Hénon-like Maps

and

Renormalisation

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The following is a dissertation submitted in partial fulfillment of the requirements for the degree Doctor of Philosophy in Mathematics awarded jointly by Rijksuniversiteit Groningen, The Netherlands and Stony Brook University, USA. It has been agreed that neither institution shall award a full doctorate. It has been agreed by both institutions that the following are to be the advisors and reading committee.

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Both institutions agree that the defense of the above degree will take place on Monday 8th December 2008 at 3:00pm at the Academiegebouw, Rijksuniversiteit, Groningen, The Netherlands and that, if successful, the degree Doctor of Philosophy in Mathematics will be awarded jointly by Rector Magnificus, dr. F. Zwarts, RuG, Groningen, and Prof. L. Martin, Graduate School Dean, Stony Brook University.

I hereby agree that I shall always describe the diplomas from Stony Brook University and Rijksuniversiteit Groningen as representing the same doctorate.

Peter Edward Hazard

Abstract of the Dissertation

Hénon-like Maps and Renormalisation

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The aim of this dissertation is to develop a renormalisation theory for the Hénon family

$$F_{a,b}(x,y) = (a - x^2 - by, x) \qquad (\dagger)$$

for combinatorics other than period-doubling in a way similar to that for the standard unimodal family $f_a(x) = a - x^2$. This work breaks into two parts. After recalling background needed in the unimodal renormalisation theory, where a space $\mathcal{U}$ of unimodal maps and an operator $\mathcal{R}_{\mathcal{U}}$ acting on a subspace of $\mathcal{U}$ are considered, we construct a space $\mathcal{H}$ –the strongly dissipative Hénon-like maps– and an operator $\mathcal{R}$ which acts on a subspace of $\mathcal{H}$. The space $\mathcal{U}$ is canonically embedded in the boundary of $\mathcal{H}$. We show that $\mathcal{R}$ is a dynamically-defined continuous operator which continuously extends $\mathcal{R}_{\mathcal{U}}$ acting on $\mathcal{U}$. Moreover the classical renormalisation picture still holds: there exists a unique renormalisation fixed point which is hyperbolic, has a codimension one stable manifold, consisting of all infinitely renormalisable maps, and a dimension one local unstable manifold.

Infinitely renormalisable Hénon-like maps are then examined. We show, as in the unimodal case, that such maps have invariant Cantor sets supporting a unique invariant probability. We construct a metric invariant, the average Jacobian. Using this we study the dynamics of infinitely renormalisable maps around a prescribed point, the 'tip'. We show, as in the unimodal case, universality exists at this point. We also show the dynamics at the tip is non-rigid: any two maps with differing average Jacobians cannot be C^1-conjugated by a tip-preserving diffeomorphism.

Finally it is shown that the geometry of these Cantor sets is, metrically and generically, unbounded in one-parameter families of infinitely renormalisable maps satisfying a transversality condition.

Dedicated to my Mother, Father

Sister and Brother

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Finally I thank my family for being my family, for their patience and understanding, and giving me all the support I needed throughout this adventure. I hope I have made them proud.

Chapter 1

Introduction

1.1 Background on Hénon-like Maps

This work aims to describe some of the dynamical properties of Hénon-like maps. These are maps of the square to itself which 'bend' at a unique place. The prototype for these maps is the Hénon family of maps, given by

$$F_{a,b}(x,y) = (a - x^2 - by, x). \tag{1.1.1}$$

In [24], Hénon gave numerical evidence which suggested, for particular values of parameters[1] a and b, there exists a strange attractor for this map (see the front cover for a picture). Since that time much work has been done in studying the properties of such maps and the bifurcations the family exhibits in the (a,b)-plane.

Showing that the attractor actually existed for certain parameter values turned out to be a significant achievement. This was first done in the work of Benedicks and Carleson [2]. They showed, for a large set of parameters that the unstable manifold is attracting and that it has a definite basin of attraction. Their breakthrough was to compare the dynamics of $F_{a,b}$ with that of the one-dimensional unimodal map $f_a(x) = a - x^2$ (their parametrisation was different but we state the equivalent formulation, see below). The tools they developed in their proof of Jakobson's Theorem (see [2] or [13, Chapter V.6]) allowed them to get very precise information about a specific point whose orbit turns out to be dense in the attractor. We will return with a precise formulation of their results later.

Let us finally remark that this application of the one-dimensional unimodal theory is one of the driving forces in current investigations of these systems. As

[1] Hénon actually studied the family

$$H_{a,b}(x,y) = (1 - ax^2 + y, by) \tag{1.1.2}$$

but the two families are affinely conjugate. He found this interesting behaviour for the parameter values $a = 1.4, b = 0.3$.

far as we are aware this was first suggested by Feigenbaum (see the book [7] by Collet and Eckmann). This is a leitmotif that drives the present work, and one which will be developed in this introduction. Before we describe Hénon-like maps in more detail let us consider the development of dynamics from a more global viewpoint.

1.2 Uniform Hyperbolicity and Topological Dynamics

First let us set up some notation. Given manifolds M and N and any $r = 0, 1, \ldots, \infty, \omega$, let $C^r(M, N)$ denote the space of all C^r-smooth maps from M to N, let $C^r_0(M, N)$ denote the subspace of maps with compact support and let $\mathrm{Emb}^r(M, N)$ denote the subspace of all C^r-embeddings from M to N. We let $\mathrm{End}^r(M)$ denote the space of C^r-endomorphisms of M and we let $\mathrm{Diff}^r(M)$ denote the space of C^r-diffeomorphisms on M. We will denote the usual C^r-norm on $C^r(M, N)$ by $|-|_{C^r(M,N)}$. If the spaces M and N are understood we will simply write $|-|_{C^r}$. In the special case when $r = 0$ and $M = N$, the sup-norm will be denoted $|-|_M$. We will reserve the notation $\|-\|$ or $\|-\|_E$ to denote the operator norm of a linear operator on the Banach space E.

Given $f \in \mathrm{Diff}^r(M)$ we will denote the set of its periodic points by $\mathrm{Per}(f)$ and the the orbit of $x \in M$ under f by $\mathrm{orb}_f(x)$. The set of non-wandering points is denoted by $\Omega(f)$. Given a periodic point $x \in M$ we will denote its stable and unstable manifolds by $W^s(x)$ and $W^u(x)$ respectively.

In the late 1950's Smale initiated the study of uniformly hyperbolic dynamical systems. The aim was to show such systems were generic and structurally stable. If this were shown a reasonable topological or differential topological classification of dynamical systems would be achieved. Systems such as *Morse-Smale*, *Kupka-Smale* and *Axiom A* were considered in detail.

Definition 1.2.1 (Kupka-Smale, Morse-Smale)**.** Let M be a manifold and $f \in \mathrm{Diff}^r(M)$ a diffeomorphism. If f satisfies the following properties,

(i) each $p \in \mathrm{Per}(f)$ is hyperbolic;

(ii) $W^u(p) \pitchfork W^s(q)$ for each $p, q \in \mathrm{Per}(f)$;

then we say f is a *Kupka-Smale* diffeomorphism on M. If f satisfies the additional properties,

(iii) $\mathrm{Per}(f)$ has finite cardinality;

(iv) $\bigcup_{p \in \mathrm{Per}(f)} W^s(p) = M$;

(v) $\bigcup_{p \in \mathrm{Per}(f)} W^u(p) = M$;

then we say f is a *Morse-Smale* diffeomorphism on M.

Definition 1.2.2 (Axiom A)**.** Let M be a manifold and $f \in \mathrm{Diff}^r(M)$ a diffeomorphism. If f satisfies the following properties,

(i) the nonwandering set $\Omega(f)$ is hyperbolic;

(ii) $\mathrm{Per}(f)$ is dense in $\Omega(f)$;

then we say f is an *Axiom A* diffeomorphism on M.

The hope was, for a long time, that, Axiom A maps would be dense. This was shown not to be the case, most conclusively by Newhouse. The following two results were shown by him in [39] and [40]. We refer the reader to chapter 6 of the book [42] by Palis and Takens for more details.

Theorem 1.2.3 (Newhouse)**.** *For any two dimensional manifold M there exists an open set $U \subset \mathrm{Diff}^2(M)$, and a dense subset $B \subset U$ such that every map $f \in B$ possesses a homoclinic tangency.*

Theorem 1.2.4 (Newhouse)**.** *For any two dimensional manifold M, and any $r \geq 2$, there exists an open set $U \subset \mathrm{Diff}^r(M)$ and a residual subset $B \subset U$ such that every map $f \in B$ has infinitely many hyperbolic periodic attractors.*

Let us also recall the following result of Katok, which acts as a nice counterpoint to the first of these two theorems.

Theorem 1.2.5 (Katok)**.** *For any compact two dimensional manifold M, let $f \in \mathrm{Diff}^{1+\alpha}(M)$ preserve the Borel probability measure μ and also satisfy the following properties,*

(i) the support of μ is not concentrated on a single periodic orbit;

(ii) μ is f-ergodic;

(iii) f has non-zero characteristic exponents with respect to μ;

then f having a transversal homoclinic point implies $h_{top}(f) > 0$, where $h_{top}(f)$ denotes the topological entropy of f.

This shows that the dense set B constructed by Newhouse lives close to the region of 'chaotic' maps. We will consider this in more detail later when outlining the renormalisation picture.

1.3 Non-Uniform Hyperbolicity and Measurable Dynamics

In the late 1960's Oseledets and Pesin, among others, initiated the study of non-uniformly hyperbolic systems, i.e. ones for which the tangent bundle does not split into factors which contract or expand at a uniform rate. The key observation was that it was the asymptotic behaviour of the action of f on elements of the tangent bundle that was significant. By considering the long term behaviour only it was discovered that there still existed a splitting, but a measure zero set of "irregular" points needed to be removed first. More precisely, Oseledets proved the following Theorem, for a proof we refer the reader to the book [31] of Mañé.

Theorem 1.3.1 (Oseledets)**.** *Let M be smooth, compact, boundary-free Riemannian manifold of dimension n. Let $f \in \mathrm{Diff}(M)$ and for each $p \in M$ let E_p^λ denote the subspace of T_pM whose elements have characteristic exponent λ. Then there exists an f-invariant Borel subset $R \subset M$ and for each $\varepsilon > 0$ a Borel function $r_\varepsilon \colon R \to (1, \infty)$ such that for all $p \in R, v \in E_p^\lambda$ and each integer n, the following properties hold,*

(i) $\bigoplus_\lambda E_p^\lambda = T_pM$;

(ii) $\frac{1}{r_\varepsilon(p)(1+\varepsilon)^{|n|}} \leq \frac{\|\mathrm{D}_p f^{\circ n}(v)\|}{\lambda^n \|v\|} \leq r_\varepsilon(p)(1+\varepsilon)^{|n|}$;

(iii) $\angle(E_p^\Lambda, E_p^{\Lambda'}) \geq r_\varepsilon(p)^{-1}$ *if* $\Lambda \cap \Lambda' = \emptyset$;

(iv) $\frac{1}{1+\varepsilon} \leq \frac{r_\varepsilon(f(p))}{r_\varepsilon(p)} \leq 1 + \varepsilon$.

Moreover R has total probability, in that $\mu(R) = 1$ for any f invariant Borel probability measure μ on M. Also, the characteristic exponents, characteristic subspaces and their dimensions are Borel functions of the base space R.

Using this result as his starting point Pesin was then able to construct much of what was known for uniformly hyperbolic systems but in a measurable context. In particular he was able to prove the following Stable Manifold Theorem: there exists a partition of the space into stable manifolds which, moreover, is absolutely continuous[2] and induce conditional measures on local unstable manifolds of almost every point. For more details we recommend [16] and [43].

1.4 The Palis Conjecture

For many properties of uniformly hyperbolic systems it is reasonable to expect they occur in other systems, at least on a large scale. For example, the property of having finitely many indecomposable sets, the so-called basic sets in the hyperbolic setting, and the property that an open dense set of orbits in each indecomposable set is attracted to a subset, called the attractor, of the indecomposable set, both hold for hyperbolic systems. These are topological notions, but the results developed by Oseledets and Pesin suggested they could be carried over to a topological/measurable framework for a larger class of systems. In [41], Palis proposed that this was indeed the case - by changing the topological notions to measurable ones in the right places he conjectures that we

[2]This means the holonomy maps which transport, locally, points from one unstable manifold to another are measurable and do not send zero measure sets to positive measure sets or vice versa.

will be able to describe all dynamical behaviour generically. We will state this conjecture more precisely below. The most topologically significant part of this conjecture is that finitude of attractors holds generically, especially since the results of Newhouse seem to suggest this should not be possible. However, the notion of attractor and basic set in the measurable setting requires careful attention. For example we have the two following definitions (see the articles [35] and [36] by Milnor).

Definition 1.4.1 (Measure Attractor)**.** Let M be a Riemannian manifold and let $f \in \mathrm{Diff}^r(M)$. A closed subset $A \subset M$ is a *measure attractor* if the following properties hold,

(i) the realm of attraction $\rho(A)$, defined to be the set of all points $x \in M$ such that $\omega(x) \subset A$, has strictly positive measure (with respect to the Riemannian volume form on M);

(ii) there is no strictly smaller closed set $A' \subset A$ such that $\rho(A')$ differs from $\rho(A)$ by a set of zero measure only.

Measure attractors are sometimes called *Milnor attractors.*

Definition 1.4.2 (Statistical Attractor)**.** A closed subset $A \subset M$ is a *statistical attractor* if the following properties hold,

(i) the orbit of almost every $x \in M$ converges statistically to A, this means $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{dist}(f^{\circ i}(x), A) = 0$;

(ii) there is no strictly smaller closed set $A' \subset A$ with the same property.

Another notion that was shown to be useful in the uniformly hyperbolic case was that of a physical measure These are also referred to as SRB, BRS, or SBR-measures, named after Sinai, Ruelle and Bowen.

Definition 1.4.3 (Physical Measure)**.** Assume we are given a measurable Borel space M and a Borel transformation $T\colon M \to M$. Endow M with a background measure μ (for example, Lebesgue). A measure ν on M is a *physical measure* if it is T-invariant and there exists a set B_ν of positive μ-measure such that $z \in B_\nu$ implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi \circ T^n(z) = \int_M \phi d\nu, \tag{1.4.1}$$

for any $\phi \in C^0(M, \mathbb{R})$. The set B_ν is called the *basin* of the physical measure ν.

We make the following remarks. Typically we require that the basin of attraction, B_ν, of the measure ν has full measure in an open set which contains it. Compare this definition with Birkhoff's Ergodic Theorem: in that situation ergodicity and measure preservation was required which allowed us to use L^1-observables ϕ but here we have removed ergodicity and measure preservation with the restriction that the observable be continuous.

Before we state the Palis Conjecture let us consider the following. Let M be a manifold, $\mathrm{End}^r(M)$ the space of C^r-endomorphisms. Let $\mathcal{P}^r(M)$ denote the subspace of $\mathrm{End}^r(M)$ consisting of maps with the following properties:

(i) there are finitely many attractors $A_0, A_1, \ldots, A_k$;

(ii) each attractor A_i supports a physical measure ν_i;

(iii) $\sum \mu(B_{\nu_i}) = \mu(M)$, where μ denotes the Riemannian volume of M;

The Palis Conjecture then states that for any manifold M and any degree of regularity $r \geq 1$ the space $\mathcal{P}^r(M)$ contains a subset D dense in $\mathrm{End}^r(M)$. Actually it states more. Firstly, given a generic, finite dimensional family f_t in $\mathrm{End}^r(M)$ assume, for the parameter value t_0, $f_{t_0} \in D$. Then there is a neighbourhood U_0 of t_0 such that for Lebesgue-almost all parameters in that neighbourhood the corresponding endomorphism also has finitely many attractors which support physical measures *and* for each attractor of the initial map there are finitely many attractors for the perturbation whose union of basins is 'nearly equal' to the basin of the initial map. Secondly each attractor is stochastically stable. By definition this means for almost every random orbit x_i (i.e. $x_i = f_{t_i}(x_{i-1})$ for some collection of f_{t_i}'s lying in a small neighbourhood of f_{t_0}) the time average is approximately the space average, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(x_i) \approx \int_M \varphi d\mu \tag{1.4.2}$$

for each continuous observable φ on M.

1.5 Renormalisation of Unimodal maps

Towards the end of the 1970's a new phenomenon in the dynamics of one dimensional unimodal maps was discovered by Feigenbaum [17], [18], and independently by Collet and Tresser [9], [10]. They observed that in many one-parameter families of unimodal maps, specifically maps with a quadratic critical point, the sequence of period doubling bifurcations accumulate to a specific parameter value and asymptotically the ratio between successive bifurcations is independent of the one-parameter family. This property was later called *universality.* See Figure 1.1 for a typical example of a bifurcation diagram. Feigenbaum's explanation of this was then (after paraphrasing) as follows:

> There exists an operator $\mathcal{R}_{\mathcal{U}}$, called the period-doubling renormalisation operator, acting on a subspace of the space of unimodal maps $\mathcal{U}$, which has a unique fixed point, which is hyperbolic with codimension-one stable manifold and dimension one local unstable manifold.

The relation to the observed phenomena is as follows. The space of unimodal maps is foliated by codimension-one manifolds whose kneading sequence

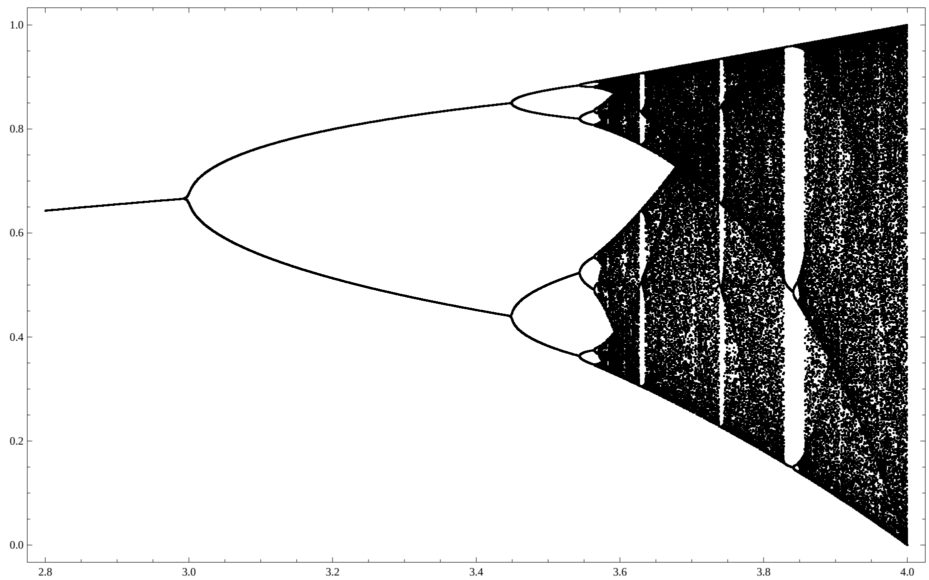

Figure 1.1: The bifurcation diagram for the family $f_\mu(x) = \mu x(1-x)$ on the interval $[0,1]$ for parameter μ . Here the attractor is plotted against the parameter μ for $2.8 \le \mu \le 4$.

is the same. The stable manifold is one of the leaves of this foliation. If the renormalisation operator is defined on one point of a leaf it is defined on the whole leaf. Moreover renormalisation will permute these leaves. Generically a one parameter family, or curve in the space of unimodal maps, intersecting the stable manifold will intersect it transversely, and hence all leaves sufficiently close will also be intersected transversely. Each period doubling bifurcation has a uniquely prescribed kneading sequence, and so they correspond to the intersection of our curve with certain singular leaves. In a neigbourhood of the fixed point each leaf, except the unstable manifold, will be pushed away from the fixed point at a geometric rate corresponding to the unstable eigenvalue. Hence these singular leaves accumulate on the unstable manifold at a geometric rate. This means the ratios between successive bifurcations will converge to the unstable eigenvalue of the renormalisation operator.

The second aspect of renormalisation, fittingly, deals with the second aspect of the bifurcation diagram such as Figure 1.1, namely what happens after the accumulation of period doubling? The picture suggests regions where the attractor consists of infinitely many points (so-called stochastic regions) and regions where there are only finitely many (regular regions). However it appears these regions are intricately interlaced. Again let us return to the kneading theoretical point of view. Firstly the period doubling bifurcations occur typically because of a monotone increase in the critical value. It was shown by Milnor and Thurston, [37], that in the particular case of the standard family, this monotone increase in critical value creates a monotone increase in the topological entropy (for details see [7] and [13]). It turns out that the onset of positive topological entropy occurs precisely at the unstable manifold of the renormalisation operator- and hence we may say renormalisation is the boundary of chaos. This is shown in two steps: first, it needs to be shown that the stochastic regions accumulate on the unstable manifold of the renormalisation operator; second, we need to show each map in this region possesses an absolutely continuous invariant measure with positive measurable entropy. Finally we invoke the variational principle.

The first conceptual proof of the first part of the Feigenbaum conjecture was given by Sullivan (see the article by Sullivan [46], or Chapter VI of the book [13] by de Melo and van Strien). In his approach he considered a renormalisation operator acting on a the space of certain quadratic-like maps which was first constructed by Douady and Hubbard in [14]. The renormalisation of a quadratic-like map which is unimodal when restricted to a real interval coincides with the usual unimodal renormalisation of the quadratic-like map restricted to this real interval. The main tools he developed were the real and complex a priori bounds, which allows us to control the geometry of central intervals and domains respectively, and the pullback argument, which allows you to construct a quasiconformal conjugacy between two maps with the same (bounded) combinatorics. We note that the pullback argument requires real a priori bounds. Using these tools he was then able to show that two infinitely renormalisable quadratic-like maps f, g with the same (bounded) combinatorics must satisfy

$$\lim_{n\to\infty} \mathrm{dist}_{J-T}(\mathcal{R}^n_{\mathcal{U}} f, \mathcal{R}^n_{\mathcal{U}} g) = 0 \tag{1.5.1}$$

where dist_{J-T} denotes the so-called Julia-Teichmüller metric.

The equivalence of the universal (real and complex a priori bounds) and rigid (pullback argument) properties were significant for many results in unimodal dynamics, see for example [26, 29, 27, 28]. Together with works such as [33], which used real methods, this culminated in a proof of the Palis Conjecture on the space of unimodal maps with quadratic critical point and negative Schwarzian derivative, see [1] and the survey article [30] for more details.

1.6 From Dimension One to Two: Hénon maps

Period-doubling cascades were also considered by Bowen and Franks at around the same time as Feigenbaum, but in a more constructive way and on the disk as well as of the interval. In [5], Bowen and Franks constructed a C^1-smooth Kupka-Smale mapping of the disk to itself such that all its periodic points were saddles. In [20], Franks and Young increased the degree of regularity to C^2-smoothness. Their motivation was a question of Smale in [44], which asked if there was a Kupka-Smale diffeomorphism of the sphere without sinks or sources. An obvious surgery, gluing two disks together, gave a map with these properties. The biggest problem with this approach was that of regularity: could this construction be extended from a C^2-smooth map to a C^∞-smooth one?

Such a map was given by Gambaudo, Tresser and van Strien in [21], but using a different strategy - instead of constructing a map combinatorially via surgery and then smoothing they considered families of maps that were already smooth and tried to locate a parameter with the desired properties. The family of maps they consider was first discussed in the paper by Collet, Eckmann and Koch [8]. Namely, they considered infinitely renormalisable unimodal maps, with doubling combinatorics, embedded in a higher dimensional space so the dynamics is preserved and examined a neighbourhood of such maps intersected with the space of embeddings. It turns out that many properties of a unimodal map are shared by those maps close by.

A complementary approach to the study of embeddings of the disk was initiated by Benedicks and Carleson in [2] at about the same time as the work by Gambaudo, Tresser and van Strien. This was done using the tools constructed by the same authors in their proof of Jakobson's Theorem on the existence of absolutely continuous invariant measures in the standard family, see [3]. As was mentioned before, their main result was the proof of the existence of an attractor for a large set of parameters. More specifically they showed the following.

Theorem 1.6.1. *Let $F_{a,b}(x,y)=(1+y-ax^2,bx)$. Let $W_{a,b}$ denote the unstable manifold of the fixed point lying in $\mathbb{R}_+\times\mathbb{R}_+$. Then for all $c<\log 2$ there exists a constant $b_0>0$ such that for all $b\in(0,b_0)$ there exists a set E_b of positive (one-dimensional) Lebesgue measure such that for all $a\in E_b$ the following holds:*

(i) There exists an open set $U_{a,b}\subset\mathbb{R}_+\times\mathbb{R}_+$ such that for all $z\in U_{a,b}$,

$$\lim_{n\to\infty}\operatorname{dist}(F^{\circ n}_{a,b}(z),\overline{W}_{a,b})=0; \tag{1.6.1}$$

(ii) There exists a point $z^0_{a,b}\in W_{a,b}$ such that $\operatorname{orb}(z^0_{a,b})$ is dense in $W_{a,b}$ and,

$$\left\|\mathrm{D}_{z^0_{a,b}}F^{\circ n}_{a,b}(0,1)\right\|\geq e^{cn}. \tag{1.6.2}$$

The first statement tells us there is a realm of attraction for the unstable manifold, and the second tells us the unstable manifold is minimal and, in some sense, expansive. The existence of a physical measure is not shown, but it is suggested by the final theorem in [21], albeit in a slightly different setting. Together these suggested the Palis Conjecture should be true for a large family of Hénon maps.

1.7 Hénon Renormalisation

In [12], de Carvalho, Lyubich and Martens constructed a period-doubling renormalisation operator for Hénon-like mappings of the form

$$F(x,y)=(f(x)-\varepsilon(x,y),x). \tag{1.7.1}$$

Here f is a unimodal map and ε was a real-valued map from the square to the positive real numbers of small size (we shall be more explicit about the maps under consideration in Sections 2 and 3). They showed that for $|\varepsilon|$ sufficiently small the unimodal renormalisation picture carries over to this case. Namely, there exists a unique renormalisation fixed point (which actually coincides with unimodal period-doubling renormalisation fixed point) which is hyperbolic with codimension one stable manifold, consisting of infinitely renormalisable period-doubling maps, and dimension one local unstable manifold. They later called this regime *strongly dissipative*.

In the period doubling case, de Carvalho, Lyubich and Martens then studied the dynamics of infinitely renormalisable Hénon-like maps F. They showed that such a map has an invariant Cantor set, $\mathcal{O}$, upon which the map acts like an adding machine. This allowed them to define the *average Jacobian* given by

$$b = \exp \int_{\mathcal{O}} \log |\mathrm{Jac}_z F| d\mu(z) \tag{1.7.2}$$

where μ denotes the unique F-invariant measure on $\mathcal{O}$ induced by the adding machine. This quantity played an important role in their study of the local behaviour of such maps around the Cantor set. They took a distinguished point, τ, of the Cantor set called the *tip*. They examined the dynamics and geometry of the Cantor set asymptotically taking smaller and smaller neighbourhoods around τ. Their two main results can then be stated as follows.

Theorem 1.7.1 (Universality at the tip)**.** *There exists a universal constant $0 < \rho < 1$ and a universal real-analytic real-valued function $a(x)$ such that the following holds: Let F be a strongly dissipative, period-doubling, infinitely renormalisable Hénon-like map. Then*

$$\mathcal{R}^n F(x, y) = (f_n(x) - b^{2^n} a(x) y(1 + \mathrm{O}(\rho^n)), x) \tag{1.7.3}$$

where b denotes the average Jacobian of F and f_n are unimodal maps converging exponentially to the unimodal period-doubling renormalisation fixed point.

Theorem 1.7.2 (Non-rigidity around the tip)**.** *Let F and $\tilde{F}$ be two strongly dissipative, period-doubling, infinitely renormalisable Hénon-like maps. Let their average Jacobians be b and $\tilde{b}$ and their Cantor sets be $\mathcal{O}$ and $\tilde{\mathcal{O}}$ respectively. Then for any conjugacy $\pi\colon \mathcal{O} \to \tilde{\mathcal{O}}$ between F and $\tilde{F}$ the Hölder exponent α satisfies*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log b}{\log \tilde{b}}\right) \tag{1.7.4}$$

In particular if the average Jacobians b and $\tilde{b}$ differ then there cannot exist a C^1-smooth conjugacy between F and $\tilde{F}$.

For a long time it was assumed that the properties satisfied by the one dimensional unimodal renormalisation theory would also be satisfied by any renormalisation theory in any dimension. In particular, the equivalence of the universal (real and complex a priori bounds) and rigid (pullback argument) properties in this setting made it natural to think that such a relation would be realised for any reasonable renormalisation theory. That is, if universality controls the geometry of an attractor and we have a conjugacy mapping one attractor to another[3] it seems reasonable to think that we could extend such a conjugacy in a "smooth" way, since the geometry of infinitesimally close pairs of orbits cannot differ too much. The above shows that this intuitive reasoning is incorrect.

In Section 3 we generalise this renormalisation operator to other combinatorial types. We show that in this case too the renormalisation picture holds if $\bar{\varepsilon}$ is sufficiently small. Namely, for any stationary combinatorics there exists a unique renormalisation fixed point, again coinciding with the unimodal renormalisation fixed point, which is hyperbolic with codimension one stable manifold, consisting of infinitely renormalisable maps, and dimension-one local unstable manifold.

We then study the dynamics of infinitely renormalisable maps of stationary combinatorial type and show that such maps have an F-invariant Cantor set $\mathcal{O}$ on which F acts as an adding machine. We would like to note that the strategy to show that the limit set is a Cantor set in the period-doubling case does not carry over to maps with general stationary combinatorics. The reason is that in both cases the construction of the Cantor set is via 'Scope Maps', defined in sections 2 and 3, which we approximate using the so-called 'Presentation function' of the renormalisation fixed point. In the period-doubling case this is known to be contracting as the renormalisation fixed point is convex (see the result of Davie [11]) and the unique fixed point lying in the interior of the interval is expanding (see the theorem of Singer [13, Ch. 3]). In the case of general combinatorics this is unlikely to be true. The work of Eckmann and Wittwer [15] suggests the convexity of fixed points for sufficiently large combinatorial types does not hold. The problem of contraction of branches of the presentation function was also asked in [25].

Once this is done we are in a position to define the average Jacobian and the tip of an infinitely renormalisable Hénon-like map in a way completely analogous to the period-doubling case. This then allows us, in Section 4, to generalise the universality and non-rigidity results stated above to the case of arbitrary combinatorics. We also generalise another result from [12], namely the Cantor set of an infinitely renormalisable Hénon-like map cannot support a continuous invariant line field. Our proof, though, is significantly different. This is because in the period-doubling case they observed a 'flipping' phenomenon was observed where orientations were changed purely because of combinatorics. Their argument clearly breaks down in the more general case where there is no control over such things.

Another facet of the renormalisation theory for unimodal maps is the notion of a priori bounds and bounded geometry. In chapter 5 we study the geometry of Cantor sets for infinitely renormalisable Hénon-like maps in more detail. Recall that, in the unimodal case, a priori bounds states there are uniform or eventually uniform bounds for the geometry of the images of the central interval at each

[3] this requires only combinatorial information

renormalisation step. Namely at each renormalisation level there is a bounded decrease in size of these interval and their gaps. More precisely if J is an image of the i-th central interval, and J' is an image of the $i+1$-st central interval contained in J, then $|J'|/|J|, |L'|/|J|$ and $|R'|/|J|$ are (eventually) uniformly bounded, where L', R' are the left and right connected components of $J \setminus J'$.

Several authors have worked on consequences of a similar notion of a priori bounds in the two dimensional case. For example, in the papers of Catsigeras, Moreira and Gambaudo [6], and Moreira [38], they consider common generalisations of the model introduced by Bowen and Franks, in [5], and Franks and Young, in [20], and of the model introduced by Gambaudo, Tresser and van Strien in [21] and [22]. In [6] it is shown that given a dissipative infinitely renormalisable diffeomorphism of the disk with bounded combinatorics and bounded geometry, there is a dichotomy: either it has positive topological entropy or it is eventually period doubling. In [38] a comparison is made between the smoothness and combinatorics of the two models using the asymptotic linking number: given a period doubling, C^∞-smooth, dissipative, infinitely renormalisable diffeomorphism of the disk with bounded geometry the convergents of the asymptotic linking number cannot converge monotonically. This should be viewed as a kind of combinatorial rigidity result which, in particular, implies that Bowen-Franks-Young maps cannot be C^∞.

We would like to note, *as of yet*, there are no known examples of infinitely renormalisable Hénon-like maps with bounded geometry. In the more general case of infinitely renormalisable diffeomorphisms of the disk considered in [6] and [38], we know of no example with bounded geometry either. In fact, at least for the Hénon-like case, we will show the following result:

Theorem 1.7.3. *Let F_b be a one parameter family of infinitely renormalisable Hénon-like maps, parametrised by the average Jacobian $b = b(F_b) \in [0, b_0)$. Then there is a subinterval $[0, b_1] \subset [0, b_0)$ for which there exists a dense G_δ subset $S \subset [0, b_1)$ with full relative Lebesgue measure such that the Cantor set $\mathcal{O}(b) = \mathcal{O}(F_b)$ has unbounded geometry for all $b \in S$.*

This is the main result of chapter 5. We conclude with a discussion of future directions of research and some open problems which the current work suggests.

1.8 Notations and Conventions

First let us introduce some standard definitions. We will denote the integers by $\mathbb{Z}$, the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. We will denote by $\mathbb{Z}_+$ the set of strictly positive integers and by $\mathbb{R}_+$ the set of strictly positive real numbers. Given real-valued functions $f(x)$ and $g(x)$ we say that $f(x)$ is $\mathrm{O}(g(x))$ if there exists $\delta > 0$ and $C > 0$ such that $|f(x)| \leq C|g(x)|$ whenever $|x| < \delta$. We say that $f(x)$ is $\mathrm{o}(g(x))$ if $\lim_{x\to 0} |f(x)/g(x)| = 0$.

Given a topological space M and a subspace $S \subset M$ we will denote its interior by $\mathrm{int}(S)$ and its closure by $\mathrm{cl}(S)$. If M is also a metric space with metric d we define the distance between subsets S and S' of M by

$$\mathrm{dist}(S, S') = \inf_{s \in S, s' \in S'} d(s, s'). \tag{1.8.1}$$

For S, S' both compact we define the *Hausdorff distance* between S and S' by

$$d_{Haus}(S, S') = \max \left\{ \sup_{s \in S} \inf_{s' \in S'} d(s, s'), \sup_{s' \in S'} \inf_{s \in S} d(s, s') \right\}. \tag{1.8.2}$$

If M also has a linear structure we denote the convex hull of S by Hull(S).

For an integer $p \geq 2$ we set $W_p = \{0, 1, \ldots, p-1\}$. When p is fixed we will simply denote this by W. We denote by W^n the space of all words of length n and by W^* the totality of all finite words over W. We will use juxtapositional notation to denote elements of W^*, so if $\mathbf{w} \in W^*$ then $\mathbf{w} = w_0 \ldots w_n$ for some $w_0, \ldots, w_n \in W$. For all $w \in W$ and $n > 0$ we will let w^n denote $w \ldots w$, where the juxtaposition is taken n times. Given $\mathbf{w} \in W$ we will denote the m-th word from the left by $\mathbf{w}(m)$ whenever it exists.

We endow W^* with the structure of a topological semi-group as follows. First endow W^* with the topology whose bases are the cylinder sets

$$[w_1 \ldots w_n]_m = \{\mathbf{w} \in W^* \colon \mathbf{w}(m) = w_1, \ldots, \mathbf{w}(m+n) = w_n\} \tag{1.8.3}$$

Now consider the map $m \colon W^* \times W^* \to \mathbb{Z}_+^*$, where $\mathbb{Z}_+^*$ denotes the set of words of arbitrary length over the positive integers $\mathbb{Z}_+$, given by $m(\mathbf{x}, \mathbf{y})(i) = \mathbf{x}(i) + \mathbf{y}(i)$. Then we define the map $s \colon \mathbb{Z}_+^* \to W^*$ inductively by

$$s(\mathbf{w})(i) = \begin{cases} \mathbf{w}(i) & \mathbf{w}(i-1) \in W_p \text{ and } \mathbf{w}(i) \in W_p \\ \mathbf{w}(i) + 1 & \mathbf{w}(i-1) \notin W_p \text{ and } \mathbf{w}(i) + 1 \in W_p \\ 0 & \text{otherwise} \end{cases} \tag{1.8.4}$$

The addition on W^* is given by $+ \colon W^* \times W^* \to W^*, \mathbf{x} + \mathbf{y} = s \circ m(\mathbf{x}, \mathbf{y})$. Let $\mathbf{1} = (1, 0, 0, \ldots)$ and let $T \colon W^* \to W^*$ be given by $T(\mathbf{w}) = \mathbf{1} + \mathbf{w}$. This map is called *addition with infinite carry*[4]. The pair (W^*, T) is called the *adding machine* over W^*. The set of all infinite words will be denoted by $\overline{W}$. Observe that T can be extended to $\overline{W}$.

Typically, we will treat the adding machine as an index set for cylinder sets of a Cantor set. The following definition[5] will also be useful.

Definition 1.8.1. Let $\mathcal{O} \subset S$ be a Cantor set, where S is a metrizable space. A *presentation* for $\mathcal{O}$ is a collection $\{B^{\mathbf{w}}\}_{\mathbf{w} \in W^*}$ of closed topological disks $B^{\mathbf{w}}$ such that, if $B^d = \bigcup_{\mathbf{w} \in W^n} B^{\mathbf{w}}$,

[4]Explicitly this is defined by

$$T(\mathbf{w}) = \begin{cases} (1 + x_0, x_1, \ldots) & x_0 < p-1 \\ (0, 0, \ldots, 0, 1 + x_k, \ldots) & x_0, \ldots, x_{k-1} = p-1, x_k \neq p-1 \end{cases}$$

[5]An equivalent definition is given in [13, Chapter VI, Section 3], the only difference being the indexing. However, their definition is more general as it allows combinatorial types other than stationary type.

(i) $\operatorname{int} B^{\mathbf{w}} \cap \operatorname{int} B^{\tilde{\mathbf{w}}} = \emptyset$ for all $\mathbf{w} \neq \tilde{\mathbf{w}} \in W^*$ of the same length;

(ii) $B^d \supset B^{d+1}$ for each $n \geq 0$;

(iii) $\bigcap_{d\geq 0} B^d = \mathcal{O}$.

For $\mathbf{w} \in W^d$ we call $B^{\mathbf{w}}$ a *piece of depth d*.

Now let us describe indexing issues in some detail. Given a presentation of a Cantor set $\mathcal{O}$ we could give the pieces the indexing above or we could have given them the ordering $B^{d,i}$, where d denotes the depth and i corresponds to a linear ordering $i = 0, \ldots, p^d - 1$ of all the pieces of depth d. Typically this ordering has the property that if $B^{d+1,i} \subset B^{d,j}$ then $B^{d+1,i+1} \subset B^{d+1,j+1}$. Let $\mathbf{q} \colon W^* \to \mathbb{Z}_+ \times \mathbb{Z}_+$ denote the correspondence between these two indexings.

Given a function F we will denote its domain by $\mathrm{Dom}(F)$. Typically this will be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. If $F \colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $z \in \mathbb{R}^n$ we will denote the derivative of F at z by $\mathrm{D}_z F$. The *Jacobian* of F is given by

$$\mathrm{Jac}_z F = \det \mathrm{D}_z F \tag{1.8.5}$$

Given a bounded region $S \subset \mathbb{R}^n$ we will define the *distortion* of F on S by

$$\mathrm{Dis}(F; S) = \sup_{z,\tilde{z}\in S} \log \left| \frac{\mathrm{Jac}_z F}{\mathrm{Jac}_{\tilde{z}} F} \right| \tag{1.8.6}$$

and the *variation* of F on S by

$$\mathrm{Var}(F; S) = \sup_{G \in C_0^1(S) : |G(z)| \leq 1} \int_S F \mathrm{div} G dz. \tag{1.8.7}$$

According to [23], when $S \subset \mathbb{R}^2$ this coincides with

$$\mathrm{Var}(F; S) = \max \left\{ \int_{S_x} \mathrm{Var}(F; S_y) dx, \int_{S_y} \mathrm{Var}(F; S_x) dy \right\}, \tag{1.8.8}$$

i.e. the integral of the one-dimensional variations, restricted to vertical or horizontal slices, is taken in the orthogonal direction.

Given a domain $S \subset \mathbb{R}^n$ and a map $F \colon S \to \mathbb{R}^n$ we will denote its i-th iterate by $F^{\circ i}$ and, if it is a diffeomorphism onto its image, its i-th preimage by $F^{\circ -i} \colon F^{\circ i}(S) \to \mathbb{R}^n$. If F is not a map we are iterating (for example if it is a change of coordinates) then we will denote its inverse by $\bar{F}$ instead. It will become clear when considering Hénon-like maps why we need to make this distinction. It is to make our indexing conventions consistent.

Now we will restrict our attention to the one- and two-dimensional cases, both real and complex. Let $\pi_x, \pi_y \colon \mathbb{R}^2 \to \mathbb{R}$ denote the projections onto the x- and y- coordinates. We will identify these with their extensions to $\mathbb{C}^2$. (In fact we will identify all real functions with their complex extensions whenever they exist.)

Given $a, b \in \mathbb{R}$ we will denote the closed interval between them by $[a, b] = [b, a]$. We will denote $[0, 1]$ by J. For any interval $T \subset \mathbb{R}$ we will denote its boundary by ∂T, its left endpoint by $\partial^- T$ and its right endpoint by $\partial^+ T$. Given two intervals $T_0, T_1 \subset J$ we will denote an affine bijection from T_0 to T_1 by $\iota_{T_0 \to T_1}$. Typically it will be clear from the situation whether we are using the unique orientation preserving or orientation reversing bijection.

Let us denote the square $[0, 1] \times [0, 1] = J^2$ by B. We call $S \subset B$ a *rectangle* if it is the Cartesian product of two intervals. Given two points $z, \tilde{z} \in B$, the *closed rectangle spanned by z and $\tilde{z}$* is given by

$$[\![z, \tilde{z}]\!] = [\pi_x(z), \pi_x(\tilde{z})] \times [\pi_y(z), \pi_y(\tilde{z})], \tag{1.8.9}$$

and the *straight line segment between z and $\tilde{z}$* is denoted by $[z, \tilde{z}]$. Given two rectangles $B_0, B_1 \subset B$ we will denote an affine bijection from B_0 to B_1 preserving horizontal and vertical lines by $I_{B_0 \to B_1}$. Again the orientations of its components will be clear from the situation.

Let S denote the interval J or the square B. Let S' be a closed subinterval or sub-square of S respectively. Let $\mathcal{D}_p^\omega(S') \subset \mathrm{End}^\omega(S)$ denote the subspace of endomorphisms F such that $F^{\circ p}(S') \subset S'$. Then the *zoom operator* $\mathcal{Z}_{S'} \colon \mathcal{D}_p(S') \to \mathrm{End}^\omega(S)$ is given by

$$\mathcal{Z}_{S'} F = I_{S' \to S} \circ F^{\circ p} \circ I_{S \to S'} \colon S \to S \tag{1.8.10}$$

where $I_{S \to S'} \colon S \to S'$ denotes the orientation-preserving affine bijection between S and S' which preserves horizontal and vertical lines. We note that in certain situations it will be more natural to change orientations but in these cases we shall be explicit.

Let $\Omega_x \subseteq \Omega_y \subset \mathbb{C}$ be simply connected domains compactly containing J and let $\Omega = \Omega_x \times \Omega_y$ denote the resulting polydisk containing B.

Chapter 2

Unimodal Maps

In this chapter we will briefly review the relevant parts of one-dimensional unimodal renormalisation theory and, in particular, the presentation function theory associated with it developed in the papers of Feigenbaum [19], Sullivan [45] and Birkhoff, Martens and Tresser [4]. The structure of this chapter will be followed macroscopically in the remainder of this work.

2.1 The Space of Unimodal Maps

Let $\beta > 0$ be a constant, which we will think of as being small. Let $\mathcal{U}_{\Omega_x,\beta}$ denote the space of maps $f \in C^\omega(J, J)$ satisfying the following properties:

(i) there is a unique critical point $c = c(f)$, which lies in $(0, 1-\beta]$;

(ii) there is a unique fixed point $\alpha = \alpha(f)$, which lies in $\mathrm{int}(J)$ and which, moreover, is expanding;

(iii) $f(\partial^+ J) = f(\partial^- J) = 0$ and $f(c) > c$;

(iv) f is orientation preserving to the left of c and orientation reversing to the right of c;

(v) f admits a holomorphic extension to the domain Ω_x, upon which it can be factored as $\psi \circ Q$, where $Q\colon \mathbb{C} \to \mathbb{C}$ is given by $Q(z) = 4z(1-z)$ and $\psi\colon Q(\Omega_x) \to \mathbb{C}$ is an orientation preserving univalent mapping which fixes the real axis;

Such maps will be called *unimodal maps*. Given any interval $T \subset \mathbb{R}$ we will say a map $g\colon T \to T$ is *unimodal on T* if there exists an affine bijection $h\colon J \to T$ such that $h^{-1} \circ g \circ h \in \mathcal{U}_{\Omega_x,\beta}$. We will identify all unimodal maps with their holomorphic extensions.

We make two observations: first, this extension will be $\mathbb{R}$-symmetric (i.e. $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega_x$) and second, the expanding fixed point will have negative multiplier.

2.2 Construction of an Operator

Definition 2.2.1. Let $p > 1$ be an integer and let $W = W_p$. A permutation, υ, on W is said to be *unimodal of length p* if there exists

(i) an order preserving embedding $\mathrm{i}\colon W \to J$;

(ii) a unimodal map $f\colon J \to J$ such that $f(\mathrm{i}(k-1)) = \mathrm{i}(k \bmod p)$.

Definition 2.2.2. Let $p > 1$ be an integer. A map $f \in \mathcal{U}_{\Omega_x,\beta}$ has a *renormalisation interval of type p* if

(i) there is a closed subinterval $J^0 \subset J$ containing the critical point such that $f^{\circ p}\left(J^0\right) \subset J^0$;

(ii) there exists an affine bijection $h\colon J \to J^0$ such that

$$\mathcal{R}_{\mathcal{U}} f = h^{-1} \circ f^{\circ p} \circ h\colon J \to J \tag{2.2.1}$$

is an element of $\mathcal{U}_{\Omega_x,\beta}$. Note there are exactly two such affine bijections, but there will only be one such that $\mathcal{R}_{\mathcal{U}} f \in \mathcal{U}_{\Omega_x,\beta}$;

The interval J^0 is called a *renormalisation interval of type p* for f.

Definition 2.2.3. Let $p > 1$ be an integer and let υ be a unimodal permutation of length p. A map $f \in \mathcal{U}_{\Omega_x,\beta}$ is *renormalisable with combinatorics υ* if

(i) f has a renormalisation interval J^0 of type p;

(ii) if we let J^w denote the connected component of $f^{\circ p-w}\left(J^0\right)$ containing $f^{\circ w}\left(J^0\right)$ then the interiors of the subintervals J^w, $w \in W$ are pairwise disjoint;

(iii) f acts on the set $\left\{J^0, J^1, \dots J^{p-1}\right\}$, embedded in the line with the standard orientation, as υ acts on the symbols in W. More precisely, if $J', J'' \in \{J^w\}_{w \in W}$ are the i-th and j-th sub-intervals from the left endpoint of J respectively. Then $f(J')$ lies to the left of $f(J'')$ if and only if $\upsilon(i) < \upsilon(j)$.

In this case the map $\mathcal{R}_{\mathcal{U}} f$ is called the *renormalisation of f* and the operator $\mathcal{R}_{\mathcal{U}}$ the *renormalisation operator* of combinatorial type υ.

Definition 2.2.4. Given a renormalisable $f \in \mathcal{U}_{\Omega_x,\beta}$ of combinatorial type υ the subinterval J^0 is called the *central interval.* This is a special case of a renormalisation interval. The collection $\{J^w\}_{w \in W}$ is called the *renormalisation cycle.* Given J^w, $w \in W$, the *maximal extension* of J^w is the largest open interval J'^w containing J^w such that $f^{\circ p-w}|J'^w$ is a diffeomorphism onto its image.

Definition 2.2.5. Let $p > 1$ be an integer. Let $0 < \gamma < 1$. Let $f \in \mathcal{U}_{\Omega_x,\beta}$ have renormalisation interval J^0 of type p. Let J^w denote the connected component of $f^{\circ p-w}\left(J^0\right)$ which contains $f^{\circ w}\left(J^0\right)$. Let c_0 denote the unique critical point of f. If

(i) $\operatorname{dist}(J^{w_0}, J^{w_1}) \geq \gamma$ for all distinct $w_0, w_1 \in W$;

(ii) $\operatorname{dist}(C_p, J^w) \geq \gamma$ for all $w \in W$;

where $C_p = f^{\circ -p}(c_0) \cup \ldots \cup f^{\circ p}(c_0)$, then we say f has the *γ-gap property.*

Remark 2.2.6. The assumption that $\mathcal{R}_{\mathcal{U}} f$ lies in $\mathcal{U}_{\Omega_x,\beta}$ implies that the boundary of J^0 consists of a p^n-periodic point and one of its preimages. Moreover, in J^0 there is no other preimage of this point, there is at most one periodic point of period p^n and none of smaller period. These will be important observations later when we consider perturbations of renormalisable unimodal maps.

Remark 2.2.7. We have hidden slightly the issue of complex renormalisation. We could have just as easily required that there exist a simply connected domain $\Omega_x^0 \subset \Omega_x$, called the *central domain*, containing the critical point and symmetric about the real axis, on which $f^{\circ p}$ is quadratic-like and for which the sub-domains Ω_x^w are pairwise disjoint. Here Ω_x^w denotes the connected component of $f^{\circ p-w}\left(\Omega_x^0\right)$ containing $f^{\circ w}\left(\Omega_x^0\right)$. See [13, Chapter VI] and [14] for more details.

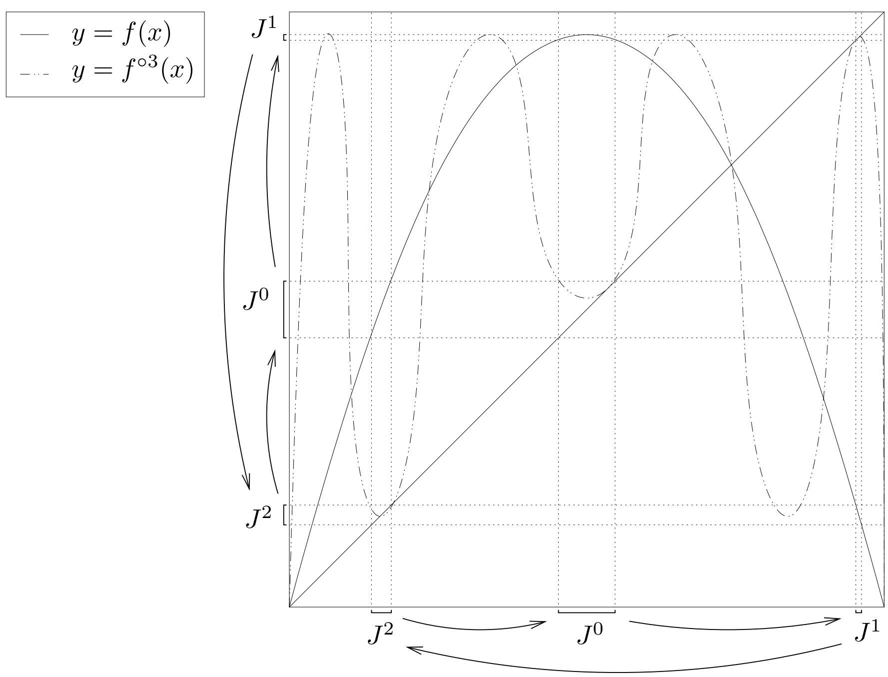

Figure 2.1: The graph of a renormalisable period-three unimodal map f with renormalisation interval J^0 and renormalisation cycle $\{J^i\}_{i=0,1,2}$. For $p = 3$ there is only one admissable combinatorial type. Observe that renormalisability is equivalent to the graph of $f^{\circ 3}$ restricted to $J^0 \times J^0$ being the graph of a map unimodal on J^0. This will be examined in more detail at the end of the chapter.

Remark 2.2.8. If f is renormalisable of combinatorial type υ there are p disjoint subintervals $J^0, \ldots, J^{p-1}$, all of which are invariant under $f^{\circ p}$. As f acts as a diffeomorphism on $J^w, w = 1, \ldots, p-1$, the map $f^{\circ p}|J^w$ will have a unique critical point in the interior, map the boundary into itself and have a unique fixed point in the interior. Hence for each $w \in W_p$ we could also consider the operator

$$\mathcal{R}_{\mathcal{U},w} f = (h^w)^{-1} \circ f^{\circ p} \circ h^w \tag{2.2.2}$$

where h^w is an affine bijection from J to J^w. Observe that this map will not be unimodal by our definition, since the factorisation property will not be satisfied. However, it will have the form $\psi^w \circ Q \circ \phi^w$ for ϕ^w and ψ^w univalently mapping to a disk around the critical point of Q and from a disk around the critical value of Q respectively. Also observe the fixed point in the interior will not necessarily be expanding, so even if we extended the definition of unimodal map to include those of the above type, it is not clear that "renormalisable around the critical point" implies "renormalisable around a critical preimage". However, in the case of period-doubling combinatorics, relations between the renormalisation fixed points of these operators were examined in [4].

Let $\mathcal{U}_{\Omega_x,\beta,\upsilon}$ denote the subspace consisting of unimodal maps $f \in \mathcal{U}_{\Omega_x,\beta}$ which are renormalisable of combinatorial type υ. If $f \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ is infinitely renormalisable there is a nested sequence $\underline{J} = \{J^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ of subintervals such that

(i) $f(J^{\mathbf{w}}) = J^{1+\mathbf{w}}$ for all $\mathbf{w} \in W^*$;

(ii) $\operatorname{int} J^{\mathbf{w}} \cap \operatorname{int} J^{\tilde{\mathbf{w}}} = \emptyset$ for all $\mathbf{w} \neq \tilde{\mathbf{w}} \in W^*$ of the same length;

(iii) $\bigcup_{w\in W} J^{\mathbf{w}w} \subset J^{\mathbf{w}}$ for each $\mathbf{w} \in W^*$.

Notation 2.2.9. If $f \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ is an infinitely renormalisable unimodal map let $f_n = \mathcal{R}^n_{\mathcal{U}} f$. Then all objects associated to f_n will also be given this subscript. For example we will denote by $\underline{J}_n = \{J_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ the nested collection of intervals constructed for f_n in the same way that $\underline{J}$ was constructed for f.

The following plays a crucial role in the renormalisation theory of unimodal maps. (See [13] for the proof and more details.)

Theorem 2.2.10 (real C^1 a priori bounds)**.** *Let $f \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ be an infinitely renormalisable unimodal map. Then there exist constants $L(f), K(f) > 1$ and $0 < k_0(f) < k_1(f) < 1$, such that for all $\mathbf{w} \in W^*, w, \tilde{w} \in W$ and each $i = 0, 1 \ldots, p^n - \mathbf{q}(\mathbf{w})$ the following properties hold,*

(i-a) $\operatorname{Dis}(f^{\circ i}; J^{\mathbf{w}}) \leq L(f)$;

(i-b) the previous bound is beau: there exists a constant $L > 1$ such that for each f as above $L(\mathcal{R}^n_{\mathcal{U}} f) < L$ for n sufficiently large;

(ii-a) $K(f)^{-1} < |J^{\mathbf{w}w}|/|J^{\mathbf{w}\tilde{w}}| < K(f)$;

(ii-b) the previous bound is beau: there exists a constant $K > 1$ such that for each f as above $K(\mathcal{R}^n_{\mathcal{U}} f) < K$ for n sufficiently large;

(iii-a) $k_0(f) < |J^{\mathbf{w}w}|/|J^{\mathbf{w}}| < k_1(f)$*;*

(iii-b) the previous bound is beau: there exist constants $0 < k_0 < k_1 < 1$ *such that for each* f *as above* $k_0 < k_0(\mathcal{R}^n_{\mathcal{U}} f) < k_1(\mathcal{R}^n_{\mathcal{U}} f) < k_1 < 1$ *for* n *sufficiently large.*

The term *beau* for such a property was coined by Sullivan - it stands for bounded eventually and universally. We note that more was actually shown: namely,

(i) The universal constant L above is uniform over all combinatorial types. However the constants K, k_0 and k_1, can only be assumed to be uniform if we restrict to combinatorics of bounded type.

(ii) This theorem was proved for the much larger class of C^1 unimodal maps whose derivative satisfies the little Zygmund condition (see [13, Chapter III] for the definition).

(iii) The set given by

$$\mathcal{O}(f) = \bigcap_{n \geq 0} \bigcup_{\mathbf{w} \in W^n} J^{\mathbf{w}} \tag{2.2.3}$$

is a Cantor set with zero Lebesgue measure. Hence the collection of subintervals $\underline{J}$ is a presentation of $\mathcal{O}$.

However, the first two of these properties will not concern us in the current work as we will only consider stationary combinatorics and all maps we consider will be analytic. The third of these properties will only tangentially concern us later in chapter 3 when we construct invariant Cantor sets for infinitely renormalisable Hénon-like maps. Now we show some properties of the renormalisation operator and renormalisable maps.

Proposition 2.2.11. *Let* $p > 1$ *be an integer. Let* $f \in \mathcal{U}_{\Omega_x,\beta}$ *have renormalisation interval* J^0 *of type* p *satisfying the following conditions,*

- $f^{\circ p}\left(J^0\right) \subsetneq J^0$*;*
- $f^{\circ p}$ *is unimodal on* J^0*.*

Then there exists an open neighbourhood $U \subset \mathcal{U}_{\Omega_x,\beta}$ *of* f *such that for any* $\tilde{f} \in U$ *the following properties hold,*

(i) $\tilde{f}$ *also has a renormalisation interval of type* p*;*

(ii) there exists a constant $C > 0$*, depending upon* f *only, such that*

$$\mathrm{dist}_{Haus}(J^0, \tilde{J}^0) < C|f - \tilde{f}|_{\Omega_x}. \tag{2.2.4}$$

Proof. By assumption ∂J^0 contains an orientation preserving expanding fixed point of $f^{\circ p}$ and a preimage whose derivative is nonzero. By Corollaries A.2.2

and A.2.3, both of these are open properties. Hence let $\tilde{J}^0$ denote the corresponding interval for $\tilde{f}$.

By assumption J^0 contains an unique orientation reversing expanding fixed point. By Corollary A.2.2 this is also an open property. Hence $f^{\circ p}$ has a unique orientation reversing expanding fixed point in the interior of $\tilde{J}^0$.

Also by assumption the only critical point of $f^{\circ p}$ contained in J^0 is c_0, which moreover is a turning point. By Proposition A.2.2, this is also an open property. Hence let $\tilde{c}_0$ denote the unique critical point of $\tilde{f}$ in $\tilde{J}^0$. Since $\mathcal{R}_{\mathcal{U}} f$ is not surjective we know that $c_p = f^{\circ p}(c_0) \in \operatorname{int} J^0$. It is clear that $c_p \in \operatorname{int} J^0$ implies $\tilde{c}_p \in \operatorname{int} \tilde{J}^0$ for $\tilde{f}$ sufficiently close to f. Therefore $\tilde{f}^p(\tilde{J}^0) \subset \tilde{J}^0$ and $\tilde{J}^0$ contains a unique non-degenerate critical point and a unique orientation reversing expanding fixed point. Moreover, by assumption $f^{\circ p}|J^0$ admits a complex analytic extension to a domain $\Omega_x^0 \subset \mathbb{C}$ containing J^0, so by Lemma A.2.4 $\tilde{f}^{\circ p}|\tilde{J}^0$ must admit a complex analytic extension to some domain $\tilde{\Omega}_x^0 \subset \mathbb{C}$ containing $\tilde{J}^0$. □

Proposition 2.2.12. *Let $p > 1$ be an integer. Let $0 < \gamma < 1$. Let υ be a unimodal permutation of length p. Let $f \in \mathcal{U}_{\Omega_x,\beta}$ have renormalisation interval J^0 of type p and satisfy the following conditions,*

- $f^{\circ p}(J^0) \subsetneq J^0$;
- *f is renormalisable with combinatorics υ;*
- *f satisfies the γ-gap property.*

Then there exists a neighbourhood $U \subset \mathcal{U}_{\Omega_x,\beta}$ of f such that for any $\tilde{f} \in U$ the following properties hold,

(i) $\tilde{f}$ is renormalisable with combinatorics υ;

(ii) there exists a constant $C > 0$, depending upon f only, such that

$$|\mathcal{R}_{\mathcal{U}} f - \mathcal{R}_{\mathcal{U}} \tilde{f}|_{\Omega_x} < C|f - \tilde{f}|_{\Omega_x}; \tag{2.2.5}$$

(iii) the operator $\mathcal{R}_{\mathcal{U}}$ is injective.

Proof. By Proposition 2.2.11 there is a neighbourhood U_0 such that any $\tilde{f} \in U_0$ has a renormalisation interval $\tilde{J}^0$. Hence to show renormalisability it only remains to show that the subintervals $\tilde{J}^w$, defined to be the connected component of $\tilde{f}^{\circ -(p-w)}(\tilde{J}^0)$ containing $\tilde{f}^{\circ w}(\tilde{J}^0)$, are pairwise disjoint. This follows since, by hypothesis, the preimages of ∂J^0 under $f^{\circ -(p-w)}$ are distinct. Moreover none of these preimages can coincide with a critical point. Therefore by Corollary A.2.3 the ordering of these preimages is an open property. Hence the preimages of $\partial \tilde{J}^0$ under $\tilde{f}$ will also be distinct and so the $\tilde{J}^w$ will be pairwise disjoint.

For the second item observe that the intervals $\tilde{J}^w$ depend continuously on $\tilde{f}$ and hence $\mathcal{Z}_{\tilde{J}^w} \tilde{f}$ depends continuously on $\tilde{f}$. (Since the zoom operator is continuous in both arguments.) As the composition operator is also continuous the claim follows. The third item can be found in [13, Chapter VI]. □

2.3 The Fixed Point and Hyperbolicity

As was noted in the introduction, real a priori bounds was an important component in Sullivan's proof of the following part of the Renormalisation conjecture. For the proof we refer the reader to the book [13, Chapter VI] by de Melo and van Strien. This also contains substantial background material and references.

Theorem 2.3.1 (existence of fixed point)**.** *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, if $\beta > 0$ is sufficiently small there exists an $f_* = f_{*,\upsilon} \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ such that*

$$\mathcal{R}_{\mathcal{U}} f_* = f_*, \tag{2.3.1}$$

i.e. f_* is an $\mathcal{R}_{\mathcal{U}}$-fixed point.

Notation 2.3.2. Henceforth we will assume that the unimodal permutation υ and the positive real number $\beta > 0$ are fixed, but β is small enough to ensure $\mathcal{U}_{\Omega_x,\beta,\upsilon}$ contains the renormalisation fixed point. We therefore will drop β from our notation.

Sullivan's proof of the above result was then strengthened by McMullen. For more information see the book [34].

Theorem 2.3.3 (weak convergence)**.** *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, there exists*

(i) a domain $\Omega'_x \Subset \Omega_x$ containing J;

(ii) an integer $N > 0$;

both dependent upon υ and Ω_x, such that for any $n > N$ if $f \in \mathcal{U}_{\Omega_x,\upsilon}$ is n-times renormalisable then

$$|\mathcal{R}^n_{\mathcal{U}} f - f_*|_{\Omega'_x} \leq \tfrac{1}{4}|f - f_*|_{\Omega'_x}. \tag{2.3.2}$$

The proof of the full renormalisation conjecture was then completed by Lyubich in the paper [27]. This used his earlier results in [26, 29] on the tower construction of McMullen.

Theorem 2.3.4 (exponential convergence)**.** *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, there exists*

(i) a domain $\Omega'_x \Subset \Omega_x$, containing J;

(ii) an $\mathcal{R}_{\mathcal{U}}$-invariant subspace, $\mathcal{U}_{adapt} \subset \mathcal{U}_{\Omega'_x,\upsilon}$;

(iii) a metric, d_{adapt}, on $\mathcal{U}_{adapt}$ which is Lipschitz-equivalent to the sup-norm on $\mathcal{U}_{\Omega'_x,\upsilon}$;

(iv) a constant $0 < \rho < 1$;

such that, for all $f \in \mathcal{U}_{adapt}$,

$$d_{adapt}(\mathcal{R}_{\mathcal{U}} f, f_*) \leq \rho d_{adapt}(f, f_*). \tag{2.3.3}$$

Theorem 2.3.5 (codimension-one stable manifold)**.** *For any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, the renormalisation operator $\mathcal{R}_{\mathcal{U}}\colon \mathcal{U}_{\Omega_x,\upsilon} \to \mathcal{U}_{\Omega_x}$ has a codimension-one stable manifold $\mathcal{W}_\upsilon$ at the renormalisation fixed point $f_{*,\upsilon}$.*

Corollary 2.3.6. *Let υ be a unimodal permutation on W. Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$ be an infinitely renormalisable unimodal map. Then the cycle, $\{J^w_n\}_{w\in W}$, of the central interval of f_n converges exponentially, in the Hausdorff topology, to the corresponding cycle, $\{J^w_*\}_{w\in W}$, of the renormalisation fixed point f_*.*

2.4 Scope Maps and Presentation Functions

Now we will rephrase the renormalisation of unimodal maps in terms of convergence of their *Scope maps* to be defined below. They were studied by Sullivan [45], Feigenbaum [19] and Birkhoff, Martens and Tresser [4] mostly in the case of the so-called *Presentation function*, which is the scope map of the renormalisation fixed point. We also note that they were examined using complex tools by Jiang, Morita and Sullivan in [25].

Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$ have cycle $\{J^w\}_{w\in W}$. Consider the functions

$$\begin{aligned} \iota_{J^0\to J}\circ f^{\circ p-w}\colon J^w \to J, &\qquad w = 1,\dots,p-1,\\ \iota_{J^0\to J}\colon J^0 \to J, &\qquad w = 0. \end{aligned} \tag{2.4.1}$$

The inverses of these maps are called the *Scope maps* of f which we denote by $\psi^w_f\colon J \to J^w$. For each $w \in W$ we will call $\psi^w_f\colon J \to J^w$ the *w-scope map.* We will denote the multi-valued function they form by $\boldsymbol{\psi}_f\colon J \to \bigcup_{w\in W} J^w$. Similarly, given an n-times renormalisable $f \in \mathcal{U}_{\Omega_x,\upsilon}$ we let $\psi^w_n = \psi^w_{f_n}$ denote the w-th scope function of f_n and the multi-valued function they form by $\boldsymbol{\psi}_n$. The multi-valued function $\boldsymbol{\psi}_* = \boldsymbol{\psi}_{f_*}$ associated to the renormalisation fixed point f_* is called the *Presentation function.*

If $f \in \mathcal{U}_{\Omega_x,\upsilon}$ is infinitely renormalisable we can extend this construction by considering, for each $\mathbf{w} = w_0\dots w_n \in W^*$, the function $\psi^{\mathbf{w}}_f = \psi^{w_0}_0 \circ \dots \circ \psi^{w_n}_n\colon J \to J^{\mathbf{w}}$ and we set $\underline{\psi_f} = \{\psi^{\mathbf{w}}_f\}_{\mathbf{w}\in W^*}$.

Proposition 2.4.1. *Let f_* denote the unimodal fixed point of renormalisation with presentation function $\boldsymbol{\psi}_*$. Then, for each $\mathbf{w} \in W^m$, the following properties hold,*

(i) $\psi^{\mathbf{w}}_* = f_*^{-\mathbf{q}(\mathbf{w})} \circ \iota_{J\to J^{0^n}}$*;*

(ii) $\psi^{\mathbf{w}}_*(\bigcup_{\mathbf{w}\in W^n} J^{\mathbf{w}}_*) \subset \bigcup_{\mathbf{w}\in W^{n+m}} J^{\mathbf{w}}_*$*;*

where $\mathbf{q}\colon W^* \to \mathbb{Z}_+$ is the correspondence between the indexing by renormalisation and the indexing by iterates More precisely, if $\mathbf{w} = w_0\dots w_n$ then $\mathbf{q}(\mathbf{w}) = \sum p^i(p-w_i)$.

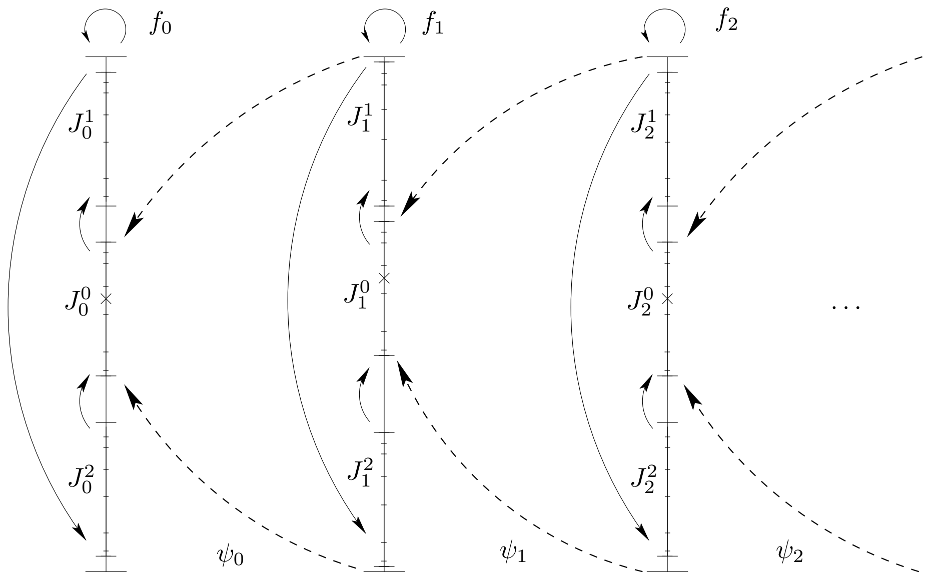

Figure 2.2: The collection of scope maps ψ_n^w for an infinitely renormalisable period-tripling unimodal map. Here f_n denotes the n-th renormalisation of f

Proof. We will show the first item inductively. Trivially it is true for $m = 0$. Assume it holds some $\mathbf{w} \in W^m$ for $m \geq 0$ and consider $w\mathbf{w} \in W^{m+1}$. Since $\mathcal{R}_{\mathcal{U}} f_* = f_*$ implies $f_*^p \circ \iota_{J \to J_*^0} = \iota_{J \to J_*^0} \circ f_*$, we find

$$\begin{aligned}
\psi_*^{w\mathbf{w}} &= \psi_*^w \circ \psi_*^{\mathbf{w}} \\
&= f_*^{\circ -(p-w)} \circ \iota_{J \to J_*^0} \circ f_*^{\circ -\mathbf{q}(\mathbf{w})} \circ \iota_{J \to J^{0^m}} \\
&= f_*^{\circ -(p-w)} \circ f_*^{\circ -p\mathbf{q}(\mathbf{w})} \circ \iota_{J \to J^0} \circ \iota_{J \to J^{0^m}} \\
&= f_*^{\circ -\mathbf{q}(w\mathbf{w})} \circ \iota_{J \to J^{0^{m+1}}}
\end{aligned} \tag{2.4.2}$$

This proves the first statement. The second statement then follows from the first since, given $\mathbf{w} \in W^n$, and $\tilde{\mathbf{w}} \in W^m$, the image of $J_*^{\mathbf{w}}$ under $\psi_*^{\mathbf{w}}$ can be expressed as a preimage of $J_*^{0^{m+n}}$ under f_*. $\square$

Taking limits then gives us the following immediate Corollary.

Corollary 2.4.2. *Let f_* denote the unimodal fixed point of renormalisation with presentation function $\boldsymbol{\psi}_*$. Let $\mathcal{O}_*$ denote the invariant Cantor set for f_*. Then, for each $\mathbf{w} \in W$, the following properties hold,*

(i) $\psi_*^{\mathbf{w}}(\underline{J}_*) \subset \underline{J}_*$;

(ii) $\psi_*^{\mathbf{w}}(\mathcal{O}_*) \subset \mathcal{O}_*$;

Lemma 2.4.3. *Let f_* denote the unimodal fixed point of renormalisation with presentation function $\boldsymbol{\psi}_*$. Then, for each $w \in W$, the following properties hold,* *(i)* ψ_*^w *has a unique attracting fixed point* α*;*

(ii) if $[\psi_*^{w^n}]$ *denotes the orientation preserving affine rescaling of* $\psi_*^{w^n}$ *to* J *then* $u_*^w = \lim_{n\to\infty}[\psi_*^{w^n}]$ *exists, and the convergence is exponential.*

Proof. It is clear that there exists a fixed point α, as ψ_*^w maps J into itself. It is also unique, since by construction $J_*^{w^{n+1}} = \psi_*^w(J_*^{w^n})$, and all images of J must contain all fixed points. However Theorem 2.2.10 implies $|J_*^{w^n}| \to 0$ as $n \to \infty$, and hence there can be only one fixed point.

Now let us show α is attracting. Theorem 2.2.10 tells us, since $J_*^{w^{n+1}} \subset J_*^{w^n}$, that $|J_*^{w^{n+1}}|/|J_*^{w^n}| < k_1 < 1$. By the Mean Value Theorem this implies there are points $\alpha_n \in J_*^{w^n}$ such that $|(\psi_*^w)'(\alpha_n)| = |J_*^{w^{n+1}}|/|J_*^{w^n}| < k_1$. Also, since $\alpha \in J_*^{w^n}$ for all $n > 0$ and $|J_*^{w^n}| \to 0$ as $n \to \infty$, we have $\alpha_n \to \alpha$. As ψ_*^w is analytic we must have $|(\psi_*^w)'(\alpha)| < k_1$. Hence α if α has multiplier σ_w, $|\sigma_w| < 1$ and so α is attracting.

For the second item let $u_n^w = \iota_{J_*^{w^n}\to J} \circ \psi_*^{w^n} \colon J \to J$. First we claim that there is a domain $U \subset \mathbb{C}$ containing J on which u_n^w has a univalent extension. This follows as α is an attracting fixed point and ψ_*^w is real-analytic on J, so there exists a domain $V \subset \mathbb{C}$ containing α on which ψ_*^w is univalent and $\psi_*^w(V) \subset V$. By Theorem 2.2.10 there exists an integer $N > 0$ such that $(\psi_*^w)^{\circ n}(J) \subset V$ for all $n \geq N$. Therefore take any domain U containing J such that $(\psi_*^w)^{\circ n}(U)$ is bounded away from the set of the first p^N critical values of f_* for all $n < N$. Then u_n^w will be univalent on U.

Observe that, letting $v_n^w = \mathcal{Z}_{J_*^{w^n}} \psi_*^w$ where $\mathcal{Z}_T$ denotes the zoom operator on the interval T, we can write

$$u_n^w = v_n^w \circ \cdots \circ v_0^w \tag{2.4.3}$$

Also observe that the argument above gives a domain W containing J on which each of these composants has a univalent extension. Therefore

$$\begin{aligned} |u_n^w - u_{n+1}^w|_W &= |v_n^w \circ \cdots \circ v_0^w - v_{n+1}^w \circ v_n^w \circ \cdots \circ v_0^w|_W \\ &= |\operatorname{id} - v_n^w|_{u_n^w(W)}. \end{aligned} \tag{2.4.4}$$

Theorem 2.2.10 implies $|J_*^{w^n}| \to 0$ exponentially as $n \to \infty$. Analyticity of ψ_*^w then implies $\operatorname{Dis}(\psi_*^w; J_*^{w^n}) \to 0$ exponentially as well. Moreover, also by analyticity, this holds on a subdomain W^n of W containing $J_*^{w^n}$. Hence, by the Mean Value Theorem,

$$\left|1 - \frac{|J_*^{w^n}|}{|J_*^{w^{n+1}}|}\psi_*^w\right|_{J^{w^n}} \to 0 \tag{2.4.5}$$

exponentially, and this will also hold on W^n if $n > 0$ is sufficiently large. Integrating then gives us

$$|\operatorname{id} - v_n^w|_{u_n^w(W)} \to 0 \tag{2.4.6}$$

exponentially, and hence $|u_n^w - u_{n+1}^w|_W \to 0$ exponentially. This implies the limit u_*^w exists and is univalent on W. □

Remark 2.4.4. In the period doubling case more precise information was given by Birkhoff, Martens and Tresser in [4]. Since the renormalisation fixed point f_* is convex in this case (see [11]), the fixed point of f_* is expanding and separates J_*^0 and J_*^1, we know that $f_*|J_*^1$ is expanding and hence ψ_*^1 is contracting. This simplified the construction of the renormalisation Cantor set for a strongly dissipative nondegenerate Hénon-like map given by de Carvalho, Lyubich and Martens in [12].

Proposition 2.4.5. *Let υ be a unimodal permutation and let $\upsilon(n)$ be the unimodal permutation satisfying $\mathcal{R}^n_{\mathcal{U},\upsilon} = \mathcal{R}_{\mathcal{U},\upsilon(n)}$. Given an n-times renormalisable $f \in \mathcal{U}_{\Omega_x,\upsilon}$ let*

$$f_{\upsilon,i} = \mathcal{R}^i_{\mathcal{U},\upsilon} f \quad \text{and} \quad f_{\upsilon(n)} = \mathcal{R}_{\mathcal{U},\upsilon(n)} f. \tag{2.4.7}$$

for all $i = 0, 1, \ldots, n$. Let $\boldsymbol{\psi}_{\upsilon,i}$ denote the presentation function for $f_{\upsilon,i}$ with respect to $\mathcal{R}_{\mathcal{U},\upsilon}$ and let $\boldsymbol{\psi}_{\upsilon(n)}$ denote the presentation function for $f_{\upsilon(n)}$ with respect to $\mathcal{R}_{\mathcal{U},\upsilon(n)}$. Then

$$\boldsymbol{\psi}_{\upsilon(n)} = \{\psi^{w_0}_{\upsilon,0} \circ \ldots \circ \psi^{w_n}_{\upsilon,n}\}_{w_0,\ldots,w_n \in W} \tag{2.4.8}$$

Proof. This follows from the fact that $\mathcal{R}_{\mathcal{U},\upsilon} f_{\upsilon,i} = f_{\upsilon,i+1}$ implies $f^p_{\upsilon,i} \circ \iota_{J\to J_i^0} = \iota_{J\to J_i^0} \circ f_{\upsilon,i+1}$ and the fact that the central interval of f under $\mathcal{R}_{\mathcal{U},\upsilon(n)}$ is equal to J^{0^n}. □

Proposition 2.4.6. *Let υ be a unimodal permutation. There exists a constant $C > 0$ such that for any $f_0, f_1 \in \mathcal{U}_{\Omega_x,\upsilon}$, and any $w \in W$,*

$$|\psi^w_{f_0} - \psi^w_{f_1}|_{\Omega_x} \leq C|f_0 - f_1|_{\Omega_x} \tag{2.4.9}$$

Proof. As both f_0 and f_1 are renormalisable let $J_0, J_1 \subset J$ denote the central interval for f_0 and f_1 respectively. Let $J_0^w = f^{\circ w}(J_0)$ and $J_1^w = f_*^{\circ w}(J_1)$ for all $w \in W$. Then by Corollary A.2.3 in the Appendix, as the boundary points of J_i^w are periodic or pre-periodic points, there exists a constant $K_0 > 0$ such that

$$\mathrm{dist}_{Haus}(J_0^w, J_1^w) < K_0|f_0 - f_1|_{\Omega_x}. \tag{2.4.10}$$

This implies, by Proposition A.2.5 in the Appendix, that there exists a constant $K_1 > 0$ such that

$$|\mathcal{Z}_{J_0^w} f_0 - \mathcal{Z}_{J_1^w} f_1|_{\Omega_x} < K_1|f_0 - f_1|_{\Omega_x}, \tag{2.4.11}$$

and also implies there is a constant $K_2 > 0$ such that

$$|\iota_{J\to J_0^w} - \iota_{J\to J_1^w}|_{\Omega_x} < K_2|f_0 - f_1|_{\Omega_x} \tag{2.4.12}$$

By Proposition A.2.6 in the Appendix, there exists a constant $K_3 > 0$ such that

$$|\mathcal{Z}_{J_0^w} f_0^{\circ p-w} \circ \iota_{J_0^w\to J} - \mathcal{Z}_{J_1^w} f_1^{\circ p-w} \circ \iota_{J_1^w\to J}|_{\Omega_x} < K_2|f_0 - f_1|_{\Omega_x}. \tag{2.4.13}$$

The result then follows by applying Proposition A.2.7 from the Appendix and observing that, as $\mathcal{U}_{\Omega_x}$ is compact the constant C can be chosen uniformly. □

Corollary 2.4.7. *There exist constants $C > 0$ and $0 < \rho < 1$ such that the following holds: given any infinitely renormalisable $f \in \mathcal{U}_{adapt}$,*

$$|\psi_n^w - \psi_*^w|_{\Omega_x} \leq C\rho^n. \tag{2.4.14}$$

Proof. From Theorem 2.3.4 we know that there are constants $C > 0, 0 < \rho < 1$ such that $|f_n - f_*| < C\rho^n$, where f_n denotes the n-th renormalisation of f. Applying Proposition 2.4.6, the result follows. □

2.5 A Reinterpretation of the Operator

Let us now consider $\mathcal{H}_\Omega(0)$, defined to be the space of maps $F \in C^\omega(B, B)$ of the form $F = (f \circ \pi_x, \pi_x)$ where $f \in \mathcal{U}_{\Omega_x}$. Let us also consider the subspace $\mathcal{H}_{\Omega,\upsilon}(0)$ of maps $F = (f \circ \pi_x, \pi_x)$ where $f \in \mathcal{U}_{\Omega_x,\upsilon}$. These will be called the space of *degenerate Hénon-like maps* and the space of *renormalisable degenerate Hénon-like maps* respectively. The reasons for this will become apparent in Section 3 when we introduce non-degenerate Hénon-like maps. For now, observe there is an imbedding $\underline{\mathrm{i}}\colon \mathcal{U}_{\Omega_x} \to \mathcal{H}_\Omega(0)$, given by $\underline{\mathrm{i}}(f) = (f \circ \pi_x, \pi_x)$, which restricts to an imbedding $\underline{\mathrm{i}}\colon \mathcal{U}_{\Omega_x,\upsilon} \to \mathcal{H}_{\Omega,\upsilon}(0)$. We will construct an operator $\mathcal{R}$, defined on $\mathcal{H}_\Omega(0)$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}_{\Omega_x,\upsilon} & \xrightarrow{\mathcal{R}_\mathcal{U}} & \mathcal{U}_{\Omega_x} \\ \downarrow{\scriptstyle \underline{\mathrm{i}}} & & \downarrow{\scriptstyle \underline{\mathrm{i}}} \\ \mathcal{H}_{\Omega,\upsilon}(0) & \xrightarrow[\mathcal{R}]{} & \mathcal{H}_\Omega(0) \end{array} \tag{2.5.1}$$

Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$, let $\{J^w\}_{w\in W}$ be its renormalisation cycle and let $\{J'^w\}_{w\in W}$ be the set of corresponding maximal extensions. Let $F = \underline{\mathrm{i}}(f)$ be the corresponding degenerate Hénon-like map, let

$$B^w = J^{w+1} \times J^w, \quad B'^w = J'^{w+1} \times J'^w \tag{2.5.2}$$

and let

$$B^w_{\mathrm{diag}} = J^w \times J^w, \quad B'^w_{\mathrm{diag}} = J'^w \times J'^w, \tag{2.5.3}$$

where $w \in W$ is taken modulo p. Observe B^w is invariant under $F^{\circ p}$ for each $w \in W$.

Consider the map $H\colon B \to B$ defined by $H = (f^{\circ p-1}, \pi_y)$. By the Inverse Function Theorem this will be a diffeomorphism onto its image on any connected open set bounded away from the critical curve $\mathcal{C}^{p-1} = \{(x,y) : (f^{\circ p-1})'(x) = 0\}$. In particular, since the box B^0 is bounded away from $\mathcal{C}^{p-1}$ whenever the maximal extensions are proper extensions, the map H will be a diffeomorphism there. We will call B^0 the *central box*. We will call the map H the *horizontal diffeomorphism*. Observe that $B^0_{\mathrm{diag}} = H(B^0)$. Recall $\bar{H}\colon B^0_{\mathrm{diag}} \to B^0$ denotes the inverse of H restricted to B^0_{diag}. The map

$$G = H \circ F^{\circ p} \circ \bar{H}\colon B^0_{\mathrm{diag}} \to B^0_{\mathrm{diag}}, \tag{2.5.4}$$

is called the *pre-renormalisation* of F around B^0. Let I denote the affine bijection from B^0_{diag} onto B such that the map

$$\mathcal{R}F = I \circ G \circ \bar{I} \colon B \to B \tag{2.5.5}$$

is again a degenerate Hénon-like map where $\bar{I}$ denotes the inverse of I. Then $\mathcal{R}F$ is called the *Hénon renormalisation* of F around B^0 and the operator $\mathcal{R}$ is called the *renormalisation operator* on $\mathcal{H}_{\Omega_x,\upsilon}(0)$. Observe that $\mathcal{R}F = (\mathcal{R}_{\mathcal{U}} f \circ \pi_x, \pi_x)$.

Remark 2.5.1. More generally, by the same argument as above, H will be a diffeomorphism onto its image when restricted to any B^w_{diag}. Since $B^w_{\mathrm{diag}} = H(B^w)$ and B^w is invariant under $F^{\circ p}$ by construction, the maps

$$G^w = H \circ F^{\circ p} \circ \bar{H} \colon B^w_{\mathrm{diag}} \to B^w_{\mathrm{diag}}. \tag{2.5.6}$$

are well defined. We will call G^w the w-th *pre-renormalisation.* There are affine bijections I^w from B^w_{diag} to B such that

$$\mathcal{R}_w F = I^w \circ G^w \circ \bar{I}^w \colon B \to B \tag{2.5.7}$$

is again a degenerate Hénon-like map where, as above, $\bar{I}^w$ denotes the inverse of I^w. Then the map $\mathcal{R}_w F$ is called the *Hénon renormalisation* of F around B^w and the operator $\mathcal{R}_w$ is called the w-th *renormalisation operator* on $\mathcal{H}_{\Omega_x,\upsilon}(0)$. Observe that $\mathcal{R}_w F = (\mathcal{R}_{\mathcal{U},w} f \circ \pi_x, \pi_x)$, where $\mathcal{R}_{\mathcal{U},w}$ denotes the renormalisation around J^w.

Remark 2.5.2. The affine bijections I^w in the remark above map squares to squares. Hence the linear part of I^w has the form

$$\pm \begin{pmatrix} \sigma^w & 0 \\ 0 & \pm\sigma^w \end{pmatrix} \tag{2.5.8}$$

for some $\sigma^w > 0$. Here the sign depends upon the combinatorial type of υ only. We call the quantity σ^w the w-th *scaling ratio* of F.

Remark 2.5.3. Since $\underline{\iota}$ is an imbedding preserving the actions of $\mathcal{R}_{\mathcal{U}}$ and $\mathcal{R}$ it is clear that $\mathcal{R}$ also has a unique fixed point F_*. Moreover, it must have the form $F_* = (f_* \circ \pi_x, \pi_x)$ where f_* is the fixed point of $\mathcal{R}_{\mathcal{U}}$. Then F_* also has a codimension one stable manifold and dimension one local unstable manifold.

Now given $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$ we let $\Psi = \bar{H} \circ \bar{I} \colon B \to B^0$ and $\Psi^w = F^{\circ w} \circ \Psi \colon B \to B^w$. Then Ψ^w is called the *w-th scope function* of F. The reason for this terminology is given by the following Proposition.

Proposition 2.5.4. *Let* $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$. *Then*

$$\Psi^w_F(x,y) = \begin{cases} (\psi^{w+1}_f(x), \psi^w_f(x)) & w > 0 \\ (\psi^{w+1}_f(x), \psi^w_f(y)) & w = 0 \end{cases}, \tag{2.5.9}$$

where ψ^w_f *denotes the w-th scope function for* f.

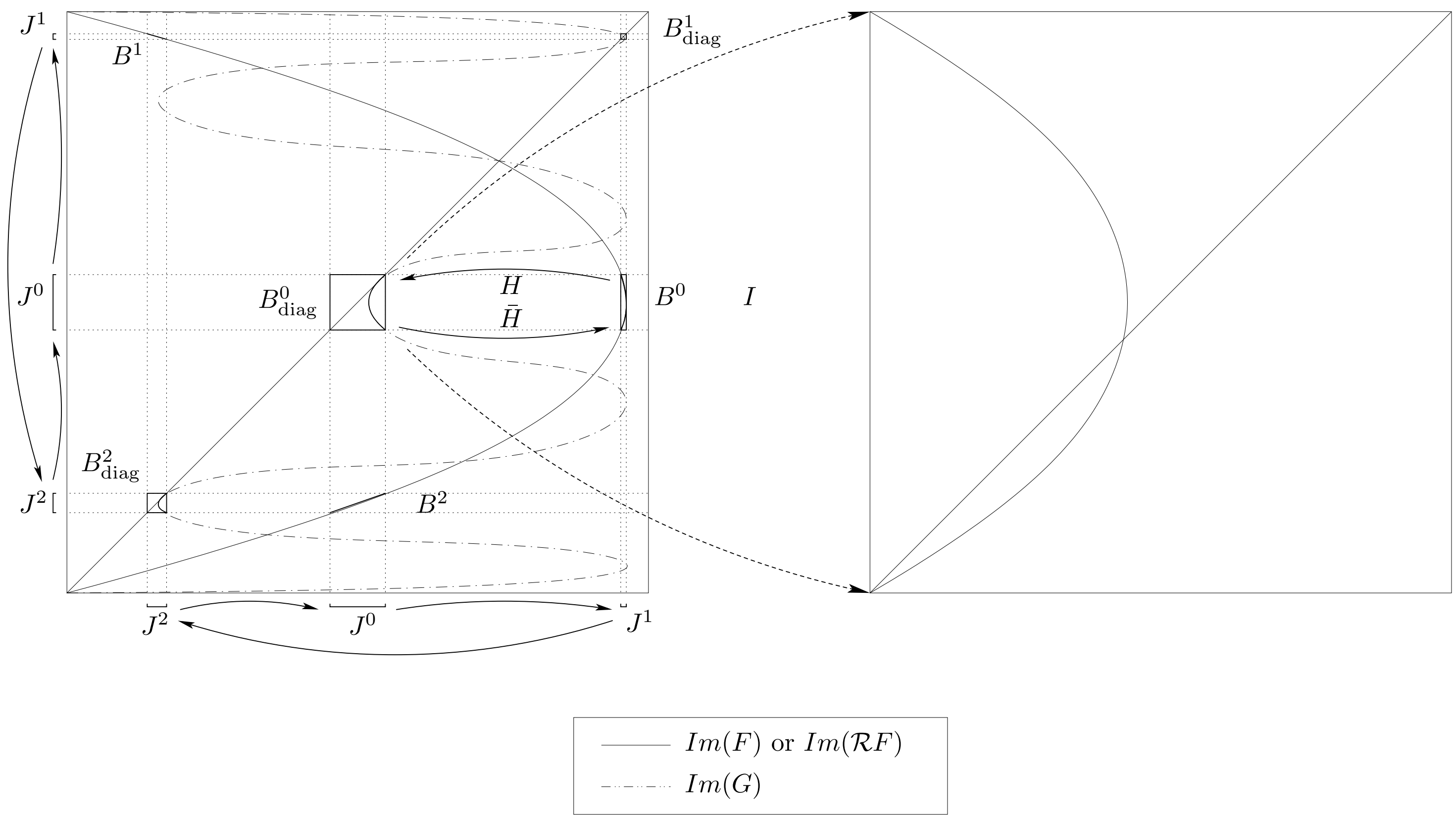

Figure 2.3: A period-three renormalisable unimodal map considered as a degenerate Hénon-like map. In this case the period is three. Observe that the image of the pre-renormalisation lies on the smooth curve $(f^{\circ 3}(x), x)$.

Proof. Observe that $\bar{H}(x,y) = (f^{\circ -p+1}(x), y)$ and

$$F^{\circ w}(x,y) = \begin{cases} (f^{\circ w}(x), f^{\circ w-1}(x)) & w > 0 \\ (x,y) & w = 0 \end{cases} \tag{2.5.10}$$

which implies

$$F^{\circ w} \circ \bar{H}(x,y) = \begin{cases} (f^{\circ -p+w+1}(x), f^{\circ -p+w}(x)) & w > 0 \\ (f^{\circ -p+1}(x), y) & w = 0 \end{cases} \tag{2.5.11}$$

where appropriate branches of $f^{\circ -p+w+1}$ and $f^{\circ -p+w}$ are chosen. Also observe $\bar{I}(x,y) = (\iota_{J\to J^0}(x), \iota_{J\to J^0}(y))$. Composing these gives us the result. □

Remark 2.5.5. Only the zero-th scope function $\Psi = \Psi^0$ is a diffeomorphism onto its image.

Now assume $F \in \mathcal{H}_{\Omega,\upsilon}$ is n-times renormalisable and denote its n-th renormalisation $\mathcal{R}^n F$ by F_n. Then for each F_n we can construct the w-th scope function $\Psi_n^w = \Psi^w(F_n)\colon \mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_n)$, where $\mathrm{Dom}(F_n) = B$ denotes the domain of F_n. Then for $\mathbf{w} = w_0 \dots w_n \in W^*$ the function

$$\Psi^{\mathbf{w}} = \Psi_0^{w_0} \circ \dots \circ \Psi_n^{w_n}\colon \mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_0) \tag{2.5.12}$$

is called the $\mathbf{w}$-scope function. Let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}$ denote the collection of all scope functions, in both the case when $n > 0$ is finite and infinite. The following Corollary is an immediate consequence of the above Proposition.

Corollary 2.5.6. *Let $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$ be an n-times renormalisable degenerate Hénon-like map. Then given a word $\mathbf{w} = w_0 \dots, w_{n-1} \in W^n$*

$$\Psi_f^{\mathbf{w}}(x,y) = \begin{cases} (\psi_f^{\mathbf{w}+1^n}(x), \psi_f^{\mathbf{w}}(x)) & \mathbf{w} \neq 0 \\ (\psi_f^{\mathbf{w}+1^n}(x), \psi_f^{\mathbf{w}}(y)) & \mathbf{w} = 0 \end{cases}, \tag{2.5.13}$$

where $\psi_f^{\mathbf{w}}$ denotes the $\mathbf{w}$-th scope function for f.

In particular we may do this for F_*, the renormalisation fixed point, giving

$$\Psi_*^w(x,y) = \begin{cases} (\psi_*^{w+1}(x), \psi_*^w(x)) & w > 0 \\ (\psi_*^{w+1}(x), \psi_*^w(y)) & w = 0 \end{cases}, \tag{2.5.14}$$

where ψ_*^w are the branches of the presentation function. We will denote the family of scope functions for F_* by $\underline{\Psi}_* = \{\Psi_*^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ where $\Psi_*^{\mathbf{w}}\colon B \to B_*^{\mathbf{w}}$ is constructed as above.

Chapter 3

Hénon-like Maps

In this chapter we generalise the construction given in [12] of a Renormalisation operator acting on a space of Hénon-like maps (defined below). We show that the standard unimodal renormalisation picture can be extended to the space of such maps if the Hénon-like maps are sufficiently dissipative. We then examine the dynamics of infinitely renormalisable maps and show, as in the unimodal case, such maps have an invariant Cantor set on which the map acts like an adding machine. We do this by introducing *Scope maps*, which are certain coordinate changes related to the renormalisation of a Hénon-like map. We then make estimates on the asymptotics of particular compositions of scope maps. It is in these bounds that we first see universal quantities from the unimodal theory appearing.

3.1 The Space of Hénon-like Maps

Let $\bar{\varepsilon} > 0$. Let $\mathcal{T}_\Omega(\bar{\varepsilon})$ denote the space of maps $\varepsilon \in C^\omega(B, \mathbb{R})$, which satisfy

(i) $\varepsilon(x, 0) = 0$;

(ii) $\varepsilon(x, y) \geq 0$;

(iii) ε admits a holomorphic extension to Ω;

(iv) $|\varepsilon|_\Omega \leq \bar{\varepsilon}$, where $|-|_\Omega$ denotes the sup-norm on Ω.

Such maps will be called *thickenings* or *$\bar{\varepsilon}$-thickenings*. Let $B' = J' \times J' \subset \mathbb{R}^2$ for some closed interval $J' \subset \mathbb{R}$. Given $\varepsilon' \in C^\omega(B', \mathbb{R})$ let $E'(x, y) = (x, \varepsilon'(x, y))$. If there is an affine bijection $I\colon B' \to B$ such that $E(x, y) = I \circ E' \circ \bar{I}(x, y) = (x, \varepsilon(x, y))$ where ε is a thickening, then we say ε' is a *thickening on B'*.

For a unimodal map $f \in \mathcal{U}_{\Omega_x}$ and a constant $\bar{\varepsilon} > 0$ let $\mathcal{H}_\Omega(f, \bar{\varepsilon})$ denote the space of $F \in \mathrm{Emb}^\omega(B, B)$ such that F is expressible as $F = (f \circ \pi_x - \varepsilon, \pi_x)$ for some $f \in \mathcal{U}_{\Omega_x}$ and $\varepsilon \in \mathcal{T}_\Omega(\bar{\varepsilon})$. Then we let

$$\mathcal{H}_\Omega(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega_x}} \mathcal{H}_\Omega(f, \bar{\varepsilon}) \tag{3.1.1}$$

and

$$\mathcal{H}_\Omega = \bigcup_{\bar{\varepsilon}>0} \mathcal{H}_\Omega(\bar{\varepsilon}). \tag{3.1.2}$$

The maps $F \in \mathcal{H}_\Omega$ will be called *parametrised Hénon-like maps* with parametrisation (f, ε). We will just write $F = (\phi, \pi_x)$ when the parametrisation is not explicit. In the current setting we will simply call them Hénon-like maps. Observe that the degenerate Hénon-like maps considered in Section 2 will lie in a subset of the boundary of $\mathcal{H}_\Omega(\bar{\varepsilon})$ for all $\bar{\varepsilon} > 0$. Given a square $B' \subset \mathbb{R}^2$ a map $F \in \mathrm{Emb}^\omega(B', B')$ is *Hénon-like on* B' if there exists an affine bijection $I\colon B' \to B$ such that $I \circ F \circ \bar{I}\colon B \to B$ is a Hénon-like map.

Given a Hénon-like map $F = (\phi, \pi_x)\colon B \to F(B)$ its inverse will have the form $F^{\circ -1} = (\pi_y, \phi^{-1})\colon F(B) \to B$ where $\phi^{-1}\colon F(B) \to J$ satisfies

$$\pi_y = \phi^{-1}(\phi, \pi_x); \quad \pi_x = \phi(\pi_y, \phi^{-1}). \tag{3.1.3}$$

More generally, given an integer $w > 0$ let us denote the w-th iterate of F by $F^{\circ w}\colon B \to B$, and the w-th preimage by $F^{\circ -w}\colon F^{\circ w}(B) \to B$. Observe that they have the respective forms $F^{\circ w} = (\phi^w, \phi^{w-1})$ and $F^{\circ -w} = (\phi^{-w+1}, \phi^{-w})$ for some functions $\phi^w\colon B \to J$ and $\phi^{-w}\colon F^{\circ w}(B) \to J$.

We then define the *w-th critical curve* or *critical locus* to be the set $\mathcal{C}^w = \mathcal{C}^w(F) = \{\partial_x \phi^w(x, y) = 0\}$.

3.2 Construction of an Operator

Let us consider the operators $\mathcal{R}_\mathcal{U}$ and $\mathcal{R}$ from Section 2.5. Observe that $\mathcal{R}_\mathcal{U}$ is constructed as some iterate under an affine coordinate change whereas $\mathcal{R}$ uses non-affine coordinate changes. That they are equivalent is a coincidence that we shall now exploit.

Our starting point is that non-trivial iterates of non-degenerate $F \in \mathcal{H}_\Omega$ will most likely not have the form $(f \circ \pi_x \pm \varepsilon, \pi_x)$ after affine rescaling. Therefore, unlike the one dimensional case, we will need to perform a 'straightening' via a non-affine change of coordinates.

Definition 3.2.1. Let $p > 1$ be an integer. A map $F \in \mathcal{H}_\Omega$ is *pre-renormalisable with combinatorics p* if the following properties hold,

(i) there exists a closed topological disk $B^0 \subset B$ with $F^{\circ p}(B^0) \subset B^0$;

(ii) there exists a diffeomorphism $H\colon B^0 \to B^0_{\mathrm{diag}}$, where B^0_{diag} is a square, symmetric about the diagonal $\{x = y\}$.

The domain B^0 is called the *pre-renormalisation domain*. The map $G = H \circ F^{\circ p} \circ \bar{H}\colon B^0_{\mathrm{diag}} \to B^0_{\mathrm{diag}}$ is called the *pre-renormalisation* of F.

Definition 3.2.2. Let $p > 1$ be an integer. A map $F \in \mathcal{H}_\Omega$ is *renormalisable with combinatorics p* if the following properties hold,

(i) F is pre-renormalisable with combinatorics p;

(ii) the domains $B^w = F^{\circ w}(B^0)$, $w \in W$, are pairwise disjoint;

(iii) if B^0_{diag} denotes the corresponding square, symmetric about the diagonal, there exists an affine map $I\colon B^0_{\text{diag}} \to B$ such that the map

$$\mathcal{R}F = I \circ G \circ \bar{I}\colon B \to B \tag{3.2.1}$$

is an element of $\mathcal{H}_\Omega$, where G denotes the pre-renormalisation of F.

Then the map $\mathcal{R}F$ is called the *Hénon-renormalisation* of F. We will denote space of all renormalisable maps by $\mathcal{H}_{\Omega,p}$. . The operator $\mathcal{R}\colon \mathcal{H}_{\Omega,p} \to \mathcal{H}_\Omega$ given by $F \mapsto \mathcal{R}F$ is called the *Hénon-renormalisation operator* or simply the *renormalisation operator* on $\mathcal{H}_\Omega$. The absolute value of the eigenvalues of the linear part of $\bar{I}$ (which coincide as it maps a square box to a square box) is called the *scaling ratio* of F.

Notation 3.2.3. We will denote the subspace of $\mathcal{H}_{\Omega,p}$ consisting of renormalisable maps expressible as $F = (f + \varepsilon, \pi_x)$, where $|\varepsilon|_\Omega < \bar{\varepsilon}$, by $\mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$ and will let $\mathcal{H}_{\Omega,v}(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega_x}} \mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$ denote their union.

Definition 3.2.4. Let $p > 1$ be an integer. Let $0 < \gamma < 1$. Let $F \in \mathcal{H}_\Omega$ have pre-renormalisation domain B^0 of type p. Let $B^w = F^{\circ w}(B)$ denote the w-th image of the pre-renormalisation domain, $w \in W$. If the following properties hold,

(i) $\operatorname{dist}(B^{w_0}, B^{w_1}) \geq \gamma$ for all distinct $w_0, w_1 \in W$;

(ii) $\operatorname{dist}(\mathcal{C}^{p-1}, B^0_{\text{diag}}) \geq \gamma$;

where $\mathcal{C}^{p-1}$ denotes the critical curve, then we say F has the *γ-gap property.*

There are, a priori, many coordinate changes which suffice. However, we will now choose one canonically which has sufficient dynamical meaning. By analogy with the degenerate case, consider the map $H = (\phi^{p-1}, \pi_y)$. The Inverse Function Theorem tells us this will be a diffeomorphism on any open set bounded away from the critical curve $\mathcal{C}^{p-1} = \{(x,y) \in B : \partial_x \phi^{p-1}(x,y) = 0\}$. Hence, abusing terminology slightly, we will call this map the *horizontal diffeomorphism* associated to F. Also consider the map $V = F^{\circ p-1} \circ \bar{H}\colon H(B^0) \to B^{p-1}$. Since $F^{\circ p-1}$ is a diffeomorphism onto its image everywhere and H is a diffeomorphism onto its image when restricted to B^0 we find that V is also a diffeomorphism onto its image. We will call V the *vertical diffeomorphism.* The reason for considering the maps H and V is given by the following Proposition.

Proposition 3.2.5. *Let $F = (\phi, \pi_x) \in \mathcal{H}_\Omega$. Assume that, for some integer $p > 1$, the following properties hold,*

(i) $B^0 \subset B$ is a subdomain on which $F^{\circ p}$ is invariant;

(ii) the horizontal diffeomorphism $H = (\phi^{p-1}, \pi_y)$ is a diffeomorphism onto its image when restricted to B^0.

Then $H \circ F^{\circ p} \circ \bar{H} \colon H(B^0) \to H(B^0)$ has the form

$$H \circ F^{\circ p} \circ \bar{H}(x, y) = (\phi^p \circ V(x, y), x) \tag{3.2.2}$$

where V is the vertical diffeomorphism described above. Moreover, the vertical diffeomorphism has the form $V(x, y) = (x, v(x, y))$ for some $v \in C^\omega(B, J)$.

Proof. Observe $\bar{H}$ has the form $\bar{H} = (\bar{\phi}^{p-1}, \pi_y)$ for some $\bar{\phi}^{p-1} \colon H(B^0) \to \mathbb{R}$. Equating $F^{\circ p-1} \circ F$ with $F^{\circ p}$ implies $\phi^{p-1}(\phi, \pi_x) = \phi^p$, while equating $H \circ \bar{H}$ and $\bar{H} \circ H$ with the identity implies

$$\pi_x = \phi^{p-1}(\bar{\phi}^{p-1}, \pi_y) = \bar{\phi}^{p-1}(\phi^{p-1}, \pi_y). \tag{3.2.3}$$

Hence, by definition of H and V we find

$$\begin{aligned} H \circ F &= (\phi^{p-1}(\phi, \pi_x), \pi_x) \\ &= (\phi^p, \pi_x) \end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned} V &= F^{\circ p-1} \circ \bar{H} \\ &= (\phi^{p-1}(\bar{\phi}^{p-1}, \pi_y), \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)) \\ &= (\pi_x, \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)). \end{aligned} \tag{3.2.5}$$

Therefore if we set $v(x, y) = \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)$ the result is shown. □

We now show that maps satisfying the hypotheses of the above Proposition exist, are numerous and in fact renormalisable in the sense described above. More precisely, we show that $\mathcal{R}$ is defined on a tubular neighbourhood of $\mathcal{H}_{\Omega,\upsilon}(0)$ in the closure of $\mathcal{H}_\Omega$. This is essentially a perturbative result. To do this we will need the following.

Proposition 3.2.6 (variational formula of the first order)**.** *Let $F \in \mathcal{H}_\Omega$ be expressible as $F = (\phi, \pi_x)$ where $\phi(x, y) = f(x) + \varepsilon(x, y)$. Then, for all $w \in W$,*

$$\phi^w(x, y) = f^{\circ w}(x) + L^w(x) + \varepsilon(x, y)(f^{\circ w})'(x) + \mathrm{O}(\bar{\varepsilon}^2) \tag{3.2.6}$$

where

$$\begin{aligned} L^w(x) =& \varepsilon(f^{\circ w-1}(x), f^{\circ w-2}(x)) + \varepsilon(f^{\circ w-2}(x), f^{\circ w-3}(x)) f'(f^{\circ w-1}(x)) \\ &+ \ldots + \varepsilon(f(x), x) \prod_{i=1}^{w-1} f'(f^{\circ i}(x)) \end{aligned} \tag{3.2.7}$$

Proof. We proceed by induction. Assume this holds for all integers $0 < i < w$ and let $L^w(x)$ be as above. Then

$$\begin{aligned} \phi^w(x, y) &= \phi(\phi^{w-1}(x, y), \phi^{w-2}(x, y)) \\ &= f(\phi^{w-1}(x, y)) + \varepsilon(\phi^{w-1}(x, y), \phi^{w-2}(x, y)) \end{aligned} \tag{3.2.8}$$

but observe, by Taylors' Theorem,

$$\begin{aligned} f(\phi^{w-1}(x,y)) &= f(f^{\circ w-1}(x) + L^{w-1}(x) + \varepsilon(x,y)(f^{\circ w-1})'(x) + \mathrm{O}(\bar{\varepsilon}^2)) \\ &= f^{\circ w}(x) + f'(f^{\circ w-1}(x))L^{w-1}(x) + \varepsilon(x,y)(f^{\circ w})'(x) + \mathrm{O}(\bar{\varepsilon}^2) \end{aligned} \tag{3.2.9}$$

and

$$\begin{aligned} \varepsilon(\phi^{w-1}(x,y), \phi^{w-2}(x,y)) &= \varepsilon((f^{\circ w-1}(x), f^{\circ w-2}(x)) + \mathrm{O}(\bar{\varepsilon})) \\ &= \varepsilon(f^{\circ w-1}(x), f^{\circ w-2}(x)) + \mathrm{O}(\bar{\varepsilon}^2), \end{aligned} \tag{3.2.10}$$

where we have used, since ε is analytic, that all derivatives of ε are of the order $\bar{\varepsilon}$. Combining these gives us the result. □

Corollary 3.2.7. *Let $F \in \mathcal{H}_\Omega$ be expressible as $F = (\phi, \pi_x)$ where $\phi(x,y) = f(x) + \varepsilon(x,y)$. For all $w \in W$ let us define the functions H^w acting on B by $H^w(x,y) = (\phi^w(x,y), y)$. Assume they have well-defined inverses $\bar{H}^w(x,y) = (\bar{\phi}^w(x,y), y)$ when restricted to some subdomain B_H of the image of H^w. Then*

$$\bar{\phi}^w(x,y) = f^{\circ -w}(x) + \bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \tag{3.2.11}$$

where

$$\bar{L}^w(x,y) = -\frac{L^w(f^{\circ -w}(x)) + \varepsilon(f^{\circ -w}(x), y)(f^{\circ w})'(f^{\circ -w}(x))}{(f^{\circ w})'(f^{\circ -w}(x)) + (L^w)'(f^{\circ -w}(x))} \tag{3.2.12}$$

Proof. Assume that

$$\bar{\phi}^w(x,y) = f^{\circ -w}(x) + \bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2), \tag{3.2.13}$$

where $\bar{L}^w = \mathrm{O}(\bar{\varepsilon})$. Then

$$f^{\circ w}(\bar{\phi}^w(x,y)) = f^{\circ w}(f^{\circ -w}(x)) + (f^{\circ w})'(f^{\circ -w}(x))\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \tag{3.2.14}$$

and

$$(f^{\circ w})'(\bar{\phi}^w(x,y)) = (f^{\circ w})'(f^{\circ -w}(x)) + (f^{\circ w})''(f^{\circ -w}(x))\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2). \tag{3.2.15}$$

while

$$L^w(\bar{\phi}^w(x,y)) = L^w(f^{\circ -w}(x)) + (L^w)'(f^{\circ -w}(x))\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \tag{3.2.16}$$

and

$$\varepsilon(\bar{\phi}^w(x,y), y) = \varepsilon(f^{\circ -w}(x), y) + \partial_x \varepsilon(f^{\circ -w}(x), y)\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2). \tag{3.2.17}$$

Now observe that

$$x = \phi^w(\bar{\phi}^w(x,y), y) = \bar{\phi}^w(\phi^w(x,y), y) \tag{3.2.18}$$

so the above Variational Formula 3.2.6 yields

$$\begin{aligned}x &= f^{\circ w}(\bar{\phi}^w(x,y)) + L^w(\bar{\phi}^w(x,y)) + \varepsilon(\bar{\phi}^w(x,y),y)(f^{\circ i})'(\bar{\phi}^w(x,y)) + \mathrm{O}(\bar{\varepsilon}^2)\\ &= x + (f^{\circ w})'(f^{\circ -w}(x))\bar{L}^w(x,y) + L^w(f^{\circ -w}(x)) + (L^w)'(f^{\circ -w}(x))\bar{L}^w(x,y)\\ &\quad + \varepsilon(f^{\circ -w}(x),y)(f^{\circ i})'(f^{\circ -w}(x)) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.19)\end{aligned}$$

Therefore, by grouping terms and making appropriate cancellations we find

$$\begin{aligned}0 = \bar{L}^w(x,y)&[(f^{\circ -w})'(x) + (L^w)'(f^{\circ -w}(x))]\\ &+ L^w(f^{\circ -w}(x)) + \varepsilon(f^{\circ -w}(x),y)(f^{\circ w})'(f^{\circ -w}(x)) + \mathrm{O}(\bar{\varepsilon}^2). \qquad (3.2.20)\end{aligned}$$

Since $(f^{\circ -w})'(x) + (L^w)'(f^{\circ -w}(x))$ is uniformly bounded from below and $\bar{\varepsilon} > 0$ is arbitrary the result follows. □

Corollary 3.2.8. *Let $F \in \mathcal{H}_\Omega$ be expressible as $F = (\phi, \pi_x)$ where $\phi(x,y) = f(x) + \varepsilon(x,y)$. For all $w \in W$ let us define the functions H^w acting on B by $H^w(x,y) = (\phi^w(x,y),y)$. Assume they have well-defined inverses $\bar{H}^w(x,y) = (\bar{\phi}^w(x,y),y)$ when restricted to some subdomain B_H of the image of H^w. Then*

$$\phi^w(\bar{\phi}^{\bar{w}}(x,y),y) = f^{\circ w-\bar{w}}(x) + L^{w,\bar{w}}(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.21)$$

where $L^{w,\bar{w}}(x,y) = \mathrm{O}(\bar{\varepsilon}^2)$.

Proof. From Corollary 3.2.7

$$f^{\circ w}(\bar{\phi}^{\bar{w}}(x,y)) = f^{\circ w}(f^{\circ -\bar{w}}(x)) + (f^{\circ w})'(f^{\circ -\bar{w}}(x))\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.22)$$

and

$$(f^{\circ w})'(\bar{\phi}^{\bar{w}}(x,y)) = (f^{\circ w}))'(f^{\circ -\bar{w}}(x)) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.23)$$

and

$$L^w(\bar{\phi}^{\bar{w}}(x,y)) = L^w(f^{\circ -\bar{w}}(x)) + (L^w)'(f^{\circ -\bar{w}}(x))\bar{L}^w(x,y) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.24)$$

while

$$\varepsilon(\bar{\phi}^{\bar{w}}(x,y),y) = \varepsilon(f^{\circ -\bar{w}}(x),y) + \mathrm{O}(\bar{\varepsilon}^2). \qquad (3.2.25)$$

Therefore, by Proposition 3.2.6,

$$\begin{aligned}\phi^w(\bar{\phi}^{\bar{w}}(x,y),y) = f^{\circ w-\bar{w}}(x) &+ (f^{\circ w})'(f^{\circ -\bar{w}}(x))\bar{L}^w(x,y)\\ &+ L^w(f^{\circ -\bar{w}}(x)) + (L^w)'(f^{\circ -\bar{w}}(x))\bar{L}^w(x,y)\\ &+ \varepsilon(f^{\circ -\bar{w}}(x),y)(f^{\circ w}))'(f^{\circ -\bar{w}}(x)) + \mathrm{O}(\bar{\varepsilon}^2) \qquad (3.2.26)\end{aligned}$$

□

Remark 3.2.9. As in [12] we note that these three results simply express the first variation of the w-th composition operator acting on $\mathcal{H}_\Omega$.

Proposition 3.2.10. *Let $p > 1$ be an integer. Let $F \in \mathcal{H}_\Omega$, let $B^0 \subset B$ be a pre-renormalisation domain of type p and let G be its pre-renormalisation. Assume*

- $\pi_x G(B^0_{\mathrm{diag}}) \subsetneq \pi_x(B^0_{\mathrm{diag}})$;
- *G is Hénon-like on B^0_{diag}.*

Then there exists a neighbourhood $U \subset \mathcal{H}_\Omega$ of F such that $\tilde{F} \in U$ implies

(i) $\tilde{F}$ has a pre-renormalisation domain with the same properties;

(ii) there exists a constant $C > 0$, depending upon f only, such that

$$\mathrm{dist}_{Haus}(B^0_{\mathrm{diag}}, \tilde{B}^0_{\mathrm{diag}}) < C|F - \tilde{F}|_\Omega; \tag{3.2.27}$$

and

$$\mathrm{dist}_{Haus}(\Omega^0_{\mathrm{diag}}, \tilde{\Omega}^0_{\mathrm{diag}}) < C|F - \tilde{F}|_\Omega; \tag{3.2.28}$$

Proof. Given $F = (\phi, \pi_x)$ satisfying our hypotheses let H denote its horizontal diffeomorphism and let V denote the vertical diffeomorphism. Let $G = (\varphi, \pi_x)$ denote its pre-renormalisation. Let $B^0_{\mathrm{diag}} = J^0 \times J^0$. Let $g_\pm(x) = \varphi(x, \partial^\pm J^0)$ be the two bounding curves of the image of G.

Similarly, given $\tilde{F} = (\tilde{\phi}, \pi_x) \in \mathcal{H}_\Omega$ let $\tilde{H}$ denote its horizontal diffeomorphism and let $\tilde{V}$ denote its vertical diffeomorphism. Let $\tilde{G} = (\tilde{\varphi}, \pi_x)$ denote its pre-renormalisation. These all depend continuously on $\tilde{F}$.

Observe that G has a fixed point $(\alpha, \alpha) \in \partial B^0_{\mathrm{diag}}$. Observe also that $\alpha \in \partial J^0$ is a fixed point for g_- which, by assumption, is expanding. Let $\beta \in \partial J^0$ be the other boundary component. Then β is a preimage of α_0 under g_- and has non-zero derivative. The image of the horizontal line through α intersects the diagonal $\{x = y\}$ tranversely at α. These properties are all open conditions. (This follows from Corollary A.2.2, as α being a fixed point is equivalent to $\varphi \circ \Delta(\alpha) = \alpha$ with $|(\varphi \circ \Delta)'(\alpha)| \neq 1$, where Δ denotes the diagonal map.) Hence there exists a neighbourhood $U_0 \subset \mathcal{H}_\Omega$ of F such that $\tilde{F} \in U_0$ implies $\tilde{F}$ also has these properties once we set $\tilde{g}_-(x) = \tilde{\varphi}(x, \tilde{\alpha})$. If we let $\tilde{J}^0 = [\tilde{\alpha}, \tilde{\beta}]$ then it is clear $\tilde{g}_-$ is unimodal on $\tilde{J}^0$.

Now let $\tilde{B}^0_{\mathrm{diag}} = \tilde{J}^0 \times \tilde{J}^0$ and $g_+(x) = \tilde{\varphi}(x, \tilde{\beta})$. Since $\pi_x(G(B^0_{\mathrm{diag}})) \subsetneq \pi_x(B^0_{\mathrm{diag}})$, the critical value of g_+ lies in $\mathrm{int}(J^0)$. Since the critical value of g_+ and ∂J^0 depend continuously on F, there exists a neighbourhood $U_1 \subset U_0$ such that $\tilde{F} \in U_1$ implies the critical value of $\tilde{g}_+$ lies in $\mathrm{int}(J^0)$. Hence $\tilde{B}^0_{\mathrm{diag}}$ is $\tilde{G}$-invariant and $\pi_x(\tilde{G}(\tilde{B}^0_{\mathrm{diag}})) \subset \pi_x(\tilde{B}^0_{\mathrm{diag}})$.

The horizontal diffeomorphism will map diffeomorphically onto B^0_{diag} as it is a local diffeomorphism on the complement of $\mathcal{C}^{p-1}$ and this set depends continuously on F. Finally, the existence of the affine bijection $\tilde{I} \colon \tilde{B}^0_{\mathrm{diag}} \to B$ is clear as long as the orientation of its components agree with those of I. □

Proposition 3.2.11. *Let $p > 1$ be an integer. Let $0 < \gamma < 1$. Let $F \in \mathcal{H}_{\Omega,p}$ be renormalisable of combinatorial type p. Let $B^0 \subset B$ be the pre-renormalisation domain of type p and let G be its pre-renormalisation. Assume*

- *$\pi_x G(B^0_{\mathrm{diag}}) \subsetneq \pi_x(B^0_{\mathrm{diag}})$;*
- *G is Hénon-like on B^0_{diag};*
- *F satisfies the γ-gap property.*

Then there exists a neighbourhood $U \subset \mathcal{H}_\Omega$ of F and a constant $C > 0$, depending upon F only, such that $F \in U$ implies

(i) $\tilde{F}$ is p-renormalisable with the same properties;

(ii) there exists a constant $C > 0$, depending upon f only, such that

$$|\mathcal{R}F - \mathcal{R}\tilde{F}|_\Omega < C|F - \tilde{F}|_\Omega. \tag{3.2.29}$$

Proof. Given $F \in \mathcal{H}_{\Omega,p}$ let H denote its horizontal diffeomorphism and, for $w \in W$, let $B^w = F^{\circ w}(H(B^0_{\mathrm{diag}}))$. Then, as F is renormalisable, these sets will be pairwise disjoint. Let U_0 denote the neighbourhood of F from Proposition 3.2.10. Given $\tilde{F} \in U_0$ let $\tilde{H}$ denote the horizontal diffeomorphism and let $\tilde{B}^w = \tilde{F}^{\circ w}(\tilde{H}(\tilde{B}^0_{\mathrm{diag}}))$.

First observe that the critical curve $\tilde{\mathcal{C}}^{p-1}$, and the domain $\tilde{B}^0_{\mathrm{diag}}$ depend continuously on $\tilde{F}$. As $\mathcal{C}^{p-1}$, and the domain B^0_{diag} are separated by a distance γ or greater, there is a neighbourhood $U_1 \subset U_0$ of F such that $\tilde{F} \in U_1$ implies $\tilde{\mathcal{C}}^{p-1}$ and $\tilde{B}^0_{\mathrm{diag}}$ are separated by a distance of $\gamma/2$ or greater.

Finally, the sets $\tilde{B}^w$ depend continuously on $\tilde{F}$ as they are the continuous images of maps which depend continuously on $\tilde{F}$. As the B^w are separated by a distance γ or greater there is a neighbourhood $U_2 \subset U_1$ of F such that $\tilde{F} \in U_2$ implies the $\tilde{B}^w$ are $\gamma/2$ separated. This, together with Proposition 3.2.10 implies the first assertion.

For the second assertion observe that $\tilde{H}$ and $\tilde{B}^0_{\mathrm{diag}}$ depend continuously on $\tilde{F}$. As B^0_{diag} is bounded away from $\tilde{\mathcal{C}}^{p-1}$ the result follows. □

Corollary 3.2.12. *Let υ be a unimodal permutation of length $p > 1$. Let $0 < \gamma < 1$. Let $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega_x,\upsilon}(0)$ satisfy the γ-gap property. Then there exist constants $C, \bar{\varepsilon}_0 > 0$ and a domain $\Omega' \subset \mathbb{C}^2$ such that for any $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ the following holds:*

(i) $\tilde{F} \in \mathcal{H}_\Omega(f, \bar{\varepsilon})$ implies $\tilde{F} \in \mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$;

(ii) $\mathcal{R}F \in \mathcal{H}_{\Omega'}(C\bar{\varepsilon}^p)$.

Proof. The first property follows from Proposition 3.2.11. We now show the second property. Let $F \in \mathcal{H}_{\Omega,p}(0)$ have parametrisation $(f, 0)$ and let $\tilde{F} \in \mathcal{H}_{\Omega,\upsilon}$ have parametrisation (f, ε). Let H and $\tilde{H}$ denote their respective horizontal diffeomorphisms. let G and $\tilde{G}$ denote their respective pre-renormalisations with parametrisations (g, δ) and $(\tilde{g}, \tilde{\delta})$. Then

$$\partial_y \delta(x, y) = \mathrm{Jac}_{(x,y)} G = \mathrm{Jac}_{\bar{H}(x,y)} F^{\circ p} \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H}. \tag{3.2.30}$$

and

$$\partial_y \tilde{\delta}(x,y) = \mathrm{Jac}_{(x,y)} \tilde{G} = \mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{F}^{\circ p} \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}}. \tag{3.2.31}$$

Now observe that $|\mathrm{Jac}_{\bar{H}(x,y)} F^{\circ p}|_\Omega = 0$ and $|\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{F}^{\circ p}|_\Omega \leq |\varepsilon|_\Omega^p$. Next recall $\mathrm{Jac}_{(x,y)} H = \partial_x \phi^{p-1}(x,y)$, so by the Variational Formula 3.2.6 there is a constant $C_0 > 0$ such that, for $|\varepsilon|_\Omega$ sufficiently small,

$$\left| \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H} - \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}} \right| \leq C_0 |\varepsilon|_\Omega. \tag{3.2.32}$$

Since

$$\left| \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H} \right| \leq \exp(\mathrm{Dis}(F; B^0_{\mathrm{diag}})) \tag{3.2.33}$$

is bounded we find there exists a constant $C_1 > 0$ such that, for $|\varepsilon|_\Omega$ sufficiently small,

$$\left| \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}} \right| < C_1. \tag{3.2.34}$$

Hence $|\partial_y \tilde{\delta}(x,y)| < C_1 |\varepsilon|_\Omega^p$. By construction the renormalisation, $\tilde{F}_1$, of $\tilde{F}$ has parametrisation $(\tilde{f}_1, \tilde{\varepsilon}_1)$ which is an affine rescaling of $(\tilde{g}, \tilde{\delta})$. There exists a constant $C_2 > 0$ such that the affine rescaling has scaling ratio $\sigma + C_2|\varepsilon|_\Omega$, where σ is the scaling ratio for F. This implies there exists a constant $C_3 > 0$ such that $|\partial_y \tilde{\varepsilon}_1|_{\Omega'} \leq C_3 |\varepsilon|_\Omega^p$. Moreover, $\tilde{\varepsilon}_1$ satisfies $\tilde{\varepsilon}_1(x,0) = 0$ by construction. Therefore $|\tilde{\varepsilon}_1|_{\Omega'} \leq C_3 |\varepsilon|_\Omega^p$ and the result is shown. □

Theorem 3.2.13. *Let υ be a unimodal permutation of length $p > 1$. Let $0 < \gamma < 1$. Then there are constants $C, \bar{\varepsilon}_0 > 0$ and a domain $\Omega' \subset \mathbb{C}$, depending upon υ and Ω, such that the following holds: for any $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ there is a subspace $\mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \subset \mathcal{H}_\Omega(\bar{\varepsilon})$ containing $\mathcal{H}_{\Omega,\upsilon}(0)$ and a dynamically defined continuous operator,*

$$\mathcal{R}\colon \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_{\tilde{\Omega}}(C\bar{\varepsilon}^p), \tag{3.2.35}$$

which extends continuously to $\mathcal{R}$ on $\mathcal{H}_{\Omega,\upsilon}(0)$. Moreover $\bar{\varepsilon}_0 > 0$ can be chosen so that

$$\mathcal{R}\colon \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_{\tilde{\Omega}}(\bar{\varepsilon}). \tag{3.2.36}$$

Proof. By Corollary 3.2.12, for each $f \in \mathcal{U}_{\Omega,\upsilon}$ there exists a $\bar{\varepsilon}_f > 0$ and a $C_f > 0$ such that $\mathcal{R}$ has an extension $\mathcal{R}\colon \mathcal{H}_\Omega(f, \bar{\varepsilon}_f) \to \mathcal{H}_\Omega(C_f \bar{\varepsilon}_f^p)$. By compactness of $\mathcal{U}_{\Omega,\upsilon}$ these constants can be chosen uniformly, so setting

$$\mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega,\upsilon}} \mathcal{H}_\Omega(f, \bar{\varepsilon}_f) \tag{3.2.37}$$

we find that $\mathcal{R}\colon \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_\Omega(C\bar{\varepsilon}^p)$. Choosing $\bar{\varepsilon}_0 > 0$ sufficiently small so that $\bar{\varepsilon} < C\bar{\varepsilon}^p$ for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ gives the final claim. □

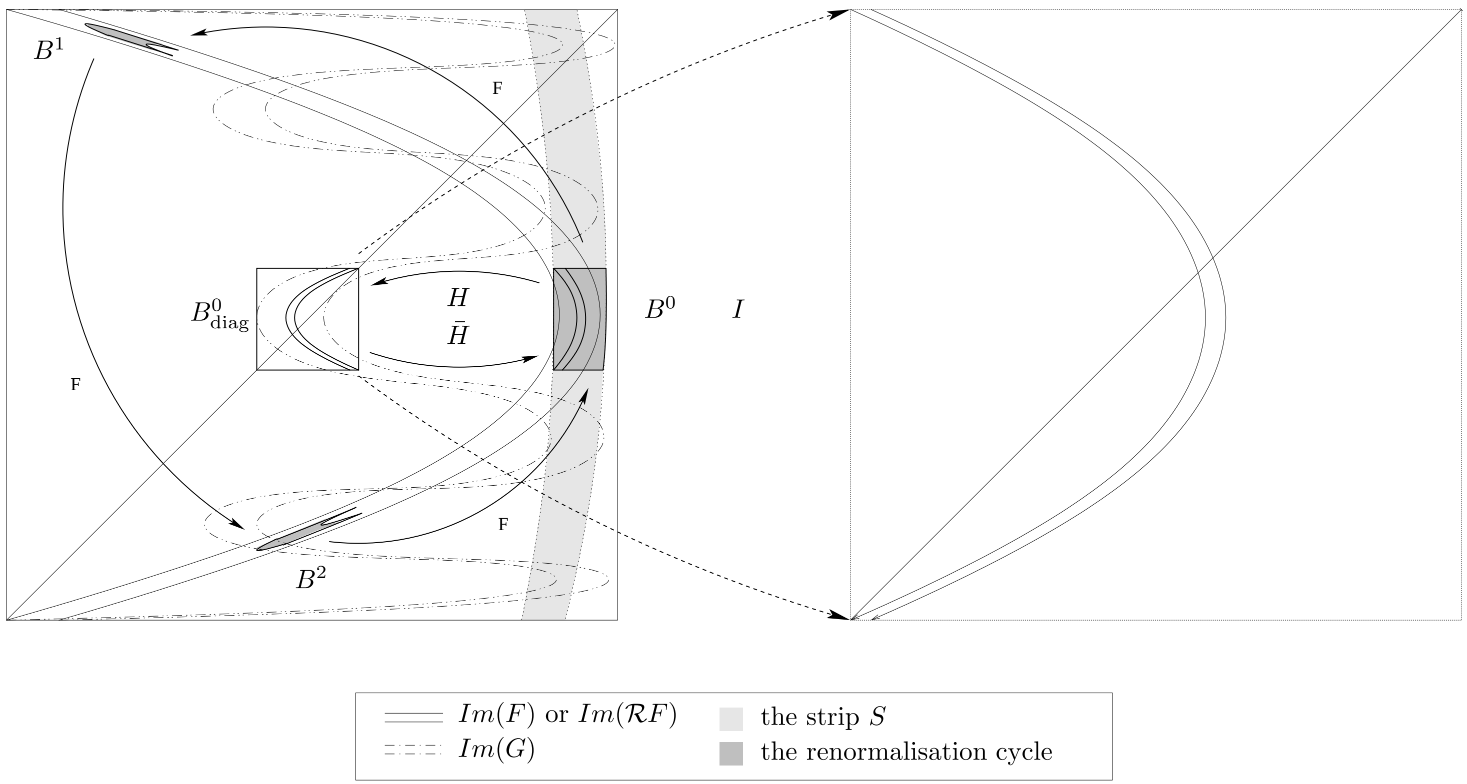

Figure 3.1: A renormalisable Hénon-like map which is a small perturbation of a degenerate Hénon-like map. In this case the combinatorial type is period tripling. Here the lightly shaded region is the preimage of the vertical strip through B^0_{diag}. The dashed lines represent the image of the square B under the pre-renormalisation G. If the order of all the critical points of $f^{\circ 2}$ is the same it can be shown that G can be extended to an embedding on the whole of B, giving the picture above.

3.3 The Fixed Point and Hyperbolicity

We will now consider Hénon-like maps that are infinitely renormalisable. Therefore throughout the rest of this chapter we will fix a unimodal permutation υ of length p. We will denote by $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ the subspace of $\mathcal{H}_\Omega(\bar{\varepsilon})$ consisting of infinitely renormalisable Hénon-like maps, where each renormalisation has the same combinatorial type υ. We call $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ the space of infinitely renormalisable Hénon-like maps with *stationary combinatorics* υ. Given any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ we write $F_n = \mathcal{R}^n F$. Throughout we will use subscripts to denote quantities associated with the n-th Hénon-renormalisation. For example $\phi_n = \phi(F_n)$ will denote the function satisfying $F_n = (\phi_n, \pi_x)$.

As was noted in Remark 2.5.3, since $\mathcal{U}_{\Omega_x,\upsilon}$ is canonically embedded in $\mathcal{H}_{\Omega_x,\upsilon}(0)$ and $\mathcal{R}$ is defined on $\mathcal{H}_{\Omega_x,\upsilon}(0)$ so that $\mathcal{R}(f\circ\pi_x, \pi_x) = (\mathcal{R}_{\mathcal{U}} f\circ\pi_x, \pi_x)$ it is clear that the fixed point of $\mathcal{R}_{\mathcal{U}}$ induces a fixed point of $\mathcal{R}$. That is, the point $F_* = (f_*\circ\pi_x, \pi_x)$ in $\mathcal{H}_{\Omega_x,\upsilon}(0)$ is a fixed point of $\mathcal{R}$, where f_* denotes the fixed point of $\mathcal{R}_{\mathcal{U}}$. It is also clear that, when restricted to $\mathcal{H}_\Omega(0)$, the fixed point if unique and hyperbolic, with codimension one stable manifold and dimension one local unstable manifold. We will now show that F_* is also hyperbolic on some extension, $\mathcal{H}_\Omega(\bar{\varepsilon})$, of $\mathcal{H}_\Omega(0)$ and, moreover, has one expanding eigendirection and all others contracting. In fact, we will show all directions transverse to $\mathcal{H}_{\Omega_x,\upsilon}$ must contract superexponentially.

It is clear from the analysis in the previous section and by compactness of $\mathcal{U}_{\Omega_x}$ that for any $n > 0$ there is a $\bar{\varepsilon} > 0$ such that for any infinitely renormalisable $f \in \mathcal{U}_{\Omega_x,\upsilon}$, any $F \in \mathcal{H}_{\Omega_x,\upsilon}(f, \bar{\varepsilon})$ is n-times renormalisable. The following shows that a converse also holds.

Lemma 3.3.1. *Let υ be a unimodal permutation of length $p > 1$ and let $\Omega \subset \mathbb{C}^2$ be a polydisk containing B. For any $n > 0$ there is a constant $\bar{\varepsilon}_0 > 0$ such that for any $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ the following holds: for any n-times renormalisable $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, there is an n-times renormalisable $\tilde{F} \in \mathcal{H}_{\Omega,\upsilon}(0)$ such that $|\mathcal{R}^n F - \mathcal{R}^n \tilde{F}|_\Omega \leq C\bar{\varepsilon}^{p^n}$.*

Proof. Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$. Then by assumption F has parametrisation (f, ε) for some $f \in \mathcal{U}_{\Omega_x,\upsilon}$ and some thickening ε satisfying $|\varepsilon|_\Omega \leq \bar{\varepsilon}$. Let (f_n, ε_n) denote the canonical parametrisation of F_n. Then $f_n \in \mathcal{U}_{\Omega_x,\upsilon}$ and ε_n is a thickening satisfying $|\varepsilon_n|_\Omega \leq C\bar{\varepsilon}^{p^n}$ for all $n \geq 0$.

From this we proceed by induction. The case $n = 1$ is already covered by Theorem 3.2.13. Fix an $n > 1$ and assume the statement is true for all $0 < m < n$. This assumption implies, if $\bar{\varepsilon} > 0$ is small enough to ensure $|\mathcal{R}_{\mathcal{U}} f_m - f_{m+1}|_\Omega \leq C\bar{\varepsilon}^{p^m}$, that $\mathcal{R}_{\mathcal{U}}^{n-m} f_m$ exists for all $0 \leq m < n$.

We will show that $\mathcal{R}_{\mathcal{U}}^n f$ is renormalisable by showing it is sufficiently close to a renormalisable map whose renormalisation is not surjective, then invoke Proposition 2.2.12. By the first claim for each $m < n$ there exists a $\kappa_m > 0$ such that for any $0 < \kappa'_m < \kappa_m$ there is a $\kappa''_m > 0$ where $|f_m - f_{m+1}| < \kappa''_m$ implies $|\mathcal{R}_{\mathcal{U}}^{n-m} f_m - \mathcal{R}_{\mathcal{U}}^{n-m-1} f_{m+1}| < \kappa'_m$. For any $K > 0$ choose $\bar{\varepsilon} > 0$ such that $|f_m - f_{m+1}| < C\bar{\varepsilon}^{p^m}$ implies $|\mathcal{R}_{\mathcal{U}}^{n-m} f_m - \mathcal{R}_{\mathcal{U}}^{n-m-1} f_{m+1}| < K/n$. Then

$$\begin{aligned}&|\mathcal{R}_{\mathcal{U}}^n f - f_n| \\ &\leq |\mathcal{R}_{\mathcal{U}}^n f - \mathcal{R}_{\mathcal{U}}^{n-1} f_1| + |\mathcal{R}_{\mathcal{U}}^{n-1} f_1 - \mathcal{R}_{\mathcal{U}}^{n-2} f_2| + \ldots + |\mathcal{R}_{\mathcal{U}} f_{n-1} - f_n| \leq K \quad (3.3.1)\end{aligned}$$

and so we may approximate $\mathcal{R}_{\mathcal{U}}^n f$ with the renormalisable f_n, which has renormalisation which is not surjective. Therefore if $\bar{\varepsilon} > 0$ is sufficiently small $\mathcal{R}_{\mathcal{U}}^n f$ is renormalisable. □

Theorem 3.3.2. *Let υ be a unimodal permutation of length $p > 1$. For any polydisk $\Omega \subset \mathbb{C}^2$ containing B there are constants $C, \bar{\varepsilon}_0 > 0$ and $0 < \rho < 1$ such that for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ the following holds: given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ there is a domain $\Omega' \Subset \Omega$, containing B in its interior, and a sequence of $\tilde{F}_{n_i} \in \mathcal{H}_{\Omega'_x}(0)$ such that*

(i) $|\tilde{F}_{n_i} - F_*|_{\Omega'} \leq C\rho^i |F - F_*|_{\Omega'}$

(ii) $|F_{n_i} - \tilde{F}_{n_i}|_{\Omega'} \leq C\bar{\varepsilon}^{p^{n_i - 1}}$

where F_{n_i} denotes the n_i-th renormalisation of F.

Proof. Recall that, by Theorem 2.3.3 we know for any domain Ω_x containing J there exists a domain $\Omega'_x \Subset \Omega_x$, also containing J, and an integer $n > 0$ such that for any n-times renormalisable $f \in \mathcal{U}_{\Omega'_x,\upsilon}$, its n-th renormalisation $\mathcal{R}_{\mathcal{U}}^n f \in \mathcal{U}_{\Omega'_x}$ and

$$|\mathcal{R}_{\mathcal{U}}^n f - f_*|_{\Omega'_x} < \tfrac{1}{4}|f - f_*|_{\Omega'_x}, \tag{3.3.2}$$

where f_* denotes the fixed point of $\mathcal{R}_{\mathcal{U}}$.

Given $F \in \mathcal{I}_{\Omega_x,\upsilon}$ let F_n denote its n-th renormalisation. For any $m > 0$ let $\tilde{F}_{mn} \in \mathcal{H}_{\Omega,\upsilon}(0)$ denote a degenerate Hénon-like which is n-times renormalisable and $|F_{mn} - \tilde{F}_{mn}|_{\Omega'} < C\bar{\varepsilon}^{p^{mn}}$. Such a map exists by Lemma 3.3.1. Then

$$\begin{aligned}|\mathcal{R}^n \tilde{F}_{(m-1)n} - \tilde{F}_{mn}|_{\Omega'} &\leq |\mathcal{R}^n \tilde{F}_{(m-1)n} - \mathcal{R}^n F_{(m-1)n}|_{\Omega'} + |F_{mn} - \tilde{F}_{mn}|_{\Omega'} \\ &\leq 2C\bar{\varepsilon}^{p^{mn}}\end{aligned} \tag{3.3.3}$$

which implies

$$\begin{aligned}|\tilde{F}_{mn} - F_*|_{\Omega'} &\leq |\mathcal{R}^n \tilde{F}_{(m-1)n} - F_*|_{\Omega'} + |\mathcal{R}^n \tilde{F}_{(m-1)n} - F_{mn}|_{\Omega'} \\ &\leq \tfrac{1}{4}|\tilde{F}_{(m-1)n} - F_*| + 2C\bar{\varepsilon}^{p^{mn}}.\end{aligned} \tag{3.3.4}$$

Now, for $\bar{\varepsilon} > 0$ sufficiently small we may assume

$$8C\bar{\varepsilon}^{p^{mn}} \leq |\tilde{F}_{(m-1)n} - F_*|_{\Omega'}, \tag{3.3.5}$$

as otherwise $|\tilde{F}_{mn} - F_*|_{\Omega'}$ decreases super-exponentially and we are done. This implies

$$\tfrac{1}{4}|\tilde{F}_{(m-1)n} - F_*|_{\Omega'} + 2C\bar{\varepsilon}^{p^{mn}} \leq \tfrac{1}{2}|\tilde{F}_{(m-1)n} - F_*|_{\Omega'} \tag{3.3.6}$$

and so by the above we find

$$|\tilde{F}_{mn} - F_*|_{\Omega'} \leq \tfrac{1}{2}|\tilde{F}_{(m-1)n} - F_*|_{\Omega'}. \tag{3.3.7}$$

Hence $|\tilde{F}_{mn} - F_*|_{\Omega'}$ decreases exponentially and by construction $|\tilde{F}_{mn} - F_{mn}|_{\Omega'} \leq C\bar{\varepsilon}^{p^{mn}}$. □

Proposition 3.3.3. *Given a polydisk $\Omega \subset \mathbb{C}^2$ containing B there exists*

(i) a domain $\Omega' \Subset \Omega$, containing B in its interior;

(ii) an $\mathcal{R}$-invariant subspace, $\mathcal{I}_{adapt} \subset \mathcal{I}_{\Omega',\upsilon}$;

(iii) a metric, d_{adapt}, on $\mathcal{I}_{adapt}$ which is Lipschitz-equivalent to the sup-norm on $\mathcal{I}_{\Omega',\upsilon}$;

(iv) a constant $0 < \rho < 1$;

such that, for all $F \in \mathcal{I}_{adapt}$,

$$d_{adapt}(\mathcal{R}F, F_*) \leq \rho d_{adapt}(F, F_*). \tag{3.3.8}$$

Proof. Given n-times renormalisable maps $F, \tilde{F} \in \mathcal{H}_{\Omega,\upsilon}$ let

$$d_{adapt}(F, \tilde{F}) = \sum_{n=0}^{N-1} \rho^{N-n} d_{sup}(\mathcal{R}^n F, \mathcal{R}^n \tilde{F}) \tag{3.3.9}$$

where $0 < \rho < 1$ is the constant from Theorem 3.3.2 above and d_{sup} denotes the metric induced by the sup-norm. Then, by the same Theorem, $d_{sup}(\mathcal{R}F, F_*) \leq \rho d_{sup}(F, F_*)$ and so

$$\begin{aligned} d_{adapt}(\mathcal{R}F, F_*) &= \sum_{n=0}^{N-2} \rho^{N-n} d_{sup}(\mathcal{R}^n F, \mathcal{R}^n \tilde{F}) + d_{sup}(\mathcal{R}^N F, F_*) \\ &\leq \rho d_{sup}(F, F_*) + \sum_{n=0}^{N-2} \rho^{N-n} d_{sup}(\mathcal{R}^n F, \mathcal{R}^n \tilde{F}) \\ &= \rho d_{adapt}(F, F_*). \end{aligned} \tag{3.3.10}$$

Therefore it remains to show d_{adapt} is Lipschitz-equivalent to d. Under the assumption that $d_{sup}(\mathcal{R}F, \mathcal{R}\tilde{F}) < d_{sup}(F, \tilde{F})$ we have

$$d_{adapt}(F, \tilde{F}) < \sum_{i=0}^{N-1} \rho^{N-i} d_{sup}(F, \tilde{F}) \leq \frac{\rho^{N+1}}{1-\rho} d_{sup}(F, \tilde{F}) \tag{3.3.11}$$

while we clearly have

$$\rho^N d_{sup}(F, \tilde{F}) \leq d_{adapt}(F, \tilde{F}), \tag{3.3.12}$$

and hence the two metrics are Lipschitz-equivalent. □

We now make some estimates on the sequence of renormalisations of an $F \in \mathcal{I}_{\Omega,\upsilon}$ that will be useful in later sections.

Proposition 3.3.4. *For $\bar{\varepsilon} > 0$ sufficiently small the following holds: given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ its renormalisations F_n have the form*

$$F_n(z) = (\phi_n(z), \pi_x(z)) \tag{3.3.13}$$

and the derivative of the maps F_n have the form

$$\mathrm{D}_z F_n = \begin{pmatrix} \partial_x \phi_n(z) & \partial_y \phi_n(z) \\ 1 & 0 \end{pmatrix}. \tag{3.3.14}$$

Let $|\partial_y \phi|_{inf} = \inf_{z\in\Omega} |\partial_y \phi(z)|$ and $|\partial_y \phi|_{sup} = \sup_{z\in\Omega} |\partial_y \phi(z)|$. Then there exist universal constants $C_0, C_1, C > 0, 0 < \rho < 1$ such that

(i) $C_0 < |\partial_x \phi_n|_{\Omega_n^0} < C_1$;

(ii) $|\partial_y \phi|_{inf}^{p^n}(1 - C\rho^n) < |\partial_y \phi_n| < |\partial_y \phi|_{sup}^{p^n}(1 + C\rho^n)$.

where $\Omega_n^0 \subset \mathbb{C}^2$ denotes the central domain for F_n.

Proof. That F_n and $\mathrm{D}_z F_n$ have these form is obvious. Given $F = (\phi, \pi_x)$ convergence of renormalisation implies ϕ_n converges and hence $|\partial_x \phi_n|_{\Omega_n^0} \to |\partial_x \phi_*|_{\Omega_*^0}$ which is bounded away from zero and infinity. Hence if $\bar{\varepsilon} > 0$ is sufficiently small, $|\partial_x \phi_n|_{\Omega_n^0}$ will also be bounded, uniformly, away from zero and infinity. The final item follows from Theorem 3.2.13, which gives us the super-exponential factor, and Theorem 3.3.3 which gives us exponential convergence $\phi_n \to \phi_*$. □

An application of the Mean Value Theorem B.1.1 in the Appendix gives us the following.

Proposition 3.3.5. *Given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let F_n denote its n-th renormalisation. For any $z, \tilde{z} \in \mathrm{Dom}(F_n)$ there exists $\xi_{z\tilde{z}}, \eta_{z\tilde{z}}, \in [\![z, \tilde{z}]\!]$, the rectangle spanned by $z, \tilde{z}$, such that*

$$\begin{aligned} \pi_x(F_n z) - \pi_x(F_n \tilde{z}) &= \partial_x \phi_n(\xi_{z\tilde{z}})(\pi_x(z) - \pi_x(\tilde{z})) + \partial_y \phi_n(\eta_{z\tilde{z}})(\pi_y(z) - \pi_y(\tilde{z})) \\ \pi_y(F_n z) - \pi_y(F_n \tilde{z}) &= \pi_x(z) - \pi_x(\tilde{z}). \end{aligned} \tag{3.3.15}$$

3.4 Scope Functions and Presentation Functions

We will now recast the renormalisation theory we have just developed for Hénon-like maps in terms of scope maps and presentation functions (defined below) in a way analogous to that in Section 2.4. Throughout this section, υ will be a fixed unimodal permutation of length $p > 1$ and $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalisation for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

Let $F \in \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon})$ be a renormalisable Hénon-like map with cycle $\{B_n^w\}_{w\in W}$. Let $H\colon B^0 \to B_{\mathrm{diag}}^0$ denote its horizontal diffeomorphism and let $G\colon B_{\mathrm{diag}}^0 \to B_{\mathrm{diag}}^0$ denote its pre-renormalisation. Let $I\colon B_{\mathrm{diag}}^0 \to B$ denote the affine rescaling such that $\mathcal{R}F = IG\bar{I}$. Then we will call the coordinate change $\Psi = \Psi(F)\colon B \to B^0$, given by $\Psi = \bar{H} \circ \bar{I}$, the *scope map* of F. More generally, for $w \in W$ we will call the map $\Psi^w = F^{\circ w} \circ \Psi\colon B \to B^w$ the *w-scope map* of F.

Assume now that F is n-times renormalisable. As in the previous section, we will denote the n-th renormalisation $\mathcal{R}^n F$ by F_n. For $w \in W$ let $\Psi^w_n = \Psi^w(F_n)\colon \mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_n)$ be the w-scope function for F_n. Then, if $\mathbf{w} = w_0 \ldots w_n \in W^*$, the function

$$\Psi^{\mathbf{w}} = \Psi^{w_0}_0 \circ \ldots \circ \Psi^{w_n}_n\colon \mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_0) \tag{3.4.1}$$

is called the $\mathbf{w}$*-scope function* for F. Let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{\mathbf{w} \in W^n}$ denote the collection of all scope functions for F.

Proposition 3.4.1. *There exist a constant $C > 0$ such that for all $0 < \bar{\varepsilon} \leq \bar{\varepsilon}_0$ the following holds: if $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ has a parametrisation (f, ε) such that f is renormalisable and $|\varepsilon| \leq \bar{\varepsilon}$, then*

$$\|\mathrm{D}_z \Psi^w_F - \mathrm{D}_z \Psi^w_f\| < C\bar{\varepsilon}, \tag{3.4.2}$$

where Ψ^w_F denotes the w-scope map of F and Ψ^w_f denotes the w-scope map of $\underline{\mathrm{i}}(f)$.

Proof. Let $F(x, y) = (\phi(x, y), x) = (f(x) - \varepsilon(x, y), x)$. Then

$$\begin{aligned}\Psi^w_F(x, y) &= F^{\circ w} \bar{H} \bar{I}_F(x, y) \\ &= \begin{cases} (\phi^w(\bar{\phi}^{p-1}(\iota_F x, \iota_F y), \iota_F y), \phi^{w-1}(\bar{\phi}^{p-1}(\iota_F x, \iota_F y), \iota_F y)) & w > 0 \\ ((\bar{\phi}^{p-1}(\iota_F(x)), \iota_F(y)), \iota_F(y)) & w = 0 \end{cases}\end{aligned} \tag{3.4.3}$$

where $\iota_F = \iota_{J \to \tilde{J}^0}$ is an affine bijection between J and $\tilde{J}^0$, and

$$\begin{aligned}\Psi^w_f(x, y) &= \Delta^w_f \bar{I}_f(x, y) \\ &= \begin{cases} (f^{\circ-(p-w)}(\iota_f(x)), f^{\circ-(p-w-1)}(\iota_f(x))) & w > 0 \\ ((f^{\circ-(p-1)}(\iota_f(x)), \iota_f(y)) & w = 0 \end{cases}\end{aligned} \tag{3.4.4}$$

where $\iota_f = \iota_{J \to J^0}$ is an affine bijection between J and J^0, the central interval of f. Let us denote the points $\bar{I}_f z$ and $\bar{I}_F z$ by z_0 and z_1 respectively.

Now we make a series of claims. First, we claim that there exists $C_1 > 0$ such that $\|\mathrm{D}_z \bar{I}_F\| < C_1$. This follows as $\bar{I}_F$ is an affine contraction.

Second, we claim that there exists a $C_2 > 0$ such that $\|\mathrm{D}_{z_0} \Delta^w_f\| < C_2$. This follows as the eigenvalues of $\mathrm{D}_{z_0} \Delta^w_f$ will be 0 and $(\psi^w)'$, in the case $w > 0$, or will be ι'_f and $(\psi^w)'$ in the case $w = 0$ (since $\mathrm{D}_{z_0} \Delta^w_f$ is triangular).

Third, we claim that there exists $C_3 > 0$ such that $\|\mathrm{D}_z \bar{I} - \mathrm{D}_z \bar{I}_F\| \leq C_3|\varepsilon|$. From Proposition 3.2.11 we know that there exists a constant $C' > 0$ such that $\mathrm{dist}(J_g, J_{\tilde{g}}) < C'|\varepsilon|$. This then implies there exists a constant $C'' > 0$ such that $|\iota_F - \iota_f|_{C^1} < C''|\varepsilon|$, therefore the eigenvalues of $\mathrm{D}_z I_F - I_f$ have the same bound. Setting $C_3 = C''$ gives us the claim.

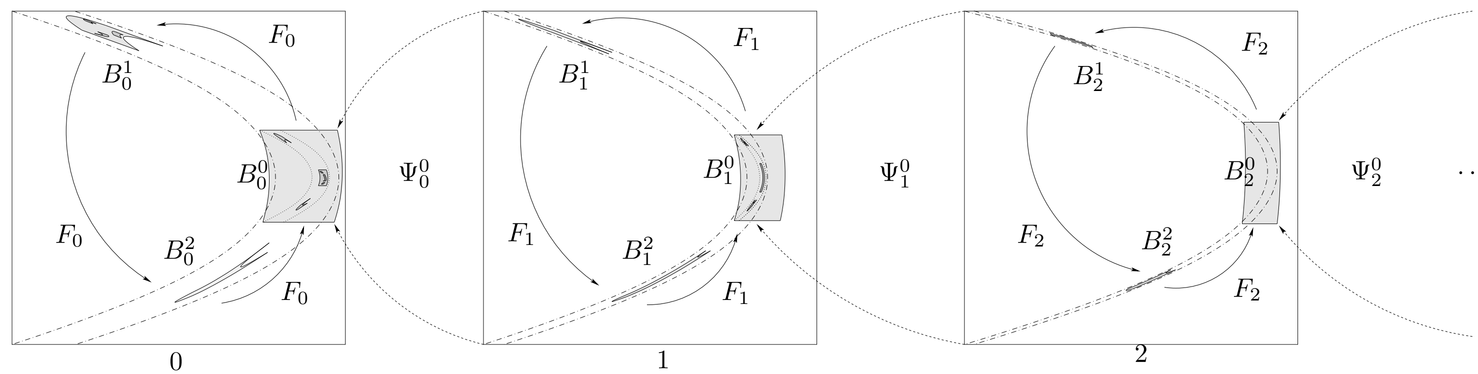

Figure 3.2: The sequence of scope maps for a period-three infinitely renormalisable Hénon-like map. In this case the maps has stationary combinatorics of period-tripling type. Here the dashed line represents the bounding arcs of the image of the square B under consecutive renormalisations F_n.

Fourth, we claim that there exists a constant $C_4 > 0$ such that $\|\mathrm{D}_{z'}\Delta_f^w - \mathrm{D}_{z'}F^{\circ w}H^{-1}\| < C_4|\varepsilon|$ for any $z' = (x', y') \in B^0_{\mathrm{diag}} = I_F(B)$. For this we use the Variational Formula 3.2.6 and its corollaries. For the sake of notation let $E(x', y') = f^{\circ-(p-w-1)}(x') - \phi^w(\bar{\phi}^{p-1}(x', y'), y')$. Observe that by Corollary 3.2.8, for $(x', y') \in B^0_{\mathrm{diag}}$,

$$\begin{aligned}
E(x', y') &= f^{\circ-(p-w-1)}(x') - \phi^w(\bar{\phi}^{p-1}(x', y'), y') \\
&= (f^{\circ w})'(f^{\circ-(p-1)}(x))\bar{L}^{p-1}(x, y) \\
&\quad + L^w(f^{\circ-(p-1)}(x)) + (L^w)'(f^{\circ-(p-1)}(x))\bar{L}^w(x, y) \\
&\quad + \varepsilon(f^{\circ-(p-1)}(x), y)(f^{\circ w}))'(f^{\circ-(p-1)}(x)) + \mathrm{O}(|\varepsilon|^2) \\
&= \mathrm{O}(|\varepsilon|).
\end{aligned} \tag{3.4.5}$$

Since all the functions under consideration are analytic the derivatives of E will also be $\mathrm{O}(|\varepsilon|)$. Hence there exists a constant $C' > 0$ such that

$$\left|\mathrm{tr}\left(\mathrm{D}_{z'}\left(\Delta_f^w - F^{\circ w}\bar{H}\right)\right)\right| \le C'|\varepsilon|, \quad \left|\det\left(\mathrm{D}_{z'}\left(\Delta_f^w - F^{\circ w}\bar{H}\right)\right)\right| \le C'|\varepsilon|^2 \tag{3.4.6}$$

Therefore by the quadratic formula, the eigenvalues of $\mathrm{D}_{z'}\left(\Delta_f^w - F^{\circ w}\bar{H}\right)$ are bounded, in absolute value by $C_4|\varepsilon|$, where[1] $C_4 = 3C'$ and hence the claim follows.

Next, we claim there exists a constant $C_5 > 0$ such that $\|\mathrm{D}_{z_0}\Delta_f^w - \mathrm{D}_{z_1}\Delta_f^w\| < C_5|\varepsilon|^2$. For this we again use the Variational Formula 3.2.6 and its corollaries. Again let $E(x, y) = f^{\circ-(p-w-1)}(x) - \phi^w(\bar{\phi}^{p-1}(x, y), y)$. By Corollary 3.2.8, since all the functions we are considering are analytic, there exists a constant $C' > 0$ such that $|\partial_{xx}E|, |\partial_{xy}E|, |\partial_{yy}E| < C'|\varepsilon|$. Hence by Proposition B.1.1 and the fact that there is a constant $C'' > 0$ with $|\bar{I}_f - \bar{I}_F| < C''|\varepsilon|$ we find

$$\begin{aligned}
|\partial_x E(x_0, y_0) - \partial_x E(x_1, y_1)| &\le |\partial_{xx}E_{y_0}||x_0 - x_1| + |\partial_{yx}E_{x_1}||y_0 - y_1| \\
&\le C'C''|\varepsilon|^2,
\end{aligned} \tag{3.4.7}$$

and similarly

$$\begin{aligned}
|\partial_y E(x_0, y_0) - \partial_y E(x_1, y_1)| &\le |\partial_{xy}E_{y_1}||x_0 - x_1| + |\partial_{yy}E_{x_0}||y_0 - y_1| \\
&\le C'C''|\varepsilon|^2,
\end{aligned} \tag{3.4.8}$$

so the same argument involving the trace and determinant in the previous claim will also work here giving a $C''' > 0$ such that

$$\left|\mathrm{tr}\left(\mathrm{D}_{z_0}\Delta_f^w - \mathrm{D}_{z_1}\Delta_f^w\right)\right| \le C'''|\varepsilon|^2, \ \left|\det\left(\mathrm{D}_{z_0}\Delta_f^w - \mathrm{D}_{z_1}\Delta_f^w\right)\right| \le C'''|\varepsilon|^4. \tag{3.4.9}$$

Hence the eigenvalues of $\mathrm{D}_{z_0}\Delta_f^w - \mathrm{D}_{z_1}\Delta_f^w$ are bounded, in absolute value by $C_5|\varepsilon|^2$, where $C_5 = 3C'''$ and so the claim follows.

[1] observe by convexity of $\sqrt{x}$, $\sqrt{a+b} < \sqrt{a} + \frac{b}{2\sqrt{a}}$ for $a, b > 0$

Finally, by the triangle inequality and the fact that for any linear operators A, B we have $\|AB\| \le \|A\|\|B\|$, we find

$$\begin{aligned}\|\mathrm{D}_z\Psi^w_f - \mathrm{D}_z\Psi^w\| &\le \|\mathrm{D}_{z_0}\Delta^w_f\|\|\mathrm{D}_z\bar{I}_f - \mathrm{D}_z\bar{I}_F\| \\ &\quad + \|\mathrm{D}_{z_0}\Delta^w_f - \mathrm{D}_{z_1}\Delta^w_f\|\|\mathrm{D}_z\bar{I}_F\| \\ &\quad + \|\mathrm{D}_{z_1}\Delta^w_f - \mathrm{D}_{z_1}F^{\circ w}\bar{H}\|\|\mathrm{D}_z\bar{I}_F\| \\ &\le C_2C_3|\varepsilon| + C_1C_5|\varepsilon|^2 + C_1C_4|\varepsilon| \\ &\le C_0|\varepsilon|, \end{aligned} \tag{3.4.10}$$

where we have set $C_0 = C_2C_3 + C_1(C_4 + C_5)$, and the result is proved. ☐

Proposition 3.4.2. *There are constants $C > 0$ and $0 \le \rho < 1$ such that for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ the following holds: For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, $w \in W$, $z \in B$ and any integer $n > 0$,*

$$\|\mathrm{D}_z\Psi^w_n - \mathrm{D}_z\Psi^w_*\| \le C\rho^n. \tag{3.4.11}$$

Remark 3.4.3. The constant $0 < \rho < 1$ above can be chosen to be constant from Theorem 3.3.2.

Proof. Let (f_n, ε_n) denote the canonical parametrisation for F_n and let $\Psi^w_{f_n}$ be the function from Proposition 3.4.1. Observe that

$$\Psi^w_{f_n}(x,y) = \begin{cases} (\psi^w_n(x), \psi^{w-1}_n(x)) & w > 0 \\ (\psi^w_n(x), \iota_n(y)) & w = 0 \end{cases}, \tag{3.4.12}$$

and

$$\Psi^w_*(x,y) = \begin{cases} (\psi^w_*(x), \psi^{w-1}_*(x)) & w > 0 \\ (\psi^w_*(x), \iota_*(y)) & w = 0 \end{cases}. \tag{3.4.13}$$

From Theorem 3.3.2 in section 3.3 we know that there are constants $C_0 > 0, 0 < \rho < 1$ such that $|f_n - f_*|_{C^2} < C_0\rho^n$. By Proposition 2.4.6 this implies there is constant $C_1 > 0$ such that $|\psi^w_n - \psi^w_*|_{C^1} < C_1\rho^n$. This then implies

$$\|\mathrm{D}_z\Psi^w_{f_n} - \mathrm{D}_z\Psi^w_*\| \le |(\psi^w_n)' - (\psi^w_*)'| \le C_1\rho^n. \tag{3.4.14}$$

By Proposition 3.4.1 there exists a constant $C_2 > 0$ such that

$$\|\mathrm{D}_z\Psi^w_n - \mathrm{D}_z\Psi^w_{f_n}\| \le C_2|\varepsilon_n|_\Omega \le C_2C_3\bar{\varepsilon}^{p^n} \tag{3.4.15}$$

where $C_3 > 0$ is the constant from Theorem 3.2.13. Therefore

$$\begin{aligned}\|\mathrm{D}_z\Psi^w_n - \mathrm{D}_z\Psi^w_*\| &\le \|\mathrm{D}_z\Psi^w_n - \mathrm{D}_z\Psi^w_{f_n}\| + \|\mathrm{D}_z\Psi^w_{f_n} - \mathrm{D}_z\Psi^w_*\| \\ &\le C_2C_3\bar{\varepsilon}^{p^n} + C_1\rho^n. \end{aligned} \tag{3.4.16}$$

Now let $C_4 > 0$ be a constant satisfying $\bar{\varepsilon}^{p^n} < C_4\rho^n$. Then, setting $C = C_1 + C_2C_3C_4$, we find

$$\|\mathrm{D}_z\Psi^w_n - \mathrm{D}_z\Psi^w_*\| < C\rho^n \tag{3.4.17}$$

as required. ☐

3.5 The Renormalisation Cantor Set

We will now show, using the scope maps considered in the previous section, that, similar to the unimodal case, infinitely renormalisable Hénon-like maps also possess an invariant Cantor set on which the Hénon-like map acts as the adding machine. The main idea is to apply the results in Appendix B.2 for general families of scope maps to the particular case when they are generated by a single map and its renormalisations. As before, throughout this section υ will be a fixed unimodal permutation of length $p > 1$ and $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalisation for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

Given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{w\in W^*}$ denote the family of scope maps associated to F. For F_n, the n-th renormalisation of F, let $\underline{\Psi_n} = \{\Psi_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ denote the family of scope maps associated with F_n. Then, for any $n \geq 0$, let $B_n^{\mathbf{w}} = \Psi_n^{\mathbf{w}}(B)$. These are closed simply-connected domains which we will call the *pieces* for F_n. Finally let $B_*^{\mathbf{w}} = \Psi_*^{\mathbf{w}}(B)$.

The following Corollary to Theorem 3.3.3 and Corollary 3.2.12 will be useful.

Corollary 3.5.1. *Let υ be a unimodal permutation of length $p > 1$. There exists a constant $\bar{\varepsilon}_0 > 0$ such that, for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$, the following holds: given any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and any $w_0, w_1 \in W$ the pieces B^{w_0} and B^{w_1} are horizontally and vertically separated.*

Moreover, for any $n > 0$, the pieces, $B_n^{w_0}, B_n^{w_1}$, for F_n are also horizontally and vertically separated and they converge exponentially, in the Hausdorff metric, to $B_*^{w_0}, B_*^{w_1}$.

Proposition 3.5.2. *Let υ be a unimodal permutation of length $p > 1$. There exists a constant $\bar{\varepsilon}_0 > 0$, depending upon υ, for which the following holds: given any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ denote its family of scope maps. Then the set*

$$\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W_p^n} \Psi^{\mathbf{w}}(B), \tag{3.5.1}$$

has the following properties:

(i) it is an F-invariant Cantor set;

(ii) F acts as the adding machine upon $\mathcal{O}$, i.e. there exists a map $h\colon \overline{W}_p \to \mathcal{O}$ such that the following diagram

$$\begin{array}{ccc} \overline{W}_p & \xrightarrow{\mathbf{w}\mapsto 1+\mathbf{w}} & \overline{W}_p \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathcal{O} & \xrightarrow[F]{} & \mathcal{O} \end{array} \tag{3.5.2}$$

commutes;

(iii) there is a unique F-invariant measure, μ, with support on $\mathcal{O}$.

The set $\mathcal{O}$ will be called the *renormalisation Cantor set* for F, or simply the Cantor set for F.

Proof. Let F_n denote the n-th renormalisation of F. Let (f_n, ε_n) denote the canonical parametrisation of F_n. Let $\tilde{F}_n = (f_n \circ \pi_x, \pi_x)$ denote the corresponding degenerate map. Let $\mathbf{\Psi} = \{\Psi^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ and $\tilde{\mathbf{\Psi}} = \{\tilde{\Psi}^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ denote the family of scope maps for the F_n and $\tilde{F}_n$ respectively.

By Theorem 3.3.3 the maps f_n converge exponentially to f_*. Hence they have renormalisation cycles with uniformly bounded geometry and the transfer maps $f_n^{\circ p-w}\colon J_n^w \to J_n^0$ have uniformly bounded distortion The transfer maps will also have positive Schwarzian derivative. Therefore by Proposition B.2.1, the set $\bigcap_{n\geq 0}\bigcup_{\mathbf{w}\in W^n}\psi^{\mathbf{w}}$ is a Cantor set. By Corollary B.2.2 the corresponding set

$$\tilde{\mathcal{O}} = \bigcap_{n\geq 0}\bigcup_{\mathbf{w}\in W^n}\tilde{\Psi}^{\mathbf{w}}(B) \tag{3.5.3}$$

is also a Cantor set. By Theorem 3.2.13 there exists a constant $C>0$ such that $|F_n - \tilde{F}_n| < C\bar{\varepsilon}^{p^n}$ and by Theorem 3.3.3 there exists a $K>0$ such that $\|\mathrm{D}_z\tilde{\Psi}_n^w\| < K$. Therefore, by Proposition B.2.3 the set

$$\mathcal{O} = \bigcap_{n\geq 0}\bigcup_{\mathbf{w}\in W^n}\Psi^{\mathbf{w}}(B) \tag{3.5.4}$$

is a Cantor set.

Next, fix $n\geq 0$ and let $\mathbf{w}\in W^n$. By our labelling convention $F(B^{\mathbf{w}}) = B^{1+\mathbf{w}}$ if $\mathbf{w}\neq (p-1)^n$. In the case $\mathbf{w} = (p-1)^n$ we find

$$\begin{aligned} F(B^{(p-1)^n}) &= F^{\circ p^n}(B^{0^n}) \\ &= \Psi_n^{0^n}\circ F_n \circ (\Psi_n^{0^n})^{-1}(B^{0^n}) \\ &= \Psi_n^{0^n}\circ F_n(B) \end{aligned} \tag{3.5.5}$$

but $F_n(B)\subset B$ and Ψ_n^w being a diffeomorphism onto its image then implies $F(B^{(\mathbf{p-1})^{\mathbf{n}}})\subset B^{0^n}$. Hence for each $n\geq 0$ the set $\bigcup_{\mathbf{w}\in W^n} B^{\mathbf{w}}$ is F-invariant, and the therefore their intersection, $\mathcal{O}$, is also.

Now observe that this also gives us the conjugation h as follows. Let $\mathbf{w} = w_0w_1\ldots\in\bar{W}$ be an infinite word. Then this defines a unique nested sequence of boxes $B^{\mathbf{w}_i}\supset B^{\mathbf{w}_{i+1}}$, where $\mathbf{w}_i = w_0w_1\ldots w_i$ denotes concatenation of the first i letters, $i\geq 0$. By the argument in the first paragraph this nest shrinks to a point. Moreover this point must be a point of $\mathcal{O}$. Label it $B^{\mathbf{w}}$. By definition of $\mathcal{O}$ each of its points is constructed in this manner hence there exists a bijection $h\colon \bar{W}\to\mathcal{O}$ given by $h(\mathbf{w}) = B^{\mathbf{w}}$.

Next we show $F(h(\mathbf{w})) = h(1+\mathbf{w})$. Choose $\mathbf{w}\in\bar{W}$ and consider the nest $B^{\mathbf{w}_i}\supset B^{\mathbf{w}_{i+1}}$. Then by our labelling convention $F(B^{\mathbf{w}_i}) = B^{1+\mathbf{w}_i}$ if $\mathbf{w}_i\neq (p-1)^i$ and from above, $F(B^{\mathbf{w}_i})\subset B^{(1+\mathbf{w})_i}$ otherwise (where we are using the addition on W^* instead of $\bar{W}$). Hence passing to the limit as i tends to infinity gives us the result.

Now we will show there exists a unique invariant measure on $\bar{W}$, which is moreover ergodic. For $\mathbf{w} \in W^*$, if we let $[\mathbf{w}]$ denote the cylinder set associated with $\mathbf{w}$ then we define the measure ν on cylinder sets by $\nu[\mathbf{w}] = p^{-length(\mathbf{w})}$. Endowing $\bar{W}$ with the sigma-algebra generated by these cylinder sets, we extend the measure to this sigma-algebra, also denoting it by ν. Pushing forward ν under h gives us a measure $\mu = h^*(\nu)$. Then since h is a bijection acting as a conjugacy between F and addition by 1, μ is also a unique invariant measure, and moreover ergodic. $\square$

Remark 3.5.3. Let us denote the cylinder sets of $\mathcal{O}$ under the action of F by $\mathcal{O}^{\mathbf{w}}$. That is $\mathcal{O}^{\mathbf{w}} = \mathcal{O} \cap \Psi^{\mathbf{w}}(B)$. Then the collection $\underline{\mathcal{O}} = \{\mathcal{O}^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ has the following structure

(i) $F(\mathcal{O}^{\mathbf{w}}) = \mathcal{O}^{1+\mathbf{w}}$ for all $\mathbf{w} \in W^*$;

(ii) $\mathcal{O}^{\mathbf{w}}$ and $\mathcal{O}^{\tilde{\mathbf{w}}}$ are disjoint for all $\mathbf{w} \neq \tilde{\mathbf{w}}$ of the same length;

(iii) the disjoint union of the $\mathcal{O}^{\mathbf{w}w}$ is equal to $\mathcal{O}^{\mathbf{w}}$, for all $\mathbf{w} \in W^*, w \in W$;

(iv) $\mathcal{O} = \bigcup_{\mathbf{w}\in W^n} \mathcal{O}^{\mathbf{w}}$ for each $n \geq 1$.

This will play an important role in studying the geometry of $\mathcal{O}$.

Remark 3.5.4. For any $n > 0$ we can construct the functions $\Psi_n^{\mathbf{w}} = \Psi^{\mathbf{w}}(F_n)$ and the sets $\mathcal{O}_n^{\mathbf{w}} = \mathcal{O}^{\mathbf{w}}(F_n)$ in exactly the same way as we did above. Let $\underline{\Psi}_n = \{\Psi_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ and $\underline{\mathcal{O}}_n = \{\mathcal{O}_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$.

The number n is called the *height* of $\Psi_n^{\mathbf{w}}$ and $\mathcal{O}_n^{\mathbf{w}}$ and the length of $\mathbf{w}$ is called the *depth*. We use the terms height and depth to reflect a kind of duality in our construction, reflected in the issue of whether to call the Ψ_n telescope maps or microscope maps. We will also use these adjectives for all associated objects.

Corollary 3.5.5. *Let υ be a unimodal permutation of length $p > 1$. There exist constants $C > 0$ and $0 < \rho < 1$ such that the following holds: Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathbf{w} \in \bar{W}$ be an arbitrary infinite word. Then the points $\mathcal{O}_n^{\mathbf{w}}$ and $\mathcal{O}_*^{\mathbf{w}}$ satisfy*

$$|\mathcal{O}_n^{\mathbf{w}} - \mathcal{O}_*^{\mathbf{w}}| < C\rho^n. \tag{3.5.6}$$

The construction of the Cantor set $\mathcal{O}$ and the measure μ now enables us to make the following definition. The *Average Jacobian* of F is

$$b_F = \exp \int_{\mathcal{O}} \log |\mathrm{Jac}_z F|\, d\mu(z). \tag{3.5.7}$$

The remainder of this work can be considered as a study of this quantity.

Lemma 3.5.6 (Distortion Lemma)**.** *Let υ be a unimodal permutation of length $p > 1$. Then there exist constants $C > 0$, and $0 \leq \rho < 1$ such that the following* *holds: Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $B^{\mathbf{w}}$ denote the piece associated to the word $\mathbf{w} \in W^*$. Then for any $B^{\mathbf{w}}$, where $\mathbf{w} \in W^n$, and any $z_0, z_1 \in B^{\mathbf{w}}$,*

$$\log \left| \frac{\operatorname{Jac}_{z_0} F^{\circ m}}{\operatorname{Jac}_{z_1} F^{\circ m}} \right| \leq C\rho^n \tag{3.5.8}$$

for all $m = 1, p, \ldots, p^n$.

Corollary 3.5.7. *Let υ be a unimodal permutation of length $p > 1$. Then there exists a universal constant $0 < \rho < 1$ such that the following holds: given $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$, let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$. Then for any integer $n \geq 0$, any $\mathbf{w} \in W^n$, and any $z \in B^{\mathbf{w}}$,*

$$\operatorname{Jac}_z F^{\circ p^n} = b_F^{p^n}(1 + \mathrm{O}(\rho^n)). \tag{3.5.9}$$

Remark 3.5.8. Here, the constant ρ may be taken as the universal constant from Theorem 3.3.3.

Proof. Observe that, as μ has support on $\mathcal{O}$,

$$\begin{aligned}
\int_{B^{\mathbf{w}}} \log|\operatorname{Jac}_z F^{\circ p^n}| d\mu(z) &= \int_{\mathcal{O}^{\mathbf{w}}} \log|\operatorname{Jac}_z F^{\circ p^n}| d\mu(z) \\
&= \int_{\mathcal{O}} \log|\operatorname{Jac}_z F| d\mu(z) \\
&= \log b_F.
\end{aligned} \tag{3.5.10}$$

Therefore, there is a $\xi \in B^{\mathbf{w}}$ such that

$$\log|\operatorname{Jac}_\xi F^{\circ p^n}| = \frac{\log b_F}{\mu(B^{\mathbf{w}})} = p^n \log b_F \tag{3.5.11}$$

so the result follows from the Lemma 3.5.6. □

Proposition 3.5.9 (Monotonicity)**.** *Let υ be a unimodal permutation of length $p > 1$. Let $F_t \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ be a one parameter family of infinitely renormalisable Hénon-like maps such that the average Jacobian $b_t = b(F_t)$ depends strictly monotonically on t. Let $\tilde{F}_t = \mathcal{R}F_t$ and let $\tilde{b}_t = b(\tilde{F}_t)$. Then $\tilde{b}_t$ is also strictly monotone in t.*

Proof. Let $\tilde{F}_t = \mathcal{R}F_t$, $\tilde{\mathcal{O}}_t = \mathcal{O}(\tilde{F}_t)$, and $\tilde{\mu}_t = \mu(\tilde{F}_t)$. Recall that, by definition,

$$\log b_t = \int_{\mathcal{O}_t} \log|\operatorname{Jac}_z F_t| d\mu_t(z), \quad \log \tilde{b}_t = \int_{\tilde{\mathcal{O}}_t} \log|\operatorname{Jac}_z \tilde{F}_t| d\tilde{\mu}_t(z). \tag{3.5.12}$$

Then by construction $\tilde{F}_t = \Psi_t^{-1} F_t^{\circ p} \Psi_t$ and $\tilde{\mathcal{O}}_t = \bar{\Psi}_t(\mathcal{O}_t^0)$, where $\mathcal{O}_t^0 = \mathcal{O}_t \cap B_t^0$. Since $\mu_t, \tilde{\mu}_t$ are determined by the adding machine actions on $\mathcal{O}_t, \tilde{\mathcal{O}}_t$ respectively we also have $\tilde{\mu}_t = p\mu_t \circ \Psi_t$. Therefore

$$\begin{aligned}
&\int_{\tilde{\mathcal{O}}_t} \log|\operatorname{Jac}_z \tilde{F}_t| d\tilde{\mu}_t(z) \\
&= p \int_{\bar{\Psi}_t(\mathcal{O}_t^0)} \log\left(\left|\operatorname{Jac}_{\Psi_t(z)} F_t^{\circ p}\right| \left| \frac{\operatorname{Jac}_z \Psi_t}{\operatorname{Jac}_{\tilde{F}_t(z)} \Psi_t} \right| \right) d(\mu_t \circ \Psi_t)(z)
\end{aligned} \tag{3.5.13}$$

hence[2]

$$\int_{\tilde{\mathcal{O}}_t} \log |\mathrm{Jac}_z \tilde{F}_t| d\tilde{\mu}_t(z) \tag{3.5.14}$$
$$= p \int_{\mathcal{O}_t^0} \log \left(\left|\mathrm{Jac}_z F_t^{\circ p}\right| \left| \frac{\mathrm{Jac}_{\bar{\Psi}_t(z)} \Psi_t}{\mathrm{Jac}_{\bar{\Psi}_t F_t^{\circ p}(z)} \Psi_t} \right| \right) d\mu_t(z)$$
$$= p \int_{\mathcal{O}_t^0} \sum_{i=0}^{p-1} \log |\mathrm{Jac}_{F_t^{\circ i}(z)} F_t| d\mu_t(z) + p \int_{\mathcal{O}_t^0} \log \left(\frac{\mathrm{Jac}_{\bar{\Psi}_t(z)} \Psi_t}{\mathrm{Jac}_{\bar{\Psi} F_t^{\circ p}(z)} \Psi_t} \right) d\mu_t(z).$$

Now observe, by definition of μ_t,

$$\int_{\mathcal{O}_t^0} \sum_{i=0}^{p-1} \log |\mathrm{Jac}_{F_t^{\circ i}(z)} F_t| d\mu_t(z) = \int_{\mathcal{O}_t} \log |\mathrm{Jac}_z F_t| d\mu_t(z) \tag{3.5.15}$$

and

$$\int_{\mathcal{O}_t^0} \log |\mathrm{Jac}_{\bar{\Psi} F_t^{\circ p}(z)} \Psi_t| d\mu_t(z) = \int_{\mathcal{O}_t^0} \log |\mathrm{Jac}_{\bar{\Psi}(z)} \Psi_t| d\mu_t(z). \tag{3.5.16}$$

Together these imply

$$\log \tilde{b}_t = \int_{\tilde{\mathcal{O}}_t} \log |\mathrm{Jac}_z \tilde{F}_t| d\tilde{\mu}_t(z) = p \int_{\mathcal{O}_t} \log |\mathrm{Jac}_z F_t| d\mu_t = p \log b_t \tag{3.5.17}$$

which depends monotonically on t if $\log b_t$ depends monotonically. Since the logarithm function is monotone the proof is complete. □

3.6 Asymptotics of Scope Functions

We study affine rescaling of scope functions and their compositions. We only consider the case when $w_i = 0$ for all $i > 0$ as this is the simplest to deal with and the most relevant in the next sections. However, we believe a large portion of the results below can be extended to the more general case. As before, unless otherwise stated, throughout this section υ will be a fixed unimodal permutation of length $p > 1$ and $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalisation for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

Proposition 3.6.1. *Let υ be a unimodal permutation of length $p > 1$. Then there exists a constant $\bar{\varepsilon}_0 > 0$ such that the following holds: given $0 \leq \bar{\varepsilon} < \bar{\varepsilon}_0$,*

[2]here we use the integral substitution fomula, namely if $(X, \mathcal{B}), (X', \mathcal{B}')$ are measurable spaces, μ is a measure on X, $T\colon X \to Y$ surjective then for all $\mu \circ T^{-1}$-measurable ϕ on Y,

$$\int_X \phi \circ T d\mu = \int_Y \phi d(\mu \circ T^{-1})$$

for any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\Psi_n \colon B \to B$ denote its n-th scope map. Explicitly, for any $(x,y) \in B$, let

$$F(x,y) = (\phi_n(x,y), x); \quad \Psi_n(x,y) = (\psi_n^1, \psi_n^0). \tag{3.6.1}$$

Then there is a constant $C > 0$, depending upon F only, such that

$$|\partial_{x^i}\psi_n^1(x,y)| < C, \quad |\partial_{x^i y^j}\psi_n^1(x,y)| < C\bar{\varepsilon}^{p^n} \tag{3.6.2}$$

for any $(x,y) \in B$ and any integers $i,j \geq 1$.

Proof. By Theorem 3.3.2 we know there exists a constant $C_0 > 0$ and, for each integer $n > 0$, a degenerate $\tilde{F}_n \in \mathcal{H}_{\Omega,\upsilon}(0)$ such that $|F_n - \tilde{F}_n|_\Omega \leq C_0\bar{\varepsilon}^{p^n}$ and F_n converges exponentially to F_*. Let $\tilde{\Psi}_n$ denote the scope function for F_n. Then this implies there exists a constant $C_1 > 0$ such that $|\Psi_n - \tilde{\Psi}_n|_\Omega \leq C_1\bar{\varepsilon}^{p^n}$ and $\tilde{\Psi}_n$ converges exponentially to Ψ_*. Since Ψ_* is analytic there exists a constant $C_2 > 0$ such that $|\partial_{x^i}\psi_*^1| < C_2$ and as F_* is degenerate $\partial_{x^i y^j}\psi_*^1 = 0$ for $j > 0$. Hence the result follows. □

The next Lemma is a simple application of Taylor's Theorem.

Lemma 3.6.2. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\Psi \colon B \to B^0$ denote its n-th scope map. Explicitly, for $(x,y) \in B$ let*

$$F(x,y) = (\phi(x,y), x); \quad \Psi(x,y) = (\psi^1(x,y), \psi^0(x,y)). \tag{3.6.3}$$

Then, for $z_0 \in B$ and $z_1 \in \mathbb{R}$ satisfying $z_0 + z_1 \in B$, Ψ can be expressed as

$$\Psi(z_0 + z_1) = \Psi(z_0) + \mathrm{D}_{z_0}\Psi \circ (\mathrm{id} + \mathrm{R}_{z_0}\Psi)(z_1) \tag{3.6.4}$$

where $\mathrm{D}_{z_0}\Psi$ denotes the derivative of Ψ at z_0 and $\mathrm{R}_{z_0}\Psi$ is a nonlinear remainder term. The maps $\mathrm{D}_{z_0}\Psi$ and $\mathrm{R}_{z_0}\Psi$ take the form

$$\mathrm{D}_{z_0}\Psi = \sigma \begin{pmatrix} s(z_0) & t(z_0) \\ 0 & 1 \end{pmatrix}; \quad \mathrm{R}_{z_0}\Psi(z_1) = \begin{pmatrix} r(z_0)(z_1) \\ 0 \end{pmatrix} \tag{3.6.5}$$

for some functions $s(z)$ and $t(z)$. Here σ denotes the scaling ratio of Ψ.

Remark 3.6.3. There are two related quantities that will henceforth play an important role. The first is the scaling ratio of F_*, defined to be the unique eigenvalue, of multiplicity two, of the affine factor of $\Psi_* = \Psi_*^0$. The second is the derivative of Ψ_*^0 at the tip τ_* of F_*. By Lemma 2.4.3 the derivative of ψ_*^1 at its fixed point is also this scaling ratio (up to sign, which depends on the combinatorics), but the fixed point of ψ^1 is the critical value, which is the projection onto the x axis of τ_*. Hence these two quantities coincide and shall be denoted by σ.

Definition 3.6.4. The functions $s(z)$ and $t(z)$ given by the Lemma 3.6.2 above are called the *squeeze* and the *tilt* of Ψ at z respectively.

Proposition 3.6.5. *Let $F \in \mathcal{H}_{\Omega,v}(\bar{\varepsilon})$ and $\tilde{F} \in \mathcal{H}_{\Omega,v}(0)$ satisfy $|F - \tilde{F}|_\Omega < \bar{\varepsilon}$. Let $\Psi = (\psi^1, \psi^0)$ and $\tilde{\Psi}_f = (\tilde{\psi}^1, \tilde{\psi}^0)$ denote their respective scope maps. Assume there is a constant $C > 1$ such that, for all $i > 0$,*

$$C^{-1} < \left|\partial_x \psi^1\right|_\Omega \quad \textit{and} \quad \left|\frac{\partial_{x^i}\psi^1}{\partial_x \psi^1}\right|_\Omega < C \tag{3.6.6}$$

Then there is a constant $K > 0$ such that, if $R(z_0)(z_1) = \mathrm{R}_{z_0}\Psi(z_1)$ is defined as above,

$$|\partial_x r(z_0)(z_1)|, |\partial_{xx} r(z_0)(z_1)| < K(1 + |F - F_*| + \bar{\varepsilon}) \tag{3.6.7}$$

and

$$|\partial_y r(z_0)(z_1)|, |\partial_{xy} r(z_0)(z_1)|, |\partial_{yy} r(z_0)(z_1)| < K\bar{\varepsilon}. \tag{3.6.8}$$

for any $z_0 \in B$ and $z_1 \in \mathbb{R}^2$ satisfying $z_0 + z_1 \in B$.

Proof. Let $z_i = (x_i, y_i)$ for $i = 0, 1$. Expanding Ψ in power series around z_0 and equating it with the above representation gives

$$r(z_0)(z_1) = \sum_{i,j \geq 0; i+j \geq 2} \binom{i+j}{j} x_1^i y_1^j \frac{\partial_{x^i y^j}\psi^1(z_0)}{\partial_x \psi^1(z_0)}. \tag{3.6.9}$$

and we get a similar expression for $\tilde{r}(z_0)(z_1)$ and $r_*(z_0)(z_1)$. We may write $r(z_0)(z_1) = A_0(x_1) + y_1 A_1(x_1) + y_1^2 A_2(x_1, y_1)$ where

$$A_0(x_1) = \sum_{i \geq 2; j=0} x_1^i \frac{\partial_{x^i}\psi^1(z_0)}{\partial_x \psi^1(z_0)} \tag{3.6.10}$$

$$A_1(x_1) = \sum_{i \geq 1; j=1} \binom{i+1}{1} x_1^i \frac{\partial_{x^i y}\psi^1(z_0)}{\partial_x \psi^1(z_0)} \tag{3.6.11}$$

$$A_2(x_1, y_1) = \sum_{i \geq 0; j \geq 2} \binom{i+j}{j} x_1^i y_1^j \frac{\partial_{x^i y^j}\psi^1(z_0)}{\partial_x \psi^1(z_0)}. \tag{3.6.12}$$

Define $\tilde{A}_0(x_1), \tilde{A}_1(x_1)$ and $\tilde{A}_2(x_1, y_1)$ for $\tilde{r}(z_0)(z_1)$ and $A_{*,0}(x_1), A_{*,1}(x_1)$ and $A_{*,2}(x_1, y_1)$ for $r_*(z_0)(z_1)$ similarly. We claim there exists a constant $K_0 > 0$ such that

$$|A_0'(x_1)|, |A_0''(x_1)| \leq K_0 \frac{|x_1|^2}{1 - |x_1|}(1 + |\tilde{F} - F_*|_\Omega + |\tilde{F} - F|_\Omega). \tag{3.6.13}$$

First, as F_* is fixed we may assume, without loss of generality, that the constant $C > 0$ satisfies $\left|\frac{\partial_{x^i}\psi_*^1}{\partial_x \psi_*^1}\right| < C$. Also observe that $C^{-1} < |\partial_x \psi^1|$ implies $|\partial_x \tilde{\psi}^1|^{-1} < C(1 + \kappa\bar{\varepsilon})$ for some $\kappa > 0$. Therefore, by Lemma A.1.5,

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x \tilde{\psi}^1} - \frac{\partial_{x^i}\psi_*^1}{\partial_x \psi_*^1}\right|_\Omega \leq C(1 + \kappa\bar{\varepsilon}) \max(1, C) \max_i \left(|\partial_{x^i}\tilde{\psi}^1 - \partial_{x^i}\psi_*^1|_\Omega\right). \tag{3.6.14}$$

The same argument, this time using the assumption $\left|\frac{\partial_{x^i}\psi^1}{\partial_x\psi^1}\right| < C$, also implies

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi^1}{\partial_x\psi^1}\right|_\Omega \leq C(1+\kappa\bar{\varepsilon})\max(1,C)\max_i\left(|\partial_{x^i}\tilde{\psi}^1 - \partial_{x^i}\psi^1|_\Omega\right), \quad (3.6.15)$$

so analyticity of $F, \tilde{F}$ and F_* implies there is a constant $C_0 > 0$ such that

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi^1_*}{\partial_x\psi^1_*}\right|_\Omega \leq C_0|\tilde{F} - F_*|_\Omega \quad (3.6.16)$$

and

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi^1}{\partial_x\psi^1}\right|_\Omega \leq C_0|\tilde{F} - F|_\Omega. \quad (3.6.17)$$

Hence, by the summation formula for a geometric progression,

$$|\tilde{A}_0(x_1) - A_{*,0}(x_1)| \leq \sum_{i\geq 2}|x_1|^i\left|\frac{\partial_{x^i}\tilde{\psi}^1(x_0)}{\partial_x\tilde{\psi}^1(x_0)} - \frac{\partial_{x^i}\psi^1_*(x_0)}{\partial_x\psi^1_*(x_0)}\right| \leq \frac{C_0|x_1|^2}{1-|x_1|}|\tilde{F} - F_*|_\Omega \quad (3.6.18)$$

and similarly

$$|\tilde{A}_0(x_1) - A_0(x_1)| \leq \sum_{i\geq 2}|x_1|^i\left|\frac{\partial_{x^i}\tilde{\psi}^1(x_0)}{\partial_x\tilde{\psi}^1(x_0)} - \frac{\partial_{x^i}\psi^1_*(x_0)}{\partial_x\psi^1_*(x_0)}\right| \leq \frac{C_0|x_1|^2}{1-|x_1|}|\tilde{F} - F|_\Omega. \quad (3.6.19)$$

Secondly, observe that analyticity and degeneracy of F_* implies there exists a constant $C_1 > 0$ such that $|A_{*,0}(x_1)| < \frac{C_1|x_1|^2}{1-|x_1|}$. Therefore there exists a $K_0 > 0$ such that

$$\begin{aligned} |A_0(x_1)| &\leq |A_{*,0}(x_1)| + |\tilde{A}_0(x_1) - A_{*,0}(x_1)| + |A_0(x_1) - \tilde{A}_0(x_1)| \quad (3.6.20)\\ &\leq \frac{K_0|x_1|^2}{1-|x_1|}\left(1 + |\tilde{F} - F_*|_\Omega + |\tilde{F} - F|_\Omega\right) \end{aligned}$$

and, by analyticity of A_0, this implies the bound on its derivatives. Next we claim there is are constants $C_2, C_3 > 0$ such that

$$|A_1(z_1)|, |A_1'(z_1)|, |A_1''(z_1)| \leq C_2\bar{\varepsilon}|z_1|, \quad (3.6.21)$$

and

$$|\partial_x A_2(z_1)|, |\partial_y A_2(z_1)|, |\partial_{xx} A_2(z_1)|, |\partial_{xy} A_2(z_1)|, |\partial_{yy} A_2(z_1)| \leq C_3\bar{\varepsilon}|z_1|. \quad (3.6.22)$$

This can be seen by observing that all the coefficients of $A_1(z_1)$ and $A_2(z_1)$ are of the form $\partial_{x^i y^j}\psi^{p-1}(z_0)/\partial_x\psi^{p-1}(z_0)$, but from the Variational Formula there exists a constant $C_4 > 0$ such that

$$C_4^{-1} < |\partial_x\psi^1| < C_4; \quad |\partial_{x^i y^j}\psi^1| < C_4\bar{\varepsilon}, \quad (3.6.23)$$

hence all coefficients are bounded by $C_4^2\bar{\varepsilon}$ in absolute value. Therefore, assuming $|z_1| \leq \gamma < 1$, the above estimates must hold by setting $C_3 = C_4^2/(1-\gamma)$.

Now differentiating $r(\Psi; z_0)$ and applying the above estimates we find there exists a $C > 0$ such that, for $|z_1| \leq \gamma < 1$,

$$\begin{aligned}
|\partial_x r(\Psi; z_0)(z_1)| &\leq |A_0'(x_1)| + |y_1||A_1'(x_1)| + |y_1|^2|\partial_x A_2(x_1, y_1)| \\
&\leq C(1 + |f - f_*| + \bar{\varepsilon}) && (3.6.24) \\
|\partial_y r(\Psi; z_0)(z_1)| &\leq |A_1(x_1)| + 2|y_1||A_2(x_1, y_1)| + |y_1|^2|\partial_y A_2(x_1, y_1)| \\
&\leq C\bar{\varepsilon} && (3.6.25) \\
|\partial_{xx} r(\Psi; z_0)(z_1)| &\leq |A_0''(x_1)| + |y_1||A_1''(x_1)| + |y_1|^2|\partial_{xx} A_2(x_1, y_1)| \\
&\leq C(1 + |f - f_*| + \bar{\varepsilon}) && (3.6.26) \\
|\partial_{xy} r(\Psi; z_0)(z_1)| &\leq |A_1'(x_1)| + 2|y_1||\partial_x A_2(x_1, y_1)| + |y_1|^2|\partial_{xy} A_2(x_1, y_1)| \\
&\leq C\bar{\varepsilon} && (3.6.27) \\
|\partial_{yy} r(\Psi; z_0)(z_1)| &\leq 2|A_2(x_1, y_2)| + 4|y_1||\partial_y A_2(x_1, y_1)| + |y_1|^2|\partial_{yy} A_2(x_1, y_1)| \\
&\leq C\bar{\varepsilon} && (3.6.28)
\end{aligned}$$

and hence the result is proved. □

3.7 Asymptotics around the Tip

As before, unless otherwise stated, throughout this section υ will be a fixed unimodal permutation of length $p > 1$ and $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalisation for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

For a given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ we now wish to study the Cantor set $\mathcal{O}$, and the behaviour of F around it, in more detail. We will do this locally around a pre-assigned point. Let

$$\tau = \tau(F) = \bigcap_{n\geq 0} B^{0^n}. \qquad (3.7.1)$$

We call this point the *tip*. The study of the orbit of this point is analogous to studying the critical orbit for a unimodal map. The remainder of our work can be viewed as the study of the behaviour of F around τ.

For $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, as usual, let F_n denote the n-th renormalisation and let $\Psi_n \colon B \to B_n^0$ denote the scope map for F_n. Explicitly, $F_n(x,y) = (\phi_n(x,y), x)$ and $\Psi_n(x,y) = (\psi_n^1(x,y), \psi_n^0(x,y))$. Now let $\Psi_{m,n} = \Psi_m \circ \ldots \circ \Psi_n$. Then $\Psi_{m,n}(x,y) = (\psi_{m,n}^1(x,y), \psi_{m,n}^0(x,y))$ from height $n+1$ to height m. By this convention we let $\Psi_{n,n} = \Psi_n$. Observe that $\psi_{m,n}^0$ is affine and depends upon y only. Let us define points τ_n inductively by $\tau_0 = \tau$ and $\tau_{n+1} = \Psi_n^{-1}(\tau_n)$. We will call τ_n the *tip at height n*. We wish to use the decompositions

$$\begin{aligned}
\Psi_n(\tau_{n+1} + z) &= \Psi_n(\tau_{n+1}) + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z) && (3.7.2) \\
&= \tau_n + \mathrm{D}_{\tau_{n+1}}\Psi_n \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z)
\end{aligned}$$

and

$$\Psi_{m,n}(\tau_{n+1}+z) = \Psi_{m,n}(\tau_{n+1}) + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z_1) \tag{3.7.3}$$
$$= \tau_m + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z_1),$$

whenever $\tau_{n+1}+z$ is in $\mathrm{Dom}(\Psi_n)$ or $\mathrm{Dom}(\Psi_{m,n})$ respectively. For notational simplicity let us denote the derivatives $\mathrm{D}_{\tau_{n+1}}\Psi_n, \mathrm{D}_{\tau_{n+1}}\Psi_{m,n}$ and remainder terms, $\mathrm{R}_{\tau_{n+1}}\Psi_n$ and $\mathrm{R}_{\tau_{n+1}}\Psi_{m,n}$, by $D_n, D_{m,n}, R_n$ and $R_{m,n}$ respectively.

It will turn out to be fruitful to change to coordinates in which the tips are situated at the origin. Therefore let $\mathbf{T}_n(z) = z - \tau_n$ and consider the maps $\hat{\Psi}_n = \mathbf{T}_n \circ \Psi_n \circ \mathbf{T}_{n+1}^{-1}$ and their composites

$$\hat{\Psi}_{m,n} = \hat{\Psi}_m \circ \cdots \circ \hat{\Psi}_n = \mathbf{T}_m \Psi_{m,n} \mathbf{T}_{n+1}^{-1}. \tag{3.7.4}$$

From Proposition B.1.2 we know, since $\mathbf{T}_n$ is a translation, that $\mathrm{R}_{z_0}\hat{\Psi}_n = \mathrm{R}_{\mathbf{T}_{n+1}^{-1}z_0}\Psi_n$. Therefore using the same decomposition as above we find,

$$\begin{aligned}\hat{\Psi}_n(z) &= \hat{\Psi}_n(0+z) \\ &= \hat{\Psi}_n(0) + \mathrm{D}_0\hat{\Psi}_n(\mathrm{id} + \mathrm{R}_0\hat{\Psi}_n)(z) \\ &= \mathrm{D}_{\tau_{n+1}}\Psi_n(\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z)\end{aligned} \tag{3.7.5}$$

and similarly

$$\begin{aligned}\hat{\Psi}_{m,n}(z) &= \hat{\Psi}_{m,n}(0+z) \\ &= \hat{\Psi}_{m,n}(0) + \mathrm{D}_0\hat{\Psi}_{m,n}(\mathrm{id} + \mathrm{R}_0\hat{\Psi}_{m,n})(z) \\ &= \mathrm{D}_{\tau_{n+1}}\Psi_{m,n}(\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z)\end{aligned} \tag{3.7.6}$$

For notational simplicity let us denote the quantities $\mathrm{D}_0\hat{\Psi}_n$, $\mathrm{D}_0\hat{\Psi}_{m,n}$, $\mathrm{R}_0\hat{\Psi}_n$ and $\mathrm{R}_0\hat{\Psi}_{m,n}$, by $\hat{D}_n, \hat{D}_{m,n}, \hat{R}_n$ and $\hat{R}_{m,n}$ respectively. Observe that, because our coordinate changes were translations, these quantities are equal to $D_n, D_{m,n}, R_n$ and $R_{m,n}$ respectively. The following follows directly from Lemma 3.6.2.

Lemma 3.7.1. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let the linear map D_n and the function $R_n(z)$ be as above. Then D_n and $R_n(z)$ have the respective forms*

$$D_n = \sigma_n \begin{pmatrix} s_n & t_n \\ 0 & 1 \end{pmatrix}; \quad R_n(z) = \begin{pmatrix} r_n(z) \\ 0 \end{pmatrix}. \tag{3.7.7}$$

Definition 3.7.2. The quantities s_n and t_n from the preceding Lemma will be called, respectively, the *squeeze* and *tilt* of Ψ_n at τ_{n+1}.

Proposition 3.7.3. *For $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $B_n^{\mathbf{w}}$ denote the box of height n with word $\mathbf{w} \in W^*$. Then*

(i) for each $\mathbf{w} \in W^*$, $\mathrm{dist}_{Haus}(B_n^{\mathbf{w}}, B_*^{\mathbf{w}}) \to 0$ exponentially;

(ii) for each $\mathbf{w} \in \overline{W}$, $\mathrm{dist}_{Haus}(\mathcal{O}_n^{\mathbf{w}}, \mathcal{O}_*^{\mathbf{w}}) \to 0$ exponentially.

Proposition 3.7.4. *There exist constants $C > 1$, and $0 < \rho < 1$ such that the following holds: given $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and for each integer $n > 0$ let σ_n, s_n, t_n be the constants and $r_n(z)$ the function defined above. Then for any $z \in B$,*

$$\sigma(1 - C\rho^n) < |\sigma_n| < \sigma(1 + C\rho^n) \tag{3.7.8}$$
$$\sigma(1 + C\rho^n) < |s_n| < \sigma(1 + C\rho^n) \tag{3.7.9}$$
$$C^{-1}\bar{\varepsilon}^{p^n} < |t_n| < C\bar{\varepsilon}^{p^n} \tag{3.7.10}$$
$$|\partial_x r_n(z)| < C|z|, |\partial_y r_n(z)| < C\bar{\varepsilon}^{p^n}|z| \tag{3.7.11}$$
$$|\partial_{xx} r_n(z)| < C|z|, |\partial_{xy} r_n(z)| < C\bar{\varepsilon}^{p^n}|z|, |\partial_{yy} r_n(z)| < C\bar{\varepsilon}^{p^n}|z| \tag{3.7.12}$$

Proof. Observe that σ_n is the eigenvalue of DI_n^{-1}, the affine bijection between $B^0_{n,diag}$ and B. By Proposition 3.7.3 there exists a constant $C_0 > 0$ such that $\mathrm{dist}_{Haus}(B^0_n, B^0_*) < C_0\rho^n$ we see that $|\sigma_n - \sigma_*| < C_0\rho^n$. Next observe that $s_n = \partial_x\psi^1_n(\tau_{n+1})$ and, by Lemma 2.4.3, $\sigma = \partial_x\psi^1_{f_*}(\tau_*)$ which implies

$$\begin{aligned} |s_n - \sigma| &\leq |\partial_x\psi^1_n(\tau_{n+1}) - \partial_x\psi^1_*(\tau_{n+1})| + |\partial_x\psi^1_*(\tau_{n+1}) - \partial_x\psi^1_*(\tau_*)| \\ &\leq |\psi^1_n - \psi^1_*|_\Omega + |\partial_{xx}\psi^1_*|_\Omega|\pi_x(\tau_{n+1}) - \pi_x(\tau_*)|. \end{aligned} \tag{3.7.13}$$

Again by Proposition 3.7.3 $|\tau_n - \tau_*| < C_0\rho^n$. Also, a consequence of Theorem 3.3.3 is that there exists a constant $C_1 > 0$ such that $|\psi^1_n - \psi^1_*|_\Omega < C_1\rho^n$. Since fixing the combinatorial type fixes the map ψ^1_*, we may assume $|\partial_{xx}\psi^1_*|_\Omega < C_2$ for some constant $C_2 > 0$. Therefore

$$|s_n - \sigma| \leq C_1\rho^n + C_0C_2\rho^n = (C_1 + C_0C_2)\rho^n. \tag{3.7.14}$$

Now for each $n > 0$ choose a $\tilde{F}_n \in \mathcal{H}_{\Omega,\upsilon}(0)$ such that $|F_n - \tilde{F}_n|_\Omega < C_3\bar{\varepsilon}^{p^n}$, where $C_3 > 0$ is the constant from Theorem 3.2.13. Applying the Proposition 3.4.1 (or rather, elements of its proof) and Convergence of Renormalisation (Theorem 3.3.3) we find there exists a constant $C_4 > 0$ such that $|\partial_y\psi^1_n| = |\partial_y\psi^1_n - \partial_y\psi^1_{f_n}| < C_4\bar{\varepsilon}^{p^n}$. This concludes the first item. For the next two items we apply Proposition 3.6.5. □

Lemma 3.7.5. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let the linear map $D_{m,n}$ and the function $R_{m,n}(z)$ be as above. Then $D_{m,n}$ and the function $R_{m,n}(z)$ have the respective form*

$$D_{m,n} = \sigma_{m,n}\begin{pmatrix} s_{m,n} & t_{m,n} \\ 0 & 1 \end{pmatrix}; \quad R_{m,n}(z) = \begin{pmatrix} r_{m,n}(z) \\ 0 \end{pmatrix}, \tag{3.7.15}$$

respectively, and so if $\tau_m = (\xi_m, \eta_n)$,

$$\Psi_{m,n}(z) = \tau_m + \sigma_{m,n}\begin{pmatrix} s_{m,n}\left((x - \xi_m) + r_{m,n}(z - \tau_m)\right) + t_{m,n}(y - \eta_m) \\ y - \eta_m \end{pmatrix}. \tag{3.7.16}$$

Moreover,

$$\sigma_{m,n} = \prod_{i=m}^{n} \sigma_i; \quad s_{m,n} = \prod_{i=m}^{n} s_i; \quad t_{m,n} = \sum_{i=m}^{n} s_{m,i-1} t_i. \tag{3.7.17}$$

Proof. From Lemma 3.7.1 we know it holds for $m = n$. For $m < n$ the chain rule $D_{m,n} = D_{m,n-1}D_n$ implies $D_{m,n}$ is again upper triangular and

$$\sigma_{m,n} = \sigma_{m,n-1}\sigma_n, \quad s_{m,n} = s_{m,n-1}s_n, \quad t_{m,n} = s_{n-1}t_n + t_{m,n-1}, \tag{3.7.18}$$

from which the lemma immediately follows by induction. □

Proposition 3.7.6. *There exist constants $C > 0$, and $0 < \rho < 1$ such that the following holds: for $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $\sigma_{m,n}, s_{m,n}, t_{m,n}$ be the constants and $r_{m,n}(z)$ the function defined above. Then*

$$\sigma^{n-m}(1 - C\rho^m) < |\sigma_{m,n}| < \sigma^{n-m}(1 + C\rho^m) \tag{3.7.19}$$

$$\sigma^{n-m}(1 - C\rho^m) < |s_{m,n}| < \sigma^{n-m}(1 + C\rho^m) \tag{3.7.20}$$

$$|t_{m,n}| < C\bar{\varepsilon}^{p^m} \tag{3.7.21}$$

$$|\partial_x r_{m,n}(z)| < C|z|, |\partial_y r_{m,n}(z)| < C\bar{\varepsilon}^{p^{m-1}}|z| \tag{3.7.22}$$

$$|\partial_{xx} r_{m,n}(z)| < C|z|, |\partial_{xy} r_{m,n}(z)| < C\sigma^{2(n-m)}\bar{\varepsilon}^{p^m}|z|, |\partial_{yy} r_{m,n}(z)| < C\bar{\varepsilon}^{p^m}|z|. \tag{3.7.23}$$

Proof. Throughout the proof $C_0 > 0$ will denote the constant from Proposition 3.7.4. From Lemma 3.7.5, Proposition 3.7.4 and Proposition A.1.1 respectively, we find there exists a constant $C_1 > 0$ such that

$$|\sigma_{m,n}| = \prod_{i=m}^{n} |\sigma_i| \le \sigma^{n-m} \prod_{i=m}^{n} (1 + C_0\rho^i) \le \sigma^{n-m}(1 + C_1\rho^m) \tag{3.7.24}$$

and similarly

$$|s_{m,n}| = \prod_{i=m}^{n} |s_i| \le \sigma^{n-m} \prod_{i=m}^{n} (1 + C_0\rho^i) \le \sigma^{n-m}(1 + C_1\rho^m). \tag{3.7.25}$$

Again by Lemma 3.7.5 and Proposition 3.7.4 above we find, for $i > m$,

$$\left|\frac{t_i}{t_m}\right| = \left|\frac{\partial_y \phi_i^{p-1}(\tau_{i+1})}{\partial_y \phi_m^{p-1}(\tau_{m+1})}\right| \left|\frac{\partial_x \phi_m^{p-1}(\tau_{m+1})}{\partial_x \phi_i^{p-1}(\tau_{i+1})}\right| \le C_0^4 \bar{\varepsilon}^{p^{i+1}-p^m}. \tag{3.7.26}$$

Therefore, by Lemma A.1.3 there exists a constant $C_2 > 0$ such that

$$\begin{aligned} |t_{m,n}| &\le |t_m| \sum_{i=m}^{n} |s_{m,i-1}| \left|\frac{t_i}{t_m}\right| \\ &\le C_0^2 \bar{\varepsilon}^{p^m} \sum_{i=m}^{n} \sigma^{i-m-1} \bar{\varepsilon}^{p^{i+1}-p^m}(1 + C_1\rho^i). \\ &\le C_2 \bar{\varepsilon}^{p^m}. \end{aligned} \tag{3.7.27}$$

This concludes the first item. For the second and third items we will proceed by induction. The case when $m = n$ is shown in Proposition 3.7.4 so, for $m+1 \leq n$, assume the inequalities hold for $r_{m+1,n}$ and consider $r_{m,n}$. Choose $z = (x, y) \in \mathbb{R}^2$ such that $\tau_{n+1} + z \in \mathrm{Dom}(\Psi_{m,n})$. Then since $\Psi_{m,n} = \Psi_m \circ \Psi_{m+1,n}$, decomposing the left hand side gives

$$\Psi_{m,n}(\tau_{n+1} + z) = \tau_m + D_{m,n}(\mathrm{id} + R_{m,n})(z) \tag{3.7.28}$$

and decomposing the right hand side and applying Proposition B.1.2 gives us

$$\begin{aligned} &\Psi_m(\Psi_{m+1,n}(\tau_{n+1} + z)) \\ &= \Psi_n(\tau_{m+1} + D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)) \\ &= \tau_m + D_m(\mathrm{id} + R_m)\left(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)\right) \\ &= \tau_m + D_{m,n}(\mathrm{id} + R_{m+1,n})(z) + D_m\left(R_m(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z))\right). \end{aligned} \tag{3.7.29}$$

Equating these and making appropriate cancellations then gives

$$R_{m,n}(z) = R_{m+1,n}(z) + D_{m+1,n}^{-1}\left(R_m(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z))\right). \tag{3.7.30}$$

By definition, $R_{m,n}(z) = (r_{m,n}(z), 0)$, $R_{m+1,n}(z) = (r_{m+1,n}(z), 0)$ and $R_m(z) = (r_m(z), 0)$. Therefore setting $z' = (x', y') = D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)$, that is

$$(x', y') = (\sigma_{m+1,n}s_{m+1,n}(x + r_{m+1,n}(x, y)) + \sigma_{m+1,n}t_{m+1,n}y, \sigma_{m+1,n}y), \tag{3.7.31}$$

we find that

$$r_{m,n}(x, y) = r_{m+1,n}(x, y) + \sigma_{m+1,n}^{-1}s_{m+1,n}^{-1}r_m(x', y'). \tag{3.7.32}$$

Differentiating this with respect to x and y gives

$$\partial_x r_{m,n}(x, y) = \partial_x r_{m+1,n}(x, y) + (1 + \partial_x r_m(x, y))\partial_x r_m(x', y') \tag{3.7.33}$$

$$\partial_y r_{m,n}(x, y) = \partial_y r_{m+1,n}(x, y) + s_{m+1,n}^{-1}\left(t_{m+1,n}\partial_x r_m(x', y') + \partial_y r_m(x', y')\right). \tag{3.7.34}$$

Now let $C_4 > 1$ be the maximum of the constant from Proposition 3.7.4 and the constant from the first item above which ensures

$$|s_{m+1,n}| > C_4^{-1}\sigma^{n-m-1}, \quad |t_{m+1,n}| < C_4\bar{\varepsilon}^{p^{m+1}}, \tag{3.7.35}$$

and

$$|\partial_x r_m(z)| < C_4|z|, \quad |\partial_y r_m(z)| < C_4\bar{\varepsilon}^{p^m}|z|, \quad |\partial_{xy} r_m(z)| < C_4\bar{\varepsilon}^{p^m}|z|. \tag{3.7.36}$$

As a consequence of our induction hypothesis, there exists a constant $C_5 > 0$ such that $|z'| < C_5\sigma^{n-m-1}|z|$. Together these imply the existence of a constant $C_6 > 0$ such that

$$\begin{aligned} |\partial_x r_{m,n}(z)| &\leq |\partial_x r_{m+1,n}(z)| + |\partial_x r_m(z')|(1 + |\partial_x r_m(z)|) \\ &\leq |\partial_x r_{m+1,n}(z)| + C_4|z'|(1 + C_4|z|) \\ &\leq |\partial_x r_{m+1,n}(z)| + C_4C_5\sigma^{n-m-1}|z|(1 + C_4|z|) \\ &\leq |\partial_x r_{m+1,n}(z)| + C_6\sigma^{n-m-1}|z| \end{aligned} \tag{3.7.37}$$

and a constant $C_7 > 0$ such that

$$|\partial_y r_{m,n}(z)| \leq |\partial_y r_{m+1,n}(z)| + |s_{m+1,n}|^{-1} \left(|\partial_x r_m(z')||t_{m+1,n}| + |\partial_y r_m(z')|\right). \tag{3.7.38}$$

$$\leq |\partial_y r_{m+1,n}(z)| + C_4\sigma^{-(n-m-1)}(C_4^2\bar{\varepsilon}^{p^{m+1}}|z'| + C_4\bar{\varepsilon}^{p^m}|z'|)$$
$$\leq |\partial_y r_{m+1,n}(z)| + C_4^2 C_5(C_4\bar{\varepsilon}^{p^{m+1}} + \bar{\varepsilon}^{p^m})|z|$$
$$\leq |\partial_y r_{m+1,n}(z)| + C_7\bar{\varepsilon}^{p^m}|z|$$

Next we consider the second order derivatives. As all functions are analytic the estimates for $\partial_{xx} r_{m,n}$ and $\partial_{yy} r_{m,n}$ follow from those of $\partial_x r_{m,n}$ and $\partial_y r_{m,n}$ respectively. Therefore we only need consider the mixed second order partial derivative. This is given by

$$\partial_{xy} r_{m,n}(z) = \partial_{xy} r_{m+1,n}(z) + \sigma_{m+1,n}\partial_{xy} r_m(z)\left(\partial_{xx} r_m(z')t_{m+1,n} + \partial_{xy} r_m(z')\right) \tag{3.7.39}$$

and hence, using the above estimates, there exists a constant $C_8 > 0$ such that

$$|\partial_{xy} r_{m,n}(z)| \tag{3.7.40}$$
$$\leq |\partial_{xy} r_{m+1,n}(z)| + |\sigma_{m+1,n}||\partial_{xy} r_m(z)|\left(|\partial_{xx} r_m(z')||t_{m+1,n}| + |\partial_{xy} r_m(z')|\right)$$
$$\leq |\partial_{xy} r_{m+1,n}(z)| + C_4^2\sigma^{n-m-1}\bar{\varepsilon}^{p^m}|z|\left(C_4^2\bar{\varepsilon}^{p^{m+1}}|z'| + C_4\bar{\varepsilon}^{p^m}|z'|\right)$$
$$\leq |\partial_{xy} r_{m+1,n}(z)| + C_8\sigma^{2(n-m)}\bar{\varepsilon}^{2p^m}|z|.$$

Therefore invoking the induction hypothesis and setting $C = \max_i C_i$ we achieve the desired result. □

Proposition 3.7.7. *There exists a constant $0 < \rho < 1$ such that that following holds: for $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $r_{m,n}(x,y)$ denote the functions constructed above for integers $0 < m < n$. Then there exists a constant $C > 0$ such that for any $(x,y) \in B$,*

$$|[x + r_{m,n}(x,y)] - v_*(x)| < C(\bar{\varepsilon}^{p^m}y + \rho^{n-m}) \tag{3.7.41}$$

and

$$|[1 + \partial_x r_{m,n}(x,y)] - \partial_x v_*(x)| < C\rho^{n-m} \tag{3.7.42}$$

where $v_*(x)$ is the affine rescaling of the universal function u_* so that its fixed point lies at the origin with multiplier 1.

Proof. Given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $F_n\colon B \to B$ denote the n-th renormalisation and let $\Psi_n\colon B \to B$ denote the n-th scope function. Let $\hat{F}_n\colon \hat{B}_{n+1} \to \hat{B}_n$ and $\hat{\Psi}_n\colon \hat{B}_{n+1} \to \hat{B}_n$ denote these maps under the translational change of coordinates described above.

First, let us consider the functions $\hat{\Psi}_m\colon \hat{B}_{m+1} \to \hat{B}_m$. By construction these preserve the x-axis, since they preserve the family of horizontal lines and the origin is a fixed point for each of them. This implies there exists a functions $\hat{\psi}_m\colon \hat{J}_{m+1} \to \hat{J}_m$ such that $\hat{\Psi}_m(x,0) = (\hat{\psi}_m(x),0)$. Lemma 3.3.1 implies there is a constant $C_0 > 0$ such that for each $n \geq 0$ there exists $f_n \mathcal{U}_{\Omega_x, \upsilon}$ satisfying $|F_n - (f_n \circ \pi_x, \pi_x)|_\Omega < C_0 \bar{\varepsilon}^{p^n}$. Let $\hat{f}_n \colon \hat{J}_n \to \hat{J}_n$ denote f_n under the translational change of coordinates and let $\hat{\psi}^1_n \colon \hat{J}_n \to \hat{J}^1_n$ be the branch of its presentation function corresponding to the interval $\hat{J}^1_n$. Proposition 2.4.6 implies there is a constant $C_1 > 0$ such that $|\hat{\psi}^1_n - \hat{\psi}_n|_{C^2} < C_1 \bar{\varepsilon}^{p^n}$ and Proposition 2.4.6 and Theorem 3.3.3 implies there is a constant $C_2 > 0$ such that $|\hat{\psi}^1_n - \hat{\psi}^1_*|_{C^2} < C_2 \rho^n$. Combining these we find there is a constant $C_3 > 0$ such that

$$|\hat{\psi}_n - \hat{\psi}^1_*|_{C^2} < C_3 \rho^n. \tag{3.7.43}$$

Now observe there exist functions $\hat{\psi}_{m,n} \colon \hat{J}_{n+1} \to \hat{J}^0_{m,n} \subset \hat{J}_m$, where $\hat{J}^0_{m,n} = \hat{\psi}_{m,n}(\hat{J}_{n+1})$, such that $\hat{\Psi}_{m,n}(x,0) = (\hat{\psi}_{m,n}(x), 0)$. Moreover, since $\hat{\Psi}_{m,n} = \hat{\Psi}_m \circ \cdots \circ \hat{\Psi}_n$ we must have $\hat{\psi}_{m,n} = \hat{\psi}_m \circ \cdots \circ \hat{\psi}_n$. Also observe that, since $\hat{\Psi}_{m,n} = \mathbf{T}_m \circ \Psi_{m,n} \circ \mathbf{T}^{-1}_{n+1}$, there are translations $\mathsf{\tau}_m$ such that $\hat{\psi}_{m,n} = \mathsf{\tau}_m \circ \psi_{m,n} \circ \mathsf{\tau}^{-1}_{n+1}$.

Now let $[\psi_{m,n}]$ and $[\psi_{*,m,n}]$ denote, respectively, the orientation preserving affine rescalings of the maps $\hat{\psi}_m \circ \cdots \circ \hat{\psi}_n$ and $\hat{\psi}_* \circ \cdots \circ \hat{\psi}_*$ to the interval J. Here the composition of $\hat{\psi}_*$ with itself is taken $n-m$ times. Then Lemma C.2.1 implies there exists a constant $C_4 > 0$ such that $|\psi_{m,n} - \psi_{*,m,n}|_{C^1} < C_4 \rho^{n-m}$. This then implies, together with the second part of Lemma 2.4.3, that there is a constant $C_5 > 0$ such that

$$\begin{aligned} |[\psi_{m,n}] - u_*|_{C^1} &\leq |[\psi_{m,n}] - [\psi_{*,m,n}]|_{C^1} + |[\psi_{*,m,n}] - u_*|_{C^1} \\ &\leq C_5 \rho^{n-m}. \end{aligned} \tag{3.7.44}$$

where u_* is the universal function from that Lemma. Next we perform an translational change of coordinates on $[\psi_{m,n}]$ and u_* so that the fixed point lies at the origin. Proposition A.2.8 then implies these coordinate changes also converge exponentially. Therefore, if $[\hat{\psi}_{m,n}]$, and $\hat{u}_*$ denote these functions in the new coordinates, there exists a constant $C_6 > 0$ such that

$$|[\hat{\psi}_{m,n}] - \hat{u}_*|_{C^1} < C_6 \rho^{n-m}. \tag{3.7.45}$$

Now observe that Proposition A.2.8 also implies difference between the multiplier $\mu_{m,n}$ of the fixed point 0 for $[\hat{\psi}_{m,n}]$ the multiplier μ_* of the fixed point 0 for $\hat{u}_*$ decreases exponentially in $n-m$ at the same rate. This implies there exists a constant $C_7 > 0$ such that

$$|\mu^{-1}_{m,n}[\hat{\psi}_{m,n}] - \mu^{-1}_* \hat{u}_*|_{C^1} < C_7 \rho^{n-m}. \tag{3.7.46}$$

Now we claim that $\mu^{-1}_{m,n}[\hat{\psi}_{m,n}] = x + r_{m,n}(x,0)$. Both come from affinely rescaling $\Psi_{m,n}$ so that the origin is fixed, the horizontal line $\{y=0\}$ is fixed and their derivatives in the x-direction are 1. Hence they are equal. Also, by definition, $\mu^{-1}_* \hat{u}_* = v_*$. This then implies, by the above and Proposition 3.7.6, that there is a constant $C > 0$ such that

$$\begin{aligned} &|[x + r_{m,n}(x,y)] - v_*(x)| &(3.7.47)\\ &\le |[x + r_{m,n}(x,y)] - [x + r_{m,n}(x,0)]| + |[x + r_{m,n}(x,0)] - v_*(x)| \\ &\le |\partial_y r_{m,n}||y| + |\mu_{m,n}^{-1}[\hat{\psi}_{m,n}] - \mu_*^{-1}\hat{u}_*|_{C^0} \\ &\le C(\bar{\varepsilon}^{p^{m-1}}|y| + \rho^{n-m}) \end{aligned}$$

which gives the first bound while

$$\begin{aligned} &|[1 + \partial_x r_{m,n}(x,y)] - \partial_x v_*(x)| &(3.7.48)\\ &\le |[1 + \partial_x r_{m,n}(x,y)] - [1 + \partial_x r_{m,n}(x,0)]| + |[1 + \partial_x r_{m,n}(x,0)] - \partial_x v_*(x)| \\ &\le |\partial_{xy} r_{m,n}||y| + |\mu_{m,n}^{-1}[\hat{\psi}_{m,n}] - \mu_*^{-1}\hat{u}_*|_{C^1} \\ &\le C(\sigma^{n-m}\bar{\varepsilon}^{p^m}|y| + \rho^{n-m}) \end{aligned}$$

which, since z lies in a bounded domain and $\bar{\varepsilon}^{p^m}$ is bounded from above, gives us the bound for the derivate. □

Proposition 3.7.8. *There exist constants $C > 0, 0 < \rho < 1$ such that the following holds: given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, for each integer $m > 0$ there exists a constant $\kappa_{(m)} = \kappa_{(m)}(F) \in \mathbb{R}$, satisfying $|\kappa_{(m)}| < C\bar{\varepsilon}^{p^m}$, such that*

$$|[x + r_{m,n}(x,y)] - [v_*(x) + \kappa_{(m)}y^2]| < C\rho^{n-m} \qquad (3.7.49)$$

Proof. Observe that, since $v_*(0) = 0$, Proposition 3.7.7 tells us there exists a constant $C_0 > 0$ and a point $\xi_{0,x} \in [0,x]$ such that

$$\begin{aligned} &|[x + r_{m,n}(x,y)] - [v_*(x) + r_{m,n}(0,y)]| &(3.7.50)\\ &= |[x + r_{m,n}(x,y) - v_*(x)] - [0 + r_{m,n}(0,y) - v_*(0)]| \\ &\le |1 + \partial_x r_{m,n}(\xi_{0,x},y) - \partial_x v_*(\xi_{0,x})||x| \\ &\le C_0\rho^{n-m}|x| \end{aligned}$$

We now claim there exists a constant $\kappa_{(m)}$ such that $|\kappa_{(m)}| < C\bar{\varepsilon}^{p^m}$ and

$$|r_{m,n}(0,y) - \kappa_{(m)}y^2| < C_1\rho^n. \qquad (3.7.51)$$

To show this we use induction. Recall that $\Psi_{m,n}(z) = \Psi_{m,n-1} \circ \Psi_n(z)$ for $z \in B$. This implies

$$R_{m,n}(z) = R_n(z) + D_n^{-1}(R_{m,n-1}(D_n(\mathrm{id} + R_n(z)))). \qquad (3.7.52)$$

Since $R_{m,n}, R_n$ and D_n have the forms given by Lemmas 3.7.1 and 3.7.5, we find that, setting $z' = \Psi_n(z)$,

$$(x',y') = (\sigma_n s_n(x + r_n(x,y)) + \sigma_n t_n y, \sigma_n y), \qquad (3.7.53)$$

where we write (x', y') for z'. This then gives us

$$r_{m,n}(x,y) = r_n(x,y) + \sigma_n^{-1} s_n^{-1} r_{m,n-1}(x', y'). \tag{3.7.54}$$

Let $\omega_n(y) = \sigma_n(s_n r_n(0,y) + t_n y)$. Then in particular, this together with the Mean Value Theorem implies there exists a $\xi \in [0, \omega_n(y)]$ such that

$$\begin{aligned} r_{m,n}(0,y) &= r_n(0,y) + \sigma_n^{-1} s_n^{-1} r_{m,n-1}(\omega_n(y), \sigma_n y) \qquad (3.7.55)\\ &= r_n(0,y) + \sigma_n^{-1} s_n^{-1} \left(r_{m,n-1}(0, \sigma_n y) + \partial_x r_{m,n-1}(\xi, \sigma_n y)\omega_n(y)\right). \end{aligned}$$

Next observe that, by construction, $r_n(x,y)$ consists of degree two terms or higher. Therefore, by the above equation, so too must $r_{m,n}(x,y)$. Thus, we may write $r_n(0,y)$ and $r_{m,n}(0,y)$ in the forms

$$r_n(0,y) = \kappa_n y^2 + K_n(y); \quad r_{m,n}(0,y) = \kappa_{m,n} y^2 + K_{m,n}(y), \tag{3.7.56}$$

where $\kappa_n, \kappa_{m,n}$ are real constants and $K_n(y), K_{m,n}(y)$ are functions of the third order in y. This implies together with equation (3.7.55), that

$$\begin{aligned} \kappa_{m,n} y^2 + K_{m,n}(y) &= \kappa_n y^2 + K_n(y) \qquad (3.7.57)\\ &\quad + \sigma_n^{-1} s_n^{-1} \left(\kappa_{m,n-1} y^2 + K_{m,n-1}(y) + \partial_x r_{m,n-1}(\xi, \sigma_n y)\omega_n(y)\right) \end{aligned}$$

By Proposition 3.7.4 there exists a constant $C_1 > 0$ such that $|\partial_y r_n(z)| < C_1 \bar{\varepsilon}^{p^n} |z|$ for all suitable z. Therefore κ_n is satisfies $|\kappa_n| < C_1 \bar{\varepsilon}^{p^n}$ and K_n satisfies $|K_n(y)| < C_1 \bar{\varepsilon}^{p^n} |y|^3$. Proposition 3.7.4 also implies there exists a constant $C_2 > 0$ such that $|\omega(y)| < C_2 \bar{\varepsilon}^{p^n} |y|$. Proposition 3.7.6 implies there exists a constant $C_3 > 0$ such that $|\partial_x r_{m,n-1}(x,y)| < C_3$. These imply, there is a constant $C_4 > 0$ such that

$$\begin{aligned} |\kappa_{m,n}| &\leq |\kappa_n| + |\sigma_n s_n^{-1}| |\kappa_{m,n-1}| + C_4 \bar{\varepsilon}^{p^n} \qquad (3.7.58)\\ &\leq 2C_4 \bar{\varepsilon}^{p^n} + (1 + C_4 \rho^n) |\kappa_{m,n-1}| \\ |K_{m,n}(y)| &\leq |K_n(y)| + |\sigma_n^2 s_n^{-1}| |K_{m,n-1}(y)| + C_4 \bar{\varepsilon}^{p^n} \qquad (3.7.59)\\ &\leq \sigma(1 + C_4 \rho^n) |K_{m,n-1}(y)| + 2C_4 \bar{\varepsilon}^{p^n} \end{aligned}$$

which implies $\kappa_{m,n}$ converges as n tends to infinity and $K_{m,n}(y)$ decreases exponentially if n is sufficiently large. Moreover, by Proposition A.1.3, $\kappa_{(m)} = \lim_{n\to\infty} \kappa_{m,n}$ satisfies $|\kappa_{(m)}| \leq C_5 \bar{\varepsilon}^{p^n}$ for some constant $C_5 > 0$. Hence the Proposition is shown. □

Chapter 4

Applications

Here we apply the results of the previous chapter to examine the local dynamics of infinitely renormalisable Hénon-like maps around their tips. We extend the results in [12] to the case of arbitrary combinatorics. First we will show that universality holds at the tip. By this we mean the rate of convergence to the renormalisation fixed point is controlled by a universal quantity. In the unimodal case this is a positive real number, but here the quantity is a real-valued real analytic function. This universality is then used to show our two other results, namely the non-existence of continuous invariant linefields on the renormalisation Cantor set and the non-rigidity of these Cantor sets.

4.1 Universality at the Tip

Theorem 4.1.1. *There exists a constant $\bar{\varepsilon}_0 > 0$, a universal constant $0 < \rho < 1$ and a universal function $a \in C^\omega(J, \mathbb{R})$ such that the following holds: Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ and let the sequence of renormalisations be denoted by F_n. Then*

$$F_n(x,y) = \Big(f_n(x) + b^{p^n} a(x) y \left(1 + \mathrm{O}\left(\rho^n\right)\right), y\Big) \tag{4.1.1}$$

where $b = b(F)$ denotes the average Jacobian of F and f_n are unimodal maps converging exponentially to f_*.

Proof. Let $F_n = (\phi_n, \pi_x)$ denote the n-th renormalisation of F. Let τ_n denote the tip of height n and let $\varsigma \in \mathrm{Dom}(F_n)$ be any other point. Applying the chain rule to $F_n = \Psi_{0,n-1}^{-1} \circ F^{\circ p^n} \circ \Psi_{0,n-1}$ at the point ς gives

$$\partial_y \phi_n(\varsigma) = \mathrm{Jac}_\varsigma F_n = \mathrm{Jac}_{\Psi_{0,n-1}(\varsigma)} F^{\circ p^n} \frac{\mathrm{Jac}_\varsigma \Psi_{0,n-1}}{\mathrm{Jac}_{F_n(\varsigma)} \Psi_{0,n-1}}. \tag{4.1.2}$$

By the Distortion Lemma 3.5.6, since $\Psi_{0,n-1}(\varsigma) \in B^{0^n}$, there exists a constant $C_0 > 0$ such that

$$\left|\mathrm{Jac}_{\Psi_{0,n-1}(\varsigma)} F^{\circ p^n}\right| \leq b^{p^n}\left(1 + C_0 \rho^n\right). \tag{4.1.3}$$

It is clear from the decomposition in Lemma 3.7.5 that

$$\mathrm{Jac}_{\varsigma}\Psi_{0,n-1} = \mathrm{Jac}_{\tau_n}\Psi_{0,n-1}\mathrm{Jac}_{\varsigma-\tau_n}\left(\mathrm{id}+R_{0,n-1}\right) \tag{4.1.4}$$

and

$$\mathrm{Jac}_{F_n(\varsigma)}\Psi_{0,n-1} = \mathrm{Jac}_{\tau_n}\Psi_{0,n-1}\mathrm{Jac}_{F_n(\varsigma)-\tau_n}\left(\mathrm{id}+R_{0,n-1}\right). \tag{4.1.5}$$

Let $\delta_n^0 = \varsigma - \tau_n$ and $\delta_n^1 = F_n(\varsigma) - \tau_n$. Observe that, by Theorem 3.3.3 and Corollary 3.5.5, there exists a constant $C_1 > 0$ such that $|\tau_n - \tau_*|, |F_n - F_*|_\Omega < C_1\rho^n$. Therefore there exists a constant $C_2 > 0$ such that, if $\varsigma_* = \tau_* + (\varsigma - \tau_n)$, $\delta_*^0 = \varsigma_* - \tau_*$ and $\delta_*^1 = F_*(\varsigma_*) - \tau_*$,

$$\left|\delta_n^0 - \delta_*^0\right| = |[\varsigma - \tau_n] - [\varsigma_* - \tau_*]| = 0 \tag{4.1.6}$$

and

$$\left|\delta_n^1 - \delta_*^1\right| = |[F_n(\varsigma) - \tau_n] - [F_*(\varsigma_*) - \tau_*]| < C_2\rho^n. \tag{4.1.7}$$

By Proposition 3.7.7 there is a constant $C_3 > 0$ such that

$$|1 + \partial_x r_{0,n-1} - v_*'|_{C^0} < C_3\rho^n. \tag{4.1.8}$$

Combining these and observing that v_* has bounded derivatives and δ_n^0 and δ_n^1 both lie in a bounded domain gives us a constant $C_4 > 0$ satisfying

$$\begin{aligned}
&\left|\mathrm{Jac}_{\delta_n^0}\left(\mathrm{id}+R_{0,n-1}\right) - v_*'\left(\pi_x\left(\delta_*^0\right)\right)\right| \\
&\le \left|\mathrm{Jac}_{\delta_n^0}\left(\mathrm{id}+R_{0,n-1}\right) - v_*'\left(\pi_x\left(\delta_n^0\right)\right)\right| + \left|v_*'\left(\delta_n^0\right) - v_*'\left(\delta_*^0\right)\right| \\
&\le \left|1 + \partial_x r_{0,n-1}\left(\delta_n^0\right) - v_*\left(\pi_x\left(\delta_n^0\right)\right)\right| |\tau_n - \tau_*| + |v_*''|_{C^0}|\tau_n - \tau_*| \\
&\le C_2C_3\rho^{2n} + C_2|v_*|_{C^2}\rho^n \\
&\le C_4\rho^n
\end{aligned} \tag{4.1.9}$$

and

$$\begin{aligned}
&\left|\mathrm{Jac}_{\delta_n^1}\left(\mathrm{id}+R_{0,n-1}\right) - v_*'\left(\pi_x\left(\delta_*^1\right)\right)\right| \\
&\le \left|\mathrm{Jac}_{\delta_n^1}\left(\mathrm{id}+R_{0,n-1}\right) - v_*'\left(\pi_x\left(\delta_n^1\right)\right)\right| + \left|v_*'\left(\pi_x\left(\delta_n^1\right)\right) - v_*'\left(\pi_x\left(\delta_*^1\right)\right)\right| \\
&\le \left|1 + \partial_x r_{0,n-1}\left(\delta_n^1\right) - v_*'\left(\pi_x\left(\delta_n^1\right)\right)\right| \left|\delta_n^1\right| + |v_*''|_{C^0}\left|\delta_n^1 - \delta_*^1\right| \\
&\le C_4\rho^n.
\end{aligned} \tag{4.1.10}$$

Observe that there exists a constant $C_5 > 0$ such that $|v_*'(x)| \ge C_5 > 0$, as v_* is a rescaling of a diffeomorphism onto its image. Observe also that there exists an $N > 0$ such that $|1 + \partial_x r_{0,n}|_{C^0} \ge \frac{1}{2}\inf|v_*'(x)| \ge C_5$ for all $n > N$. Therefore there exists a constant $C_6 > 1$ such that for all $n > N$,

$$\max\left(1, \left|\frac{v_*'\left(\pi_x\left(\delta_*^0\right)\right)}{v_*'\left(\pi_x\left(\delta_*^1\right)\right)}\right|\right) < C_6; \qquad C_6^{-1} < \left|\mathrm{Jac}_{\delta_n^1}\Psi_{0,n}\right|. \tag{4.1.11}$$

Therefore, applying Lemma A.1.5 we find

$$\left|\frac{\mathrm{Jac}_{\delta_n^0}\Psi_{0,n-1}}{\mathrm{Jac}_{\delta_n^1}\Psi_{0,n-1}}-\frac{v_*'\left(\pi_x\left(\delta_*^0\right)\right)}{v_*'\left(\pi_x\left(\delta_*^1\right)\right)}\right|=\left|\frac{\mathrm{Jac}_{\delta_n^0}\left(\mathrm{id}+R_{0,n-1}\right)}{\mathrm{Jac}_{\delta_n^1}\left(\mathrm{id}+R_{0,n-1}\right)}-\frac{v_*'\left(\pi_x\left(\delta_*^0\right)\right)}{v_*'\left(\pi_x\left(\delta_*^1\right)\right)}\right| \quad (4.1.12)$$

$$\leq C_6^2\max_{i=0,1}\left(\left|1+\partial_x r_{0,n-1}\left(\delta_n^i\right)-v_*'\left(\pi_x\left(\delta_*^i\right)\right)\right|\right)$$

$$\leq C_4C_6^2\rho^n.$$

Together with equation 4.1.2 and 4.1.3 this implies,

$$\partial_y\phi_n\left(\varsigma\right)=b^{p^n}a(\xi)\left(1+\mathrm{O}\left(\rho^n\right)\right) \quad (4.1.13)$$

where $\varsigma=(\xi,\eta)$ and

$$a(\xi)=\frac{v_*'\left(\xi-\pi_x\left(\tau_*\right)\right)}{v_*'\left(f_*\left(\xi\right)-\pi_x\left(\tau_*\right)\right)}. \quad (4.1.14)$$

This implies that, if $z=(x,y)\in B$, upon integrating with respect to the y-variable we find

$$\phi_n(x,y)=g_n(x)+yb^{p^n}a(x)\left(1+\mathrm{O}\left(\rho^n\right)\right), \quad (4.1.15)$$

for some function g_n independent of y. But now let (f_n,ε_n) be any parametrisation of F_n such that $|\varepsilon_n|\leq C_7\bar{\varepsilon}^{p^n}$ and $|f_n-f_*|<C_8\rho^n$. Here $C_7>0$ is the constant from Theorem 3.2.13 and $C_8>0$ is the constant from Theorem 3.3.2. Then there is a constant $C_9>0$ such that $|g_n-f_n|=|\varepsilon_n-b^{p^n}\pi_y\circ a|\leq C_9\rho^n$. Therefore, for $n>0$ sufficiently large g_n will also be unimodal and $|g_n-f_*|\leq|g_n-f_n|+|f_n-f_*|\leq(C_9+C_8)\rho^n$. Hence we may absorb their difference into into the $\mathrm{O}(\rho^n)$ term. □

The following is an immediate consequence of the proof of above Theorem.

Proposition 4.1.2. *Let $F\in\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $\Psi_{m,n}$ denote the scope function from height $n+1$ to height m. Let $t_{m,n}$ denote the tilt of $\Psi_{m,n}$ and let τ_{m+1} denote the tip at height $m+1$. Let $a=a(\tau_*)$ where $a(x)$ is the universal function from Theorem 4.1.1 above. Then exists constants $C>0$ and $0<\rho<1$ such that for all $0<m<n$ sufficiently large,*

$$ab^{p^m}(1-C\rho^m)<|t_m(\tau_{m+1})\ |<ab^{p^m}(1+C\rho^m) \quad (4.1.16)$$

$$ab^{p^m}(1-C\rho^m)<|t_{m,n}(\tau_{n+1})|<ab^{p^m}(1+C\rho^m). \quad (4.1.17)$$

Moreover $t_{m,*}=\lim_{n\to\infty}t_{m,n}(\tau_{n+1})$ exists and the convergence is exponential.

Proof. Let $\tau_m=(\xi_m,\eta_m)$. Recall that

$$t_m=\pm\frac{\partial_y\phi_m^{p-1}(\tau_m)}{\partial_x\phi_m^{p-1}(\tau_m)}, \quad (4.1.18)$$

but by the Variational Formula 3.2.6 we know

$$\phi_m^{p-1}\left(\xi_m,\eta_m\right)=f_m^{\circ p-1}\left(\xi_m\right)+L_m^{p-1}\left(\xi_m\right)+\varepsilon_m\left(\xi_m,\eta_m\right)\left(f_m^{\circ p-1}\right)'\left(\xi_m\right)+\mathrm{O}(\bar{\varepsilon}^{2p^m}) \quad (4.1.19)$$

which implies

$$\partial_x \phi_m^{p-1}(\xi_m, \eta_m) = \left(f_m^{\circ p-1}\right)'(\xi_m) + \left(L_m^{p-1}\right)'(\xi_m) + \partial_x \varepsilon_m(\xi_m, \eta_m)\left(f_m^{\circ p-1}\right)'(\xi_m)$$
$$+ \varepsilon_m(\xi_m, \eta_m)\left(f_m^{\circ p-1}\right)''(\xi_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}) \tag{4.1.20}$$

$$\partial_y \phi_m^{p-1}(\xi_m, \eta_m) = \partial_y \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})'(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}) \tag{4.1.21}$$

Therefore, by the fact that $(f_m^{\circ p-1})'(\xi_m)$ is uniformly bounded from zero if n is sufficiently large,

$$\frac{\partial_y \phi_m^{p-1}(\xi_m, \eta_m)}{\partial_x \phi_m^{p-1}(\xi_m, \eta_m)} = \frac{\partial_y \varepsilon_m(\xi_m, \eta_m)\left(f_m^{\circ p-1}\right)'(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m})}{\left(f_m^{\circ p-1}\right)'(\xi_m) + \mathrm{O}(\bar{\varepsilon}^{p^n})} \tag{4.1.22}$$
$$= \left(\partial_y \varepsilon_m(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m})\right)\left(1 + \mathrm{O}(\bar{\varepsilon}^{p^m})\right)$$
$$= \partial_y \varepsilon_m(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}).$$

Theorem 4.1.1 above and observing that the $\mathrm{O}(\bar{\varepsilon}^{2p^m})$ term can be absorbed into the $\mathrm{O}(\rho^m)$ then tells us

$$|t_m(\tau_{m+1})| = a(\xi_m)\, b^{p^m}(1 + \mathrm{O}(\rho^m)), \tag{4.1.23}$$

but by Proposition 3.7.3 we know that ξ_m converges to ξ_* exponentially and so analyticity of a implies $a(\xi_m) = a(\xi_*)(1+\mathrm{O}(\rho^m))$. Hence we get the first claim. Secondly, observe by Lemma 3.7.5,

$$t_{m,n-1}(\tau_n) = \sum_{i=m}^{n-1} s_{m,i-1}(\tau_i) t_i(\tau_{i+1}) \tag{4.1.24}$$
$$= t_m(\tau_{m+1}) \sum_{i=m}^{n-1} s_{m,i-1}(\tau_i)\left(\frac{\partial_x \phi_m^{p-1}(\tau_m)}{\partial_x \phi_i^{p-1}(\tau_i)}\right)\left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)}\right)$$
$$= t_m(\tau_{m+1}) \sum_{i=m}^{n-1} s_{m+1,i}(\tau_{i+1})\left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)}\right)$$

Therefore we can write $t_{m,n-1}(\tau_n) = t_m(\tau_{m+1})(1 + K_{m,n-1}(\tau_{n+1}))$ where

$$K_{m,n-1} \sum_{i=m+1}^{n-1} s_{m+1,i}(\tau_{i+1})\left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)}\right). \tag{4.1.25}$$

By Proposition 3.7.6 and the Variational Formula 3.2.6, there exists a constant $C_7 > 0$ such that $|K_{m,n-1}(\tau_{n+1})| \geq C\bar{\varepsilon}^{p^{m+1}-p^m}$. Absorbing this error into the $\mathrm{O}(\rho^m)$ term gives us the second claim. The third claim follows as the terms in $K_{m,n}$ decrease super-exponentially as n tends to infinity, but τ_m only converges exponentially to τ_*. □

Proposition 4.1.3. *Let $F \in \mathcal{I}_{\Omega,v}(\bar{\varepsilon})$ be as above, let τ_n denote the tip of F_n and let $\varsigma_n = F_n^{\circ p}(\tau_n)$. Then there exists a constant $C > 1$ for all $0 < m < n$*

$$C^{-1}|s_{m,n-1}(\tau_n)| \leq |s_{m,n-1}(\varsigma_n)| \leq C|s_{m,n-1}(\tau_n)| \tag{4.1.26}$$

$$C^{-1}|t_{m,n-1}(\tau_n)| \leq |t_{m,n-1}(\varsigma_n)| \leq C|t_{m,n-1}(\tau_n)| \tag{4.1.27}$$

$$|s_{m,n-1}(\varsigma_n) - s_{m,n-1}(\tau_n)| > C^{-1}|\varsigma_n - \tau_n| \tag{4.1.28}$$

$$|t_{m,n-1}(\varsigma_n) - t_{m,n-1}(\tau_n)| > C^{-1}|\varsigma_n - \tau_n| \tag{4.1.29}$$

Proof. These follow from the estimates on the second order terms (i.e. the functions $r_{m,n}$) given by Proposition 3.7.6 and the observation that $\tau_n, \varsigma_n \in B_n^0$ implies, for n sufficiently large, that the derivatives of $s_{m,n-1}, t_{m,n-1}$ in the rectangle spanned by τ_n, ς_n will be uniformly bounded. □

4.2 Invariant Line Fields

Let $F \in \mathcal{I}_{\Omega,v}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalisation Cantor set. We will now consider the space of F-invariant line fields on $\mathcal{O}$. As we are considering line fields, let us projectivise all the transformations under consideration. Let us take the projection onto the line $\{y = 1\}$, and let us denote the projected coordinate by X. Then the maps $D(\Psi_{m,n}; z)$ and $D(F_n^{\circ p}; z)$ induce the transformations

$$\widetilde{\mathrm{D}}_z\Psi_{m,n}(X) = s_{m,n}(z)X + t_{m,n}(z) \tag{4.2.1}$$

$$\widetilde{\mathrm{D}}_z F_n^{\circ p}(X) = \zeta_n(z)\frac{X + \eta_n(z)}{X + \theta_n(z)}. \tag{4.2.2}$$

where $s_{m,n}(z), t_{m,n}(z)$ are as in Section 3.4 and $\zeta_n(z), \eta_n(z), \theta_n(z)$ are given by

$$\zeta_n(z) = \frac{\partial_x \phi_n^p(z)}{\partial_x \phi_n^{p-1}(z)}, \quad \eta_n(z) = \frac{\partial_y \phi_n^p(z)}{\partial_x \phi_n^p(z)}, \quad \theta_n(z) = \frac{\partial_y \phi_n^{p-1}(z)}{\partial_x \phi_n^{p-1}(z)}. \tag{4.2.3}$$

Proposition 4.2.1. *Let $F \in \mathcal{I}_{\Omega,v}(\bar{\varepsilon})$ be as above. Then there exists a constant $C > 1$ such that for all $n > 0$*

$$C^{-1} < |\zeta_n(\tau_n)| < C \tag{4.2.4}$$

$$|\eta_n(\tau_n)| < C\bar{\varepsilon}^{p^{n+1}} \tag{4.2.5}$$

$$|\theta_n(\tau_n)| < C\bar{\varepsilon}^{p^n} \tag{4.2.6}$$

Proof. Let (f_n, ε_n) be a parametrisation for F_n. Let v_n denote the critical value of f_n. Observe, by convergence of renormalisation 3.3.3, that v_n and $\pi_x\tau_n$ are exponentially close and so there is a constant $C_0 > 0$ such that

$$\left|\left(f_n^{\circ p-1}\right)'(v_n)\right|, \left|\left(f_n^{\circ p-1}\right)'(\pi_x\tau_n)\right| > C_0, \tag{4.2.7}$$

if $n > 0$ is sufficiently large. Therefore by the variational formula, there is a constant $C_1 > 0$ such that

$$\left|\frac{\partial_x \phi_n^p(\tau_n)}{\partial_x \phi_n^{p-1}(\tau_n)} - \frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)}\right| \leq C_1 \bar{\varepsilon}^{p^n}. \tag{4.2.8}$$

Now observe, by Theorem 3.3.2, that there is a $C_2 > 0$ such that

$$\left|\frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)} - \frac{(f_*^{\circ p})'(v_*)}{(f_*^{\circ p-1})'(v_*)}\right| < C_2 \rho^n. \tag{4.2.9}$$

Therefore there exists a $C_3 > 0$ such that

$$\begin{aligned} &\left|\zeta_n(\tau_n) - f_*'(f_*^{\circ p}(v_*))\right| \qquad (4.2.10)\\ &\leq \left|\frac{\partial_x \phi_n^p(z)}{\partial_x \phi_n^{p-1}(z)} - \frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)}\right| + \left|\frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)} - \frac{(f_*^{\circ p})'(v_*)}{(f_*^{\circ p-1})'(v_*)}\right|\\ &\leq C_3 \rho^n. \end{aligned}$$

Since $f_*'\left(f_*^{\circ p}(v_*)\right) \neq 0$ (infinitely renormalisable maps are never postcritically finite), this implies for $n > 0$ sufficiently large the first item is true.

For the second item, taking the Jacobian of $F_n^{\circ p}$ at τ_n, applying Proposition 3.7.4 and making the same observation regarding $f_*'\left(f_*^{\circ p}(v_*)\right) \neq 0$ as above, gives us the result.

The third item follows directly from Proposition 3.7.4. □

Theorem 4.2.2. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalisation Cantor set. Then there do not exist any continuous invariant line fields on $\mathcal{O}$. More precisely, if X is an invariant line field then it must be discontinuous at the tip, τ, of F.*

Proof. Let X be a continuous invariant line field on $\mathcal{O}$. Let τ_n denote the tip of F_n and let $\varsigma_n = F_n^{\circ p}(\tau_n)$ denote its first return, under F_n, to B_n^0.

Before we begin let us define some constants that shall help our exposition. Let $C_0 > 0$ satisfy $|\theta_n(\tau_n)| < C_0 \bar{\varepsilon}^{p^n}$ and $|\eta_n(\tau_n)| < C_0 \bar{\varepsilon}^{p^{n+1}}$ for all $n > 0$. Such a constant exists by Proposition 4.2.1. Let $C_1 > 1$ satisfy $C_1^{-1} < |\zeta_n(\tau_n)| < C_1$ for all $n > 0$. Such a constant exists by Proposition 4.2.1. Let $C_2 > 0$ satisfy $|t_m(\tau_{m+1})|, |t_{m,n-1}(\tau_n)| < C_2 \bar{\varepsilon}^{p^m}$ for all $0 < m < n$. Such a constant exists by Propositions 3.7.4 and 3.7.6. Let $C_3 > 0$ satisfy $|s_{m,n-1}(\tau_n)| > C_3 \sigma^{n-m-1}$ for all $0 < m < n$. Finally let $C_4 > 1$ satisfy

$$C_4^{-1}|s_{m,n-1}(\tau_n)| \leq |s_{m,n-1}(\varsigma_n)| \leq C_4 |s_{m,n-1}(\tau_n)| \tag{4.2.11}$$

$$C_4^{-1}|t_{m,n-1}(\tau_n)| \leq |t_{m,n-1}(\varsigma_n)| \leq C_4 |t_{m,n-1}(\tau_n)| \tag{4.2.12}$$

$$|s_{m,n-1}(\varsigma_m) - s_{m,n-1}(\tau_m)| > C_4^{-1}|\varsigma_m - \tau_m| \tag{4.2.13}$$

$$|t_{m,n-1}(\varsigma_m) - t_{m,n-1}(\tau_m)| > C_4^{-1}|\varsigma_m - \tau_m| \tag{4.2.14}$$

for all $0 < m < n$. Such a constant exists by Proposition 4.1.3 above.

Observe that X induces continuous invariant line fields X_n for F_n on $\mathcal{O}_n$, the induced Cantor sets. Thus

$$X_m(\tau_m) = \widetilde{\mathrm{D}}_\tau \Psi_{0,m}^{-1} X(\tau) = \left(X(\tau) - t_{0,m}(\tau_m)\right)/s_{0,m}(\tau_m). \tag{4.2.15}$$

There are two possibilities: either $X(\tau) = t_{0,*}(\tau_*) = \lim t_{0,m-1}(\tau_m)$, and so $X_m(\tau_m)$ converges to zero (since $t_{0,m}$ converges super-exponentially to $t_{0,*}$ but $s_{0,m}$ converges only exponentially to 0), or $X(\tau) \neq t_{0,*}(\tau_*)$, and so $X_m(\tau_m)$ tends to infinity.

First, let us show the second case cannot occur. Let $K, \kappa > 0$ be constants. Choose $M > 0$ such that $|X_m(\tau_m)| > K$ for all $m > M$. Fix such an $m > M$. By continuity of X_m there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|X_m(x) - X_m(y)| < \kappa$ for any $x, y \in \mathcal{O}_m$. Choose $N > m$ such that, for all $n > N$, $|\Psi_{m,n-1}(\tau_n) - \Psi_{m,n-1}(\varsigma_n)| < \delta$. This then implies $|X_m(\Psi_{m,n-1}(\tau_n)) - X_m(\Psi_{m,n-1}(\varsigma_n))| < \kappa$.

By invariance of the X_n,

$$|X_n(\varsigma_n)| = \left|\widetilde{\mathrm{D}}_{\tau_n} F_n^{\circ p}\left(X_n(\tau_n)\right)\right| = |\zeta_n(\tau_n)| \left|\frac{X_n(\tau_n) + \eta_n(\tau_n)}{X_n(\tau_n) + \theta_n(\tau_n)}\right|. \tag{4.2.16}$$

By our above hypotheses we know $|\theta_n(\tau_n)|, |\eta_n(\tau_n)| < C_0 \bar{\varepsilon}^{p^n}$. Since $n > m$, we also know $|X_n(\tau_n)| > K$. Therefore

$$\begin{aligned}\left|\frac{X_n(\tau_n) + \eta_n(\tau_n)}{X_n(\tau_n) + \theta_n(\tau_n)}\right| &\leq \frac{1 + |\eta_n(\tau_n)/X_n(\tau_n)|}{1 - |\theta_n(\tau_n)/X_n(\tau_n)|} \\ &\leq \frac{1 + C_0\bar{\varepsilon}^{p^n}/K}{1 - C_0\bar{\varepsilon}^{p^n}/K}\end{aligned} \tag{4.2.17}$$

Therefore, combining this with the above equation 4.2.16 and the hypotheses of the second paragraph we find

$$|X_n(\varsigma_n)| \leq C_1 \left(\frac{1 + C_0\bar{\varepsilon}^{p^n}/K}{1 - C_0\bar{\varepsilon}^{p^n}/K}\right) \tag{4.2.18}$$

Now we apply $\widetilde{\mathrm{D}}_{\varsigma_n}\Psi_{m,n-1}$. Then by the definition of the constant $C_4 > 0$ in the second paragraph and Proposition 3.7.6

$$\begin{aligned}|X_m(\Psi_{m,n-1}(\varsigma_n))| &= |s_{m,n-1}(\varsigma_n) X_n(\varsigma_n) + t_{m,n-1}(\varsigma_n)| \\ &\leq |s_{m,n-1}(\varsigma_n)|\,|X_n(\varsigma_n)| + |t_{m,n-1}(\varsigma_n)| \\ &\leq C_4\left(|s_{m,n-1}(\tau_n)|\,|X_n(\varsigma_n)| + |t_{m,n-1}(\tau_n)|\right) \\ &\leq C_4\sigma^{n-m-1}(1 + |X_n(\varsigma_n)|)\end{aligned} \tag{4.2.19}$$

and hence

$$\begin{aligned}&|X_m(\Psi_{m,n-1}(\tau_n)) - X_m(\Psi_{m,n-1}(\varsigma_n))| \\ &\geq \big||X_m(\Psi_{m,n-1}(\tau_n))| - |X_m(\Psi_{m,n-1}(\varsigma_n))|\big| \\ &\geq K - C_4\sigma^{n-m-1}\left[1 + C_1\left(\frac{1 + C_0\bar{\varepsilon}^{p^n}/K}{1 - C_0\bar{\varepsilon}^{p^n}/K}\right)\right].\end{aligned} \tag{4.2.20}$$

But, by our continuity assumption, this must be less than κ. For $K > 0$ sufficiently large this cannot happen.

So now let us assume $X(\tau) = t_{0,*}$. Then the induced line fields must satisfy $X_m(\tau_m) = t_{m,*}$, for all $m > 0$. The idea is, as before, to look at the first returns under F_m of B^0_m. We will apply $\widetilde{\mathrm{D}}_{\tau_m} F_m^{\circ p}$ to the line $X_m(\tau_m) = t_{m,n}$ and take the limit as n tends to infinity.

Proposition 4.1.2 implies, as $t_m(\tau_{m+1}) = \pm\partial_y\phi_m^{p-1}(\tau_m)/\partial_x\phi_m^{p-1}(\tau_m) = \pm\eta_m$, that there exists a constant $C_5 > 0$ for which

$$|t_{m,n-1}(\tau_n) + \theta_m(\tau_m)| \leq |t_m(\tau_{m+1})|\,|K_{m,n-1}(\tau_{n+1})| \leq C_5\bar{\varepsilon}^{p^{m+1}}. \tag{4.2.21}$$

On the other hand, we know $|\eta_m(\tau_m)| < C_0\bar{\varepsilon}^{p^{m+1}}$ and $|t_{m,n-1}(\tau_n)| < C_2\bar{\varepsilon}^{p^m}$ and hence

$$|t_{m,n-1}(\tau_n) + \eta_m(\tau_m)| \geq \big||t_{m,n-1}(\tau_n)| - |\eta_m(\tau_m)|\big| \geq C_2\bar{\varepsilon}^{p^m} - C_0\bar{\varepsilon}^{p^{m+1}} \tag{4.2.22}$$

We also know $|\zeta_m(\tau_m)| > C_1^{-1}$. Therefore there exists a constant $C_6 > 0$ such that

$$\begin{aligned}\left|\widetilde{\mathrm{D}}_{\tau_m} F_m^{\circ p}(t_{m,n-1}(\tau_n))\right| &= |\zeta_m(\tau_m)|\left|\frac{t_{m,n-1}(\tau_n)+\eta_m(\tau_m)}{t_{m,n-1}(\tau_n)+\theta_m(\tau_m)}\right| \\ &\geq C_1^{-1}C_5^{-1}\bar{\varepsilon}^{-p^{m+1}}(C_2\bar{\varepsilon}^{p^m} - C_0\bar{\varepsilon}^{p^{m+1}}) \\ &\geq C_6\bar{\varepsilon}^{-p^{m+1}}.\end{aligned} \tag{4.2.23}$$

Now recall $|t_{m,n-1}(\tau_n)| < C_2\bar{\varepsilon}^{p^m}$. Also observe that both of these estimates are independent of n. Therefore they still hold when passing to the limit, as n tends to infinity, giving

$$|X_m(\varsigma_m)| > C_6\bar{\varepsilon}^{-p^{m+1}}, \quad |X_m(\tau_m)| < C_2\bar{\varepsilon}^{p^m}. \tag{4.2.24}$$

Finally, applying $\Psi_{0,m-1}$ and setting $\varsigma = \Psi_{0,m-1}(\varsigma_m)$ we find that

$$\begin{aligned}&|X(\varsigma) - X(\tau)| \\ &= \big|[s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) + t_{0,m-1}(\varsigma_m)] - [s_{0,m-1}(\tau_m)X_m(\tau_m) + t_{0,m-1}(\tau_m)]\big| \\ &\geq \big||s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) - s_{0,m-1}(\tau_m)X_m(\tau_m)| - |t_{0,m-1}(\varsigma_m) - t_{0,m-1}(\tau_m)|\big|\end{aligned} \tag{4.2.25}$$

but by our assumptions in the second paragraph

$$\begin{aligned}&|s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) - s_{0,m-1}(\tau_m)X_m(\tau_m)| \\ &\geq \big||s_{0,m-1}(\varsigma_m)|\,|X_m(\varsigma_m) - X_m(\tau_m)| - |s_{0,m-1}(\varsigma_m) - s_{0,m-1}(\tau_m)|\,|X_m(\tau_m)|\big| \\ &\geq C_4^{-1}\,|s_{0,m-1}(\tau_m)|\,|X_m(\varsigma_m) - X_m(\tau_m)| - C_4\,|\varsigma_m - \tau_m|\,|X_m(\tau_m)|\end{aligned} \tag{4.2.26}$$

and

$$|t_{0,m-1}(\varsigma_m) - t_{0,m-1}(\tau_m)| \leq C_4|\varsigma_m - \tau_m|\,. \tag{4.2.27}$$

Therefore again by our assumptions in the second paragraph, $|s_{0,m-1}(\tau_m)| > C_3\sigma^m$. Hence, by our bounds on $|X_m(\varsigma_m)|$ and $|X_m(\tau_m)|$ and the above we find

$$\begin{aligned} &|X(\varsigma) - X(\tau)| \qquad (4.2.28)\\ &\geq C_4^{-1}C_3\sigma^m |X_m(\varsigma_m) - X_m(\tau_m)| - C_4 |\varsigma_m - \tau_m| |X_m(\tau_m)| - C_4 |\varsigma_m - \tau_m| \\ &\geq C_4^{-1}C_3\sigma^m \left(C_2\bar{\varepsilon}^{-p^{m+1}} - C_6\bar{\varepsilon}^{p^m}\right) - C_4 |\varsigma_m - \tau_m| \left(1 + C_6\bar{\varepsilon}^{p^m}\right)\end{aligned}$$

However, since $|\varsigma_m - \tau_m|$ is bounded from above there is a constant $C_7 > 0$ such that

$$|X(\varsigma) - X(\tau)| \geq C_7\sigma^m\bar{\varepsilon}^{-p^{m+1}}. \qquad (4.2.29)$$

Therefore, as we increase $m > 0$ the points τ and ς get exponentially closer but the distance between $X(\tau)$ and $X(\varsigma)$ diverges superexponentially. In particular X cannot be continuous at τ as required. □

We now need to define the following type of convergence, which is stronger than Hausdorff convergence.

Definition 4.2.3. Let $\mathcal{O}_* \subset M$ be a Cantor set, embedded in the metric space M, with presentation $\underline{B}_* = \{B_*^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$. Let $\mathcal{O}_*^{\mathbf{w}}$ denote the cylinder set for $\mathcal{O}_*$ associated to the word $\mathbf{w} \in \overline{W}$. Let $\mathcal{O}_n \subset M$ denote a sequence of Cantor sets, also embedded in M, with presentations $\underline{B}_n = \{B_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ combinatorially equivalent to $\underline{B}_*$. Then we say $\mathcal{O}_n$ *strongly converges* to $\mathcal{O}_*$ if, for each $\mathbf{w} \in \overline{W}$, $\mathcal{O}_n^{\mathbf{w}} \to \mathcal{O}_*^{\mathbf{w}}$.

Definition 4.2.4. Let X_n be a line field on $\mathcal{O}_n$. Then we say X_n *strongly converges* to a line field X_* on $\mathcal{O}_*$ if, for each $\mathbf{w} \in \overline{W}$, $X_n(\mathcal{O}_n^{\mathbf{w}})$ converges to $X_*(\mathcal{O}_*^{\mathbf{w}})$ in the projected coordinates.

Proposition 4.2.5. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalisation Cantor set. Given any invariant line field X on $\mathcal{O}$ the induced line fields X_n on $\mathcal{O}_n$ do not strongly converge to the tangent line field X_* on $\mathcal{O}_*$.*

Proof. Let us denote the correspondence between elements of $\mathcal{O}_n$ and $\mathcal{O}_*$ by π_n. Then a sequence of line fields X_n strongly converges to X_* if $X_n\circ\pi_n$ converges to X_*, where we have identified the line fields with their projectivised coordinates.

Assume convergence holds and let $\epsilon > 0$ and choose $N > 0$ such that $|X_n \circ \pi_n - X_*|_{\mathcal{O}_*} < \epsilon$ for all $n > N$. Take any $m > N$ and let $n > m$ be chosen so that $\sigma^{n-m+1} \leq b^{p^m} \leq \sigma^{n-m}$. Then

$$|X_m(\tau_m) - X_*(\tau_*)|\,,\ |X_m(F_m(\tau_m)) - X_*(F_*(\tau_*))| < \epsilon, \qquad (4.2.30)$$

and the same holds if we replace m by n. Let us denote the points $F_i(\tau_i)$ by ς_i.

Observe that $X_*(\varsigma_*) = \partial_x\phi_*(\tau_*)$. Therefore, as convergence of renormalisation implies $|\partial_x\phi_m(\tau_m) - \partial_x\phi_*(\tau_*)| < C\rho^m$, this tells us

$$|X_m(\varsigma_m) - \partial_x\phi_m(\tau_m)| < \epsilon + C\rho^m. \qquad (4.2.31)$$

We will now show they must differ by a definite constant and achieve the required contradiction. We will show this by evaluating X_m at a point near to ς_m. Consider the points $\varsigma = \Psi_{m,n}(\varsigma_n)$ and $\varsigma' = F_m\Psi_{m,n}(\varsigma_n)$. First let us evaluate X_m at ς'. By invariance this must be

$$\widetilde{\mathrm{D}}_{\varsigma_n} F_m \Psi_{m,n}\left(X_n(\varsigma_n)\right) = \partial_x\phi_m(\varsigma) + \frac{\partial_y\phi_m(\varsigma)}{s_{m,n}(\varsigma_n) + t_{m,n}(\varsigma_n)X_n(\varsigma_n)}. \tag{4.2.32}$$

The second term must be bounded away from zero as $X_n(\varsigma_n)$ is bounded from above if n is sufficiently large and the hypothesis on m,n tells us $s_{m,n}$ and $t_{m,n}$ are both comparable to b^{p^m}, as is the numerator $\partial_y\phi_m(\varsigma)$. It is clear this bound can be made uniform in m.

Second, observe that $|\varsigma' - \varsigma_m|$ can be made arbitrarily small by choosing m and $n-m$ sufficiently large, by the assumption that $\mathcal{O}_n$ converges strongly to $\mathcal{O}_*$. Combining these gives us the required contradiction, as our hypothesis implies increasing m leads to an exponential increase in n. □

4.3 Failure of Rigidity at the Tip

Using the same method as for the period doubling case we show that given two Cantor attractors $\mathcal{O}$ and $\tilde{\mathcal{O}}$ for some $F, \tilde{F} \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ with average Jacobian $b, \tilde{b}$ respectively, there is a bound on the Holder exponent of any conjugacy that preserves 'tips'.

Theorem 4.3.1. *Let $F, \tilde{F} \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ be two infinitely renormalisable Hénon-like maps with respective renormalisation Cantor sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$, and tips τ and $\tilde{\tau}$. If there is a conjugacy $\pi\colon \tilde{\mathcal{O}} \to \mathcal{O}$ mapping $\tilde{\tau}$ to τ then the Hölder exponent α of π satisfies*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log \tilde{b}}{\log b}\right) \tag{4.3.1}$$

Proof. We will denote all objects associated with F without tilde's and all objects associated with $\tilde{F}$ with them. For example Ψ and $\tilde{\Psi}$ will denote the scope function for F and $\tilde{F}$ respectively.

Let $K > 0$ be a positive constant which we will think of as being large. Let us choose an integer $m > 0$ which ensures that $\tilde{b}^{p^m} > Kb^{p^m}$ and take an integer $n > m$ which satisfies $\sigma^{n-m+1} \leq b^{p^m} < \sigma^{n-m}$. This will be the depth of the Cantor sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$ that we will consider. So let us consider F and $\mathcal{O}$. Let us denote the tip of F_{n+1} by τ and let ς be its image under F_{n+1}. Let $\dot{\tau}$ and $\dot{\varsigma}$ be the respective images of these points under $\Psi_{m,n}$. Let $\ddot{\tau}$ and $\ddot{\varsigma}$ be the respective images of $\dot{\tau}$ and $\dot{\varsigma}$ under F_m. Let $\dddot{\tau}$ and $\dddot{\varsigma}$ be the respective images of $\ddot{\tau}, \ddot{\varsigma}$ under $\Psi_{0,m-1}$. The equivalent points for $\tilde{F}$ will be denoted by with tilde's. Finally, τ_* denotes the tip of F_* and ς_* denotes its image under F_*.

Observe that τ_* and ς_* will not lie on the same vertical or horizontal line. Therefore we know that the following constant

$$C_0 = \tfrac{1}{2}\min\left(\left|\pi_x(\varsigma_*) - \pi_x(\tau_*)\right|, \left|\pi_y(\varsigma_*) - \pi_y(\tau_*)\right|\right) \tag{4.3.2}$$

is positive. By Theorem 3.3.2 there exists an integer $N > 0$ such that

$$\left|\pi_x(\varsigma) - \pi_x(\tau)\right|, \left|\pi_y(\varsigma) - \pi_y(\tau)\right|, \left|\pi_x(\tilde{\varsigma}) - \pi_x(\tilde{\tau})\right|, \left|\pi_y(\tilde{\varsigma}) - \pi_y(\tilde{\tau})\right| > C_0 > 0, \tag{4.3.3}$$

for all integers $m > N$. Let $\delta = (\delta_x, \delta_y) = \varsigma - \tau$ and $\tilde{\delta} = \left(\tilde{\delta}_x, \tilde{\delta}_y\right) = \tilde{\varsigma} - \tilde{\tau}$. Clearly we also have an upper bound for each of these quantities, namely $C_1 = \operatorname{diam}(B)$.

First we will derive an upper bound for the distance between $\dddot{\varsigma}$ and $\dddot{\tau}$, then we will derive a lower bound for the distance between $\dddot{\tilde{\varsigma}}$ and $\dddot{\tilde{\tau}}$.

Applying $\Psi_{m,n}$ to ς and τ gives $\dot{\varsigma} - \dot{\tau} = D_{m,n}\left(\mathrm{id} + R_{m,n}\right)(\varsigma - \tau)$. Let $\dot{\delta} = \left(\dot{\delta}_x, \dot{\delta}_y\right) = \dot{\varsigma} - \dot{\tau}$. Hence by Proposition 3.7.6 and the above paragraph there exists a constant $C_2 > 0$ such that,

$$\begin{aligned} \left|\dot{\delta}_x\right| &= \left|\sigma_{m,n} s_{m,n}\left[\delta_x + r_{m,n}\left(\delta_x, \delta_y\right)\right] + \sigma_{m,n} t_{m,n} \delta_y\right| \\ &\leq C_2 \sigma^{n-m}\left(\sigma^{n-m} + b^{p^m}\right) \end{aligned} \tag{4.3.4}$$

$$\begin{aligned} \left|\dot{\delta}_y\right| &= \left|\sigma_{m,n} \delta_y\right| \\ &\leq C_2 \sigma^{n-m} \end{aligned} \tag{4.3.5}$$

Next we apply $F_m = (\phi_m, \pi_x)$ which gives $\ddot{\varsigma} - \ddot{\tau} = F_m(\dot{\varsigma}) - F_m(\dot{\tau})$. Let $\ddot{\delta} = \left(\ddot{\delta}_x, \ddot{\delta}_y\right) = \ddot{\varsigma} - \ddot{\tau}$. First observe that by convergence of renormalisation, i.e. Theorem 3.3.2, there is a constant $C_2 > 0$ such that $|\partial_x \phi_m| < C_2$. Second observe, by Theorem 3.2.13 there exists a constant $C_3 > 0$ such that $|\partial_y \phi_m| < C_3 b^{p^m}$. Then by the Mean Value Theorem, if $\xi = \left(\pi_x(\dot{\tau}), \pi_y(\dot{\varsigma})\right)$, there exist points $\xi_y \in [\dot{\varsigma}, \xi], \xi_x \in [\xi, \dot{\tau}]$ such that

$$\begin{aligned} \left|\ddot{\delta}_x\right| &= \left|\partial_x \phi_m\left(\xi_y\right) \dot{\delta}_x + \partial_y \phi_m\left(\xi_x\right) \dot{\delta}_y\right| \\ &\leq C_3\left|\dot{\delta}_x\right| + C_4 b^{p^m}\left|\dot{\delta}_y\right| \\ &\leq C_2 \sigma^{n-m}\left(C_2\left(\sigma^{n-m} + b^{p^m}\right) + C_3 b^{p^m}\right) \\ &\leq C_5 \sigma^{n-m}\left(\sigma^{n-m} + b^{p^m}\right) \end{aligned} \tag{4.3.6}$$

$$\begin{aligned} \left|\ddot{\delta}_y\right| &= \left|\dot{\delta}_x\right| \\ &\leq C_2 \sigma^{n-m}\left(\sigma^{n-m} + b^{p^m}\right) \end{aligned} \tag{4.3.7}$$

Now we apply $\Psi_{0,m}$ which gives $\dddot{\varsigma} - \dddot{\tau} = D_{0,m}\left(\mathrm{id} + R_{0,m}\right)(\ddot{\varsigma} - \ddot{\tau})$. Let $\dddot{\delta} = (\dddot{\delta}_x, \dddot{\delta}_y) = \dddot{\varsigma} - \dddot{\tau}$. Hence, by Proposition 3.7.6 and the above paragraph, there is a constant $C_6 > 0$ such that

$$|\dddot{\delta}_x| = \left|\sigma_{0,m}s_{0,m}\left[\ddot{\delta}_x + r_{0,m}\left(\ddot{\delta}_x, \ddot{\delta}_y\right)\right] + \sigma_{0,m}t_{0,m}\ddot{\delta}_y\right| \tag{4.3.8}$$
$$\leq C_2\sigma^m \left|\sigma^m\left[\left|\ddot{\delta}_x\right| + |\partial_x r_{0,m}|\left|\ddot{\delta}\right|\right] + b^{p^m}\left|\ddot{\delta}_y\right|\right|$$
$$\leq C_6\sigma^{2m}\sigma^{n-m}\left(\sigma^{n-m} + b^{p^m}\right) + C_2^2\sigma^n b^{p^m}\left(\sigma^{n-m} + b^{p^m}\right)$$
$$\leq \left(C_6\sigma^{n+m} + C_2^2\sigma^n b^{p^m}\right)\left(\sigma^{n-m} + b^{p^m}\right)$$
$$|\dddot{\delta}_y| = \left|\sigma_{0,m}\ddot{\delta}_y\right| \tag{4.3.9}$$
$$\leq C_2^2\sigma^n\left(\sigma^{n-m} + b^{p^m}\right)$$

From the second inequality we find there exists a constant $C_7 > 0$ such that $\mathrm{dist}(\dddot{\varsigma}, \dddot{\tau}) \leq C_7\sigma^{2n-m}$.

Now we wish to a find a lower bound for $\mathrm{dist}(\dddot{\tilde{\varsigma}}, \dddot{\tilde{\tau}})$. Applying $\tilde{\Psi}_{m,n}$ to these points gives $\dot{\tilde{\varsigma}} - \dot{\tilde{\tau}} = \tilde{D}_{m,n}(\mathrm{id} + \tilde{R}_{m,n})(\tilde{\varsigma} - \tilde{\tau})$. Let $\dot{\tilde{\delta}} = (\dot{\tilde{\delta}}_x, \dot{\tilde{\delta}}_y) = \dot{\tilde{\varsigma}} - \dot{\tilde{\tau}}$. Hence, as before, by Proposition 3.7.6 and the second paragraph there exists a constant $C_2 > 0$ such that, $|\dot{\tilde{\delta}}_y| = |\tilde{\sigma}_{m,n}\tilde{\delta}_y| \leq C_2\sigma^{n-m}$. Let $C_8 > 1$ be constants satisfying

$$|\tilde{\sigma}_{m,n}| > C_8^{-1}\sigma^{n-m}, \quad \left|\tilde{t}_{m,n}\right| > C_8^{-1}\tilde{b}^{p^m}, \quad |s_{m,n}| < C_8\sigma^{n-m}, \quad |\tilde{r}_{m,n}| < C_8. \tag{4.3.10}$$

But, since $\tilde{b}^{p^m} > K\sigma^{n-m+1}$, Proposition 3.7.6 tells us

$$\left|\dot{\tilde{\delta}}_x\right| = \left|\tilde{\sigma}_{m,n}\tilde{s}_{m,n}\left[\tilde{\delta}_x + \tilde{r}_{m,n}\left(\tilde{\delta}_x, \tilde{\delta}_y\right)\right] + \tilde{\sigma}_{m,n}\tilde{t}_{m,n}\tilde{\delta}_y\right| \tag{4.3.11}$$
$$\geq |\tilde{\sigma}_{m,n}|\left||\tilde{s}_{m,n}|\left|\tilde{\delta}_x + \tilde{r}_{m,n}\left(\tilde{\delta}_x, \tilde{\delta}_y\right)\right| - \left|\tilde{t}_{m,n}\tilde{\delta}_y\right|\right|$$
$$\geq C_8^{-1}\sigma^{n-m}\left(C_8^{-1}C_0 b^{p^m} - C_8\left(C_0 + C_8\right)\sigma^{n-m}\right)$$
$$\geq C_8^{-1}\sigma^{n-m}b^{p^m}\left(C_8^{-1}C_0 - K^{-1}\sigma^{-1}C_8\left(C_0 + C_8\right)\right).$$

Since $K > 0$ was assumed to be large (and the constants C_8 had no dependence upon m and n) we find there exists a constant $C_9 > 0$ such that $\left|\dot{\tilde{\delta}}_x\right| > C_9 b^{p^m}\sigma^{n-m}$.

Applying $\tilde{F}_m$ to $\dot{\tilde{\varsigma}}$ and $\dot{\tilde{\tau}}$ gives $\ddot{\tilde{\varsigma}} - \ddot{\tilde{\tau}} = F_m\left(\dot{\tilde{\varsigma}}\right) - F_m\left(\dot{\tilde{\tau}}\right)$. Let $\ddot{\tilde{\delta}} = \left(\ddot{\tilde{\delta}}_x, \ddot{\tilde{\delta}}_y\right) = \ddot{\tilde{\varsigma}} - \ddot{\tilde{\tau}}$. Then, ignoring the difference in the x-direction, we find $\left|\ddot{\tilde{\delta}}_y\right| = \left|\dot{\tilde{\delta}}_x\right| \geq C_{11}b^{p^m}\sigma^{n-m}$.

Now we apply $\tilde{\Psi}_{0,m}$ which gives $\dddot{\tilde{\varsigma}} - \dddot{\tilde{\tau}} = \tilde{D}_{0,m}\left(\mathrm{id} + \tilde{R}_{0,m}\right)\left(\ddot{\tilde{\varsigma}} - \ddot{\tilde{\tau}}\right)$. Let $\dddot{\tilde{\delta}} = \left(\dddot{\tilde{\delta}}_x, \dddot{\tilde{\delta}}_y\right) = \dddot{\tilde{\varsigma}} - \dddot{\tilde{\tau}}$. Then from Lemma 3.7.5 we find $\left|\dddot{\tilde{\delta}}_y\right| = \left|\tilde{\sigma}_{0,m}\ddot{\tilde{\delta}}_y\right|$. But Proposition 3.7.6 implies there exists a constant $C_{10} > 0$ such that $|\tilde{\sigma}_{0,m}| \geq C_{10}\sigma^m$, so combining this with the estime from preceding paragraph gives $\left|\dddot{\tilde{\delta}}_y\right| \geq C_9C_{10}\sigma^n b^{p^m}$.

Now let us combine these upper and lower bounds. Let $C_{11}, C_{12} > 0$ be constants satisfying $\operatorname{dist}\left(\dddot{\tilde{\varsigma}}, \dddot{\tilde{\tau}}\right) > C_1 1 \sigma^n b^{p^m}$ and $\operatorname{dist}\left(\dddot{\varsigma}, \dddot{\tau}\right) < C_{12}\sigma^{2n-m}$. Then, assuming the Hölder condition holds for some $C_{13}, \alpha > 0$ we have

$$C_{11}\sigma^n \tilde{b}^{p^m} \leq \operatorname{dist}\left(\dddot{\tilde{\tau}}, \dddot{\tilde{\varsigma}}\right) \leq C \operatorname{dist}\left(\dddot{\tau}, \dddot{\varsigma}\right)^\alpha \leq C_{13} C_{12}^\alpha (\sigma^{2n-m})^\alpha \tag{4.3.12}$$

which implies, after collecting all constant factors, that there is a $C > 0$ such that

$$\sigma^m b^{p^m} \tilde{b}^{p^m} \leq C \left(\sigma^m b^{p^m} b^{p^m}\right)^\alpha \tag{4.3.13}$$

and hence after taking the logarithm of both sides and passing to the limit gives

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log \tilde{b}}{\log b}\right). \tag{4.3.14}$$

and hence the theorem is shown. $\square$

Chapter 5

Unbounded Geometry Cantor Sets

We outline the structure of this. In the following section we define *boxings* of the Cantor set. These are nested sequences of pairwise disjoint simply connected domains that 'nest down' to the Cantor set $\mathcal{O}$ and are invariant under the dynamics. We then introduce our construction and the mechanism that will destroy the geometry of our boxings, namely *horizontal overlapping*. Then we give a condition in terms of the average Jacobian for horizontal overlapping of boxes to occur. We show this condition is satisfied for a dense G_δ set of parameters with full Lebesgue measure. This last part is purely analytical and has no dynamical content.

Definition 5.0.2. We say that two planar sets $S, \tilde{S} \subset \mathbb{R}^2$ *horizontally overlap* if they mutually intersect a vertical line, which is equivalent to saying their projections onto the x-axis intersect, i.e. $\pi_x(\mathrm{Hull}(S)) \cap \pi_x(\mathrm{Hull}(\tilde{S})) \neq \emptyset$. If they do not horizontally overlap we say they are *horizontally separated*. Similarly we say two planar sets $S, \tilde{S} \subset \mathbb{R}^2$ *vertically overlap* or are *vertically separated* if, respectively, they mutually intersect a horizontal line or do not.

5.1 Boxings and Bounded Geometry

Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ and let $\underline{\mathcal{O}}$ and $\underline{\Psi}$ be as in Section 3. A collection of simply connected open sets $\underline{B} = \{B^{\mathbf{w}}\}_{\mathbf{w} \in W^*}$ is called a *boxing* of $\underline{\mathcal{O}}$ with respect to F if

(B-1) $F(B^{\mathbf{w}}) \subset B^{1+\mathbf{w}}$ for all $\mathbf{w} \in W^*$,

(B-2) $B^{\mathbf{w}}$ and $B^{\tilde{\mathbf{w}}}$ are disjoint for all $\mathbf{w} \neq \tilde{\mathbf{w}}$ of the same length,

(B-3) the disjoint union of the $B^{\mathbf{w}w}$, $w \in W$, is a subset of $B^{\mathbf{w}}$, for all $\mathbf{w} \in W^*$, (B-4) $\mathcal{O}^{\mathbf{w}} \subset B^{\mathbf{w}}$ for all $\mathbf{w} \in W^*$,

The sets $B^{\mathbf{w}}$ are called the *pieces* of the boxing and the *depth* of the piece $B^{\mathbf{w}}$ is the length of the word $\mathbf{w}$. The scope functions give us a boxing $\underline{B}_{can} = \{B^{\mathbf{w}}_{can}\}_{\mathbf{w}\in W^*}$, where $B^{\mathbf{w}}_{can} = \Psi^{\mathbf{w}}(B)$, which we will call the *canonical boxing*.

Observe that the since the scope functions $\underline{\Psi_n} = \{\Psi^{\mathbf{w}}_n\}_{\mathbf{w}\in W^*}$ for F_n can be written as $\Psi^{\mathbf{w}}_n = \Psi^{-1}_{0,n} \circ \Psi_{0,n} \circ \Psi^{\mathbf{w}}_n$ and $\Psi_{0,n} \circ \Psi^{\mathbf{w}}_n \in \underline{\Psi}$, the canonical boxing $\underline{B_{n,can}}$ for F_n is the preimage under $\Psi_{0,n}$ of all the pieces contained in $\Psi_{0,n}(B)$. Hence the scope maps preserve the canonical boxings of various heights.

There is also another 'standard' boxing, which we call the *topological boxing*. The pieces are simply connected domains whose boundary consists of two arcs, one of which is a segment of the unstable manifold of a particular periodic point and the other consisting of a segment of stable manifold of a different periodic point of the same period. These boxings in the period doubling case were first considered in [12].

Definition 5.1.1. We say that a boxing $\underline{B} = \{B^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ has *bounded geometry* if there exist constants $C > 1, 0 < \kappa < 1$ such that for all $\mathbf{w} \in W^*, w, \tilde{w}, \in W$,

$$C^{-1}\operatorname{dist}(B^{\mathbf{w}w}, B^{\mathbf{w}\tilde{w}}) < \operatorname{diam}(B^{\mathbf{w}w}) < C\operatorname{dist}(B^{\mathbf{w}w}, B^{\mathbf{w}\tilde{w}}) \tag{5.1.1}$$

$$\kappa\operatorname{diam}(B^{\mathbf{w}}) < \operatorname{diam}(B^{\mathbf{w}w}) < (1-\kappa)\operatorname{diam}(B^{\mathbf{w}}) \tag{5.1.2}$$

We will say that $\mathcal{O}$ has *bounded geometry* if there exists a boxing $\underline{B}$ of $\underline{\mathcal{O}}$ with bounded geometry. Otherwise we will say $\mathcal{O}$ has unbounded geometry.

Remark 5.1.2. As the results we will prove are actually stronger than mere unbounded geometry. We will show that Property 5.1.1 is violated almost everywhere in one-parameter families of infinitely renormalisable Hénon-like maps. We believe that any breakdown of Property 5.1.2 is much more dependent upon the choice of boxings - in principle we could take any boxing and just enlarge the one containing the tip. The only thing to show would then be whether the return of this box is contained in the original box.

We will use the assumption below in the following sections for expositional simplicity. Its necessity will become clear in Section 5.2 when we describe the construction.

(B-5) $B^{\mathbf{w}w} \subset B^{\mathbf{w}}_{can}$ for all $w \in W$ and all sufficiently large $\mathbf{w} \in W^*$.

This will allow us, given any boxing $\underline{B}$ of $\underline{\mathcal{O}}$, to construct induced boxings $\underline{B}_n$ at all sufficiently great heights. However below, in Lemma 5.1.3, we show this assumption is redundant.

Lemma 5.1.3. *Given a boxing $\underline{B}$ of $\underline{\mathcal{O}}$ there is a boxing $\hat{\underline{B}}$ satisfying Property (B-5) above such that if $\hat{\underline{B}}$ has unbounded geometry then $\underline{B}$ has unbounded geometry.*

Proof. Given a boxing $\underline{B}$ of $\underline{\mathcal{O}}$ define $\hat{\underline{B}}$ to be the collection $\{\hat{B}^{\mathbf{w}}\}_{w\in W^*}$ where

$$\hat{B}^{\mathbf{w}w} = B^{\mathbf{w}w} \cap B^{\mathbf{w}}_{can}, \quad w \in W, \mathbf{w} \in W^*$$

It is clear that

$$\operatorname{dist}(B^{\mathbf{w}}, B^{\tilde{\mathbf{w}}}) \leq \operatorname{dist}(\hat{B}^{\mathbf{w}}, \hat{B}^{\tilde{\mathbf{w}}})$$

and

$$\operatorname{diam}(B^{\mathbf{w}}) \geq \operatorname{diam}(\hat{B}^{\mathbf{w}}).$$

□

5.2 The Construction

Now let us introduce the construction and set-up some notation that shall be used throughout the remainder of the paper. Firstly, for any infinitely renormalisable Hénon-like map, we will change coordinates for each renormalisation so that the n-th tip, τ_n, lies at the origin. As this coordinate change is by translations only, this will not affect the geometry of the Cantor set. The new scope maps will have the form

$$\hat{\Psi}_{m,n}(z) = D_{m,n} \circ (\operatorname{id} + R_{m,n})(z).$$

Secondly, the following quantities will prove to be useful. Given $z = (x, y), \tilde{z} = (\tilde{x}, \tilde{y}) \in \operatorname{Dom}(F_{n+1})$ let

$$\Upsilon_*(z, \tilde{z}) = \frac{v_*(\tilde{x}) - v_*(x)}{\tilde{y} - y},$$

where v_* is the universal function given by Proposition 3.7.6. Given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ and points $z, \tilde{z} \in B_{n+1}$ let

$$\Upsilon_m(z, \tilde{z}) = \Upsilon_*(z, \tilde{z}) - c_m \frac{\tilde{y}^2 - y^2}{\tilde{y} - y}$$

where $c_m = c_m(F)$ are the constants given by Proposition 3.7.6.

Remark 5.2.1. A technicality that was not present in [12] is the following: the quantity $t_{m,n}/s_{m,n}$ (where $t_{m,n}$ and $s_{m,n}$ are tilt and the squeeze of $\Psi_{m,n}$ as given by Proposition 3.7.6) is important in controlling horizontal overlap of pieces of a boxing. The sign of this will determine which boxes we take to ensure their images horizontally overlap. Observe that the combinatorial type υ determines whether the sign of $t_{m,n}/s_{m,n}$ alternates or remains constant. This is due to the sign of $t_{m,n}$ being always negative, but the sign of s_i will asymptotically depend upon the sign of the derivative of the presentation function at its fixed point so, as $s_{m,n}$ is the product of s_i, the sign of $s_{m,n}$ will either be $(1)^{n-m}$ or $(-1)^{n-m}$. Consequently we will restrict ourselves to considering sufficiently large $m, n \in 2\mathbb{N}$ or $2\mathbb{N}+1$ to ensure $t_{m,n}/s_{m,n}$ is negative. Our method would also work for the other case, but this would require choosing more words and points below and doing a case analysis, which adds to the complications.

Definition 5.2.2. Given words $\mathbf{w}, \tilde{\mathbf{w}}$ the points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$, and $\tilde{z}_*^0 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ are *well placed* if

(i) $x_*^0 < x_*^1 < \tilde{x}_*^0, \quad y_*^0 < y_*^1 < \tilde{y}_*^0$;

(ii) $\Upsilon_*(z_*^0, \tilde{z}_*^0) < \Upsilon_*(z_*^0, z_*^1)$.

A pair of words $\mathbf{w}, \tilde{\mathbf{w}}$ are called *well chosen* if

(i) there exist well placed points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$, and $\tilde{z}_*^0 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$;

(ii) $\mathbf{w}$ and $\tilde{\mathbf{w}}$ differ only on the last letter, i.e. $\mathbf{w} = w_0 \dots w_{n-1} w_n$ and $\tilde{\mathbf{w}} = w_0 \dots w_{n-1} \tilde{w}_n$ for some $w_0, \dots, w_n, \tilde{w}_n \in W$ and some integer $n > 0$.

Remark 5.2.3. Observe Property (i) will occur for certain words as $\mathcal{O}_*^{\mathbf{w}}$ and $\mathcal{O}_*^{\tilde{\mathbf{w}}}$ are horizontally and vertically separated if $\mathbf{w}$ and $\tilde{\mathbf{w}}$ have the same length. If the $t_{m,n}/s_{m,n}$ were positive we would change the ordering above.

Lemma 5.2.4. *Well chosen pairs of words exist.*

Proof. First we wish to find well-placed points, then it will become clear from our argument that we can assume they boxes with well chosen words. Recall that we have changed coordinates so that the tip τ_* lies at the origin. Let $\hat{f}_*$ denote the translation f_* that agrees with this coordinate change. Observe that points in $\mathcal{O}_*$ have the form $z = (\hat{f}_*(y), y)$ where y lies in the one-dimensional Cantor attractor for $\hat{f}_*$ in the interval. Therefore given points $z_*^0, z_*^1, \tilde{z}_* \in \mathcal{O}_*$ we have

$$\Upsilon_*(z_*^0, z_*^1) = \frac{v_* \circ \hat{f}_*(y_*^1) - v_* \circ \hat{f}_*(y_*^0)}{y_*^1 - y_*^0}, \quad \Upsilon_*(z_*^0, \tilde{z}_*) = \frac{v_* \circ \hat{f}_*(\tilde{y}_*) - v_* \circ \hat{f}_*(y_*^0)}{\tilde{y}_* - y_*^0}. \tag{5.2.1}$$

Since v_* and $\hat{f}_*$ are analytic so is the function $v_* \circ \hat{f}_*$. Since the derivative of $v_* \circ \hat{f}_*$ is zero at the critical point c_* analyticity implies there exists a neighbourhood V around c_* on which $v_* \circ \hat{f}_*$ is concave or convex. Therefore if $z_*^0, z_*^1, \tilde{z}_* \in \mathcal{O}_*$ are any points whose y-projections lie in V then Property 1 implies Property 2, by the Mean Value Theorem for example. But choosing y_*^0, y_*^1 and $\tilde{y}_*$ to lie all either to the left of c_* or to the right will give us Property 1.

Finally choosing the largest disjoint cylinder sets $\mathcal{O}_*^{\mathbf{w}}, \mathcal{O}_*^{\tilde{\mathbf{w}}}$ of $\mathcal{O}_*$, of the same depth, such that $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$ and $\tilde{z}_* \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ gives us the desired well-chosen words. ∎

We can now make the following assumptions. There exist words $\mathbf{w}, \tilde{\mathbf{w}}$, of the same length, and points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}, \tilde{z}_*^0, \tilde{z}_*^1 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$, which we now fix, satisfying

(i) $x_*^0 < x_*^1 < \tilde{x}_*^0 < \tilde{x}_*^1, \quad y_*^0 < y_*^1 < \tilde{y}_*^0 < \tilde{y}_*^1$;

(ii) the points $z_*^0, z_*^1, \tilde{z}_*^0$ are well placed.

Given these points let us now define some quantities which shall prove to be useful. Let

$$\kappa_0 = |\Upsilon_*(z_*^0, z_*^1) - \Upsilon_*(\tilde{z}_*^0, \tilde{z}_*^1)|, \quad \kappa_1 = \frac{|y_*^1 - y_*^0|}{|\tilde{y}_*^0 - y_*^0|},$$

and

$$\kappa_2 = |\tilde{y}_*^0 - y_*^0|, \quad \kappa_3 = |y_*^1 - y_*^0|, \quad \kappa_4 = |\tilde{y}_*^1 - \tilde{y}_*^0|.$$

These are all well-defined nonzero quantities by Lemma 5.2.4. For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let the points

$$z_n^0 = (x_n^0, y_n^0), z_n^1 = (x_n^1, y_n^1) \in \mathcal{O}_n^{\mathbf{w}}$$

and

$$\tilde{z}_n^0 = (\tilde{x}_n^0, \tilde{y}_n^0), \tilde{z}_n^1 = (\tilde{x}_n^1, \tilde{y}_n^1) \in \mathcal{O}_n^{\tilde{\mathbf{w}}}$$

have the same respective addresses in $\mathcal{O}_n$ (see subsection 2 to recall the definition) as those of $z_*^0, z_*^1, \tilde{z}_*^0, \tilde{z}_*^1$ in $\mathcal{O}_*$. Let

$$M = \left[\frac{\Upsilon_*(z_*^0, \tilde{z}_*^0) - \frac{\kappa_1}{2}\Upsilon_*(z_*^0, z_*^1)}{1 - \frac{\kappa_1}{2}}, \Upsilon_*(z_*^0, \tilde{z}_*^0)\right]. \tag{5.2.2}$$

This is a well defined interval because z_*^0, z_*^1 and $\tilde{z}_*^0$ are well placed which implies $\Upsilon_*(z_*^0, z_*^1) > \Upsilon_*(z_*^0, \tilde{z}_*^0)$ and hence

$$\Upsilon(z_*^0, \tilde{z}_*^0) - \frac{\kappa_1}{2}\Upsilon_*(z_*^0, z_*^1) < \Upsilon_*(z_*^0, \tilde{z}_*^0)(1 - \frac{\kappa_1}{2}) \tag{5.2.3}$$

Dividing by $1 - \frac{\kappa_1}{2}$ and recalling $0 < \kappa_1/2 < 1$ gives us the claim. Fix a $\delta > 0$ such that

$$M_\delta = \left[\frac{\Upsilon_*(z_*^0, \tilde{z}_*^0) - \frac{\kappa_1}{2}\Upsilon_*(z_*^0, z_*^1)}{1 - \frac{\kappa_1}{2}} + \frac{\delta}{3}\left(\frac{3 - \frac{\kappa_1}{2}}{1 - \frac{\kappa_1}{2}}\right), \Upsilon_*(z_*^0, \tilde{z}_*^0) - \delta\right]. \tag{5.2.4}$$

is a well defined interval. Choose $N > 0$ sufficiently large so that

$$4C\rho^N < \frac{\kappa_2}{2}\left(1 - \frac{\kappa_1}{2}\right)\frac{\delta}{3} \tag{5.2.5}$$

and

$$4C\rho^N(1/\kappa_3 + 1/\kappa_4) < \kappa_0/8. \tag{5.2.6}$$

Let $\mathcal{A} \subset \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ denote the subspace of all infinitely renormalisable Hénon-like maps F such that, for all $n > m > 0, n - m > N$:

(A-1) $x_{n+1}^0 < x_{n+1}^1 < \tilde{x}_{n+1}^0 < \tilde{x}_{n+1}^1, \quad y_{n+1}^0 < y_{n+1}^1 < \tilde{y}_{n+1}^0 < \tilde{y}_{n+1}^1$;

(A-2) $1 > |y_{n+1}^1 - y_{n+1}^0|/|\tilde{y}_{n+1}^0 - y_{n+1}^0| > \kappa_1/2$;

(A-3) $|\tilde{y}_{n+1}^0 - y_{n+1}^0| > \kappa_2/2,\ |y_{n+1}^1 - y_{n+1}^0| > \kappa_3/2,\ |\tilde{y}_{n+1}^1 - \tilde{y}_{n+1}^0| > \kappa_4/2$;

(A-4) $|\Upsilon_m(z_{n+1}^0, z_{n+1}^1) - \Upsilon_m(\tilde{z}_{n+1}^0, \tilde{z}_{n+1}^1)| > \kappa_0/2$;

(A-5) $|(x + r_{m,n}(z)) - (v_*(x) - c_m y^2)| < C\rho^{n-m}$ for all $z \in B_{n+1}$;

(A-6) $|\Upsilon_m(z_{n+1}^0, z_{n+1}^1) - \Upsilon_*(z_*^0, z_*^1)|, |\Upsilon_m(z_{n+1}^0, \tilde{z}_{n+1}^0) - \Upsilon_*(z_*^0, \tilde{z}_*^0)| < \delta/3$;

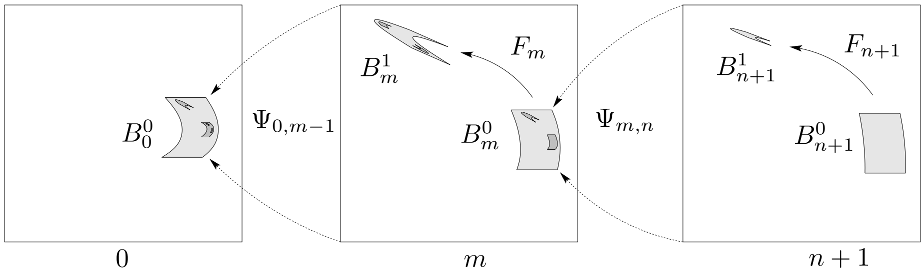

Figure 5.1: The Construction. We take a pair of boxes of depth $n-m$ around the tip and then 'perturb' them by the dynamics of F_m, the m-th renormalisation, before mapping to height zero

(A-7) $t_{m,n}/s_{m,n} < 0$ and moreover

$$\left| \frac{t_{m,n}}{s_{m,n}} + a\frac{b^{p^m}}{\sigma^{n-m}} \right| < \delta/3;$$

where $\sigma_{m,n}, s_{m,n}, t_{m,n}$ are respectively the scaling ratio, squeeze and tilt from height $n+1$ to height m, σ is the universal scaling ratio, c_m is the constant and v_* the univeraal function from inequality (3.7.49), a is the universal constant from inequality (4.1.16) and $C > 0$ and $0 < \rho < 1$ are chosen so that all estimates from the preceding section hold.

Proposition 5.2.5. *Given a family $F_b \in \mathcal{I}_{\Omega,v}(\bar{\varepsilon}_0)$ parametrised by the average Jacobian, there exists an integer $N_0 > 0$ and $0 < b_0 < 1$ such that $\mathcal{R}^n F_b \in \mathcal{A}$ for all $n > N_0, 0 \leq b \leq b_0$.*

Proof. This follows as $\mathcal{R}^n(F_b)$ converges exponentially to F_* which lies in $\mathcal{A}$, so we may choose the $N_0 > 0$ so that $\mathcal{R}^n(F_0) \in \mathcal{A}$ for all $n > N_0$. Then it is clear there exists a $b_0 > 0$ such that $\mathcal{R}^{N_0}(F_b) \in \mathcal{A}$ for all $0 \leq b \leq b_0$ since $\mathcal{A}$ is open. It is also clear $\mathcal{A}$ is invariant under $\mathcal{R}$ so the Proposition follows. ☐

We now describe the construction. This was used in [12] to prove several negative results, such as non-existence of continuous invariant line fields (see these two references for further details). Let $F \in \mathcal{A}$ and let us fix $n, m \in 2\mathbb{N}$ or $2\mathbb{N}+1$ as per remark 5.2.1 such that $n > m > 0$ and $n - m > N$. Consider the maps $\Psi_{0,m-1}, F_m, \Psi_{m,n}$. In reverse order, these map from height $n+1$ to height m, from height m to itself and from height m to height 0 respectively (see figure 5.1).

We will adopt the following notation convention: if we have a quantity Q in the domain of $\Psi_{m,n}$ we will denote its images under $\Psi_{m,n}, F_m$ and $\Psi_{0,m-1}$ by $\dot{Q}, \ddot{Q}$ and $\dddot{Q}$ respectively.

5.3 Horizontal Overlapping Distorts Geometry

Recall that in the previous section we fixed well chosen words $\mathbf{w}, \tilde{\mathbf{w}} \in W^*$ with points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$ and $\tilde{z}_*^0, \tilde{z}_*^1 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ so that z_*^0, z_*^1 and $\tilde{z}_*^0$ are well-placed. We make the following definition.

Definition 5.3.1. Given a boxing $\underline{B}$ of a Cantor set we will say it satisfies the property $\mathrm{Hor}_{\mathbf{w},\tilde{\mathbf{w}}}(m,n)$ if the pieces $B_{n+1}^{\mathbf{w}}, B_{n+1}^{\tilde{\mathbf{w}}} \in \underline{B}_{n+1}$ have images $B_m^{0^{n-m}\mathbf{w}}$, and $B_m^{0^{n-m}\tilde{\mathbf{w}}}$, under $\Psi_{m,n}$, which horizontally overlap.

Throughout the rest of the section we will assume the boxing $\underline{B}$ is fixed.

Lemma 5.3.2 (Key Lemma)**.** *Given a constant $K > 0$, there is a constant $C > 0$ such that the following holds: given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$, if there are points $z, \tilde{z} \in \mathrm{Dom}(F_{n+1})$ satisfying*

$$|\pi_y(z) - \pi_y(\tilde{z})| > K \tag{5.3.1}$$

$$\left|\pi_x(\dot{z}) - \pi_x\left(\dot{\tilde{z}}\right)\right| = 0 \tag{5.3.2}$$

then

$$|\Upsilon_*(z,\tilde{z})| - C\max(\rho^m,\rho^{n-m}) < \frac{ab^{p^m}}{\sigma^{n-m}} < |\Upsilon_*(z,\tilde{z})| + C\max(\rho^m,\rho^{n-m}) \tag{5.3.3}$$

Proof. Equality (3.7.16) from Proposition 3.7.6 tells us if $\dot{z}, \dot{\tilde{z}}$ lie on the same vertical line then

$$0 = s_{m,n}([x + r_{m,n}(x,y)] - [\tilde{x} + r_{m,n}(\tilde{x},\tilde{y})]) + t_{m,n}(y - \tilde{y}). \tag{5.3.4}$$

Dividing by $s_{m,n}(y - \tilde{y})$, which is nonzero, gives us

$$-\frac{t_{m,n}}{s_{m,n}} = \frac{[x + r_{m,n}(z)] - [\tilde{x} + r_{m,n}(\tilde{z})]}{y - \tilde{y}}. \tag{5.3.5}$$

By inequality (3.7.49) in Proposition 3.7.6 implies

$$|\Upsilon_m(z,\tilde{z})| - \frac{C\rho^{n-m}}{|\tilde{y} - y|} < \left|\frac{t_{m,n}}{s_{m,n}}\right| < |\Upsilon_m(z,\tilde{z})| + \frac{C\rho^{n-m}}{|\tilde{y} - y|}. \tag{5.3.6}$$

Again by inequality (3.7.49) in Proposition 3.7.6 and the definition of Υ_m we know

$$|\Upsilon_*(z,\tilde{z})| - C\bar{\varepsilon}_0^{p^m} < |\Upsilon_m(z,\tilde{z})| < |\Upsilon_*(z,\tilde{z})| + C\bar{\varepsilon}_0^{p^m}. \tag{5.3.7}$$

By inequalities (3.7.20) and (4.1.16) in Proposition 3.7.6 we know there is a constant $C' > 0$ such that

$$\left|\frac{t_{m,n}}{s_{m,n}}\right|(1 - C'\rho^m) < \frac{ab^{p^m}}{\sigma^{n-m}} < \left|\frac{t_{m,n}}{s_{m,n}}\right|(1 + C'\rho^m). \tag{5.3.8}$$

Combining inequalities (5.3.6), (5.3.7) and (5.3.8), together with our first assumption and the observation $\bar{\varepsilon}_0^{p^m} = \mathrm{O}(\rho^m)$, gives us the result. ∎

Corollary 5.3.3. *There exists a constant $C > 0$ such that the following holds: let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $z_{n+1}^0, \tilde{z}_{n+1}^0 \in \mathcal{O}_n$ have the same respective addresses as $z_*^0, \tilde{z}_*^0 \in \mathcal{O}_*$. If $|\pi_x(\dot{z}_{n+1}^0) - \pi_x(\dot{\tilde{z}}_{n+1}^0)| = 0$ then*

$$C^{-1}\sigma^{n-m} < b^{p^m} < C\sigma^{n-m} \tag{5.3.9}$$

Proof. This follows as $z_{n+1}^0, \tilde{z}_{n+1}^0$ can be taken to be arbitrarily close to $z_*^0, \tilde{z}_*^0$ and so the constant $K > 0$ in Lemma 5.3.2 will eventually only depend upon the vertical distance between these points, which is fixed. □

Proposition 5.3.4. *For any words $\mathbf{w}, \tilde{\mathbf{w}} \in W^*$ there exists a $C_0 > 0$ such that the following holds: for any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ and any boxing $\underline{B}$ of F, if points $z \in B_{n+1}^{\mathbf{w}}$ and $\tilde{z} \in B_{n+1}^{\tilde{\mathbf{w}}}$ satisfy $|\pi_x(\dot{z}) - \pi_x(\dot{\tilde{z}})| = 0$ then*

$$\operatorname{dist}\left(\dddot{z}, \dddot{\tilde{z}}\right) < C_0\sigma^{2m}b^{p^m}\sigma^{n-m}. \tag{5.3.10}$$

Proof. Let $z = (x, y), \tilde{z} = (\tilde{x}, \tilde{y}), \dot{z} = (\dot{x}, \dot{y}), \dot{\tilde{z}} = (\dot{\tilde{x}}, \dot{\tilde{y}})$ and so on. Then by Proposition 3.7.6 and our hypothesis that $\dot{z}, \dot{\tilde{z}}$ lie on the same vertical line, we know

$$\begin{aligned} \left|\dot{\tilde{x}} - \dot{x}\right| &= 0 \\ \left|\dot{\tilde{y}} - \dot{y}\right| &= |\sigma_{m,n}|\,|\tilde{y} - y|\,. \end{aligned} \tag{5.3.11}$$

Applying Lemma 3.3.5 we then know there exists $\eta \in [\![\dot{z}, \dot{\tilde{z}}]\!]$ such that

$$\begin{aligned} \left|\ddot{\tilde{x}} - \ddot{x}\right| &= |\partial_y\phi_m(\eta)||\sigma_{m,n}||\tilde{y} - y| \\ \left|\ddot{\tilde{y}} - \ddot{y}\right| &= 0. \end{aligned} \tag{5.3.12}$$

Then Proposition 3.7.6 once more implies

$$\begin{aligned} &\left|\dddot{\tilde{x}} - \dddot{x}\right| = |\sigma_{0,m-1}|\,|s_{0,m-1}|\,\left|\left[\ddot{\tilde{x}} + r_{0,m-1}\left(\ddot{\tilde{z}}\right)\right] - \left[\ddot{x} + r_{0,m-1}\left(\ddot{z}\right)\right]\right| \\ &\left|\dddot{\tilde{y}} - \dddot{y}\right| = 0. \end{aligned} \tag{5.3.13}$$

But, by the Mean Value Theorem and that $\ddot{\tilde{y}} = \ddot{y}$, we find there is a $\xi \in [\ddot{x}, \ddot{\tilde{x}}]$ such that

$$\begin{aligned} &\left|\left[\ddot{\tilde{x}} + r_{0,m-1}\left(\ddot{\tilde{z}}\right)\right] - \left[\ddot{x} + r_{0,m-1}\left(\ddot{z}\right)\right]\right| = |1 + \partial_x r_{0,m-1}\left(\xi, \ddot{y}\right)|\left|\ddot{\tilde{x}} - \ddot{x}\right| \\ &= |1 + \partial_x r_{0,m-1}\left(\xi, \ddot{y}\right)|\,|\partial_y\phi_m(\eta)|\,|\sigma_{m,n}|\,|\tilde{y} - y|\,. \end{aligned} \tag{5.3.14}$$

It follows from Propositions 3.7.6 and 3.3.4 that there exist three constants $C', C'', C''' > 0$, independent of m, n, such that

$$|1 + \partial_x r_{0,m-1}(\xi, \ddot{y})| < C', \quad |\partial_y\phi_m(\eta)| < C''b^{p^m}, \quad |\sigma_{m,n}| < C'''\sigma^{n-m}. \tag{5.3.15}$$

Hence it follows from (5.3.13), (5.3.14) and (5.3.15) that there is a $C_0 > 0$ such that

$$\operatorname{dist}\left(\dddot{z}, \dddot{\tilde{z}}\right) = \left|\dddot{\tilde{x}} - \dddot{x}\right| < C_0\sigma^{2m}b^{p^m}\sigma^{n-m} \tag{5.3.16}$$

□

Proposition 5.3.5. *For well chosen words* $\mathbf{w}$ *and* $\tilde{\mathbf{w}}$ *and points* $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$ *and* $\tilde{z}_*^0, \tilde{z}_*^1 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ *so that* $z_*^0, z_*^1, \tilde{z}_*^0$ *and* $\tilde{z}_*^0, \tilde{z}_*^1, z_*^1$ *are well-placed triples, there exists a constant* $C_1 > 0$*, depending upon* Ω, υ *and the above words and points only, such that the following holds: Let* $F \in \mathcal{A}$ *and let* $\underline{B}$ *be a boxing for* F*. Then there exist points* $z^0, z^1 \in B_{n+1}^{\mathbf{w}}, \tilde{z}^0, \tilde{z}^1 \in B_{n+1}^{\tilde{\mathbf{w}}}$ *such that either*

$$\operatorname{dist}(\dddot{z}_0, \dddot{z}_1) > C_1 \sigma^m \sigma^{2(n-m)} \quad \text{or} \quad \operatorname{dist}\left(\dddot{\tilde{z}}_0, \dddot{\tilde{z}}_1\right) > C_1 \sigma^m \sigma^{2(n-m)}. \tag{5.3.17}$$

Proof. Let $z^0 = z_{n+1}^0, z^1 = z_{n+1}^1$ and $\tilde{z}^0 = \tilde{z}_{n+1}^0, \tilde{z}^1 = \tilde{z}_{n+1}^1$. By Proposition 3.7.6

$$\begin{aligned} &\left|\dot{x}^1 - \dot{x}^0\right| \\ &= |\sigma_{m,n}| \left|s_{m,n}([x^1 + r_{m,n}(z^1)] - [x^0 + r_{m,n}(z^0)]) + t_{m,n}(y^1 - y^0)\right| \end{aligned} \tag{5.3.18}$$

Applying Proposition 3.3.5 we then get

$$\begin{aligned} \left|\ddot{y}^1 - \ddot{y}^0\right| &= \left|\dot{x}^1 - \dot{x}^0\right| \\ &= |\sigma_{m,n}| \left|s_{m,n}([x^1 + r_{m,n}(z^1)] - [x^0 + r_{m,n}(z^0)]) + t_{m,n}(y^1 - y^0)\right|. \end{aligned} \tag{5.3.19}$$

Then again applying Proposition 3.7.6 we have

$$\begin{aligned} &\left|\dddot{y}^1 - \dddot{y}^0\right| \\ &= |\sigma_{0,m-1}| \, |\sigma_{m,n}| \left|s_{m,n}([x^1 + r_{m,n}(z^1)] - [x^0 + r_{m,n}(z^0)]) + t_{m,n}(y^1 - y^0)\right|. \end{aligned} \tag{5.3.20}$$

By the same argument a similar expression holds for $|\dddot{\tilde{y}}^1 - \dddot{\tilde{y}}^0|$. It follows from Properties (A-3) that

$$\begin{aligned} 2C\rho^{n-m} > \Big| &\left([x^1 + r_{m,n}(z^1)] - [x^0 + r_{m,n}(z^0)]\right) \\ &- \left([\upsilon_*(x^1) + c_m(y^1)^2] - [\upsilon_*(x^0) + c_m(y^0)^2]\right)\Big| \end{aligned} \tag{5.3.21}$$

and

$$\begin{aligned} 2C\rho^{n-m} > \Big| &\left([\tilde{x}^1 + r_{m,n}(\tilde{z}^1)] - [\tilde{x}^0 + r_{m,n}(\tilde{z}^0)]\right) \\ &- \left([\upsilon_*(\tilde{x}^1) + c_m(\tilde{y}^1)^2] - [\upsilon_*(\tilde{x}^0) + c_m(\tilde{y}^0)^2]\right)\Big|. \end{aligned} \tag{5.3.22}$$

Then dividing by $|y^1 - y^0|$ and applying (A-5) gives us

$$\left|\frac{\left[x^1 + r_{m,n}\left(z^1\right)\right] - \left[x^0 + r_{m,n}\left(z^0\right)\right]}{y^1 - y^0} - \Upsilon_m\left(z^0, z^1\right)\right| < \frac{4C}{\kappa_3}\rho^{n-m} \tag{5.3.23}$$

and similarly

$$\left|\frac{\left[\tilde{x}^1 + r_{m,n}\left(\tilde{z}^1\right)\right] - \left[\tilde{x}^0 + r_{m,n}\left(\tilde{z}^0\right)\right]}{\tilde{y}^1 - \tilde{y}^0} - \Upsilon_m\left(\tilde{z}^0, \tilde{z}^1\right)\right| < \frac{4C}{\kappa_4}\rho^{n-m}. \tag{5.3.24}$$

But by Properties (A-4) and (5.2.6) this implies

$$\begin{aligned} &\frac{\kappa_0}{4} < \\ &\left|\frac{\left[x^1 + r_{m,n}\left(z^1\right)\right] - \left[x^0 + r_{m,n}\left(z^0\right)\right]}{y^1 - y^0} - \frac{\left[\tilde{x}^1 + r_{m,n}\left(\tilde{z}^1\right)\right] - \left[\tilde{x}^0 + r_{m,n}\left(\tilde{z}^0\right)\right]}{\tilde{y}^1 - \tilde{y}^0}\right| \end{aligned} \tag{5.3.25}$$

and therefore either

$$\frac{\kappa_0}{8} < \left| \frac{\left[x^1 + r_{m,n}\left(z^1\right)\right] - \left[x^0 + r_{m,n}\left(z^0\right)\right]}{y^1 - y^0} + \frac{t_{m,n}}{s_{m,n}} \right| \tag{5.3.26}$$

or

$$\frac{\kappa_0}{8} < \left| \frac{\left[\tilde{x}^1 + r_{m,n}\left(\tilde{z}^1\right)\right] - \left[\tilde{x}^0 + r_{m,n}\left(\tilde{z}^0\right)\right]}{\tilde{y}^1 - \tilde{y}^0} + \frac{t_{m,n}}{s_{m,n}} \right| \tag{5.3.27}$$

or possibly both. Now by Proposition 3.7.6 there are constants $C', C'', C''' > 0$ such that

$$|\sigma_{0,m-1}| > C'\sigma^m, \quad |\sigma_{m,n}| > C''\sigma^{n-m}, \quad |s_{m,n}| > C'''\sigma^{n-m}. \tag{5.3.28}$$

This, together with Property (A-3), equality (5.3.20) and the estimate in the previous paragraph, implies there is a constant $C_1 > 0$ such that either

$$\operatorname{dist}\left(\ddddot{z}^0, \ddddot{z}^1\right) > C_1\sigma^m\sigma^{2(n-m)} \tag{5.3.29}$$

or

$$\operatorname{dist}\left(\ddddot{\tilde{z}}^0, \ddddot{\tilde{z}}^1\right) > C_1\sigma^m\sigma^{2(n-m)}. \tag{5.3.30}$$

□

We distill these three results into the following.

Proposition 5.3.6. *For any* $\mathbf{w}, \tilde{\mathbf{w}} \in W^*$ *well chosen there exist constants* $C_0, C_1 > 0$*, depending upon* υ *and* Ω *only, such that given* $F \in \mathcal{A}$ *the following holds: for any boxing* $\underline{B}$ *satisfying property* $\mathrm{Hor}_{\mathbf{w},\tilde{\mathbf{w}}}(m,n)$ *the pieces* $B_0^{0^m 10^{n-m}\mathbf{w}}$*,* $B_0^{0^m 10^{n-m}\tilde{\mathbf{w}}} \in \underline{B}_0$ *of depth* $n + length(\mathbf{w})$ *satisfying*

$$\operatorname{dist}\left(B_0^{0^m 10^{n-m}\mathbf{w}}, B_0^{0^m 10^{n-m}\tilde{\mathbf{w}}}\right) < C_0\sigma^{2m}b^{2p^m} \tag{5.3.31}$$

and

$$\operatorname{diam}\left(B_0^{0^m 10^{n-m}\mathbf{w}}\right) \text{ or } \operatorname{diam}\left(B_0^{0^m 10^{n-m}\tilde{\mathbf{w}}}\right) > C_1\sigma^m b^{2p^m} \tag{5.3.32}$$

Proof. Propositions 5.3.4 implies

$$\operatorname{dist}\left(B_0^{0^m 10^{n-m}\mathbf{w}}, B_0^{0^m 10^{n-m}\tilde{\mathbf{w}}}\right) < C_0\sigma^m b^{p^m}\sigma^{n-m}, \tag{5.3.33}$$

while Proposition 5.3.5 implies one of

$$\operatorname{diam}\left(B_0^{0^m 10^{n-m}\mathbf{w}}\right) > C_1\sigma^m\sigma^{2(n-m)}, \quad \operatorname{diam}\left(B_0^{0^m 10^{n-m}\tilde{\mathbf{w}}}\right) > C_1\sigma^m\sigma^{2(n-m)}. \tag{5.3.34}$$

is true. However Corollary 5.3.3 implies b^{p^m} and σ^{n-m} are comparable. Hence the result follows. □

Remark 5.3.7. Observe these bounds have no dependence upon n, the height at which the overlapping boxes 'originate'. This suggests that only the overlapping distorts the geometry and not that they are close to the tip, τ_m, of F_m, which is a crucial part of our estimate.

5.4 A Condition for Horizontal Overlap

Now we wish to show that this horizontal overlapping behaviour occurs sufficiently often. Recall that in the previous section we fixed well chosen words $\mathbf{w}, \tilde{\mathbf{w}} \in W^*$ with points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}$ and $\tilde{z}_*^0, \tilde{z}_*^1 \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ so that z_*^0, z_*^1 and $\tilde{z}_*^0$ are well-placed.

Proposition 5.4.1. *Given well chosen words $\mathbf{w}, \tilde{\mathbf{w}} \in W^*$ with well placed points $z_*^0, z_*^1 \in \mathcal{O}_*^{\mathbf{w}}, \tilde{z} \in \mathcal{O}_*^{\tilde{\mathbf{w}}}$ there exist constants $0 < A_0 < A_1$, depending upon υ and Ω also, such that the following holds: given $F \in \mathcal{A}$ and any boxing $\underline{B}$, if*

$$A_0 < \frac{b_F^{p^m}}{\sigma^{n-m}} < A_1 \tag{5.4.1}$$

then property $\mathrm{Hor}_{\mathbf{w},\tilde{\mathbf{w}}}(m,n)$ is satisfied. That is, $B_m^{0^{n-m}\mathbf{w}}$ and $B_m^{0^{n-m}\tilde{\mathbf{w}}}$ horizontally overlap.

Proof. Let $z^0 = (x^0, y^0) = z_{n+1}^0, z^1 = (x^1, y^1) = z_{n+1}^1$ and $\tilde{z} = (\tilde{x}, \tilde{y}) = \tilde{z}_{n+1}^0$. As we will take m, n to be fixed integers for notational simplicity we also denote $\sigma_{m,n}, r_{m,n}, s_{m,n}, t_{m,n}, \Upsilon_m$ and c_m by $\sigma, r, s, t, \Upsilon$ and c respectively. We will still denote the limits of Υ_m and c_m by Υ_* and c_*. Observe that $B_m^{0^{n-m}\mathbf{w}}$ and $B_m^{0^{n-m}\tilde{\mathbf{w}}}$ horizontally overlap if $\dot{x}^0 < \dot{\tilde{x}} < \dot{x}^1$ or, equivalently,

$$0 < \dot{\tilde{x}}^0 - \dot{x}^0 < \dot{x}^1 - \dot{x}^0. \tag{5.4.2}$$

For $i = 0, 1$, Proposition 3.7.6 implies that

$$\dot{x}^i = \sigma(s[x^i + r(z^i)] + ty^i), \quad \dot{\tilde{x}} = \sigma(s[\tilde{x} + r(\tilde{z})] + t\tilde{y}), \tag{5.4.3}$$

and therefore

$$\dot{\tilde{x}} - \dot{x}^0 = \sigma\left(s\left([\tilde{x} + r(\tilde{z})] - \left[x^0 + r\left(z^0\right)\right]\right) + t\left(\tilde{y} - y^0\right)\right) \tag{5.4.4}$$

$$\dot{x}^1 - \dot{x}^0 = \sigma\left(s\left(\left[x^1 + r\left(z^1\right)\right] - \left[x^0 + r\left(z^0\right)\right]\right) + t\left(y^1 - y^0\right)\right). \tag{5.4.5}$$

By Property (A-5), there is a constant $C > 0$ such that

$$\begin{aligned} 2C\sigma s\rho^{n-m} > &\Big|\left[\dot{\tilde{x}} - \dot{x}^0\right] \\ &-\sigma\left(s\left(\left[v_*(\tilde{x}) - v_*\left(x^0\right)\right] + c\left[(\tilde{y})^2 - \left(y^0\right)^2\right]\right) + t\left(\tilde{y} - y^0\right)\right)\Big| \end{aligned} \tag{5.4.6}$$

$$\begin{aligned} 2C\sigma s\rho^{n-m} > &\Big|\left[\dot{x}^1 - \dot{x}^0\right] \\ &-\sigma\left(s\left(\left[v_*\left(x^1\right) - v_*\left(x^0\right)\right] + c\left[\left(y^1\right)^2 - \left(y^0\right)^2\right]\right) + t\left(y^1 - y^0\right)\right)\Big|. \end{aligned} \tag{5.4.7}$$

Hence sufficient conditions for (5.4.2) to hold are

$$0 < \sigma\left(s\left(\left[v_*(\tilde{x}) - v_*\left(x^0\right)\right] + c\left[(\tilde{y})^2 - \left(y^0\right)^2\right]\right) + t\left(\tilde{y} - y^0\right)\right) - 2C\sigma s\rho^{n-m} \tag{5.4.8}$$

and

$$\sigma\left(s\left(\left[v_*\left(\tilde{x}\right)-v_*\left(x^0\right)\right]+c\left[\left(\tilde{y}\right)^2-\left(y^0\right)^2\right]\right)+t\left(\tilde{y}-y^0\right)\right) \tag{5.4.9}$$
$$<\sigma\left(s\left(\left[v_*\left(x^1\right)-v_*\left(x^0\right)\right]+c\left[\left(y^1\right)^2-\left(y^0\right)^2\right]\right)+t\left(y^1-y^0\right)\right)-4C\sigma s\rho^{n-m}.$$

Our initial hypotheses imply $\sigma, s > 0$, and by Property (A-1) we know $\tilde{y}-y>0$, so dividing both of these inequalities by $\sigma s(\tilde{y}-y)$ and applying hypothesis (A-3) gives us

$$\frac{4C\rho^{n-m}}{\kappa_2}<\frac{2C\rho^{n-m}}{\tilde{y}-y}<\Upsilon\left(\tilde{z},z^0\right)+\frac{t}{s} \tag{5.4.10}$$

and

$$\Upsilon\left(\tilde{z},z^0\right)+\frac{t}{s}<\frac{\kappa_1}{2}\left(\Upsilon\left(z^1,z^0\right)+\frac{t}{s}\right)-\frac{4C\rho^{n-m}}{\tilde{y}-y}<\frac{\kappa_1}{2}\left(\Upsilon\left(z^1,z^0\right)+\frac{t}{s}\right)-\frac{8C\rho^{n-m}}{\kappa_2} \tag{5.4.11}$$

Hence if

$$\frac{4C\rho^{n-m}}{\kappa_2}<\Upsilon\left(\tilde{z},z^0\right)+\frac{t}{s} \tag{5.4.12}$$

and

$$\Upsilon\left(\tilde{z},z^0\right)+\frac{t}{s}<\frac{\kappa_1}{2}+\frac{t}{s}<\frac{\kappa_1}{2}\left(\Upsilon\left(z^1,z^0\right)+\frac{t}{s}\right)-\frac{8C\rho^{n-m}}{\kappa_2} \tag{5.4.13}$$

then (5.4.2) is satisfied and so there is horizontal overlap. Now let us show that there exists constants $0<A_0<A_1$ such that (5.4.1) implies inequalities (5.4.12) and (5.4.13). Let us treat inequality (5.4.12) first. We claim that

$$\frac{ab^{p^m}}{\sigma^{n-m}}<\Upsilon_*\left(\tilde{z}_*,z_*^0\right)-\delta \tag{5.4.14}$$

implies (5.4.12). By Property (5.2.5),

$$\left|\frac{t}{s}\right|<\frac{ab^{p^m}}{\sigma^{n-m}}+\frac{\delta}{3} \tag{5.4.15}$$

and by Property (5.2.4),

$$\Upsilon_*\left(\tilde{z}_*,z_*^0\right)<\Upsilon\left(\tilde{z},z^0\right)+\frac{\delta}{3}. \tag{5.4.16}$$

Combining these gives us

$$\left|\frac{t}{s}\right|<\Upsilon\left(\tilde{z},z^0\right)-\frac{\delta}{3}. \tag{5.4.17}$$

By Property (A-6) and Property (A-2) we know $\frac{8C\rho^{n-m}}{\kappa_2}<\frac{\delta}{3}$. Hence

$$\left|\frac{t}{s}\right|<\Upsilon\left(\tilde{z},z^0\right)-\frac{8C\rho^{n-m}}{\kappa_2}. \tag{5.4.18}$$

Finally recall that $t/s < 0$, so multiplying by -1 and reversing the above inequality gives (5.4.12) as required. Next we claim that

$$\frac{\Upsilon_*\left(\tilde{z}_*, z_*^0\right) - \frac{\kappa_1}{2}\Upsilon_*\left(z_*^1, z_*^0\right)}{1-\frac{\kappa_1}{2}} + \frac{\delta}{3}\frac{2}{1-\frac{\kappa_1}{2}} < \frac{ab^{p^m}}{\sigma^{n-m}} - \frac{\delta}{3} \tag{5.4.19}$$

implies inequality (5.4.13). From Property (A-6) we know that $\frac{8C\rho^{n-m}}{\kappa_2\left(1-\frac{\kappa_1}{2}\right)} < \frac{\delta}{3}$ and from Property (5.2.4) we know

$$\begin{aligned}\frac{\Upsilon\left(\tilde{z}, z^0\right) - \frac{\kappa_1}{2}\Upsilon\left(z^1, z^0\right)}{1-\frac{\kappa_1}{2}} &< \frac{\left[\Upsilon_*\left(\tilde{z}_*, z_*^0\right) + \frac{\delta}{3}\right] - \frac{\kappa_1}{2}\left[\Upsilon_*\left(z_*^1, z_*^0\right) - \frac{\delta}{3}\right]}{1-\frac{\kappa_1}{2}} \\ &= \frac{\Upsilon_*\left(\tilde{z}_*, z_*^0\right) - \frac{\kappa_1}{2}\Upsilon_*\left(z_*^1, z_*^0\right)}{1-\frac{\kappa_1}{2}} + \frac{\delta}{3}\left(\frac{1+\frac{\kappa_1}{2}}{1-\frac{\kappa_1}{2}}\right)\end{aligned} \tag{5.4.20}$$

Together these imply

$$\frac{\Upsilon\left(\tilde{z}, z^0\right) - \frac{\kappa_1}{2}\Upsilon\left(z^1, z^0\right)}{1-\frac{\kappa_1}{2}} + \frac{8C\rho^{n-m}}{\kappa_2\left(1-\frac{\kappa_1}{2}\right)} < \frac{\Upsilon_*\left(\tilde{z}_*, z_*^0\right) - \frac{\kappa_1}{2}\Upsilon_*\left(z_*^1, z_*^0\right)}{1-\frac{\kappa_1}{2}} + \frac{\delta}{3}\frac{2}{1-\frac{\kappa_1}{2}}. \tag{5.4.21}$$

By Property (5.2.5) we know

$$\frac{ab^{p^m}}{\sigma^{n-m}} - \frac{\delta}{3} < \left|\frac{t}{s}\right| \tag{5.4.22}$$

so the above two inequalities (5.4.21) and (5.4.22) imply

$$\frac{\Upsilon\left(\tilde{z}, z^0\right) - \frac{\kappa_1}{2}\Upsilon\left(z^1, z^0\right)}{1-\frac{\kappa_1}{2}} + \frac{8C\rho^{n-m}}{\kappa_2\left(1-\frac{\kappa_1}{2}\right)} < \left|\frac{t}{s}\right|. \tag{5.4.23}$$

Since $1-\frac{\kappa_1}{2} > 0$, this is equivalent to

$$\Upsilon\left(\tilde{z}, z^0\right) - \frac{\kappa_1}{2}\Upsilon\left(z^1, z^0\right) < \left|\frac{t}{s}\right|(1-\frac{\kappa_1}{2}) - \frac{8C\rho^{n-m}}{\kappa_2}. \tag{5.4.24}$$

Recalling that $t/s < 0$ then tells us

$$\frac{t}{s}(1-\frac{\kappa_1}{2}) + \frac{8C\rho^{n-m}}{\kappa_2} < \frac{\kappa_1}{2}\Upsilon\left(z^1, z^0\right) - \Upsilon\left(\tilde{z}, z^0\right). \tag{5.4.25}$$

which, upon rearranging, gives us

$$\Upsilon\left(\tilde{z}, z^0\right) + \frac{t}{s} + \frac{8C\rho^{n-m}}{\kappa_2} < \frac{\kappa_1}{2}\left(\Upsilon\left(z^1, z^0\right) + \frac{t}{s}\right) \tag{5.4.26}$$

which, by moving the error term to the right of the inequality sign, gives us inequality (5.4.13) as required. Finally set

$$A_0 = a^{-1}\left[\left(\frac{\Upsilon_*\left(\tilde{z}_*, z_*^0\right) - \frac{\kappa_1}{2}\Upsilon_*\left(z_*^1, z_*^0\right)}{1-\frac{\kappa_1}{2}}\right) + \frac{\delta}{3}\left(\frac{3-\frac{\kappa_1}{2}}{1-\frac{\kappa_1}{2}}\right)\right] \tag{5.4.27}$$

$$A_1 = a^{-1}\left[\Upsilon_*\left(\tilde{z}_*, z_*^0\right) - \delta\right]. \tag{5.4.28}$$

The interval $[A_0, A_1]$ is well defined by Property (A-5.2.5). Then inequality (5.4.1) implies, since $a > 0$, together with (5.4.14) and (5.4.19) that inequalities (5.4.12) and inequality (5.4.13) hold and therefore the boxes overlap. □

5.5 Construction of the Full Measure Set

We will now prove the following result which will show that set of parameters satisfying our overlap condition is large.

Theorem 5.5.1. *Given any $0 < A_0 < A_1$, $0 < \sigma < 1$ and any $p \geq 2$ the set of parameters $b \in [0,1]$ for which there are infinitely many $0 < m < n$ satisfying*

$$A_0 < \frac{b^{p^m}}{\sigma^{n-m}} < A_1 \tag{5.5.1}$$

is a dense G_δ set with full Lebesgue measure.

Remark 5.5.2. We note that this result is purely analytical; it has no dynamical content and as such is quite separate from the other sections.

We introduce the following notation, setting

$$d = n - m; \quad \delta = \delta(m) = 1/p^m; \quad \alpha_i = \log A_i / \log \sigma = \log_\sigma A_i. \tag{5.5.2}$$

and letting $I_{d,\delta}$ be the set of b which satisfy inequality (5.5.1). That is

$$I_{d,\delta} = \left[\sigma^{d\delta} A_0^\delta, \sigma^{d\delta} A_1^\delta\right]. \tag{5.5.3}$$

The following two lemmas are an easy calculation and are left to the reader.

Lemma 5.5.3. *(i)* $\operatorname{diam}(I_{d,\delta}) = \sigma^{d\delta}(A_1^\delta - A_0^\delta)$.

(ii) If $I_{d+1,\delta}, I_{d,\delta}$ are disjoint then $I_{d+1,\delta}$ lies to the left of $I_{d,\delta}$.

(iii) If $I_{d',\delta'}, I_{d,\delta}$ are disjoint and $I_{d',\delta'}$ lies to the left of $I_{d,\delta}$ then

$$\operatorname{dist}(I_{d,\delta}, I_{d',\delta'}) = \sigma^{d\delta}(A_0^\delta - \sigma^{d'\delta' - d\delta} A_1^{\delta'}).$$

Remark 5.5.4. In the proof of Proposition 5.5.9 we will see there is a dichotomy: either, for a fixed $\delta > 0$, $I_{d,\delta}, I_{d+1,\delta}$ are always disjoint or they always intersect, for all $d > 0$, and moreover if property holds for one δ then it also holds for every choice of δ. This depends on whether $A_1\sigma < A_0$ holds or not.

Lemma 5.5.5. *Let $I_{d,\delta}, I_{d',\delta'}, I_{d'',\delta''}$ be pairwise disjoint and assume $I_{d',\delta'}$ lies to the left of $I_{d,\delta}$. Then $I_{d'',\delta''}$ lies to the right of $I_{d',\delta'}$ when*

$$d'' \leq \frac{\delta'}{\delta''}(d' + \alpha_1) - \alpha_0 \tag{5.5.4}$$

and $I_{d'',\delta''}$ lies to the left of $I_{d,\delta}$ when

$$d'' \geq \frac{\delta}{\delta''}(d + \alpha_0) - \alpha_1. \tag{5.5.5}$$

Lemma 5.5.6. *Suppose $I_{d,\delta}, I_{d',\delta'}$ are disjoint and $I_{d',\delta'}$ lies to the left of $I_{d,\delta}$. Let $0 < \delta'' < \min(\delta, \delta')$. Let $d''_{min} \le d'' \le d''_{max}$ be the range of all d'' for which $I_{d'',\delta''}$ lies strictly between $I_{d,\delta}$ and $I_{d',\delta'}$. If the $I_{d'',\delta''}$ are pairwise disjoint then*

$$\left| \bigcup_{d''=d''_{min}}^{d''_{max}} I_{d'',\delta''} \right| = (A_1^{\delta''} - A_0^{\delta''}) \frac{\sigma^{d''_{min}\delta''} - \sigma^{(d''_{max}+1)\delta''}}{1 - \sigma^{\delta''}} \tag{5.5.6}$$

Proof. If the $I_{d'',\delta''}$ are pairwise disjoint then

$$\left| \bigcup_{d''=d''_{min}}^{d''_{max}} I_{d'',\delta''} \right| = \sum_{d''=d''_{min}}^{d''_{max}} |I_{d'',\delta''}|. \tag{5.5.7}$$

Consequently, Lemma 5.5.3 and the summation formula for geometric series implies the result. □

Remark 5.5.7. By Lemma 5.5.5 we know that d''_{max} and d''_{min} have the form

$$d''_{max} = \lfloor \frac{\delta'}{\delta''}(d' + \alpha_1) - \alpha_0 \rfloor; \quad d''_{min} = \lceil \frac{\delta}{\delta''}(d + \alpha_0) - \alpha_1 \rceil. \tag{5.5.8}$$

Lemma 5.5.8. *Assume $\sigma A_1 < A_0$. Then there exists a constant $0 < L \le 1$ such that the following holds: choose any admissable $\delta, \delta', d, d' > 0$ such that $I_{d,\delta}$, and $I_{d',\delta'}$ are disjoint and $I_{d',\delta'}$ lies to the left of $I_{d,\delta}$. Then there exists a $\bar{\delta} < \delta, \delta'$ such that for any admissable $0 < \delta'' = \delta(m'') < \bar{\delta}$,*

$$L \operatorname{dist}(I_{d,\delta}, I_{d',\delta'}) < \sum_{d''=d''_{min}}^{d''_{max}} |I_{d'',\delta''}|. \tag{5.5.9}$$

Moreover we can take $L = \frac{1}{4} \left| \frac{1}{\log \sigma} \right| \left(1 - \frac{A_0}{A_1}\right) \le 1$.

Proof. First observe that

$$\operatorname{dist}(I_{d,\delta}, I_{d',\delta'}) = A_0^{\delta} \sigma^{d\delta} - A_1^{\delta'} \sigma^{d'\delta'} \tag{5.5.10}$$

and

$$\sum_{d''=d''_{min}}^{d''_{max}} |I_{d'',\delta''}| = (A_1^{\delta''} - A_0^{\delta''}) \frac{\sigma^{d''_{min}\delta''} - \sigma^{(d''_{max}+1)\delta''}}{1 - \sigma^{\delta''}}. \tag{5.5.11}$$

We wish to approximate this last quantity. By Lemma 5.5.5 we know that

$$\delta(d + \alpha_0) - \alpha_1 \delta'' < d''_{min} \delta'' < \delta(d + \alpha_0) - \alpha_1 \delta'' + \delta'' \tag{5.5.12}$$

and

$$\delta'(d' + \alpha_1) - \delta'' \alpha_0 < (d''_{max} + 1)\delta'' < \delta'(d' + \alpha_1) - \delta'' \alpha_0 + \delta''. \tag{5.5.13}$$

Hence

$$A_0^\delta\sigma^{\delta d}\frac{\sigma^{\delta''}}{A_1^{\delta''}} - A_1^{\delta'}\sigma^{\delta' d'}\frac{1}{A_0^{\delta''}} < \sigma^{d''_{min}\delta''} - \sigma^{(d''_{max}+1)\delta''} < A_0^\delta\sigma^{\delta d}\frac{1}{A_1^{\delta''}} - A_1^{\delta'}\sigma^{\delta' d'}\frac{\sigma^{\delta''}}{A_0^{\delta''}}. \tag{5.5.14}$$

We also know, by the Mean Value Theorem and the concavity of $x \mapsto x^\delta$ for $\delta < 1$, that

$$\delta'' A_1^{\delta''-1}\frac{A_1 - A_0}{1-\sigma^{\delta''}} < \frac{A_1^{\delta''} - A_0^{\delta''}}{1-\sigma^{\delta''}} < \delta'' A_0^{\delta''-1}\frac{A_1 - A_0}{1-\sigma^{\delta''}}. \tag{5.5.15}$$

Together these imply

$$K\left(A_0^\delta\sigma^{\delta d}\sigma^{\delta''} - A_1^{\delta'}\sigma^{\delta' d'}\frac{A_1^{\delta''}}{A_0^{\delta''}}\right) < \sum_{d''=d''_{min}}^{d''_{max}} |I_{d'',\delta''}| \tag{5.5.16}$$

where

$$K = K(\delta'') = \left(1 - \frac{A_0}{A_1}\right)\left(\frac{\delta''}{1-\sigma^{\delta''}}\right). \tag{5.5.17}$$

Now observe that $\sigma A_1 < A_0$ implies

$$A_0^\delta\sigma^{\delta d}\sigma^{\delta''} - A_1^{\delta'}\sigma^{\delta' d'}\sigma^{-\delta''} < A_0^\delta\sigma^{\delta d}\sigma^{\delta''} - A_1^{\delta'}\sigma^{\delta' d'}\frac{A_1^{\delta''}}{A_0^{\delta''}}. \tag{5.5.18}$$

Therefore Lemma A.1.6 tells us, substituting $A_0^\delta\sigma^{\delta d}, A_1^{\delta'}\sigma^{\delta' d'}$ and δ' for P, Q and s respectively, there exists a constant $\delta_0 > 0$ such that for all $\delta'' < \delta_0$,

$$\frac{1}{2} < \frac{A_0^\delta\sigma^{\delta d}\sigma^{\delta''} - A_1^{\delta'}\sigma^{\delta' d'}(A_1/A_0)^{\delta''}}{A_0^\delta\sigma^{\delta d} - A_1^{\delta'}\sigma^{\delta' d'}}. \tag{5.5.19}$$

Also observe that, by l'Hôpital's rule,

$$\lim_{\delta''\to 0}\frac{\delta''}{1-\sigma^{\delta''}} = \lim_{\delta''\to 0} -\frac{1}{\sigma^{\delta''}\log\sigma} = \left|\frac{1}{\log\sigma}\right|, \tag{5.5.20}$$

and hence there exists a constant $\delta_1 > 0$ such that for all $\delta'' < \delta_1$

$$K(\delta'') = \left(1 - \frac{A_0}{A_1}\right)\left(\frac{\delta''}{1-\sigma^{\delta''}}\right) > \frac{1}{2}\left|\frac{1}{\log\sigma}\right|\left(1 - \frac{A_0}{A_1}\right). \tag{5.5.21}$$

Therefore, if we let $\bar{\delta} = \min_{i=0,1}\delta_i$, inequalities (5.5.19) and (5.5.21) tell us that for any $\delta'' < \bar{\delta}$,

$$\frac{1}{4}\left|\frac{1}{\log\sigma}\right|\left(1 - \frac{A_0}{A_1}\right)\operatorname{dist}(I_{d,\delta}, I_{d',\delta'}) < K(\delta'')\left(A_0^\delta\sigma^{\delta d}\sigma^{\delta''} - A_1^{\delta'}\sigma^{\delta' d'}\frac{A_1^{\delta''}}{A_0^{\delta''}}\right). \tag{5.5.22}$$

Therefore by inequality (5.5.16) the Proposition follows. □

Proposition 5.5.9. *There exists a dense G_δ subset of $[0, b_0]$ with full relative Lebesgue measure such that each point lies in infinitely many $I_{d,\delta}$.*

Proof. There are two cases. The first is when $A_1\sigma \geq A_0$. Then

$$\left(A_0\sigma^{d+1}\right)^\delta < \left(A_0\sigma^d\right)^\delta \leq \left(A_1\sigma^{d+1}\right)^\delta < \left(A_1\sigma^d\right)^\delta, \tag{5.5.23}$$

that is, the right endpoint of $I_{d+1,\delta}$ lies to the right of the left endpoint of $I_{d,\delta}$. Therefore $I_{d+1,\delta}$ and $I_{d,\delta}$ overlap for all $d, \delta > 0$. Hence for each point $x \in (0, b_0]$ and any admissible $\delta > 0$ there exists an integer $d = d(x, \delta) > 0$ such that $x \in I_{d(x,\delta),\delta}$. Therefore x lies in infinitely many $I_{d,\delta}$ and clearly $(0, b_0]$ is a dense G_δ with full relative Lebesgue measure in $[0, b_0]$.

The second case is when $A_1\sigma < A_0$. Then observe that $I_{d+1,\delta}$ and $I_{d,\delta}$ will be pairwise disjoint for all $d, \delta > 0$. For any such pair let

$$J_{d,\delta} = \left[\left(\sigma^{d+1}A_1\right)^\delta, \left(\sigma^d A_0\right)^\delta\right] \tag{5.5.24}$$

denote the corresponding gap. The idea is to construct an infinite sequence of full measure sets, each a countable union of intervals $I_{d,\delta}$. We do this by the following inductive process. For a given δ we take the union of all $I_{d,\delta}$, this gives us gaps which we fill with $I_{d',\delta'}$, which leads to further gaps and so on. We can fill these gaps by a definite amount each time by Lemma 5.5.8. Hence the resulting set will have full Lebesgue measure.

Now let us proceed with the proof. First let us introduce the following notation. Given a union $T \subset [0, b_0]$ of disjoint intervals we will denote by T_δ the union of all $J_{d,\delta}$ strictly contained in T. We will use the notation $T_{\delta,\delta'} = (T_\delta)_{\delta'}$, $T_{\delta,\delta'\delta''} = (T_{\delta,\delta'})_{\delta''}$, and so on. We will denote the complement of $T_{\delta,\delta',\ldots}$ by $S_{\delta,\delta',\ldots}$.

Let $0 < b_1 < b_0$. We will show that there is a dense G_δ subset of full relative Lebesgue measure in $[b_1, b_0]$ with the required properties and then send b_1 to zero. Therefore let $T = [b_1, b_0]$. Let $\Delta = \{\delta(m)\}_{m\in\mathbb{N}}$ denote the set of all admissible δ's ordered decreasingly. Let us construct an infinite subset Δ_0 of Δ with infinite complement as follows. First choose $\delta_0^{(0)}$ to be arbitrary. Assume $\Delta_0^{(n)} = \{\delta_0, \ldots, \delta^{(n)}\}$ is given. Then Lemma 5.5.8 tells us there is a $\delta > 0$ such that for any $\delta_0^{(n+1)} < \delta$,

$$\left|T_{\delta_0,\ldots,\delta_0^{(n)},\delta_0^{(n+1)}}\right| < (1 - L_0)\left|T_{\delta_0,\ldots,\delta_0^{(n)}}\right|. \tag{5.5.25}$$

where L_0 is the contraction constant given by the same Lemma. We may do this as there are only finitely many gaps in $T_{\delta_0,\ldots,\delta_0^{(n-1)}}$. It is clear that by this process we can choose the $\Delta_0^{(n)}$ such that their limit Δ_0 has complement with infinite cardinality. Also observe that, inductively

$$\left|T_{\delta_0,\ldots,\delta_0^{(n)},\delta_0^{(n+1)}}\right| < (1 - L_0)^{n+1}|T|, \tag{5.5.26}$$

so the limiting set T_0 will have zero measure since $0 < L_0 < 1$. Hence its complement, S_0, which is a dense countable union of open intervals by construction, will have full relative Lebesgue measure.

Now assume we are given pairwise disjoint subsets $\Delta_0, \ldots, \Delta_N \subset \Delta$ whose union has infinite cardinality and we have the subsets $T_0, \ldots T_N \subset T$. Construct $\Delta_{N+1} = \{\delta^{(n)}_{N+1}\}_{n\in\mathbb{N}} \subset \Delta$ disjoint from all these sets such that

$$\left|T_{\delta_{N+1},\ldots,\delta^{(n-1)}_{N+1},\delta^{(n)}_{N+1}}\right| < (1 - L_0)\left|T_{\delta_{N+1},\ldots,\delta^{(n)}_{N+1}}\right| \tag{5.5.27}$$

for all $n > 0$ and such that the union of $\Delta_0, \ldots, \Delta_N, \Delta_{N+1}$ has complement with infinite cardinality. We can do this by the same argument as in the preceding paragraph. Also by the preceding paragraph it is clear that $T_{N+1} = \lim_{n\to\infty} T_{\delta^{(0)}_{N+1},\ldots,\delta^{(n)}_{N+1}}$ has zero measure and its complement S_{N+1} is a dense countable union of open intervals with full relative Lebesgue measure. Therefore we construct a sequence of subsets $S_0, \ldots, S_n, \ldots \subset T$ which are dense countable unions of open intervals with full relative Lebesgue measure, implying their common intersection $S = \bigcup_{n\geq 0} S_n$ is a dense G_δ with full relative Lebesgue measure.

Now let us show that any $x \in S$ is contained in infinitely many $I_{d,\delta}$'s. For each $n \geq 0$, x is contained S_n. But S_n is the union of $I_{d,\delta}$'s with $\delta \in \Delta_n$ and so x lies in one of these. Since the Δ_n are pairwise disjoint, if $x \in I_{d_n,\delta_n} \cap I_{d_m,\delta_m}$ for $\delta_n \in \Delta_n, \delta_m \in \Delta_m, m \neq n$ then $\delta_n \neq \delta_m$. Hence x is contained in infinitely many $I_{d,\delta}$'s. □

5.6 Proof of the Main Theorem

All the result so far have been for individual maps $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$. We will need the following lemma to make these statements about single maps applicable to one parameter families parametrised by b.

Lemma 5.6.1. *Let $F_b \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ be a one-parameter family parametrised by the average Jacobian $b = b(F_b) \in [0, b_0)$. Then there is an $N > 0$ and $0 < b_1 < b_0$ such that $\mathcal{R}^N F_b \in \mathcal{A}$ for all $b \in [0, b_1]$.*

Proof. The set $\mathcal{A}$ is an open neighbourhood of F_* in the closure of $\mathcal{H}_\Omega$. We know that $\operatorname{dist}(\mathcal{R}^n F_b, F_*) < \rho^n \operatorname{dist}(F_b, F_*)$, where dist denotes the adapted metric. Therefore there is an $N > 0$ such that $\mathcal{R}^n F_b \in \mathcal{A}$ for all integers $n > N$. □

We are now in a position to prove the main theorem of this chapter.

Theorem 5.6.2. *Let F_b be a one-parameter family, parametrised by the average Jacobian $b = b(F_b) \in [0, b_0)$, of infinitely renormalisable Hénon-like maps. Then there is a subinterval $[0, b_1] \subset [0, b_0)$ for which there exists a dense G_δ-subset $S \subset [0, b_1)$ with full relative Lebesgue measure such that the Cantor set $\mathcal{O}(b) = \mathcal{O}(F_b)$ has unbounded geometry for all $b \in S$.*

Proof. By Lemma 5.6.1 there is an integer $N > 0$ and a $b_1 > 0$ such that $\mathcal{R}^n F_b \in \mathcal{A}$ for all $n > N, b \in [0, b_1]$. Let $\tilde{F}_b = \mathcal{R}^N F_b$.

Proposition 5.4.1 implies if $\tilde{F}_b \in A$ then for every b satisfying inequality (5.4.1), $\tilde{F}_b$ has property $\mathrm{Hor}_{\mathbf{w},\tilde{\mathbf{w}}}(m, n)$. By Theorem 5.5.1 the set, $\tilde{S}$, of parameters b for which $\mathrm{Hor}_{\mathbf{w},\tilde{\mathbf{w}}}(m, n)$ is satisfied for infinitely many m, n has full Lebesgue measure. But then by Proposition 5.3.6 if b lies in this set then $\tilde{F}_b$ has unbounded geometry.

Now we retrieve the statement for F_b as follows. First observe that mapping $\mathcal{O}(\tilde{F}_b)$ under $\Psi_{0,N}(F_b)$ we get a subset of $\mathcal{O}(F_b)$. The maps $\Psi_{0,N}(F_b)$ have bounded distortion by Proposition 3.7.6. Hence if $\mathcal{O}(\tilde{F}_b)$ has unbounded geometry so will $\mathcal{O}(F_b)$. Secondly we need to show

$$S \subset \Big\{ b : \mathcal{O}(\tilde{F}_b) \text{ has unbounded geometry} \Big\} \tag{5.6.1}$$

is a dense G_δ with full relative Lebesgue measure. This follows as $b(\tilde{F}_b) = b^{p^N}$, but $b \mapsto b^{p^N}$ preserves these properties, so by comparability and injectivity the map $b(F_b) \mapsto b(\tilde{F}_b)$ must also preserve these properties. Since $\tilde{S}$ is a dense G_δ with full relative Lebesgue measure S must also. □

Chapter 6

Directions for Further Research

We have seen that there similarities and differences between the renormalisation pictures for unimodal maps and Hénon-like maps. Here we will discuss what this may imply. This first collection of problems is related to the construction of our renormalisation operator. The underpinning theme in these questions is if we can find more renormalisation operators which tell us more dynamical information about Hénon-like maps, particularly about invariant sets.

(i) The horizontal diffeomorphism H is an important part of our renormalisation operator as it acts as a 'straightening map', taking the first return map to a Hénon-like map on a square about the diagonal. Can we find another straightening map for which the associated renormalisation operator behaves differently or do all reasonable renormalisations behave in the same way? In some sense the horizontal diffeomorphism is defined on a vertical strip, not a square and it seems that the expansion rate of the straightening map in the vertical direction could play a role.

(ii) If all reasonable straightening maps give us the same renormalisation picture, do they give us the same stable and local unstable manifolds of the renormalisation fixed point? Let $\mathcal{R}$ and $\mathcal{R}'$ be two different renormalisation operators. First, assume the stable manifolds for two different renormalisations intersect. Given an F in their intersection are the Cantor sets, under each operator, the same? Second, assume the stable manifolds do not intersect. Then we would like to classify dynamically the obstructions to a map being renormalisable with respect to one operator but not the other.

(iii) Using the horizontal diffeomorphism H we constructed the pre-renormalisation G and restricted it to the central box B^0_{diag}. We could also have considered the pre-renormalisation restricted to another box B^w_{diag}. Does the same renormalisation picture hold for the renormalisation operators defined in this way? In [4] this was investigated for the period-doubling unimodal renormalisation operator. In this case it is clear that the fixed point for the central interval renormalisation operator induces a fixed point for the renormalisation operator on the other interval as the renormalisations are just affine rescalings of the first return map. However, as our renormalisation operator uses non-affine coordinate changes the same property for Hénon-like maps is not obvious.

(iv) Given two unimodal permutations υ_1 and υ_2 in the unimodal case there is a unimodal permutation υ such that $\mathcal{R}_{\mathcal{U},\upsilon_1} \circ \mathcal{R}_{\mathcal{U},\upsilon_2} = \mathcal{R}_{\mathcal{U},\upsilon}$ (renormalising once at a deep level coincides with renormalising twice at shallower levels). Again this is because the coordinate changes between the renormalisation and the first return map are affine. However in the Hénon-like case a choice is made concerning the domain of the pre-renormalisation and it is not clear that when renormalising once at a deep level and twice at shallower levels these domains will match up. What seems to be likely is that we instead need to define the notion of a germ of renormalisation, where two renormalisations are equivalent if the domains of their pre-renormalisations overlap (note that taking the largest such domain may not yield a Hénon-like map).

An issue which touches on the problems above is how our renormalisation operator, which we could call the 'dynamic' or 'analytic' renormalisation operator, relates to the 'topological' Hénon renormalisation operator defined in [12] and the 'geometric' renormalisation operator, or class of operators, also suggested there. The topological renormalisation is defined in terms of stable and unstable manifolds of fixed points. The geometric renormalisation requires a horizontal and vertical foliation to be given, then the Hénon-like maps are those sending vertical leaves to horizontal leaves to parabolic leaves. The renormalisation then requires a 'straightening map', such as the horizontal diffeomorphism, to ensure the renormalisation has the same property. It seems that these two operators will play a larger role when we increase the average Jacobian beyond the strongly dissipative threshold.

The second collection of problems all concern themselves with the extendibility of renormalisation outside of the strongly dissipative maps. When considering only our renormalisation operator we note that there are three confluent issues here: the critical locus, the distortion of the horizontal diffeomorphism and the existence of an invariant domain. The problem with the first is that when the critical curve develops a 'kink' or when the connected components of the critical locus cross it becomes more difficult to find a domain on which we can define the pre-renormalisation. The distortion of horizontal diffeomorphism is related more to the contraction property: if we start with a thickening of size

$\bar{\varepsilon}$ the renormalisation will have thickening of size no greater than $C\bar{\varepsilon}^p$. Here the constant C is a bound on this distortion, so if we allow the distortion to increase we may no longer get a super-exponential contraction. The existence of invariant domains is self explanatory but we would not that this seems very unlikely to happen in the conservative case, so maybe a new notion of renormalisation is necessary in this case or close to this case.

(i) Can the renormalisation operator we constructed be extended outside the space of strongly dissipative Hénon-like maps and up to the space of conservative Hénon-like maps?

(ii) Let $\mathcal{W}_\upsilon$ denote the stable manifold of the renormalisation fixed point of type υ. For $p > 1$ consider the collection of all $\mathcal{W}_\upsilon$ such that υ is of length p. We would like to know if and where any of the $\mathcal{W}_\upsilon$ intersect as we increase the average Jacobian.

(iii) Can the renormalisation horseshoe be extended throughout the space of strongly dissipative maps or is there a threshold where the unimodal horseshoe degenerates? Is this threshold $\bar{\varepsilon} = 0$?

(iv) Similarly, can the lamination in the space of unimodal maps constructed by Lyubich be extended to the space of Hénon-like maps?

The final collection of problems all come from the study in the last two chapters of universal and rigid phenomena for infinitely renormalisable maps on their renormalisation Cantor sets.

(i) Can we find a canonical point of the Cantor set of an infinitely renormalisable Hénon-like, different from the tip, where universality does not hold. Can we find

(ii) Can we also find a canonical point of the Cantor set of an infinitely renormalisable Hénon-like, different from the tip, where rigidity does hold. More specifically can we find a pair of infinitely renormalisable Hénon-like maps, F and $\tilde{F}$, and an address $\mathbf{w} \in \overline{W}$ for which there is a C^1-conjugacy $\pi\colon \mathcal{O} \to \tilde{\mathcal{O}}$ which sends $\mathcal{O}^{\mathbf{w}}$ to $\tilde{\mathcal{O}}^{\mathbf{w}}$?

(iii) If we remove the restriction that tips are preserved by conjugacy does a form of rigidity hold?

(iv) If the average Jacobians of two infinitely renormalisable Hénon-like maps are equal is there a C^1-conjugacy between their Cantor sets? Can this be extended to a higher degree of smoothness?

(v) Does there exist an infinitely renormalisable Hénon-like map whose Cantor set has bounded geometry, either locally around the tip or globally? Our proof of almost everywhere unbounded geometry showed that if $A_1\sigma \geq A_0$ then there cannot be a strongly dissipative map

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Appendices

Appendix A

Elementary Results

A.1 Some Estimates

In this section we simply connect together several elementary analytic results that are used in various places throughout our work. We state them here for completeness and because many are required in several independent proofs. They will be given without proof when we think the proofs are straightforward. Apart from the final two results everything may be seen as a study of the interplay between the exponential expansion that could exist for a fixed unimodal map and the super-exponential contractions that are achieved by thickening them.

Proposition A.1.1. *Let $C > 0$ and $0 \leq \rho \leq \delta < 1$. Then the product $\prod_{i=0}^{\infty}(1 + C\rho^i)$ converges and, moreover there exists a $C_0 > 0$ such that*

$$\prod_{i=m}^{\infty}(1 + C\rho^i) < 1 + C_0\rho^m \tag{A.1.1}$$

Proof. First let us show convergence. Observe that concavity of log implies $\log(1 + C\rho^i) < \log(1) + \log'(1)C\rho^i = C\rho^i$. Therefore taking logarithms gives

$$\log\left[\prod_{0}^{n}(1 + C\rho^i)\right] \leq \sum_{i=0}^{n}\log(1 + C\rho^i) \leq C\sum_{i=0}^{n}\rho^i \leq \frac{C}{1-\rho}. \tag{A.1.2}$$

Therefore, since the partial convergents are increasing, Bolzano-Weierstrass implies $\log\left[\prod_{i=0}^{\infty}(1 + C\rho^i)\right]$ exists. Hence, applying exp gives us convergence.

Now let $F_{m,n}(\rho) = \prod_{i=m}^{n}(1 + C\rho^i)$. Observe that, by the product rule,

$$\frac{d}{d\rho}F_{m,n}(\rho) = \prod_{i=m}^{n}(1 + C\rho^i)\sum_{i=m}^{n}\frac{Ci\rho^{i-1}}{1 + C\rho^i} \tag{A.1.3}$$

$$= CF_{m,n}(\rho)\rho^{m-1}\sum_{i=0}^{n-m}\frac{(m+i)\rho^i}{1 + C\rho^i} \tag{A.1.4}$$

but since $C, \rho > 0$,

$$\sum_{i=0}^{n-m} \frac{(m+i)\rho^i}{1+C\rho^i} \le m \sum_{i=0}^{n-m} \rho^i + \rho \sum_{i=0}^{n-m} i\rho^{i-1} \tag{A.1.5}$$
$$\le m \sum_{i=0}^{\infty} \rho^i + \rho \frac{d}{d\rho}\left(\sum_{i=0}^{\infty} \rho^i\right)$$
$$\le \frac{m}{1-\rho} + \frac{\rho}{(1-\rho)^2}$$

So, setting $M = CF_{m,n}(\delta)\left(\frac{m}{1-\delta} + \frac{\delta}{(1-\delta)^2}\right)$ and $G_m(\rho) = (1 + \frac{M}{m}\rho^m)$, we find

$$\frac{d}{d\rho} F_{m,n}(\rho) \le M\rho^{m-1} \le \frac{d}{d\rho} G_m(\rho). \tag{A.1.6}$$

Hence, as $F_{m,n}(0) = 0 = G_m(0)$ the result follows by setting $C_0 = M/m$. □

Lemma A.1.2. *Let $C > 0$ and $0 < \rho < 1$. Then there exists a constant $C_0 > 0$ such that*

$$\frac{1+C\rho}{1-C\rho} < 1 + C_0\rho^2. \tag{A.1.7}$$

Lemma A.1.3. *Given constants $0 \le \bar{\varepsilon}, \rho, \sigma < 1$ and $C_0, C_1 > 0$ and a fixed integer $p > 1$ there exists a constant $C > 0$ such that for all integers $0 < m < M$,*

- *(i)* $C_0\bar{\varepsilon}^{p^m} + C_1\bar{\varepsilon}^{p^{m+1}} \le C\bar{\varepsilon}^{p^m}$;
- *(ii)* $C_0\bar{\varepsilon}^{p^m} + C_1\rho^m \le C\rho^m$
- *(iii)* $\sum_{m<n<M} \sigma^{i-m-1}\bar{\varepsilon}^{p^n - p^m}(1 + C_0\rho^n) < C$
- *(iv)* $\sum_{n>M} \bar{\varepsilon}^{p^n} \le C\bar{\varepsilon}^{p^M}$

Proposition A.1.4. *Given any $\rho > 0$ there exists a $\overline{\varepsilon} > 0$ such that $\sum_{i>0} \rho^i \varepsilon^{p^i}$ converges for all $\varepsilon < \overline{\varepsilon}$. Moreover for $0 < \overline{\overline{\varepsilon}} < \overline{\varepsilon}$ there exists a constant $C = C(\overline{\overline{\varepsilon}}) > 0$ such that $\sum_{i>0} \rho^i \varepsilon^{p^i} < C\varepsilon$ for all $0 < \varepsilon < \overline{\overline{\varepsilon}}$.*

Lemma A.1.5. *Let $P, Q, P', Q' \in \mathbb{R}$ with P, Q' non-zero. Then*

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| \le C \max\left(|P - P'|, |Q - Q'|\right) \tag{A.1.8}$$

where $C = 2|Q|^{-1}\max(1, |P'/Q'|)$.

Proof. Observe that

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| = \left|\frac{1}{Q}(P - P') + P'\left(\frac{1}{Q} - \frac{1}{Q'}\right)\right| \tag{A.1.9}$$
$$\le \frac{1}{|Q|}\left[|P - P'| + \left|\frac{P'}{Q'}\right||Q' - Q|\right].$$

from which the claim is immediate. □

Lemma A.1.6. *Let $\sigma, P, Q \in \mathbb{R}$ satisfy $0 < \sigma \leq 1$ and $0 < P < Q$. Then there exists a a positive real number $\bar{s} > 0$ such that for all $0 < s < \bar{s}$ we have*

$$\tfrac{1}{2} < \frac{\sigma^s P - \sigma^{-s} Q}{P - Q}.$$

Proof. Consider the quadratic polynomial in R,

$$R^2 P - \tfrac{1}{2}(P - Q)R - Q. \tag{A.1.10}$$

Take the neighbourhood of $R = 1$ for which this is greater than $\frac{1}{4}(P - Q)$. Then substituting $R = \sigma^s$ gives

$$\tfrac{1}{2}(P - Q)\sigma^s < \sigma^{2s} P - Q \tag{A.1.11}$$

so dividing by $\sigma^{-s}(P - Q)$ gives the result. □

A.2 Perturbation Results

In this section we collect together several elementary results on perturbations of smooth maps. In particular we consider how periodic points, critical points and preimages behave under perturbation.

Lemma A.2.1. *Let $f \in C^1(J)$ and let α be a zero of f that is non-critical. (That is, $f(\alpha) = 0, f'(\alpha) \neq 0$.) Then there exists a neighbourhood $U \subset C^1(J)$ of f such that each $\tilde{f} \in U$ has a point $\tilde{\alpha}$ with the same property. Moreover there exists a constant $C > 0$ such that $|\alpha - \tilde{\alpha}| < C|f - \tilde{f}|_{C^1}$ for all $\tilde{f} \in U$.*

Proof. Let $V = (a, b)$ be an open neighbourhood of α such that $|f'|_V > K$ for some $K > 0$. In particular this means f is strictly monotone of V. Choose a constant $\bar{\varepsilon} > 0$ such that $\bar{\varepsilon} < \min\{|f(a)|, |f(b)|\}$ and $\bar{\varepsilon} < K$. Then $|f - \tilde{f}|_{C^1} < \bar{\varepsilon}$ implies,

(i) $\tilde{f}(a), \tilde{f}(b)$ have differing signs and hence by the Intermediate Value Theorem $\tilde{f}$ must have a zero $\tilde{\alpha}$ in V;

(ii) $\tilde{\alpha}$ must be unique, for if there exists another zero $\tilde{\beta}$ for $\tilde{f}$ then by the Mean Value Theorem there exists a ξ in $(\tilde{\alpha}, \tilde{\beta})$, and hence in V, such that $f'(\xi) = 0$, a contradiction.

For any $0 < \varepsilon < \bar{\varepsilon}$ let a_ε and b_ε be the zeroes of $f - \varepsilon$ and $f + \varepsilon$ respectively. If $|V|$ is sufficiently small these will be unique for all ε sufficiently small. Let $\tilde{f}$ satisfy $|f\tilde{f}|_{C^1} = \varepsilon$. Then the Mean Value Theorem implies there exist $\xi_\varepsilon \in (a_\varepsilon, \alpha)$ and $\eta_\varepsilon \in (\alpha, b_\varepsilon)$ such that

$$|f(a_\varepsilon) - f(\alpha)| = \varepsilon = |f'(\xi)||a_\varepsilon - \alpha| \tag{A.2.1}$$

and

$$|f(\alpha) - f(\beta_\varepsilon)| = \varepsilon = |f'(\eta)||\alpha - b_\varepsilon|. \tag{A.2.2}$$

But we know $\tilde{\alpha} \in (a_\varepsilon, b_\varepsilon)$ since, for example, the graph of $\tilde{f}$ lies in an ε-neighbourhood of the graph of f. Therefore

$$|\alpha - \tilde{\alpha}| \leq \max\{|a_\varepsilon - \alpha|, |\alpha - b_\varepsilon|\} \leq \frac{1}{\inf_{x\in V} |f'(x)|} |f - \tilde{f}|_{C^1} \tag{A.2.3}$$

and hence the result is shown. ☐

Corollary A.2.2. *Let $f \in C^2(J)$ and let α be one of the following*

(i) a hyperbolic periodic point;

(ii) a nondegenerate critical point.

Then there exists a neighbourhood $U \subset C^2(J)$ of f such that each $\tilde{f} \in U$ has a point $\tilde{\alpha}$ with the same property. Moreover there exists a constant $C > 0$ such that $|\alpha - \tilde{\alpha}| < C|f - \tilde{f}|_{C^2}$ for all $\tilde{f} \in U$.

Proof. This follows by applying Lemma A.2.1 to the functions $f^p(x) - x$ and $f'(x)$. ☐

Corollary A.2.3. *Let $f \in C^2(J)$ and let α be one of the following*

(i) the image or preimage of a nondegenerate critical point;

(ii) the image or preimage of a hyperbolic periodic point.

If α is not a critical point then there exists a neighbourhood $U \subset C^2(J)$ of f such that each $\tilde{f} \in U$ has a point $\tilde{\alpha}$ with the same property. Moreover there exists a constant $C > 0$ such that $|\alpha - \tilde{\alpha}| < C|f - \tilde{f}|_{C^2}$ for all $\tilde{f} \in U$.

Lemma A.2.4. *Let $f \in C^\omega(J)$ have an invariant subinterval J' on which it admits a complex analytic extension to a domain $\Omega' \subset \mathbb{C}$. Then there is a neighbourhood $U \subset C^\omega(J)$ such that if $\tilde{f} \in U$ has a corresponding invariant subinterval $\tilde{J}'$ then $\tilde{f}$ admits a complex analytic extension to some domain $\tilde{\Omega}'$ containing J'.*

Proposition A.2.5. *Let $f, g \in C^2(J)$ and let $J_f, J_g \subset J$ be two dynamically defined intervals of the same type (their boundaries are images of the critical point or periodic points or pre-periodic points). Then there exists a constant $C > 0$ such that*

$$|\mathcal{Z}_{J_f} f - \mathcal{Z}_{J_g} g|_{C^1} \leq C|f - g|_{C^2} \tag{A.2.4}$$

Proposition A.2.6. *Let $f_i, g_i \in \mathrm{Diff}^3_+(J)$, $i = 1, \ldots, n$. Then there exists a constant $C > 0$ such that*

$$|f_1 \circ \cdots \circ f_n - g_1 \circ \cdots \circ g_n| < C \max_{i=1,\ldots,n} |f_i - g_i| \tag{A.2.5}$$

Proposition A.2.7. *If $f, g \in \mathrm{Diff}^2(J)$ then*

$$|f^{-1} - g^{-1}|_{C^0} \leq \frac{1}{\inf_J |df|} |f - g|_{C^0} \tag{A.2.6}$$

Proof. Assuming $f, g \in \mathrm{Diff}^2_+(J)$, we know

$$|x - f(g^{-1}(x))| = |f(f^{-1}(x)) - f(g^{-1}(x))| \geq \left(\inf_{x\in J} |df(x)|\right) |f^{-1}(x) - g^{-1}(x)|. \tag{A.2.7}$$

But $|x - f(g^{-1}(x))| = |g(g^{-1}(x)) - f(g^{-1}(x))| \leq |f - g|_{C^0}$, which implies

$$|f^{-1} - g^{-1}|_{C^0} \leq \frac{1}{\inf_J |df|} |f - g|_{C^0}. \tag{A.2.8}$$

□

Proposition A.2.8. *Let $f_n, f_* \in C^2(J)$ such that $|f_n - f_*|_{C^1} < C\rho^n$ for some $C > 0$ and $0 < \rho < 1$. Assume f has hyperbolic fixed point $\alpha_* \in \mathrm{int}(J)$. Then there is a constants C_0 such that, for $n > 0$ sufficiently large, f_n also has a hyperbolic fixed point α_n, whichs satisfies $|\alpha_n - \alpha_*|, |f_n'(\alpha_n) - f_*'(\alpha_*)| < C_0\rho^n$.*

Appendix B

Stability of Cantor Sets

This appendix shows, among other things, that given a sequence of Scope Maps acting on the square whose limit set is a Cantor set a small perturbation of those Scope Maps will also have a Cantor set for a limit set.

B.1 Variational Properties of Composition Operators

In this section we derive properties of the composition operator. We show how the remainder term from Taylor's Theorem behaves under composition and we derive the first variation of the n-fold composition operator. Although we only state these for maps on $\mathbb{R}^2$ or $\mathbb{C}^2$ these work in full generality.

Proposition B.1.1. *Given $E\colon \mathbb{R}^2 \to \mathbb{R}$ and point $z = (x, y), z' = (x', y') \in \mathbb{R}^2$*

$$|E(z) - E(z')| = |\partial_x E(\xi_x, y)(x - x')| + |\partial_y E(x', \xi_y)(y - y')|. \tag{B.1.1}$$

Proof. Write

$$E(x, y) - E(x', y') = E(x, y) - E(x', y) + E(x', y) - E(x', y') \tag{B.1.2}$$

and apply the one-dimensional Mean Value Theorem. □

Proposition B.1.2. *Let $F, G \in \mathrm{Emb}^2(\mathbb{R}^2, \mathbb{R}^2)$. For any $z_0, z_1 \in \mathbb{R}^2$, consider the decompositions*

$$F(z_0 + z_1) = F(z_0) + \mathrm{D}_{z_0}F(\mathrm{id} + \mathrm{R}_{z_0}F)(z_1) \tag{B.1.3}$$

and

$$G(z_0 + z_1) = G(z_0) + \mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1). \tag{B.1.4}$$

Then

$$\mathrm{R}_{z_0}FG(z_1) = \mathrm{R}_{z_0}G(z_1) + \mathrm{D}_{z_0}G^{-1}\mathrm{R}_{G(z_0)}F(\mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1)) \tag{B.1.5}$$

Proof. Observe that

$$FG(z_0 + z_1) = FG(z_0) + \mathrm{D}_{z_0}FG(z_1) + \mathrm{D}_{z_0}FG(\mathrm{R}_{z_0}FG)(z_1) \tag{B.1.6}$$

must be equal to

$$\begin{aligned} F(G(z_0 + z_1)) &= F(G(z_0) + \mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1)) \qquad (B.1.7)\\ &= F(G(z_0)) + \mathrm{D}_{G(z_0)}F(\mathrm{id} + \mathrm{R}_{G(z_0)}F)(\mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1))\\ &= F(G(z_0)) + \mathrm{D}_{G(z_0)}F\mathrm{D}_{z_0}G(z_1) + \mathrm{D}_{G(z_0)}F\mathrm{D}_{z_0}G(\mathrm{R}_{z_0}G(z_1))\\ &+ \mathrm{D}_{G(z_0)}F\mathrm{R}_{G(z_0)}F(\mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1)). \end{aligned}$$

This implies that

$$\mathrm{R}_{z_0}FG(z_1) = \mathrm{R}_{z_0}G(z_1) + \mathrm{D}_{z_0}G^{-1}\mathrm{R}_{G(z_0)}F(\mathrm{D}_{z_0}G(\mathrm{id} + \mathrm{R}_{z_0}G)(z_1)) \tag{B.1.8}$$

and hence the Proposition is shown. □

Proposition B.1.3. *For each integer $n > 0$ let $C_n \colon C^\omega(B,B)^n \to C^\omega(B,B)$ denote the n-fold composition operator*

$$C_n(G_1, \ldots, G_n) = G_1 \circ \cdots \circ G_n. \tag{B.1.9}$$

For $i = 1, \ldots, n$ assume we are give $F_i, G_i \in C^\omega(B,B)$ and let E_i be defined by $G_i = F_i + E_i$. Then

$$C(G_1, \ldots, G_n) = C(F_1, \ldots, F_n) + \delta C_n(F_1, \ldots, F_n; E_1, \ldots, E_n) + \mathrm{O}(|E_i||E_j|) \tag{B.1.10}$$

where

$$\delta C_n(F_1, \ldots, F_n; E_1, \ldots, E_n) = \sum_{i=1}^{n-1} \mathrm{D}_{F_{i+1,\ldots,n}(z)}F_{1,\ldots,i}(E_{i+1}(F_{i+2,\ldots,n}(z))) \tag{B.1.11}$$

where we have set $F_\emptyset, E_{n+1} = \mathrm{id}$.

Proof. For notational simplicity let $F_{1,\ldots,n} = F_1 \circ \cdots \circ F_n, G_{1,\ldots,n} = G_1 \circ \cdots \circ G_n$ and let $E_{1,\ldots,n}$ satisfy $G_{1,\ldots,n} = F_{1,\ldots,n} + E_{1,\ldots,n}$. Then equating $G_{1,2,\ldots,n}$ with $G_1 \circ G_{2,\ldots,n}$ and using the power series expansion of G_1 gives

$$\begin{aligned} G_{1,\ldots,n}(z) &= G_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z)) \qquad (B.1.12)\\ &= F_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z)) + E_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z))\\ &= F_1(F_{2,\ldots,n}(z)) + \mathrm{D}_{F_{2,\ldots,n}(z)}F_1(E_{2,\ldots,n}(z)) + \mathrm{O}(|E_{2,\ldots,n}|^2)\\ &+ E_1(F_{2,\ldots,n}(z)) + \mathrm{O}(|\mathrm{D}E_1||E_{2,\ldots,n}|) \end{aligned}$$

while equating $G_{1,2,\ldots,n}$ with $G_{1,\ldots,n-1} \circ G_n$ and using the power series expansion of $G_{1,\ldots,n-1}$ gives

$$\begin{aligned} G_{1,\ldots,n}(z) &= G_{1,\ldots,n-1}(F_n(z) + E_n(z)) \qquad (B.1.13)\\ &= F_{1,\ldots,n-1}(F_n(z) + E_n(z)) + E_{1,\ldots,n-1}(F_n(z) + E_n(z))\\ &= F_{1,\ldots,n}(z) + \mathrm{D}_{F_n(z)}F_{1,\ldots,n-1}(E_n(z)) + \mathrm{O}(|E_n|^2)\\ &+ E_{1,\ldots,n-1}(F_n(z)) + \mathrm{O}(|\mathrm{D}E_{1,\ldots,n}||E_n|). \end{aligned}$$

From the second of these expressions, inductively we find, setting $F_\emptyset, E_{n+1} = \mathrm{id}$, that

$$E_{1,\ldots,n}(z) = \sum_{i=1}^{n-1} \mathrm{D}_{F_{i+1,\ldots,n}(z)} F_{1,\ldots,i}(E_{i+1}(F_{i+2,\ldots,n}(z))) + \mathrm{O}(|E_i||E_j|) \quad \text{(B.1.14)}$$

□

B.2 Cantor Sets generated by Scope Maps

In this section we examine the limit set induced by collections of scope functions, both one- and two-dimensional. We show that, in a certain sense, the property that the limit set is a Cantor set is stable under perturbations if suitable conditions are made on the perturbation. Although we apply these results to show infinitely renormalisable Hénon-like maps possess invariant Cantor sets we believe this method has other applications, such as the study of pseudo-trajectories of the renormalisation operator that, in a sense, is the content of our proof of the existence of the renormalisation fixed point in section 3.

Proposition B.2.1. *Let $f_n \in \mathcal{U}_{\Omega_x,v}$ be a sequence of renormalisable unimodal maps and let $\boldsymbol{\psi}_n = \{\psi_n^w\}_{w\in W}$ denote the presentation function of f_n. Assume*

(i) the central cycle $\{J_n^w\}_{w\in W}$ has uniformly bounded geometry;

(ii) $\mathrm{Dis}(\psi_n^w; z)$ is uniformly bounded;

(iii) there exists an integer $N > 0$ such that $\mathrm{S}_{\psi_n^w} > 0$ for all $n > N, w \in W$.

Then

$$\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \psi^{\mathbf{w}}(J) \quad \text{(B.2.1)}$$

is a Cantor set.

Proof. Given closed intervals $M \subset T$, with M properly containd in T, consider their cross-ratio,

$$D(M,T) = \frac{|M||T|}{|L||R|} \quad \text{(B.2.2)}$$

where L and R are the left and right connected components of $T\backslash M$ respectively. We recall the following properties:

(i) maps with positive Schwarzian derivative contract the cross-ratio;

(ii) for all $K > 0$ there exists a $0 < K' < 1$ such that $D(J,T) < K$ implies $\frac{|J|}{|T|} < K'$.

The first assumption implies $D(J_n^w, J) < K$ for all $w \in W, n \in \mathbb{N}$ and some $K > 0$. The third assumption implies the intervals $J_N^{w_N \dots w_n} = \psi_N^{w_N} \circ \dots \circ w_n^{w_n}(J)$ are images of J_n^w under positive Schwarzian maps for all $n > N$. Hence the first property of the cross-ratio implies $D(J_N^{w_N \dots w_n}, J_N^{w_N \dots w_{n-1}}) < K$ for all $n > N$. By the second property of the cross ratio this implies $\frac{|J_N^{w_N \dots w_n}|}{|J_N^{w_N \dots w_{n-1}}|} < K' < 1$. The same argument applies to the images of the gaps between the J_n^w. Therefore

$$\mathcal{O}_N = \bigcap_{n \geq N} \bigcup_{\mathbf{w} \in W^n} \psi_N^{w_N \dots w_n}(J) \tag{B.2.3}$$

is a Cantor set. By the second assumption $\psi_0^{w_0 \dots w_{N-1}}$ has bounded distortion for all $w_0 \dots w_{N-1} \in W^N$. The image of a Cantor set under a map with bounded distortion is still a Cantor set. Hence

$$\mathcal{O} = \bigcap_{n \geq 0} \bigcup_{\mathbf{w} \in W^n} \psi^{\mathbf{w}}(J) \tag{B.2.4}$$

is a Cantor set and the result is shown. □

The following is an immediate Corollary of the above Proposition. It simply rephrases the above in terms of scope maps for degenerate Hénon-like maps instead of scope maps for unimodal maps.

Corollary B.2.2. *Let $F_n = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,v}$ be a sequence of renormalisable degenerate Hénon-like maps and let $\mathbf{\Psi}_n = \{\Psi_n^w\}_{w \in W}$ denote the presentation function of F_n and let $\boldsymbol{\psi}_n = \{\psi_n^w\}_{w \in W}$ is the presentation function for f. Assume*

(i) the central cycle $\{B_n^w\}_{w \in W}$ has uniformly bounded geometry;

(ii) $\mathrm{Dis}(\Psi_n^w; z)$ is uniformly bounded;

(iii) there exists an integer $N > 0$ such that $\mathrm{S}_{\psi_n^w} > 0$ for all $n > N, w \in W$.

Then

$$\mathcal{O} = \bigcap_{n \geq 0} \bigcup_{\mathbf{w} \in W^n} \Psi^{\mathbf{w}}(B) \tag{B.2.5}$$

is a Cantor set.

We are now in a position to prove the following. This is the main result of this section. It states that, under suitable conditions, a perturbation of a family of scope maps whose limit set is a Cantor set will also have a limit set which is a Cantor set.

Proposition B.2.3. *Let $F_n \in \mathcal{H}_{\Omega,v}$ be a sequence of renormalisable Hénon-like maps and let $\mathbf{\Psi}_n = \{\Psi_n^w\}_{w \in W}$ denote the presentation function of F_n. Assume*

(i) the set $\mathcal{O} = \bigcap_{n \geq 0} \bigcup_{\mathbf{w} \in W^n} \Psi^{\mathbf{w}}(B)$ is a Cantor set;

(ii) for $\mathbf{w} = w_0 w_1 \ldots \in W^*$ *the cylinder sets* $\Psi^{w_0,\cdots w_n}(B)$ *'nest down exponentially': there exists a constant* $0 < \delta < 1$ *such that* $\mathrm{diam}(\Psi^{w_0,\cdots w_n}(B)) < \delta\, \mathrm{diam}(\Psi^{w_0,\cdots w_{n-1}}(B))$ *for all* $n > 0$*;*

(iii) $\|\mathrm{D}_z \Psi_n^w\| < K$ *for all* $z \in \Omega, w \in W$ *and* $n > 0$.

Then there exists an $\bar{\varepsilon} > 0$ *such that for any sequence* $\tilde{F}_n \in \mathcal{H}_{\Omega,v}$ *of renormalisable Hénon-like maps satisfying* $|F_n - \tilde{F}_n|_\Omega < C\bar{\varepsilon}^{p^n}$ *the set*

$$\tilde{\mathcal{O}} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \tilde{\Psi}^{\mathbf{w}}(B) \tag{B.2.6}$$

is also a Cantor set.

Proof. It is clear that $\tilde{\mathcal{O}}$ is closed and non-empty, hence we are just required to show it is totally disconnected and contains no isolated points. Before we begin let us introduce some notation. First let us define E_n by $F_n = \tilde{F}_n + E_n$. Then $|E_n|_\Omega \leq C_0\bar{\varepsilon}^{p^n}$. This implies we can write $\Psi_n^w = \tilde{\Psi}_n^w + \Lambda_n^w$ where $\tilde{\Psi}_n^w$ is the w-th presentation function for $\tilde{F}_n$ and $|\Lambda_n^w|_\Omega \leq C_1\bar{\varepsilon}^{p^n}$. For $\mathbf{w} = w_0 \ldots w_n \in W^*$ let $\Psi^{w_0\cdots w_n} = \Psi_0^{w_0} \circ \cdots \circ \Psi_n^{w_n}$ and $\tilde{\Psi}^{w_0\cdots w_n} = \tilde{\Psi}_0^{w_0} \circ \cdots \circ \tilde{\Psi}_n^{w_n}$. Then define $\Lambda^{w_0\cdots w_n}$ to be the function satisfying $\Psi^{w_0\cdots w_n} = \tilde{\Psi}^{w_0\cdots w_n} + \Lambda^{w_0\cdots w_n}$. From the variational analysis above we find for $z \in B$, after setting $z_i = \tilde{\Psi}^{w_i\cdots w_n}(z)$ and $\tilde{\Psi}^{\emptyset} = \mathrm{id}$, that

$$\Lambda^{w_0\cdots w_n}(z) = \sum_{i\geq 1} \mathrm{D}_{z_i}\tilde{\Psi}^{w_0\cdots w_{i-1}}(\Lambda^{w_i}(z_{i+1})) + \mathrm{O}(|\mathrm{D}\Lambda^{w_i}||\Lambda^{w_j}|, |\Lambda^{w_i}|^2). \tag{B.2.7}$$

Now let $z, z' \in B$ be any distinct pair of points and let $z_i = \tilde{\Psi}^{w_i\cdots w_n}(z)$ and $z_i' = \tilde{\Psi}^{w_i\cdots w_n}(z')$. First observe that by hypothesis there exist constants $C_2 > 0$ and $0 < \delta < 1$ such that $|z_m - z_m'| \leq C_2\delta^{n-m}$. Second, by hypothesis $\|\mathrm{D}_z\tilde{\Psi}_i^{w_i}\| \leq K$ for all $z \in B$ and $|\Lambda_i^{w_i}|_\Omega \leq C_1\bar{\varepsilon}^{p^i}$. This together with (B.2.7) implies $|\Lambda^{w_m\cdots w_n}(z) - \Lambda^{w_m\cdots w_n}(z')| \leq C_1\sum_{i\geq 1} K^i\bar{\varepsilon}^{p^i}$, which by Proposition A.1.4 implies there exists a constant $C_3 > 0$ such that $|\Lambda^{w_m\cdots w_n}(z) - \Lambda^{w_m\cdots w_n}(z')| \leq C_3K^m\bar{\varepsilon}^{p^m}$. Thirdly, consider

$$|\Psi^{w_0\cdots w_n}(z) - \Psi^{w_0\cdots w_n}(z')| \leq \sup_{\xi\in B}\|\mathrm{D}_\xi\Psi^{w_0\cdots w_{m-1}}\||\Psi^{w_m\cdots w_n}(z) - \Psi^{w_m\cdots w_n}(z')|. \tag{B.2.8}$$

By hypothesis

$$\sup_{\xi\in B}\|\mathrm{D}_\xi\Psi^{w_0\cdots w_{m-1}}\| \leq \prod_{i=0}^{m-1}\sup_{\xi_i\in B}\|\mathrm{D}_{\xi_i}\Psi^{w_i}\| \leq K^m \tag{B.2.9}$$

and from the above

$$\begin{aligned} |\Psi^{w_m\cdots w_n}(z) - \Psi^{w_m\cdots w_n}(z') &\leq |z_m - z_m'| + |\Lambda^{w_m\cdots w_n}(z) - \Lambda^{w_m\cdots w_n}(z')| \\ &\leq C_2\delta^{n-m} + C_3K^{n-m}\bar{\varepsilon}^{p^m}. \end{aligned} \tag{B.2.10}$$

Hence we find that

$$|\Psi^{w_0\cdots w_n}(z) - \Psi^{w_0\cdots w_n}(z')| \leq K^m(C_2\delta^{n-m} + C_3K^{n-m}\bar{\varepsilon}^{p^m}) \tag{B.2.11}$$

This can be made arbitrarily small by choosing $0 < m < n$ sufficiently large. Therefore cylinder sets consist of single points. Next we show that $\mathcal{O}$ does not have any isolated points. Assume there is a word $\mathbf{w} = w_0w_1\ldots \in W^*$ for which the associated cylinder set $B^{\mathbf{w}}$ is isolated. Then $\mathrm{dist}(B^{\mathbf{w}}, B^{\tilde{\mathbf{w}}}) > \rho$ for some $\rho > 0$ which we may assume satisfies $\rho < 1$. We know that for any $0 < \rho < 1$ there is an integer $N > 0$ such that for all $n > N$ $\mathrm{diam}(B^{w_0\cdots w_n}) < \rho$. In particular $\mathrm{dist}(B^{w_0\cdots w_n w_{n+1}}, B^{w_0\cdots w_n\tilde{w}}) < \rho$ for any $\tilde{w} \in W$, a contradiction. Hence $\mathcal{O}$ does not have any isolated points. $\square$

Appendix C

Sandwiching and Shuffling

In this appendix we give a slightly more general version of a Sandwich Lemma given in [12]. We use this result to show that the nonlinear remainders of the scope functions behave well under compositions.

C.1 The Shuffling Lemma

Before we begin let us recall some definitions. Let df denote the derivative of the diffeomorphism f of the interval. The *Nonlinearity* f is given by

$$\mathrm{N}_f = \frac{d^2 f}{df}, \tag{C.1.1}$$

and the *Schwarzian derivative* is given by

$$\mathrm{S}_f = \frac{d^3 f}{df} - \frac{3}{2}\left(\frac{d^2 f}{df}\right)^2 \tag{C.1.2}$$

Observe that $\mathrm{S}_f = \mathrm{DN}_f - \frac{1}{2}\mathrm{N}_f^2$. In this section we simply state the following Shuffling Lemma from the appendix in [32]. This will be useful in the next section.

Lemma C.1.1. *For every $B > 0$ there exists a $K > 0$ such that the following holds: for $m = 1, \ldots, n$ and $i = 0, 1$ let $\phi_m^i \in \mathrm{Diff}_+^3(J), m = 1, \ldots, n$ and let*

$$\Phi^i = \phi_n^i \circ \cdots \circ \phi_2^i \circ \phi_1^i. \tag{C.1.3}$$

If, for $i = 0, 1$,

$$\sum_{j=1}^{n} |\mathrm{N}_{\phi_j^i}|_{C^1} \le B, \tag{C.1.4}$$

then

$$\mathrm{dist}_{C^2}(\Phi^0, \Phi^1) \le K \sum_{j=1}^{n} |\mathrm{N}_{\phi_j^0} - \mathrm{N}_{\phi_j^1}|_{C^0}. \tag{C.1.5}$$

C.2 The Sandwich Lemma

We are now in a position to prove the following result. In particular, it tells us that limiting scope maps behave well under perturbations: if a limit exists for a sequence of scope maps then a small perturbation of those composants maps will also give a limit.

Lemma C.2.1. *Let $0 < \rho < 1, C > 1$. For $m = 1, \ldots, n$ let $\phi_m, \psi_m \in C^3(J)$, and let*

$$\phi_{m,n} = \phi_m \circ \cdots \circ \phi_n, \quad \psi_{m,n} = \psi_m \circ \cdots \circ \psi_n \tag{C.2.1}$$

and define $J^\phi_m = \phi_{m+1,n}(J)$ and $J^\psi_m = \psi_{m+1,n}(J)$. Assume that the following properties are satisfied,

$$|\phi_m - \psi_m|_{C^3} < C\rho^m; \tag{C.2.2}$$

$$|J^\phi_m|/|J^\phi_{m+1}|, |J^\psi_m|/|J^\psi_{m+1}| < \rho; \tag{C.2.3}$$

$$|\mathrm{N}_{\phi_m}|_{C^1}, |\mathrm{N}_{\psi_m}|_{C^1} < C; \tag{C.2.4}$$

$$C^{-1} < |d\phi_m|_{C^0}, |d\psi_m|_{C^0} < C; \tag{C.2.5}$$

for all $m = 1, \ldots, n$. Then, letting[1]

$$[\phi_{m,n}] = \iota_{J^\phi_{m-1} \to J} \circ \phi_{m,n} \colon J \to J, \quad [\psi_{m,n}] = \iota_{J^\psi_{m-1} \to J} \circ \psi_{m,n} \colon J \to J, \tag{C.2.6}$$

there is a constant $C_0 > 0$, depending upon C and ρ only, such that for all $m = 1, \ldots, n$,

$$|[\phi_{m,n}] - [\psi_{m,n}]|_{C^2} \leq C_0 \rho^{n-m}. \tag{C.2.7}$$

Proof. For $m = 1, \ldots, n$ let x^ϕ_m, y^ϕ_m and x^ψ_m, y^ψ_m be the unique points satisfying $J = [x^\phi_n, y^\phi_n] = [x^\psi_n, y^\psi_n]$, $J^\phi_m = [x^\phi_m, y^\phi_m]$ and $J^\psi_m = [x^\psi_m, y^\psi_m]$. Let $\Delta x_m = x^\phi_m - x^\psi_m$ and $\Delta y_m = y^\phi_m - y^\psi_m$. Let $\iota^\phi_m = \iota_{J \to J^\phi_m}$ and $\iota^\psi_m = \iota_{J \to J^\psi_m}$, and let

$$[\phi_m] = (\iota^\phi_{m-1})^{-1} \circ \phi_m \circ \iota^\phi_m, \quad [\psi_m] = (\iota^\psi_{m-1})^{-1} \circ \psi_m \circ \iota^\psi_m. \tag{C.2.8}$$

We make the following assertions. First, there is a constant $C_1 > 0$ such that, for $m = 1, \ldots, n$,

$$|\Delta x_m|, |\Delta y_m| \leq C_1 \rho^m. \tag{C.2.9}$$

To see this first observe that, by our initial hypothesis, $|x^\phi_{n-1} - x^\psi_{n-1}| = |\phi_n(x_n) - \psi_n(x_n)| \leq C\rho^n$. Proceeding inductively, if $\Delta x_m < C'\rho^{m+1}$ then, using $x^\phi_{m-1} = \phi_m(x^\phi_m)$ and $x^\psi_{m-1} = \psi_m(x^\psi_m)$, we find

$$\begin{aligned} |\Delta x_{m-1}| &\leq |\phi_m(x^\phi_m) - \phi_m(x^\psi_m)| + |\phi_m(x^\psi_m) - \psi_m(x^\psi_m)| \\ &\leq |d\phi_m|_{C^0} |\Delta x_m| + |\phi_m - \psi_m|_{C^0} \\ &\leq CC'\rho^{m+1} + C\rho^m, \end{aligned} \tag{C.2.10}$$

[1] i.e. the affine rescaling of $\phi^i_{m,n}$ so that $[\phi^i_{m,n}] \in \mathrm{Diff}^3_+(J)$.

and a similar estimate holds for Δy_m. Second, there is a constant $C_2 > 0$ such that, for all $m = 1, \ldots, n$,

$$|J_m^\phi|, |J_m^\psi| \le C_2 \rho^{n-m}. \tag{C.2.11}$$

This follows straightforwardly from our initial hypotheses. Third, there is a constant $C_3 > 0$ such that for all $m = 1, \ldots n$

$$|\iota_m^\phi - \iota_m^\psi|_{C^1} \le C_3 \min\left(\rho^m, \rho^{n-m}\right). \tag{C.2.12}$$

To see this, from above it follows that

$$\begin{aligned} |\iota_m^\phi - \iota_m^\psi|_{C^0} &= \sup_{x\in J}\left|x_m^\phi + (x - x_n)|J_m^\phi|/|J| - x_m^\psi - (x - x_n)|J_m^\psi|/|J|\right| \\ &\le |\Delta x_m| + \left||J_m^\phi| - |J_m^\psi|\right| \\ &\le |\Delta x_m| + \min\left\{|\Delta x_m| + |\Delta y_m|, |J_m^\phi| + |J_m^\psi|\right\} \end{aligned} \tag{C.2.13}$$

and

$$\begin{aligned} \left|d\iota_m^\phi - d\iota_m^\psi\right|_{C^0} &= \left||J_m^\phi|/|J| - |J_m^\psi|/|J|\right| \\ &\le |J|^{-1} \min\left\{|\Delta x_m| + |\Delta y_m|, |J_m^\phi| + |J_m^\psi|\right\} \end{aligned} \tag{C.2.14}$$

from which the claim follows immediately from the preceding two statements.

Fourth, there is a constant $C_4 > 0$ such that for all $m = 1, \ldots, n$,

$$\left|\mathrm{N}_{\phi_m} - \mathrm{N}_{\psi_m}\right|_{C^0} < C_4 \rho^m. \tag{C.2.15}$$

This follows as

$$\begin{aligned} \mathrm{N}_{\phi_m} - \mathrm{N}_{\psi_m} &= \frac{d^2\phi_m}{d\phi_m} - \frac{d^2\psi_m}{d\psi_m} \\ &\le \frac{1}{d\psi_m}\left(\mathrm{N}_{\phi_m}(d\psi_m - d\phi_m) + (d^2\phi_m - d^2\psi_m)\right) \end{aligned} \tag{C.2.16}$$

implies

$$\left|\mathrm{N}_{\phi_m} - \mathrm{N}_{\psi_m}\right|_{C^0} \le \frac{1 + |\mathrm{N}_{\phi_m}|_{C^0}}{\inf_{x\in J}|d\psi_m(x)|}\left|\psi_m - \phi_m\right|_{C^2} \tag{C.2.17}$$

but by our initial hypotheses

$$\frac{1 + |\mathrm{N}_{\phi_m}|_{C^0}}{\inf_{x\in J}|d\psi_m(x)|} \le C(1 + C) \tag{C.2.18}$$

and so the claim follows.

We now apply these assertions to show the result. Firstly, it follows from the chain rule for nonlinearities that $\mathrm{N}_{[\phi_m]} = d\iota_m^\phi \mathrm{N}_{\phi_m} \circ \iota_m^\phi$ and $\mathrm{N}_{[\psi_m]} = d\iota_m^\psi \mathrm{N}_{\psi_m} \circ \iota_m^\psi$, and hence

$$\begin{aligned}\left|\mathrm{N}_{[\phi_m]} - \mathrm{N}_{[\psi_m]}\right|_{C^0} &= |d\iota_m^\phi \mathrm{N}_{\phi_m} \circ \iota_m^\phi - d\iota_m^\psi \mathrm{N}_{\psi_m} \circ \iota_m^\psi|_{C^0} \\ &\leq |d\iota_m^\phi - d\iota_m^\psi|_{C^0} |\mathrm{N}_{\phi_m}|_{C^0} \\ &+ |d\iota_m^\psi|_{C^0} |\mathrm{N}_{\phi_m} - \mathrm{N}_{\psi_m}|_{C^0} \\ &+ |d\iota_m^\psi|_{C^0} |d\mathrm{N}_{\psi_m}|_{C^0} |\iota_m^\phi - \iota_m^\psi|_{C^0} \\ &\leq |\iota_m^\phi - \iota_m^\psi|_{C^1} \left(|\mathrm{N}_{\phi_m}|_{C^0} + |d\iota_m^\psi|_{C^0} |d\mathrm{N}_{\psi_m}|_{C^0}\right) \\ &+ |d\iota_m^\psi|_{C^0} |\mathrm{N}_{\phi_m} - \mathrm{N}_{\psi_m}|_{C^0}.\end{aligned} \tag{C.2.19}$$

From the above assertions we find

$$\begin{aligned}\left|\mathrm{N}_{[\phi_m]} - \mathrm{N}_{[\psi_m]}\right|_{C^0} &\leq C_3 \min(\rho^m, \rho^{n-m})(C + C\rho^{n-m}) + C_4 \rho^{n-m} \rho^m \\ &\leq C_5 \min(\rho^m, \rho^{n-m}).\end{aligned} \tag{C.2.20}$$

From this it follows that there is a $C_6 > 0$ such that, for $m = 1, \ldots, n$,

$$\sum_{j=m}^{n} \left|\mathrm{N}_{[\phi_j]} - \mathrm{N}_{[\psi_j]}\right|_{C^0} \leq C_6 \rho^{n-m}. \tag{C.2.21}$$

Applying the Shuffling Lemma C.1.1 to the $[\phi_j]$ and $[\psi_j]$ and observing

$$[\phi_{m,n}] = [\phi_m] \circ \cdots \circ [\phi_n], \quad [\psi_{m,n}] = [\psi_m] \circ \cdots \circ [\psi_n] \tag{C.2.22}$$

then gives us the result. □