of maps with "large enough" domain of definition which is invariant under this renormalization. More precisely, consider $P(z) = z(1+z)^2 : U \to \mathbb{C}$, where U is a certain domain containing the 0 fixed point and the $-1/3$ critical point. The *Inou-Shishikura* class, $\mathcal{IS}$, is defined as the class of maps $P \circ \varphi^{-1} : U_\varphi \to \mathbb{C}$, where $\varphi : U \to U_\varphi$ is a conformal map with $\varphi(0) = 0$, $\varphi'(0) = e^{-2\pi\alpha i}$. They show that there exists a constant $N > 0$ such that every map in this class with $\alpha \in Irr_N$ is infinitely many times renormalizable, and all the renormalizations belong to this class. This result has proved to be useful in studying local dynamics of maps with a neutral fixed point. For example, it was used in Buff and Chéritat's proof of existence of positive area Julia sets to control post-critical set of such maps.

The idea of the proof of above theorem is to construct infinitely many "gates" with the 0 fixed point on their boundary such that almost every point in the Julia set has to go through them. We can also control diameter of gates in terms of the *Brjuno function* and conclude that these diameters shrink to zero once α is a non-Brjuno number. The orbit of the critical point goes through all these gates (without any assumption on the area of the Julia set). Therefore, a corollary of the above theorem is the following:

Corollary 1.4. *If α is a non-Brjuno number in Irr_N (same N as in the above theorem) then every map in the Inou-Shishikura class with the fixed point of multiplier $e^{2\pi\alpha i}$ at 0 is non-linearizable. In particular, if f is a rational map of the form $e^{2\pi\alpha i} \cdot P \circ h$, with $\alpha \in Irr_N$, where h is an arbitrary rational map of the Riemann sphere satisfying*

- $h(0) = 0$, $h'(0) = 1$,

- *no critical value of h belongs to U (domain of P),* *Then f is not linearizable at* 0.

For a rational map of the Riemann sphere f, the post-critical set $\mathcal{PC}(f)$ is defined as closure of orbits of all critical points of f. It is proved by Lyubich [Ly83b] that the *post-critical set* of a rational map is the *measure theoretic attractor* of points in the Julia set of that map. That is, for every neighborhood of the post-critical set, orbit of almost every point in the Julia set eventually stays in that neighborhood. A central result in our work on this class of quadratic maps is the following:

Theorem 1.5. *There exists a constant N such that for every non-Bruno $\alpha \in Irr_N$, the post-critical set of P_α is connected and has zero area.*

A straight corollary of the above theorem is the following:

Corollary 1.6. *Almost every point in the Julia set of P_α with a non-Brjuno $\alpha \in Irr_N$ is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on the Julia set.*

Chapter 2

Preliminaries

2.1 Julia and Fatou sets

Let U be an open subset of the Riemann sphere $\hat{\mathbb{C}}$ and $\mathcal{F}$ be a family of holomorphic maps defined on U. The family $\mathcal{F}$ is called normal in U, if every infinite sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, contains a subsequence which converges uniformly on all compact subsets of U to a holomorphic map defined on U. Given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we can consider the family of n-fold compositions of f by itself, $\{f^n\}_{n=0}^{\infty}$, defined on $\hat{\mathbb{C}}$. The Fatou set of f, denoted by $F(f)$, is defined as the largest open subset of $\hat{\mathbb{C}}$ on which the sequence of iterates $\{f^n\}_{n=0}^{\infty}$ is normal. The Julia set $J(f)$ is defined as the complement of the Fatou set. By definition, these two sets are invariant.

The post-critical set of f, which plays an important role in understanding the dynamics of f, is defined as

$$\mathcal{PC}(f) := \bigcup_{c:f'(c)=0} \overline{\bigcup_{n=1}^{\infty} f^n(c)},$$

that is, topological closure of forward images $f^k(c)$ with $k > 0$, of critical points of f.

A point $z \in \hat{\mathbb{C}}$ is called periodic of period p if $f^p(z) = z$ and $f^j(z) \neq z$ for $1 < j < p$. It's *multiplier* is defined as $\lambda = (f^p)'(z)$. By definition, periodic points are characterized as *attracting*, *repelling*, or *neutral*, depending on if $|\lambda| < 1$, $|\lambda| > 1$, or $|\lambda| = 1$, respectively. It is called *super attracting* if $\lambda = 0$. It is easy to see that every attracting periodic point is contained in the Fatou set and every repelling one is contained in the Julia set. Indeed, by works of Julia and Fatou, the Julia set is equal to topological closure of repelling periodic points.

A neutral periodic point is called *parabolic* if its multiplier is $e^{2\pi \mathbf{i} \frac{p}{q}}$ for a rational number p/q. Parabolic periodic points belong to the Julia set. A fixed point z with multiplier $e^{2\pi\alpha\mathbf{i}}$ is called *linearizabale*, if there exists a change of coordinate ϕ defined on some neighborhood of z with $\phi(z) = 0$, and

$$\phi^{-1} \circ f \circ \phi = z \mapsto e^{2\pi\alpha\mathbf{i}} z$$

on that neighborhood. If there is no such a coordinate, z is called a *Cremer fixed point*. In the linearizable case, the maximal domain of linearization is called a *Siegel disk*. Linearizable periodic points and Cremer periodic points are defined similarly. Every linearizable periodic point belongs to Fatou set and every Cremer one belongs to Julia set.

It turns out that the linearizability of a neutral fixed point with an irrational rotation depends on the arithmetic nature of the rotation. Every irrational number can be written as a continued fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

which produces best rational approximants

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \frac{1}{a_n}}}},$$

for every $n \geq 1$. The following theorem gives a combinatorial condition on the rotation which implies linearizability at a neutral fixed point.

Theorem 2.1 (Siegel, Brjuno)**.** *If*

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty,$$

then every holomorphic germ with a fixed point of multiplier $e^{2\pi\alpha i}$ is locally linearizable.

In the other direction we have,

Theorem 2.2 (Yoccoz)**.** *If the above sum is divergent, then the quadratic polynomial $z \to e^{2\pi\alpha \mathbf{i}} z + z^2$ is not linearizable at* 0*.*

By a theorem of Sullivan [Su85] every component of Fatou set is eventually periodic. Let U be a periodic component of Fatou set of a rational map f of period p. There are only five possibilities for f^p on U listed below.

1. **Supper attracting case:** there exists a super attracting periodic point $z \in U$, attracting orbit of every point in U under f^p.

2. **Attracting case:** there exists a periodic point $z \in U$, $0 < |(f^p)'(z)| < 1$, attracting orbit of every point in U under f^p.

3. **Parabolic case:** there exits a periodic point $z \in \partial U$ with $(f^p)'(z) = 1$, attracting orbit of every point in U under f^p.

4. **Siegel disk:** U is conformally isomorphic to a disk, and f^p acts as an irrational rotation on U.

5. **Herman Ring:** U is conformally isomorphic to an annulus, and f^p acts as an irrational rotation on U.

Finally, M. Shishikura [Sh87] proves a conjeture on an upper bound for number of priodic Fatou components.

2.2 Dynamics of polynomials

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree d, $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d$, ∞ is a super attracting fixed point of f and its *basin of attraction* is defined as

$$D_f(\infty) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

Its complement is called the *filled Julia* set; $K(f) = \mathbb{C} \setminus D_f(\infty)$. The Julia set $J(f)$ is the common boundary of $K(f)$ and $D_f(\infty)$. It is known that the Julia set and the filled Julia set of a polynomial are connected if and only if all critical points stay bounded under iteration.

With f as above, there exists a conformal change of coordinate, *Böttcher coordinate*, B_f which conjugates f to the *dth* power map $z \mapsto z^d$ throughout some neighborhood of infinity U_f. It is defined as,

$$\begin{aligned} &B_f : U_f \to \{z \in \mathbb{C} : |z| > r_f \geq 1\}, \\ &B_f(z) := \lim_{n\to\infty} \sqrt[d^n]{f^n(z)} \end{aligned} \tag{2.1}$$

where d^nth root is chosen such that B_f is tangent to the identity map at infinity. By definition, it satisfies the *equivariance relation* $B_f(f(z)) = (B_f(z))^d$, and $B_f(z) \sim z$ as $z \to \infty$.

In particular, if the filled Julia set is connected, B_f coincides with the Riemann mapping of $D_f(\infty)$ onto the complement of the closed unit disk, normalized to be tangent to the identity map at infinity.

The *external ray* $R^\theta = R^\theta_f$ of angle θ is defined as

$$B_f^{-1}\left(\{re^{\mathbf{i}\theta} : r_f < r < \infty\}\right).$$

The *equipotential* $E^r = E^r_f$ of level $r > r_f$ is defined as

$$B_f^{-1}\left(\{re^{\mathbf{i}\theta} : 0 \leq \theta \leq 2\pi\}\right).$$

It follows from equivariance relation that $f(R^\theta) = R^{d\theta}$, and $f(E^r) = E^{r^d}$.

A ray R^θ is called periodic ray of period p if $f^p(R^\theta) = R^\theta$. A ray is fixed ($p = 1$) if and only if θ is a rational number of the form $2\pi j/(d-1)$. By definition, a ray R^θ lands at a well defined point z in $J(f)$ if the limiting value of the ray R^θ (as $r \to r_f$) exists and equals to z. Such a point $z \in J(f)$ is called the *landing point* of R^θ. The following theorem characterizes landing points of periodic rays. See [DH82] for further discussions.

Theorem 2.3. *Let f be a polynomial of degree $d \geq 2$ with connected Julia set. Every periodic ray lands at a well defined periodic point which is either repelling or parabolic. Vice versa, every repelling or parabolic periodic point is the landing point of at least one, and at most finitely many periodic rays with the same ray period.*

In particular, this theorem implies that the external rays landing at a periodic point are organized in several cycles. Let $\overline{a} = \{a_k\}_{k=0}^{p-1}$ be a repelling or parabolic cycle of f and let $\mathfrak{R}(a_k)$ be union of all external rays landing at a_k. The configuration

$$\mathfrak{R}(\overline{a}) = \bigcup_{k=0}^{p-1}(\mathfrak{R}(a_k))$$

with the rays labeled by their external angles, is called the *periodic point portrait* of f associated to the cycle $\overline{a}$.

2.3 Unicritical family and the connectedness locus

Any degree d polynomial with only one critical point is affinely conjugate to $P_c(z) = z^d + c$ for some complex number c. A case of especial interest is the following fixed point portrait. The $d-1$ fixed rays $R^{2\pi j/(d-1)}$ land at $d-1$ fixed points called β_j, and moreover, these are the only rays that land at β_j's. These fixed points are *non-dividing*, that is, $K(F) \setminus \beta_j$ is connected for every j. If the other fixed point, called α, is also repelling, there are at least 2 rays landing at it. Thus, α-fixed point is *dividing* and by Theorem 2.3, the rays landing at α-fixed point are permuted under the dynamics. The following statement has been shown in [M95] for quadratic polynomials. The same ideas apply to prove it for degree d unicritical polynomials.

Proposition 2.4. *If at least* 2 *rays land at the* α *fixed point of* P_c*, we have:*

- *The component of* $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ *containing the critical value is a sector bounded by two external rays.*
- *The component of* $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ *containing the critical point is a region bounded by* $2d$ *external rays landing in pairs at the points* $e^{2\pi j/d}\alpha$*, for* $j = 0, 1, \ldots, d-1$.

The *Connectedness locus* $\mathcal{M}_d$ of degree d is defined as the set of parameters c in $\mathbb{C}$ for which $J(P_c)$ is connected, or equivalently, the critical point of P_c does not escape to infinity under iteration of P_c. In particular, $\mathcal{M}_2$ is the well-known *Mandelbrot* set. See Figures 2.1 and 2.2. A well-known result due to Douady and Hubbard [DH82] shows that the connectedness loci are connected. Their argument is based on constructing an explicit conformal isomorphism

$$B_{\mathcal{M}_d} : \mathbb{C} \setminus \mathcal{M}_d \to \{z \in \mathbb{C} : |z| > 1\}$$

given by $B_{\mathcal{M}_d}(c) = B_c(c)$, where B_c is the Böttcher coordinate for P_c.

By means of conformal isomorphism $B_{\mathcal{M}_d}$, the parameter external rays R_θ and equipotentials E_r are defined, similarly, as the $B_{\mathcal{M}_d}$-preimages of the straight rays going to infinity and round circles around 0.

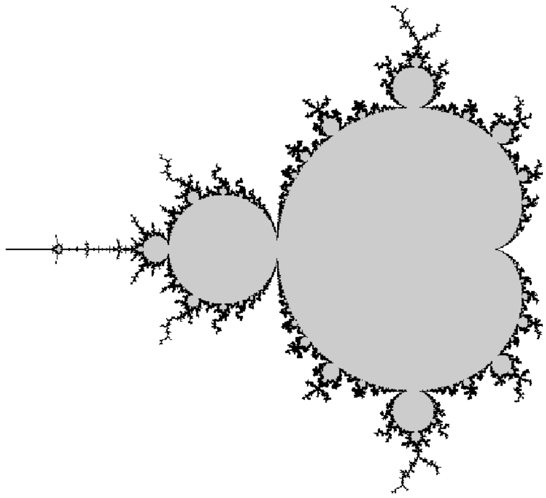

Figure 2.1: The Mandelbrot set

A polynomial P_c (and the corresponding parameter c) is called *hyperbolic* if P_c has an attracting periodic point. The set of hyperbolic parameters in $\mathcal{M}_d$, topologically open by definition, is the union of some components of $\operatorname{int}\mathcal{M}_d$. These components are called *hyperbolic components*.

The main hyperbolic component is defined as the set of parameter values c for which P_c has an attracting fixed point. Outside of the closure of this set all fixed points become repelling. Now, consider a hyperbolic component $\mathscr{H} \subset \operatorname{int}\mathcal{M}_d$, and suppose $\overline{b}_c$ is the corresponding attracting cycle with period k. On the boundary of $\mathscr{H}$ this cycle becomes neutral, and there are $d-1$ parameters $c_i \in \partial\mathscr{H}$ where P_{c_i} has a parabolic cycle with multiplier equal to one. One of these parameters c_i, denoted by c_{root}, which divides the connectedness locus into two pieces, is called *root* of $\mathscr{H}$ (See [DH82] for quadratic polynomials and [Sch04] for arbitrary degree unicritical polynomials). Indeed, any hyperbolic component has one root and $d-2$ *co-roots*. The root is the landing point of two parameter rays, while every co-root is the landing point of a single parameter ray, See Figure 2.3.

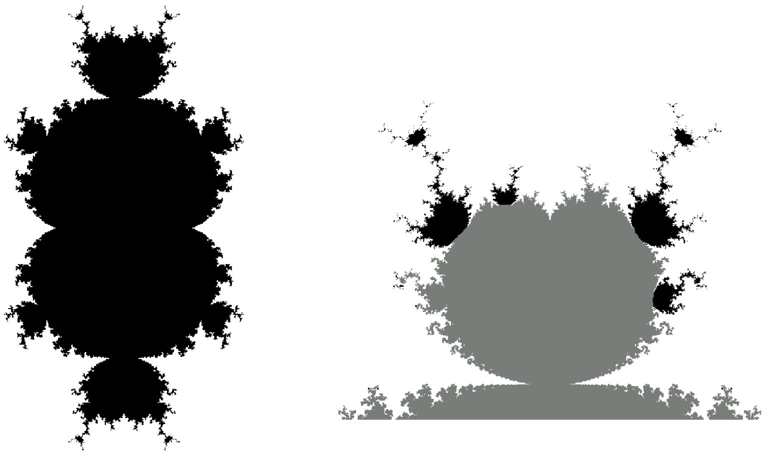

Figure 2.2: Figure on the left shows the connectedness locus $\mathcal{M}_3$. The figure on the right is an enlargement of a primary limb in $\mathcal{M}_3$. The dark regions show some of the secondary limbs

Let c belong to a hyperbolic component $\mathscr{H}$, different from main component of the connectedness locus, with attracting cycle $\bar{b}_c$. The *basin of attraction* of $\bar{b}_c$, denoted by A_c, is defined as the set of points $z \in \mathbb{C}$ with $P_c^n(z)$ converges to the cycle $\bar{b}_c$. The boundary of the component of A_c containing c is a jordan curve which we denote it by D_c. The map P_c^k on D_c is topologically conjugate to $\theta \mapsto d\theta$ on the unit circle. Therefore, there are $d-1$ fixed points of P_c^k on this jordan curve which are repelling periodic points (of P_c) of period dividing period of $\bar{b}_c$ (its period can be strictly less than period of b_c). Among all rays landing at these repelling periodic points, let θ_1 and θ_2 be the angles of the external rays bounding the sector containing the critical value of P_c (See Figure 2.3). The following theorem makes a connection between external rays R^{θ_1}, R^{θ_2} and the parameter external rays R_{θ_1}, R_{θ_2} on the parameter plane. See [DH82] or [Sch04] for proofs.

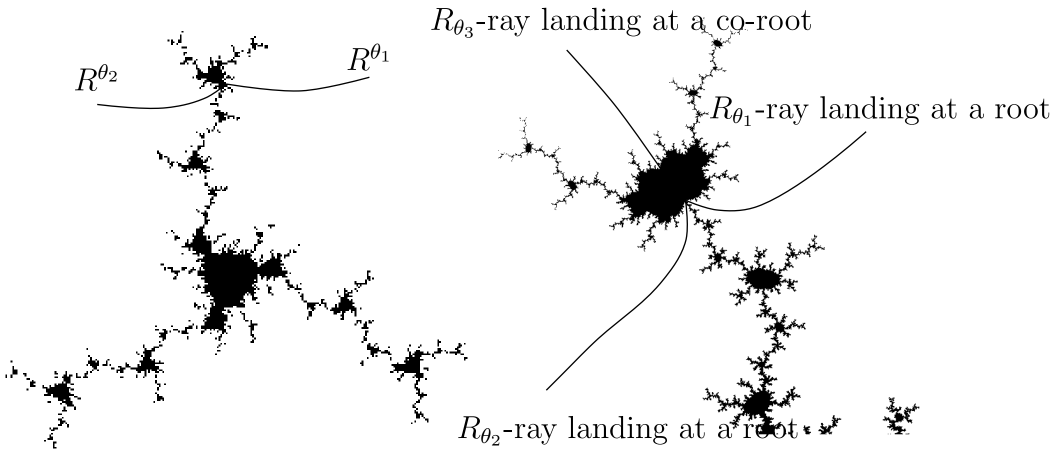

Figure 2.3: Figure on the left shows a primitive renormalizable Julia set and the external rays R^{θ_1} and R^{θ_2} landing at the corresponding repelling periodic point. The figure on the right is the corresponding primitive little multibrot copy. It also shows the parameter external rays R_{θ_1} and R_{θ_2} landing at the root point.

Theorem 2.5. *The parameter external rays R_{θ_1} and R_{θ_2} land at the root point of $\mathscr{H}$, and these are the only rays that land at this point.*

Closure of the two parameter external rays R_{θ_1} and R_{θ_2} cut the plane into two components. The one containing the component $\mathscr{H}$, with the root point attached to it, is called the *wake* $W_{\mathscr{H}}$. So, a wake is an open set with a root point attached to its boundary. For a wake $W_{\mathscr{H}}$ and an equipotential of level η, E_η, the *truncated wake* $W_{\mathscr{H}}(\eta)$ is the bounded component of $W_{\mathscr{H}} \setminus E_\eta$. Part of the connectedness locus contained in the wake $W_{\mathscr{H}}$ with the root point attached to it is called the *limb* $\mathscr{L}_{\mathscr{H}}$ of the connectedness locus *originated* at $\mathscr{H}$. In other words, the limb $\mathscr{L}_{\mathscr{H}}$ is part of $\mathcal{M}_d$ contained in $W_{\mathscr{H}}$. By definition, every limb is a closed set.

The wakes attached to the main hyperbolic component of $\mathcal{M}_d$ are called *primary wakes* and a limb associated to such a primary wake will be called *primary limb*. If $\mathscr{H}$ is a hyperbolic component attached to the main hyperbolic component, all the wakes attached to such a component $\mathscr{H}$ (except $W_{\mathscr{H}}$ itself) are called *secondary wakes*. Similarly, a limb associated to a secondary wake will be called *secondary limb*. A *truncated limb* is obtained from a limb by removing a neighborhood of its root. Some secondary limbs are shown in Figure 2.2.

Given a parameter c in $\mathscr{H}$, we have the attracting cycle $\overline{b}_c$ as above, and the associated repelling cycle $\overline{a}_c$ which is the landing point of the external rays R^{θ_1} and R^{θ_2}. The following result gives the dynamical meaning of the parameter values in the wake $W_{\mathscr{H}}$ which is bounded by parameter external rays R_{θ_1} and R_{θ_2} (See [Sch04] for the proof).

Theorem 2.6. *For parameters c in $W_{\mathscr{H}} \setminus \{$root point$\}$, the repelling cycle $\overline{a}_c$ stays repelling and moreover, the isotopic type of the ray portrait $\mathfrak{R}(\overline{a}_c)$ is fixed throughout $W_{\mathscr{H}}$.*

2.4 Quasi-conformal mappings

There are several equivalent definitions of quasi-conformal mappings convenient in different situations. A good source on the subject is Ahlfor's book [Ah66]. An analytic definition is more appropriate for us. Let Ω be an open subset of the complex plane. A homeomorphism $\phi : \Omega \to \mathbb{C}$ is called absolutely continuous on lines if for every rectangle R contained in Ω with sides parallel to x and y axis, ϕ is absolutely continuous on almost every horizontal and almost every vertical line in R. Such function has partial derivatives almost every where in Ω. Let ϕ_z denote $\frac{1}{2}(\phi_x - \mathbf{i}\phi_y)$, and $\phi_{\bar{z}}$ denote $\frac{1}{2}(\phi_x + \mathbf{i}\phi_y)$, with $z = x + \mathbf{i}y$.

An orientation preserving homeomorphism $\phi : \Omega \to \mathbb{C}$ is called K-quasi-conformal, K-q.c. for short, if

- ϕ is absolutely continuous on lines,
- $|\phi_{\bar{z}}| \leq k|\phi_z|$ almost every where in Ω, where $k = \frac{K-1}{K+1}$.

The measurable function $\mu_\phi = \phi_{\bar{z}}/\phi_z$ is called *complex dilatation* of ϕ and the smallest value of K satisfying above definition is called *dilatation* of ϕ. Some properties of q.c. mappings which will be used in our work is listed in the following theorem. One may refer to [Ah66] for their proofs.

Theorem 2.7. *Quasi-conformal mappings satisfy the following properties:*

- *Composition of a K_1-q.c. and a K_2-q.c. map is a K_1K_2-q.c. map.*
- *The space of K-q.c. mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ fixing three points 0, 1, and ∞ is a compact class (under uniform convergence on compact sets).*
- *A 1-q.c. map is conformal.*

We will be interested in solving Beltrami equation $\mu_\phi = \mu$ for a given measurable function $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Solution to this problem is referred to as measurable Riemann mapping theorem.

Theorem 2.8. *For any measurable $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$, there exists a unique normalized q.c. mapping ϕ^μ with complex dilatation μ that leaves 0, 1, ∞ fixed.*

Any topological annulus is conformally isomorphic to one of $\mathbb{C}\setminus\{0\}$, $B(0,1)\setminus\{0\}$, or $B(0,r) \setminus B(0,1)$ for some $r > 1$. It's *modulus*, denoted as $\operatorname{mod} A$, is defined as ∞ in the first two cases and $\frac{1}{2\pi}\log r$ in the last case. It turns out that this quantity is *quasi-invariant* under q.c. mappings. That is, given a K-q.c. mapping $\phi : A \to \mathbb{C}$, we have $\frac{1}{K} \mod A \leq \mod \phi(A) \leq K \mod A$.

The following theorem makes the connection between q.c. mappings on Ω and maps on boundary of Ω. For convenience we will assume that Ω is upper half plane.

Theorem 2.9 (boundary correspondence)**.** *Let ϕ be a K-q.c. mapping of upper half plane onto itself which maps ∞ to itself. Then ϕ induces a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ which satisfies*

$$M(K)^{-1} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M(K),$$

for some constant $M(K)$ depending only on K. Vice versa, every homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(\infty) = \infty$ that satisfies above inequality for some M, extends to a q.c. mapping of the upper half plane with dilatation depending only on M.

2.5 Polynomial-like maps

A holomorphic proper branched covering of degree d, $f : U' \to U$, where U and U' are simply connected domains with U' compactly contained in U, is called a *polynomial-like map.* Reader can consult [DH85] for the following material on polynomial-like maps. Every polynomial can be viewed as a polynomial-like map once restricted to an appropriate neighborhood of its filled Julia set. In what follows we will only consider polynomial-like maps with one branched point of degree d (which is assumed to be at zero after normalization) and refer to them as *unicritical polynomial-like* maps.

The filled Julia set $K(f)$ of a polynomial-like map $f : U' \to U$ is naturally defined as

$$K(f) = \{z \in \mathbb{C} : f^n(z) \in U', n = 0, 1, 2, \ldots\}.$$

The Julia set $J(f)$ is defined as the boundary of $K(f)$. They are connected if and only if $K(f)$ contains critical point of f.

Two polynomial-like maps f and g are said *topologically* (*quasi-conformally*, *conformally*, *affinely*) *conjugate* if there is a choice of domains $f : U' \to U$ and $g : V' \to V$ and a homeomorphism $h : U \to V$ (quasi-conformal, conformal, or affine isomorphism, respectively) such that $h \circ f|U' = g \circ h|U'$.

Two polynomial like maps f and g are *hybrid or internally equivalent* if there is a q.c. conjugacy h between f and g with $\overline{\partial} h = 0$ on $K(f)$. The following theorem due to Douady and Hubbard [DH85] makes the connection between polynomial-like maps and polynomials.

Theorem 2.10 (Straightening)**.** *Every polynomial-like map f is hybrid equivalent to (suitable restriction of) a polynomial P of the same degree. Moreover, P is unique up to affine conjugacy when $K(f)$ is connected.*

In particular, any unicritical polynomial-like map with connected Julia set corresponds to a unique (up to affine conjugacy) unicritical polynomial $z \mapsto z^d + c$ with c in the connectedness locus $\mathcal{M}_d$. Note that $z^d + c$ and $z^d + c/\lambda$ are conjugate via $z \mapsto \lambda z$ for every $d - 1$th root of unity λ.

Given a polynomial-like map $f : U' \to U$, we can consider the *fundamental annulus* $A = U \setminus U'$. It is not canonic because any choice of $V' \Subset V$ such that $f : V' \to V$ is a polynomial-like map with the same Julia set will give a different annulus. However we can associate a real number, *modulus of* f, to any polynomial-like map f as follows:

$$\mathrm{mod}(f) = \sup \mathrm{mod}(A)$$

where the sup is taken over all possible fundamental annuli A of f.

It can be seen from the proof of the straightening theorem that dilatation of the q.c. conjugacy between f and the corresponding polynomial $P_{c(f)} : z \mapsto z^d + c(f)$ can be controlled by modulus of f.

Proposition 2.11. *If mod$(f) \geq \mu > 0$ then dilatation of the q.c. conjugacy obtained in the straightening theorem depends only on μ.*

2.6 Holomorphic motions

In this section we consider analytic deformations of sets in $\hat{\mathbb{C}}$. Let A be a subset of $\hat{\mathbb{C}}$. A holomorphic motion of A parametrized on a complex manifold M is a map $\Phi : M \times A \to \hat{\mathbb{C}}$ such that

- For any fixed $z \in A$, the map $\lambda \to \Phi(\lambda, z)$ is holomorphic in M.
- For any fixed $\lambda \in M$, the map $z \to \Phi(\lambda, z)$ is an injection.
- There exists a point $\lambda_0 \in M$ with $\Phi(\lambda_0, z) = z$, for every $z \in A$.

The following remarkable result due to Mané-Sad-Sullivan [MSS83] relates holomorphic dynamics with q.c. mappings.

Theorem 2.12 (λ-Lemma)**.** *If $\Phi : M \times A \to \hat{\mathbb{C}}$ is a holomorphic motion, then Φ has an extension to $\tilde{\Phi} : M \times \bar{A} \to \hat{\mathbb{C}}$, with*

- *$\tilde{\Phi}$ is a holomorphic motion of $\bar{A}$.*
- *Each $\tilde{\Phi}(\lambda, \cdot) : \bar{A} \to \hat{\mathbb{C}}$ is quasi-conformal.*

An important result about holomorphic motions is Slodkowski's generalized λ-lemma [Sl91] stating that a holomorphic motion of a set A extends to a holomorphic motion of the whole Riemann sphere.

Chapter 3

Combinatorial Rigidity in the Unicritical Family

3.1 Modified principal nest

3.1.1 Yoccoz puzzle pieces

Recall that for a parameter $c \in \mathcal{M}_d$ outside of the main component of the connectedness locus, P_c has a unique dividing fixed point α_c. The $q \geq 2$ external rays $\mathfrak{R}(\alpha_c)$ landing at this fixed point together with an arbitrary equipotential E^r, cut the domain inside E^r into q closed topological disks $Y_j^0, j = 0, 1, \ldots, q-1$, called *puzzle pieces of level zero*. That is, Y_j^0's are the closures of the bounded components of $\mathbb{C} \setminus \{E^r \cup \mathfrak{R}(\bar{\alpha}_c) \cup \{\alpha_c\}\}$. The main property of this partition is that $P_c(\partial Y_j^0)$ does not intersect interior of any piece Y_i^0.

Now the *puzzle pieces* Y_i^n *of level or depth* n are defined as the closures of the connected components of $f^{-n}(\mathrm{int}(Y_j^0))$. They partition the neighborhood of the filled Julia set bounded by equipotential $f^{-n}(E^r)$ into finite number of closed disks. By definition all puzzle pieces are bounded by piecewise analytic curves. The *label* of each puzzle piece is the set of the angles of external rays bounding that puzzle piece. If the critical point does not land on the α_c-fixed point, there is a unique puzzle piece Y_0^n of level n containing the critical point.

The family of all puzzle pieces of P_c of all levels has the following *Markov property*:

- Puzzle pieces are disjoint or nested. In the latter case, the puzzle piece of higher level is contain in the puzzle piece of lower level.

- Image of any puzzle piece of level $n \geqslant 1$ is a puzzle piece of level $n-1$. Moreover, $P_c : Y_j^n \to Y_k^{n-1}$ is d-to-1 branched covering or univalent, depending on whether Y_j^n contains the critical point or not.

On the first level, there are $d(q-1)+1$ puzzle pieces. One critical piece Y_0^1, $q-1$ ones, denoted by Y_i^1, attached to the fixed point α_c, and the $(d-1)(q-1)$ symmetric ones, denoted by Z_i^1, attached to $P_c^{-1}(\alpha_c) \setminus \{\alpha_c\}$. Moreover $f|Y_0^1$ d-to-1 covers Y_1^1, $f|Y_i^1$ univalently covers Y_{i+1}^1, for $i = 1, \dots, q-2$, and $f|Y_{q-1}^1$ univalently covers, $Y_0^1 \cup \bigcup_{i=1}^{(d-1)(q-1)} Z_i^1$. Thus $f^q(Y_0^1)$ truncated by $f^{-1}(E^r)$ is the union of Y_0^1 and Z_i^1's.

We will assume after this that $P_c^n(0) \neq \alpha_c$ for all n, so that the critical puzzle pieces of all levels are well defined. As it will be apparent in a moment, this condition is always the case for renormalizable polynomials.

3.1.2 The complex bounds in the favorite nest and renormalization

For a puzzle piece $V \ni 0$, let $R_V : \mathrm{Dom}\, R_V \subseteq V \to V$ denote *the first return map* to V. It is defined at every points z in V for which there exists a positive integer t with $P_c^t(z) \in \operatorname{int} V$. Then $R_V(z)$ is defined as $P_c^t(z)$ where t is the first positive moment when $P_c^t(z) \in \operatorname{int} V$. Markov property of puzzle pieces implies that any component of Dom R_V is contained in V and the restriction of this return map (P_c^t, for some t) to such a component is d-to-1 or 1-to-1 proper map onto V. The component of the Dom R_V which contains the critical point is called the *central component* of R_V. If the image of critical point under the first return map belongs to the central component, the return is called *central return*.

The *first landing map* L_V to a puzzle piece $V \ni 0$ is defined at all points $z \in \mathbb{C}$ for which there exists an integer $t \geq 0$ with $P_c^t(z) \in \operatorname{int} V$. It is the identity map on the component V and univalently maps each component of Dom L_V onto V.

Consider a puzzle piece $Q \ni 0$. If the critical point returns back to Q under iteration of P_c, the central component $P \subset Q$ of R_Q is the pullback of Q by P_c^p along the orbit of the critical point, where p is the first moment when critical orbit enters $\operatorname{int} Q$. This puzzle piece P is called the *first child* of Q. Recall that $P_c^p : P \to Q$ is a proper map of degree d.

The *favorite child* Q' of Q is constructed as follows; Let $p > 0$ be the first moment when $R_Q^p(0) \in \operatorname{int}(Q \setminus P)$ (if it exists) and $q > 0$ be the first moment (if it exists) when $R^{p+q}(0) \in \operatorname{int} P$ ($p + q$ is the moment of the first return back to P after the first escape of the critical point from P under iterates of R_Q). Now Q' is defined as the pullback of Q under R_Q^{p+q} containing the critical point. Markov property of puzzle pieces implies that the map $P_c^k = R_Q^{p+q}$ (for an appropriate $k > 0$) from Q' to Q is proper of degree d. The main property of the favorite child is that the image of the critical point under the map $P_c^k : Q' \to Q$ belongs to the first child P.

Consider a unicritical polynomial P_c with q rays landing at its α fixed point and form the corresponding Yoccoz puzzle pieces. The map P_c is called *satellite renormalizable*, or *immediately renormalizable* if

$$P_c^{lq}(0) \in Y_0^1, \quad \text{for } l = 0, 1, 2, \ldots.$$

The map $P_c^q : Y_0^1 \to P_c^q(Y_0^1)$ is proper of degree d but its domain is not compactly contained in its range. By slight "thickening" of Y_0^1 so that Y_0^1 is compactly contained in $P_c^q(Y_0^1)$ (see [M00]), P_c^q can be turned into a unicritical polynomial-like map. Note that the above condition implies that the critical point does not escape its domain of definition and therefore the corresponding little Julia set is connected.

If P_c is not satellite renormalizable, then there is a first moment k such that $P_c^{kq}(0)$ belongs to some Z_i^1. Define Q^1 as the pullback of this Z_i^1 under P_c^{kq}. By the above construction we form the first child, P^1, and the favorite child ,Q^2, of Q^1. Repeating the above process we get a nest of puzzle pieces

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^n \supset P^n \supset \cdots \tag{3.1}$$

where P^i is the first child of Q^i, and Q^{i+1} is the favorite child of Q^i.

The above process stops if and only if one of the following happens:

- The map P_c is combinatorially non-recurrent, that is, the critical point does not return to some critical puzzle piece.
- The critical point does not escape the first child P^n under iterates of R_{Q^n} for some n, or equivalently, returns to all critical puzzle pieces of level bigger than n are central.

In the former case, combinatorial rigidity in the critically non-recurrent case has been taken care of in [M00].

In the latter case, $R_{Q^n} = P_c^k : P^n \to Q^n$ (for an appropriate k) is a unicritical polynomial-like map of degree d with P^n compactly contained in Q^n. The map P is called *primitively renormalizable* in this case. Note that the corresponding little Julia set is connected because all the returns of critical point to Q^n are central by definition.

A map $P_c : \mathbb{C} \to \mathbb{C}$ is called *renormalizable* if it is satellite or primitively renormalizable.

To deal with non-renormalizable and recurrent polynomials, the following *a priori* bounds has been proved in [AKLS05].

Theorem 3.1. *There exists* $\delta > 0$ *such that for every* $\varepsilon > 0$ *there exists* $n_0 = n_0(\varepsilon) > 0$ *with the following property. For the nest of puzzle pieces*

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^m \supset P^m \supset \cdots$$

as above, if $\operatorname{mod}(Q^1 \setminus P^1) > \varepsilon$ *and* $n_0 < n < m$ *then* $\operatorname{mod}(Q^n \setminus P^n) > \delta$.

If a map P_c is combinatorially recurrent, the critical point does not land at α-fixed point, therefore puzzle pieces of all levels are well defined. The *combinatorics of* P_c up to level n is an equivalence relation on the set of angles of puzzle pieces up to level n. Two angles θ_1 and θ_2 are equivalent if the corresponding rays R^{θ_1} and R^{θ_2} land at the same point. One can see that the combinatorics of a map up to level $n+t$ determines the puzzle piece Y_j^n of level n containing the critical value $f^t(0)$. Two non-renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics up to arbitrary level n,that is, they have the same set of labels of puzzle pieces and the same equivalence relation on them. Combinatorics of a renormalizable map will be defined in section 3.1.3.

Two unicritical polynomials P_c and $P_{\tilde{c}}$ with the same combinatorics up to level n are called *pseudo-conjugate (up to level n)* if there is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that $H(Y_j^0) = \widetilde{Y}_j^0$ for $j = 0, 1, 2, \dots$ and $H \circ P_c = P_{\tilde{c}} \circ H$ outside of the critical puzzle piece Y_0^n. A pseudo-conjugacy H is said to *match the Böttcher marking* if near infinity it becomes identity in the Böttcher coordinates for P_c and $P_{\tilde{c}}$. Thus a pseudo-conjugacy is the identity map in the Böttcher coordinate outside of $\cup_j Y_j^n$ by its equivariance property.

Let q_m and p_m be the levels of the puzzle pieces Q^m and P^m, that is, $Q^m = Y_0^{q_m}$ and $P^m = Y_0^{p_m}$. The following statement is the main technical result of [AKLS05] which will be used frequently in our construction.

Theorem 3.2. *Assume that a (finite) nest of puzzle pieces*

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^m \supset P^m \tag{3.2}$$

is obtained for P_c. If $P_{\tilde{c}}$ is combinatorially equivalent to P_c, then there exists a K-q.c. pseudo-conjugacy H (up to level q_m where $Q^m = Y_0^{q_m}$) between P_c and $P_{\tilde{c}}$ which matches the Böttcher marking.

Proposition 3.3. *Assume that the nest of puzzle pieces in the above theorem is defined using equipotential of level η, then dilatation of the q.c. pseudo-conjugacy, $K = K(c, \tilde{c})$, obtained in that theorem depends only on the distance between c and $\tilde{c}$ in the primary wake truncated by equipotential of level η and equipped with the Poincaré metric.*

Proof. To prove the proposition we need a brief sketch of the proof of the above theorem. For more details refer to [AKLS05]. Combinatorial equivalence of P_c and $P_{\tilde{c}}$ up to level zero implies that the parameters c and $\tilde{c}$ belong to the same truncated wake $W(\eta)$ attached to the main component of the Connectedness locus. Inside $W(\eta)$, the q external rays $\mathfrak{R}(\alpha)$ and the equipotential $E(h)$ (for every $h > \eta$) move holomorphically in $\mathbb{C} \setminus 0$. That is, there exists a holomorphic motion Φ of $\mathfrak{R}(\alpha) \cup E(h)$, given by $B_{\tilde{c}}^{-1} \circ B_c$ in the second coordinate, parametrized on $W(\eta)$ such that

$$\phi(\tilde{c}, \mathfrak{R}(\alpha) \cup E(h)) = (\tilde{c}, \widetilde{\mathfrak{R}}(\alpha) \cup \widetilde{E}(h)).$$

Outside of equipotential $E(h)$, this holomorphic motion extends to a motion holomorphic in both variables (c, z) which is coming from the Böttcher coordinate near ∞. By [Sl91] the map $\phi^{\tilde{c}} \circ (\phi^c)^{-1}$ extends to a K_0 q.c. map $G_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, where K_0 only depends on the hyperbolic distance between c and $\tilde{c}$ in the truncated wake $W(\eta)$. This gives a q.c. map $G_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which conjugates P_c and $P_{\tilde{c}}$ outside of puzzle pieces of level zero.

By adjusting the q.c. map G_0 inside equipotential $E(h)$ such that it sends c to $\tilde{c}$, we get a q.c. map (not necessarily with the same dilatation) G_0'. By lifting G_0' via P_c and $\tilde{f}$ we get a new q.c. map G_1. Repeating this process, for $i = 1, 2, \ldots, n = q_m$, which is adjusting the q.c. map G_i inside union of puzzle pieces of level $i+1$ so that it sends c to $\tilde{c}$ and lifting it, we obtain a q.c. map G_{i+1} (not with the same dilatation) conjugating P_c and $P_{\tilde{c}}$ outside union of puzzle pieces of level $i+1$. At the end we will have a q.c. map G_n which conjugates P_c and $P_{\tilde{c}}$ outside of equipotential $E(h/d^n)$.

The nest of puzzle pieces

$$\widetilde{Q}^1 \supset \widetilde{P}^1 \supset \widetilde{Q}^2 \supset \widetilde{P}^2 \supset \cdots \supset \widetilde{Q}^m \supseteq \widetilde{P}^m$$

for $P_{\tilde{c}}$ is defined as the image of the nest of puzzle pieces in (3.2) under the map G_n. Combinatorial equivalence of P_c and $P_{\tilde{c}}$ implies that this new nest has the same properties as the one for P_c. In other words, $\widetilde{Q}^{i+1}$ is the favorite child of $\widetilde{Q}^i$ and $\widetilde{P}^i$ is the first child of $\widetilde{Q}^i$. Hence Theorem 3.1, applies to this nest too. By properties of these nests, one constructs a K-q.c. map H_n from the critical puzzle piece Q^n to the corresponding one $\widetilde{Q}^n$ where K only depends on the *a priori* bounds δ and the hyperbolic distance between c and $\tilde{c}$ in the truncated wake W^η. The pseudo-conjugacy H_n is obtained from univalent lifts of H_n onto other puzzle pieces. □

If P_c is renormalizable, the process of constructing modified principal nest stops at some level and returns to the critical puzzle pieces after this level are central. One can see that critical puzzle pieces do not shrink to 0.

3.1.3 Combinatorics of a map

If $P_{c_0} : z \mapsto z^d + c_0$ is renormalizable, there is a homeomorphic copy $\mathcal{M}_d^1 \ni c_0$ of the connectedness locus within the connectedness locus satisfying the following properties (see[DH85]): For $c \in \mathcal{M}_d^1 \setminus \{\text{the root point}\}$, $P_c : z \mapsto z^d + c$ is renormalizable, and there is a holomorphic motion of the dividing fixed point α_c and the rays landing at it on a neighborhood of $\mathcal{M}_d^1 \setminus \{\text{the root point}\}$, such that the renormalization of P_c is associated to this fixed point and external rays. In other words, all parameters in this copy have Yoccoz puzzle pieces of all levels with the same labels. This homeomorphism is not unique because of the symmetry in the connectedness locus. However, we define it uniquely by sending the unique root point of the copy to the landing point of the parameter external ray of angle 0.

We show below that among all renormalizations of P_c, there is a unique one denoted by $\mathcal{R}P_c$ which corresponds to a maximal copy (not included in any other copy except $\mathcal{M}_d$ itself) of the connectedness locus inside the connectedness locus.

Assume that $\mathcal{R}P_c$ is equal to $P_c^j : U \to U'$, for some positive integer j and topological disk U compactly contained in U'. By straightening theorem, $\mathcal{R}P_c$ is conjugate to a unicritical polynomial $P_{c'}$. That is, there exists a q.c. mapping $\chi : U' \to \mathbb{C}$, with $\chi \circ P_c^j(z) = P_{c'} \circ \chi(z)$, for every $z \in U$. This polynomial $P_{c'}$ is determined up to conformal equivalence in the theorem. However, there are only $d-1$ conformally equivalent polynomials in each class. We make this parameter unique by choosing the image of c under the homeomorphism uniquely determined above.

If $P_{c'}$ is also renormalizable, P_c is called *twice renormalizable*. Let positive integer k, and topological disks V and V' be such that $P_{c'}^k : V \to V'$ is a renormalization of $P_{c'}$. Define $\widetilde{V}$ and $\widetilde{V}'$ as χ-preimage of V and V', respectively. One can see that χ conjugates $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ with $P_{c'}^k : V \to V'$. Therefore, $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ is also a polynomial-like map. We denote this map by $\mathcal{R}^2 P_c$.

Above process may be continued to associate a canonical finite or infinite sequence $P_c, \mathcal{R}P_c, \mathcal{R}^2 P_c, \ldots$, of polynomial-like maps to P_c, and accordingly, call P_c *at most finitely*, or *infinitely renormalizable*. Equivalently, there is a finite or infinite sequence $\tau(P_c) := \langle \mathcal{M}_d^1, \mathcal{M}_d^2, \ldots \rangle$, of maximal connectedness locus copies associated to P_c, where $\mathcal{M}_d^n$ corresponds to the renormalization $\mathcal{R}^n P_c$ of $\mathcal{R}^{n-1} P_c$. In the infinitely renormalizable situation, $\tau(P_c)$ is called the *combinatorics* of P_c.

Two infinitely renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics, i.e. correspond to the same sequence of maximal connectedness locus copies.

We say an infinitely renormalizable map P_c satisfies the *secondary limbs condition*, if all of the connectedness copies in the combinatorics $\tau(P_c)$ belong to a finite number of truncated secondary limbs. Let $\mathcal{SL}$ stand for the class of infinitely unicritical polynomial-like maps satisfying the secondary limbs condition.

An infinitely renormalizable map P_c has *a priori bounds*, if there is an $\varepsilon > 0$ with $\mod(\mathcal{R}^m P_c) \geq \varepsilon$, for all $m \geq 0$.

3.2 Proof of the rigidity theorem

In this section we begin to prove the rigidity theorem in a slightly more general form as follows.

3.2.1 Reductions

Theorem 3.4 (Rigidity theorem)**.** *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical polynomial-like maps satisfying $\mathcal{SL}$ condition with a priori bounds. If f and $\tilde{f}$ are combinatorially equivalent, then they are hybrid equivalent.*

Remark. If the two maps f and $\tilde{f}$ in the above theorem are polynomials, then hybrid equivalence becomes conformal equivalence. That is because the Böttcher coordinate, which conformally conjugates the two maps on the complement of the Julia sets, can be glued to the hybrid conjugacy on the Julia set. See Proposition 6 in [DH85] for a precise proof of this.

The proof breaks into following steps:

$$\begin{gathered}\text{combinatorial equivalence}\\ \Downarrow\\ \text{topological equivalence}\\ \Downarrow\\ \text{q.c. equivalence}\\ \Downarrow\\ \text{hybrid equivalence}\end{gathered}$$

It has been shown in [J00] that any *unbranched* infinitely renormalizable map with *a priori* bounds has locally connected Julia set. A renormalization

$f^n : U \to V$ is called unbranched if $\mathcal{PC}(f) \cap U = \mathcal{PC}(f^n : U \to V)$. Here, unbranched condition follows from our combinatorial condition and *a priori* bounds (see [Ly97] Lemma 9.3). Then the first step, topological equivalence of combinatorially equivalent maps, follows from the local connectivity of the Julia sets by the Carathéodory theorem. Indeed by [Do93] there is a topological model for the Julia set of these maps based on their combinatorics.

The last step in general follows from McMullen's Rigidity Theorem [Mc94] (Theorem 10.2). He has shown that an infinitely renormalizable quadratic polynomial-like map with *a priori* bounds does not have any nontrivial invariant line field on its Julia set. The same proof works for degree d unicritical polynomial-like maps. It follows that any q.c. conjugacy h between f and $\tilde{f}$ satisfies $\overline{\partial} h = 0$ almost everywhere on the Julia set. Therefore, h is a hybrid conjugacy between f and $\tilde{f}$. However, if all infinitely renormalizable unicritical maps in a given combinatorial class satisfy *a priori* bounds condition, it is easier to show that q.c. conjugacy implies hybrid conjugacy for that class rather than showing that there is no nontrivial invariant line field on the Julia set. Since we are finally going to apply our theorem to combinatorial classes for which *a priori* bounds have been established, we will prove it in Proposition 3.20.

So assume that f and $\tilde{f}$ are topologically conjugate. We want to show the following:

Theorem 3.5. *Let f and $\tilde{f}$ be infinitely renormalizable unicritical polynomial-like maps satisfying a priori bounds and $\mathcal{SL}$ conditions. If f and $\tilde{f}$ are topologically conjugate then they are q.c. conjugate.*

3.2.2 Thurston equivalence

Suppose two unicritical polynomial-like maps $f : U_2 \to U_1$ and $\tilde{f} : \widetilde{U}_2 \to \widetilde{U}_1$ are topologically conjugate. A q.c. map

$$h : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

is a *Thurston conjugacy* if it is homotopic to a topological conjugacy

$$\psi : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

between f and $\tilde{f}$ relative $\partial U_1 \cup \partial U_2 \cup \mathcal{PC}(f)$. Note that a Thurston conjugacy is not a conjugacy between two maps. It is a conjugacy on the postcritical set and homotopic to a conjugacy on the complement of the postcritical set. We will see in the next lemma that it is in the "right" homotopy class.

The following result is due to Thurston and Sullivan [S92] which originates the "Pull-Back Method" in holomorphic dynamics.

Lemma 3.6. *Thurston conjugate unicritical polynomial-like maps are q.c. conjugate.*

Proof. Assume $h_1 : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$ is a Thurston conjugacy homotopic to a topological conjugacy $\Psi : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$ relative $\partial U_1 \cup \partial U_2 \cup \mathcal{PC}(f)$.

As $f : U_2 \setminus \{0\} \to U_1 \setminus \{f(0)\}$ and $\tilde{f} : \widetilde{U}_2 \setminus \{0\} \to \widetilde{U}_1 \setminus \{\tilde{f}(0)\}$ are covering maps, $h_1 : U_1 \setminus \{f(0)\} \to \widetilde{U}_1 \setminus \{\tilde{f}(0)\}$ can be lifted to a homeomorphism $h_2 : U_2 \setminus \{0\} \to \widetilde{U}_2 \setminus \{0\}$. Moreover, since h_1 satisfies the *equivariance relation* $h_1 \circ f = \tilde{f} \circ h_1$ on the boundary of U_2, h_2 can be extended onto $U_1 \setminus U_2$ by h_1. It also extends to the critical point by sending it to the critical point of $\tilde{f}$. Let us denote this new map by h_2. For the same reason, every homotopy h_t between

Ψ and h_1 can be lifted to a homotopy between Ψ and h_2. As f and $\tilde{f}$ are holomorphic maps, h_2 has the same dilatation as dilatation of h_1. This implies that the new map h_2 is also a Thurston conjugacy with the same dilatation as the one of h_1. By definition, the new map h_2 satisfies the equivariance relation on the annulus $U_2 \setminus f^{-1}(U_2)$.

Repeating the same process with h_2, we obtain a q.c. map h_3 and so on. Thus, we have a sequence of K-q.c. maps h_n from U_1 to $\widetilde{U}_1$ which satisfies equivariance relation on the annulus $U_2 \setminus f^{-n}(U_2)$. All these maps can be extended onto complex plane (see Theorem 2.9) with a uniform bound on their dilatation. This family of q.c. maps is normalized at points $0, f(0), f^2(0), \dots$ by mapping them to the corresponding points $0, \tilde{f}(0), \tilde{f}^2(0), \dots$. Compactness of this class, Theorem 2.7, implies that there is a subsequence h_{n_j} which converges to a K-q.c. map H on U_1.

For every z outside of the Julia set, the sequence $h_{n_j}(z)$ stabilizes and, by definition, eventually $h_{n_j} \circ f(z) = f \circ h_{n_j}(z)$. Taking limit of both sides will imply that $H \circ f(z) = f \circ H(z)$ for every such z. As filled Julia set of an infinitely renormalizable unicritical map has empty interior, conjugacy relation for an arbitrary z on the Julia set follows from continuity of H. $\square$

By the *a priori* bounds assumption in the theorem, there are topological disks $V_{n,0} \Subset U_{n,0}$ containing 0 such that $\mathcal{R}^n f = f^{t_n} : V_{n,0} \to U_{n,0}$ is a unicritical degree d polynomial-like map and $\mod (U_{n,0} \setminus V_{n,0}) \geq \varepsilon$. By going several levels down, i.e. considering $f^{t_n} : f^{-kt_n}(V_n) \to f^{-kt_n}(U_n)$ for some positive integer k, we may assume that $\mod (U_n \setminus V_n)$ and $\mod (\widetilde{U}_n \setminus \widetilde{V}_n)$ are proportional. Also by slightly shrinking the domains, if necessary, we may assume that these domains have smooth boundaries. Thus we have

- For every $n \geq 1$, we have $\mod (U_{n,0} \setminus V_{n,0}) \geq \varepsilon$, and $\mod (\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0}) \geq \varepsilon$,

- There exits a constant M such that for every $n \geq 1$,

$$\frac{1}{M} \leq \frac{\text{mod } (U_n \setminus V_n)}{\text{mod } (\widetilde{U}_n \setminus \widetilde{V}_n)} \leq M,$$

- For every $n \geq 1$, $U_{n,0}$ and $\widetilde{U}_{n,0}$ have smooth boundaries.

We use the following notations throughout the rest of this note.

$f : V_0 \to U_0, K_{0,0} = K(f)$,

$\mathcal{R}f = f^{t_1} : V_{1,0} \to U_{1,0}, K_{1,0} = K(\mathcal{R}f)$,

$\mathcal{R}^2 f = f^{t_2} : V_{2,0} \to U_{2,0}, K_{2,0} = K(\mathcal{R}^2 f)$,

$\vdots$

$\mathcal{R}^n f = f^{t_n} : V_{n,0} \to U_{n,0}, K_{n,0} = K(\mathcal{R}^n f)$,

$\vdots$

The domain $V_{n,i}$, for $i = 1, 2, \ldots, t_n - 1$, is defined as the pullback of $V_{n,0}$ under f^{-i} containing the *little Julia set* $J_{n,i} := f^{t_n - i}(J_{n,0})$ and $U_{n,i}$ as the component of $f^{-i}(U_{n,0})$ containing $V_{n,i}$ so that $f^{t_n} : V_{n,i} \to U_{n,i}$ is a polynomial-like map. The domain $W_{n,i}$ is defined as the preimage of $V_{n,i}$ under the map $f^{t_n} : V_{n,i} \to U_{n,i}$.

Accordingly, $K_{n,i}$ is defined as the component of $f^{-i}(K_{n,0})$ inside $V_{n,i}$. Note that $\mathcal{R}^n f : V_{n,i} \to U_{n,i}$ is a polynomial-like map with the filled Julia set $K_{n,i}$ which is conjugate to $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$ by conformal isomorphism $f^i : U_{n,i} \to U_{n,0}$. It has been proved in [Ly97] (Lemma 9.2) that there is always definite space in between Julia sets in the primitive case for parameters under our assumption. Compare the proof of Lemma 3.13. It has been shown in [Mc96] that definite space between little Julia sets implies that there exist choice of domains $U_{n,i}$ which are disjoint for different i's and moreover the annuli $U_{n,i} \setminus V_{n,i}$ have definite moduli. So we will assume that on the primitive levels, the domains $U_{n,i}$ are disjoint for different i's.

In all of the above notation, the first lower subscripts denote the level of renormalization and the second lower subscripts run over little filled Julia sets, Julia sets and their neighborhoods accordingly. In what follows all corresponding objects for $\tilde{f}$ will be marked with a tilde and any notation introduced for f will be automatically introduced for $\tilde{f}$ too.

To build a Thurston conjugacy, we first introduce multiply connected domains $\Omega_{n(k),i}$ (and $\widetilde{\Omega}_{n(k),i}$) in $\mathbb{C}$ for an appropriate subsequence $n(k)$ of the renormalization levels and a sequence of q.c. maps with uniformly bounded distortion

$$h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$$

for $k = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, t_{n(k)} - 1$. These domains will satisfy the following properties:

- Each $\Omega_{n(k),j}$, for $n(k) \geq 0$, is a topological disk minus $n(k+1)$ topological disks $D_{n(k+1),i}$.

- Each $\Omega_{n(k),i}$, for $n(k) \geq 1$, is well inside $D_{n(k),i}$ which means that the modulus of the annulus obtained from $D_{n(k),i} \setminus \Omega_{n(k),i}$ is uniformly bounded below for all $n(k)$ and i.

- Every *little postcritical set* $J_{n(k),i} \cap \mathcal{PC}(f)$ is well inside $D_{n(k),i}$.

- Every $D_{n(k),i}$ is the pullback of $D_{n(k),0}$ under f^{-i} containing $J_{n(k),i} \cap \mathcal{PC}(f)$ and every $\Omega_{n(k),i}$ is the component of $f^{-i}(\Omega_{n(k),0})$ inside $D_{n(k),i}$.

Finally, we construct a Thurston conjugacy by an appropriate gluing of the maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$ together on the complement of all these multiply connected domains (which is a union of annuli). See Figure 3.1.

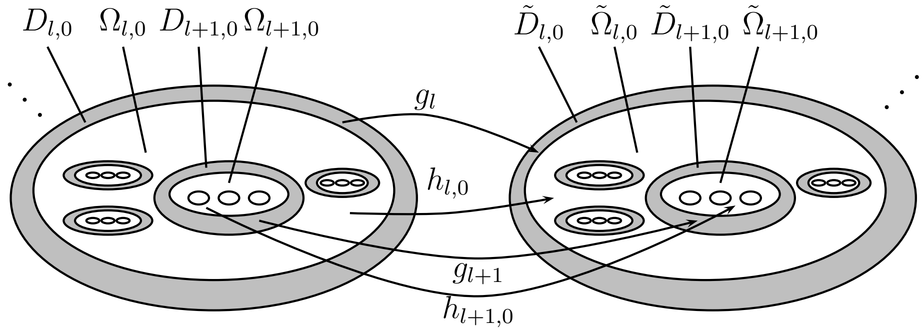

Figure 3.1: The multiply connected domains and the buffers

3.2.3 The domains $\Omega_{n,j}$ and the maps $h_{n,j}$

By applying the straightening theorem to the polynomial-like map

$$\mathcal{R}^{n-1}f : V_{n-1,0} \to U_{n-1,0}$$

we get a $K_1(\varepsilon)$-q.c. map and a unicritical polynomial $\mathbf{f}_{c_{n-1}}$, such that

$$S_{n-1} : (U_{n-1,0}, V_{n-1,0}, 0) \to (\Upsilon^0_{n-1}, \Upsilon^1_{n-1}, 0), \tag{3.3}$$
$$S_{n-1} \circ \mathcal{R}^{n-1} f = \mathbf{f}_{c_{n-1}} \circ S_{n-1}.$$

See Figure 3.2.

Remark. To make notations easier to follow, we will drop the second subscript whenever it is zero and it does not create any confusion. Also, all objects on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ (the ones after straightening) will be denoted by **bold** face of notations used for objects on the dynamic planes of f and $\tilde{f}$.

To define $\Omega_{n-1,j}$ and $h_{n-1,j}$, because of of the difference in type of renormalizations, we will consider the following three cases:

$\mathscr{A}$. $\mathcal{R}^{n-1}f$ is primitively renormalizable.

$\mathscr{B}$. $\mathcal{R}^{n-1}f$ is satellite renormalizable and $\mathcal{R}^{n}f$ is primitively renormalizable.

$\mathscr{C}$. $\mathcal{R}^{n-1}f$ is satellite renormalizable and $\mathcal{R}^{n}f$ is also satellite renormalizable.

For a given infinitely renormalizable map f_c, the renormalization on each level is of primitive or satellite type. Therefore, we can associate a word

$$P \ldots PS \ldots SP \ldots$$

of P and S where a P or a S in the i's place means that the i's renormalization of f_c is of primitive or satellite type, respectively. Corresponding to any such word, we define a word of cases $\mathscr{A}^{m_1}\mathscr{B}^{m_2}\mathscr{C}^{m_3}\ldots$ (for non-negative m_j's) defined as follows. Starting from left, a P is replaced by $\mathscr{A}$, SP by $\mathscr{B}$, and SS by $\mathscr{C}S$. By repeating this process, we obtain a word of cases which is used to decide which case to pick at each step.

Case $\mathscr{A}$:

We need the following lemma to show that there are equipotentials of sufficiently high level $\eta(\varepsilon)$ inside $S_{n-1}(W_{n-1,0})$ and $\tilde{S}_{n-1}(\widetilde{W}_{n-1,0})$ in the dynamic planes of the maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$.

Lemma 3.7. *If $P_c : U' \to U$ is a unicritical polynomial with connected Julia set and* $\operatorname{mod}(U \setminus U') \geq \varepsilon$*, then U' contains equipotentials of level less than $\eta(\varepsilon)$ depending only on ε.*

Proof. The map P_c on the complement of $K(P_c)$ is conjugate to P_0 on the complement of the closed unit disk D_1 by Böttcher coordinate B_c. Since levels of equipotentials are preserved under this map and modulus is conformal invariant, it is enough to prove the statement for $P_0 : V' \to V$ for V' compactly

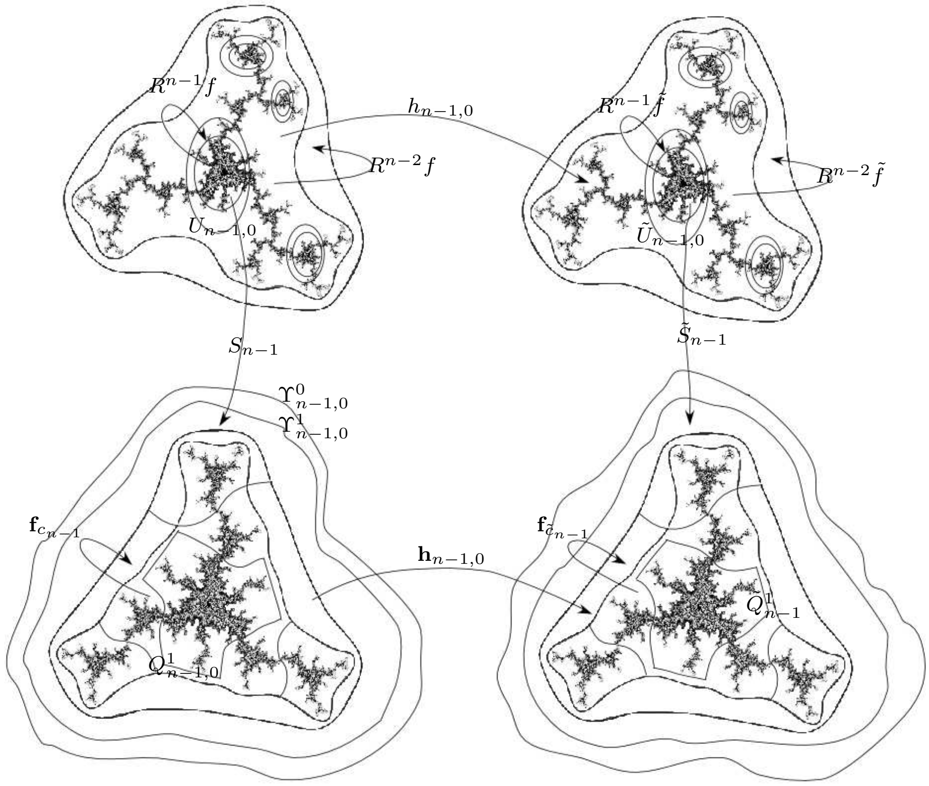

Figure 3.2: Primitive case

contained in V and $\bmod (V \setminus V') \geq \varepsilon$. As $P_0 : P_0^{-1}(V \setminus V') \to (V \setminus V')$ is a covering of degree d, modulus of the annulus $P_0^{-1}(V \setminus V')$ is ε/d which implies that modulus of $V' \setminus D_1 \geq \varepsilon/d$. By Grötzsch problem in [Ah66] (Section A in Chapter III) we conclude that $V' \setminus D_1$ contains a round annulus $D_{\eta(\varepsilon)} \setminus D_1$. $\square$

By considering the equipotentials of level $\eta(\varepsilon)$ (obtained in the previous lemma) and the external rays landing at the dividing fixed points α_{n-1} and $\tilde{\alpha}_{n-1}$ of the maps $\mathbf{f}_{c_{n-1}}$, and $\mathbf{f}_{\tilde{c}_{n-1}}$, we can form the favorite nest of puzzle pieces (3.1) introduced in Section 3.1.2. The hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ in the truncated primary wake containing c_{n-1} and $\tilde{c}_{n-1}$, $W(\eta(\varepsilon))$, is bounded by some $M(\varepsilon)$ depending only on ε and the combinatorial class $\mathcal{SL}$. That is because c_{n-1} and $\tilde{c}_{n-1}$ belong to a finite number of truncated limbs which is a compact subset of this wake. Therefore, by Proposition 3.3, dilatation of the pseudo-conjugacy obtained in Theorem 3.2 is uniformly bounded by some constant $K_2(\varepsilon)$.

Let $Q^{\chi_n}_{n,0} = Y^{q_{\chi_n}}_{n,0}$ and $P^{\chi_n}_{n,0}$ denote the last critical puzzle pieces obtained in the nest (3.1), and $\mathbf{h}_{n-1} = \mathbf{h}_{n-1,0}$ denote the corresponding $K_2(\varepsilon)$-q.c. pseudo-conjugacy obtained in Theorem 3.2. Components of $\mathbf{f}^{-i}_{c_{n-1}}(Q^{\chi_n}_{n,0})$ and $\mathbf{f}^{-i}_{c_{n-1}}(P^{\chi_n}_{n,0})$ containing the little Julia sets $J_{n,i}$, for $i = 0, 1, 2, \ldots, t_n/t_{n-1} - 1$, are denoted by $Q^{\chi_n}_{n,i}$ and $P^{\chi_n}_{n,i}$, respectively. Note that that t_n/t_{n-1} is the period of the first renormalization of $\mathbf{f}_{c_{n-1}}$.

As the polynomials $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ also satisfy our combinatorial condition and *a priori* bounds assumption, there is a topological conjugacy, denoted by $\boldsymbol{\psi}_{n-1}$, between them which is obtained from extending $B^{-1}_{\tilde{c}_{n-1}} \circ B_{c_{n-1}}$ onto the Julia set.

Now we would like to adjust $\mathbf{h}_{n-1} : Q^{\chi_n}_n \to \widetilde{Q}^{\chi_n}_n$ using dynamics of

$$\mathbf{f}^{t_n/t_{n-1}}_{c_{n-1}} : P^{\chi_n}_n \to Q^{\chi_n}_n, \text{ and } \mathbf{f}^{t_n/t_{n-1}}_{\tilde{c}_{n-1}} : \widetilde{P}^{\chi_n}_n \to \widetilde{Q}^{\chi_n}_n.$$

Let A^0_n denote the closure of the annulus $Q^{\chi_n}_n \setminus P^{\chi_n}_n$, and A^k_n, for $k = 0, 1, 2, \ldots,$ denote the component of $\mathbf{f}^{-kt_n/t_{n-1}}_{c_{n-1}}(A^0_n)$ around $J_{n,0}$ by A^k_n. We can lift $\mathbf{h}_{n-1}$ by $\mathbf{f}^{t_n/t_{n-1}}_{c_{n-1}} : Q^{\chi_n}_n \to \mathbb{C}$ and $\mathbf{f}^{t_n/t_{n-1}}_{\tilde{c}_{n-1}} : \widetilde{Q}^{\chi_n}_n \to \mathbb{C}$ to obtain a K_2-q.c. map $g : A^0_n \to \widetilde{A}^0_n$ which is homotopic to $\boldsymbol{\psi}_{n-1}$ relative boundaries of A^0_n. That is because by the external rays connecting $\partial P^{\chi_n}_n$ to $\partial Q^{\chi_n}_n$, the annulus A^0_n is partitioned into some topological disks and the two maps coincide on the boundaries of these topological disks.

As $\mathbf{f}^{kt_n/t_{n-1}}_{c_{n-1}} : A^k_n \to A^0_n$ and $\mathbf{f}^{kt_n/t_{n-1}}_{\tilde{c}_{n-1}} : \widetilde{A}^k_n \to \widetilde{A}^0_n$ are holomorphic unbranched coverings, g can be lifted to a K_2-q.c. map from A^k_n to $\widetilde{A}^k_n$, for every $k \geq 1$. All these lifts are the identity map in the Böttcher coordinate on the boundaries of these annuli. Hence, they match together to K_2-q.c. conjugate the two maps

$$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P_n^{\chi_n} \setminus J_n \to Q_n^{\chi_n} \setminus J_n, \text{ and } \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{P}_n^{\chi_n} \setminus \tilde{J}_n \to \widetilde{Q}_n^{\chi_n} \setminus \tilde{J}_n.$$

Finally, we would like to extend this map further onto little Julia set J_n. This is a especial case of a more general argument presented below.

Given a polynomial f with connected Julia set J, the *rotation* of angle θ on $\mathbb{C} \setminus J$ is defined as the rotation of angle θ in the Bötcher coordinate on $\mathbb{C} \setminus J$, that is, $B_c^{-1}(e^{\mathbf{i}\theta} \cdot B_c)$. By means of straightening, one can define rotations on the complement of the Julia set of a polynomial-like map. It is not canonical as it depends on the choice of straightening map. However, its effect on the landing points of external rays is canonical.

Proposition 3.8. *Let $f : V_2 \to V_1$ be a polynomial-like map with connected Julia set J. If $\phi : V_1 \setminus J \to V_1 \setminus J$ is a homeomorphism which commutes with f, then there exists a rotation of angle $2\pi j/(d-1)$, ρ_j, such that $\rho_j \circ \phi$ extends as the identity map onto J.*

Proof. Consider an external ray R landing at the non-dividing fixed point β_0 of f. As this ray is invariant under f and ϕ commutes with f, we have $f \circ \phi(R) = \phi \circ f(R) = \phi(R)$. Thus, $\phi(R)$ is also invariant under f which implies that $\phi(R)$ lands at a non-dividing fixed point β_j of f. Now, choose ρ_j such that $\rho_j(\phi(R))$ lands at β_0. Let ψ denote $\rho_j \circ \phi$ and R' denote $\rho_j(\phi(R))$. For such a rotation ρ_j, ψ also commutes with f and R' is also invariant under f.

The external ray R cuts the annulus $V_1 \setminus V_2$ into a quadrilateral $I_{0,1}$. The preimage $f^{-1}(I_{0,1})$ produces d quadrilaterals denoted by

$$I_{1,1}, I_{1,2}, \ldots, I_{1,d},$$

ordered clockwise starting with R. Similarly, the f^n-preimage of $I_{0,1}$ produces d^n quadrilaterals $I_{n,1}, I_{n,2}, \ldots, I_{n,d^n}$ (also ordered clockwise starting with R). In the same way, the external ray R' produces quadrilaterals denoted by $I'_{n,j}$ ordered clockwise starting with R'. First we will show that the Euclidean diameter of $I_{n,j}$ (and $I'_{n,j}$) goes to zero as n tends to infinity.

Denote $f^{-i}(V_1)$ by V_{i+1} and let d_{i+1} denote the hyperbolic distance on the annulus $V_{i+1} \setminus J$. As $I_{n,j} \subseteq V_{n+1}$, and the intersection of the nest of topological disks V_n is equal to J, the quadrilaterals $I_{n,j}$ converge to the boundary of $V_1 \setminus J$ as n goes to infinity. In order to show that the Euclidean diameter of these quadrilaterals go to zero, it is enough to show that their hyperbolic diameters in $(V_1 \setminus J, d_1)$ stay bounded. Since $f^{n-1} : (V_n \setminus J, d_n) \to (V_1 \setminus J, d_1)$ is an unbranched covering of degree d^{n-1}, therefore an isometry, and closure of $f^{n-1}(I_{n,j})$ is a compact subset of $V_1 \setminus J$, we conclude that $I_{n,j}$ has bounded hyperbolic diameter in $(V_n \setminus J, d_n)$. Finally, contraction of the inclusion map from $(V_n \setminus J, d_n)$ to $(V_1 \setminus J, d_1)$ implies that $I_{n,j}$ has bounded hyperbolic diameter in $(V_1 \setminus J, d_1)$. With the same type of argument, one can show that the hyperbolic distance between $I_{n,j}$ and $I'_{n,j}$ inside $(V_1 \setminus J, d_1)$ is also uniformly bounded.

Since ψ is a conjugacy it sends $I_{n,j}$ to $I'_{n,j}$, and therefore, as w converges to J, w and $\psi(w)$ belong to $I_{n,j}$ and $I'_{n,j}$ with larger and larger index n. Combining with the above argument, we conclude that the Euclidean distance between these two points tends to zero. This implies that ψ can be extended as the identity map on the Julia set. □

A particular case of above proposition is that every homeomorphism which commutes with a quadratic polynomial-like map can be extended onto Julia set.

Applying the above lemma to $\boldsymbol{\psi}_{n-1}^{-1} \circ g$ with $V_1 = Q_n^{\chi_n}$, $V_2 = P_n^{\chi_n}$, and an external ray connecting $\partial Q_n^{\chi_n}$ to $J_{n,0}$ we conclude that g can be extended as $\boldsymbol{\psi}_{n-1}$ onto $J_{n,0}$. It also follows from proof of the above lemma that g and $\boldsymbol{\psi}_{n-1}$ are homotopic relative $J_n \cup \partial Q_n^{\chi_n}$. That is because the quadrilaterals obtained in the above lemma cut the puzzle piece $Q_n^{\chi_n}$ into infinite number of topological disks such that g and $\boldsymbol{\psi}_{n-1}$ are equal on their boundaries.

Similarly, $\mathbf{h}_{n-1}$ can be adjusted on the other puzzle pieces $Q_{n,i}^{\chi_n}$, and moreover, $\mathbf{h}_{n-1}$ is homotopic to $\boldsymbol{\psi}_{n-1}$ on $Q_{n,i}^{\chi_n}$ relative $J_{n,i} \cup \partial Q_{n,i}^{\chi_n}$. We will denote the map obtained from extending $\mathbf{h}_{n-1}$ onto little Julia sets $J_{n,i}$ with the same notation $\mathbf{h}_{n-1}$.

The final argument is to prepare $\mathbf{h}_{n-1}$ for the next step of the process. It is stated in the following lemma.

Lemma 3.9. *The map $\mathbf{h}_{n-1}$ can be adjusted (through a homotopy) on a neighborhood of $\cup_i J_{n,i}$ to a q.c. map $\mathbf{h}'_{n-1}$ which maps $\mathbf{V}_{n,i} = S_{n-1}(V_{n,i})$ onto $\widetilde{\mathbf{V}}_{n,i} = \tilde{S}_{n-1}(\widetilde{V}_{n,i})$. Moreover, dilatation of $\mathbf{h}'_{n-1}$ is uniformly bounded by a constant $K_2(\varepsilon)$ depending only on ε.*

Proof. The basic idea is move all of $\mathbf{h}_{n-1}(\mathbf{V}_{n,i})$ simultaneously close enough to little Julia sets $J_{n,i}$, and then move them back to $\widetilde{\mathbf{V}}_{n,i}$. We will do this more precisely below. Let $\mathbf{U}_{n,i}$ denote $S_{n-1}(U_{n,i})$ and $\widetilde{\mathbf{U}}_{n,i}$ denote $\widetilde{S}_{n-1}(\widetilde{U}_{n,i})$.

The annuli $\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{J}_{n,i}$ and $\widetilde{\mathbf{V}}_{n,i} \setminus \widetilde{J}_{n,i}$ have moduli bigger that ε/dK where K is the dilatation of $\mathbf{h}_{n-1}$. Therefore, there exist topological disks $\widetilde{L}_{n,i}$ with smooth boundaries and a constant $r > 1$ satisfying the following properties

- $L_{n,i} \subset \mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \cap \widetilde{\mathbf{V}}_{n,i}$
- $\mathrm{mod}\ (\widetilde{L}_{n,i} \setminus \widetilde{J}_{n,i}) \geq r - 1$
- $\mathrm{mod}\ (\widetilde{\mathbf{V}}_{n,i} \setminus L_{n,i}) \geq \varepsilon/2dK$ and $\mathrm{mod}\ (\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{L}_{n,i}) \geq \varepsilon/2dK$.

Now we claim that there exist q.c. maps

$$\chi_i : \big(\mathbf{h}_{n-1}(\mathbf{U}_{n,i}), \mathbf{h}_{n-1}(\mathbf{V}_{n,i}), \widetilde{L}_{n,i}, J_{n,i}\big) \to \big(D_5, D_3, D_2, D_1\big)$$

with uniformly bounded dilatation. That is because all the annuli $L_{n,i} \setminus J_{n,i}$, $\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{L}_{n,i}$, and $\mathbf{h}_{n-1}(\mathbf{U}_{n,i}) \setminus \mathbf{h}_{n-1}(\mathbf{V}_{n,i})$ have moduli uniformly bounded from below and above. The homotopy $g_t : \mathrm{Dom}\,(h) \to \mathbb{C}$, for $t \in [0,1]$, is defined as

$$g_t(z) := \begin{cases} \mathbf{h}_{n-1}(z) & \text{if } z \notin \bigcup_i \mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \\ \chi_i^{-1}\Big((\frac{-t}{3} \sin \frac{(|\chi_i \circ \mathbf{h}_{n-1}(z)| - 1)\pi}{4} + 1) \cdot \chi_i \circ \mathbf{h}_{n-1}(z) \Big) & \text{if } z \in \mathbf{h}_{n-1}(\mathbf{V}_{n,i}). \end{cases}$$

It is straight to see that $g_0 \equiv \mathbf{h}_{n-1}$, g_t is a well defined homeomorphism for every fixed t, and depends continuously on t fro every fixed z. For every $z \in \partial \mathbf{V}_{n,i}$, at time $t = 1$, we have $g_1(z) = \chi_i^{-1}(\frac{2}{3} \cdot \chi_i \circ \mathbf{h}_{n-1}(z)) \in \partial \widetilde{L}_{n,i}$. That is, g_1 maps $\partial V_{n,i}$ to $\partial \widetilde{L}_{n,i}$. For the returning part, we consider a q.c. map

$$\Theta_i : \big(\widetilde{\mathbf{U}}_{n,i}, \widetilde{\mathbf{V}}_{n,i}, L_{n,i}, J_{n,i}\big) \to \big(D_5, D_3, D_2, D_1\big),$$

and define $g_{t+1} : \mathrm{Dom}\, \mathbf{h}_{n-1} \to \mathbb{C}$, for $t \in [0,1]$, as

$$g_{t+1}(z) := \begin{cases} g_1(z) & \text{if } z \notin g_1^{-1}(\bigcup_i \widetilde{\mathbf{U}}_{n,i}) \\ \Theta_i^{-1}\Big((\frac{t}{\sqrt{2}} \sin \frac{(|\chi_i \circ g_1(z)| - 1)\pi}{4} + 1) \cdot \Theta_i \circ g_1(z) \Big) & \text{if } z \in g_1^{-1}(\widetilde{\mathbf{U}}_{n,i}). \end{cases}$$

The homotopy g_t for $t \in [0,2]$ is the desired adjustment and the map $g_2 : \mathrm{Dom}\, \mathbf{h}_{n-1} \to \mathrm{Range}\, \mathbf{h}_{n-1}$ is denoted by $\mathbf{h}'_{n-1}$. □

Let $\Delta_{n-1,0}$ denote the S_{n-1}-preimage of the domain bounded by the equipotential of level $\eta(\varepsilon)$ in the dynamic plane of $\mathbf{f}_{c_{n-1}}$. The multiply connected domain $\Omega_{n-1,0}$ is defined as

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}} V_{n,it_{n-1}}.$$

The domains $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, \ldots, t_{n-1}$, are defined as the pull back of $\Delta_{n-1,0}$ and $\Omega_{n-1,i}$, respectively, under f^{-i} along the orbit of the critical point.

Consider the map

$$h_{n-1,0} := \tilde{S}_{n-1}^{-1} \circ \mathbf{h}'_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0}, \tag{3.4}$$

and then,

$$h_{n-1,i} := \tilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i}. \tag{3.5}$$

As these maps are compositions of two $K_1(\varepsilon)$-q.c., a $K_2(\varepsilon)$-q.c., and possibly some holomorphic maps, they are q.c. with a uniform bound on their dilatation. By our adjustment in Lemma 3.9 we have $h_{n-1,i}$ maps $\Omega_{n-1,i}$ onto $\widetilde{\Omega}_{n-1,i}$.

Finally, the annulus $V_{n-1,0} \setminus W_{n-1,0}$, with modulus bigger than $\varepsilon/2$ encloses $\Omega_{n-1,0}$ and is contained in $V_{n-1,0}$. This proves that the domain $\Omega_{n-1,0}$ is well inside the disk $V_{n-1,0}$. Similarly, appropriate preimages of $V_{n-1,0} \setminus W_{n-1,0}$ under the conformal maps f^{-i} introduce definite annuli around $\Omega_{n-1,i}$ which are contained in $V_{n-1,i}$. In this case, the topological disks $D_{n,i}$ are the domains $V_{n,i}$ which contain the little Julia sets $J_{n,i}$ well inside themselves.

Case $\mathscr{B}$:

Here, $\mathbf{f}_{c_{n-1}}$ is satellite renormalizable and its second renormalization is of primitive type. Let α_{n-1} denote the dividing fixed point of $\mathbf{f}_{c_{n-1}}$, and α_n the dividing fixed point of it's first renormalization $\mathcal{R}\mathbf{f}_{c_{n-1}}$. In this situation, the Julia set of the primitive renormalization $\mathcal{R}\mathbf{f}_{c_{n-1}}$, and its forward images under $\mathbf{f}_{c_{n-1}}$ can get arbitrarily close to α_{n-1} (which is a non-dividing fixed point of $\mathcal{R}\mathbf{f}_{c_{n-1}}$). Our idea is to skip the satellite renormalization in this case which essentially imposes the secondary limbs condition on us.

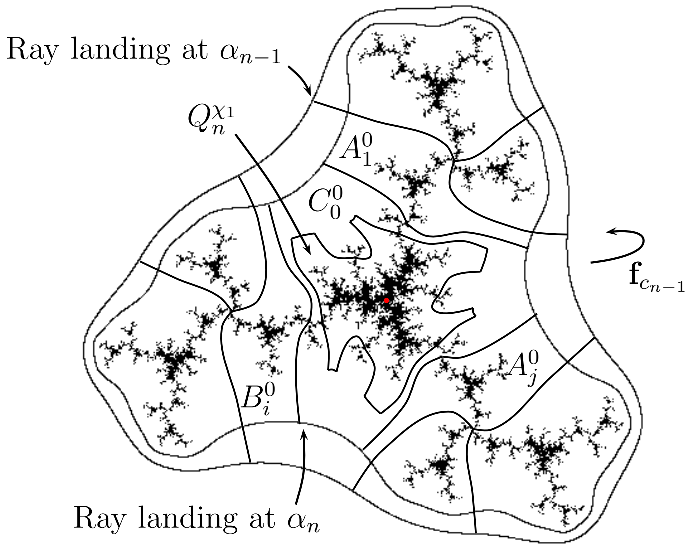

Figure 3.3: Figure of an infinitely renormalizable Julia set. The first renormalization is of satellite type and the second one is of primitive type. The puzzle piece at the center, $Q_n^{\chi_1}$, is the first puzzle piece in the favorite nest.

Now, using the above rays and equipotential, we introduce some new puzzle pieces. Let Y_0^0, as before, denote the puzzle piece containing the critical point and bounded by $E(\eta)$, the external rays landing at α_{n-1} and the external rays landing at $\mathbf{f}_{c_{n-1}}$-preimages of α_{n-1} (i.e. at all $\omega\alpha_{n-1}$ with ω a dth root of unity). The External rays landing at α_n and their g-preimages, cut the puzzle piece Y_0^0 into finitely many pieces. Let us denote the one containing the critical point by C_0^0, the non-critical ones which have a boundary ray landing at α_n by B_i^0 and the rest of them by A_j^0 (these ones have a boundary external ray landing at some $\omega\alpha_n$).

The g-preimage of Y_0^0 along the postcritical set is contained in itself. As all processes of making modified principal nest and the pseudo-conjugacy in Theorem 3.2 are based on pullback arguments, the same ideas are applicable here. The only difference is that we do not have equipotentials for the second renormalization. As we will see in a moment, certain external rays and part of the equipotential bounding Y_0^0 will play the role of an equipotential for the second renormalization of $\mathbf{f}_{c_{n-1}}$.

By definition of satellite and primitive renormalizability, every $g^n(0)$ belongs to Y_0^0, for $n \geq 0$, and there is a first moment t with $g^t(0) \in A_1^0$ (by rearranging the indices if required). Pulling back A_1^0 under g^t along the critical orbit, we obtain a puzzle piece $Q_n^{\chi_1} \ni 0$, such that $C_0^0 \setminus Q_n^{\chi_1}$ is a non-degenerate annulus. That is because C_0^0 is bounded by the external rays landing at α_n and their g-preimage. Therefore, if $Q_n^{\chi_1}$ intersects ∂C_0^0 at some point on the rays, orbit of this intersection under g^k, for $k \geq 1$, stays on the rays landing at α_n. This implies that image of $Q_n^{\chi_1}$ can not be A_1^0. Also, they can not intersect at equipotentials as they have different levels. Now, consider the first moment $m > t$ when $g^m(0)$ returns back to $Q_n^{\chi_1}$, and pullback $Q_n^{\chi_1}$ under g^m along the critical orbit to obtain $P_n^{\chi_1}$. The map g^m is a unicritical degree d branched covering from $P_n^{\chi_1}$ onto $Q_n^{\chi_1}$. This introduces the first two pieces in the favorite nest. The rest of the process to form the whole favorite nest is the same as in Section 3.1.2.

Consider the map $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : Y_0^0 \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0)$, and the corresponding tilde one. One applies Theorem 3.2 to these maps, using the favorite nests introduced in the above paragraph, to obtain a q.c. pseudo-conjugacy

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0) \to \widetilde{\mathbf{f}}_{c_{n-1}}^{t_n/t_{n-1}}(\widetilde{Y}_0^0).$$

The equipotential of level $\eta(\varepsilon)$, the external rays landing at α_{n-1}, and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of α_n depend holomorphically on the parameter within the secondary wake $W(\eta)$ containing the parameter c_{n-1}. Therefore, by Proposition 3.3, the dilatation of $\mathbf{h}_{n-1}$ depends on the hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ within one of the finite secondary wakes $W(\eta)$ under our combinatorial assumption. Thus, it only depends on the *a priori* bounds ε and the combinatorial condition.

As $\mathbf{f}_{c_{n-1}}^j : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^0) \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^0)$, for $j = 1, 2, \ldots, t_n/t_{n-1} - 1$, is univalent, we can lift $\mathbf{h}_{n-1}$ on other puzzle pieces as

$$\mathbf{h}_{n-1} := \widetilde{\mathbf{f}}_{c_{n-1}}^{-j} \circ \mathbf{h}_{n-1} \circ \mathbf{f}_{c_{n-1}}^j : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^0) \to \widetilde{\mathbf{f}}_{c_{n-1}}^{t_n/t_{n-1}-j}(\widetilde{Y}_0^0)$$

for these j's. Since all these maps match the Böttcher coordinates, they fit together to build a q.c. map from a neighborhood of $J(\mathbf{f}_{c_{n-1}})$ to a neighborhood of $J(\widetilde{\mathbf{f}}_{c_{n-1}})$. Further, it can be extended as the identity map in the Böttcher coordinates to a q.c. map from the domain bounded by equipotential $E^{\eta(\varepsilon)}$ to the corresponding tilde domain.

Finally, by Lemma 3.15, we adjust $\mathbf{h}_{n-1}$ to obtain a q.c. map $\mathbf{h}'_{n-1,0}$ that satisfies

$$\mathbf{h}'_{n-1}(S_{n-1}(V_{n+1,it_{n-1}})) = \tilde{S}_{n-1}(\widetilde{V}_{n,it_{n-1}}), \text{ for } i = 0, 1, 2, \ldots, t_{n+1}/t_{n-1} - 1.$$

Now, $\Delta_{n-1,0}$ is defined as S_{n-1}-pullback of the domain bounded by the equipotential $E^{\eta(\varepsilon)}$. The domain $\Omega_{n-1,0}$ is

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} V_{n+1,it_{n-1}}.$$

The regions $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, 3, \ldots, t_{n-1}$, are pullbacks of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ under f^i, respectively. Like in the previous case, $h_{n-1,i}$ is defined as in Equations (3.4) and (3.5) and satisfies

$$h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}, \text{ for } i = 1, 2, \ldots, t_{n-1}.$$

For the same reason as in Case $\mathscr{A}$, $\Omega_{n-1,i}$ is well inside the disk $D_{n-1,i} := V_{n-1,i}$.

Case $\mathscr{C}$:

Here, $\mathbf{f}_{c_{n-1}}$ is twice satellite renormalizable. The argument in this case relies more on the compactness of the parameters under consideration rather than a dynamical discussion.

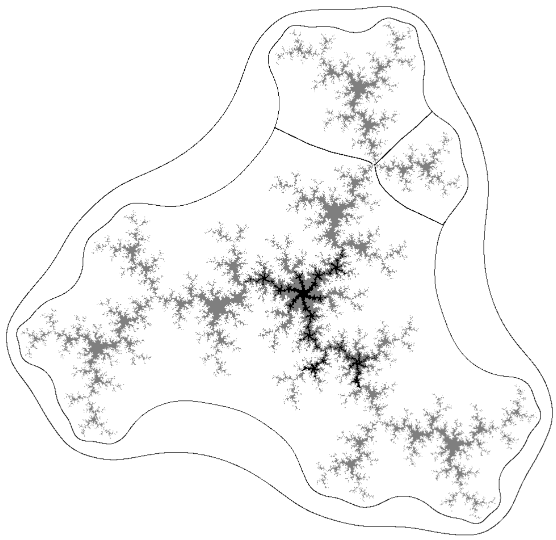

Figure 3.4: A twice satellite renormalizable Julia set drawn in grey. The dark part is the Julia Bouquet $\mathbf{B}_{2,0}$.

Little Julia sets $\mathbf{J}_{1,i}$ of the first renormalization of $\mathbf{f}_{c_{n-1}}$ touch at the dividing fixed point α_{n-1} of $\mathbf{f}_{c_{n-1}}$. Note that α_{n-1} is one of the non-dividing fixed points of the first renormalization of $\mathbf{f}_{c_{n-1}}$. The union of these little Julia sets is called *Julia bouquet* and denoted by $\mathbf{B}_{1,0}$. Similarly, the little Julia sets $\mathbf{J}_{2,i}$, for $i = 0, 1, 2, \ldots, (t_{n+1}/t_{n-1}) - 1$, of the second renormalization of $\mathbf{f}_{c_{n-1}}$ are organized in pairwise disjoint *bouquets*, $\mathbf{B}_{2,j}$, touching at a periodic point. That is, each $\mathbf{B}_{2,j}$ consists of t_{n+1}/t_n little Julia sets $\mathbf{J}_{2,i}$ touching at one of their non-dividing fixed points. As usual, $\mathbf{B}_{2,0}$ denotes the bouquet containing the critical point. See Figure 3.4.

By an equipotential of level $\eta(\varepsilon)$ (which encloses $S_{n-1}(W_{n-1,0})$) and the external rays landing at α_{n-1}, we form the puzzle pieces of level zero. Recall that Y_0^0 denotes the one containing the critical point. The following lemma states that the bouquets are well apart form each other.

Lemma 3.10. *For parameters in a finite number of truncated secondary limbs, modulus of the annulus $Y_0^0 \setminus \mathbf{B}_{2,0}$ is uniformly bounded above and below.*

To prove this lemma we first need to recall some definitions. Let X and Y be compact subsets of $\mathbb{C}$ equipped with the Euclidean metric d. The *Hausdorff distance* between X and Y is defined as

$$d_H(X, Y) := \inf_{\varepsilon} \{\varepsilon : Y \subset B_\varepsilon(X), \text{ and } X \subset B_\varepsilon(Y)\}.$$

The space of all compact subsets of $\mathbb{C}$ endowed with this metric is a complete metric space.

A set valued map $c \mapsto X_c$, with X_c compact in $\mathbb{C}$, is *upper semi-continuous* if $c_n \to c$ implies that X_c contains the Hausdorff limit of X_{c_n}.

A family of simply connected domains U_λ depends *continuously* on λ if there exit choices of uniformizations $\psi_\lambda : D_1 \to U_\lambda$ continuous in both variables.

A family of polynomial-like maps $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$, parametrized on a topological disk Λ, depends continuously on λ if U_λ is a continuous family of simply connected domains in $\mathbb{C}$, and for every fixed $z \in \mathbb{C}$, $P_\lambda(z)$ depends continuously on λ where ever it is defined.

Proposition 3.11. *Let $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$ be a continuous family of polynomial like maps with connected filled Julia sets K_λ. The map $\lambda \mapsto K_\lambda$ is upper semi-continuous.*

Proof. Assume that $\lambda_n \to \lambda$, $K_{\lambda_n} := K(P_{\lambda_n})$, and $K_\lambda := K(P_\lambda)$. To prove that $K := \lim K_{\lambda_n}$ is contained in K_λ, it is enough to show that for every $\varepsilon > 0$, $K_{\lambda_n} \subseteq B_\varepsilon(K_\lambda)$ for sufficiently large n.

To see this, assume that $z \notin B_\varepsilon(K_\lambda)$. If $z \notin V_\lambda$, then by continuous dependence of V_λ on λ, $z \notin K_{\lambda_n}$ for large n. If $z \in V_\lambda$, then there exists a positive integer l with $P_\lambda^l(z) \in U_\lambda \setminus V_\lambda$. As $P_{\lambda_n} : V_{\lambda_n} \to U_{\lambda_n}$ converges to $P_\lambda : V_\lambda \to U_\lambda$, $P_n^l(z)$ or $P_n^{l+1}(z)$ belongs to $U_{\lambda_n} \setminus V_{\lambda_n}$. Therefore, $z \notin K_{\lambda_n}$. ☐

Proof of Lemma 3.10. Let c_{n-1} be a twice satellite renormalizable parameter in a truncated secondary limb. Consider the external rays landing at the dividing fixed point α_n of $\mathcal{R}\mathbf{f}_{c_{n-1}}$ and it's preimages under $\mathcal{R}\mathbf{f}_{c_{n-1}}$. Let X_0^0 denote the puzzle piece obtained from these rays which contains the critical point. As $\mathbf{f}_{c_{n-1}}$ is twice satellite renormalizable, $\mathcal{R}^2\mathbf{f}_{c_{n-1}} : X_0^0 \to \mathbb{C}$ is a branched covering map over its image. One can consider a continuous thickening of X_0^0, described in Section 3.1.2, to form a continuous family of polynomial-like maps parametrized over this truncated limb. The little Julia set of this map was denoted by $\mathbf{J}_{2,0}$ and the Julia bouquet $\mathbf{B}_{2,0}$ is the connected component of

$$\bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} \mathbf{f}_{c_{n-1}}(\mathbf{J}_{2,0})$$

containing the critical point.

For c_{n-1} in a finite number of truncated secondary limbs, the Julia bouquet $B_{2,0}$ is unoin of a finite number of little Julia sets. Hence, by above lemma, it depends upper semi-continuously on c_{n-1}. As $\mathbf{B}_{2,0}$ is contained well inside the interior of Y_0^0 for such c_{n-1} which belongs to a compact set of parameters, we conclude that modulus of $Y_0^0 \setminus B_{2,j}$ is uniformly bounded below.

To see that this modulus is uniformly bounded above, one only needs to observe that α_n and 0 belong to $\mathbf{B}_{2,0}$ and are strictly apart from each other for these parameters. □

By Lemma 3.10 there are simply connected domains $\mathbf{L}'_{n-1} \subseteq \mathbf{L}_{n-1}$ and $\widetilde{\mathbf{L}}'_{n-1} \subseteq \widetilde{\mathbf{L}}_{n-1}$ such that moduli of the annuli

$$\begin{array}{lll} Y_0^0 \setminus \mathbf{L}_{n-1}, & \mathbf{L}_{n-1} \setminus \mathbf{L}'_{n-1}, & \mathbf{L}'_{n-1} \setminus \mathbf{B}_{2,0} \\ \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_{n-1}, & \widetilde{\mathbf{L}}_{n-1} \setminus \widetilde{\mathbf{L}}'_{n-1}, & \widetilde{\mathbf{L}}'_{n-1} \setminus \widetilde{\mathbf{B}}_{2,0} \end{array}$$

are bigger than some constant depending only on the combinatorial class $\mathcal{SL}$. Moreover, the ratios

$$\frac{\operatorname{mod}(Y_0^0 \setminus \mathbf{L}_{n-1})}{\operatorname{mod}(\widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_{n-1})}, \quad \frac{\operatorname{mod}(\mathbf{L}_{n-1} \setminus \mathbf{L}'_{n-1})}{\operatorname{mod}(\widetilde{\mathbf{L}}_{n-1} \setminus \widetilde{\mathbf{L}}'_{n-1})}, \quad \frac{\operatorname{mod}(\mathbf{L}'_{n-1} \setminus \mathbf{B}_{2,0})}{\operatorname{mod}(\widetilde{\mathbf{L}}'_{n-1} \setminus \widetilde{\mathbf{B}}_{2,0})}$$

are also uniformly bounded below and above independent of n.

Theorem 2.9, combined with above data, implies that there exits a q.c. map $\mathbf{h}_{n-1} : Y_0^0 \setminus \mathbf{L}_{n-1} \to \tilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_{n-1}$, with a uniform bound on its dilatation, which matches the Böttcher marking on the boundary of Y_0^0. We further lift $\mathbf{h}_{n-1}$ via $\mathbf{f}_{c_{n-1}}^{-i}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{-i}$ to extend $\mathbf{h}_{n-1}$ to q.c. maps

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{-i}(Y_0^0 \setminus \mathbf{L}_{n-1}) \to \tilde{\mathbf{f}}_{c_{n-1}}^{-i}(\tilde{Y}_0^0 \setminus \tilde{\mathbf{L}}_{n-1}), \text{ for } i = 1, 2, \ldots, \frac{t_n}{t_{n-1}} - 1$$

with the same dilatation. The domain of each such map is a puzzle piece Y_j^0 (with $j = t_n/t_{n-1} - i$) cut off by the equipotential of level η/d^l and a component of $\mathbf{f}_{c_{n-1}}^{-i}(\partial \mathbf{L}_{n-1})$. As all these maps match the Böttcher marking on the boundaries of Y_j^0, they can be glued together. Finally, one extends this map as the identity in the Böttcher coordinates onto spaces between equipotential of level η and equipotentials of level η/d^l. we denote this extended map with the same notation $\mathbf{h}_{n-1}$. As it is lifted under and extended by holomorphic maps, there is a uniform bound on its dilatation.

Let $\Delta_{n-1,0}$ be the S_{n-1}-preimage of the domain inside equipotential $E(\eta)$ and $L_{n-1,i}$ be the component of the S_{n-1}-preimage of $\mathbf{f}_{c_{n-1}}^{-i}(\mathbf{L}_{n-1}))$ enclosing the little Julia set $J_{n-1,i}$. We define the multiply connected regions

$$\Omega_{n-1,0} := \Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}-1} L_{n-1,it_{n-1}}$$

Like before, $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$ are defined as f^i-preimage of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ containing or enclosing $J_{n-1,i}$, respectively.

We have

$$h_{n-1,0} := \widetilde{S}_{n-1}^{-1} \circ \mathbf{h}_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0}, \text{ with } h_{n-1,0}(\Omega_{n-1,0}) = \widetilde{\Omega}_{n-1,0}.$$

Also,

$$h_{n-1,i} := \widetilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i}, \text{ with } h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}.$$

As the equipotential $\eta(\varepsilon)$ is contained in $S_{n-1}(W_{n-1,0})$, $\Omega_{n-1,0}$ is contained in $W_{n-1,0}$. Therefore, $\Omega_{n-1,0}$ is well inside $V_{n-1,0}$. Conformal invariance of modulus implies that the other domains $\Omega_{n-1,i}$ are also well inside $V_{n-1,i}$. This completes the construction in Case $\mathscr{C}$.

To fit together the multiply connected domains $\Omega_{n(k),i}$ and the q.c. maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$, we follow the word of cases introduced at the beginning of the construction. In Cases $\mathscr{A}$ and $\mathscr{B}$, we have adjusted $\mathbf{h}_{n-1}$ such that it sends $\partial V_{n-1,0}$ to $\partial \tilde{V}_{n-1,0}$. Therefore, if any of the three cases follows Case $\mathscr{A}$ or $\mathscr{B}$, we consider $\mathcal{R}^n f : W_{n,0} \to V_{n,0}$ and straighten it with these choices of domains (instead of $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$). If a case of construction on level n is following Case $\mathscr{C}$, the set $\mathbf{\Delta}_{n,0}$ introduced on level n is replaced by $\mathbf{\Delta}_{n,0} \cap S_n(L'_{n-1})$, $\mathbf{h}_{n,0}$ is restricted to this set, and adjusted so that it sends $S_n(L'_{n-1,i})$ to $\tilde{S}_n(\tilde{L}'_{n-1,i})$. The annulus $L_{n-1} \setminus L'_{n-1}$ provides a definite annulus separating $\Omega_{n,0}$ and $L_{n-1,0}$.

In the following two sections, we will denote the holes of $\Omega_{n,i}$ by $\mathbb{V}_{n+1,j}$, that is, $\mathbb{V}_{n+1,j}$ is $V_{n+1,j}$, if n belongs to Case $\mathscr{A}$ or $\mathscr{B}$ or $\mathbb{V}_{n+1,j}$ is $S_n^{-1}(L_{n,j})$ if n belongs to Case $\mathscr{C}$.

3.2.4 The gluing maps $g^i_{n(k)}$

In this section we build $K'(\varepsilon)$-q.c. maps

$$g^i_{n(k)} : \mathbb{V}_{n(k),i} \setminus \Delta_{n(k),i} \to \widetilde{\mathbb{V}}_{n(k),i} \setminus \widetilde{\Delta}_{n(k),i}.$$

Every $g^i_{n(k)}$ has to be identical with $h_{n(k),i}$ on $\partial \mathbb{V}_{n(k),i}$ and with $h_{n(k+1),i}$ on $\partial \Delta_{n(k),i}$ (which is outer boundary of $\Omega_{n(k),i}$). Then gluing all these maps $g^i_{n(k)}$ and $h_{n(k),i}$ produces a q.c. map H with dilatation bounded by maximum of K and $K'(\varepsilon)$. In what follows, for simplicity of notation, we use index n instead of $n(k)$, and assume that n runs over subsequence $n(k)$. So for all n, means for all $n(k)$'s.

Like previous steps, it is enough to build g^0_n for all n, and pull them back by f^{-i} to obtain g^i_n. Properties of the maps $h_{n,i}$ and the domains $\mathbb{V}_{n,i}$ and $\Omega_{n,i}$ implies that these maps glue together on the boundaries of their domain's of definition. Again, we drop the superscript index if it is 0, i.e. g_n denotes the map g^0_n. To build a q.c. map from an annulus to another annulus with given boundary conditions, there is a choice in the number of "twists" one may make. Moreover, to have a uniform bound on the dilatation of such a map, the two annuli must have proportional moduli bounded below, and the number of twists has to be uniformly bounded. Note that the number of twists effects on the homotopy class of the final map H.

In this section, we show that the corresponding annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ have proportional moduli with a constant depending only on ε. In the next section we specify the number of twists needed for the isotopy class of a Thurston conjugacy.

Lemma 3.12. *Let U', U, and $\widetilde{U}$ be three annuli whose inner boundaries are D_1 such that* $\text{mod }(U \setminus U') \geq \varepsilon$, *and* $\text{mod }(U) \leq M$, *for some constant M. Let ψ be a K-q.c. map from U onto $\widetilde{U}$. For every r with $D_r \subset U' \cap \psi(U')$, moduli of $U \setminus D_r$ and $\widetilde{U} \setminus D_r$ are proportional with a constant depending only on M, K, and ε.*

Proof. By properties of q.c. maps we have

$$\varepsilon \leq \text{mod }(U \setminus D_r) \leq M,$$

$$\frac{\varepsilon}{K} \leq \text{mod }(\widetilde{U} \setminus D_r) \leq KM$$

which implies the lemma. □

Lemma 3.13. *Moduli of the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ are proportional with a constant depending only on ε.*

Proof. If level n follows Case $\mathscr{A}$ or $\mathscr{B}$, one proves this by applying the above lemma to images of $\mathbb{V}_{n,0}$, $\mathbb{W}_{n,0} = f^{-t_n}(\mathbb{V}_{n,0})$ and $\Delta_{n,0}$ under $B_{c(\mathcal{R}^n f)} \circ S_n$ and the corresponding tilde objects, where $B_{c(\mathcal{R}^n f)}$ is the Böttcher coordinate for P_{c_n}. Note that $\Delta_{n,0}$ is mapped to D_r. To obtain an upper bound M, it is enough to go several levels lower than the fundamental annulus to get an annulus with bounded modulus.

If level n follows Case $\mathscr{C}$, by definition, $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ is $L_n \setminus L'_n$ and $\tilde{\mathbb{V}}_{n,0} \setminus \tilde{\Delta}_{n,0}$ is $\tilde{L}_n \setminus \tilde{L}'_n$ which were chosen to be proportional. □

Given a curve $\ell \subset U$ which is given as $\gamma : [a, b] \to U$ with $\gamma(a)$ on the inner boundary of U (corresponding to the bounded component of $\mathbb{C} \setminus U$) and $\gamma(b)$ on the outer boundary of U (corresponding to the unbounded one), define the *wrapping number* $\omega(\ell)$ as

$$\frac{1}{2\pi} \int_a^b \frac{\partial \pi_2(\phi(\gamma(t)))}{\partial \theta} \, dt.$$

where ϕ is a uniformization of the annulus U by a round annulus, and $\pi_2(z)$ is the polar angle of the point z. Basically, $\omega(\ell)$ is total turning of the curve ℓ in the uniformized coordinate. Note that $\omega(\ell)$ is invariant under the automorphism group of U. So, it is independent of the choice of uniformization and just like winding number, it is constant over the homotopy class of all curves with same boundary points.

Let $A(r)$ denote the round annulus $D_r \setminus D_1$.

Proposition 3.14. *Given fixed constants $K \geq 1$ and $r > 1$, there exists a constant N such that, for every K-q.c. map $\psi : A(r) \to A(r')$, the wrapping number of the curve $\psi(t), t \in [1, r]$, belongs to the interval $[-N, N]$.*

Proof. Consider the curve family L consisting of all ray segments in $A(r)$ obtained from rotating the real line segment $[1, r]$ about the origin, that is, the radial lines in $A(r)$.

We denote by $\Gamma(F)$ the extremal length of a given curve family F. The inequality $\Gamma(L)/K \leq \Gamma(\psi(L)) \leq K\Gamma(L)$ holds for the K-q.c. map ψ. See [Ah66] for more details on curve families and extremal length properties.

It is easy to see that if the interval $[1, r]$ is mapped to a curve with wrapping number T, then every curve in L is mapped to a curve with wrapping number between $T + 1$ and $T - 1$.

By definition of extremal length and choosing conformal metric ρ as the

Euclidean metric, we obtain

$$K\Gamma(L) \geq \Gamma(\psi(L)) = \sup_{\rho} \frac{\inf \ell_\rho(\psi(\gamma))^2}{S_\rho} \geq \frac{4\pi(T-1)^2}{R'^2 - 1},$$

and $\Gamma(L) = \log R/2\pi$. By properties of q.c. mappings, we have $R' \leq R^K$ which implies

$$T \leq \frac{1}{\pi}\sqrt{\frac{K \log(R)(R^{2K}-1)}{8}} + 1$$

□

Lemma 3.15. *Fix round annuli $A(r)$, $A(r')$, a positive constant K_1, and an integer k with* $\operatorname{mod} A(r')/K_1 \leq \operatorname{mod} A(r) \leq K_1 \operatorname{mod} A(r')$ *and* $\operatorname{mod} A(r) \geq \delta$. *If homeomorphisms $h_1 : \partial D_r \to \partial D_{r'}$ and $h_2 : \partial D_1 \to \partial D_1$ have K_2-q.c. extensions to some neighborhood of these circles for some K_2, then there exists a K-q.c. map $h : A(r) \to A(r')$ such that*

- *$h(z) = h_1(z)$ for every $z \in \partial D_r$, and $h(z) = h_2(z)$ for every $z \in \partial D_1$*
- *The curve $h(t)$, for $t \in [1, r]$, has wrapping number $\theta(h_1(r)) - \theta(h_2(1)) + k$.*

Moreover, K depends only on K_1, K_2, k and δ.

Proof. One proves this statement by explicitly building such maps for every k. More details left to reader. □

Applying the above lemma to the uniformization of the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$, the induced maps from $h_{n-1,0}$ and $h_{n,0}$ on their boundaries and a number k_n, which is specified later, gives the required K'-q.c. maps g_n. In the next section we specify some especial numbers k_n ,which are bounded by a constant depending on ε, in order to make the K-q.c. map H, obtained after gluing all these maps, homotopic to a topological conjugacy relative the postcritical set.

Definite moduli of the annuli $\mathbb{V}_{n,i} \setminus \Delta_{n,i}$ implies that the holes $\mathbb{V}_{n,i}$ shrink to points in the postcritical set. Therefore, H can be extended to a well defined K-q.c. map on the postcritical set. See [St55] for a detailed proof of this and further results on quasi-conformal removability.

3.2.5 Isotopy

Denote by $\boldsymbol{\psi}_n$ the topological conjugacy between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ obtained from extending the identity map in the Böttcher coordinates onto Julia set. The lift $\psi_{n,0} = \tilde{S_n}^{-1} \circ \boldsymbol{\psi}_n \circ S_n$ is a topological conjugacy between $\mathcal{R}^n f$ and $\mathcal{R}^n \tilde{f}$ on a neighborhood of the little Julia set $J_{n,0}$. Note that this neighborhood covers the domain $\Omega_{n,0}$. In the dynamic plane of $\mathbf{f}_{c_n}$, let $U(\eta)$ denote the domain enclosed by the equipotential of level $\eta(\varepsilon)$.

Lemma 3.16. *If level n belongs to one of Case $\mathscr{A}$ or $\mathscr{B}$, then the q.c. map $h_{n,i} : \Delta_{n,i} \to \widetilde{\Delta}_{n,i}$, for $i = 0, 1, \ldots t_n - 1$, is homotopic to*

$$\psi_{n,i} := f^{-i} \circ \tilde{S}_n^{-1} \circ \boldsymbol{\psi}_n \circ S_n \circ f^i : \Delta_{n,i} \to \mathbb{C}$$

relative the little Julia sets $J_{n+1,j}$ of level $n+1$ inside $\Delta_{n,i}$.

Note that $\psi_{n,i}(\Delta_{n,i})$ is a neighborhood of the little Julia sets $\tilde{J}_{n+1,i+t_n j}$ contained in $\widetilde{\Delta}_{n,i}$.

Proof. First assume that level n follows a Case $\mathscr{A}$ or $\mathscr{B}$. From the definition of the domains $\Delta_{n,i}$, $\mathbb{V}_{n,i}$ and the q.c. maps $h_{n,i}$, it is enough to prove the statement for $i = 0$. For other ones one pulls the homotopies back by f^i, or constructs them in the same way as for $i = 0$).

As $\Delta_{n,0}$, $\psi_{n,0}$, and the q.c. map $h_{n,0}$ are lifts of $\boldsymbol{\Delta}_{n,0}$, $\boldsymbol{\psi}_n$, and $\mathbf{h}'_{n,0}$ by the straightening map, it is enough to make the homotopy on the dynamic planes of $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ and then transfer it to the dynamic planes of $\mathcal{R}^n f$ and $\mathcal{R}^n \tilde{f}$ by the straightening map. Recall that in our construction, $\mathbf{h}'_{n,0}$ is an adjustment of $\mathbf{h}_{n,0}$ through a homotopy relative the little Julia sets of the next level. Thus, to prove the lemma, we only need to prove that $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_n$ are homotopic relative the little Julia sets.

Assume that level n belongs to Case $\mathscr{A}$. The idea of the proof, presented in detail below, is to divide the domain $\mathbf{\Delta}_{n,0}$, by means of rays and equipotential arcs, into some topological disks and one annulus such that $\boldsymbol{\psi}_n$ and $\mathbf{h}_{n,0}$ are identical on the boundaries of these domains.

Consider the puzzle piece $Q^{\chi_n}_{n,0}$ ($Q^{\chi_n}_{n,0} = Y_0^{q_{\chi_n}}$). The equipotential $\mathbf{f}_{c_n}^{-\chi_n}(E(\eta))$, and the rays of the puzzles $Q^{\chi_n}_{n,i}$ up to equipotential $\mathbf{f}_{c_n}^{-\chi_n}(E(\eta))$, cut the domain $\mathbf{\Delta}_{n,0}$ into one annulus $\mathbf{\Delta}_{n,0} \setminus \mathbf{f}^{-\chi_n}(U(\eta))$ and some topological disks. The topological disks which do not intersect the little Julia sets of level n, the puzzle pieces $Q^{\chi_n}_{n,i}$, and the remaining annulus $E(\eta) \setminus f^{-\chi_n}(E(\eta))$, are the appropriate domains. By Theorem 3.2 the maps $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_n$ are identical in the Böttcher coordinate on the boundaries of these domains. Indeed, the topological conjugacy $\boldsymbol{\psi}_n$ between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ is identity in the Böttcher coordinate and the pseudo-conjugacy $h_{n,0}$ obtained in Theorem 3.2 matches the Böttcher marking. This proves that the two maps are homotopic outside of the puzzle pieces $Q^{\chi_n}_{n,i}$.

To find a homotopy inside the pieces $Q^{\chi_n}_{n,0}$, recall that we started with a q.c. map on $Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0}$, which was homotopic to $\boldsymbol{\psi}_n$ relative the boundaries. So all the pullbacks of this map on the annuli $A_n^k \setminus A_n^{k+1}$ are homotopic to $\boldsymbol{\psi}_n$ relative boundaries. As the two maps are identical on the little Julia sets, we conclude that the two maps are homotopic relative the little Julia sets.

If level n belongs to Case $\mathscr{B}$, we repeat above argument on puzzle pieces of level zero.

The same proof works if level n follows Case $\mathscr{C}$. The only difference is that the domain of definition of the homeomorphisms are restricted to a smaller set. Thus, we may restrict the homotopy onto that set. $\square$

In the following paragraphs we assign the number of twists k_n for the gluing maps g_n.

If level n belongs to Case $\mathscr{A}$ or $\mathscr{B}$, and it follows a Case $\mathscr{A}$ or $\mathscr{B}$, consider the uniformizations $\phi_1 : A(s) \to (\mathbb{V}_{n,0} \setminus K_{n,0})$, $\phi_2 : A(r) \to (\Delta_{n,0} \setminus K_{n,0})$, $\widetilde{\phi}_1 : A(\tilde{s}) \to (\widetilde{\mathbb{V}}_{n,0} \setminus \tilde{K}_{n,0})$ and $\widetilde{\phi}_2 : A(\tilde{r}) \to (\widetilde{\Delta}_{n,0} \setminus \tilde{K}_{n,0})$ by round annuli. The q.c. maps $h_{n-1,0}$ and $h_{n,0}$ lift via ϕ_i and $\widetilde{\phi}_i$ to q.c. maps $\hat{h}_{n-1,0} : A(s) \to V(\tilde{s})$ and $\hat{h}_{n,0} : A(r) \to V(\tilde{r})$ with the same dilatation. By composing the uniformizations with rotations, we may assume that the point one is mapped to the point one by these two maps. By Proposition 3.14, the image of the segment $[1, s]$ under the q.c. map $\hat{h}_{n-1}$ has wrapping number ω_{1n} bounded by some N and image of the segment $[1, r]$ under the q.c. map $\hat{h}_{n,0}$ has wrapping number ω_{2n} bounded by N which depends only on ε. Define k_n as $\omega_{1n} - \omega_{2n}$, and note that gluing $\hat{h}_{n,0}$ and $\hat{h}_{n-1,0}$ in Lemma 3.15 by such a choice makes the image of the segment $[1, s]$ under the two maps g_n and $\hat{h}_{n,0}$ glued together, homotopic to image of the segment $[1, s]$ under $\hat{h}_{n-1,0}$ relative two boundary circles. This homotopy lifts to a homotopy between $h_{n-1,0}$ and the two maps g_n and $h_{n,0}$ glued together.

Before we define k_n for the other cases, we need to show that the q.c. map $\mathbf{h}'_{n-1}$, built in Case $\mathscr{C}$, has a q.c. extension over the topological disk $\mathbf{L}_{n-1}$.

Lemma 3.17. *The q.c. map $\mathbf{h}'_{n-1}$ introduced in Case $\mathscr{C}$ has a q.c. extension onto $\mathbf{L}_{n-1}$ with a uniform bound on its dilatation depending only on ε. Moreover, this extension can be made homotopic to $\boldsymbol{\psi}_{n-1}$ relative the Julia bouquet $B_{2,0}$.*

Proof. Consider the fundamental annuli $S_{n-1}(U_{n,0} \setminus V_{n,0})$ and $\tilde{S}_{n-1}(\tilde{U}_{n,0} \setminus \tilde{V}_{n,0})$ for the first renormalizations of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. Let

$$g_n : S_{n-1}(U_{n,0} \setminus V_{n,0}) \to \tilde{S}_{n-1}(\tilde{U}_{n,0} \setminus \tilde{V}_{n,0})$$

be a q.c. map which satisfies the equivariance relation on the boundaries of these annuli. By lifting g_n on the preimages of these annuli we obtain a q.c. map g_n from complement of the little Julia set $J_{1,0}$ to the complement of the little Julia set $\tilde{J}_{1,0}$ on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. By Lemma 3.8, g_n (or some rotation of it) can be extended as $\boldsymbol{\psi}_{n-1}$ onto the Julia set $J_{1,0}$. Moreover these two maps are homotopic relative this little Julia set. We then adjust g_n so that it maps $\mathbf{L}'_{n-1}$ to $\widetilde{\mathbf{L}}'_{n-1}$.

Consider the three annuli $Y_0^0 \setminus \mathbf{L}_{n-1}$, $\mathbf{L}_{n-1} \setminus \mathbf{L}'_{n-1}$, and $\mathbf{L}'_{n-1} \setminus B_{2,0}$, as well as the corresponding tilde ones. We have $\mathbf{h}_{n-1} : Y_0^0 \setminus \mathbf{L}_{n-1} \to \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_{n-1}$ and $g_n : \mathbf{L}'_{n-1} \setminus B_{2,0} \to \widetilde{\mathbf{L}}'_{n-1} \setminus \widetilde{B}_{2,0}$. To glue these two maps on the middle annulus, we use the above argument to find the right number of twists on this annulus. Consider a curve γ connecting a point a on the bouquet $B_{2,0}$ to a point d on the boundary of Y_0^0 such that it intersects the boundaries of $\mathbf{L}'_{n-1}$ and $\mathbf{L}_{n-1}$ only once which are denoted by b and c. Lets denote by γ_{ab}, γ_{bc}, and γ_{cd} each segment of this curve cut off by these four points. The real number $\omega(\boldsymbol{\psi}(\gamma)) - \omega(\mathbf{h}_{n-1}(\gamma_{cd})) - \omega(g_n(\gamma_{ab}))$ is uniformly bounded, by Proposition 3.14, depending only on ε. If we glue $\mathbf{h}_{n-1}$ and g_n by such a number of twists (see Lemma 3.15), the resulting map will be homotopic to $\boldsymbol{\psi}_{n-1}$ relative $B_{2,0} \cup \partial Y_0^0$. Note that the two maps $\mathbf{h}'_{n-1}$ and $\boldsymbol{\psi}_{n-1}$ are identical on the boundary of Y_0^0. Thus, one can extend this map over the other topological disks $\mathbf{L}_{n-1,i}$. We will denote the final q.c. map by $\mathbf{h}'_{n-1}$. □

If a Case $\mathscr{C}$ follows a Case $\mathscr{A}$ or $\mathscr{B}$, the number of twists k_n is defined as the one introduced above. If level $n-1$ belongs to Case $\mathscr{C}$ and level n is any of the three cases, we use the uniformization of the annuli $\mathbb{V}_{n,0} \setminus B_{n,0}$ and $\Delta_{n,0} \setminus B_{n,0}$, as well as the corresponding tilde ones, instead of the above annuli, to define k_n.

The following lemma is frequently used in the final proof of isotopy. Its prove is left to reader.

Lemma 3.18. *Let U and $\widetilde{U}$ be closed annuli with $\partial U = \gamma_1 \cup \gamma_2$, and $\partial \widetilde{U} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$. Also, let $h_i^t : \gamma_i \to \tilde{\gamma}_i$, for $t \in [0,1]$, $i = 1,2$, be two continuous families of homeomorphisms, and the homeomorphism $G^0 : U \to \widetilde{U}$ be an interpolation of h_1^0 and h_2^0 on U. Then G^0 can be extended to a continuous family of interpolations of h_1^t and h_2^t, for $t \in [0,1]$.*

Proposition 3.19. *The K-q.c. map H obtained from gluing all the maps g_n^i and $h_{n,k}$ is homotopic to the topological conjugacy Ψ relative the postcritical set $\mathcal{PC}(f)$.*

Proof. Let H_n denote the map obtained from gluing the maps g_1^0,$g^k, \ldots, g_{n-1}^l$ and $h_{1,0}, h_{2,i}, \ldots, h_{n,j}$, for all possible indices i, j, k and l up to level n.

First, we show that H_1 is homotopic to the topological conjugacy Ψ between $\mathbf{f}_c$ and $\mathbf{f}_{\tilde{c}}$ relative the little Julia sets $J_{1,i}$. Then we show that each H_{n-1} is homotopic to H_n relative the little Julia sets of level $n+1$.

The map H_1 is just $h_{1,0}$ which is homotopic to $\psi_{1,0}$, by Lemma 3.16 or Lemma 3.17. It is homotopic to Ψ by Lemma 3.8.

Recall that the two maps H_{n-1} and H_n are identical on the complement of $\mathbb{V}_{n,j}$. Inside $\Delta_{n,0}$, H_{n-1} and H_n are $h_{n-1,0}$ and $h_{n,0}$, respectively.

The domain $\mathbb{V}_{n,0}$ is divided into annulus $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and the topological disk $\Delta_{n,0}$. On $\Delta_{n,0}$, $h_{n,0}$ and $h_{n-1,0}$ are homotopic to $\psi_{n,0}$ relative $\cup_i J_{n+1,i}$ by Lemmas 3.16 and 3.17. Thus, there exists a homotopy h_n^t, for t in $[0,1]$, which starts with $h_{n,0}$ and ends with $h_{n-1,0}$, such that it maps $\partial\Delta_{n,0}$ to $\partial\widetilde{\Delta}_{n,0}$, for all

$t \in [0,1]$. At time zero consider the map $h_{n,0}$ on the inner boundary of this annulus, $h_{n-1,0}$ on the outer boundary of this annulus and the interpolation $G_n^0 = g_n^0$ between them. Applying above lemma to the fixed homeomorphism $h_{n-1,0}$ on the outer boundary, and h_n^t on the inner boundary, we obtain a continuous family of interpolations G_n^t between them. The map G_n^1 is a homeomorphism from the annulus to itself which is an interpolation of $h_{n-1,0}$ on the boundaries, but this interpolation has to be homotopic to $h_{n-1,0}$ on the annulus. Indeed, these two maps send a curve joining the two different boundaries to two curves (joining the two boundaries) which are homotopic relative end points. This comes from our choice of wrapping numbers for the gluing maps.

Let $t_0 = 0 < t_1 < t_2 < \cdots < 1$, be an increasing sequence of numbers in $[0,1]$. Assume that H^t, for t in $[t_0, t_1]$, denotes the homotopy obtained above between Ψ and H_1 relative the little Julia sets $J_{1,i}$. Also, let H^t, for $t \in [t_n, t_{n+1}]$, and $n = 1, 2, \ldots$, denote the homotopy between H_n and H_{n+1} relative the little Julia sets of level $n+2$.

It is clear from the construction that $H^t(z)$ eventually stabilizes for any fixed z, and equals to $H(z)$. Indeed, *a priori* bounds assumption implies that the diameter of the topological disks $\mathbb{V}_{n,i}$ tends to zero as $n \to \infty$. Therefore, the uniform distance between H^t and H is going to zero as $t \to 1$. We conclude that H^t for t in $[0,1]$ is a homotopy between Ψ and the Thurston conjugacy H relative the postcritical set. $\square$

3.2.6 Promotion to hybrid conjugacy

Proposition 3.20. *Suppose all infinitely renormalizable unicritical polynomials in a given combinatorial class $\tau = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots\}$, satisfying $\mathcal{SL}$ condition, enjoy a priori bounds. Then q.c. conjugacy implies hybrid conjugacy* *for maps in this class.*

Proof. Assume that this is not correct and there are polynomials P_1 and P_2 in this class which are q.c. equivalent but not hybrid equivalent. Define the set

$$\begin{aligned}\Omega &= \{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_1\}\\ &= \{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_2\}.\end{aligned}$$

We show that the set Ω is both open and closed in $\mathbb{C}$ which is not possible.

Theorem 3.4 implies that q.c. conjugacy is equivalent to combinatorial conjugacy for maps in this class. Since every combinatorial class is an intersection of closed sets (connectedness locus copies), Ω is closed.

Consider a point P in Ω. The polynomial P is not hybrid equivalent to both of P_1 and P_2 by assumption. Assume that it is not hybrid equivalent to P_1 (for the other case just change P_1 to P_2). Let $\phi_1 : \mathbb{C} \to \mathbb{C}$, K-q.c. conjugates P to P_1, i.e. $\phi_1 \circ P = P_1 \circ \phi_1$. By pulling back the standard complex structure μ_0 on $\mathbb{C}$ under ϕ_1, we obtain a complex structure μ on $\mathbb{C}$ with dilatation bounded by $\frac{K-1}{K+1}$. Consider the complex structures $\mu_\lambda := \lambda \cdot \mu$, for λ in the disk of radius $\frac{K+1}{K-1}$ around origin.

By applying measurable Riemann mapping Theorem 2.8 , there are q.c. maps ϕ_λ which map complex structure μ_λ to μ_0, and leave the points 0 and ∞ fixed. The maps $P_\lambda := \phi_\lambda^{-1} \circ P_1 \circ \phi_\lambda$, for λ in the disk of radius $\frac{K+1}{K-1}$ around origin, are holomorphic maps of the same degree as the degree of P_1, and send infinity to infinity with the same local degree as the one of P_1. Thus, they are polynomials. For $\lambda = 1$ we obtain the polynomial P, and for $\lambda = 0$ we obtain P_1. By analytic dependence of the solution of the measurable Riemann mapping theorem on the complex structure, P_λ cover a neighborhood of P in Ω. This shows that P is an interior point in Ω. As P was an arbitrary point in Ω, we conclude that Ω is open. □

3.2.7 Dynamical description of the combinatorics

In this section we give a dynamical definition of some combinatorial classes. It has been proved in [KL05] that infinitely renormalizable parameters satisfying *decorations* enjoy *a priori* bounds. Let c be an infinitely renormalizable parameter with renormalizations $f_n := \mathcal{R}^n(P_c)$, their straightening $\mathbf{f}_{c_n}$, and critical puzzle piece of level 1 denoted by $Y_0^1(n)$. The dynamical meaning of a parameter c satisfying the decoration condition is that there exists a constant M such that for every $n \geq 0$ there are integers t_n and q_n, both bounded by M, with

- $\mathbf{f}_n^{kq_n}(0) \in Y_0^1(n)$, for every $k < t_n$, and every $n \geq 1$,
- $\mathbf{f}_n^{t_n q_n}(0) \notin Y_0^1$, for every $n \geq 1$.

In particular, this condition implies that the number of rays landing at the dividing fixed point of $\mathbf{f}_n$ (here q_n) is uniformly bounded. An infinitely renormalizable parameter is of bounded type if the relative return times t_{n+1}/t_n of the renormalizations $\mathcal{R}^n(f) = f^{t_n}$ are bounded by some constant M. It follows from definition that the decoration condition includes infinitely primitive renormalizable parameters of bounded type.

In Section 3.1.3, we associated a sequence of maximal connectedness locus copies $\tau(f) = \langle \mathcal{M}^1, \mathcal{M}^2, \dots \rangle$ to every infinitely renormalizable unicritical polynomial-like map f. Let $\pi_n(\tau(f)) = \mathcal{M}^n$ denote the projection map onto n's component. Define

$$\tau(f, n) = \left\{ c \in \mathcal{M}_d \;\middle|\; \begin{array}{l} c \text{ is at least } n \text{ times renormalizable, and} \\ \pi_i(\tau(f)) = \pi_i(\tau(P_c)), \text{ for } i = 1, 2, \dots, n \end{array} \right\}.$$

In other words, the connectedness locus copy $\tau(f, n)$ is the set of at least n times renormalizable parameters with the same combinatorics as of f up to level n.

Given an infinitely renormalizable map f, and an increasing sequence of positive integers $\langle n_i \rangle$, we define a sequence of relative connectedness locus copies of $\mathcal{M}_d$ as follows:

$$(\tilde{\tau}(f), \{n_i\}) := \langle \tilde{\mathcal{M}}^{n_1}, \tilde{\mathcal{M}}^{n_2}, \ldots, \tilde{\mathcal{M}}^{n_k}, \ldots \rangle,$$

where,

$$\tilde{\mathcal{M}}^{n_k} := \tau(\mathcal{R}^{n_{k-1}} f, n_k - n_{k-1}).$$

One can see that there is a one to one correspondence between the two sequences $\tau(f)$ and $(\tilde{\tau}(f), \{n_i\})$ for every increasing sequence $\langle n_i \rangle$. Thus, one may take the latter one as definition of the combinatorics of f.

Consider the main hyperbolic component of the connectedness locus $\mathcal{M}_d$. There are infinitely many primary hyperbolic components attached to this component (corresponding to rational numbers). Similarly, there are infinitely many hyperbolic components, secondary ones, attached to these primary components, and so on. Consider the set of all hyperbolic components obtained this way. That is, the ones which can be connected to the main hyperbolic component by a chain of hyperbolic components bifurcating one from another. The closure of this set after adding all possible components of its complement is called the *molecule* $\mathscr{M}_d$.

We say that an infinitely renormalizable map f satisfies the molecule condition, if there exists a positive constant $\eta > 0$ and an increasing sequence of positive integers $\langle n_i \rangle_{i=1}^{\infty}$, such that for every $i \geq 1$, $\mathcal{R}^{n_i} f$ is a primitive renormalization of $\mathcal{R}^{n_i - 1} f$, and moreover, the Euclidean distance between $\tilde{\mathcal{M}}_d^{n_i}$ and the molecule $\mathcal{M}_d$ is at least η. Note that for a map satisfying this condition, there may be infinitely many satellite renormalizable maps in the sequence $\{\mathcal{R}^n f\}$.

However, the condition requires that there are infinite number of primitive levels with the corresponding relative connectedness locus copies uniformly away from the molecule. One can see that the parameters in the decoration condition satisfy the molecule condition.

For every $\varepsilon \geq 0$, and every hyperbolic component of the connectedness locus, there are at most finite number of limbs attached to this hyperbolic component with diameter bigger than ε (see [H93]). This implies that for every $\eta > 0$, all the secondary limbs except finite number of them are contained in the η neighborhood of the molecule. This implies that the parameters satisfying the molecule condition also satisfy $\mathcal{SL}$ condition, Therefore, we have the following.

Corollary 3.21. *Let f and $\tilde{f}$ be two infinitely renormalizable quadratic polynomials satisfying the molecule condition. If f and $\tilde{f}$ are combinatorially equivalent, then they are conformally equivalent.*

Chapter 4

Typical Trajectories of complex quadratic polynomials

4.1 Accumulation on the fixed point

4.1.1 Post-critical set as an attractor

Let $f\colon U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Given a point $z \in U$, if $f(z)$ belongs to U we can define $f^2(z) = f \circ f(z)$. Similarly, if $f^2(z)$ also belongs to U, $f^3(z)$ is defined and so on. *Orbit of* z, denoted by $\mathcal{O}(z)$, is the sequence, $z, f(z), f^2(z), \dots$, as long as it is defined. So it might be a finite or an infinite sequence. We say that $\mathcal{O}(z)$ *eventually stays* in a given set $E \subset \hat{\mathbb{C}}$, if there exists an integer k such that $f^i(z) \in E$, for every integer $i \geq k$.

Distortion of a map $f\colon U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the supremum of $\log(|f'(z)/f'(w)|)$ (in the spherical distance) for all z and w in U, which might be finite or infinite.

We say that a simply connected domain $U \subset \mathbb{C}$, different from $\mathbb{C}$, has *bounded eccentricity*, if there exists a univalent onto map $\psi\colon B(0,1) \to U$ (a uniformization) with bounded distortion. One can see that if a simply connected domain U has bounded distortion M, then ratio of radii of smallest circle containing U, and largest circle contained in U, is less than some constant depending only on M.

We frequently use the following distortion theorem due to Koebe [P75] to transfer areas under holomorphic maps.

Theorem 4.1. *(Koebe distortion theorem) Suppose that $f\colon B(0,1) \to \mathbb{C}$ is a univalent function with $f(0) = 0$, and $f'(0) = 1$. For every $z \in B(0,1)$ we have the following estimates*

(1) $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$,

(2) $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$,

(3) $\frac{1-|z|}{1+|z|} \leq |zf'(z)/f(z)| \leq \frac{1+|z|}{1-|z|}$.

This implies the $1/4$ *theorem: The domain* $f(B(0,1))$ *contains* $B(0,1/4)$*.*

The following result in [Ly83b] shows that the post-critical set of a rational map attracts orbit of almost every point in the Julia set.

Proposition 4.2. *Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with $J \neq \hat{\mathbb{C}}$ and V be an arbitrary neighborhood of $\mathcal{PC}(f)$. Then, the orbit of almost every point in the Julia set of f eventually stays in V.*

Here, we give a simple argument based on the Montel's normal family theorem for readers convenience. Also, we will use this approach in the next section, with iterating a renormalization instead of the map itself, to examine the Lebesgue measure of certain post-critical sets.

Proof. For a given domain $V \supset \mathcal{PC}(f)$, define the set

$$\Gamma := \{z \in J \mid \text{for infinitely many integer } k > 0, f^k(z) \notin V\}.$$

If area of Γ is not zero, let z be a Lebesgue density point of Γ. Let n_k be an increasing sequence of positive integers with $f^{n_k}(z) \notin V$, and let y be an accumulation point of the sequence $\langle f^{n_k}(z)\rangle$. As $y \notin V$, it has a definite distance δ from $\mathcal{PC}(f)$. For sufficiently large n_k, let E_{n_k} denote the component of $f^{-n_k}(B(y, \delta/2))$ containing z. As $B(y, \delta/2)$ does not intersect $\mathcal{PC}(f)$, $f^{n_k} \colon E_{n_k} \to B(y, \delta/2)$ is univalent, and in addition, its inverse has a univalent extension over the larger domain $B(y, \delta)$. By the Koebe distortion theorem, all the domains E_{n_k} have bounded eccentricity, and the maps $f^{n_k} \colon E_{n_k} \to B(y, \delta/2)$ have uniformly bounded distortion.

If E_{n_k}'s do not shrink to z as $n_k \to \infty$, their uniformly bounded eccentricity implies that E_{n_k}'s contain a ball $B(z, r)$, for some constant $r > 0$. Thus, every member of the sequence $\langle f^{n_k}\rangle$ maps $B(z, r)$ into $B(y, \delta)$. This implies that $\{f^{n_k}\}$ is a normal family by Montel's theorem, contradicting the choice of z in the Julia set. Therefore, diameter of E_{n_k} tends to 0.

The family E_{n_k} *shrink regularly* to z, i.e. there exists a constant $c > 0$ such that for each E_{n_k}, there exists a round ball B with

$$E_{n_k} \subset B, \text{ and } \operatorname{area}(E_{n_k}) \geq c \cdot \operatorname{area} B.$$

As z is a Lebesgue density point of Γ (and so of J), Lebesgue's Theorem implies that,

$$\lim_{n_k \to \infty} \frac{area(E_{n_k} \bigcap \Gamma)}{area(E_{n_k})} = 1.$$

As f^{n_k}'s have bounded distortion, and Γ is f invariant, we have

$$\lim_{n_k \to \infty} \frac{area(f^{n_k}(E_{n_k} \bigcap \Gamma))}{area(f^{n_k}(E_{n_k}))} = \lim_{n_k \to \infty} \frac{area(B(y, \delta/2) \bigcap \Gamma)}{area(B(y, \delta/2))} = 1.$$

One concludes from the last equality, and that $\Gamma \subseteq J$, to get $B(y, \delta/2) \subseteq J$. This implies that $J = \hat{\mathbb{C}}$, contradicting our assumption. □

4.1.2 The Inou-Shishikura class and the near-parabolic renormalization

Continued fractions: We use a slightly different type of continued fractions defined as follows. Any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ can be written as an accelerated continued fraction of the form:

$$\alpha = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots}}}$$

where $a_n \in \mathbb{Z}$ and $\varepsilon_n = \pm 1$, for $n = 0, 1, 2, \ldots$. For any real number $\alpha \in \mathbb{R}$, define $\|\alpha\| := \min\{|x - n| : n \in \mathbb{Z}\}$. Let $\alpha_0 = \|\alpha\|$ and a_0 be the closest integer to α, so that $\alpha = a_0 \pm \alpha_0$. For every $n \geq 0$, let $\alpha_{n+1} = \|\frac{1}{\alpha_n}\|$ and a_{n+1} be the closest integer to $\frac{1}{\alpha_n}$. Then the signs ε_n are determined by $\frac{1}{\alpha_{n-1}} = a_n + \varepsilon_n \alpha_n$. Note that α_n belongs to $(0, 1/2)$ for every $n \geq 1$.

Consider a map $h\colon \mathrm{Dom}\,(h) \to \mathbb{C}$, where $\mathrm{Dom}\,(h)$ denotes domain of h. Given a compact set $K \subset \mathrm{Dom}\,(h)$ and an $\varepsilon > 0$, a neighborhood of h is defined as

$$\mathcal{N}(h, K, \varepsilon) := \{g\colon \mathrm{Dom}\,(g) \to \mathbb{C} \mid K \subset \mathrm{Dom}\,(g), \text{ and } \sup_{z \in K} |g(z) - h(z)| < \varepsilon\}.$$

By a sequence $h_n\colon \mathrm{Dom}\,(h_n) \to \mathbb{C}$ (not necessarily defined on the same set) *converges* to h we mean that given an arbitrary neighborhood of h defined as above, h_n is contained in this neighborhood for sufficiently large n.

Inou-Shishikura class: Consider the cubic polynomial $P(z) = z(1+z)^2$. This polynomial has a *parabolic* fixed point at 0, a critical point at $-1/3$ which is mapped to the critical value at $-4/27$, and another critical point at -1 which is mapped to 0. See Figure 4.1

Define

$$U := P^{-1}(B(0, \frac{4}{27}e^{4\pi})) \setminus ((-\infty, -1] \cup B) \tag{4.1}$$

where B is the component of $P^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$ containing -1.

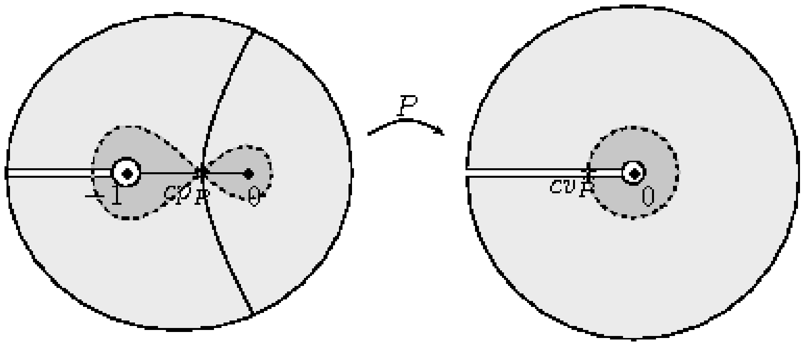

Figure 4.1: A schematic presentation of the Polynomial P, its domain, and range. Similar colors and linestyles describe the map.

Define the class of maps

$$\mathcal{IS} := \left\{ P \circ \varphi^{-1} \colon U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi \colon U \to U_f \text{ is univalent}, \varphi(0) = 0,\ \varphi'(0) = 1, \textit{and} \\ \phi^{-1} \text{ extends onto } \bar{U}_f \text{ as a continuous function} \end{array} \right\}.$$

For a positive real number α_*, consider the following class

$$\mathcal{IS}[\alpha_*] := \{e^{2\pi\alpha\mathbf{i}} \cdot f \mid f \in \mathcal{IS}, \text{ and } \alpha \in [0, \alpha_*]\}.$$

As the class $\mathcal{IS}[\alpha_*]$ is identified with the space of univalent maps on the unit disk with a neutral fixed point at 0, it is a compact class in the above topology.

Any map $h = e^{2\pi\alpha\mathbf{i}} f_0$ in $\mathcal{IS}[\alpha_*]$, with $\alpha \neq 0$ and $f_0 \in \mathcal{IS}$, has a fixed point at 0 with multiplier $e^{2\pi\alpha\mathbf{i}}$, and another fixed point $\sigma_h \neq 0$. The σ_h fixed point has asymptotic expansion $\sigma_h = -4\pi\alpha\mathbf{i}/f_0''(0) + o(\alpha)$, when h converges to f_0 in a fixed neighborhood of 0. Clearly, $\sigma_h \to 0$ as $\alpha \to 0$.

See Figure 4.2 for the contents of the following theorem.

Theorem 4.3. *(Inou-Shishikura [IS06]) There exist a real number* $\alpha_* > 0$*, and positive integers* $k, \hat{k}$ *such that the class* $\mathcal{IS}[\alpha_*]$ *satisfies the following:*

(1) $h''(0) \neq 0$ *for any map* $h \in \mathcal{IS}[\alpha_*]$.

(2) For any map $h \colon U_h \to \mathbb{C}$ *in* $\mathcal{IS}[\alpha_*]$*, there exist a domain* $\mathcal{P}_h \subset U_h$*, bounded by piecewise smooth curves, and a univalent map* $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ *with the following properties:*

(a) $\mathcal{P}_h$ *is compactly contained in* U_h*. Moreover, it contains the critical point* $\mathrm{cp}_h := \varphi(-\frac{1}{3})$ *in its interior as well as* 0 *and* σ_h *on its boundary.*

(b) There exists a continuous branch of argument defined on $\mathcal{P}_h$ *such that*

$$\max_{w,w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi\hat{k}$$

(c) $\Phi_h(\overline{\mathcal{P}_h}) = \{w \in \mathbb{C} \mid 0 \leq \mathrm{Re}(w) \leq \lfloor 1/\alpha \rfloor - k\}$*. Also,* $\mathrm{Im}\,\Phi_h(z) \to +\infty$ *when* $z \in \mathcal{P}_h \to 0$ *and* $\mathrm{Im}\,\Phi_h(z) \to -\infty$ *when* $z \in \mathcal{P}_h \to \sigma_h$.

(d) Φ_h *satisfies the Abel functional equation, that is,*

$$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$

Moreover, Φ_h *is unique once normalized to send* cp_h *to* 0.

(e) The map Φ_h *depends continuously on* h.

We refer to the univalent map Φ_h obtained in the above theorem as *Perturbed Fatou coordinate* or sometimes *Fatou Coordinate* for short.

Remark. Parts (b) and (c) in the above theorem (existence of uniform k and $\hat{k}$) are not stated in [IS06] but it follows from their work. These are consequences of the compactness of the class $\mathcal{IS}[\alpha_*]$ and will become clear by Lemma 4.16 which we prove later.

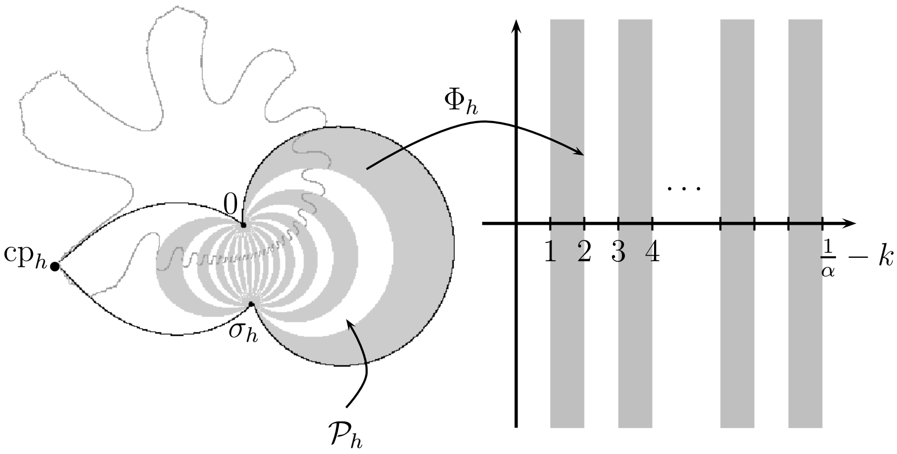

Figure 4.2: An example of a perturbed Fatou coordinate and its domain.

Renormalization: Consider a map $h\colon U_h \to \mathbb{C}$ in $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$ with $\alpha \le \alpha^*$ (α^* is obtained in Theorem 4.3) and let $\Phi_h\colon \mathcal{P}_h \to \mathbb{C}$ be the normalized Fatou coordinate obtained in that theorem. Define

$$\begin{aligned} \mathcal{C} &:= \{z \in \mathcal{P}_h : 1/2 \le \mathrm{Re}(\Phi_h(z)) \le 3/2\,,\ -2 < \mathrm{Im}\,\Phi_h(z) \le 2\}, \text{and} \\ \mathcal{C}^\sharp &:= \{z \in \mathcal{P}_h : 1/2 \le \mathrm{Re}(\Phi_h(z)) \le 3/2\,,\ 2 \le \mathrm{Im}\,\Phi_h(z)\}. \end{aligned} \tag{4.2}$$

By definition, $\mathcal{C}$ contains the critical value of h in its interior, and $\mathcal{C}^\sharp$ contains 0 (fixed point of h) on its boundary. For integers $k > 0$, let $(\mathcal{C}^\sharp)^{-k}$ denote the unique connected component of $h^{-k}(\mathcal{C}^\sharp)$ with 0 on its boundary. Similarly, if there exists a unique connected component of $h^{-k}(\mathcal{C})$ which has non-empty intersection with $(\mathcal{C}^\sharp)^{-k}$, it will be denoted by $\mathcal{C}^{-k}$. Let k_h be the smallest positive integer (if it exists) for which the sets $\mathcal{C}^{-k_h}$ and $(\mathcal{C}^\sharp)^{-k_h}$ are contained in the set

$$\{z \in \mathcal{P}_h \mid 0 < \mathrm{Re}\,\Phi_h(z) < 1/\alpha - k - 1/2\}.$$

We define the set S_h as,

$$S_h := \mathcal{C}^{-k_h} \cup (\mathcal{C}^\sharp)^{-k_h}.$$

Consider the map

$$\Phi_h \circ h^{k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \to \mathbb{C}. \tag{4.3}$$

By equivariance property of Φ_h (Abel functional equation), this map projects via $z = \frac{-4}{27} e^{2\pi \mathbf{i} w}$ to a map of the form $z \mapsto e^{2\pi \frac{-1}{\alpha}\mathbf{i}} z + O(z^2)$, defined on some neighborhood of the origin.

Further conjugating this map by $s\colon z \mapsto \bar{z}$, to make the rotation number at 0 positive, we obtain a map $\mathcal{R}(h)$ of the form $z \mapsto e^{\frac{2\pi}{\alpha}\mathbf{i}} z + O(z^2)$. The map $\mathcal{R}(h)$ is called the *near parabolic renormalization* of h by Inou and Shishikura. We simply refer to it as *renormalization* of h. One can see (Lemma 4.5) that one time iterating $\mathcal{R}(h)$ corresponds to several times iterating the map h, or in other words, many times iterating h is equal to composition of two changes of coordinate and one iterate of $\mathcal{R}(h)$.

The following theorem in [IS06] states that the above definition of near parabolic renormalization $\mathcal{R}$ can be carried out for maps in $\mathcal{IS}$. For a given positive integer N, let Irr_N denote the set of real numbers $\alpha = [a_0, a_1, a_2, \dots]$ with $a_i \geq N$.

Theorem 4.4. *(Inou-Shishikura) There exist an integer $N > 0$ such that if $h \in e^{2\pi\alpha \boldsymbol{i}} \cdot \mathcal{IS}$ with $\alpha \in Irr_N$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}[1/N]$ (i.e. it can be written of the form $\mathcal{R}(h) = e^{\frac{2\pi}{\alpha}\boldsymbol{i}} \cdot P \circ \psi^{-1}$).*

The same conclusion holds for the map $P_\alpha(z) = e^{2\pi\alpha\boldsymbol{i}} z + z^2$, that is, $\mathcal{R}(P_\alpha)$ is well-defined and belongs to $\mathcal{IS}[1/N]$ provided α is small enough and belongs to Irr_N.

Although quadratic polynomials $P_\alpha = e^{2\pi\alpha\mathbf{i}} z + z^2$ do not belong to the class

$\mathcal{IS}[\alpha_*]$, the theorem states that one can define the Fatou coordinate for this map and renormalize it by the above definition. Hence, the theorem guaranties that for α in Irr_N, the sequence of renormalizations

$$f_n := \mathcal{R}^n(P_\alpha) : U_n \to \mathbb{C}$$

are defined and belong to the class $\mathcal{IS}[1/N]$. For simplicity of notation, we let $f_0 = P_\alpha$ and $\alpha_0 = \alpha$. Each map f_n has a fixed point at 0 with multiplier $e^{2\pi\alpha_n i}$.

4.1.3 Sectors around the post-critical set

Here we introduce a sequence of subsets of $\mathbb{C}$ containing 0 on their boundary, such that a.e. z in the Julia set of P_α has to visit these sets. From now on we will assume that N is large enough or $\alpha_* = 1/N$ is small enough so that the class $\mathcal{IS}[1/N]$ satisfies the conditions in Theorems 4.3 and 4.4. Moreover, for technical reasons, we will assume that

$$\alpha_* \leq \frac{1}{k + \hat{k}} \tag{4.4}$$

for k and $\hat{k}$ obtained in Theorem 4.3.

Change of coordinates: For every $n \geq 0$, let $\Phi_n = \Phi_{f_n}$ denote the Fatou coordinate of the map $f_n \colon U_n \to \mathbb{C}$ defined on the set $\mathcal{P}_n := \mathcal{P}_{f_n}$ introduced in Theorem 4.3. By part (b) of Theorem 4.3 and our assumption (4.4), there are holomorphic inverse branches, $\eta_n \colon \mathcal{P}_n \to \mathbb{C}$, of the map

$$\mathbb{E}xp(z) := z \longmapsto \frac{-4}{27} s \circ e^{2\pi \mathbf{i} z} : \mathbb{C} \to \mathbb{C}^*, \text{ where } s(z) = \bar{z},$$

with $\eta_n(\mathcal{P}_n) \subset \Phi_{n-1}(\mathcal{P}_{n-1})$. There may be several choices for this map but we choose one of them for each n and fix this choice whenever we refer to this map.

Now we can define $\psi_n := \Phi_{n-1}^{-1} \circ \eta_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}$. Note that each ψ_n can be continuously extended to 0, on the boundary of $\mathcal{P}_n$, by mapping it to 0. Consider the maps

$$\Psi_n := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n \colon \mathcal{P}_n \to \mathcal{P}_0$$

with values in the dynamical plane of the polynomial f_0.

For each f_n, $n = 0, 1, 2, \ldots$, let $\mathcal{C}_n$ and $\mathcal{C}_n^\sharp$ be the sets obtained in (4.2) for f_n. Let k_n be the smallest positive integer for which $\mathcal{C}_n^{-k_n}$ and $(\mathcal{C}_n^\sharp)^{-k_n}$ are contained in $\mathcal{P}_n$. We define the sector S_n^0 as

$$S_n^0 := \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \mathcal{P}_n.$$

By definition, the critical value of f_n is contained in $f_n^{k_n}(S_n^0)$.

For every $n \geq 0$ define

$$S_n^1 := \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n.$$

In general for $i \geq 2$, let

$$S_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

All these sectors contain 0 on their boundary. We will mainly consider S_0^i, S_1^i, and S_i^0 for $i = 0, 1, 2, \ldots$. See Figure 4.3.

Lemma 4.5. *Let $z \in \mathcal{P}_{n-1}$ be a point with $w := \mathbb{E}xp \circ \Phi_{n-1}(z) \in \mathrm{Dom}\,(f_n)$. There exists an integer ℓ_z with $2 \leq \ell_z \leq \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k + k_{n-1} + 1$ such that*

- $f_{n-1}^{\ell_z}(z) \in \mathcal{P}_{n-1}$,
- $\mathbb{E}xp \circ \Phi_{n-1}(f_{n-1}^{\ell_z}(z)) = f_n(w)$,
- $z, f_{n-1}(z), f_{n-1}^2(z), \ldots, f_{n-1}^{\ell_z}(z) \in \bigcup_{i=0}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1} f_{n-1}^i(S_{n-1}^0)$,

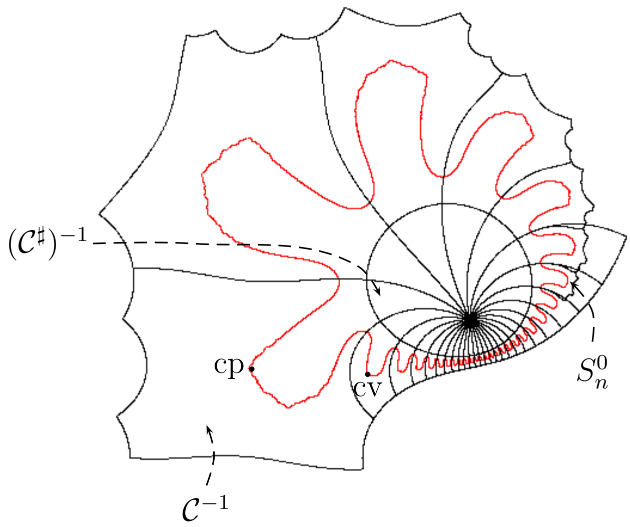

Figure 4.3: Figure shows the first generation of sectors. The grey curve approximates orbit of the critical point.

Moreover, if $w \in \mathrm{int}(\mathrm{Dom}\,(f_n))$, *then*

$$z, f_{n-1}(z), f_{n-1}^2(z), \ldots, f_{n-1}^{\ell_z}(z)$$

belong to the interior of $\bigcup_{i=0}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k-2} f_{n-1}^i(S_{n-1}^0)$

Proof. As $w \in \mathrm{Dom}\,(f_n)$, by definition of renormalization $\mathcal{R}(f_{n-1}) = f_n$, there are

$$\zeta \in \Phi_{n-1}(S_{n-1}^0), \quad \zeta' \in \Phi_{n-1}(\mathcal{P}_{n-1})$$

such that

$$\mathbb{E}xp(\zeta) = w, \quad \mathbb{E}xp(\zeta') = f_n(w), \quad \zeta' = \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\zeta).$$

Since $\mathbb{E}xp(\Phi_{n-1}(z)) = w$ too, and $\zeta \in \Phi_{n-1}(S_{n-1}^0)$, then there exists an integer l_1 with $-k_{n-1} + 1 \le l_1 \le \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k$ such that $\Phi_{n-1}(z) + l_1 = \zeta$ (k is the constant obtained in Theorem 4.3).

By equivariance property of Φ_{n-1}, we have

$$\begin{aligned}\zeta' &= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\zeta) \\ &= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\Phi_{n-1}(z) + l_1) \\ &= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}+l_1} \circ \Phi_{n-1}^{-1}(\Phi_{n-1}(z)) \\ &= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}+l_1}(z).\end{aligned}$$

If we let $\ell_z := k_{n-1} + l_1 + 1$ then we have

$$2 \leq \ell_z \leq k_{n-1} + \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k + 1, \quad f_{n-1}^{\ell_z}(z) = \Phi_{n-1}^{-1}(\zeta' + 1) \in \mathcal{P}_{n-1},$$

and

$$\mathbb{E}xp \circ \Phi_{n-1}(f_{n-1}^{\ell_z}(z)) = \mathbb{E}xp \circ \Phi_{n-1}(\Phi_{n-1}^{-1}(\zeta' + 1)) = \mathbb{E}xp(\zeta' + 1) = f_n(w)$$

This finishes the first two properties. For the last property, we can see that one of the following two occurs

– there exists a positive integer j with $f_{n-1}^{j}(z) \in S_{n-1}^{0}$,

– there exists a non-negative integer j with $z \in f_{n-1}^{j}(S_{n-1}^{0})$.

If the first case occurs (this is when l is positive), then

$$\begin{aligned} z, f_{n-1}(z), \ldots, f_{n-1}^{j-1}(z) &\in \bigcup_{i=k_{n-1}}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1} f_{n-1}^{i}(S_{n-1}^{0}), \\ f_{n-1}^{j}(z), \ldots, f_{n-1}^{\ell_z}(z) &\in \bigcup_{i=0}^{k_{n-1}+1} f_{n-1}^{i}(S_{n-1}^{0}). \end{aligned}$$

If the second case occurs (when l is negative), then

$$z, f_{n-1}(z), \ldots, f_{n-1}^{\ell_z}(z) \in \bigcup_{i=j}^{k_{n-1}+1} f_{n-1}^{i}(S_{n-1}^{0}).$$

The final statement follows from open mapping property of holomorphic maps, that is, image of every open set under a holomorphic map is open. For example, if $w \in \operatorname{int}(\operatorname{Dom}(f_n))$ then $\zeta \in \operatorname{int}(\Phi_{n-1}(S^0_{n-1}))$ which implies that z belongs to the interior of the above union. ☐

In the above lemma there are clearly many choices for ℓ_z. Indeed, there are $\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1$ choices for ℓ_z, however, in the following two lemmas we will make a specific choice of $f^{\ell_z}_{n-1}(z)$ in order to control ℓ_z.

Lemma 4.6. *For every $n \geq 1$, the two maps*

$$f_n \colon \mathcal{P}_n \to f_n(\mathcal{P}_n) \qquad \textit{and} \qquad f_0^{q_n} \colon \Psi_n(\mathcal{P}_n) \to f_0^{q_n}(\Psi_n(\mathcal{P}_n))$$

are conjugate by Ψ_n, that is, the following diagram

$$\begin{array}{ccc} \Psi_n(\mathcal{P}_n) & \xrightarrow{f_0^{q_n}} & f_0^{q_n}(\Psi_n(\mathcal{P}_n)) \\ \Psi_n \uparrow & & \uparrow \Psi_n \\ \mathcal{P}_n & \xrightarrow{f_n} & f_n(\mathcal{P}_n) \end{array}$$

commutes wherever it is defined.

Similarly for every $n > m$, $f_n \colon \mathcal{P}_n \to f_n(\mathcal{P}_n)$ is conjugate to some iterate of f_m defined on the set $\psi_{m+1} \circ \cdots \circ \psi_n(\mathcal{P}_n)$.

Proof. By definition of renormalization $\mathcal{R}$, this property holds near 0 (fixed point). so, by analytic continuation, they hold on their domain of definition. The integers q_n are the closest return times for the rotation of angle α_0 near the 0 fixed point. ☐

Consider the map

$$f_n^{k_n} : S_n^0 \to f_n^{k_n}(S_n^0) \subseteq \mathcal{P}_n.$$

The following lemma translates dynamics of this map to the dynamic plane of f_0.

Lemma 4.7. *The map* $f_n^{k_n}\colon S_n^0 \to f_n^{k_n}(S_n^0)$ *is conjugate to*

$$f_0^{k_n q_n + q_{n-1}}\colon \Psi_n(S_n^0) \to \Psi_n(f_n^{k_n}(S_n^0)),$$

by Ψ_n.

Proof. Proof of this lemma is similar to the previous one. In this case $k_n q_n + q_{n-1}$ is the return time for $\Psi_n(S_n^0)$ back to $\Psi_n(\mathcal{P}_n)$ under f_0. □

On each level $j \geq 0$, we consider union of sectors

$$\Omega_j^0 := \bigcup_{i=0}^{k_j + \lfloor \frac{1}{\alpha_j} \rfloor - k - 1} f_j^i(S_0^j).$$

Using the two previous lemmas, we transfer these sets to the dynamic plane of f_0 to obtain,

$$\Omega_0^n := \bigcup_{i=0}^{q_{n+1} + (k_n - k - 1) q_n} f_0^i(S_0^n),$$

for every $n \geq 0$.

To transfer the sectors in Ω_n^0 from level n to level 0, the first k_n sectors give $k_n q_n + q_{n-1}$ sectors by Lemma 4.7, and the $\lfloor \frac{1}{\alpha_n} \rfloor - k - 1$ remaining ones produce $q_n(\lfloor \frac{1}{\alpha_n} \rfloor - k - 1)$ by Lemma 4.6. Thus, totally we obtain

$$\begin{aligned}(k_n q_n + q_{n-1}) + q_n(\lfloor \frac{1}{\alpha_n} \rfloor - k - 1) &= q_n(\lfloor \frac{1}{\alpha_j} \rfloor + q_{n-1}) + q_n(k_n - k - 1) \\ &\leq q_{n+1} + q_n(k_n - k - 1),\end{aligned}$$

by formula $q_{n+1} = a_{n+1} q_n + q_{n-1}$.

Proposition 4.8. *For every* $n \geq 0$*, we have the following:*

1. Ω_0^{n+1} *is compactly contained in the interior of* Ω_0^n

2. *The post critical set of* f_0 *is contained in the interior of* Ω_0^n

Proof.

Part (1) : To show that $\Omega_0^{n+1} \subset \Omega_0^n$, it is enough to show that for every $z \in S_0^{n+1}$ there are points $z_1, z_2, \dots, z_m$ in S_0^n as well as non-negative integers $t_1, t_2, \dots, t_{m+1}$, for some positive integer m (indeed $m = k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1$), with the following properties:

- $f_0^{t_1}(z_1) = z$ and $f_0^{t_{m+1}}(z_m) = f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(z)$,
- $f_0^{t_j}(z_{j-1}) = z_j$, for $j = 2, 3, \dots, t_{m-1}$,
- $t_j \le q_{n+1} + (k_n - k - 1)q_n$, for every $j = 1, 2, \dots, m+1$.

For a given $z \in S_0^{n+1}$, let $\zeta := \Psi_{n+1}^{-1}(z) \in S_{n+1}^0$. By definition of S_{n+1}^0, the iterates

$$\zeta, f_{n+1}(\zeta), f_{n+1}^2(\zeta), \dots, f_{n+1}^{k_{n+1}+\lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1}(\zeta)$$

are defined and belong to the domain of f_{n+1}.

By Lemma 4.5 for $\psi_{n+1}(\zeta)$, there are two points ξ_1 (take $\xi_1 = \psi_{n+1}(\zeta)$) and ξ_2 in $\mathcal{P}_n$ as well as a positive integer ℓ_0 with $\mathbb{E}xp \circ \Phi_n(\xi_1) = \zeta$, $\mathbb{E}xp \circ \Phi_n(\xi_2) = f_{n+1}(\zeta)$ and $f_n^{\ell_0}(\xi_1) = \xi_2$. Let $\sigma_1 \in S_n^0$ and an integer ℓ_1 with $1 \le \ell_1 \le k_n + a_{n+1} - k - 1$ be such that $f_n^{\ell_1}(\sigma_1) = \xi_1$. With the same lemma, there is a point σ_2 in the orbit

$$\xi_1, f_n(\xi_1), f_n^2(\xi_1), \dots, f_n^{\ell_0 - 1}(\xi_1), \xi_2$$

which belongs to S_n^0. Let ℓ_2 be the positive integer with $1 \le \ell_2 \le k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$ that $f_n^{\ell_2}(\sigma_1) = \sigma_2$.

By the same argument for ξ_2 with $\mathbb{E}xp \circ \Phi_n(\xi_2) = f_{n+1}(\zeta)$, we obtain points $\sigma_3 \in S_n^0$, $\xi_3 \in \mathcal{P}_n$ and a positive integer ℓ_3 with $1 \le \ell_3 \le k_n + a_{n+1} - k - 1$, such that $f_n^{\ell_3}(\sigma_2) = \sigma_3$.

Repeating this argument for $\xi_3, \xi_4, \dots, \xi_m$, for $m = k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1$, one obtains a sequence

$$\sigma_1, \sigma_2, \dots, \sigma_m$$

of points in S_n^0 and positive integers

$$\ell_1, \ell_2, \dots, \ell_{k_{n+1}+\lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1}, \ell_{m+1},$$

all bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$, which satisfy:

- $f_n^{\ell_{j+1}}(\sigma_j) = \sigma_{j+1}$ for all $j = 2, 3, \dots, m-1$
- $f_n^{\ell_1}(\sigma_1) = \xi_1$ and $f_n^{\ell_{m+1}}(\sigma_m) = \xi_m$

Now we define $z_i := \Psi_n(\sigma_i) \in S_0^n$, for $j = 1, 2, \dots, m$. By definition $\Psi_n(\xi_1) = z$. We claim that $\Psi_n(\xi_m) = f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(z)$. This is because

$$\mathbb{E}xp \circ \Phi_n(\xi_m) = f_{n+1}^{k_{n+1}+\lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1}(\zeta)$$

which is mapped to $f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(z)$ by Ψ_{n+1} using Lemmas 4.6 and 4.7 for f_0 and f_{n+1}.

By Lemmas 4.6 and 4.7, ℓ_j times iterating f_n corresponds to t_j times iterating f_0, for each $j = 1, 2, \dots, \ell_{m+1}$. With the same lemmas, as ℓ_j is bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$, each t_j is bounded by

$$k_n q_n + q_{n-1} + (\lfloor \frac{1}{\alpha_n} \rfloor - k - 1) q_n \leq q_{n+1} + q_n(k_n - k - 1).$$

To show that Ω_0^{n+1} is compactly contained in the interior of Ω_0^n, we use the open mapping property of holomorphic maps. So if z' is a point in the closure of Ω_0^{n+1}, as f_0 is a polynomial defined on the whole complex plane, there exists a point z in the closure of S_0^{n+1} with $f_0^{t_0}(z) = z'$, for a non-negative integer t_0 less than $q_{n+2} + (k_n - k - 1)q_{n+1}$. The last statement in Lemma 4.5 implies that all the points σ_j in the above argument can be chosen in the interior of S_n^0. Hence, all the z_i are contained in the interior of S_0^n.

Part (2) : First we claim that for every $n \geq 0$, the critical point of f_0 belongs to Ω_0^n and can be iterated at least $(a_{n+1} - k - 1)q_n$ times with in this set.

To prove the claim, note that $f_n \colon S_n^0 \to f_n^{k_n}(S_n^0)$ is a degree two map. Thus, by Lemma 4.7, $f_0^{k_n q_n + q_{n+1}} \colon S_0^n \to \Psi_n(f_n^{k_n}(S_n^0))$ is also a degree two map. That means that the critical point of f_0 is contained in the union $\cup_{i=0}^{k_n q_n + q_{n-1}} f_0^i(S_0^n)$. Therefore, by definition of Ω_0^n, the critical point can be iterated at least

$$q_{n+1} + (k_n - k - 1)q_n - k_n q_n - q_{n-1} = (a_{n+1} - k - 1)q_n$$

times with in Ω_0^n.

As $a_{n+1} - k - 1 \geq 1$ and q_n growth (exponentially) to infinity as n goes to infinity, combining with part (1), the critical point of f_0 can be iterated infinite number of times with in each Ω_0^n.

For every $n \geq 0$, Ω_0^n contains closure of Ω_0^{n+1} in its interior and Ω_0^{n+1} contains orbit of the critical point. Therefore, the post-critical set is contained in Ω_0^n. □

In the next lemma we show that all the sectors contained in the union Ω_0^n are visited by almost every point in the Julia set of f_0, that is,

Lemma 4.9. *Let n and ℓ be positive integers with $0 \leq \ell \leq q_{n+1} + (k_n - k - 1)q_n$. Then for almost all z in the Julia set of f_0, there exists a non-negative integer ℓ_z such that $f_0^{\ell_z}(z) \in f_0^{\ell}(S_0^n)$.*

Proof. Obviously, it is enough to prove the lemma for $\ell = 0$. We claim that for every $n \geq 0$, the set of points which visit Ω_0^{n+1} contains the set of points which visit S_0^n. Assuming the claim for a moment, because the set of points in the Julia set that visit Ω_0^{n+1} has full measure by Propositions 4.2 and 4.8, we can conclude the lemma.

To prove the claim, let z be an arbitrary point in J for which there exists an integer $t_1 \geq 0$ with $f_0^{t_1}(z) \in \Omega_0^{n+1}$. Let $t_2 \geq t_1$ be a positive integer with

$$f_0^{t_2}(z) \in f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(S_0^{n+1}).$$

Let ξ denote the point $\Psi_n^{-1}(f_0^{t_2}(z))$ in $\mathcal{P}_n$. As $\zeta := \Psi_{n+1}^{-1}(f_0^{t_2}(z)) = \mathbb{E}xp \circ \Phi_n(\xi)$ belongs to $\mathcal{P}_{n+1}$, $f_{n+1}(\zeta)$ is defined. Hence, by Lemma 4.5, there exists a non-negative integer j with the points $\xi, f_n(\xi), f_n^2(\xi), \ldots, f_n^j(\xi)$ contained in $\mathcal{P}_n$ and the last point, $f_n^j(\xi)$, contained in S_n^0. By Lemma 4.6, $f_0^{t_2+q_n j}(z)$ belongs to S_0^n. □

4.1.4 Size of the sectors

Now we want to control size of certain sectors contained in the unions Ω_0^n in terms of Brjuno function. The following two lemmas are our main technical tools. Their proof comes at the end of this section.

Lemma 4.10. *There exists a constant $M \geq 1$ such that for every integer $n \geq 1$ there exists an integer $\tau(n)$ with $k_n \leq \tau(n) \leq a_{n+1} - k - 2$, and*

$$\operatorname{diam}(f_n^{\tau(n)}(S_n^0)) \leq M\alpha_n.$$

Lemma 4.11. *There exists a constant $M \geq 1$ such that for every integer $n \geq 1$, there exists an integer $\kappa(n)$, with $0 \leq \kappa(n) \leq a_{n+1} - k - 1$ that satisfies the following:*

For every $w \in \mathcal{P}_{n+1}$,

(1) $f_n^{\kappa(n)} \circ \psi_{n+1}(w) \in \mathcal{P}_n$,

(2) $|f_n^{\kappa(n)} \circ \psi_{n+1}(w)| \le M\alpha_n |w|^{\alpha_n}$.

From now on we assume that M denotes a constant which satisfies these two lemmas.

Proposition 4.12. *There exists a constant C such that for every $m \ge 1$, there exist non-negative integers $\gamma(m)$ and $\gamma'(m) \le q_{n+1} + (k_n - k - 1)q_n$ for which the following holds*

(1) $\operatorname{diam}(f_1^{\gamma(m)}(S_1^m)) \le C \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \dots \alpha_m^{\alpha_1 \dots \alpha_{m-1}}$.

(2) $f_1^{\gamma(m)}(S_1^m) \subseteq \mathcal{P}_1$

(3) $\psi_1(f_1^{\gamma(m)}(S_1^m)) = f_0^{\gamma'(m)}(S_0^{m+1})$*, that is,* $\psi_1(f_1^{\gamma(m)}(S_1^m))$ *is among the sectors in the union* Ω_0^{m+1}.

Proof. For the constant M obtained for the two Lemmas 4.10 and 4.11, let

$$\begin{aligned} C &= M \cdot M^{\alpha_1} \cdot M^{\alpha_1\alpha_2} \cdot M^{\alpha_1\alpha_2\alpha_3} \dots \\ &= M^{1+\alpha_1+\alpha_1\alpha_2+\alpha_1\alpha_2\alpha_3+\dots} \\ &\le M^{1+1/2+1/2^2+1/2^3+\dots} = M^2 < \infty. \qquad (\text{as } \alpha_i < 1/2) \end{aligned}$$

Given $m \ge 1$, by Lemma 4.10, there exists $\tau(m)$ with $k_m \le \tau(m) \le k_m - k - 2$, and

$$\operatorname{diam}(f_m^{\tau(m)}(S_m^0)) \le M \cdot \alpha_m.$$

By Lemma 4.11 with $n = m-1$, and $w \in f_m^{\tau(m)}(S_m^0)$, we obtain

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)} \circ \psi_m(f_m^{\tau(m)}(S_m^0))) \le M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}}.$$

Now by Lemma 4.6 we have

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)} \circ f_{m-1}^{\tau(m)a_{m+1}+1}(\psi_m(S_m^0)) \le M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}}$$

or equivalently

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)+\tau(m)a_{m+1}+1}(S_{m-1}^1) \leq M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}}.$$

Again applying Lemma 4.11 with $n = m-2$, the last inequality implies that

$$\operatorname{diam}(f_{m-2}^{\kappa(m-2)} \circ \psi_{m-1}(f_{m-1}^{\kappa(m-1)+\tau(m)a_{m+1}+1}(S_{m-1}^1)) \leq \\ M \cdot \alpha_{m-2} \cdot (M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}})^{\alpha_{m-2}}.$$

which is equivalent, by Lemma 4.6, to

$$\operatorname{diam}(f_{m-2}^{\kappa(m-2)+(\kappa(m-1)+\tau(m)a_{m+1}+1)a_m+1}(\psi_{m-1}(S_{m-1}^1))) \leq \\ M \cdot \alpha_{m-2} \cdot (M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}})^{\alpha_{m-2}}.$$

Repeatedly using Lemma 4.11 with $n = m-3, m-4, \ldots, 1$, one obtains

$$\operatorname{diam}(f_1^{\gamma(m)}(S_1^m)) \leq M \cdot \alpha_1 \cdot [M \cdot \alpha_2[M \cdot \alpha_3[\ldots[M \cdot \alpha_m]^{\alpha_{m-1}}]^{\alpha_{m-2}} \ldots]^{\alpha_2}]^{\alpha_1}$$

for some integer $\gamma(m)$. Therefore $f_1^{\gamma(m)}(S_1^m)$ has diameter less than

$$M \cdot M^{\alpha_1} \cdot M^{\alpha_1\alpha_2} \ldots M^{\alpha_1\alpha_2\ldots\alpha_{m-1}} \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \ldots \alpha_m^{\alpha_1\ldots\alpha_{m-1}} \leq \\ C \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \ldots \alpha_m^{\alpha_1\ldots\alpha_{m-1}}.$$

this finishes the first Part of the proposition.

The second part of the proposition follows from above argument when Lemma 4.11 was used with $m = 1$.

To see the third statement in the proposition, first note that $\tau(m)$ is chosen strictly less than $a_{m+1} + k_m - k - 1$. Therefor, $\psi_1(f_1^{\gamma(m)}(S_1^m))$ is among the sectors in the union Ω_0^{m+1}. Indeed, one can see that $\gamma'(m)$ is strictly between $k_m q_m + q_{m-1}$ and $q_{m+1} + (k_m - k - 1)q_m$. □

Lemma 4.13. *The sequence*

$$\{\alpha_1 \alpha_2^{\alpha_1} \alpha_3^{\alpha_1 \alpha_2} \alpha_4^{\alpha_1 \alpha_2 \alpha_3} \dots \alpha_k^{\alpha_1 \dots \alpha_{k-1}}\}_k$$

converges to zero as $k \to \infty$, if and only if the Brjuno sum

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}$$

is divergent.

This is a purely combinatorial lemma and one may refer to [Yoc95] (page 13) for its proof.

Proof of Theorem 1.3. The set of points in the Julia set of f_0 which accumulate on the 0 fixed point is equal to the intersection of the sets

$$A_n = \{z \in J : \mathcal{O}(z) \cap B(0, 1/n) \neq \varnothing\}.$$

for $n = 1, 2, \dots$. To prove the theorem, it is enough to show that every A_n has full Lebesgue measure in the Julia set. As the map $\psi_1 \colon \mathcal{P}_1 \to \mathcal{P}_0$ has continuous extension to the boundary point 0, there exists a $\delta_n > 0$ such that if $|w| < \delta_n$, for some $w \in \mathcal{P}_1$, then $|\psi_1(w)| < 1/n$.

By Lemma 4.13, there exists an integer $m > 0$, for which

$$C \cdot \alpha_1 \alpha_2^{\alpha_1} \alpha_3^{\alpha_1 \alpha_2} \alpha_4^{\alpha_1 \alpha_2 \alpha_3} \dots \alpha_k^{\alpha_1 \dots \alpha_{k-1}}$$

is less than δ_n, where C is the constant obtained in Proposition 4.12. Now by part (1) and (2) of Proposition 4.12, $\psi_1(f_1^{\gamma(m)}(S_1^m))$ is contained in $B(0, 1/n)$. Part (3) of Proposition 4.12 and Lemma 4.9 implies that this set is visited by almost every point in the Julia set of f_0. This completes our proof of Theorem 1.3. □

4.1.5 Perturbed Fatou coordinate

In order to prove Lemmas 4.10 and 4.11 in this section, we will give an approximate formula for the Fatou coordinate Φ_h with a bound on its error.

Assume $h(z) = e^{2\pi\alpha\mathbf{i}} \cdot P \circ \varphi^{-1}(z) \colon \varphi(U) \to \mathbb{C}$ belongs to the class $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$, and σ_h denotes its non-zero fixed point. In [IS06], N was chosen large enough so that $h(z)$ has only two fixed points 0 and σ_h in its domain of definition. Therefore, one can write $h(z)$ as

$$h(z) = z + z(z - \sigma_h)u_h(z)$$

where $u_h(z)$ is a non-zero holomorphic function defined on the set $\varphi(U)$. Differentiating both sides of this equation at 0, one obtains

$$\sigma_h = \frac{1 - e^{2\pi\alpha\mathbf{i}}}{u_h(0)}. \tag{4.5}$$

Note also that the map $u_h(z) = (h(z) - z)/(z(z - \sigma_h))$ depends continuously on the map $h(z)$.

Let

$$\tau_h(w) := \frac{\sigma_h}{1 - e^{-2\pi\mathbf{i}\alpha w}}$$

be the universal covering of the Riemann sphere minus two points 0 and σ_h that has deck transformation group generated by

$$T_\alpha(w) := w + \frac{1}{\alpha}.$$

One can see that $\tau_h(w) \to 0$, as $\operatorname{Im}(\alpha w) \to \infty$, and $\tau_h(w) \to \sigma_h$, as $\operatorname{Im}(\alpha w) \to -\infty$.

Define the map

$$F_h(w) := w + \frac{1}{2\pi\alpha\mathbf{i}} \log\big(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}\big), \qquad \text{with } z = \tau_\alpha(w)$$

on the set of points w with $\tau_h(w) \in \mathrm{Dom}\,(h)$. The branch of log in the above formula is determined by $-\pi < \operatorname{Im}\log(\cdot) < \pi$. The map F_h is defined on the inverse image of $\mathrm{Dom}\,(h)$ under τ_α.

It is immediate calculation to see that

$$h \circ \tau_h = \tau_h \circ F_h, \quad \text{and} \quad T_\alpha \circ F_h = F_h \circ T_\alpha.$$

Indeed, F_h was defined using this relations.

We will see in a moment that the Fatou coordinate of a map in the class $\mathcal{IS}[\alpha_*]$ is "essentially" equal to τ_α for our purposes. Hence, we wish to further control τ_h on certain domains.

For every real number $R > 0$, let $\Theta(R)$ denote the set

$$\Theta(R) := \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} T_\alpha^n(B(0, R)).$$

Lemma 4.14. *There exists a positive constant C_1 such that*

(1) For every $Y > 0$, there exists $\varepsilon_Y > 0$, such that for every $h \in e^{2\pi\alpha i}\mathcal{IS}$, with $\alpha < \varepsilon_Y$, we have

$$\forall w \in \Theta(Y), \quad |\tau_h(w)| \le C_1/Y$$

(2) For every $r \in (0, 1/2)$, $w \in \Theta(\frac{r}{\alpha})$, and every $h \in \mathcal{IS}[\alpha_*]$ we have

$$|\tau_h(w)| \le C_1 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

Proof. The σ_h fixed point has the form (4.5) in terms of u_h, where u_h is a non-zero function in a compact class. Therefore, there exists a constant C' such that for every $h \in e^{2\pi\alpha \mathbf{i}}\mathcal{IS}$, $|\sigma_h| < C'\alpha$. The rest follows from analyzing the explicit formula $1/(1 - e^{-2\pi \mathbf{i}\alpha w})$ on $\Theta\,(Y)$.

To see Part (1), fix an arbitrary $Y > 0$. If $w \in \Theta(Y)$, then $1 - e^{2\pi \mathbf{i}\alpha w}$ belongs to complement of the ball of radius $e^{-2\pi\alpha Y} - 1$ centered at 0 in $\mathbb{C}$. Therefore

$$|\frac{\sigma_h}{1 - e^{-2\pi\alpha Y}}| < \frac{C'\alpha}{3\pi\alpha Y} = \frac{C'}{3\pi Y}$$

for small α.

To see part (2), one can observe that for such a w, $|1 - e^{-2\pi\mathbf{i}\alpha w}| \geq e^{2\pi r} - 1$, and conclude that there exists a constant C'' with

$$|1 - e^{-2\pi\mathbf{i}\alpha w}| \geq C'' r e^{2\pi\alpha \operatorname{Im} w}.$$

This implies that

$$|\tau_h(w)| \leq \frac{C'}{C''}\frac{\alpha}{r}e^{-2\pi\alpha \operatorname{Im} w}.$$

Now we can take C_1 as the maximum of $\frac{C'}{3\pi}$ and $\frac{C'}{C''}$. □

Lemma 4.15. *There exist positive constants $\varepsilon_0 > 0$, C_2, C_3, C_4 as well as a positive integer j_0, such that for every map $h \in e^{2\pi\alpha i}\mathcal{IS}$, with $\alpha \leq \varepsilon_0$, the induced map F_h is defined and is univalent on $\Theta(C_2)$, and moreover*

(1) For all $w \in \Theta(C_2)$, we have

$$|F_h(w) - (w+1)| < 1/4, \quad \textit{and} \quad |F_h'(w) - 1| < 1/4.$$

(2) For every $r \in (0, 1/2)$, and $w \in \Theta(\frac{r}{\alpha} + 1)$, we have

$$|F_h(w) - (w+1)| < C_3\frac{\alpha}{r}e^{-2\pi\alpha \operatorname{Im} w}, \quad \textit{and} \quad |F_h'(w) - 1| < C_3\frac{\alpha}{r}e^{-2\pi\alpha \operatorname{Im} w}.$$

(3) $\mathrm{cp}_h \in B(0,2) \setminus B(0,.22)$. If $i(h)$ is the smallest non-negative integer with $\operatorname{Re} F_h^{i(h)}(\mathrm{cp}_{F_h}) \geq C_2$, then $i(h) \leq \min\{j_0, 1/\alpha\}$.

(4) For every positive integer $j \leq j_0 + \frac{2}{3\alpha}$, we have

$$|\operatorname{Im} F_h^j(\mathrm{cp}_{F_h})| \leq C_4(1 + \log j), \quad \textit{and} \quad |\operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) - j| \leq C_4(1 + \log j).$$

Proof.

Parts (1) *and* (2) : Consider a map $h = e^{2\pi\alpha\mathbf{i}}P \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{C}$ in $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$. As $\mathrm{cp}_P = -1/3 \notin B(0,1/3)$, by Koebe Distortion Theorem, $\mathrm{cp}_h = \phi(-1/3) \notin B(0,1/12)$. So, every h in the above class is defined and univalent on $B(0,1/12)$. Applying part (1) of Lemma 4.14 with $Y = 12C_1$, we obtain an $\varepsilon_0 > 0$ such that if $\alpha \leq \varepsilon_0$, then

$$\tau_h(\Theta(Y)) \subset B(0,1/12).$$

Therefore, the induced map F_h is defined and univalent on $\Theta(Y)$.

For $w \in \Theta(Y)$, using notation $\lambda = e^{2\pi\alpha\mathbf{i}}$, we have

$$\begin{aligned} F_h(w) &= w + \frac{1}{2\pi\alpha\mathbf{i}} \log\Big(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}\Big), \quad \text{with} \quad z = \tau_h(w) \\ &= w + 1 + \frac{1}{2\pi\alpha\mathbf{i}} \log\Big(\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\Big). \end{aligned}$$

Now assume we want to show that

$$|F_h(w) - (w+1)| = \Big|\frac{1}{2\pi\alpha} \log\Big(\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\Big)\Big| < A$$

for some A with $0 < A < 1/4$. As $2\pi\alpha A < 1$, it is enough to prove

$$\frac{1}{2\pi\alpha}\Big|\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}) - 1\Big| < \frac{A}{e}.$$

As $|\lambda| = 1$, it is enough to show that

$$\frac{1}{2\pi\alpha}\Big|1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)} - \lambda\Big| < \frac{A}{e}.$$

Replacing σ_h by its value from equation (4.5), and using $|1 - \lambda| < 2\pi\alpha$, we obtain

$$\frac{1}{2\pi\alpha}\Big|(1-\lambda)(1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)})\Big| < \Big|1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)}\Big|.$$

Since u_h belongs to a compact class, it is possible to make

$$1 - \frac{u_h(z)}{(1 + zu_h(z))u_h(0)} \tag{4.6}$$

less than $\frac{1}{4e}$ by restricting z to a sufficiently small disk of radius δ around 0. So,

$$\left|1 - \frac{u_h(z)}{(1 + zu_h(z))u_h(0)}\right| < \frac{1}{4e} = \frac{1/4}{e}, \text{ on } B(0, \delta).$$

By Lemma 4.14, part (1), there is a constant $C'(\delta) \geq Y$ such that $|z| = |\tau_\alpha(w)| < \delta$ holds for every $w \in \Theta(C'(\delta))$. With this constant $C'(\delta)$ (for C_2), we have the first inequality in part (1) of the lemma.

The second inequality in part (1) follows from the Cauchy estimate (integral formula) applied to $F_h(w) - w - 1$, once we restrict w to smaller domain $\Theta(C'(\delta) + 1)$. Hence, for $C_2 := C'(\delta) + 1$ we have both inequalities.

For the first inequality in Part (2), using Taylor's Theorem for Expression (4.6), one obtains

$$1 - \frac{u_h(z)}{(1 + zu_h(z))u_h(0)} < 2 \cdot |\frac{u_h'(0)}{u_h(0) - 1}||z|.$$

Moreover, as u_h belongs to a compact class,

$$|\frac{u_h'(0)}{u_h(0) - 1}| < D'$$

for some constant D'.

Using part (2) of Lemma 4.14, we conclude that for every $w \in \Theta(\frac{r}{\alpha})$, we have

$$\begin{aligned} |1 - \frac{u_h(z)}{(1 + zu_h(z))u_h(0)}| &< 2D'C_1 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w} \\ &= \frac{\frac{2D'C_1}{e} \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}}{e}. \end{aligned}$$

This proves the first inequality by introducing $C_3 := 2D'C_1/e$.

The other inequality in (2) is also a consequence of Cauchy estimate, once we restrict w to $\Theta(\frac{r}{\alpha}+1)$.

Part (3) : By explicit calculation one can see that $e^{-2\pi\alpha}h$ is univalent on the ball $B(0,1-\sqrt{8/27}e^{-2\pi}) \supset B(0,2/3)$. Koebe distortion Theorem applied to this map on $B(0,2/3)$ implies that $\mathrm{cp}_h \in B(0,2)\setminus B(0,.22)$.

By above argument, there is a choice of cp_{F_h} in $\tau_\alpha^{-1}(\mathrm{cp}_h)$ that belongs to a compact subset of $\mathbb{C}$ (independent of α). Since h converges to maps in the compact class $\mathcal{IS}$ as $\alpha \to 0$, cp_{F_h} visits $\Theta(C_2)$ in a finite number of iterates $i(h)$, uniformly bounded by some constant j_0 independent of h. For the same reason,

$$|F_h^i(\mathrm{cp}_{F_h})| \leq C', \text{ for } i = 0, 1, \ldots i(h)$$

for some constant C'.

Part (4) :

It is enough to prove the inequalities for small values of α. For larger α, there are only finite number of iterates to consider. Therefore, by our previous argument in part (3), the inequalities hold for large enough C_4. So, in the following we assume that $\alpha \leq \min\{\frac{C_2}{3}+\frac{5j_0+5}{24}, \frac{1}{8C_2+41}\}$.

By the first part of this lemma, at each step j with $i(h) \leq j \leq j_0 + \frac{2}{3\alpha}$, we have $F_h^j(\mathrm{cp}_{F_h}) \in \Theta(C_2)$,

$$C' + \frac{3j}{4} \leq \operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) \leq C' + \frac{5}{4} + \frac{5j}{4}.$$

and,

$$-C' - \frac{1}{6\alpha} \leq -C' - \frac{j}{4} \leq \operatorname{Im} F_h^j(\mathrm{cp}_{F_h}) \leq C' + \frac{j}{4} \leq C' + \frac{1}{6\alpha}.$$

Now, one can use part (2) with $r_j = \frac{j\alpha}{6}$ at $F_h^j(\mathrm{cp}_{F_h})$, for $j = i(h), \ldots, j_0 + \frac{2}{3\alpha}$, to obtain:

$$
\begin{aligned}
|F_h(F_h^j(\mathrm{cp}_{F_h})) - F_h^j(\mathrm{cp}_{F_h}) - 1| &\le C_3 \frac{6}{j} e^{2\pi\alpha(C' + \frac{1}{6\alpha})} \\
&\le 6C_3 e^{\pi(C'+\frac{1}{3})} \frac{1}{j}
\end{aligned}
$$

Putting above inequalities together using triangle inequality, we obtain the following estimates for every $j \le j_0 + \frac{2}{3\alpha}$:

$$
\begin{aligned}
|\operatorname{Im} F_h^j(\mathrm{cp}_{F_h})| &\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})} \sum_{m=i(h)}^{j} \frac{1}{m} \\
&\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log j) \\
&\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log \frac{2}{3\alpha}),
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) \le C' + (j - i(h)) + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log j), \text{ and} \\
&\operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) \ge -C' + (j - i(h)) - 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log |j - j_0|)
\end{aligned}
$$

This finishes part (4) by introducing the appropriate constant C_4. □

The following lemma is basically repeating existence of Fatou coordinate in the second part of Theorem 4.3. There is a standard argument based on the measurable Riemann mapping theorem to construct such a coordinate. Since we need to further analyze this coordinate, we repeat this argument in the next lemma.

For real constant $Q > 0$, let Σ_Q denote the set

$$
\begin{aligned}
\Sigma_Q := \{w \in \mathbb{C} : Q \le \operatorname{Re}(w) \le \frac{1}{\alpha} - Q\} \cup \\
\{w \in \mathbb{C} : \operatorname{Re}(w) \le Q, \text{ and } |\operatorname{Im} w| \ge -\operatorname{Re}(w) + 2Q\} \cup \\
\{w \in \mathbb{C} : \operatorname{Re}(w) \ge \frac{1}{\alpha} - Q, \text{ and } |\operatorname{Im} w| \ge \operatorname{Re}(w) - \frac{1}{\alpha} + 2Q\}.
\end{aligned}
$$

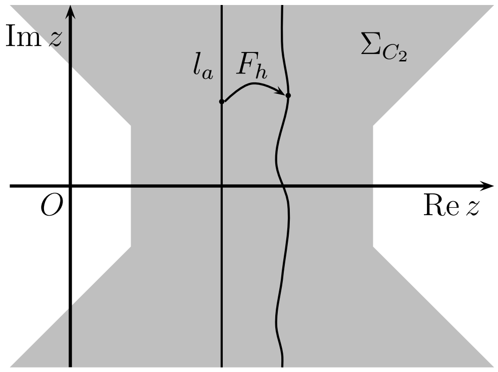

Figure 4.4: The gray region shows the domain Σ_{C_2}.

Lemma 4.16. *For every map $h \in e^{2\pi\alpha \mathbf{i}}\mathcal{IS}$ with α less than ε_0 (obtained in Lemma 4.15), there is a univalent map $L_h\colon \operatorname{Dom}(L_h) \to \mathbb{C}$ with the following properties:*

(1) $\Sigma_{C_2} \cup \{\mathrm{cp}_{F_h}\} \subset \operatorname{Dom}(L_h)$ and

$$\{w \in \mathbb{C} : 0 \leq \operatorname{Re}(w) \leq \lfloor 1/\alpha \rfloor - k\} \subseteq L_h(\operatorname{Dom}(L_h))$$

(same k as in Theorem 4.3).

(2) L_h satisfies

$$L_h(F_h(w)) = L_h(w) + 1 \qquad \text{(Abel functional equation)}$$

whenever both sides are defined. Moreover, L_h is unique once normalized by mapping the critical value of F_h to 1.

Proof. Let l_a denote the vertical line $\{a + \mathbf{i}t : -\infty < t < +\infty\}$, for a in $[C_2, \frac{1}{\alpha} - C_2 - \frac{5}{4}]$. If $\alpha \leq \varepsilon_0$, by Lemma 4.15–(1), image of l_a under F_h does not intersect itself. By the same lemma, the two curves l_a and $F_h(l_a)$ cut the complex plane into three connected components. Denote closure of the one with bounded real part by $\mathcal{K}_h$.

Consider the homeomorphism

$$g\colon \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq 1\} \to \mathcal{K}_h$$

defined as

$$g(s + \mathbf{i}t) := (1 - s)(a + \mathbf{i}t) + sF_h(a + \mathbf{i}t).$$

The partial derivatives of g exist everywhere and can be calculated as

$$\begin{aligned}
\frac{\partial g}{\partial w}(s + \mathbf{i}t) &= \frac{1}{2}[\frac{\partial g}{\partial s} - \mathbf{i}\frac{\partial g}{\partial t}](s + \mathbf{i}t) \\
&= \frac{1}{2}[F_h(a + \mathbf{i}t) - (a + \mathbf{i}t) + 1 + s(F_h'(a + \mathbf{i}t) - 1)], \\
\frac{\partial g}{\partial \bar{w}}(s + \mathbf{i}t) &= \frac{1}{2}[\frac{\partial g}{\partial s} + \mathbf{i}\frac{\partial g}{\partial t}](s + \mathbf{i}t) \\
&= \frac{1}{2}[F_h(a + \mathbf{i}t) - (a + \mathbf{i}t) - 1 + s(1 - F_h'(a + \mathbf{i}t))].
\end{aligned} \tag{4.7}$$

By the inequalities in part (1) of Lemma 4.15, dilatation of the map g, $|g_{\bar{w}}/g_w|$, is bounded by $1/3$. Thus, it is a quasi-conformal map onto $\mathcal{K}_h$. Moreover,

$$\forall w \in l_a,\ g^{-1}(F_h(w)) = g^{-1}(w) + 1.$$

The Beltrami differential

$$\nu(w) := \frac{\partial g/\partial \bar{w}}{\partial g/\partial w}(w)\frac{d\bar{w}}{dw}$$

is the pull back of the standard complex structure on $\mathbb{C}$ by g. Using $\nu(T_1(w)) = \nu(w)$, we can extend $\nu(w)$ over the whole complex plane $\mathbb{C}$. By measurable Riemann mapping theorem ([Ah66], Ch V, Theorem 3), there exists a unique quasi-conformal mapping $g_1\colon \mathbb{C} \to \mathbb{C}$ which solves the Beltrami differential equation $\frac{\partial}{\partial \bar{w}}g_1 = \nu \cdot \frac{\partial}{\partial w}g_1$ and leaves the points 0 and 1 fixed.

As $g_1 \circ T_1 \circ g_1^{-1}$ is quasi-conformal and $\partial(g_1 \circ T_1 \circ g_1^{-1})/\partial\bar{w} = 0$, by explicit calculation, Weyl's Lemma ([Ah66], Ch II, Corollary 2) implies that this map is a conformal map of the complex plane. As it is conjugate to T_1, it can not have any fixed point. Therefore, it is a translation of the plane. Finally, $g_1(0) = 1$ implies that $g_1 \circ T_1 \circ g_1^{-1} = T_1$, or in other words, $g_1(w+1) = g_1(w) + 1$.

For the same reason, the map $L_h := g_1 \circ g^{-1} : \mathcal{K}_h \to \mathbb{C}$ is conformal and by previous arguments satisfies $L_h(F_h(w)) = L_h(w) + 1$ on l_a. This relation can be used to extend L_h on a larger domain. By part (1) of Lemma 4.15, for every $w \in \Sigma_{C_2}$ there is an integer j_w for which $F_h^{j_w}(w) \in \mathcal{K}_h$. Thus domain of L_h contains at least Σ_{C_2} and by definition satisfies the Abel functional equation on its domain of definition.

Note that $L_h(a) = 0$. Given any simply connected domain in $\mathbb{C} \setminus \{0, \sigma_h\}$, there is a continuous inverse branch of τ_h defined on this domain. Further, if image of such a domain under this branch, τ_h^{-1}, is contained in domain of L_h, composition of this branch and L_h is a Fatou coordinate for h. By uniqueness in Theorem 4.3, there exists a constant t_h such that $L_h \circ \tau_\alpha^{-1} + t_h = \Phi_h$ is the unique Fatou coordinate which sends cp_h to zero. This would imply that L_h extends over a larger domain containing cp_{F_h} on its boundary. Moreover, its image contains the set

$$\{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \lfloor 1/\alpha \rfloor - k\}$$

for the constant k obtained in that theorem. □

To control Fatou coordinate of a given map h, which is of the from $L_h \circ \tau_h^{-1}$, we need to control L_h. First we give a rough estimate on derivative of L_h.

Lemma 4.17. *There exists a positive constant C_5 such that for every $h \in e^{2\pi\alpha i}\mathcal{IS}$, and every ζ with $1 \leq \mathrm{Re}\,\zeta \leq 1/\alpha - k$, we have $1/C_5 \leq |(L_h^{-1})'(\zeta)| \leq C_5$.*

Proof. Let $G : (0, 1/\alpha - k) \times (-\infty, \infty) \to \mathbb{C}$, denote the map L_h^{-1} through this proof. We will consider two separate cases. First assume $\xi := G(\zeta) \in \Theta(C_2)$ and $\operatorname{Im} \zeta \in (1.5, 1/\alpha - k - 1.5)$. So, G is defined and univalent on $B(\zeta, 1.5)$ and $F_h(\xi) \in B(\xi + 1, 1/4)$. Now, by 1/4 Theorem, $|G'(\zeta)|/4 \leq (1 + 1/4)$ which implies $G'(\zeta) \leq 5$. For the other direction, by Koebe distortion theorem, we have

$$\forall w \in B(\zeta, 1),\ |G'(w)/G'(\zeta)| \leq 45.$$

By comparing distances $d(\zeta, \zeta + 1)$ and $d(\xi, F_h(\xi))$, we obtain

$$45|G'(\zeta)| \geq 1 \cdot \sup_{w \in B(\zeta,1)} |G'(w)| \geq 1 - 1/4 = 3/4,$$

which implies, $|G'(\zeta)| \geq 1/36$. This proves the lemma in this case.

Now if $\xi \in \Theta(C_2)$ and $\operatorname{Im} L_h(\xi) \in (1, 1/\alpha - k)$. By Abel functional equation in Lemma 4.16, at least one of ξ, $F_h(\xi)$, $F_h^2(\xi)$, $F_h^{-1}(\xi)$, $F_h^{-2}(\xi)$ satisfies above condition. Differentiating Abel functional equation and using Lemma 4.15, part (1), we see

$$3/4 \leq |L_h'(F_h(\xi))|/|L_h'(\xi)| = |F_h'(\xi)| \leq 5/4,$$

which takes care of this case.

Finally, if $\xi \notin \Theta(C_2)$ then ξ belong to a compact subset of $\mathbb{C}$. As the normalized Fatou coordinate L_h is univalent and depends continuously on h in the compact open topology, this case follows from compactness of the class $\mathcal{IS}[\alpha_*]$. □

Finally, the following is our fine control of L_h. Let $C_6 > 1$ be a positive constant that satisfies $C_4(1 + \log \frac{5}{4\alpha}) + 2C_5 \leq C_6/\alpha$.

Lemma 4.18. *There exists a positive constant C_7 such that for every map L_h with $\alpha(h) < \varepsilon_0$, every $r \in (0, 1/2)$, and every $w_1, w_2 \in \text{Dom } L_h$ with*

- $\operatorname{Re} w_1 = \operatorname{Re} w_2$, *and* $\operatorname{Im} w_1, \operatorname{Im} w_2 > -C_6/\alpha$,
- *for all* $t \in (0, 1)$, $tw_1 + (1-t)w_2 \in \Theta(\frac{r}{\alpha} + 1)$,

we have,

(1) $|\operatorname{Re}(L_h(w_1) - L_h(w_2))| \leq C_7/r$

(2) $|\operatorname{Im}(L_h(w_1) - L_h(w_2)) - \operatorname{Im}(w_1 - w_2)| \leq C_7/r$

Proof. Given w_1, w_2 satisfying the conditions in the lemma, choose a vertical line l with $w_1, w_2 \in \mathcal{K}_h$ in the construction of the map g in Lemma 4.16. Let $F_h^{i(h)}(\mathrm{cp}_{F_h})$ be the first visit of cp_{F_h} to Σ_{C_2}, and let $w^* := F_h^{i(h)+j}(\mathrm{cp}_{F_h})$ be the first visit of this point to $\mathcal{K}_h$. By part (1) of Lemma 4.15, $j \leq \frac{1/2\alpha}{3/4} \leq \frac{2}{3\alpha}$. Hence, (4) of the same lemma implies that

$$\operatorname{Im} w^* \in [-C_4(1 + \log \frac{2}{3\alpha}), C_4(1 + \log \frac{2}{3\alpha})]. \tag{4.8}$$

Let t_h be a complex constant with imaginary part in this set and

$$(L_h + t_h)(w^*) = i(h) + j.$$

Then by Abel functional equation we conclude that cp_{F_h} is mapped to 0 under $L_h + t_h$. We will denote this map by the same notation L_h, thus $L_h(w^*) = i(h) + j$.

We have the following simple inequalities for the quasi-conformal map g^{-1} constructed using the choice of vertical line l:

$$|\operatorname{Im}(g^{-1}(w_1) - g^{-1}(w_2)) - \operatorname{Im}(w_1 - w_2)| \leq 1/2,$$
$$|\operatorname{Re}(g^{-1}(w_1)) - \operatorname{Re}(g^{-1}(w_2))| \leq 1/2.$$

To prove similar results for L_h, we will compare it to g^{-1} using Green's integral formula. Choose t_1 and t_2 so that w_1 and w_2 are contained in the curves $s \mapsto g(s + \mathbf{i}t_1)$, and $s \mapsto g(s + \mathbf{i}t_2)$, for $0 \leq s \leq 1$, respectively. Using notations $\zeta = s + \mathbf{i}t$, $d\zeta = ds + \mathbf{i}dt$ and $d\bar{\zeta} = ds - \mathbf{i}dt$, by Green's Theorem applied to the map $g_1(\zeta) = L_h \circ g$ on the rectangle

$$\mathcal{D} := \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Re}(\zeta) \leq 1,\ t_1 \leq \operatorname{Im}\zeta \leq t_2\}$$

we have

$$\int_{\partial\mathcal{D}} g_1(\zeta)\, d\zeta = \iint_{\mathcal{D}} -\frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}. \qquad \text{(Green's formula)}$$

If we let $w = g(\zeta)$, the complex chain rule for $g_1(\zeta)$, using the Cauchy-Riemann equation $\frac{\partial L_h}{\partial \bar{w}} = 0$, can be written as

$$\frac{\partial g_1}{\partial\bar{\zeta}} = \frac{\partial(L_h \circ g)}{\partial\bar{\zeta}} = (\frac{\partial L_h}{\partial w} \circ g)\frac{\partial g}{\partial\bar{\zeta}}.$$

Therefore,

$$\begin{aligned}\Big|\iint_{\mathcal{D}} \frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}\Big| &\leq \int_{t_1}^{t_2}\int_0^1 2\Big|\frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\Big|\, dsdt \\ &\leq \int_{t_1}^{t_2}\int_0^1 \frac{4}{3}\cdot \sup|L_h'| \cdot C_3\frac{\alpha}{r}e^{-2\pi\alpha \operatorname{Im} g(s+\mathbf{i}t)}\, dsdt.\end{aligned}$$

The last inequality follows from (4.7) and Lemma 4.15–(2). By our assumption on w_1 and w_2, the last integral is less than or equal to

$$\begin{aligned}&\int_{-C_6/\alpha}^{\infty} \frac{4}{3}C_5C_3\frac{\alpha}{r}e^{-2\pi\alpha(t-1/4)}\, dt \\ &\leq \frac{4C_5C_3}{3\pi r}e^{-2\pi\alpha(-C_6/\alpha-1/4)} \\ &\leq \frac{4C_5C_3e^{\pi(2C_6+1)}}{3\pi}\frac{1}{r},\end{aligned}$$

which is bounded independent of α.

If we parametrize boundary of $\mathcal{D}$ as

$$\vartheta_1(\ell) := \mathbf{i}\ell,\ \ell \in [t_1, t_2] \qquad \vartheta_2(\ell) := \ell + \mathbf{i}t_2,\ \ell \in [0,1]$$

$$\vartheta_3(\ell) := 1 + \mathbf{i}(t_1 + t_2 - \ell),\ \ell \in [t_1, t_2] \qquad \vartheta_4(\ell) := 1 - \ell,\ \ell \in [0,1]$$

the left hand side of the (Green's formula) can be written as

$$\int_{t_1}^{t_2} g_1(\mathbf{i}\ell)\mathbf{i}\, d\ell + \int_0^1 g_1(\ell + \mathbf{i}t_2)\, d\ell +$$
$$\int_{t_1}^{t_2} g_1(1 + \mathbf{i}(t_1 + t_2 - \ell))(-\mathbf{i})\, d\ell + \int_0^1 -g_1(1-\ell)\, d\ell.$$

Replacing $g_1(\zeta+1)$ by $g_1(\zeta)+1$ and making a change of coordinate in the third integral, we obtain

$$-i(t_2 - t_1) + \int_0^1 g_1(\ell + \mathbf{i}t_2)\, d\ell + \int_0^1 -g_1(1-\ell)\, d\ell.$$

Now we show that the above two integrals are in bounded distance of $L_h(w_2)$ and $-L_h(w_1)$, as follows:

$$\begin{aligned} \Big| \int_0^1 g_1(\ell + \mathbf{i}t_2)\, d\ell - L_h(w_2) \Big| &\leq \int_0^1 |g_1(\ell + \mathbf{i}t_2) - L_h(w_2)|\, d\ell \\ &= \int_0^1 |g_1(\ell + \mathbf{i}t_2) - g_1(\ell_1 + \mathbf{i}t_2)|\, d\ell, \\ &\leq \int_0^1 \sup_{\zeta \in [0,1]+\mathbf{i}t_2} |g_1'(\zeta)|\, d\ell \leq \frac{5}{4} C_5, \end{aligned}$$

for some $\ell_1 \in [0,1]$. Similarly

$$\Big| \int_0^1 -g_1(1-\ell)\, d\ell + L_h(w) \Big| \leq \frac{5}{4} C_5.$$

Now one infers parts (1) and (2) of the lemma by considering real part and imaginary part of Green's formula). □

4.1.6 Proof of main technical lemmas

Proof of Lemma 4.10. It is enough to prove the statement for small values of α_n. That is because the sector $f_n^{k_n}(S_n^0)$ is contained in Dom $f_n \subset B(0, 4/27e^{4\pi})$.

Therefore, it has uniformly bounded diameter. Now, one can choose a large enough M to satisfy the inequality in the lemma once α_n is not too small.

Let L_n denote the linearizing map L_{f_n} corresponding to f_n that can be obtained in Lemma 4.16. Consider the half-line

$$\gamma(t) := F_n^{\lfloor 1/2\alpha_n \rfloor}(\mathrm{cp}_{F_n}) + \mathbf{i}t : [-1/\alpha_n - 4C_7, \infty) \to \mathbb{C}.$$

By part (4) of Lemma 4.15,

$$\operatorname{Re} F_n^{\lfloor 1/2\alpha_n \rfloor}(\mathrm{cp}_{F_n}) \in [\lfloor 1/2\alpha_n \rfloor - C_4(1+\log\frac{1}{2\alpha_n}), \lfloor 1/2\alpha_n \rfloor + C_4(1+\log\frac{1}{2\alpha_n})].$$

Thus, for sufficiently small α_n, one can use Lemma 4.18, with $r = 1/4$, $w_1 = F_n^{\lfloor 1/2\alpha_n \rfloor}(\mathrm{cp}_{F_n})$ and w_2 any point on γ, to conclude that

$$\operatorname{diam}\{\operatorname{Re} L_n(\gamma(t)) - \frac{1}{2\alpha_n} : t \in \operatorname{Dom}\,\gamma\} \leq 4C_7,$$
$$\operatorname{Im} L_n(\gamma(-1/\alpha_n - 4C_7)) \leq -1/\alpha_n.$$

Hence, the set

$$\bigcup_{t \in \operatorname{Dom}\,\gamma} B(L_n(\gamma(t)), 4C_7 + 2)$$

contains the half-strip

$$A := \{\zeta \in \mathbb{C} : \lfloor \frac{1}{2\alpha_n} \rfloor - 1/2 \leq \operatorname{Re}\zeta \leq \lfloor \frac{1}{2\alpha_n} \rfloor + 1/2, \operatorname{Im}\zeta \geq -\frac{1}{\alpha_n}\}. \tag{4.9}$$

Now, by Lemma 4.17, image of this strip under L_n^{-1} must be contained in the set

$$\bigcup_{t \in \operatorname{Dom}\,\gamma} B(\gamma(t), C_5(4C_7 + 2)),$$

which is, by Lemma 4.15 part (4), a subset of the half-strip

$$B := \Big\{\zeta \in \mathbb{C} : |\operatorname{Re}\zeta - \lfloor \frac{1}{2\alpha_n} \rfloor| \leq C_4(1+\log\frac{1}{2\alpha_n}) + C_5(4C_7+2),$$
$$\operatorname{Im}\zeta \geq \frac{-1}{\alpha_n} - 4C_7 - C_4(1+\log\frac{1}{2\alpha_n}) - C_5(4C_7+2)\Big\}.$$

By definition of S_n^0,

$$f_n^{k_n}(S_n^0) = \{z \in \mathcal{P}_n : 1/2 \leq \operatorname{Re} \Phi_n(z) \leq 3/2, \operatorname{Im} \Phi_n(z) \geq -2\}.$$

Equivariance relation, (Theorem 4.3–b)), implies that

$$f_n^{k_n + \lfloor 1/2\alpha_n \rfloor - 1}(S_n^0)$$
$$= \{z \in \mathcal{P}_n : \lfloor \frac{1}{2\alpha_n} \rfloor - 1/2 \leq \operatorname{Re} \Phi_n(z) \leq \lfloor \frac{1}{2\alpha_n} \rfloor + 1/2, \operatorname{Im} \Phi_n(z) \geq -2\}.$$

Since $\Phi_n^{-1} = \tau_n \circ L_n^{-1}$, to conclude the lemma, It is enough to bound $\operatorname{diam} \tau_n(B)$. For small α_n, Lemma 4.14 with $r = 1/4$ applies and we obtain $\operatorname{diam} \tau_n(B) \leq M\alpha_n$, where $M = 4C_1 e^{\pi(2+4C_7+C_5(4C_7+2)+3C_4)}$. We have further shown that:

$$\forall \zeta \in A, \ \ |\tau_n(\zeta)| \leq M\alpha_n \tag{4.10}$$

Which will be used later. □

Proof of Lemma 4.11. If α_n is large and $|w|$ is also bounded below then one can make choose the constant M large enough. So we only consider other cases.

First assume that α_n is small so that the following argument works. Recall that η_{n+1} is an arbitrarily chosen inverse branch of $\mathbb{E}xp$ on $\mathcal{P}_{n+1}$. So we may assume that $\operatorname{Re}(\eta_{n+1}(\mathcal{P}_{n+1}) \subset [0, \hat{k}]$ by Theorem 4.3. If we let $\zeta = \eta_{n+1}(w)$, then $\operatorname{Im} \zeta = \frac{-1}{2\pi} \log \frac{27|w|}{4}$. Now, let $\kappa(n) := \lfloor \frac{1}{2\alpha_n} \rfloor$.

It follows from Lemma 4.15 and 4.18 with $r = 1/4$ (for small enough α_n) that $L_n^{-1}(\zeta + \kappa(n))$ satisfies the following:

$$\frac{1}{4\alpha_n} \leq \operatorname{Re} L_n^{-1}(\zeta + \kappa(n)) \leq \frac{3}{4\alpha_n},$$
$$\operatorname{Im} L_n^{-1}(\zeta + \kappa(n)) \geq \frac{-1}{2\pi} \log \frac{27|w|}{4} - 4C_7 - C_4(1 + \log \frac{1}{2\alpha_n}).$$

Now one uses Lemma 4.14 part (2), with $r = 1/4$, to obtain

$$\begin{aligned}|f_n^{\kappa(n)}(\psi_{n+1}(w))| &= |\tau_n(L_n^{-1}(\zeta + \kappa(n)))| \\ &\leq 4C_1\alpha_n e^{-2\pi\alpha_n \operatorname{Im} L_n^{-1}(\zeta)} \\ &\leq 27C_1 e^{(4C_7+2)\pi} \cdot \alpha_n |w|^{\alpha_n}.\end{aligned}$$

which proves the lemma in this case.

The lemma for larger α_n and sufficiently small $|w|$ follows from compactness of the class $\mathcal{IS}[\alpha_*]$. Indeed, f_n belongs to the class $\mathcal{IS}[\alpha_*]$ with $\alpha_n \in [\varepsilon, \alpha^*]$ for some ε and one can see that the associated map F_n converges geometrically to $z \to z + 1$ as $\operatorname{Im} z \to \infty$. This implies that the linearizing map L_n is bounded away from a translation at points with large imaginary part. Now one uses continuous dependence of linearizing map L_h^{-1} on the map F_h on a compact set $[\varepsilon, \alpha^*] \times [-2, \text{large number}]$ to conclude that the translation constant must have a bounded absolute value. Therefore, if $\zeta = \eta_{n+1}(w)$, then $|\operatorname{Im} L_n^{-1}(\zeta) - \operatorname{Im} \zeta| \leq M'$ for some constant M'. Like above argument this implies that for any choice of $\kappa(n) \in [0, \frac{1}{\alpha_n} - k - 1]$ we have

$$|f_n^{\kappa(n)} \circ \psi_{n+1}(w)| \leq M'' \cdot \alpha_n \cdot |w|^{\alpha_n},$$

for some constant M''. □

A corollary of our proof of Theorem 1.3 is the following.

Corollary 4.19. *Let $P(z) = z(1+z)^2$, U be the domain defined in* (4.1)*, and α be a non-Brjuno number in Irr_N. If h is a rational map of the Riemann sphere with the following properties*

- *$h(0) = 0$, $h'(0) = 1$,*
- *$h(c) \notin U$, if c is a critical point of h.*

Then the rational map $g(z) := e^{2\pi\alpha i} \cdot P \circ h$ *is non-linearizable at* 0.

Note that we do not assume in the above corollary that the corresponding Julia set has positive area.

Proof. Above conditions on h implies that g restricted to U belongs to $\mathcal{IS}[\alpha_*]$. All the sectors S_0^n are defined for g and satisfy the estimates in Proposition 4.12. Thus, the critical point which visits all the sectors, by definition, must accumulate on the fixed point. Hence, g is not linearizable at 0. $\square$

4.2 Measure and topology of the attractor

In this section we consider Lebesgue measure (area) and topology of the post-critical set of quadratic polynomials with a non-Brjuno multiplier of high return times.

We will show that intersection of the sets Ω_0^n, which contains the post-critical set by Proposition 4.8, has area zero, by showing that it does not contain any Lebesgue density point. Strategy of our proof is to show that given any point z in this intersection, one can find balls of arbitrarily small size but comparable to their distance to z in the complement of the intersection. The balls will be introduced in domains of the renormalized maps f_n and then transferred through our changes of coordinates to the dynamic plane of P_α. We will use the Koebe distortion theorem to derive required properties about shape, size, and the distance to z of the image balls.

4.2.1 Balls in the complement

The following lemma guarantees the complementary balls within the domain of each f_n.

Lemma 4.20. *There are positive constants δ_1 and r^* such that for every $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \le \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}$, and $\mathbb{E}xp(\zeta) \in \Omega_{n+1}^0$ for some integer $n \ge 1$, there exists a line segment $\gamma_n : [0,1] \to \mathbb{C}$ with $\gamma_n(0) = \zeta$ satisfying the following properties:*

(1) $\mathbb{E}xp\big(B(\gamma_n(1), r^*)\big) \cap \Omega_{n+1}^0 = \varnothing$, $f_{n+1}\big(\mathbb{E}xp(B(\gamma_n(1), r^*))\big) \cap \Omega_{n+1}^0 = \varnothing$,

(2) $\mathbb{E}xp\big(B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])\big) \subseteq \operatorname{Dom}(f_{n+1}) \setminus \{0\}$,

(3) $\operatorname{diam}\operatorname{Re}\big(B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])\big) \le 1 - \delta_1$,

(4) mod $B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]) \setminus (B(\gamma_n(1), r^*) \cup \gamma_n) \geq \delta_1$.

Proof. First assume $\alpha_{n+1} \leq \min\{\frac{1}{k''+k}, \frac{1}{8(k+1)}, \varepsilon_0\}$, where ε_0 is the constant obtained in Lemma 4.15. Consider the line segment

$$\vartheta(t) := t - (2 + t/2)\mathbf{i} : [2, \frac{1}{2\alpha_{n+1}}] \to \mathbb{C}$$

between the two points $2-3\mathbf{i}$, and $\frac{1}{2\alpha_{n+1}} - (2 + \frac{1}{4\alpha_{n+1}})\mathbf{i}$. For every $t \in [2, \frac{1}{2\alpha_{n+1}}]$, under our assumption $\alpha_{n+1} \leq \frac{1}{4(k+1)}$, we have

$$\begin{aligned} B(\vartheta(t), t/2) &\subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k, \mathrm{Im}\, w \leq -2\}, \\ B(\vartheta(t), t/2) + 1 &\subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k, \mathrm{Im}\, w \leq -2\}. \end{aligned} \tag{4.11}$$

Similarly (when $\alpha_{n+1} \leq 1/8k$) one can see that

$$B(\vartheta(t), 3t/4) \subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k\} = \Phi_{n+1}(\mathcal{P}_{n+1})$$

which gives the following lower bound for conformal modulus:

$$\text{mod } (\Phi_{n+1}(\mathcal{P}_{n+1}) \setminus B(\vartheta(t), t/2)) \geq \frac{1}{2\pi} \log \frac{3}{2}. \tag{4.12}$$

The idea of the proof is to show that lifts of $\Phi_{n+1}^{-1}(B(\vartheta(t), t/2))$ via $\mathbb{E}xp$ provide balls satisfying the required properties in the lemma. First we will consider lifts of the curve $\Phi_{n+1}^{-1} \circ \vartheta$, via $\mathbb{E}xp$ and show that they start from a bounded height and reach the needed height $\frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}}$. Then we consider lifts of $\Phi_{n+1}^{-1}(B(\vartheta(t), t/2))$ and show that they contain balls of a definite size.

Recall that $\Phi_{n+1}^{-1} = \tau_{n+1} \circ L_{n+1}^{-1}$. By Lemma 4.17 we have

$$\begin{aligned} |L_{n+1}^{-1}(\vartheta(2)) - cv_{F_{n+1}}| &\leq \sup |L_{n+1}^{-1}| \cdot |(2, -3) - (1, 0)| \\ &\leq C_5 \sqrt{10}. \end{aligned}$$

Since τ_{n+1} maps the critical value $cv_{F_{n+1}}$ to $-4/27$, one can see that every point in $\mathbb{E}xp^{-1}(\Phi_{n+1}^{-1}(\vartheta(2)))$ has imaginary part uniformly bounded above by some constant δ.

For the other end point, $\vartheta(\frac{1}{2\alpha_{n+1}})$ belongs to the half-strip A defined in (4.9). Thus by (4.10), we have

$$|\Phi_{n+1}^{-1}(\vartheta(\frac{1}{2\alpha_{n+1}}))| \leq M\alpha_{n+1}.$$

This implies that every point in $\mathbb{E}xp^{-1}(\Phi_{n+1}^{-1}(\vartheta(\frac{1}{2\alpha_{n+1}})))$ has imaginary part bigger than $\frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}} - \frac{1}{2\pi}\log\frac{27M}{4}$.

To transfer the balls, consider the map

$$\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1} : \Phi_{n+1}(\mathcal{P}_{n+1}) \to \mathbb{C} \tag{4.13}$$

where η_{n+1} is an arbitrary inverse branch of $\mathbb{E}xp$ defined on $\mathbb{C}$ minus a ray landing at 0.

We claim that there exists a constant M' such that derivative of the above map at every point $t-2\mathbf{i} \in \mathbb{C}$, $t \in \text{Dom}\,\vartheta$, is at least M'/t. By compactness of the class $\mathcal{IS}[\alpha_*]$ and continuous dependence of linearizing map in the compact-open topology it is enough to prove the claim for values of t bigger than some constant (indeed when $C_4(1+\log t) \leq t/2$)). Also by Koebe distortion theorem, it is enough to prove this for integer values of t bigger than that constant. For such t's, $L_{n+1}^{-1}(t) = F_{n+1}^t(cv_{F_{n+1}})$. So by Lemma 4.15 part (4), we have

$$|\operatorname{Im} L_{n+1}^{-1}(t)| \leq C_4(1+\log t), \text{ and } \quad |\operatorname{Re} L_{n+1}^{-1}(t) - t| \leq C_4(1+\log t).$$

Hence, by Lemma 4.17,

$$\begin{aligned} &|\operatorname{Im} L_{n+1}^{-1}(t-2\mathbf{i})| \leq C_4(1+\log t) + 2C_5, \\ &|\operatorname{Re} L_{n+1}^{-1}(t-2\mathbf{i}) - t| \leq C_4(1+\log t) + 2C_5. \end{aligned} \tag{4.14}$$

Define the set O_t as follows:

$$O_t := \{\xi \in \mathbb{C} : |\operatorname{Im}\zeta| \leq C_4(1+\log t)+2C_5, \text{and} |\operatorname{Re}\zeta - t| \leq C_4(1+\log t)+2C_5\}$$

The point t belongs to O_t, and by an explicit calculation one can see than

$$(\eta_{n+1} \circ \tau_{n+1})'(t) \geq 1/2t.$$

As $\operatorname{mod}(\mathbb{C}^* \setminus O_t)$ is bounded below independent of t (indeed, it is increasing in terms of t), Koebe distortion theorem implies that there exists a constant M'' such that for every $\xi \in O_t$, we have

$$(\eta_{n+1} \circ \tau_{n+1})'(\xi) \geq M''/t.$$

Since $L_{n+1}^{-1}(t - 2\mathbf{i}) \in O_t$, by (4.14), combining with Lemma 4.17 we have

$$(\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1})'(t - 2\mathbf{i}) \geq \frac{M''}{C_5}\frac{1}{t}.$$

Again Koebe distortion theorem, using (4.12), implies that

$$\forall \xi \in B(\vartheta(t), t/2) \text{ we have,} \quad (\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1})'(\xi) \geq M'/t$$

for some constant M' independent of t and α_{n+1}. Therefore, image of the ball $B(\vartheta(t), t/2)$ under the map (4.13) contains a ball of constant radius r^* around $\mathbb{E}xp$ pre-images of $\Phi_{n+1}^{-1}(\vartheta(t))$.

Domain of f_{n+1} contains the ball of radius .22, therefore, every point in $\mathbb{C}$ with positive imaginary part is mapped into Dom f_{n+1} under $\mathbb{E}xp$. To associate a curve γ_n to the given point ζ, we consider the following two separate cases:

If $\operatorname{Im}\zeta \geq 1$, by previous argument, there exists a point ζ' in a lift of $\Phi_{n+1}^{-1}(\vartheta)$ (under $\mathbb{E}xp$) which satisfies $\operatorname{Re}(\zeta - \zeta') \leq 1/2$ and $\operatorname{Im}(\zeta - \zeta') \leq \max\{\delta, \frac{1}{2\pi}\log\frac{27M}{4}\}$. Define $\gamma_n : [0,1] \to \mathbb{C}$ as the straight line segment between ζ and ζ' with $\gamma_n(0) = \zeta$ and $\gamma_n(1) = \zeta'$. Thus, $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ projects into Dom f_{n+1}. Moreover, if r^* is chosen less than 1/4, we have

$$\operatorname{diam}(\operatorname{Re}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])) \leq 3/4.$$

Hence, $\mathbb{E}xp$ is univalent on the 1/4 neighborhood of $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$.

Part (1) of the lemma follows from (4.11) and that (when $\alpha_{n+1} \leq \frac{1}{k''+k}$)

$$\bigcup_{j=0}^{k_{n+1}-1} f_{n+1}^j(S_{n+1}^0) \cap \{w \in \mathcal{P}_{n+1} : \operatorname{Im} \Phi_{n+1}(w) < -2\} = \varnothing.$$

Parts (2) and (3) follows form definition.

Now assume that $\operatorname{Im} \zeta$ is uniformly bounded above, or α_{n+1} is bounded below (which implies $\operatorname{Im} \zeta$ is bounded above). In this case our argument is based on the compactness of the class $\mathcal{IS}[\alpha_*]$. Indeed, there exists a $\delta' > 0$ such that

$$B_{\delta'}(\Omega_{n+1}^0) \subset \operatorname{Dom} f_{n+1}. \tag{4.15}$$

Since $\xi = \mathbb{E}xp(\zeta)$ is away from 0 and has uniformly bounded diameter, there exists a (uniformly bounded) real number $s > 1$, with

$$B(s\xi, \delta'/2) \cap \Omega_{n+1}^0 = \varnothing, \text{ and } f_{n+1}(B(s\xi, \delta'/2)) \cap \Omega_{n+1}^0 = \varnothing.$$

Now, define $\gamma'(t) := t\xi + (1-t)s\xi : [0,1] \to \operatorname{Dom} f_{n+1}$. The curve γ_n is defined as the lift of γ' starting at ζ. Thus, $\mathbb{E}xp^{-1}(B(s\xi, \delta'/2))$ contains a ball of radius r^* satisfying the lemma in this case as well. ☐

Lemma 4.21. *There exists a real constant $\delta_2 \leq \delta_1$ such that for every $\xi \in \mathbb{C}$ with $\mathbb{E}xp(\xi) \in \Omega_{n+1}^0$, we have*

- $\mathbb{E}xp(B(\xi, \delta_2)) \subset \operatorname{Dom}(f_{n+1})$,
- $\forall n \in \mathbb{Z},\ \mathbb{E}xp(B(n, \delta_2)) \subset \operatorname{int} f_n^{k_n}(S_n^0) \subset \Omega_n^0$.

Proof. It follows from continuous dependence of the Fatou coordinate in the compact-open topology, that there exists a real constant $\delta > 0$ such that for every $n \geq 0$,

$$B(-4/27, \delta) \subset f_n^{k_n}(S_n^0) = \{\xi \in \mathbb{C} : \operatorname{Im} \xi > -2, 1/2 \leq \operatorname{Re} \xi \leq 3/2\}.$$

The first inclusion in the lemma follows from (4.15) and the second one follows from above observation. □

For every integer $n \geq 1$ and every integer j, with $0 \leq j < \frac{1}{\alpha_n} - k$, we define curves $I_{n,j}$ as follows

$$I_{n,j} := \Phi_n^{-1}\{\xi \in \mathbb{C} : \mathrm{Re}\,\xi = j,\ \mathrm{Im}\,\xi > -2\}.$$

Each $I_{n,j}$ is a smooth curve contained in Ω_n^0, and connects boundary of Ω_n^0 to 0. Also one can see that for every such n and j, every closed loop (image of a continuous curve with the same initial and terminal point) contained in $\Omega_n^0 \setminus I_{n,j}$ is contractible in $\mathbb{C}^*$. This implies that there is a continuous inverse branch of $\mathbb{E}xp$ defined on every $\Omega_n^0 \setminus I_{n,j}$.

By compactness of the class $\mathcal{IS}[\alpha_*]$ there exists a positive integer k' such that

$$\forall j \text{ with } 0 \leq j < \tfrac{1}{\alpha_n} - k, \quad \sup_{z \in \Omega_n^0 \setminus I_{n,j}} \arg(z) \leq 2\pi k', \tag{4.16}$$

for every continuous branch of argument defined on $\Omega_n^0 \setminus I_{n,j}$. We assume the following technical condition on α_n's

$$\alpha_n \leq \frac{1}{2k' + k} \tag{4.17}$$

during this section.

4.2.2 Going down the renormalization tower

Fix an arbitrary point z_0 in $\bigcap_{n=0}^{\infty} \Omega_0^n$ different from 0. We associate a sequence of quadruples

$$\{(z_i, w_i, \zeta_i, \sigma(i))\}_{i=0}^{\infty} \tag{4.18}$$

to z_0, where z_i and w_i are points in $\mathrm{Dom}\,(f_i)$, ζ_i is a point in $\Phi_i(\mathcal{P}_i)$ and $\sigma(i)$ is a non-negative integer. This sequence will serve us as a guide to transfer the balls in the previous lemma to the dynamic plane of f_0.

The sequence of quadruples (4.18) is defined inductively as follows: Since $z_0 \in \cup_{j=0}^{1/\alpha_0-k+k_n-1} f_0^j(S_0^0)$, we have one of the following two possibilities:

Case I $z_0 \in \mathcal{P}_0$, and one of the following two occurs:

– $\mathrm{Re}\,\Phi_0(z_0) \in [k'+1/2, 1/\alpha_0 - k]$,

– $\Phi_0(z_0) \in B(j, \delta_2)$ for some $j = 1, 2, \ldots, k'$

Case II $z_0 \in \mathcal{P}_0$, $\mathrm{Re}\,\Phi_0(z_0) \in [0, k'+1/2)$, and $\Phi_0(z_0) \notin B(j, \delta_2)$, for $j = 1, 2, \ldots, k'$.

Or,

$z_0 \notin \mathcal{P}_0$.

If case I occurs, define $w_0 := z_0$, $\sigma(0) := 0$, and $\zeta_0 := \Phi_0(w_0)$.

If case II occurs, let $w_0 \in S_0^0$, and positive integer $\sigma(0) \leq k_0 + k'$ be such that $f_0^{\sigma(0)}(w_0) = z_0$. The point w_0 satisfying this property is not necessarily unique, however, one can take any of them. The positive integer $\sigma(0)$ is uniquely determined. Indeed when $\sigma(0) \leq k_0 - 1$ or $|z_0|$ is small enough, such w_0 is unique , otherwise, there are at most two choices for w_0. The point ζ_0 is defined as $\Phi_0(w_0)$. This defines the first quadruple $(z_0, w_0, \zeta_0, \sigma(0))$.

Now, let $z_1 := \mathbb{E}xp(\zeta_0)$. Since z_0 belongs to Ω_0^1, One can see that z_1 belongs to $\cup_{j=0}^{1/\alpha_1-k+k_1-1} f_1^j(S_1^0)$. Thus, we can repeat the above process (replacing 0 by 1 in the above cases) to define the quadruple $(z_1, w_1, \zeta_1, \sigma(1))$ and so on. In general, for every $l \geq 0$,

$$\begin{aligned}
&z_l = \mathbb{E}xp(\zeta_{l-1}),\ z_l \in \cup_{j=0}^{1/\alpha_l - k + k_l - 1} f_l^j(S_l^0), \\
&f_l^{\sigma(l)}(w_l) = z_l,\ \Phi_l(w_l) := \zeta_l, \\
&0 \leq \sigma(l) \leq k_l + k' \leq k'' + k'
\end{aligned} \tag{4.19}$$

where k'' is a uniform bound on the integers k_l.

By definition of this sequence, for every $n \geq 0$ we have

$$\begin{aligned}
k' + 1/2 \leq \operatorname{Re}\zeta_n \leq \frac{1}{\alpha_n} - k,\ \text{or}, \\
\zeta_n \in B(j, \delta_2),\ \text{for some } j \in \{1, 2, \ldots, k'\}.
\end{aligned} \tag{4.20}$$

The following lemma guarantees that some of ζ_j in the above sequence reach the balls provided in the previous lemma.

Lemma 4.22. *Assume that $z_0 \in \cap_{n=0}^{\infty} \Omega_0^n \setminus \{0\}$, and α is a non-Brjuno number in Irr_N. If $\{\zeta_j\}_{j=0}^{\infty}$ is the above sequence associated to z_0, then there are arbitrarily large positive integers m with*

$$\operatorname{Im}\zeta_m \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}}.$$

To see this, we need the following lemma. Let D_1 be a constant such that

$$\frac{D_1}{\alpha_{n+1}} \geq \frac{1}{4\alpha_{n+1}} + C_2 + 4C_7 + C_4(1 + \log \frac{1}{\alpha_{n+1}})$$

for every $\alpha_{n+1} \in [1/2, \infty)$, where the constants C_2, C_4, and C_7 were introduced in Lemmas 4.15 and 4.18.

Lemma 4.23. *There exists a positive constant D_2 such that for every $n > 0$, we have*

$$\begin{aligned}
&\textit{if}\ \operatorname{Im}\zeta_{n+1} \geq \frac{D_1}{\alpha_{n+1}}, \\
&\qquad \textit{then}\ \operatorname{Im}\zeta_{n+1} \leq \frac{1}{\alpha_{n+1}} \operatorname{Im}\zeta_n - \frac{1}{2\pi\alpha_{n+1}} \log \frac{1}{\alpha_{n+1}} + \frac{D_2}{\alpha_{n+1}}.
\end{aligned} \tag{4.21}$$

Proof. Given ζ_{n+1} with $\operatorname{Im}\zeta_{n+1} \geq D_1/\alpha_{n+1}$, there is an integer i with $\frac{-1}{\alpha_{n+1}} \leq i \leq \frac{1}{\alpha_{n+1}}$ such that $\operatorname{Re} L_{n+1}^{-1}(\zeta_{n+1}+i) \in [\frac{1}{2\alpha_{n+1}}, \frac{1}{2\alpha_{n+1}}+2]$. By part (4) of Lemma 4.15, and Lemma 4.18, both with $r=1/4$, we obtain

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) \geq \operatorname{Im}(\zeta_{n+1}+i) - 4C_7 - C_4(1+\log\frac{1}{\alpha_{n+1}}).$$

By our assumption on $\operatorname{Im}\zeta_{n+1}$, this implies that

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) \geq \frac{1}{4\alpha_{n+1}} + C_2.$$

Now, one uses part (1) of Lemma 4.15, to conclude that i iterates of $L_{n+1}^{-1}(\zeta_{n+1}+i)$ under F_{n+1} stay in $\Theta(C_2)$, and moreover,

$$\begin{aligned}
\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}) &= \operatorname{Im} F_{n+1}^{-i}(L_{n+1}^{-1}(\zeta_{n+1}+i)) \\
&\geq \operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) - \frac{i}{4} \\
&\geq \operatorname{Im}\zeta_{n+1} - 4C_7 - C_4(1+\log\frac{1}{\alpha_{n+1}}) - \frac{1}{4\alpha_{n+1}}.
\end{aligned}$$

Using Lemma 4.14 with $r=1/4$ at $L_{n+1}^{-1}(\zeta_{n+1})$ implies that

$$|\tau_{n+1}(L_{n+1}^{-1}(\zeta_{n+1}))| \leq 4C_1\alpha_{n+1}e^{-2\pi\alpha_{n+1}\left(\operatorname{Im}\zeta_{n+1}-4C_7-C_4(1+\log\frac{1}{\alpha_{n+1}})-\frac{1}{4\alpha_{n+1}}\right)}.$$

Hence, $\Phi_{n+1}(w_{n+1}) = \zeta_{n+1}$ implies

$$\begin{aligned}
|w_{n+1}| &= |\Phi_{n+1}^{-1}(\zeta_{n+1})| \\
&\leq 4C_1 e^{2\pi\alpha_{n+1}(+4C_7+C_4(1+\log\frac{1}{\alpha_{n+1}})+\frac{1}{4\alpha_{n+1}})}\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}} \\
&\leq C\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}},
\end{aligned}$$

for some constant C.

As w_{n+1} is mapped to z_{n+1} in a bounded number of iterates $\sigma(n)$ under f_{n+1} which belongs to a compact class, $|z_{n+1}| \leq C'|w_{n+1}|$ for some constant C'. Therefore,

$$\begin{aligned}
4/27e^{-2\pi\operatorname{Im}\zeta_n} &= |-4/27e^{-2\pi i\zeta_n}| \\
&= |z_{n+1}| \leq CC'\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}}.
\end{aligned}$$

Multiplying by 27/4 and then taking log of both sides, one obtains Inequality (4.21) for some constant D_2. □

Proof of Lemma 4.22. Given integer $\ell \geq 1$, we will show that there exists $m \geq \ell$ satisfying the inequality in the lemma. For arbitrary α, one of the following two occurs

($*$) There exists a positive integer $n_0 \geq \ell$ such that for every $j \geq n_0$, we have $\operatorname{Im} \zeta_j \geq \frac{D_1}{\alpha_j}$.

($**$) There are infinitely many integers j, $j \geq \ell$, with $\operatorname{Im} \zeta_j < \frac{D_1}{\alpha_j}$.

Assume conclusion of the lemma is not correct, that is, for every m greater than or equal to ℓ we have $\operatorname{Im} \zeta_m > \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}}$. We will show that each of the above cases leads to a contradiction.

If ($*$) holds, we can use Lemma 4.23 for every $j \geq n_0$. So, for every integer n bigger than n_0, using Relation (4.21) repeatedly, we obtain

$$\begin{aligned} \operatorname{Im} \zeta_n \leq & \frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} \operatorname{Im} \zeta_{n_0 - 1} - \frac{1}{2\pi \alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} \log \frac{1}{\alpha_{n_0}} \\ & - \frac{1}{2\pi \alpha_n \alpha_{n-1} \cdots \alpha_{n_0+1}} \log \frac{1}{\alpha_{n_0+1}} \cdots - \frac{1}{2\pi \alpha_n} \log \frac{1}{\alpha_n} \\ & + D_2 \left(\frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} + \frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0 - 1}} + \cdots + \frac{1}{\alpha_n} \right). \end{aligned} \tag{4.22}$$

Let $\beta_{-1} := 1$, and $\beta_j := \alpha_0 \alpha_1 \cdots \alpha_j$, for every $j \geq 0$. Using our contradiction assumption and then multiplying both sides of the above inequality by $2\pi\beta_n$, we see

$$\begin{aligned} \sum_{j=n_0-1}^{n} \beta_j \log \frac{1}{\alpha_{j+1}} & \leq 2\pi \beta_{n_0-1} \operatorname{Im} \zeta_{n_0-1} + 2\pi D_2 \left(\beta_{n_0-1} + \beta_{n_0} + \cdots + \beta_{n-1} \right) \\ & \leq 2\pi \operatorname{Im} \zeta_{n_0-1} + 2\pi D_2. \end{aligned}$$

Since n was an arbitrary integer, this contradicts α being a non-Brjuno number.

Now assume (**) holds. Let $n_1 < m_2 \le n_2 < m_3 \le n_3 < \cdots$ be an increasing sequence of positive integers with the following properties

- For every integer j with $m_i \le j \le n_i$, we have $\operatorname{Im} \zeta_j < \frac{D_1}{\alpha_j}$
- For every integer j with $n_i < j < m_{i+1}$, we have $\operatorname{Im} \zeta_j \ge \frac{D_1}{\alpha_j}$.

Estimate (4.22) holds for $j = n_i + 1, n_i + 2, \cdots, m_{i+1} - 1$, where Lemma 4.23 can be used, and implies that for every $i \ge 2$:

$$\sum_{j=n_i}^{m_{i+1}-1} \beta_j \log \frac{1}{\alpha_{j+1}} \le 2\pi \beta_{n_i} \operatorname{Im} \zeta_{n_i} + 2\pi D_2 \left(\beta_{n_i} + \beta_{n_i+1} + \cdots + \beta_{m_{i+1}-2}\right).$$

Hence,

$$\begin{aligned}
\sum_{j=m_2}^{\infty} \beta_j \log \frac{1}{\alpha_{j+1}} &= \sum_{j;\, m_i \le j < n_i} \beta_j \log \frac{1}{\alpha_{j+1}} + \sum_{j;\, n_i \le j < m_{i+1}} \beta_j \log \frac{1}{\alpha_{j+1}} \\
&\le \sum_{j;\, m_i \le j < n_i} \beta_j \log \frac{1}{\alpha_{j+1}} + 2\pi \sum_{i=2}^{\infty} \beta_{n_i} \operatorname{Im} \zeta_{n_i} + 2\pi D_2 \sum_{j;\, n_i \le j < m_{i+1}-1} \beta_j.
\end{aligned}$$

In the first and the second sums we have used $\frac{1}{2\pi} \log \frac{1}{\alpha_{j+1}} < \operatorname{Im} \zeta_j < \frac{D_1}{\alpha_j}$. Therefore, the whole sum is less than

$$2\pi D_1 \sum_{j;\, m_i \le j < n_i} \beta_{j-1} + 2\pi D_1 \sum_{i=2}^{\infty} \beta_{n_i - 1} + 2\pi D_2 \sum_{j=n_2}^{\infty} \beta_j$$

which contradicts α being a non-Brjuno number. □

4.2.3 Going up the renormalization tower

Recall the sectors $\mathcal{C}_n^{-i} \cup (\mathcal{C}_n^\sharp)^{-i}$, for $i = 1, 2, \ldots, k_n$, introduced in definition of the renormalization (for f_n), where $S_n^0 = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n}$. If $k_n < k' + 1$, by our assumption (4.17) on k', we can consider further pre-images for $i = k_n + 1, \cdots, k' + 1$ as

$$\mathcal{C}_n^{-i} := \Phi_n^{-1}(\Phi_n(\mathcal{C}_n^{-k_n}) - (i - k_n)),$$
$$(\mathcal{C}_n^{\sharp})^{-i} := \Phi_n^{-1}(\Phi_n((\mathcal{C}_n^{\sharp})^{-k_n}) - (i - k_n)).$$

Let $\mathcal{D}_n$ denote the sector $\mathcal{C}_n^{-k'-1} \cup (\mathcal{C}_n^{\sharp})^{-k'-1}$, and observe that $f_n^{k'+1} : \mathcal{D}_n \to f_n^{k_n}(S_n^0)$.

For every integer $n \geq 0$, define the set $\mathcal{P}_n^{\natural}$ as:

$$\mathcal{P}_n^{\natural} := \bigcup_{j=0}^{k'} f_n^j(\mathcal{D}_n).$$

We define a map $\Phi_n^{\natural} : \mathcal{P}_n^{\natural} \to \mathbb{C}$, using the dynamics of f_n, as follows. For $z \in \mathcal{P}_n^{\natural}$, there is an integer j with $0 \leq j \leq k' + 1$, such that $f_n^j(z) \in \mathcal{P}_n$. Now, let

$$\Phi_n^{\natural}(z) := \Phi_n(f_n^j(z)) - j.$$

As Φ_n satisfies the Abel functional equation, one can see that $\Phi_n^{\natural}$ matches on the boundary of above sectors and gives a well defined holomorphic map on $\mathcal{P}_n^{\natural}$. The map $\Phi_n^{\natural}$ is not univalent, however, it still satisfies the Abel Functional equation on $\mathcal{P}_n^{\natural}$. It has critical points at the critical point of f_n and its pre-images within $\mathcal{P}_n^{\natural}$. The $k' + 1$ critical points of $\Phi_n^{\natural}$ are mapped to $0, 1, \ldots, k'$.

The map $\Phi_n^{\natural}$ is a natural extension of Φ_n to a multi-valued holomorphic map on $\mathcal{P}_n^{\natural} \cup \mathcal{P}_n$. However, the two maps

$$\Phi_n^{\natural} : \bigcup_{j=0}^{k'} f_n^j(\mathcal{D}_n) \to \mathbb{C}, \quad \text{and} \quad \Phi_n : \bigcup_{j=k'+1}^{1/\alpha_n + k' - k - 1} f_n^j(\mathcal{D}_n) \to \mathbb{C}$$

provide a well-defined holomorphic map on every $k' + 1$ consecutive sectors of the form $f_n^j(\mathcal{D}_n)$. We denote this map by $\Phi_n^{\natural} \amalg \Phi_n$. More precisely, for every l

with $0 \leq l < \frac{1}{\alpha_n} - k$,

$$\Phi_n^\natural \amalg \Phi_n : \bigcup_{j=0}^{k'} f_n^{l+j}(\mathcal{D}_n) \to \mathbb{C}$$

is defined as

$$\Phi_n^\natural \amalg \Phi_n = \begin{cases} \Phi_n^\natural(z), & \text{if } z \in f_n^i(\mathcal{D}_n) \text{ and } i < k'+1; \\ \Phi_n(z), & \text{if } z \in f_n^i(\mathcal{D}_n) \text{ and } i \geq k'+1. \end{cases}$$

Consider the Sequence (4.18) and assume that $\operatorname{Im} \zeta_n \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}}$ holds for some positive integer n. Let $\mathcal{A}_n$ denote the topological disk $B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ with γ_n and r^* were introduced in Lemma 4.20. We will define domains $V_n, V_{n-1}, \ldots, V_1$ and holomorphic maps $g_{n+1}, g_n, \ldots, g_1$ satisfying the following diagram

$$\mathcal{A}_n \xrightarrow{g_{n+1}} V_n \xrightarrow{g_n} V_{n-1} \xrightarrow{g_{n-1}} \cdots V_1 \xrightarrow{g_1} V_0 := B(\Omega_0^0, 1) \tag{4.23}$$

where,

$$\begin{aligned} &V_m = \Omega_m^0 \setminus I_{m,j(m)}, \text{ for some } j(m) \text{ with } 0 \leq j(m) < \tfrac{1}{\alpha_m} - k, \\ &g_{n+1} : \mathcal{A}_n \to V_n, g_1 : V_1 \to \Omega_0^0, \\ &g_m : V_m \to V_{m-1}, \text{ for every } m \text{ with } 1 \leq m \leq n, \\ &g_m(z_m) = z_{m-1} \text{ for every } m = 1, 2, \ldots, n. \end{aligned} \tag{4.24}$$

The idea is to use an inductive process to define pairs $g_i + 1, V_i$, starting with $i = n$ and ending with $i = 0$.

Base step $i = n$

We have $\zeta_n \in \mathcal{A}_n$ and satisfies (4.20). As $\operatorname{diam}(\mathcal{A}_n) \leq 1 - \delta_1$, and $\delta_2 < \delta_1$, there exists an integer j, with $0 \leq j \leq 1$, such that

$$\operatorname{Re}(\mathcal{A}_n - j) \subset (0, \frac{1}{\alpha_n} - k).$$

We define $g_{n+1} : \mathcal{A}_n \to \mathbb{C}$ as

$$g_{n+1}(\zeta) := f_n^{j+\sigma(n)}(\Phi_n^{-1}(\zeta - j)).$$

By Lemma 4.20, $\mathbb{E}xp(\mathcal{A}_n - j)$ is contained in Dom f_{n+1}. So Lemma 4.5 implies that $f_n^{j+\sigma(n)}$ is defined at every point in $\Phi_n^{-1}(\mathcal{A}_n - j)$, that is, the above map is well defined. Using the same lemma, as $j + \sigma(n) \leq 1 + k_n + k'$ by (4.19), we conclude that $g_n(\mathcal{A}_n)$ is contained in Ω_n^0.

Because $\mathcal{A}_n - j$ is contained in $\operatorname{int} \Phi_n(\mathcal{P}_n)$, it does not intersect the vertical line $\{\xi \in \mathbb{C} : \operatorname{Re} \xi = 0\}$. Therefore, $\Phi_n^{-1}(\mathcal{A}_n - j)$ does not intersect the curve $I_{n,0}$. One can see from this that, $g_{n+1}(\mathcal{A}_n) = f_n^{j+\sigma(n)}(\Phi_n^{-1}(\mathcal{A}_n - j))$ does not intersect the curve $I_{n,j'}$, where j' is $j + \sigma(n)$ module $\lfloor \frac{1}{\alpha_n} \rfloor - k$. We define V_n as $\Omega_n^0 \setminus I_{n,j'}$.

Finally, by equivariance property of Φ_n,

$$g_{n+1}(\zeta_n) = f_n^{j+\sigma(n)}(\Phi_n^{-1}(\zeta_n - j)) = f_n^{\sigma(n)}(w_n) = z_n.$$

Induction step

Assume (g_{i+1}, V_i) is defined and we want to define (g_i, V_{i-1}). Since every closed loop in V_i is contractible in $\mathbb{C}^*$, there exists an inverse branch of $\mathbb{E}xp$, denoted by η_i, defined on V_i with $\eta_i(z_i) = \zeta_{i-1}$. Now consider the following two cases,

case i $\operatorname{Re}(\eta_i(V_i)) \subset (1/2, \infty)$,

case ii $\operatorname{Re}(\eta_i(V_i)) \cap (-\infty, 1/2] \neq \varnothing$.

Case i

Since $\zeta_n \in \eta_i(V_i)$ satisfies (4.20) and $\operatorname{diam} B_{\delta_2}(\eta_i(V_i)) \leq k' + 1/2$ by (4.16), there exists an integer j, $0 \leq j \leq k' + 1$, with

$$B_{\delta_2}(\eta_i(V_i)) - j \subset \{\xi \in \mathbb{C} : \tfrac{1}{4} \leq \operatorname{Re} \xi \leq \tfrac{1}{\alpha_{i-1}} - k - \tfrac{1}{2}\}. \qquad (4.25)$$

We define $\widetilde{g}_i : B_{\delta_2}(\eta_i(V_i)) \to \mathbb{C}$ as

$$\widetilde{g}_i(\zeta) := f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(\zeta - j)), \tag{4.26}$$

and let

$$g_i(z) := \widetilde{g}_i \circ \eta(z).$$

By Lemma 4.21, $\mathbb{E}xp(B_{\delta_2}(\eta_i(V_i)))$ is contained in Dom f_i. Thus, Lemma 4.5 and condition (4.17) implies that $f_{i-1}^{j+\sigma(i-1)}$ is defined on $\Phi_{i-1}^{-1}(B_{\delta_2}(\eta_i(V_i)) - j)$, that is, the above map is well defined.

By equivariance property of Φ_{i-1} (Theorem 4.3), we have

$$g_i(z_i) = f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(\eta_i(z_{i-1}) - j)) = f_{i-1}^{\sigma(i-1)}(w_{i-1}) = z_{i-1}.$$

One also concludes from (4.25) that $B_{\delta_2}(\eta_i(V_i)) - j$ does not intersect the vertical line $\{\xi \in \mathbb{C} : \mathrm{Re}\,\xi = 0\}$. Therefore,

$$\widetilde{g}_i(B_{\delta_2}(\eta(V_i)) - j) = f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(B_{\delta_2}(\eta_i(V_i)) - j))$$

does not intersect the curve $I_{i-1,j'}$, where j' is $j + \sigma(i-1)$ module $\lfloor \frac{1}{\alpha_{i-1}} \rfloor - k$. Hence, by defining $V_{i-1} := \Omega_{i-1}^0 \setminus I_{i-1,j'}$, we have

$$\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i)) - j) \subset V_{i-1}, \tag{4.27}$$

which will be used later.

Case ii

Because $\mathrm{diam}(\eta_i(V_i)) \leq k'$ and $\eta_i(V_i)$ contains ζ_{i-1} which satisfies (4.20), we must have $\zeta_{i-1} \in B(j, \delta_2)$ for some j in $\{1, 2, \ldots, k'\}$.

We claim that

$$B_{\delta_2}(\eta_i(V_i)) \cap \{0, -1, -2, \ldots, -k'\} = \varnothing. \tag{4.28}$$

As $\mathbb{E}xp(\mathbb{Z}) = -4/27$, it is equivalent to see that $\eta_i(-4/27) \notin \{0, -1, \dots, -k'\}$. But by inclusion in the Lemma (4.21), for every integer n, we have

$$\mathbb{E}xp(B(n, \delta_2)) \subset \operatorname{int} V_i.$$

This implies that $\eta_i(-4/27) \in \{1, 2, \cdots, k'\}$.

The set $B_{\delta_2}(\eta_i(V_i))$ has diameter strictly less than $k'+1$, so it can intersect at most $k'+1$ vertical strips of width 1. More precisely, $B_{\delta_2}(\eta_i(V_i))$ is contained in the $k'+1$ consecutive sets in the list

$$\Phi^{\natural}_{i-1}(\mathcal{D}_{i-1}), \Phi^{\natural}_{i-1}(f_{i-1}(\mathcal{D}_{i-1})), \dots, \Phi^{\natural}_{i-1}(f_{i-1}^{k'-1}(\mathcal{D}_{i-1})),$$
$$\Phi_{i-1}(f_{i-1}^{k'}(\mathcal{D}_{i-1})), \dots, \Phi_{i-1}(f_{i-1}^{2k'+1}(\mathcal{D}_{i-1})).$$

Thus, by the above argument about $\Phi^{\natural}_{i-1} \amalg \Phi_{i-1}$, and that every closed loop in $B_{\delta_2}(\eta_i(V_i))$ is contractible in the complement of the critical values of $\Phi^{\natural}_{i-1} \amalg \Phi_{i-1}$, there exists an inverse branch of this map, denoted by $\widetilde{g}_i$, defined on $B_{\delta_2}(\eta_i(V_i))$. We let

$$g_i(z) := \widetilde{g}_i(\eta_i(z)) : V_i \to \Omega^0_{i-1}.$$

In this case $\sigma(i-1) = 0$, $\Phi_{i-1}(w_{i-1}) = \zeta_{i-1}$, and $w_{i-1} = z_{i-1}$. So $g_i(z_i) = z_{i-1}$.

Like previous case, one can see that $\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i)))$ does not intersect the curve $I_{i-1,j}$ for $j = \sup\{\operatorname{Re}(B_{\delta_2}(\eta_i(V_i)))\} + 1$. We can define $V_{i-1} := \Omega^0_{i-1}$ and obtain $g_i : V_i \to V_{i-1}$. Indeed, we have

$$\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i))) \subset V_{i-1}. \tag{4.29}$$

This finishes definition of the domains and maps satisfying (4.24).

Each domain $V_n, V_{n-1} \dots, V_0$, is a hyperbolic Riemann surface. Let ρ_i denote the Poincaré metric on V_i, that is, $\rho_i(z)|dz|$ is the complete metric of constant negative curvature on V_i. Similarly, ρ_{n+1} denotes the Poincare metric on $\mathcal{A}_n$. The following two lemmas are natural consequence of our construction of the chain (4.23).

Lemma 4.24. *Each map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$, for $i = n, n-1, \dots, 1$, is uniformly contracting. More precisely, for every $z \in V_i$, we have*

$$\rho_{i-1}(g_i(z)) \cdot |g_i'(z)| \leq \delta_3 \cdot \rho_i(z),$$

for $\delta_3 = \frac{2k'+1}{2k'+1+\delta_2}$.

Proof. Let $\tilde{\rho}_i(z)|dz|$ and $\hat{\rho}_i(z)|dz|$ denote the Poincaré metric on the domains $\eta_i(V_i)$ and $B_{\delta_2}(\eta_i(V_i))$, respectively. By definition of g_i and properties (4.27) and (4.29) we can decompose the map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$ as follows:

$$(V_i, \rho_i) \xrightarrow{\eta_i} (\eta_i(V_i), \tilde{\rho}_i) \xhookrightarrow{inc.} (B_{\delta_2}(\eta_i(V_i)), \hat{\rho}_i) \xrightarrow{\tilde{g}_i} (V_{i-1}, \rho_{i-1}).$$

By Schwartz-Pick lemma the first map, and the last map are non-expanding, i.e.,

$$\tilde{\rho}_i(\eta_i(\zeta))\,|\eta_i'(\zeta)| \leq \rho_i(\zeta), \text{ and } \rho_{i-1}(\tilde{g}_i(\zeta))\,|\tilde{g}_i'(\zeta)| \leq \hat{\rho}_i(\zeta).$$

To show that the inclusion map is uniformly contracting in the respective metrics, fix an arbitrary point ζ_0 in $\eta_i(V_i)$, and define

$$H(\zeta) := \zeta + (\zeta - \zeta_0)\frac{\delta_2}{(\zeta - \zeta_0 + 2k' + 1)} : \eta_i(V_i) \to \mathbb{C}.$$

Since $\operatorname{diam} \eta_i(V_i) \leq k'$, we have $|\operatorname{Re}(\zeta - \zeta_0))| \leq k'$ for every $\zeta \in \eta_i(V_i)$ and also $H(\zeta_0) = \zeta_0$. This implies that $|\frac{\zeta - \zeta_0}{\zeta - \zeta_0 + 2k'+1}| < 1$. Thus,

$$|H(\zeta) - \zeta| = \delta_2 |\frac{\zeta - \zeta_0}{\zeta - \zeta_0 + 2k' + 1}| < \delta_2,$$

which implies that $H(\zeta)$ is a holomorphic map from $\eta_i(V_i)$ into $B_{\delta_2}(\eta_i(V_i))$. By Schwartz-Pick lemma, H is non-expanding. In particular at ζ_0, we obtain

$$\tilde{\rho}_i(\zeta_0)|H'(\zeta_0)| = \tilde{\rho}_i(\zeta_0)(1 + \frac{\delta_2}{2k' + 1}) \leq \hat{\rho}_i(\zeta_0).$$

That is, $\hat{\rho}_i(\zeta_0) \leq \delta_3 \cdot \tilde{\rho}_i(\zeta_0)$ for $\delta_3 = \frac{2k'+1}{2k'+1+\delta_2} < 1$. Putting all this together gives the inequality in the lemma. ☐

Lemma 4.25. *There exists a positive constant δ_4 such that for every $i = 1, 2, \ldots, n+1$, the following holds*

- *The map $g_i : V_i \to V_{i-1}$ is univalent or has only one simple critical point .*
- *The map $g_i : V_i \to V_{i-1}$ is univalent on the hyperbolic ball*

$$B_{\rho_i}(z_i, \delta_4) := \{z \in V_i \mid d_{\rho_i}(z, z_i) < \delta_4\}.$$

Proof. Each map g_i is composition of at most four maps; η_i, a translation by an integer j, Φ_{i-1}^{-1}, and $f_{i-1}^{j+\sigma(i-1)}$. The first three maps are univalent. The map $f_{i-1}^{j+\sigma(i-1)}$ is univalent or has at most one simple critical in $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$. To see this, first note that the critical points of $f_{i-1}^{j+\sigma(i-1)}$ are

$$\{\mathrm{cp}_{f_{i-1}}, f_{i-1}^{-1}(\mathrm{cp}_{f_{i-1}}), \cdots f_{i-1}^{-j-\sigma(i-1)}(\mathrm{cp}_{f_{i-1}})\}.$$

All of them are non-degenerate and, by our technical assumption, $j+\sigma(i-1) \leq 2k'+1$. If $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$ contains more than one point in the above list, by equivariance property of Φ_{i-1}, there must be a pair of points ξ, $\xi+m$ (for some integer m) in $\eta_i(V_i) - j$. As this set is a lift of a simply connected domain in $\mathbb{C}^*$ under $\mathbb{E}xp$, that is not possible.

The maps g_i introduced in *case ii* are univalent, therefore, to see the second part, we only need to consider maps introduced in *case i* in the above inductive process. First we claim that there exists a real constant $\delta > 0$, such that the ball

$$\{z \in V_{i-1} : d_{\rho_{i-1}}(z, z_{i-1}) < \delta\}$$

is simply connected and does not contain critical value of g_i (if there is any critical value).

Assuming the claim for a moment, one can take $\delta_4 := \delta$. Because, by the previous lemma, image of $B_{\rho_i}(z_i, \delta_4)$ is contained in the above ball. As $B_{\rho_{i-1}}(z_{i-1}, \delta)$ is simply connected and does not contain any critical value, one can find a univalent inverse branch for g_i on this ball. Therefore, g_i is univalent on the ball $B_{\rho_i}(z_i, \delta_4)$.

To prove the claim, note that by definition (4.26) of $\widetilde{g}_{i-1}$, and condition $0 \leq j \leq k' + 1$, a possible critical value of g_{i-1} can only be one of

$$-4/27, f_{i-1}(-4/27), \ldots, f_{i-1}^{k'}(-4/27).$$

First we show that if $\mathrm{cv}_{g_{i-1}}$ belongs to $\Phi_{i-1}^{-1}(B(l, \delta_2))$ for some $l \in \{1, 2, \ldots, k'\}$, then z_{i-1} does not belong to this set. To see this, we consider *case I* and *case II* in the definition of quadruples (4.18) separately.

If *case I* holds, then we have $\sigma(i-1) = 0$, $z_{i-1} = w_{i-1}$, and $\zeta_{i-1} = \Phi_{i-1}(z_{i-1})$. If $\mathrm{Re}\, \zeta_{i-1} \geq k' + 1/2$, then z_{i-1} does not belong to any of the balls $\Phi_{i-1}^{-1}(B(l, \delta_2))$ for $l \in \{1, 2, \ldots, k'\}$. If $\zeta_{i-1} \in B(l, \delta_2)$ for some $l \in \{1, 2 \ldots, k'\}$, then by (4.26) there is no critical value of g_{i-1} in any of $\Phi_{i-1}^{-1}(B(l, \delta_2))$.

If *case II* holds, then, by definition of the quadruples, z_{i-1} does not belong to any of $\Phi_{i-1}(B(l, \delta_2))$ for $l = 1, 2, \ldots, k'$.

Finally, we need to show that each set $\Phi_{i-1}^{-1}(B(l, \delta_2))$, for $l = 1, 2 \ldots, k'$, contains a hyperbolic ball of radius δ independent of l and i. Fix such an l, and observe that

$$\mathrm{mod}\left(\Phi_{i-1}(\mathcal{P}_{i-1}) \setminus \{B(l, \delta_2) \cup \{l - \mathbf{i}t \mid t \in [0, 2]\}\right) \geq c$$

for some constant $c > 0$. By Koebe distortion theorem for Φ_{i-1}^{-1}, we conclude that

$$\frac{\text{Euclidean diameter}\,(\Phi_{i-1}^{-1}(B(l,\delta_2)))}{\text{Euclidean distance between }\Phi_{i-1}^{-1}(l)\text{ and }\partial V_{i-1}} \geq c'$$

As ρ_{i-1} is proportional to one over distance to the boundary in V_{i-1}, the set $\Phi_{i-1}^{-1}(B(l,\delta_2))$ contains a round hyperbolic ball of radius uniformly bounded below. It is clear that each of these balls is simply connected. □

Let $\mathcal{G}_n$ denote the map

$$\mathcal{G}_n := g_1 \circ g_2 \cdots \circ g_{n+1} : \mathcal{A}_n \to \Omega_0^0.$$

Recall that γ_n is the line segment obtained in Lemma 4.20, and $\gamma_n(0) = \zeta_n$. So, $\mathcal{G}_n(\gamma_n(0)) = z_0$. The following lemma guarantees that $\mathcal{G}_n$ safely transfers the ball from level $n+1$ to the dynamic plane of f_0.

Lemma 4.26. *There exists a positive constant D_3 such that for every $\mathcal{G}_n$ introduced above, there is a positive constant r_n with the following properties,*

(1) $\mathcal{G}_n(B(\gamma_n(1), r^*)) \cap \Omega_0^{n+1} = \varnothing$,

(2) $B(\mathcal{G}_n(\gamma_n(1)), r_n) \subset \mathcal{G}_n(B(\gamma_n(1), r^*))$, *and* $|\mathcal{G}_n(\gamma_n(1)) - z_0| \leq D_3 \cdot r_n$,

(3) $r_n \leq D_3(\delta_3)^n$.

Proof.

Part (1): By Lemma 4.20, for every $z \in B(\gamma_n(1), r^*)$ we have

$$\mathbb{E}xp(z) \notin \Omega_{n+1}^0, \text{ and } f_{n+1}(\mathbb{E}xp(z)) \notin \Omega_{n+1}^0.$$

We claim that this implies

$$g_{n+1}(z) \notin \Omega_n^1, \text{ and } f_n(g_{n+1}(z)) \notin \Omega_n^1,$$

where

$$\Omega_n^1 = \bigcup_{i=0}^{\lfloor 1/\alpha_n \rfloor (k_{n+1}+1/\alpha_{n+1}-k-1)+1} f_n^i(\psi_{n+1}(S_{n+1}^0)).$$

That is because if $g_{n+1}(z) \in \Omega_n^1$, then by definition of renormalization and definition of Ω_n^1, there is $a \in \mathcal{P}_n \cap \Omega_n^1$, and $b \in \mathcal{P}_n \cap \Omega_n^1$, such that $f_n^{i_1}(a) = g_{n+1}(z)$, $f_n^{i_2}(g_{n+1}(z)) = b$, $\mathbb{E}xp(\Phi_n(a)) = z$, and $\mathbb{E}xp(\Phi_n(b)) = f_{n+1}(z)$ for non-negative integers i_1 and i_2. One can see from this that $\mathbb{E}xp(\Phi_n(a)) = z \in \Omega_{n+1}^0$ and $\mathbb{E}xp(\Phi_n(b) = f_{n+1}(z) \in \Omega_{n+1}^0$ which contradicts our assumption.

The same argument implies the following statement for every $i = n, n-1, \dots, 1$,

$$\text{If } w \notin \Omega_i^{n-i+1}, \text{ and } f_i(w) \notin \Omega_i^{n-i+1}$$
$$\text{then } g_i(w) \notin \Omega_{i-1}^{n-i+2}, \text{ and } f_{i-1}(g_i(w)) \notin \Omega_{i-1}^{n-i+2}$$

where Ω_l^k is defined accordingly. One infers from these, with an induction argument, that $\mathcal{G}_{n+1}(z) \notin \Omega_{n+1}^0$.

Part (2): It follows from part (4) of Lemma 4.20 that $B\left(\gamma_n(1), r^*\right) \cup \gamma_n[0,1]$ has hyperbolic diameter (with respect to ρ_{n+1} in $\mathcal{A}_n$) uniformly bounded by some constant C (independent of n). Let l be the smallest non-negative integer with

$$C \cdot (\delta_3)^l \leq \delta_4/2.$$

We decompose the map $\mathcal{G}_{n+1}$ into two maps

$$\mathcal{G}_{n+1}^1 := g_{n-l+1} \circ g_{n-l+2} \circ \cdots \circ g_{n+1} \text{ and } \mathcal{G}_{n+1}^2 := g_1 \circ g_{n-l+2} \circ \cdots \circ g_{n-l}.$$

By Lemma 4.24 and our choice of l, we have

$$\mathcal{G}_{n+1}^1(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]) \subseteq B_{\rho_{n-l}}(z_{n-l}, \delta_4/2).$$

Since each map g_i is uniformly contracting and univalent on $B_{\rho_i}(z_i, \delta_4)$, by Lemmas 4.24 and 4.25, $\mathcal{G}_{n+1}^2$ is univalent on $B_{\rho_{n-l}}(z_{n-l}, \delta_4)$. Moreover, by Koebe distortion theorem, it has bounded distortion on $\mathcal{G}_{n+1}^1(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])$.

We claim that $\mathcal{G}^1_{n+1}$ belongs to a compact class. That is because it is composition of l maps (uniformly bounded independent of n) g_i, for $i = n + 1, \cdots, n - l + 1$, where each of them is composition of three maps as

$$g_i = f_i^{\sigma(i)+j} \circ \tilde{g}_i \circ (\eta_i - j).$$

The map η_i is univalent on V_i and, by Koebe distortion theorem, has uniformly bounded distortion on sets of bounded hyperbolic diameter. The map $\tilde{g}_i$ extends over the larger set $B_\delta(\eta_i(V_i))$ (see Equations (4.27) and (4.29)), so it also has uniformly bounded distortion. Finally, $f_i^{\sigma(i)+j}$ is a finite iterate of a map in a compact class.

Putting all these together, one infers that Euclidean diameter of the domain $\mathcal{G}^1_{n+1}(B(\gamma_n(1), r^*))$ is proportional to the Euclidean distance between two points $\mathcal{G}^1_{n+1}(\gamma_n(1))$ and $z_0 = \mathcal{G}_{n+1}(\gamma_n(0))$. Similarly, $\mathcal{G}^1_{n+1}(B_{r^*})$ contains a round ball of Euclidean radius proportional to its diameter. By previous argument about $\mathcal{G}^2_{n+1}$, this finishes part (2) of the lemma.

Part (3): First observe that $\mathcal{G}_{n+1}(B(\gamma_n(1), r^*))$ is contained in the compact subset Ω^0_0 of V_0 where the Euclidean and the hyperbolic metrics are proportional. The uniform contraction in Lemma 4.24 with respect to the hyperbolic metric implies the statement in this part. □

4.2.4 Proof of main results

Corollary 4.27. *There exists a positive integer N such that for every non-Brjuno number $\alpha \in Irr_N$ the post-critical set of P_α has area zero.*

Proof. If $\mathcal{PC}(P_\alpha) = \mathcal{PC}(f_0)$ has positive area, by Lebesgue density point theorem, for almost every point z in this set we must have

$$\lim_{r\to 0} \frac{area(B(z,r) \cap \mathcal{PC}(f_0))}{area B(z,r)} = 1.$$

Let z_0 be an arbitrary point different from zero in $\mathcal{PC}(f_0)$. By Proposition 4.8, z_0 is contained in the intersection of Ω_0^n, for $n = 0, 1, \cdots$. Thus, we can define the sequence of Quadruples (4.18). Lemma 4.22 provides us with an strictly increasing sequence of integers n_i for which we have

$$\operatorname{Im} \zeta_{n_i} \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{n_i+1}}.$$

With Lemma 4.20 at levels n_i, we obtain curves γ_{n_i} and balls $B(\gamma_{n_i}(1), r^*)$ enjoying the properties in that lemma. One introduces the sequence $\mathcal{G}_{n_i+1}$ which, by Lemma 4.26, provides us with a sequence of balls $B(\gamma_{n_i}(1), r_{n_i})$ satisfying

$$B(\gamma_{n_i}(1), r_{n_i}) \cap \Omega^0_{n_i+1} = \varnothing,\ |\mathcal{G}_{n_i+1}(\gamma_{n_i}(1)) - z_0| \leq D_3 \cdot r_{n_i},\ \text{and } r_{n_i} \to 0.$$

With $s_i := r_{n_i} + D_3 \cdot r_{n_i}$ we have

$$\begin{aligned}\frac{\operatorname{area}(B(z_0, s_i)) \cap \mathcal{PC}(f_0))}{\operatorname{area}(B(z_0, s_i))} &\leq \frac{\pi(s_i)^2 - \pi(r_{n_i})^2}{\pi(s_i)^2} \\ &\leq \frac{(D_3)^2 + 2D_3}{(D_3)^2 + 2D_3 + 1} \\ &< 1.\end{aligned}$$

which contradicts z_0 being a Lebesgue density point of $\mathcal{PC}(f_0)$. □

Corollary 4.28. *There exists a positive integer N such that, if $\alpha \in Irr_N$ is a non-Brjuno number, then Lebesgue almost every point in the Julia set of P_α is non-recurrent.*

In particular, there is no finite absolutely continuous (with respect to the Lebesgue measure) invariant measure supported on the Julia set.

Proof. By Propositions 4.8 and 4.2 almost every point in the complement of Ω_0^n is non- recurrent. As area Ω_0^n shrinks to zero, we conclude the first part in the lemma. The second part follows from the first part and the Poincaré recurrence theorem. □

Corollary 4.29. *There exists a positive integer N such that for every non-Brjuno number $\alpha \in Irr_N$ the post-critical set of P_α is connected.*

Proof. We claim that $\mathcal{PC}(P_\alpha) = \bigcap_{n=0}^{\infty} \Omega_n^0$. As each set Ω_n^0 is connected and intersection of a nest of connected sets is connected, the corollary follows from this claim.

To prove the claim, let $z \neq 0$ be an arbitrary point in the above intersection. We can build the sequence of Quadruples (4.18) corresponding to z. By Lemma 4.22, there is an infinite sequence of positive integers n_i satisfying $\operatorname{Im} \zeta_{n_i} \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{n_i+1}}$. It follows from proof of Lemma 4.20 that there exists a point ζ'_{n_i} in the lift $\eta_{n_i}(\Phi_{n_i}^{-1}(\lfloor \frac{1}{2\alpha_{n_i}} \rfloor))$ such that, $|\operatorname{Re}(\zeta_{n_i} - \zeta'_{n_i})| \leq 1/2$, and $|\zeta_{n_i} - \zeta'_{n_i}|$ is uniformly bounded. One transfers these two point to the dynamic plane of f_0 by $\mathcal{G}_{n_i+1}$ and concludes from Lemma 4.24 that $|\mathcal{G}_{n_i+1}(\zeta'_{n_i}) - z|$ goes to zero as n_i tends to infinity. The point $\Phi_{n_i+1}^{-1}(\lfloor \frac{1}{2\alpha_{n_i+1}}) \rfloor)$ belongs to the orbit of critical value of f_{n_i+1}, therefore by definition of renormalization, see Lemma 4.5, $\mathcal{G}_{n_i+1}(\zeta'_{n_i+1})$ belongs to the orbit of the critical value of f_0. Thus, $\mathcal{G}_{n_i+1}(\zeta'_{n_i+1}) \in \mathcal{PC}(f_0)$ and converges to z. This finishes proof of the claim. ☐

A corollary of above proof is the following:

Corollary 4.30. *There are positive constants M, N, and $\mu < 1$ such that for every $\alpha \in Irr_N$ and every $z \in \Omega_0^{n+1}$ we have*

$$\| P_\alpha^{q_n}(z) - z \| \leq M\mu^n.$$

In particular this holds on the post-critical set.

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