# A monotonicity conjecture for the entropy of Hubbard trees

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# Abstract of the Dissertation

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This thesis is composed of two parts. In the first part, we extend Douady's result about the monotonicity of the entropy of the real quadratic map along the real axis. In fact, we prove the monotonicity of topological entropy of post-critically finite quadratic polynomials acting on Hubbard trees. In the second part, we study the parameter space of cubic polynomials with one critical point fixed. In particular, there is a similar result on the entropy of post-critically finite maps restricted on the Hubbard tree as in the quadratic case.

To my parents,

my son Justin Lee and

my wife Yuefeng Tang

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This thesis is dedicated to my parents, my son and my wife.

# Chapter 1

# Introduction

In this thesis, we study the combinatorial properties of two families of complex polynomials: the quadratic family and the cubic family with one critical point fixed. As a measurement of complexity of dynamical systems, topological entropy is our main object. Our goal is to prove the monotonicity property of topological entropy restricted to the Hubbard trees for these two families.

In this chapter, we first give some review of topological entropy for real quadratic polynomials acting on the real line. Afterwards, we state the main results and discuss some directions of further study.

## 1.1 The context

Complex Polynomials is a main subject for the study of complex dynamics. There is a well-known result which states that topological entropy for complex polynomials of degree $d$ is $\log d$. Let $f$ be a complex polynomial of degree $d$ and $X \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ compact and invariant under $f$, denote the topological entropy of $f$ acting on $X$ by $h(X, f)$. (see §2.4 for the definition of topological entropy.) Then on the complex plane or restricted to the Julia Set of $f$, we have the following result:

$$h(\hat{\mathbb{C}}, f) = h(K(f), f) = h(J(f), f) = \log d.$$

where $K(f)$ stands for the filled Julia set of $f$ (See §2.1.1).

But how does the entropy vary when we restrict the polynomials to other invariant subsets? For $c \in \mathbb{R}$, let us consider the real quadratic polynomials $f_c : x \mapsto x^2 + c$. Denote $h(c)$ the entropy of $f_c$ acting on $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$. Douady showed in [D2] that the entropy function $h : \mathbb{R} \mapsto [0, \log 2]$ is (weakly) decreasing and continuous. He also gives the necessary and sufficient conditions on when entropy is greater than zero or when two entropies are equal.

Douady's main tool is an external ray argument. Douady pointed out in [D2] that the entropy of a quadratic polynomial acting on $\overline{\mathbb{R}}$, which is identical to the entropy on $J(f) \cap \mathbb{R}$, equals to the entropy of the doubling map of $\mathbb{T}$ acting on the set of external arguments of points in $J(f) \cap \mathbb{R}$. The external arguments, closely connecting dynamical plane and parameter plane, can then be used to determine and compare topological entropies.

It is worthwhile to mention that to use the external ray argument, we need to make sure that the Julia set of the polynomial is locally connected. To overcome this difficulties on infinitely renormalizable maps and calculate the entropy on all real quadratic polynomials, Douady used kneading sequence and kneading angle. After Douady's paper [D2], Levin and Strien published their well-known paper [LS2] which states that the Julia set for all real quadratic polynomials is either locally connected or totally disconnected. With the aid of local connectivity for real quadratic polynomials with connected Julia set, we can get the monotonicity result for topological entropy by using the external ray argument only.

## 1.2 Results

There are two parts in this thesis. The first part deals with quadratic post-critically finite maps acting on their Hubbard trees. The second part deals with cubic cases.

Given a post-critically finite polynomial map $f$, the (minimal) Hubbard tree is defined as the smallest regulated tree in the filled Julia set $K(f)$ containing the critical orbits. The Hubbard tree is compact and invariant, so we can talk about the topological entropy of maps acting on the Hubbard tree. Given any map $f_c$ with parameter $c$, denote the (minimal) Hubbard tree by $T_0(c)$ or $T_0(f_c)$. The end-number $N(c)$ of the Hubbard tree $T_0(c)$ is defined as the cardinality of the set of end points of $T_0(c)$.

Given a real post-critically finite quadratic polynomial $f_c$, the topological entropy of $f_c$ acting on the real axis is equal to the entropy of $f_c$ acting on the Hubbard tree:

$$h(\overline{\mathbb{R}}, f_c) = h(K(f_d) \cap \overline{\mathbb{R}}, f_c) = h(T_0(f_c), f_c).$$

So it is a natural question to study the topological entropy of general post-critically finite polynomials acting on their Hubbard trees. The Mandelbrot

Set $\mathcal{M}$ is defined as the set of parameters $c$ such that $f_c$ is connected. Our main goal is to prove the monotonicity of topological entropy of post-critically finite polynomials with parameters in $\mathcal{M}$ acting on Hubbard trees.

Given the Mandelbrot Set $\mathcal{M}$, define $\mathcal{M}^0 \subset \mathcal{M}$ as the set of all parameters in $\mathcal{M}$ such that the corresponding maps are post-critically finite. And then we can define partial order "$\prec$" on $\mathcal{M}^0$ as follows: given two different parameters $c_1$ and $c_2$ in $\mathcal{M}^0$, we say that $c_2 \prec c_1$ if $c_2$ (or the root point of the hyperbolic component which contains $c_2$) separates $c_1$ from 0 on $\mathcal{M}$. It's easy to check that "$\prec$" is transitive: Given three different parameters $c_i$ for $i = 1, 2, 3$ such that $c_3 \prec c_2$ and $c_2 \prec c_1$, then $c_3 \prec c_1$.

We will prove:

**Theorem 3.1** *Given any two parameters $c_2$ and $c_1$ on $\mathcal{M}^0$. If $c_2 \prec c_1$, then $h(T_0(c_2), f_{c_2}) \leq h(T_0(c_1), f_{c_1})$.*

J. Milnor first studied the cubic polynomials with one critical point fixed in 1991.([M4]). Following [M4], the cubic polynomial with critical fixed point $a \in \mathbb{C}$ can be normalized as $f_a(z) = z^3 - 3a^2 z + 2a^3 + a$. Denote the associated connectedness locus by $\mathcal{C}$. It has many interesting properties and been studied by many authors in later years.

The second part of this thesis still deals with post-critically finite maps. We define $\mathcal{C}^0 \subset \mathcal{C}$ as all parameters in $\mathcal{C}$ such that the corresponding maps are post-critically finite. Again we can define partial order "$\prec$" on $\mathcal{C}^0$ in the following sense: given two different parameters $a_1$ and $a_2$ in $\mathcal{C}^0$, we say that $a_2 \prec a_1$ if $a_2$ (or the root point of the hyperbolic component which contains $a_2$) separates $a_1$ from 0 on $\mathcal{C}$. Again it is easy to check that "$\prec$" is transitive: Given three different parameters $a_i$ for $i = 1, 2, 3$ such that $a_3 \prec a_2$ and $a_2 \prec a_1$, then $a_3 \prec a_1$.

Our result is the following, where $N(a_i)$ again denotes the end-number of the Hubbard trees $T_0(a_i)$ for $i = 1, 2$.

**Theorem 4.3** *Given any two parameters $a_2 \prec a_1$ in $\mathcal{C}^0$. If $N(a_1) = N(a_2)$, then $h(T_0(a_2), f_{a_2}) \leq h(T_0(a_1), f_{a_1})$.*

The imaginary axis on $\mathcal{C}$ corresponds to the real axis on $\mathcal{M}$. Given $a \in \mathcal{C} \cap i\mathbb{R}$, the closure of imaginary axis $\overline{i\mathbb{R}} = i\mathbb{R} \cup \{\infty\}$ is $f_a$-invariant. So we can talk about the entropy acting on $\overline{i\mathbb{R}}$. Since $f_a$ conjugate to $f_{-a}$ via involution $\mathcal{I} : z \mapsto -z$, we will only consider the maps $f_a$ with $a \in \mathcal{C} \cap i\mathbb{R}^+$. Our next result is the following.

**Theorem 4.21** *The topological entropy for the maps $f_a$ with $a \in \mathcal{C} \cap i\mathbb{R}^+$ acting on $\overline{i\mathbb{R}}$ is monotone.*

The main tool to prove these theorems is an external ray argument. To apply this idea, we need to make sure that the end-numbers of the two Hubbard trees are the same (it turns out that if $c_2 \prec c_1$ and $N(c_2) = N(c_1)$, then $T_0(c_1)$ is topologically homeomorphic to $T_0(c_2)$). When the end-number are different, we can still prove the monotonicity in the quadratic cases using some other combinatorial tools (over-Markov packing, see §2.18).

## 1.3 Further study

Our main goal is to prove the monotonicity of entropy on any limb either on $\mathcal{M}^0$ or on $\mathcal{C}^0$. We have done this completely in the quadratics cases. We have did some experimental calculation, it is rather reasonable that the conditions on the end-number of Hubbard trees in Theorem 4.3 can be removed. Thus our conjecture is the following:

**Conjecture 1.1.** *Given any two parameters* $a_1$ *and* $a_2$ *on* $\mathcal{C}^0$*. If* $a_2 \prec a_1$*, then* $h(T_0(a_2), f_{a_2}) \leq h(T_0(a_1), f_{a_1})$*.*

# Chapter 2

# Background

## 2.1 Dynamics of Complex Polynomials

### 2.1.1 The Julia set

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$, $f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d$.

A point $z$ is a periodic point of $f$ if $f^p(z) = z$ for some $p \geq 1$ . The smallest such $p$ is called the period of $z$.

A periodic point $z$ with period $p$ is *repelling* if $|(f^p)'(z)| > 1$, *indifferent* if $|(f^p)'(z)| = 1$, *attracting* if $|(f^p)'(z)| < 1$ and *super-attracting* if $|(f^p)'(z)| = 0$.

The ***filled Julia set*** $K_f$ is defined as the set of points with bounded orbit under $f$

$$K_f = \{z \in \mathbb{C} \mid f^{\circ n}(z) \nrightarrow \infty\}.$$

Define the ***attracting basin of*** $\infty$, $A_f(\infty)$, as the set of points with unbounded orbit

$$A_f(\infty) = \mathbb{C}\backslash K_f = \{z \in \mathbb{C} \mid f^{\circ n}(z) \to \infty\}.$$

The common boundary $\partial K_f = \partial A_f(\infty) = J_f$ is called the ***Julia set***. Define the ***interior of*** $K_f$ by $int(K_f) = K_f \setminus J_f$. The ***Fatou set*** is defined as the complement of the Julia set $F(f) = \mathbb{C}\backslash J_f$. Note that the critical orbit is bounded if the critical point belongs to $K_f$ and unbounded if the critical point belongs to $A_f(\infty)$. When $f = f_c$, denote the filled Julia set and the Julia set of $f_c$ by $K_c$ and $J_c$ respectively.

Define the Green function $G_f : \mathbb{C} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log_+(|f^{\circ n}(z)|)$$

where $\log_+(|z|)$ =max{0,log($|z|$)}. The map $G_f$ is continuous on $\mathbb{C}$, harmonic on $\mathbb{C}\backslash K_f$ and equal to 0 on $K_f$. Moreover, $G_f(z)$ =log($|z|$) + $O(1)$ when $|z| \to \infty$, and $G_f(f(z)) = dG_f(z)$.

For any $\eta > 0$. The set $G_f^{-1}(\eta)$ is called the ***equipotential*** of value $\eta$. Note that $f$ maps each equipotential $G_f^{-1}(\eta)$ to the equipotential $G_f^{-1}(d\eta)$ by a $d$-to-one fold covering map.

### 2.1.2 The Böttcher Theorem

Consider the dynamics of a holomorphic map in some small neighborhood of a super-attracting fixed point 0. The map has the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

with $n \geq 2$ and $a_n \neq 0$.

**Theorem 2.1** (Böttcher). *With $f$ as above, there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, which conjugates $f$ to the $n$-th power map $w \mapsto w^n$ throughout some neighborhood of zero. Furthermore, $\phi$ is unique up to multiplication by an $(n-1)$-st root of unity.*

Thus, near any critical fixed point, $f$ is conjugate to a map of the form $\phi \circ f \circ \phi^{-1} : w \mapsto w^n$, with $n \geq 2$. We can apply this theorem to polynomials of degree $d \geq 2$ acting on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for which $\infty$ is a super-attracting fixed point.

For a monic polynomial $f$ of degree $d$, set $U_f = \{z \in \mathbb{C} \ : \ G_f(z) \ > \ G_f(c)\}$. Then, there exists a unique analytic isomorphism

$$\varphi_f : U_f \longrightarrow \mathbb{C} \backslash \bar{D}_{\exp G_{f(c)}}$$

satisfying $\varphi_f(z)/z \longrightarrow 1$ as $|z| \rightarrow \infty$ and conjugating $f$ to the polynomial $f_0(z) = z^d$, i.e. $\varphi_f \circ f = f_0 \circ \varphi_f$. If all the critical points of $f$ are contained in $K_f$ then $U_f = \mathbb{C} \backslash K_f$ and $K_f$ are connected.

**Theorem 2.2.** *Let $f$ be a polynomial of degree $d \geq 2$. If the filled Julia set $K_f$*

*contains all the finite critical points of $f$, then both $K_f$ and $J_f$ are connected, and the complement of $K_f$ is conformally isomorphic to the exterior of the closed unit disk $\overline{\mathbb{D}}$ under an isomorphism*

$$\varphi_f : \mathbb{C}\backslash K_f \longrightarrow \mathbb{C}\backslash\overline{\mathbb{D}}$$

*which conjugate $f$ on $\mathbb{C}\backslash K_f$ to the $d$-th power map $w \mapsto w^d$. Furthermore, $\varphi_f^{-1}$ extends continuously to a map $\varphi_f^{-1} : \mathbb{C}\backslash\mathbb{D} \longrightarrow \mathbb{C}\backslash K_f \cup J_f$, and restriction on the boundary gives a continuous surjection $\varphi_f^{-1} : \mathbb{T} \longrightarrow J_f$ (the Carathédory loop).*

*On the other hand, if at least one critical point of $f$ belongs to $\mathbb{C}\backslash K_f$, then both $K_f$ and $J_f$ have uncountably many connected components.*

### 2.1.3 External Rays

Suppose that the set $K_f$ is connected. Let $\varphi_f : \widehat{\mathbb{C}}\backslash K_f \longrightarrow \widehat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ be the isomorphism as above. The orthogonal trajectories $\{z : \arg(\varphi_f(z)) = \text{constant}\}$ to the family of equipotentials curves are called ***external rays*** for $K_f$. The ray of ***external argument*** $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $\theta \in \mathbb{R}/\mathbb{Z}$, is defined by $R_f(\theta) = \varphi_f^{-1}(\{re^{2\pi i\theta}|r > 1\})$.

An external ray $R_f(\theta)$ is called ***rational*** if its angle $\theta \in \mathbb{R}/\mathbb{Z}$ is rational; and ***periodic*** if $\theta$ is periodic under multiplication by the degree $d$ so that $d^p\theta \equiv \theta(mod 1)$ for some $p \geq 1$.

## 2.2 The Quadratic Polynomials

Consider the quadratic polynomials $f_c(z) = z^2 + c$, $c \in \mathbb{C}$. The dynamical behavior of $c$ plays a crucial role in determining the dynamics of $f_c$ and the topology of $K_c$. By Theorem 2.2, if $f_c^n(0) \to \infty$ as $n \to \infty$, the Julia set $J_c$ is totally disconnected. Otherwise, $f_c^n(0)$ is bounded and the Julia set is connected. This dichotomy is reflected in the definition of the Mandelbrot set $M$.

### 2.2.1 The Mandelbrot set

A complex polynomial map $f$ is called ***hyperbolic*** if the orbit of every critical point converges to an attracting cycle. $f$ is said to be ***post-critically finite*** if the critical orbit is finite. Given a family of polynomial maps, the ***connectedness locus*** is defined as the set of parameters such that the Julia set of the corresponding map is connected, or equivalently the set of parameters such that the orbit of every critical point is bounded. For the family of quadratic polynomials $f_c : z \mapsto z^2 + c$, $c \in \mathbb{C}$, the connectedness locus is known as the ***Mandelbrot set*** $\mathcal{M}$:

$$\mathcal{M} = \{c \in \mathbb{C} \mid J(f_c) \textit{ is connected } \}.$$

The **main cardioid** is the hyperbolic component in $\mathcal{M}$ which contains 0. Thus, $c \in \mathcal{M}$ if and only if 0 does not belong to the basin of attraction of the super-attracting fixed point at $\infty$. If $c \in \mathbb{C} \backslash \mathcal{M}$, then $J(f_c)$ is a Cantor set.

The Mandelbrot set is connected which was proved by Douady and Hub-

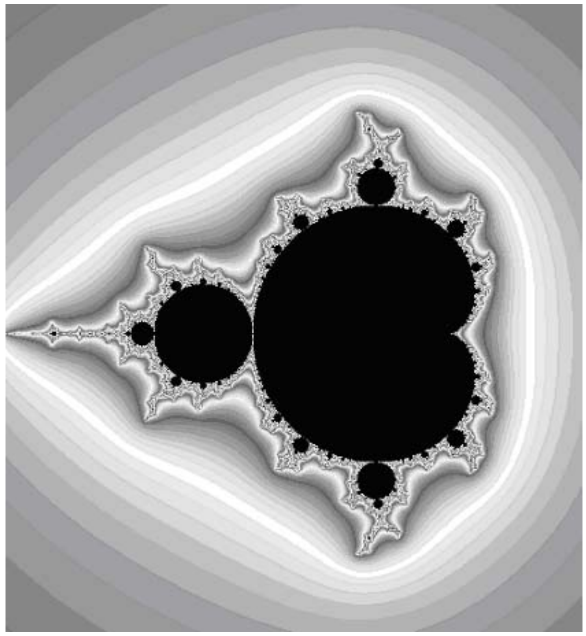

Figure 2.1: The Mandelbrot set

bard (see [DH1]). In fact, it is proven by constructing explicitly the Riemann mapping $\Phi_{\mathcal{M}} : \mathbb{C} \setminus \mathcal{M} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ defined as

$$\Phi_{\mathcal{M}}(c) = \varphi_c(c),$$

where $\varphi_c$ is the Böttcher function of $f_c$.

### 2.2.2 Limbs and Wakes of the Mandelbrot set

The boundary of each hyperbolic component $H$ of the Mandelbrot set $\mathcal{M}$ is a Jordan curve and can be parameterized by a map $\gamma_H : [0, 1) \longrightarrow \partial H$ such that if $c = \gamma_H(t)$, then $f_c$ has an indifferent periodic orbit of multiplier $e^{2\pi it}$. The point $c = \gamma_H(0)$ is called the ***root*** of the hyperbolic component $H$. Denote the main cardioid by $H_0$. At each boundary point $\gamma_{H_0}(p/q)$, there is an attached hyperbolic component $H_{p/q}$ of period $q$. We define the $p/q$-***limb*** of $\mathcal{M}$, $\mathcal{M}_{p/q}$, as the union of $c = \gamma_{H_0}(p/q)$ and the connected component of $\mathcal{M}\setminus\overline{H}_0$ attached to the main cardioid at the point $c = \gamma_{H_0}(p/q)$.

Using the isomorphism $\Phi_{\mathcal{M}}$, we can define the ***parameter external rays*** and ***equipotentials*** respectively as the preimages of the straight lines going to $\infty$ and round circles centered at 0. Define the parameter external ray of external argument $\theta$ as $\mathcal{R}_{\mathcal{M}}(\theta) = \Phi_{\mathcal{M}}^{-1}(\{re^{2\pi i\theta} : 1 < r < \infty\})$. If $\mathcal{R}_{\mathcal{M}}(\theta)$ has a limit $c \in \partial\mathcal{M}$ when $r \to 1$, we say that $\mathcal{R}_{\mathcal{M}}(\theta)$ lands at $c$. It is known that all external rays with rational arguments land at either a root of a hyperbolic component or at a ***Misiurewicz*** point, i.e. a parameter value $c \in \partial\mathcal{M}$ for which $w = 0$ is strictly preperiodic under $f_c$.

There are exactly two external rays landing at each root point in $\mathcal{M}$ (except at $c = 1/4$). Given $p/q \in (0,1)\cap\mathbb{Q}$, we denote by $\theta^-_{p/q}$ and $\theta^+_{p/q}$ the arguments of the two external rays landing at the root point of $H_{p/q}$, i.e., at $\gamma_{H_0}(p/q) \in \partial H_0$. Then, we define the $p/q$-*wake* of $\mathcal{M}$, $W_{p/q}$, as the open subset of $\mathbb{C}$ that contains the $p/q$-limb of $\mathcal{M}$ and is bounded by these two rays.

## 2.3 Hubbard Tree

Let $f : z \mapsto z^2 + c$ be a post-critically finite quadratic polynomial and $K$ be its filled Julia set. An ***embedded arc*** in $K$ is any subset of $K$ which is homeomorphic to the closed interval $[0, 1] \subset \mathbb{R}$. Since $K$ is locally connected, given any two points $x, y \in K$, there exists an embedded arc $\gamma$ in $K$ which connects $x$ and $y$. If $c$ is a Misiurewicz point, then $int(K)$ is empty set and

$\gamma$ is uniquely determined by the two end points. If $c$ is a center of some hyperbolic component of $\mathcal{M}$, since $K$ has an non-empty interior, $\gamma$ is not uniquely determined. We will show in the following how to choose a canonical embedded arc between any two points in the filled Julia set.

Since $c$ is a center of some hyperbolic component of $\mathcal{M}$, every component $U$ of $int(K)$ is a bounded Fatou component whose closure $\overline{U}$ is homeomorphic to the closed disk $\overline{\mathbb{D}}$, and every such component eventually maps to the periodic component $U_0$ which is the immediate basin of critical point 0. ***The center*** $c(U)$ is defined as the unique backward image of 0 in $U$. In particular, $c(U_0) = 0$.

Given any bounded Fatou component $U$, there exists a homeomorphism $\phi : \overline{U} \mapsto \overline{\mathbb{D}}$ which is holomorphic in $U$ with $\phi(c(U)) = 0$. ***A radial arc*** means an arc in $\overline{U}$ of the form $\phi^{-1}\{re^{i\eta} \ : 0 \leq r \leq 1\}$. Since $\phi$ is unique up to post-composition with a rotation of $\overline{\mathbb{D}}$, radial arcs are well-defined.

Following [DH1], an embedded arc $I$ is ***regulated*** if, for every bounded Fatou component $U$, the intersection $I \bigcap \overline{U}$ is either empty, a point or consists of radial arcs in $\overline{U}$.

The following lemma is from [Z1] Lemma 1.

**Lemma 2.3.** *Given any two points $x, y \in K$, there exists a unique regulated arc $I$ in $K$ with endpoints $x, y$. Furthermore, if $\eta$ is any embedded arc in $K$ which connects $x$ to $y$, then $I \bigcap J \subset \eta \bigcap J$.*

Denote the regulated arc $I$ in the above lemma by $[x, y]$. The open arc $(x, y)$ is defined as $[x, y] \setminus \{x, y\}$. Similarly we can define a semi-open arc $[x, y)$, etc.

More generally, given finitely many points $x_1, x_2, \cdots, x_n$ in $K$, there is a unique smallest connected set $[x_1, x_2, \cdots, x_n] \subset K$ which consists of regulated arcs and contains all these points. It is always a finite topological tree. (see [DH1]). We call $[x_1, x_2, \cdots, x_n]$ the ***regulated tree*** generated by $\{x_1, x_2, \cdots, x_n\}$.

Given a regulated tree $T$, a point $x \in T$ is called an ***end point*** if $T \setminus \{x\}$ is connected. The set of end points is denoted by $\partial T$ and the cardinality of $\partial T$ is called ***end-number*** of the tree. A point $x \in T$ is called a ***branch point*** if $T_0(c) \setminus \{x\}$ has more than two components. The set of branch points is denoted by $Br(T)$.

**Lemma 2.4.** *Let $\eta$ be a regulated arc containing no critical points, except for it's end points. Then $f|_\eta$ is injective and $f(\eta)$ is a regulated arc.*

See [P] Lemma 1.8 for the proof.

Given a post critically finite quadratic polynomial $f_c : z \to z^2 + c$, the (minimal) ***Hubbard tree*** is defined as the smallest regulated tree containing the critical orbit $Orb(0)$. A point $x \in T_0(c)$ is a ***vertex*** of $T_0(c)$ if $x \in Orb(0)$ or $x \in Br(T_0(c))$. Denote the set of vertices by $V(T_0(c))$. Then $V(T_0(c)) = Orb(0) \bigcup Br(T_0(c))$. The set of vertices $V(T_0(c))$ cut the Hubbard tree $T_0(c)$ up into a number of open topological intervals. The closure $I_j$ of these open intervals are called the ***edges*** of the Hubbard tree $T_0 c$. Finally denote the end-number of $T_0(c)$ by $N(c)$.

**Proposition 2.5.** *$T_0(f)$ is invariant under $f$, that is $f(T_0(f) = T_0(f)$.*

*Proof.* $T_0(f)$ is the union of regulated arcs of the form $[x_1, x_2]$ not containing critical points except possibly for their end points. By lemma 2.4, $f(T_0(f))$

is the union of regulated arcs $[f(x_1), f(x_2)]$. Since $f(T_0(f))$ is connected and contains all points of $Orb(0)$, by definition it equals $T_0(f)$. $\square$

**Proposition 2.6** (Expansivity)**.** *Given a post-critically finite quadratic polynomial $f_c$. For any $x \neq y \in V(T_0(c))$, there exists a integer $n \geq 0$ such that $0 \in f^n([x, y])$.*

*Proof.* For the proof, see [BS] §3. $\square$

**Lemma 2.7.** *Suppose that $f = z^2 + c$ is a post-critically finite map. Let $N(c)$ be the end-number of $T_0(f)$. Then $\{c, f(c), \cdots, f^{N(c)-1}(c)\}$ are exactly the only end points of the Hubbard tree $T_0(f)$.*

*Proof.* First suppose that the critical point 0 is an end point of $T_0(f)$. Then $f$ restricted on $T_0(f)$ is a homeomorphism onto itself. Then the critical orbit are exactly the set of end points. Then the statement was proved.

Next we suppose that 0 is not end point of $T_0(f)$. Since $f$ is locally one to one except at critical point, thus the critical point $\{0\}$ is the only non-end point which maps to an end point of the Hubbard tree. $\square$

## 2.4 Topological Entropy

In this section, we go over the definition and some properties of topological entropy. See [D2] for more details.

Let $X$ be a compact metric space, $\mathscr{U}$ an open cover of $X$ and $f$ a continuous map on $X$. Define the ***efficient cardinality*** $\sharp^*\mathscr{U}$ as the minimum cardinality of possible finite subcovers of $\mathscr{U}$. Set $f^*\mathscr{U} = \{f^{-1}(U)\}_{U \in \mathscr{U}}$. Given any two open covers $\mathscr{U}$ and $\mathscr{V}$ of $X$, set $\mathscr{U} \vee \mathscr{V} = \{U \cap V\}_{U \in \mathscr{U}, V \in \mathscr{V}}$ and $\bigvee^n \mathscr{U} = \mathscr{U} \vee f^*\mathscr{U} \vee \cdots \vee (f^{n-1})^*\mathscr{U}$. Note that a non empty element of $\bigvee^n \mathscr{U}$ corresponds to a given $n$-itinerary in $\mathscr{U}$. Now we can define the ***entropy*** of $f$ on $X$ with respect to $\mathscr{U}$ as $h(X, \mathscr{U}, f) = \lim \frac{1}{n} \log(\sharp^* \bigvee^n \mathscr{U})$. Note that the limit always exists. Finally we can define the ***topological entropy*** as $h(X, f) = \sup_{\mathscr{U}} h(X, \mathscr{U}, f)$. Sometimes we write simply $h(f)$ when the choice of X is clear.

We have the following properties.

**Proposition 2.8.** $h(f^k) = k \cdot h(f)$

**Proposition 2.9.** *If $X = X_1 \bigcup X_2$, with $X_i$ compact and $f$-invariant for $i = 1, 2$, then $h(X, f) = \sup(h(X_1, f), h(X_2, f))$.*

**Proposition 2.10.** *Let $Y$ be a closed f-invariant subset of $X$. If*

$$\lim_{n \to \infty} d(f^n(x), Y) = 0, \forall x \in X,$$

*then $h(X, f) = h(Y, f)$, where $d(\cdot, \cdot)$ stands for the metric.*

We will not give proofs of the above three properties. The first two are easy to prove. You can find a proof of the third property in Douady's paper [D2].

**Proposition 2.11.** *Consider the following semi-conjugacy diagram where $f, g$ are continuous maps respectively on compact metric spaces $X$ and $Y$, and $\pi : Y \longrightarrow X$ is surjective. Then $h(f) \leq h(g)$. Furthermore, suppose that the cardinality for any fiber $\pi^{-1}(x)$ is bounded by a finite number $m$, then*

$h(g) = h(f)$.

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

*Proof.* The first part is easy. Let's prove the second part.

First we claim that $h(g) \leq h(f) + \log m$

Proof of the claim: For each open covering $\mathscr{V}$ of $Y$, we can find an open covering $\mathscr{U} = (U_i)_{i \in I}$ of $X$, and an open covering $\mathscr{W} = (W_{i,j})_{i \in I, j \in \{1,2,\cdots,m_i\}}$ finer than $\mathscr{V}$, with $m_i \leq m$ and $W_{i,j} \subset \pi^{-1}(U_i)$. Then for each integer $n$, we have $\sharp^* \bigvee^n \mathscr{W} \leq m^n \cdot \sharp^* \bigvee^n \mathscr{U}$. By taking the logarithm, the claim is proved.

To finish the proof the property, we only need to know that for each integer $n$, we have a similar inequality $h(g^n) \leq h(f^n) + \log m$, and then $h(g) \leq h(f) + \frac{1}{n} \log m$. $\square$

**Proposition 2.12.** *Let $c \in M$ such that $f_c$ is post-critically finite, then*

$$h(\hat{\mathbb{C}}, f_c) = h(K_c, f_c) = h(J_c, f_c) = \log 2.$$

*Proof.*

1. First let us prove $h(\hat{\mathbb{C}}, f_c) = h(K_c, f_c)$. Since all the points of $\hat{\mathbb{C}} \setminus K_c$ are attracted by $\infty$, by proposition 2.10, $h(\hat{\mathbb{C}}, f_c) = h(K_c \cup \infty, f_c) = h(K_c, f_c)$.

2. Next we will show that $h(K_c, f_c) = h(J_c, f_c)$. If $int(K_c) = \emptyset$, then

$K_c = J_c$. If $int(K_c) \neq \emptyset$, then there is an attracting or parabolic cycle $\xi$ attracting $int(K_c)$. Again $h(K_c, f_c) = h(J_c \cup \xi, f_c) = h(J_c, f_c)$.

3. Finally we will prove that $h(J_c, f_c) = \log 2$. Let $\gamma_c : \mathbb{T} \to J_c$ be the Carathédory loop and $q$ the doubling map $t \mapsto 2t$ on the unit circle $\mathbb{T}$. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{T} \arrow[r, "q"] \arrow[d, "\gamma_c"] & \mathbb{T} \arrow[d, "\gamma_c"] \\ J(f_c) \arrow[r, "f_c"] & J(f_c) \end{tikzcd}$$

Since $K_c$ is locally connected, $\gamma_c$ is surjective. Also for all $x \in J_c$, the cardinality of $\gamma_c^{-1}(x)$ is bounded by a finite integer $m$ which is determined by $f_c$. Then by proposition 2.11, $h(J_c, f_c) = h(\mathbb{T}, q) = \log 2$.

□

**Remark 2.13.** The proposition still holds even $f_c$ is not post-critically finite. And more general, for complex polynomial $f$ with degree d, we have the following result: $h(\hat{\mathbb{C}}, f) = h(K(f), f) = h(J(f), f) = \log d$.

**Lemma 2.14.** ***(PERRON-FROBENIUS)*** *Let $A$ be a $k \times k$ matrix with entries $0$ or $1$. Then there exists a real eigenvalue $\lambda_1$ such that all other eigenvalues $\lambda_i$ satisfy $|\lambda_i| \leq \lambda_1$, for $i = 2, \cdots, k$. Moreover, we have the eigenvalue estimate $\min_i \sum_j a_{ij} \leq \lambda_1 \leq \max_i \sum_j a_{ij}$.*

*Proof.* For the proof, see [H1] §8. □

Let $f$ be a post-critically finite quadratic polynomial and $T_0(f)$ be its Hubbard tree. For any edge $I$ of $T_0(f)$, $f$ is injective on $I$ and $f(I)$ is a union of edges of $T_0(f)$. Define the Markov matrix $M = (M_{i,j})$ of $f$ in the following way: Let $I_1, I_2, \cdots, I_k$ be the edges of $T_0(f)$, set $M_{i,j} = 1$ if $I_j \subset f(I_i)$ and 0 otherwise. Obviously $M$ is a $k \times k$ matrix with entries 0 or 1.

By the Perron-Frobenius Lemma, there exists a leading eigenvalue $\lambda_1$. Since for any edge $a_i$, $f(a_i)$ is always a union of some edges, and then $\sum_j a_{ij} \geq 1$. Thus $\lambda_1 \geq \min_i \sum_j a_{ij} \geq 1$.

We claim that the topological entropy of $f$ acting on $T_0(f)$ is the natural logarithm of $\lambda_1$.

To prove the above statement, we have to introduce subdivisions on the Hubbard Tree. Again let $f$ be a post-critically finite quadratic polynomial and $T_0(f)$ be its Hubbard tree. $\mathcal{S} = \{I_1, \cdots, I_k\}$ is called a subdivision of $T_0(f)$ if $I_i$ is a closed regulated arc, $\bigcup_{i=1}^k I_i = T_0(f)$ and $I_i \bigcap I_j$ is either an empty set or consists of a single point, for any $1 \leq i < j \leq k$. Also define the boundary $\partial\mathcal{S}$ as the set of all end points of $\{I_1, \cdots, I_k\}$. Define the critical subdivision $\mathcal{S}_f$ be the set of all closed edges of $T_0(f)$, then $\partial\mathcal{S}_f$ is the set of vertices of $T_0(f)$.

A subdivision $\mathcal{S}$ is finer than $\mathcal{S}_f$ if each element of $\mathcal{S}$ is a subset of some element in $\mathcal{S}_f$.

For any subdivision $\mathcal{S}$ on $T_0(f)$ and integer $k > 1$, define the subdivision $(f^k)^*\mathcal{S}$ by $\partial(f^k)^*\mathcal{S} = f^{-k}(\partial\mathcal{S}) \bigcap T_0(f)$. If $\mathcal{S}$ and $\mathcal{S}'$ are two subdivisions of $T_0(f)$, define $\mathcal{S} \vee \mathcal{S}'$ by $\partial(\mathcal{S} \vee \mathcal{S}') = \partial\mathcal{S} \bigcup \partial\mathcal{S}'$. Finally we set

$$\bigvee^n \mathcal{S} = \mathcal{S} \vee f^*\mathcal{S} \vee \cdots \vee (f^{n-1})^*\mathcal{S}.$$

Note that since the critical point 0 is not a branch point and $f$ is locally homeomorphic, the set of branch points in $T_0(f)$ is backward invariant under $f|_{T_0(f)}$. Then $\partial \bigvee^n \mathcal{S}_f$ is the union of all $n$-preimage of points in Orb(0) in $T_0(f)$ and all branch points of $T_0(f)$. The entropy of $f$ acting on $T_0(f)$ defined via $\mathcal{S}$ is

$$h(T_0(f), \mathcal{S}, f) = \lim \frac{1}{n} \log \ \sharp \bigvee^n \mathcal{S}.$$

For $\mathcal{S}$ finer than $\mathcal{S}_f$, the limit exists and is also the infimum.

We have the following important lemma.

**Lemma 2.15.** *$h(T_0(f), \mathcal{S}_f, f) = h(T_0(f), f)$.*

*Proof.* See Douady [D2] for a proof of the corresponding statement for a continuous piecewise monotone map of the interval. His proof extends easily to piecewise monotone maps from a tree to itself. □

**Lemma 2.16.** *Let $f$ be a post-critically finite quadratic polynomial. Also let $M$ be the corresponding $k \times k$ Markov matrix and $\lambda$ be the leading eigenvalue. Then $h(T_0(f), f) = \log \lambda$.*

*Proof.* Since $f$ is post-critically finite and the set of branch points is backward invariant under $f|_{T_0(f)}$. Then

$$\partial \mathcal{S}_f \subset \partial(f^* \mathcal{S}_f) \subset \cdots \subset \partial((f^{n-1})^* \mathcal{S}_f).$$

and we have $\bigvee^n \mathcal{S}_f = (f^{n-1})^* \mathcal{S}_f$.

Let $\mathcal{S}_f = \{I_1, \cdots, I_k\}$. For any subdivision $\mathcal{S}$ finer than $\mathcal{S}_f$, define the ***Markov vector*** $\upsilon_{\mathcal{S}} = (\upsilon_1, \upsilon_2, \cdots, \upsilon_k)$ where $\upsilon_i$ is the number of elements of $\mathcal{S}$ which is in $I_i$. Apparently, $\upsilon_{\mathcal{S}_f} = (1, 1, \cdots, 1)$.

By the definition of Markov matrix $M$, we can check that $\upsilon_{f^*\mathcal{S}} = M^* \cdot \upsilon_{\mathcal{S}}$ where $M^*$ stands for the transpose of $M$. And then it follows that $\upsilon_{\bigvee^n \mathcal{S}_f} = (M^*)^{n-1} \cdot \upsilon_{\mathcal{S}_f}$.

Given any vector $\upsilon = (\upsilon_1, \upsilon_2, \cdots, \upsilon_k)$ such that $\upsilon_i > 0$ for all $i$, set $\|\upsilon\| = \sum \upsilon_i$. It is known that there exists positive constants $0 < C_1 < C_2$ such that $C_1 \cdot \lambda^n \leq \|(M^*)^n\| \cdot \upsilon \leq C_2 \cdot \lambda^n$. Also by definition, $\|\upsilon_{\mathcal{S}}\| = \sharp\mathcal{S}$, then $C_1 \cdot \lambda^n \leq \sharp \bigvee^n \mathcal{S}_f \leq C_2 \cdot \lambda^n$.

Taking the logarithm, dividing by $n$ letting $n \to \infty$, it follows that $h(h(T_0(f), \mathcal{S}_f, f)) = \log \lambda$. Together with Lemma 2.15, this proves Lemma 2.16. $\square$

In general, we have the following property.

Let $M$ be a $k \times k$ matrix with entries 0 and 1 such that its leading eigenvalue $\lambda \geq 1$. Let $X$ be a connected compact space and $f : X \mapsto X$ a continuous map. Also let $\mathcal{A} = (A_1, A_2, \cdots, A_k)$ be a set of compact subsets of $X$ such that the interior $int(A_i)$'s are non-empty and mutually disjoint. We say that $\mathcal{A}$ is an **over-markov packing** with matrix $M$ if $A_j \subset f(A_i)$ whenever $M_{i,j} = 1$. $\{A_{i_1}, A_{i_2}, \cdots, A_{i_n}\}$ is called ***$n$-itinerary in $X$ with respect to $\mathcal{A}$*** if $A_{i_{j+1}} \subset f(A_{i_j})$ for $j = 1, 2, \cdots, n-1$ where $1 \leq i_j \leq n$ for all $j$.

**Proposition 2.17.** *If $\mathcal{A} = (A_1, A_2, \cdots, A_k)$ is an over-markov packing with matrix $M$ , then $h(X, f) \geq \log \lambda$, where $M$ and $\lambda$ is defined as above.*

*Proof.* [Comparing [D2]] Let $\upsilon_0 = (1, 1, \cdots, 1)$ and $\upsilon_n = M^n \cdot \upsilon_0$. The number of possible $n$-itineraries in $X$ with respect to $\mathcal{A}$ is greater than $\|\upsilon_{n-1}\|$. Let $U_i = X - (\bigcup_{j \neq i} A_j)$ and $\mathcal{U} = \{U_i\}$. If $x$ has n-itinerary $(A_{i_0}, A_{i_1}, \cdots, A_{i_{n-1}})$ with respect to $\mathcal{A}$, then $(U_{i_0}, U_{i_1}, \cdots, U_{i_{n-1}})$ is its only n-itinerary with respect to $\mathcal{U}$. In other words, the number of possible $n$-itineraries in $X$ with respect to $\mathcal{U}$ is greater than the number of possible n-itineraries in $X$ with respect to $\mathcal{A}$. Finally, we can see that the efficient cardinality $\sharp^*\bigvee^n\mathcal{U} \geq \|\upsilon_{n-1}\|$. By the definition of entropy, it then follows that $h(X, f) \geq \log\lambda$. $\square$

# Chapter 3

# Quadratic Case

In this chapter, we shall always assume that $f$ is a quadratic polynomial.

We will study the topological entropy of the post-critically finite quadratic maps. The following is the main theorem. It's an extension of Douady's result, as described in §1.1.

**Theorem 3.1.** *Given any two parameters $c_2$ and $c_1$ on $\mathcal{M}^0$. If $c_2 \prec c_1$, then $h(T_0(c_2), f_{c_2}) \le h(T_0(c_1), f_{c_1})$.*

## 3.1 External Rays landing on the Hubbard tree

In this section, we show that the entropy on the Hubbard tree for the given map is the same as the entropy on some subset of $\mathbb{T}$ for the angle doubling map. It is based on [D2].

**Definition 3.2.** *Given any integer $k$ and angles $\theta_i \in \mathbb{R}/\mathbb{Z}$ for $1 \le i \le k$. The angles $\{\theta_1, \theta_2, \theta_3, \cdots, \theta_k\}$ are in* ***positive cyclic order*** *if there exists a* *lift $\hat{\theta}_i$ of $\theta_i$ on $\mathbb{R}$ such that*

$$\hat{\theta}_1 < \hat{\theta}_2 < \cdots < \hat{\theta}_k < \hat{\theta}_1 + 1$$

**Definition 3.3.** *Given $0 \leq \theta \neq \theta' < 1$, the* **open angle interval** $(\theta, \theta')$ *means the the set of $\eta$ such that $\{\theta, \eta, \theta'\}$ are in positive cyclic order. We can define the closed angle interval $[\theta, \theta']$ or the half-open angle interval similarly.*

Let $f$ be a post-critically finite polynomial. Given $x \in T_0(f)$, define $\theta(x)$ to be the angle of the external ray which lands on $x$ if $x \in J(f)$, or the angle of the external ray which lands on the root point of the Fatou component which contains $x$ if $x \notin J(f)$. We call $R_f(\theta(x))$ the **external ray associated to** $x$ **in** $T_0(f)$. If there are more than one external rays landing on $x$ (or the root point of the Fatou component which contains $x$), specify them by $R_f(\theta(x)^-)$, $R_f(\theta(x)^+)$ , $R_f(\theta(x)^1)$, $R_f(\theta(x)^2)$, etc.

Let $f$ be a post-critically finite polynomial and $T_0 = T_0(f)$ be the Hubbard tree. Define trees $T_0 \subset T_1 \subset T_2 \subset \cdots$ by $T_{n+1} = f^{-1}(T_n)$.

**Definition 3.4.** *Let $f = f_c$ be a post-critically finite quadratic map and $x \neq y \in T_1(f) \setminus \{0\}$. We say that $x \prec y$ if $x \in (0, y)$ and $\theta(x) \neq \theta(y)$.*

**Definition 3.5** (***external rays and angle interval associated to regulated arc***)**.** *Let $f = f_c$ be a post-critically finite quadratic polynomial. Suppose that $x \prec y$. Let $\{\theta(x)^-, \theta(x)^+, \theta(y)^-, \theta(y)^+\}$ be the angles of the four external rays such that the angle intervals $[\theta(x)^-, \theta(y)^-]$ and $[\theta(x)^+, \theta(y)^+]$ are the two smallest angle intervals which contain all external rays landing on $[x, y]$. We* *call*

$$\{R_f(\theta(x)^-), R_f(\theta(y)^-), R_f(\theta(y)^+), R_f(\theta(x)^+)\}$$

*the* **external rays associated to** $(x, y]$*. Also define the angle interval* $(\theta(x)^-,$ $\theta(x)^+)$ *to be the* **angle interval associated to** $(x, y]$ *(or* $[x, y]$*). Note that when there is only one external ray landing on* $y$*,* $\theta(y)^- = \theta(y)^+ = \theta(y)$*. In that case, denote the external rays associated to* $(x, y]$ *by* $\{R_f(\theta(x)^-), R_f(\theta(y)),$ $R_f(\theta(x)^+)\}$*. See the following Figure 3.1 for the illustration of external rays and angle interval associated to regulated arcs.*

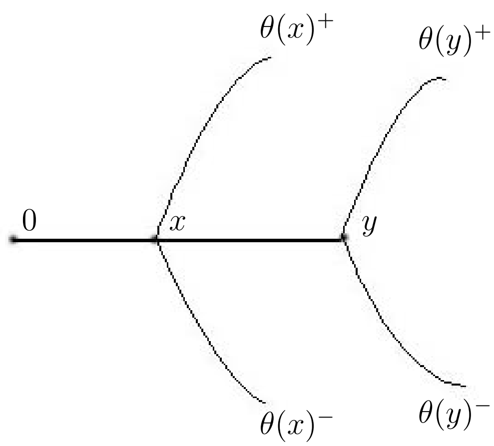


Figure 3.1: The illustration of angle interval associated to $[x, y]$.

**Lemma 3.6.** *Let* $f = f_c$ *be a post-critically finite quadratic map. If* $x \prec y$*, then*

$$\{\theta(x)^-, \theta(y)^-, \theta(y)^+, \theta(x)^+\}$$

*is in positive cyclic order. In particular, "*$\prec$*" is a partial order.*

*Proof.* The first statement follows from the definition. And the second part is from the first statement. $\square$

**Definition 3.7** (**Characteristic Angle Intervals $\mathscr{U}(f)$ associated to the Hubbard tree $T_0(f)$**)**.** *Let $f$ be a post-critically finite polynomial and $T_0 = T_0(f)$ be the Hubbard tree.*

*Since we can write $T_1 \setminus T_0$ uniquely as a union of finitely many half open regulated arcs, denote it by $T_1 \setminus T_0 = \bigcup_{k=1}^{n} I_k$ where $I_k = (x_k, y_k], x_k \in T_0, y_k \in \partial T_1$.*

*Let $\{R_f(\theta(x_k)^-), R_f(\theta(y_k)^-), R_f(\theta(y_k)^+), R_f(\theta(x_k)^+)\}$ be the external rays associated to $(x_k, y_k]$. Define the* **angle interval associated** *to $I_k$ by $U_k = (\theta(x_k)^-, \theta(x_k)^+)$. Finally, we define the characteristic angle intervals $\mathscr{U}(f)$ associated to the Hubbard tree $T_0$ as $\mathscr{U}(f) = \bigcup_{k=1}^{n} U_k$. If the function $f$ is parameterized by $f_c$, $f = f_c$, we simply denote the corresponding characteristic angle intervals by $\mathscr{U}(c)$.*

**Remark 3.8.** Two such regulated arcs in $T_1 \setminus T_0$ defined above may have a non-empty intersection. If $I_k \bigcap I_l \neq \emptyset$, then either $U_k = U_l$ or $U_k \bigcap U_l = \emptyset$.

**Example 3.9.** The following Figure 3.2 shows the filled Julia set of $f_c$ where $c = -0.15652 - 1.032245i$. The critical orbit and the external rays associated to it have been marked. The $\alpha$-fixed point and $-\alpha$ are also been marked. It's easy to see that $T_1(c) \setminus T_0(c) = (f_c^3(c), -f_c^2(c)] \cup (-\alpha, -c]$, where $-c$, $-f_c^2(c)$ are the co-images of $c$, $-f_c^2(c)$ respectively.

Since the angle interval associated to $(f_c^3(c), -f_c^2(c)]$ is $(\frac{4}{5}, \frac{1}{15})$ and the angle interval associated to $(-\alpha, -c]$ is $(\frac{9}{14}, \frac{11}{14})$. Then

$$\mathscr{U}(c) = (\frac{4}{5}, \frac{1}{15}) \bigcup (\frac{9}{14}, \frac{11}{14}).$$

**Lemma 3.10.** *Suppose that $f$ is a post-critically finite quadratic map. Then $R_f(\theta)$ lands on $T_0$ if and only if the orbit of $\theta$ under the doubling map $q$ never hits $\mathscr{U}(f) = \bigcup U_k$.*

*Proof.* Let us prove that the condition is necessary. Since $T_0$ is invariant, $f(T_0) = T_0$, if $R_f(\theta)$ lands on $T_0$, then $R_f(q(\theta))$ still lands on $T_0$. And then the orbit of $\theta$ never hits $\mathscr{U}(f)$.

let us prove the condition is sufficient. Suppose $R_f(\theta)$ lands on $x \in J \setminus T_0$, define

$$\alpha^+ = \inf\{\alpha \geq \theta \mid R_f(\alpha) \textit{ lands on } T_0\}$$

and

$$\alpha^- = \sup\{\alpha \leq \theta \mid R_f(\alpha) \textit{ lands on } T_0\}.$$

Since $J$ is locally connected and path connected, $T_0$ is a closed set, it is clear that external rays with angle $\alpha^+$, $\alpha^-$ must land on $T_0$, and $\theta \subset (\alpha^+, \alpha^-)$.

We will show that there exists an integer $k$ such that $q^k(\alpha^+, \alpha^-) \subset \mathscr{U}(f)$. In fact, since $T_0 \subset T_1 \subset T_2 \subset \cdots$ and $\bigcup_{n=0}^{\infty} T_n$ is dense in $J$, the image of $(\alpha^+, \alpha^-)$ under the iteration of $q$ must hit $\gamma_c^{-1}(T_1)$, i.e., $\exists k \in \mathbb{N}$, such that $q^k(\alpha^+, \alpha^-) \cap \gamma_c^{-1}(T_1) \neq \varnothing$, $q^{k-1}(\alpha^+, \alpha^-) \cap \gamma_c^{-1}(T_1) = \varnothing$, where $\gamma_c : \mathbb{T} \to J_c$ is the Carathédory map.

Since $T_1 = f^{-1}(T_0) = T_0 \cup (T_1 \setminus T_0)$, $q^k(\alpha^+, \alpha^-)$ will hit $\gamma_c^{-1}(T_1 \setminus T_0)$ before it hits $\gamma_c^{-1}(T_0)$. Then by the construction of $\mathscr{U}(f)$, $q^k(\alpha^+, \alpha^-) \subset \mathscr{U}(f)$. Finally,

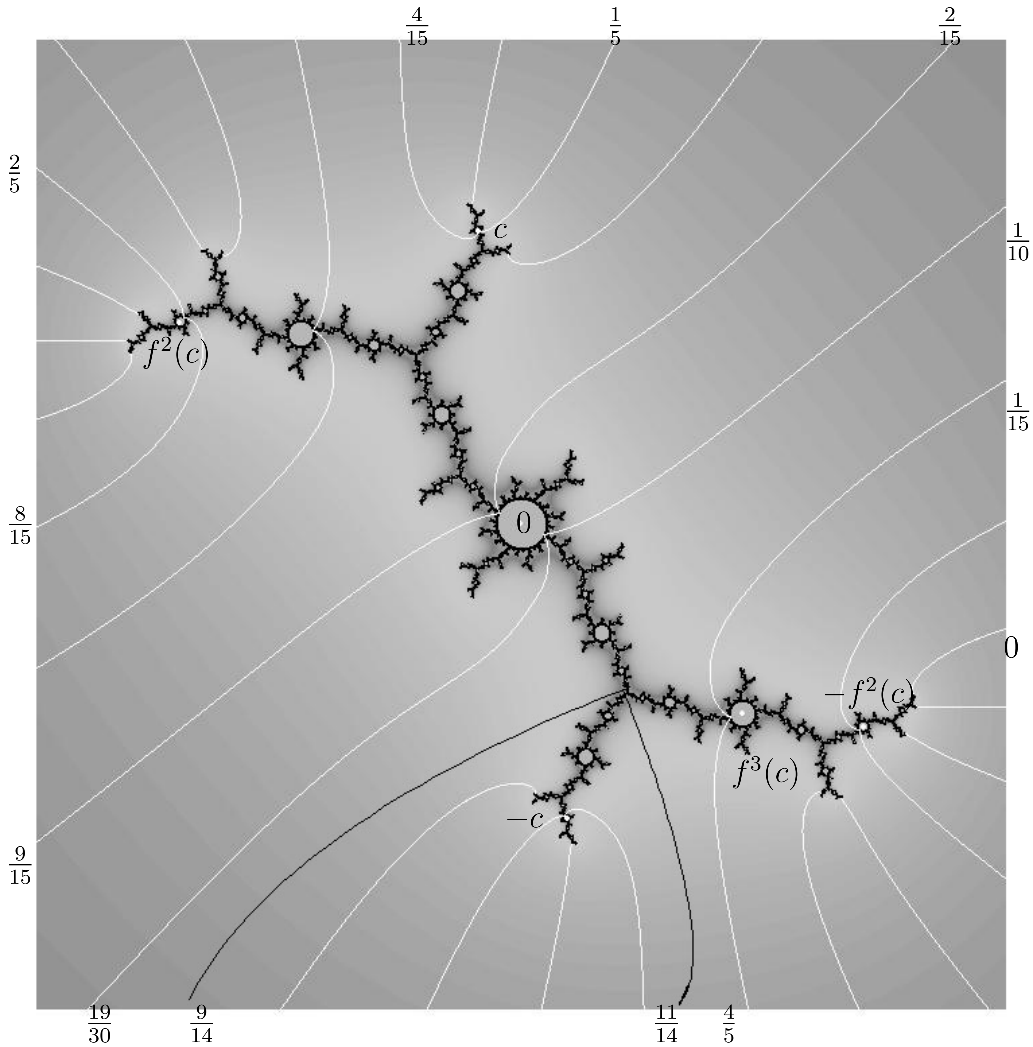


Figure 3.2: The filled Julia Set of $f_c$ with $c = -0.15652 - 1.032245i$

since $\theta \subset (\alpha^+, \alpha^-)$, this finishes the proof of the lemma. $\square$

**Corollary 3.11.** $h(T_0(f), f) = h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(f)), q)$

*Proof.* We have the following commutative diagram, where $\mathbb{T} = \{e^{2\pi i\alpha} | \alpha \in \mathbb{R}\}$ is the unit circle, $q$ is the angle doubling map on the circle, and $\gamma_c : \mathbb{T} \to J_c$ is the Carathédory loop.

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "q"] \arrow[d, "\gamma_c"] & \mathbb{T} \arrow[d, "\gamma_c"] \\
J(f_c) \arrow[r, "f_c"] & J(f_c)
\end{tikzcd}$$

By Lemma 3.10, $\gamma_c^{-1}(T_0(f)) = \mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(f))$, thus we have the following commutative diagram,

$$\begin{tikzcd}
\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_i))) \arrow[r, "q"] \arrow[d, "\gamma_{c_i}"] & \mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_i))) \arrow[d, "\gamma_{c_i}"] \\
T_0(c_i) \arrow[r, "f_{c_i}"] & T_0(c_i)
\end{tikzcd}$$

and we get

$$h(T_0(c_i), f_{c_i}) = h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_i)), q)$$

$\square$

## 3.2 Entropy on the Hubbard tree, case I

In this section we will prove theorem 3.1 under the condition that the two Hubbard trees have same end-number. In fact, it states that given two post-critically finite parameters $c_1$ and $c_2$, if $c_2 \prec c_1$ and $N(c_1) = N(c_2)$, then $h(T_0(c_2), f_{c_2}) \leq h(T_0(c_1), f_{c_1})$.

**Definition 3.12.** *Define $\mathcal{M}^0$ to be all parameters on $\mathcal{M}$ such that the corresponding maps are post-critically finite:*

$$\mathcal{M}^0 = \{c \in \mathcal{M} \mid f_c \ is \ postcritically \ finite\}.$$

**Definition 3.13** (Partial order $\prec$ on $\mathcal{M}^0$)**.** *Let $c_1$, $c_2$ be two different parameters on $\mathcal{M}^0$. We say that $c_2 \prec c_1$ if $c_2$ (or the root point of the hyperbolic component which contains $c_2$) separates $c_1$ from* $0$.

From the above definition it is easy to see that "$\prec$" is transitive: if $c_2 \prec c_1$ and $c_3 \prec c_2$, then $c_3 \prec c_1$.

**Lemma 3.14.** *(***Orbit forcing***) Given any two parameters $c_2$ and $c_1$ in $\mathcal{M}^0$ with $c_2 \prec c_1$. Then there exists a unique $x = x(c_1, c_2) \in T_0(c_1)$ which has the same orbit as $c_2$ (or as the root point of the immediate basin which contains $c_2$). And furthermore, if an external ray $R_{c_2}(\theta)$ lands on $c_2$ (or on the root point of the Fatou component which contains $c_2$) in the dynamical plane of $f_{c_2}$ , then the external ray $R_{c_1}(\theta)$ with same argument lands on $x$ in the dynamical plane of $f_{c_1}$.*

*Proof.* For the proof, see [M2] §7 or [BS] §6. □

**Lemma 3.15.** *Given any two parameters $c_2$ and $c_1$ in $\mathcal{M}^0$. If $c_2 \prec c_1$, then $N(c_2) \leq N(c_1)$.*

*Proof.* First suppose that $c_2$ is a Misiurewicz point, external rays with angles $\theta_i,\ i = 1, 2, ...k$ landing on it in the parameter space. Then on the dynamical plane of $f_{c_2}$, external rays with same angles $\theta_i,\ i = 1, 2, ...k$ land on the critical value $c_2$. Since $c_2$ is preperiodic, the angle set $\{\theta_1, \cdots, \theta_k\}$ is preperiodic under the angle doubling map $q$ and external rays with angle $2^m\theta_i,\ i = 1, 2, ...k$ land on $f_{c_2}^m(c_2)$.

By Lemma 3.14, on the dynamical plane of $f_{c_1}$, external rays with the same angles $\theta_i,\ i = 1, 2, ...k$ land on some point $x$ on the Hubbard tree of $f_{c_1}$, i.e., $x \in T_0(f_{c_1})$. Then there is a bijection between the orbit of $x \in J(f_{c_1})$ and the orbit of $c_2 \in J(f_{c_2})$ by the external rays landing on them. And so the regulated path generated by orbit of $x$ has $N(c_2)$ end points, $\mathbf{Orb}(x) = [x, f_{c_1}(x), \cdots, f_{c_1}^{N(c_2)}(x)]$. Finally the fact that this regulated path is a subset of the Hubbard tree $T_0(c_1)$ gives us the conclusion that $N(c_2) \leq N(c_1)$ in the first case.

Second, in the case that $f_{c_2}$ has a periodic critical point, the proof is similar. External rays with angles $\theta_i,\ i = 1, 2$ land on the root point of the hyperbolic component which contains $c_2$ in the parameter space. Then on the dynamical plane of $f_{c_2}$, external rays with same angles $\theta_i,\ i = 1, 2$ land on the root point of the Fatou component which contains critical value $c_2$. Since $c_2$ is periodic, the angle set $\{\theta_1, \theta_2\}$ is periodic under the angle doubling map $q$. For any integer $m$, external rays with angle $2^m\theta_i,\ i = 1, 2$ land on root point of Fatou component which contains $f_{c_2}^m(c_2)$.

By Lemma 3.14, on the dynamical plane of $f_{c_1}$, external rays with the same angles $\theta_i,\ i = 1, 2$ land on some point $x$ on the Hubbard tree of $f_{c_1}$, i.e., $x \in T_0(c_1)$. Then there is a bijection between the orbit of $x \in T_0(c_1)$ and the orbit of root point of Fatou component which contain $c_2$ by the external rays landing on them. And so the regulated path generated by orbit of $x$ has $N(c_2)$ end points, $\mathbf{Orb}(x) = [x, f_{c_1}(x), \cdots, f_{c_1}^{N(c_2)}(x)]$. Finally the fact that this regulated path is a subset of Hubbard tree $T_0(c_1)$ give us the conclusion that $N(c_2) \leq N(c_1)$ in the second case. $\square$

**Lemma 3.16.** *Let $f_{c_i}$ be post-critically finite maps with $c_2 \prec c_1$ and $N(c_1) = N(c_2)$. Also let $x = x(c_1, c_2)$ as defined in Lemma 3.14. Then*

$$T_0(c_1) \cong T_0(c_2) \cong [x, f_{c_1}(x), \cdots, f_{c_1}^{N-1}(x)],$$

*where "$\cong$" means "topologically homeomorphic".*

*Proof.* For simplicity, denote the end-number number by $N(c_1) = N(c_2) = N$ and $f_{c_1} = f$.

By the definition of $x$, the external rays landing on $x$ in $T_0(c_1)$ are exactly the same as the external rays landing on $c_2$ in $T_0(c_2)$. Then $\{x, f(x), \cdots, f^{N-1}(x)\}$ are exactly the end points of $[x, f(x), \cdots, f^{N-1}(x)]$ and

$$T_0(c_2) \cong [x, f(x), \cdots, f^{N-1}(x)].$$

We will prove inductively that there are no branch points and critical points in $(f^k(x), f^k(c))$ for $k = 0, 1, \cdots, N-1$.

First obviously $0 \notin (x, c_1)$ and there are no branch points in $(x, c_1)$. (Otherwise, it will be easy to see that $N(c_1) > N(c_2)$.) Suppose now that there are no branch points and critical points in $(f^k(x), f^k(c_1))$ for all $k \leq k_0 < N-1$. Since $f$ is injective on a neighborhood of any regulated arc not containing the critical point, it follows that $(f^{k_0+1}(x), f^{k_0+1}(c_1))$ also does not contain branch points. If $0 \in (f^{k_0+1}(x), f^{k_0+1}(c_1))$, since $N(c_1) = N(c_2)$, there exists $k_0 + 1 < l < N$ such that $f^l(x) \in (0, f^{k_0+1}(c_1))$. But then $f^{k_0+1}(x)$ is not the end point of regulated tree $[x, f(x), \cdots, f^{N-1}(x)]$. This is a contradiction.

Since there are no branch points in $(f^k(x), f^k(c))$ for $k = 0, 1, \cdots, N-1$, then $T_0(c_1) \cong [x, f(x), \cdots, f^{N-1}(x)]$. And furthermore, $f^k(x)$ is the only point which separates $f^k(c_1)$ from 0. $\square$

**Lemma 3.17.** *Let $f = f_c$ be a post-critically finite quadratic map. Given $x, y \in T_0(c) \cap J_c$. Suppose that $f^i(x) \prec f^i(y)$ for $0 \le i \le k$, then the following holds.*

1. *If $f^k(y) \prec -y$ in $T_1(c)$, then $f^k(x) \prec -x$ in $T_1(c)$.*

2. *If $-y \prec f^k(y)$ in $T_1(c)$, then $-x \prec f^k(x)$ in $T_1(c)$.*

*Proof.* First note that $T_1(c)$ is the union of Hubbard tree $T_0(c)$ and its 180 degree rotation: $T_1(c) = T_0(c) \bigcup \{-T_0(c)\}$.

Since $f^i(x) \prec f^i(y)$ for $0 \le i \le k$, then $0 \notin [f^i(x), f^i(y)]$ and $f : [f^i(x), f^i(y)] \to [f^{i+1}(x), f^{i+1}(y)]$ is a homeomorphism for $0 \le i \le k-1$. Then $f^k : [x, y] \to [f^k(x), f^k(y)]$ is also a homeomorphism. Also from the fact $x \prec y$, we can get $0 \notin [-x, -y]$ and $-x \prec -y$. Since $[-x, -y]$ is the symmetric arc of $[x, y]$ with respect to 0, we still have the homeomorphism $f^k : [-x, -y] \to [f^k(x), f^k(y)]$.

Now let us prove the first statement by contradiction. Suppose that $f^k(y) \prec -y$ and $-x \prec f^k(x)$ in $T_1(c)$, then we have

$$-x \prec f^k(x) \prec f^k(y) \prec -y.$$

It follows that $[f^k(x), f^k(y)] \subset [-x, -y]$. Then $f^k$ restricted on $[-x, -y]$ is a homeomorphism onto its proper subset. This is a contradiction with the expansivity of $f_c$ on $J_c$ since $-x, -y \in J_c$.

The second statement can be proved similarly. $\square$

**Theorem 3.18.** *Suppose that $f_{c_1}$, $f_{c_2}$ are post-critically finite maps. If $c_2 \prec c_1$ and $N(c_1) = N(c_2)$, then $\mathscr{U}(c_1) \subset \mathscr{U}(c_2)$.*

*Proof.* Both $\mathscr{U}(c_1)$ and $\mathscr{U}(c_2)$ are a disjoint union of angle intervals each of which is associated to a regulated arc in $T_1(c_1)$ and $T_1(c_2)$. To prove the theorem, we show in the following that for each such angle interval $U \subset \mathscr{U}(c_1)$, there exists an angle interval $U' \subset \mathscr{U}(c_2)$ such that $U \subset U'$.

In the following, for simplicity For simplicity denote $c_1 = c$, $N(c_1) = N(c_2) = N$ and $f_{c_1} = f$.

Let $T_1(c) \setminus T_0(c) = \bigcup_{k=1}^{l} I_k$ which is a union of finite regulated arcs. Given any regulated arc $I_k$,

1. If $I_k = (f^{k_1}(c), -f^{k_2}(c)]$ for some $1 \leq k_1, k_2 < N$, then $f^{k_1}(c) \prec -f^{k_2}(c)$. (Obviously $k_1 \neq k_2$. )

   If $\{R_c(\theta(f^{k_1}(c))^-), R_c(\theta(-f^{k_2}(c))^-), R_c(\theta(-f^{k_2}(c))^+), R_c(\theta(f^{k_1}(c))^+)\}$ are the external rays associated to $[f^{k_1}(c), -f^{k_2}(c)]$, then

   $$(\theta(f^{k_1}(c))^-, \theta(f^{k_1}(c))^+) \subset \mathscr{U}(c_1)$$

   is an angle interval of $\mathscr{U}(c_1)$.

   Also by Lemma 3.17, $f^{k_1}(x) \prec -f^{k_2}(x)$.(Note: If $k_1 > k_2$ apply first statement of lemma 3.17; and If $k_1 < k_2$ apply second statement of

Lemma 3.17) Since external rays landing on $f^{k_1}(x)$ in regulated tree $T_1(c_1)$ are exactly the same external rays landing on $f_{c_2}(c_2)$ in regular tree $T_1(c_2)$. Then similar to above,

$$(\theta(f^{k_1}(x))^-, \theta(f^{k_1}(x))^+) \subset \mathscr{U}(c_2)$$

is an angle interval of $\mathscr{U}(c_2)$.

But by Lemma 3.16, $T_0(c_1) \cong [x, f_{c_1}(x), \cdots, f_{c_1}^{N-1}(x)]$, and $f^{k_1}(x) \prec f^{k_1}(c)$. And furthermore, $f^{k_1}(c), -f^{k_2}(x) \in [f^{k_1}(x), -f^{k_2}(c)]$. Then

$$(\theta(f^{k_1}(x))^-, \theta(f^{k_1}(x))^+) \subset (\theta(f^{k_1}(c))^-, \theta(f^{k_1}(c))^+).$$

2. Otherwise, we have $I_k = (z, -f^{k_1}(c)]$ for some $0 \le k_1 \le N$, where $z \notin Orb(0)$ is not in the orbit of critical point. Since $T_0(c_1) \cong [x, f_{c_1}(x), \cdots, f_{c_1}^{N-1}(x)]$, then $[z, -f^{k_1}(c)] \bigcap [x, f_{c_1}(x), \cdots, f_{c_1}^{N-1}(x)] = \{z\}$. And it follows that the angle interval $U_k$ associated to $I_k$ is also an angle interval of $\mathscr{U}(c_2)$.

The above two cases give the proof of the theorem. □

**Corollary 3.19.** *Suppose that $f_{c_1}$, $f_{c_2}$ are postcritically finite maps. If $c_2 \prec c_1$ and $N(c_1) = N(c_2)$, then $h(T_0(c_2), f_{c_2}) \le h(T_0(c_1), f_{c_1})$.*

*Proof.* Given the condition, by Theorem 3.18, we know that $\mathscr{U}(c_1) \subset \mathscr{U}(c_2)$ and then $\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_2)) \subset \mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_1))$. So we have the following inequality on the entropy:

$$h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_2)), q) \leq h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_1)), q). \tag{3.1}$$

Also from Corollary 3.11 we have

$$h(T_0(c_i), f_{c_i}) = h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} q^{-k}(\mathscr{U}(c_i)), q)$$

Thus we get $h(T_0(c_2), f_{c_2}) \leq h(T_0(c_1), f_{c_1})$. $\square$

The Corollary 3.19 has proved the main theorem under the condition that the two parameters have the same end-number. Since the end-number is finite on every limb of the Mandelbrot Set, this can apply in many cases. But we still need to consider the case when the two parameters have different end-numbers. This is done in next section.

## 3.3 Entropy on the Hubbard tree, case II

In this section we will prove the main theorem 3.1 without the condition that the two Hubbard tree are homeomorphic (without considering the map). In fact, it states that given two post-critically finite parameters $c_1$ and $c_2$, if $c_2 \prec c_1$ and $N(c_2) < N(c_1)$, then the entropy of $f_{c_2}$ acting on its Hubbard tree $T_0(c_2)$ is not bigger than the entropy of $f_{c_2}$ acting on $T_0(c_1)$. Combining the result in the last section, we obtain the complete proof of the main theorem.

There are three subsections in this section. The first subsection shows some examples, the second subsection deals with the entropy of topological tuning which is very useful to simplify the proof of the main theorem. The theorem is proved in the last subsection.

### 3.3.1 One Example

We can see in the following example that if the two Hubbard tree are not topologically homeomorphic, then theorem 3.18 may not hold. So we have to use a new method to prove the main theorem in this case.

By graph theory, it will turn out that if $c_2 \prec c_1$ and $T_0(c_1)$ has more edges than $T_0(c_2)$, then the entropy of $f_{c_1}$ on its Hubbard tree is (weakly) bigger than the corresponding entropy of $fc_2$.

We first look at one simple case.

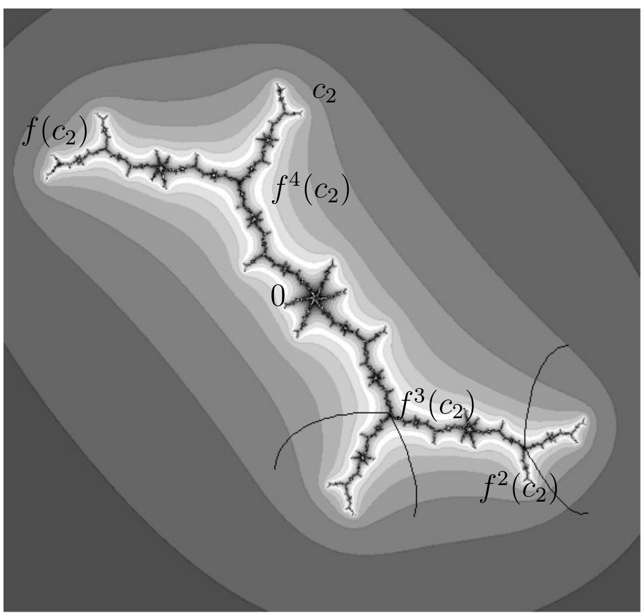


Figure 3.3: The filled Julia set of $f_{c_2}$

**Example 3.20.** In the following figures, Figure 3.3 and 3.5 are the Julia set and Hubbard tree respectively with parameter value $c_2 = -0.17595297 + 1.08659342i$. Figure 3.4 and 3.6 are the Julia set and Hubbard tree respectively

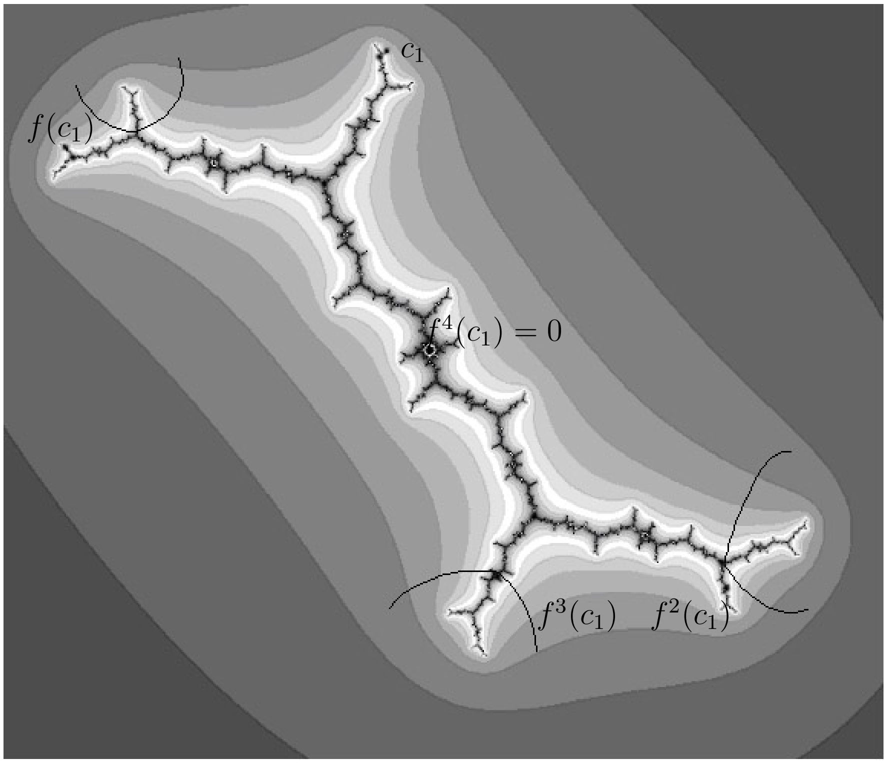


Figure 3.4: The filled Julia set of $f_{c_1}$

with parameter value $c_1 = -0.16356193 + 1.09778922i$. The marked points are critical orbits.

The critical orbits are marked with numbers on each Julia set. Although $c_2 \prec c_1$, it's clear from the pictures that $\mathscr{U}(c_1)$ has three open intervals whereas $\mathscr{U}(c_2)$ has two open intervals and $\mathscr{U}(c_1) \nsubseteq \mathscr{U}(c_2)$. And then we can not apply the result of last section to compare their entropy.

The following is the markov matrix of $T_{c_2}$

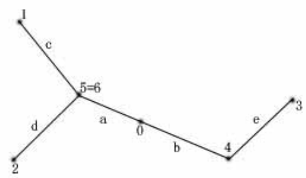


Figure 3.5: Hubbard tree of $f_{c_2}$

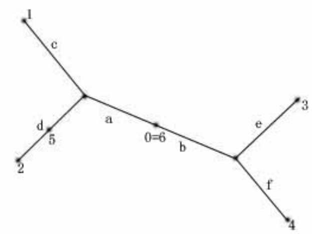


Figure 3.6: Hubbard tree of $f_{c_1}$

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and it's easy to see that there exists $\mathscr{A}$ on Hubbard tree $T_1$ which is an over-Markov packing with $M$.

### 3.3.2 Topological tuning

Let $f_0$ be a quadratic map with attracting periodic critical point of period $k$. Also let $g$ be any quadratic polynomial.

We define the topological tuning $f = f_0 * g$ in the following way.

Let the critical orbit of $f_0$ be $\{c_i\}_{i=0}^{k-1}$, where $c_0 = 0$ is the critical point. The regulated arcs of $c_i$ will mean the intersection of the Hubbard tree with the immediate basin $U(c_i)$ of $c_i$. We can get the Hubbard tree $T_0(f)$ of $f = f_0 * g$

from the Hubbard tree $T_0(f_0)$ by replacing the regulated arcs of all $c_i$'s with a copy of $T_0'(g)$, where $T_0'(g)$ is an extension of $T_0(g)$ such that it can be matched with the finite set $T_0(f_0) \cap U(c_i)$ on the boundary of $U(c_i)$. $f$ maps the $i$-th copy of $T_0(g)$ to $(i+1)$-th copy of $T_0(g)$ and $f^k$ conjugates to $g$ restricted to any of these copies. Finally $f$ conjugates $f_0$ on the remaining of $T_0(f_0)$.

The following is an example of topological tuning (by modification).

**Definition 3.21.** *Given a hyperbolic component $C$ and one of its iterated satellites $C'$, we say that $C'$ has level $n$ if there are total $n-1$ hyperbolic components which separate $C'$ from $C$. By this definition, the satellites which directly attach to $C$ have level* 1*.*

**Lemma 3.22.** *Let $f = f_0 * g$ be a tuned quadratic map where $f_0$ and $g$ are maps described as above. Then $h(T_0(f), f) = \sup(h(T_0(f_0), f_0), \frac{1}{k}h(T_0(g), g))$.*

*Proof.* We have showed that $T_0(f)$ consists of two parts: $T_0(f) = A \bigcup B$ where $A$ is $k$ copies of Hubbard tree $T_0(f_0)$, and $B$ is the Hubbard tree of $f_0$ removing the $k$ open regular arcs of critical orbit.

First since $f^k$ conjugates $g$ on each copy of $T_0(f_0)$, $h(A, f^k) = h(T_0(g), g)$, and then $h(A, f) = \frac{1}{k}h(T_0(g), g)$ and $h(T_0(f), f) = \sup(h(B, f_0), \frac{1}{k}h(T_0(g), g))$.

To prove the lemma, we must show that $h(B, f_0) = h(T_0(f_0), f_0)$. In fact, we can apply similar idea as above to $f_0$ itself: $T_0(f) = A' \bigcup B$ where $A'$ is regular paths of critical orbit of $f_0$, and B is Hubbard tree of $f_0$ removing the k regular paths of critical orbit. Since the $k$-iterate of $f_0$ restricted on any of these regular paths conjugates to standard square map on interval and hence has zero entropy, i.e. $h(A', f_0) = 0$ , $h(B, f_0) = h(T_0(f_0), f_0)$. This finishes the proof. □

**Lemma 3.23.** *For the center of any iterated satellite of the main cardioid, the corresponding map acting on the Hubbard tree has zero entropy.*

*Proof.* We prove this inductively. First we can verify this lemma for any hyperbolic components which directly attached to the main cardioid. Suppose $C$ is directly attached to the main cardioid on the $p/q$ limb. Then the Hubbard tree for the center of $C$ looks like a "Y" but has $q$ edges: Those edges have and only have one common end, and the map iterates on those edges in one cycle. The $q$ iterate of the map $f$ is then a homeomorphism on the Hubbard tree and then $h(T_0(f), f^k) = 0$. Thus $h(T_0(f), f) = \frac{1}{k}h(T_0(f), f^k) = 0$.

Now suppose that for any iterated satellite of the main cardioid with level at most $n$, the corresponding Hubbard tree for the it's center has zero entropy. Let $C$ be the iterated satellite of the main cardioid with level $n+1$. Then $C$ is directly attached to another iterated satellite of the main cardioid with level $n$, denote it by $C_0$. We also denote the corresponding map the centers by $f$ and $f_0$. Then $f = f_0 * g$ where $g$ represents a quadratic map with parameter is the center of some hyperbolic components which directly attached to the main cardioid. Obviously $h(T_0(g), g) = 0$ by the first paragraph of proof this lemma.

We have assumed that $h(T_0(f_0), f_0) = 0$, by Lemma 3.22, $h(T_0(f), f) = \sup(h(T_0(f_0), f_0), \frac{1}{k}h(T_0(g), g)) = 0$. □

**Theorem 3.24.** *Given any hyperbolic component and any of its iterated satellite, the corresponding Hubbard trees for the centers of the two components have the same entropy.*

*Proof.* Suppose $C$ is any hyperbolic component and $C_0$ is one of its iterated satellite with level $n$. Again denote the corresponding maps of the centers by $f$ and $f_0$. Then $f = f_0 * g$ where $g$ represents a quadratic map with parameter is the center of some level $n$ iterated satellite of the main cardioid. By Lemma 3.23, $h(T_0(g), g) = 0$. And then finally by Lemma 3.22, $h(T_0(f), f) = h(T_0(f_0), f_0)$. □

### 3.3.3 Proof of the main theorem

**Proposition 3.25.** *Given two postcritically finite parameters $c_1$ and $c_2$, if $c_2 \prec c_1$ and $N(c_2) < N(c_1)$, then $h(T_0(c_2), f_{c_2}) \leq h(T_0(c_1), f_{c_1})$.*

*Proof.* By Lemma 3.15 , the end-number function $N$ is a finite integer function and non-decreasing on each limb where it is defined. Without loss of generality, we can always assume that there is no post-critically finite parameter $c$ which separates $c_1$ and $c_2$ such that $N(c_1) > N(c) > N(c_2)$.

First assume that $c_2$ is a Misiurewicz point.

Given Hubbard trees $T_0(c_i)$ for $i = 1, 2$, let $x = x(c_1, c_2)$ as before. Consider the regulated tree $T(x)$ generated by $Orb(x)$. $T(x)$ is a subtree of $T_0(c_1)$ which is not invariant under $f_{c_1}$. We choose the vertices as the union of critical point, orbit of $x$ and all other branch points of $T(x)$ Then there is a bijection between the vertices of $T(x)$ and $T_0(c_2)$. (Note that $Orb(x) \bigcap \{0\} = \emptyset$.) In particular, similar to Lemma 3.16, $T(x)$ is topologically homeomorphic to the Hubbard tree $T_0(c_2)$.

Suppose that $T_0(c_2)$ has $l$ edges and label them by $a_1, a_2, \cdots, a_l$. Also denote the corresponding Markov matrix by $M(c_2)$ with entries $M(i, j)$. We known from Chapter 2 that $M(c_2)$ has a leading simple eigenvalue $\lambda \geq 1$. By

Lemma 2.16 we have $h(T_0(c_2), f_{c_2}) = \log\lambda$.

Since $T(x)$ is topologically homeomorphic to $T_0(c_2)$, we can also label the edges of $T(x)$ by $b_1, b_2, \cdots, b_l$. It satisfies the condition that $f_{c_1}(b_i) \supset b_j$ (acting on $T_0(c_1)$!) whenever $M(i,j) = 1$.

Then by the above construction, $\mathcal{B} = \{b_1, b_2, \cdots, b_l\}$ is an over-Markov packing with matrix $M(c_2)$. By Proposition 2.17, $h(T_0(c_1), f_{c_1}) \geq \log\lambda$. And then $h(T_0(c_1), f_{c_1}) \geq h(T_0(c_2), f_{c_2})$. This finishes the proof of the first case.

Now let $c_2$ be the center of some hyperbolic component $C$. By Theorem 3.24, for any hyperbolic component and any of its iterated satellite, the entropy of the maps corresponding to the centers acting on their Hubbard trees are equal. So we can assume $C$ is not a iterated satellite of any other hyperbolic component.

Given Hubbard trees $T_0(c_i)$ for $i = 1, 2$, let $x = x(c_1, c_2)$ as before. Consider the regulated tree $T(x)$ generated by $Orb(x)$. $T(x)$ is a subtree of $T_0(c_1)$ which is not invariant under $f_{c_1}$. We choose the vertices as the union of orbit of $x$ and all other branch points of $T(x)$. Similarly, $T(x)$ is topologically homeomorphic to Hubbard tree $T_0(c_2)$. We can label the edges of $T(x)$ as in the first case. Note that critical point 0 is not end point of $T(x)$, but we can still get an over-Markov packing with matrix $M(c_2)$. The remaining is same as in the first case. $\square$

**Remark 3.26.** When we label the edges of a Hubbard tree, we exclude the simple cases for which the critical point 0 is an end point of the Hubbard tree. if the critical point 0 is an end point of the Hubbard tree, the topological entropy is zero. Any Hubbard tree which is not in the simple case has at least

$N(c) + 2$ edges.

Finally we are able to prove the main theorem 3.1.

**Theorem 3.1** *Given any two parameters* $c_2$ *and* $c_1$ *on* $\mathcal{M}^0$*. If* $c_2 \prec c_1$*, then* $h(T_0(c_2), f_{c_2}) \leq h(T_0(c_1), f_{c_1})$.

*Proof.* By Lemma 3.15, $N(c_2) \leq N(c_1)$. The main theorem 3.1 is proved in two possible cases. Proposition 3.19 proves the case that $N(c_2) = N(c_1)$ and Proposition 3.25 proves the case that $N(c_2) < N(c_1)$. $\square$

# Chapter 4

# Cubic Case

We will study the topological entropy of post-critically finite cubic polynomials with one critical fixed point acting on the Hubbard tree.

Following [M4], the cubic polynomial with critical fixed point $a \in \mathbb{C}$ can be normalized as $f_a(z) = z^3 - 3a^2 z + 2a^3 + a$. When $a \neq 0$, $f_a$ has two critical points $\{\pm a\}$ and $a$ is also a fixed point: $f_a(a) = a$. The **co-critical point** of $-a$ is $2a$ which satisfies $f_a(2a) = f_a(-a) = 4a^3 + a$. (Similarly, we can check that the co-critical point of $a$ is $-2a$.)

The ***connectedness locus*** can be defined as

$$\mathcal{C} = \{\ a \in \mathbb{C} \mid\ J(f_a) \textit{ is connected }\}.$$

See figure 4.1. The ***principal hyperbolic component*** $\mathcal{H}_0$ is the hyperbolic component which contains the parameter 0.

**Definition 4.1.** *Define $\mathcal{C}^0$ as all parameters on $\mathcal{C}$ such that the corresponding*

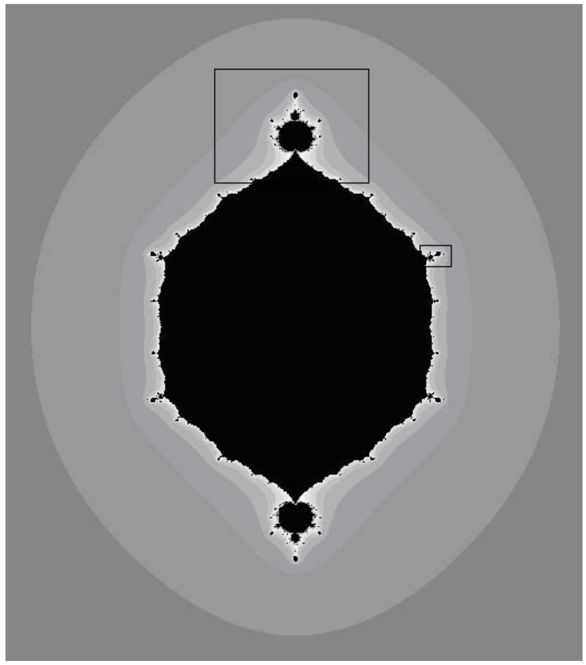

Figure 4.1: The connectedness locus $\mathcal{C}$ with blow up in later figures

*maps are post-critically finite:*

$$\mathcal{C}^0 = \{a \in \mathcal{C} \mid f_a \text{ is postcritically finite}\}.$$

**Definition 4.2** (Partial order $\prec$ on $\mathcal{C}^0$)**.** *Let $a_1$, $a_2$ be two different parameters on $\mathcal{M}^0$. We say that $a_2 \prec a_1$ if $a_2$ or the root point of the hyperbolic component which contains $a_2$ separates $a_1$ from 0.*

It's easy to see that the above definition is transitive: if $a_2 \prec a_1$ and $a_3 \prec a_2$, then $a_3 \prec a_1$.

Our main theorem is the following.

**Theorem 4.3.** *Given any two parameters $a_2 \prec a_1$ on $\mathcal{C}^0$. If $N(a_1) = N(a_2)$, then $h(T_0(a_2), f_{a_2}) \leq h(T_0(a_1), f_{a_1})$.*

## 4.1 The dynamics of $f_a$

Given $a \in \mathcal{C}$, then $K(f_a)$ is compact and connected.

We can apply the Böttcher Theorem as in the quadratic case. By Theorem 2.2, there exists a unique analytic isomorphism

$$\varphi_a : \mathbb{C} \backslash K(f_a) \longrightarrow \mathbb{C} \backslash \overline{\mathbb{D}}$$

satisfying $\varphi_a(z)/z \longrightarrow 1$ as $|z| \to \infty$ and conjugating $f_a$ to the polynomial $f_0(z) = z^3$, i.e. $\varphi_a \circ f = f_0 \circ \varphi_a$. And then we can define (dynamical) equipotentials and external rays $R_a(\theta)$ as usual.

Following [M4] and [F1], let $V_a$ be the basin of attraction of $a$, that is the set of points which converge to $a$ under forward iteration of $f_a$. Also let $U_a$ be the immediate basin of $a$, which is the connected component of $V_a$ containing $a$.

It has been shown in [F1] and [R] that for any $a \in \mathcal{C}$, the boundary of every connected component of $V_a$ is a Jordan curve.

## 4.2 The parameter plane

The connectedness locus $\mathcal{C}$ is compact and connected. Furthermore, $\mathbb{C} \setminus \mathcal{C}$ is analytically isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$. In fact, we can define the isomorphism

$$\Phi_{\mathcal{C}}(a) : \mathbb{C} \setminus \mathcal{C} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$$

by $\Phi_{\mathcal{C}}(a) = \varphi_a(2a)$, where $\varphi_a$ is the Böttcher function of $f_a$ for $a \in \mathbb{C} \setminus \mathcal{C}$. (See [M4].) Based on this fact, we can define the parameter external ray $\mathcal{R}_a(\theta)$ respectively. In particular, if the parameter external ray $\mathcal{R}_{\mathcal{C}}(\xi)$ lands on $a \in \mathcal{C}$ (or on the root point of the hyperbolic component which contains $a$), then on the dynamical plane of $f_a$, the external ray $R_a(\xi)$ lands on $2a$ (or the root point of the Fatou component which contains $2a$).

Milnor describes in [M4] four possible type of bounded hyperbolic components in any family of polynomials with two critical points.

**Adjacent component** (type A) is a hyperbolic component for which the two critical points are in the same component of the immediate basin of an attracting periodic orbit. ***Bi-transitive component***(type B) is a hyperbolic component for which the two critical points are in different components of the immediate basin of an attracting periodic orbit. ***Capture component*** (type C) [1] is a hyperbolic component for which only one critical point is in the immediate basin of an attracting periodic orbit, although both critical points are attracted by it. Finally, ***Disjoint component*** (type D) is a hyperbolic component for which there are two distinct attracting periodic orbits. (Each of which must attract one critical point!)

In our cases, Bi-transitive components do not occur in $\mathcal{C}$ since $a$ is a super attracting fixed point of $f_a$. Also only the principal hyperbolic component $\mathcal{H}_0$ is an Adjacent component. See Figure 4.2 for some examples of hyperbolic components in $\mathcal{C}$.

There are infinitely many hyperbolic component of type C and D in $\mathcal{C}$. A

[1] Caution: The term "capture component" is used with a very different meaning in the work of Ben Wittner and Mary Rees.

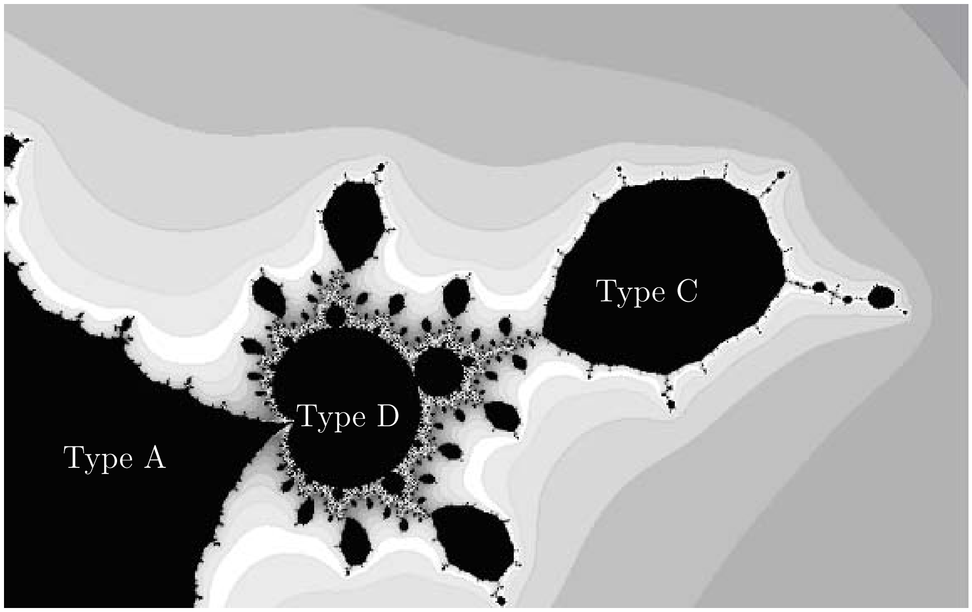


Figure 4.2: The hyperbolic components in $\mathcal{C}$

type D component always occurs in a small Mandelbrot set copy in $\mathcal{C}$. It is conjectured in [M4] and has been proved in [F1] and [R] that the boundary of every bounded hyperbolic component is a Jordan curve.

It has been shown in [M4] that there are countably many small Mandelbrot set copies directly attached to the boundary of $\mathcal{H}_0$ and there is a Capture component directly attached to any tip of each such copy. It is conjectured and has been proved by Roesch in [R] that except in the above cases, any two closure of any Capture components or small Mandelbrot set copies are disjoint.

## 4.3 Hubbard tree of $f_a$

Given $a \in \mathcal{C}^0$, then $f_a$ is a post-critically finite cubic map with a super-attracting fixed point $a$. We will define the Hubbard tree $T_0(a)$ of $f_a$ in the following way.

Since $a$ is a super-attracting fixed point, the interior $int(K_a)$ is non-empty. Every component $U$ of $int(K)$ is a bounded Fatou component whose closure $\overline{U}$ is homeomorphic to the closed disk $\overline{\mathbb{D}}$, and every such component eventually maps to the component $U_a$ if $-a$ is pre-periodic (or the periodic component $U_{-a}$ if $-a$ is periodic where $U_{-a}$ is the immediate basin of critical point $-a$). ***The center*** $c(U)$ is defined as the unique backward image of $a$ (or $-a$) in $U$. In particular, $c(U_a) = a$.

It was shown in [M4] or [F1] that given any bounded Fatou component $U$, there exists a homeomorphism $\phi : \overline{U} \to \overline{\mathbb{D}}$ which is holomorphic in $U$ with $\phi(c(U)) = 0$. ***A radial arc*** means an arc in $\overline{U}$ of the form $\phi^{-1}\{re^{i\eta} \ : 0 \leq r \leq 1\}$. Since $\phi$ is unique up to post-composition with a rotation of $\overline{\mathbb{D}}$, radial arcs are well-defined.

Then we can define embedded arc and regulated arc as in the quadratic case. An ***embedded arc*** in $K_a$ is any subset of $K_a$ which is homeomorphic to the closed interval $[0, 1] \subset \mathbb{R}$. An embedded arc $I$ is ***regulated*** if, for every bounded Fatou component $U$, the intersection $I \cap \overline{U}$ is either empty, a point or consists of radial arcs in $\overline{U}$.

Now Lemma 2.3 and Lemma 2.4 still hold for regulated arcs of $K_a$. ***Regulated trees*** are defined exactly as in the quadratic case. The (minimal) ***Hubbard tree*** $T_0(a)$ is defined as the smallest regulated tree generated by the critical orbit $Orb(-a)$ and the super-attracting fixed point $a$.

Similar to the quadratic case, given a regulated tree $T$ constructed from $K_a$, the filled Julia set of $f_a$, a point $x \in T$ is called an ***end point*** if $T \setminus \{x\}$ is connected. The set of end points is denoted by $\partial T$ and the cardinality of $\partial T$ is called ***end-number*** of the tree. A point $x \in T$ is called a ***branch point*** if $T \setminus \{x\}$ has more than two components. The set of branch points is denoted by $Br(T)$.

Given a Hubbard tree $T_0(a)$, a point $x \in T_0(a)$ is a ***vertex*** of $T_0(a)$ if $x = a$, $x \in Orb(-a)$ or $x \in Br(T_0(a))$, where $Br(T_0(a))$ is the set of branch points of $T_0(a)$ . Define the set of vertices as $V(T_0(a))$. Then $V(T_0(a)) = \{a\} \bigcup Orb(-a) \bigcup Br(T_0(a))$. Finally denote the end-number of $T_0(a)$ by $N(a)$.

Just as in proposition 2.5, the tree $T_0(a)$ is $f_a$-invariant: $f_a(T_0(a)) = T_0(a)$.

**Lemma 4.4.** *Given cubic polynomial $f_a$ as above. Let $N(a)$ be the end-number of $T_0(a)$. Suppose that $a \notin \partial(T_0(a))$. Then $\{f(-a), \cdots, f^{N(a)}(-a)\}$ are exactly the only end points of the Hubbard tree $T_0(f)$.*

*Proof.* Suppose that $a \neq 0$ and the critical point $-a$ is an end point of $T_0(a)$. Any non-end point which maps to a end point must be a critical point. But $a$ is a fixed point, there is no non-end point which can map to $-a$, then $-a$ must be periodic and the critical orbit of $-a$ are exactly the set of end points. Thus the statement was proved.

Now suppose that $a \neq 0$ and the critical point $-a$ is a non-end point of $T_0(a)$. Since $f$ is locally one to one except at the critical points and since the critical point $a$ is a fixed point, then the other critical point $\{-a\}$ is the only non-endpoint which maps to an end point of the Hubbard tree. $\square$

## 4.4 Some simple cases

Suppose $f_a$ is post-critically finite as above.

In this section, we will discuss the topological entropy of $f_a$ acting on the Hubbard tree $T_0(a)$ for which $a$ is in the small Mandelbrot set copy which directly attached $\mathcal{H}_\prime$ or in the closure of capture component.

### 4.4.1 Small Mandelbrot set copy $f_t * \mathcal{M}$

According to [M4], Starting from the principal component $\mathcal{H}_0$, along any direction $t \in \mathbb{T}$ which is rational with odd denominator, there is a small Mandelbrot set copy directly attached to $\mathcal{H}_0$. Denote the center of the small Mandelbrot set copy by $t$, then the small Mandelbrot set copy can be described as $f_t * \mathcal{M}$.

Given $f_t$ as above, suppose that $t$ has period $n$ under doubling map. Then the Hubbard tree $T_0(t)$ consists of $k$ edges radiating out from a center vertex at angles $t, 2t, \cdots$. Thus $f_t$ acting on $T_0(t)$ conjugates to the rotations on those edges. Then $h(T_0(t), f_t) = 0$.

Any map $f_a$ with $a \in f_t * \mathcal{M}$ has the form $f_t * f_c$ where $f_c$ is some quadratic polynomial with $c \in \mathcal{M}$. As in §4.3 of Douady's paper [D2], the filled Julia set $K_a$ of $f_a$ can be obtained from the filled Julia set $K_t$ of $f_t$ in the following way: for each component $U$ of $int(K_t)$ which eventually maps to the immediate basin $U_{-a}$ of $-a$, replace the closure of $U$ by a copy of $K_{f_c}$.

Lemma 3.22 in Chapter 3 can be modified slightly to apply to this case. Thus we have

$$h(T_0(a), f_a) = \sup(h(T_0(f_t), f_t), \frac{1}{k} h(T_0(c), f_c))$$

where $k$ is the period of free critical point for $f_t$. Since $h(T_0(f_t), f_t) = 0$, we have $h(T_0(a), f_a) = \frac{1}{k}h(T_0(c), f_c)$.

In particular, starting from $\mathcal{H}_0$, along the pure imaginary axis, there is one small Mandelbrot set copy $f_{t_0} * \mathcal{M}$. The Hubbard tree of $f_{t_0}$ only consists of 1 edges with the two vertices are the two critical points of $f_{t_0}$. Thus given $f_a = f_{t_0} * f_c$, we can get

$$h(i\overline{\mathbb{R}}, f_a) = h(T_0(a), f_a) = h(T_0(c), f_c).$$

### 4.4.2 The capture component

In [M4], Milnor also showed that for any embedded Mandelbrot set $f_t * \mathcal{M}$, there is a capture component $C$ directly attached to $f_t * \mathcal{M}$ at any of its tip.

Let $a$ be any tip of $f_t * \mathcal{M}$, then $f_a = f_t * f_c$ where $f_t$ is the center map of the Mandelbrot set copy and $c$ is the corresponding tip of the Mandelbrot set. Again we have $h(T_0(a), f_a) = \frac{1}{k}h(T_0(c), f_c)$ where $k$ is the period of free critical point for $f_t$.

**Proposition 4.5.** *Let $a_0$ be the center of some capture component $C$. Let $a \in \partial C$ such that $-a$ is pre-periodic for $f_a$. Then $h(T_0(a), f_a) = h(T_0(a_0), f_{a_0})$.*

*Proof.* Given $a \in \partial C$ such that $-a$ is pre-periodic for $f_a$, then there exists a smallest integer $n$ such that $f_a^n(-a)$ is periodic and $f_a^n(-a) \in \partial C$. (See [R].) Also let $k$ be the period of $f_a^n(-a)$.

Let $l \leq n$ be the smallest integer such that $f_a^l(-a) \in \partial C$. Then $U_a \cap T_0(a) = \{f_a^l(-a), f_a^{l+1}(-a), \cdots, f_a^n(-a), f_a^{n+1}(-a), f_a^{n+k-1}(-a)\}$. In other words, the orbit of $f_a^l(-a)$ consists of $n+k-l$ points and all of them are on the boundary of $C$.

Let $T$ be the regulated tree generated by the orbit of $f_a^l(-a)$. It consists $n+k-l$ edges which has the common vertex $a$. Let $b_i$ be the edge connecting $a$ and $f_a^i(-a)$ for $l \leq i \leq n+k-1$. Then $f_a(b_i) = b_{i+1}$ for $l \leq i < n+k-1$ and $f_a(b_{n+k-1} = b_n$. We can check that $h(T, f_a) = 0$.

We can get the Hubbard tree $T_a$ from the Hubbard tree $T_{a_0}$ by replacing the critical point $a$ with the regulated tree $T$ described as above. Since $h(T, f_a) = 0$, then $h(T_0(a), f_a) = \sup h(T_0(a_0), f_{a_0}), h(T, f_a) = h(T_0(a_0), f_{a_0})$. $\square$

In particular, by the above proposition, the Hubbard tree for the center of any capture component has the same entropy as the Hubbard tree for the root point of the capture component.

## 4.5 External rays landing on the Hubbard tree

Based on [D2] again, the entropy on the Hubbard tree for the given map $f_a$ is same as the entropy on some subset of $\mathbb{T}$ for the angle tripling map. In the following commutative diagram, $\mathbb{T} = \{e^{2\pi i\alpha} | \alpha \in \mathbb{R}\}$ is circle, $p$ is the angle tripling map on the unit circle and $\gamma_c :\ \mathbb{T} \to J(f_a)$ is the Carathédory loop.

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "p"] \arrow[d, "\gamma_a"] & \mathbb{T} \arrow[d, "\gamma_a"] \\
J(f_a) \arrow[r, "f_a"] & J(f_a)
\end{tikzcd}$$

Thus $h(T_0(a), f_a) = h(\gamma_a^{-1}(T_0(a), p)$.

Since $a$ is a super-attracting fixed point, there is a similar but different descriptions of external rays which land on the Hubbard tree.

Let $f_a$ be a post-critically finite cubic polynomial with the critical point $a$ fixed, where $a \in \mathcal{C}$. Given $x \in T_0(a)$, define $\theta(x)$ as the angle of the external ray which lands on $x$ if $x \in J(f_a)$, or the angle of the external ray which lands on the root point of the Fatou component which contains $x$ if $x \notin J(f_a)$. We call $R_a(\theta(x))$ the **external ray associated to $x$ in $T_0(a)$**. If there are more than one external rays land on $x$ or the root point of the Fatou component which contains $x$, we will clarify them by writing $R_a(\theta(x)^-)$, $R_a(\theta(x)^+)$ , $R_a(\theta(x)^1)$, $R_a(\theta(x)^2)$, etc.

Let $f_a$ be above and $T_0 = T_0(a)$ be the Hubbard tree. Define trees $T_0 \subset T_1 \subset T_2 \subset \cdots$ by $T_{n+1} = f_a^{-1}(T_n)$.

**Definition 4.6.** *Given $f_a$ as above. Let $[x.y]$ be a regulated arc such that $[x, y] \cap [a, -a] = \emptyset$. We say that $x \prec y$ if $x \in (a, y)$ and $\theta(x) \neq \theta(y)$.*

Given $f_a$ as above. Let $[x.y]$ be a regulated arc such that $\pm a \notin [x, y]$. Let $\{\theta(x)^-, \theta(x)^+, \theta(y)^-, \theta(y)^+\}$ be the angles of the four external rays such that angle intervals $[\theta(x)^-, \theta(y)^-]$ and $[\theta(x)^+, \theta(y)^+]$ are the two smallest angle intervals which contain all external rays landing on $[x, y]$. We call

$$\{R_a(\theta(x)^-), R_a(\theta(y)^-), R_a(\theta(y)^+), R_a(\theta(x)^+)\}$$

the ***external rays associated to*** $[x, y]$. Note that when there is only one external ray landing on $y$, $\theta(y)^- = \theta(y)^+ = \theta(y)$. In that case, denote $\{R_a(\theta(x)^-), R_a(\theta(y)), R_a(\theta(x)^+)\}$ the external rays associated to $[x, y]$.

**Lemma 4.7.** *Given post-critically finite cubic polynomial $f_a$ as above. if $x \prec$*

*$y$, then*

$$\{\theta(x)^-, \theta(y)^-, \theta(y)^+, \theta(x)^+\}$$

*is in positive cycllic order. In particular, "$\prec$" is a partial order.*

*Proof.* The first statement follows easily from the definition. And the second part is from the first statement. $\square$

**Definition 4.8** (**Characteristic Angle Intervals $\mathscr{U}(a)$ associated to Hubbard tree** $T_0(a)$)**.** *Let $f_a$ be above and $T_0 = T_0(a)$ be the Hubbard tree. Assume that $-a \notin \partial T_0 a$.*

*Since we can write $T_1 \backslash T_0$ uniquely as a union of finitely many half open regulated arcs, denote it as $T_1 \setminus T_0 = \bigcup_{k=1}^n I_k$ where $I_k = (x_k, y_k], x_k \in V(T_1), y_k \in \partial T_1$.*

*For $x_k \neq a$, let $\{R_a(\theta(x_k)^-), R_a(\theta(y_k)^-), R_a(\theta(y_k)^+), R_a(\theta(x_k)^+)\}$ be the external rays associated to $[x_k, y_k]$. Define $U_k = (\theta(x_k)^-, \theta(x_k)^+)$.*

*On the other hand, let $T_0(a) \cap \partial U_a = \{z_1, z_2, \cdots, z_m\}$ be the set of points in $T_0(a)$ which intersect the boundary of $U_a$. Denote the external rays landing on $z_l$ by $R_a(\xi_l)^{\pm}$, for $1 \leq l \leq m$. Define*

$$\mathscr{V}(a) = \mathbb{T} \setminus \bigcup_{l=1}^{m} (\xi_l^-, \xi_l^+).$$

*Finally, we define the characteristic angle intervals $\mathscr{U}(a)$ associated to Hubbard tree $T_0$ as*

$$\mathscr{U}(a) = \mathscr{V}(a) \cup (\bigcup_{k=1, x_k \neq a}^{n} U_k).$$

**Remark 4.9.** 1. Any two such regulated arcs in $T_1 \setminus T_0$ defined above may have non-empty intersection. If $I_k \bigcap I_l \neq \emptyset$, then either $U_k = U_l$ or $U_k \bigcap U_l = \emptyset$.

2. If $a \neq 0$ and $a \in \partial T_0(a)$, then we can check that $a \in f_{t_0} * \mathcal{M}$ or $a$ is the center of some capture component which directly attached to $f_{t_0} * \mathcal{M}$. Where $f_{t_0} * \mathcal{M}$ is the small Mandelbrot set copy which attaches to $\mathcal{H}_0$ along the pure imaginary axis direction.

**Example 4.10.** The following Figure 4.3 shows the filled Julia set of $f_a$ with $a = 0.5313 + 0.2625i$.

$T_0(a) \cap \overline{U}_a$ is a period 2 orbit on which the external rays $\{\frac{1}{8}, \frac{1}{4}\}$ and $\{\frac{3}{8}, \frac{3}{4}\}$ land. So $\mathscr{V}(f_a) = (\frac{1}{4}, \frac{3}{8}) \cup (\frac{3}{4}, \frac{1}{8})$

On the other hand, there is only one regulated arc in $T_1 \setminus T_0$ which is disjoint from $\overline{U}_a$ and is bounded by external rays with arguments $\frac{10}{24}$ and $\frac{17}{24}$. So

$$\mathscr{U}(f) = \mathscr{V}(f) \cup U_1 = (\frac{1}{4}, \frac{3}{8}) \cup (\frac{3}{4}, \frac{1}{8}) \cup (\frac{10}{24}, \frac{17}{24})$$

**Lemma 4.11.** *Given $f_a$ as above. Then a dynamical external ray $R_a(\theta)$ lands on $T_0(a)$ if and only if the orbit of $\theta$ under tripling never hits $\mathscr{U}(a)$.*

*Proof.* Let $T_0(a) = T_0$. We first prove that the condition is necessary. Since $T_0$ is invariant, $f(T_0) = T_0$, if $R_a(\theta)$ lands on $T_0$, then $R_a(q(\theta))$ still lands on $T_0$. And then the orbit of $\theta$ never hit $\mathscr{U}(a)$.

Next let's prove the condition is sufficient. Suppose $R_a(\theta)$ lands on $x \in J \setminus T_0$, define

$\alpha^+ = \inf\{\alpha \geq \theta \mid R_a(\alpha)$ with angle $\alpha$ lands on $T_0\}$ and

$\alpha^- = \sup\{\alpha \leq \theta \mid R_a(\alpha)$ with angle $\alpha$ lands on $T_0\}$.

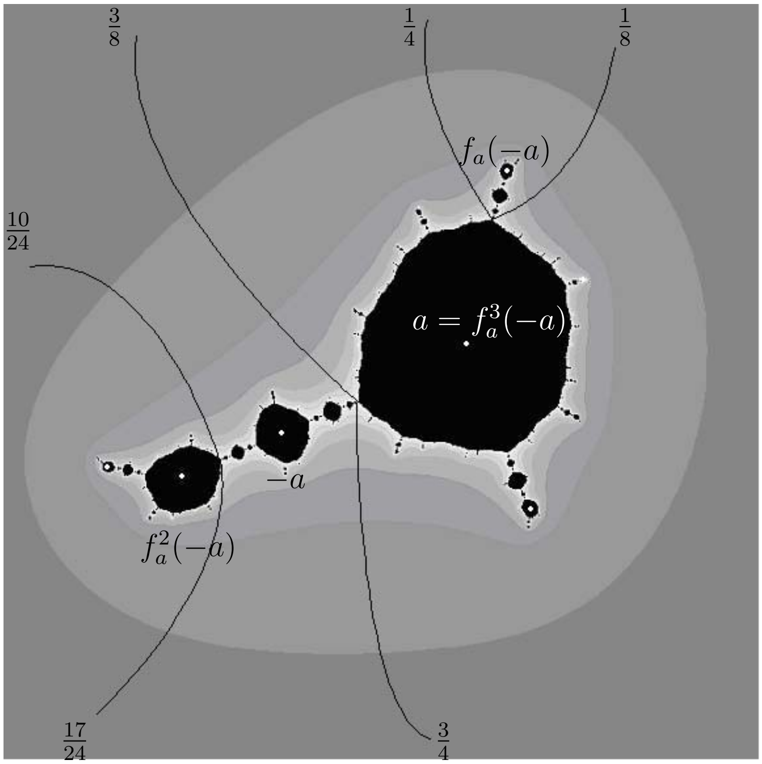


Figure 4.3: The filled Julia Set of $f_a$ with $a = 0.5313 + 0.2625i$

Since $J(f_a)$ is locally connected and path connected, $T_0$ is a closed set, it's clear that external rays with angle $\alpha^+$, $\alpha^-$ must land on $T_0$, and $\theta \subset (\alpha^+, \alpha^-)$.

We will show that there exists an integer $k \geq 0$ such that $q^k(\alpha^+, \alpha^-) \subset \mathscr{U}(a)$. In fact, since $T_0 \subset T_1 \subset T_2 \subset \cdots$ and $\bigcup_{n=0}^{\infty} T_n$ is dense on $J$, so the image of $(\alpha^+,\ \alpha^-)$ under the iteration of $q$ must hit $\gamma_a^{-1}(T_1)$, i.e., $\exists k \in \mathbb{N}$, such that $q^k(\alpha^+, \alpha^-) \cap \gamma_a^{-1}(T_1) \neq \varnothing$, $q^{k-1}(\alpha^+, \alpha^-) \cap \gamma_a^{-1}(T_1) = \varnothing$. Where $\gamma_a : \mathbb{T} \to J_{f_a}$ is the Carathédory map.

Since $T_1 = f^{-1}(T_0) = T_0 \cup (T_1 \setminus T_0)$, $q^k(\alpha^+, \alpha^-)$ will hit $\gamma_a^{-1}(T_1 \setminus T_0)$ before it hit $\gamma_a^{-1}(T_0)$. Then by the construction of $\mathscr{U}(a)$,$p^k(\alpha^+, \alpha^-) \subset \mathscr{U}(a)$. Finally, since $\theta \subset (\alpha^+, \alpha^-)$, this finishes the proof of the lemma. □

## 4.6 Entropy on the Hubbard tree

In this section we will prove the theorem 4.3 under the condition that the two Hubbard trees have the same end-number. In fact, it states that given two post-critically finite parameters $a_1$ and $a_2$, if $a_2 \prec a_1$ and $N(a_1) = N(a_2)$, then $h(T_0(a_2), f_{a_2}) \le h(T_0(a_1), f_{a_1})$.

**Lemma 4.12.** *(***Orbit forcing***) Given any two parameters $a_2$ and $a_1$ in $\mathcal{C}^0$ with $a_2 \prec a_1$. Then there exists a unique $x = x(a_1, a_2) \in T_1(c_1)$ which has the same orbit as $2a_2$ (or as the root point of the immediate basin which contains $2a_2$). And furthermore, if an external ray $R_{a_2}(\theta)$ lands on $2a_2$ (or on the root point of the Fatou component which contains $2a_2$) in the dynamical plane of $f_{a_2}$ , then the external ray $R_{a_1}(\theta)$ with same argument lands on $x$ in the dynamical plane of $f_{a_1}$.*

*Proof.* For the proof, see [K] §5. □

**Lemma 4.13.** *Given any two parameters $a_2$ and $a_1$ in $\mathcal{C}^0$. If $a_2 \prec a_1$, then $N(a_2) \le N(a_1)$.*

*Proof.* The proof is similar to the quadratic case.

First suppose that $-a_2$ is pre-periodic , and that external rays with angles $\theta_i,\ i = 1, 2, ...k$ land on $a_2$ in the parameter space. Then on the dynamical plane of $f_{a_2}$, external rays with same angles $\theta_i,\ i = 1, 2, ...k$ landing on the co-critical point $2a_2$. Since $-a_2$ is preperiodic, the angle set $\{\theta_1, \cdots, \theta_k\}$ is preperiodic under angle doubling map $q$ and external rays with angle $2^m\theta_i,\ i = 1, 2, ...k$ land on $f^m_{a_2}(2a_2)$.

By Lemma 4.12, on the dynamical plane of $f_{a_1}$, external rays with the same angles $\theta_i,\ i = 1, 2, ...k$ land on some point $x$ on the Julia set of $f_{a_1}$. Then there is a bijection between the orbit of $x \in J(f_{a_1})$ and the orbit of $a_2 \in J(f_{a_2})$ by the external rays landing on them. And so the regulated path generated by orbit of $f_{a_1}(x)$ has $N(a_2)$ end points, $\mathbf{Orb}(x) = [f_{a_1}(x), \cdots, f^{N(a_2)}_{a_1}(x)]$. Finally the fact that this regulated path is a subset of Hubbard tree $T_0(a_1)$ give us the conclusion that $N(a_2) \leq N(a_1)$ in the first case.

In the case that $f_{a_2}$ has periodic critical point, the proof is similar. external rays with angles $\theta_i,\ i = 1, 2$ landing on the root point of the hyperbolic component which contains $c_2$ in the parameter space. Then on the dynamical plane of $f_{a_2}$, external rays with same angles $\theta_i,\ i = 1, 2$ landing on the root point of the Fatou component which contains critical value $a_2$. Since $a_2$ is periodic, the angle set $\{\theta_1, \theta_2\}$ is periodic under angle doubling map $q$. For any integer $m$, external rays with angle $2^m\theta_i,\ i = 1, 2$ land on root point of Fatou component which contains $f^m_{a_2}(a_2)$.

By Lemma 4.12, on the dynamical plane of $f_{a_1}$, external rays with the same angles $\theta_i,\ i = 1, 2$ land on some point $x$ on the Hubbard tree of $f_{a_1}$, i.e., $x \in T_0(c_1)$. Then there is a bijection between the orbit of $x \in T_0(c_1)$ and the orbit of root point of Fatou component which contain $a_2$ by the external rays landing on them. And so the regulated path generated by orbit of $x$ has $N(a_2)$ end points, $\mathbf{Orb}(x) = [x, f_{a_1}(x), \cdots, f^{N(a_2)}_{a_1}(x)]$. Finally the fact that this regulated path is a subset of Hubbard tree $T_0(a_1)$ give us the conclusion that $N(a_2) \leq N(a_1)$ in the second case. $\square$

**Lemma 4.14.** *Let $f_{a_i}$ be a post-critically finite maps with $a_2 \prec a_1$ and $N(a_1) = N(a_2)$. Also let $x = x(a_1, a_2)$ as in Lemma 4.12. Then*

$$T_0(a_1) \cong T_0(a_2) \cong [f_{a_1}(x), \cdots, f_{a_1}^N(x)].$$

*where "$\cong$" means "topologically homeomorphic".*

*Proof.* The proof is similar to Lemma 3.16.

For simplicity, denote the end-number number by $N(a_1) = N(a_2) = N$ and $f_{a_1} = f$.

By definition of $x$, the external rays landing on $x$ in $T_0(a_1)$ are exactly the same external rays landing on $-a_2$ in $T_0(a_2)$. Then $\{x, f(x), \cdots, f^{N-1}(x)\}$ are exactly the end pints of $[x, f(x), \cdots, f^{N-1}(x)]$ and

$$T_0(a_2) \cong [x, f(x), \cdots, f^{N-1}(x)].$$

We will prove inductively that there are no branch points and critical points in $(f^k(x), f^k(a_1))$ for $k = 0, 1, \cdots, N-1$.

First obviously $0 \notin (x, a_1)$ and there are no branch points in $(x, a_1)$. (Otherwise, it will be easy to see that $N(a_1) > N(a_2)$.) Suppose now that there are no branch points and critical point in $(f^k(x), f^k(a_1))$ for all $k \leq k_0 < N-1$. Since $f$ is injective on regular arcs not containing critical point, it follows that $(f^{k_0+1}(x), f^{k_0+1}(a_1))$ also doesn't contain branch points. If $0 \in (f^{k_0+1}(x), f^{k_0+1}(a_1))$, since $N(a_1) = N(a_2)$, there exists $k_0 + 1 < l < N$ such that $f^l(x) \in (0, f^{k_0+1}(a_1)$. But then $f^{k_0+1}(x)$ is not the end point of regulated tree $[f_{a_1}(x), \cdots, f_{a_1}^N(x)]$. This is a contradiction.

Since there are no branch points in $(f^k(x), f^k(-a))$ for $k = 1, 2, \cdots, N$, then $T_0(a_1) \cong [f_{a_1}(x), \cdots, f_{a_1}^N(x)]$. And furthermore, $f_{a_1}^k(x)$ are the only point which separates $f_a^k(-a)$ from 0. $\square$

**Lemma 4.15.** *Given $f = f_a$ as above. Let $x, y \in T_0(a) \cap J_a$. Suppose that $f^i(x) \prec f^i(y)$ for $0 \leq i \leq k$, then the following holds.*

1. *If $f^k(y) \prec y'$ in $T_1(a)$, then $f^k(x) \prec x'$ in $T_1(a)$.*

2. *If $y' \prec f^k(y)$ in $T_1(a)$, then $x' \prec f^k(x)$ in $T_1(a)$.*

*Where $y'$ is any preimage of $f_a(y)$ which is different from $y$. $x'$ is the preimage of $f_a(x)$ such that $a, -a \notin [x', y']$.*

*Proof.* First note that $T_1(a)$ is the union of Hubbard tree $T_0(a)$ and its preimage: $T_1(a) = T_0(a) \bigcup f^{-1}(T_0(a))$.

Since $f^i(x) \prec f^i(y)$ for $0 \leq i \leq k$, then $a, -a \notin [f^i(x), f^i(y)]$ and $f : [f^i(x), f^i(y)] \to [f^{i+1}(x), f^{i+1}(y)]$ is a homeomorphism for $0 \leq i \leq k-1$. Then $f^k : [x, y] \to [f^k(x), f^k(y)]$ is also a homeomorphism. Also from the fact $x \prec y$, we can get $0 \notin [x', y']$ and $x' \prec y'$. By definitions of $x'$ and $y'$, we still have the homeomorphism $f^k : [x', y'] \to [f^k(x), f^k(y)]$.

Now let us prove the first statement by contradiction. Suppose that $f^k(y) \prec y'$ and $x' \prec f^k(x)$ in $T_1(a)$, then we have

$$x' \prec f^k(x) \prec f^k(y) \prec y'.$$

It follows that $[f^k(x), f^k(y)] \subset [x', y']$. Then $f^k$ restricted on $[x', y']$ is a homeomorphism onto its proper subset. This is a contradiction with the expansivity of $f$ on $J_a$ since $x', y' \in J_a$.

The second statement can be proved similarly. $\square$

**Theorem 4.16.** *Suppose that $f_{a_1}$, $f_{a_2}$ are post-critically finite maps. If $a_2 \prec a_1$ and $N(a_1) = N(a_2)$, then $\mathscr{U}(a_1) \subset \mathscr{U}(a_2)$.*

*Proof.* First we can see that $\mathscr{V}(a_1) = \mathscr{V}(a_2)$.

Both $\mathscr{U}(a_1)$ and $\mathscr{U}(a_2)$ are a disjoint union of angle intervals each of which is associated to an regulated arc in $T_1(a_1)$ and $T_1(a_2)$. To prove the theorem, we will show in the following that for each such angle interval $U \subset \mathscr{U}(a_1)$ with bounding external rays not landing on $\partial U_a$, there exists an angle interval $U' \subset \mathscr{U}(a_2)$ such that $U \subset U'$.

In the following, for simplicity denote $a_1 = a$, $N(a_1) = N(a_2) = N$ and $f_{a_1} = f$.

Let $T_1(a) \setminus T_0(a) = \bigcup_{k=1}^{l} I_k$ which is a union of finite regulated arcs. Given any regulated arc $I_k$ such that $I_k \cap \partial U_a = \emptyset$,

1. If $I_k$ is in the form $(f^{k_1}(-a), -f^{k_2}(-a)]$ for some $1 \leq k_1, k_2 < N$, then $f^{k_1}(-a) \prec -f^{k_2}(-a)$. (Obviously $k_1 \neq k_2$. )

   If $\{R_a(\theta(f^{k_1}(-a))^-), R_a(\theta(-f^{k_2}(-a))^-), R_a(\theta(-f^{k_2}(-a))^+),$
   $R_a(\theta(f^{k_1}(-a))^+)\}$ are the external rays associated to $[f^{k_1}(-a), -f^{k_2}(-a)]$, then

   $$(\theta(f^{k_1}(-a))^-, \theta(f^{k_1}(-a))^+) \subset \mathscr{U}(a_1)$$

   is an angle interval of $\mathscr{U}(a)$.

   Also by lemma 4.15, $f^{k_1}(x) \prec -f^{k_2}(x)$.(Note: If $k_1 > k_2$ apply first case of lemma 4.15; and If $k_1 < k_2$ apply second case of lemma 4.15) Since external rays landing on $f^{k_1}(x)$ in regulated tree $T_1(a_1)$ are exactly the same external rays landing on $f_{a_2}^{k_1}(-a_2)$ in regular tree $T_1(a_2)$. Then similar to above,

$$(\theta(f^{k_1}(x))^-, \theta(f^{k_1}(x))^+) \subset \mathscr{U}(a_2)$$

is an angle interval of $\mathscr{U}(a_2)$.

But by lemma 4.14, $T_0(a_1) \cong [f_{a_1}(x), \cdots, f_{a_1}^N(x)]$, and $f^{k_1}(x) \prec f^{k_1}(-a)$. and furthermore, $f^{k_1}(-a), -f^{k_2}(x) \in [f^{k_1}(x), -f^{k_2}(-a)]$. Then

$$(\theta(f^{k_1}(x))^-, \theta(f^{k_1}(x))^+) \subset (\theta(f^{k_1}(-a))^-, \theta(f^{k_1}(-a))^+).$$

2. Otherwise, $I_k$ must be in the form $(z, -f^{k_1}(-a)]$ for some $0 \leq k_1 \leq N$, where $z \notin Orb(-a)$ is not in the orbit of critical point. Since $T_0(a_1) \cong [f_{a_1}(x), \cdots, f_{a_1}^N(x)]$, then $[z, -f^{k_1}(-a)] \bigcap [f_{a_1}(x), \cdots, f_{a_1}^N(x)] = \{z\}$. And it follows that the angle interval $U_k$ associated to $I_k$ is also an angle interval of $\mathscr{U}(a_2)$.

The above two cases gives the proof of the theorem. □

Now we can give the proof of the main theorem.

**Theorem 4.3** *Suppose that $f_{a_1}$, $f_{a_2}$ are post-critically finite maps. If $a_2 \prec a_1$ and $N(a_1) = N(a_2)$, then $h(T_0(a_1), f_{a_1}) \geq h(T_0(a_2), f_{a_2})$.*

*Proof.* Given the condition, by theorem 4.16, we know that $\mathscr{U}(a_1) \subset \mathscr{U}(a_2)$ and then $\mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(a_2)) \subset \mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(a_2))$. so we have the following inequality on the entropy:

$$h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(a_2)), p) \leq h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(a_1)), p) \tag{4.1}$$

But by the following diagram,

$$\begin{array}{ccc}
\mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(c_i))) & \xrightarrow{p} & \mathbb{T} \setminus \bigcup_{k=1}^{\infty} p^{-k}(\mathscr{U}(a_i))) \\
\Big\downarrow \gamma_{a_i} & & \Big\downarrow \gamma_{a_i} \\
T_0(a_i) & \xrightarrow{f_{a_i}} & T_0(a_i)
\end{array}$$

we can get

$$h(T_0(a_i), f_{a_i}) = h(\mathbb{T} \setminus \bigcup_{k=1}^{\infty} a^{-k}(\mathscr{U}(a_i)), p) \tag{4.2}$$

From the above (4.1) and (4.2), we can see that $h(T_0(a_2), f_{a_2}) \leq h(T_0(a_1), f_{a_1})$. □

## 4.7 Entropy of cubic maps with pure imaginary parameter

In this section, we will prove that for $a \in i\mathbb{R}$, the topological entropy of $f_a$ acting on the pure imaginary axis $\overline{i\mathbb{R}}$ is monotone. The idea is based on [D2].

Let $e_0$ be the top tip of $f_{t_0} * \mathcal{M}$ along the imaginary direction. $e_0 \approx 0.852687i$. Let $e_1$ be the top tip of $\mathcal{C}$ along the imaginary axis. $e_1 \approx 0.884646i$

**Definition 4.17.** *Given $a \in [e_0, e_1]$, according to [F1], the filled Julia set $K_a$ is connected and locally connected. Denote $Y_a = \gamma_a^{-1}(\overline{i\mathbb{R}} \cap J_a)$ and $X_a = \gamma_a^{-1}(T_0(a) \cap J_a)$. For $1/2 < \theta < 3/4$, Set $\Theta_\theta$ and $X_\theta$ as subsets of $\mathbb{T}$ in the* *following way:*

$$\Theta_\theta = (\theta, \frac{3}{2} - \theta) \cup (0, \frac{1}{6}) \cup (\frac{1}{3}, \frac{1}{2}) \tag{4.3}$$

*and*

$$X_\theta = \{t \in \mathbb{T} | \quad (\forall n \geq 0) \quad p^n(t) \notin \Theta_\theta\} \tag{4.4}$$

*The set $X_\theta$ is closed and has no interior points. Indeed it is invariant under tripling map $p$.*

**Definition 4.18.** *Given $a \in [e_0, e_1] \setminus \mathcal{H}_0$, then we can define $1/6 < \xi = \xi_a < 1/4$ and $1/2 < \theta = \theta_a < 3/4$ in the following way:*

1. *If $-a \in J_a$, let $\xi < 1/4$ be the argument of one of the external rays which land on $2a$, then $1/4 - \xi$ is the argument of the other external ray which lands on $2a$ and $\theta = 3\xi$, $3/4 - \theta$ are the arguments of the two external rays which land on $f_a(2a) = f_a(-a)$.*

2. *If $-a \in intK_a$, let $\xi$ be the argument of one of the external rays which land on root point $x$ of Fatou component which contains $2a$, then again $1/4 - \xi$ is the argument of the other external ray which lands on $x$ and $\theta = 3\xi$, $3/4 - \xi$ are the arguments of the two external rays which land on the root point $f_a(x)$ which contain $f_a(2a) = f_a(-a)$.*

**Example 4.19.** Let $a = 0.8809411i$, it's the center of a capture component and $f_a^4(-a) = a$. On parameter plane, $\mathcal{R}_\mathcal{C}(\frac{40}{162})$, $\mathcal{R}_\mathcal{C}(\frac{41}{162})$ land on root point of capture component with center $a$.

Fifure 4.4 is the filled Julia set of $f_a$. The critical orbits and co-critical point was marked. We can see that $R_a(\frac{40}{162})$, $R_a(\frac{41}{162})$ land on root point of Fatou component which contains $2a$. $R_a(\frac{40}{54})$, $R_a(\frac{41}{54})$ land on root point of Fatou component which contains $f_a(-a)$. The external rays land on $-a$, $f_a^2(-a)$ and $f_a^3(-a)$ are also been specified.

In this example, $\xi_a = \frac{40}{162}$, $\theta_a = \frac{40}{54}$ and

$$\Theta_\theta = (\frac{40}{54}, \frac{41}{54}) \bigcup (0, \frac{1}{6}) \bigcup (\frac{1}{3}, \frac{1}{2}).$$

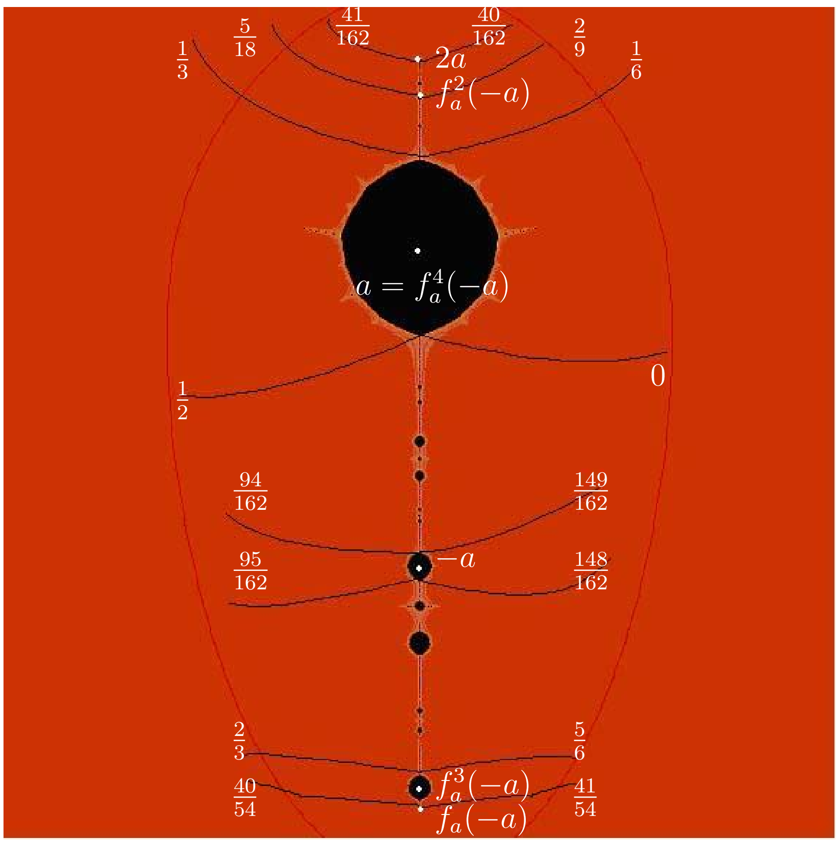


Figure 4.4: The filled Julia Set of $f_a$ with $a = 0.8809411i$

**Theorem 4.20.** *Given $a \in [e_0, e_1] \setminus \mathcal{H}_0$, let $\xi = \xi_a$ and $\theta = \theta_a$ be as in definition 4.18. then*

1. $X_a \subset X_\theta \subset Y_a$

*2.* $h(\overline{i\mathbb{R}}, f_a) = h(J_a \cap \overline{i\mathbb{R}}, f_a) = h(Y_a, p) = h(X_\theta, p) = h(X_a, p)$

*Proof.*

1. Since $T_0(a)$ is invariant set under $f_a$, and $\gamma_a^{-1}(T_0(a)) \cap \Theta_\theta = \gamma_a^{-1}(T_0(a) \cap J_a) \cap \Theta_\theta = \varnothing$, then it's clearly that $X_a \subset X_\theta$.

   On the other hand, we must show that all external ray in $X_\theta$ will land on the imaginary axis. First by definition of $X_\theta$, open arcs $(0, 1/6)$, $(1/3, 1/2)$ which bound external rays landing on Fatou component $U(a)$ and open arc $(\theta, 3/2 - \theta)$ which bound all external rays landing on and below Fatou component $U(f_a(-a))$ are excluded from $X_\theta$. Since $J_a$ is locally connected, given any external ray $R_a(\xi)$ which doesn't land on imaginary axis, there exists a Fatou component $U$ which intersect the imaginary axis, let $R_a(\xi_1) < R_a(\xi_2)$ be the two external rays land on $U \cap i\mathbb{R}$, i.e., any external ray bounded by $R_a(\xi_1)$ and $R_a(\xi_2)$ will not land on imaginary axis. Now by no wandering domain theorem, there exists $k > 0$, such that $f^k(U) = U(a)$ or $f^k(U) = U(f_a(-a))$. then The whole arc $\xi_1, \xi_2$ under $p^k$ either goes to $(0, \frac{1}{6}) \bigcup (\frac{1}{3}, \frac{1}{2})$ in the first case or goes to $(\theta, \frac{3}{2} - \theta)$ in the second case.

2. This is straightforward.

$\square$

Now we can prove the monotonicity result.

**Theorem 4.21.** *The topological entropy for all maps $f_a$ with $a \in \mathcal{C} \cap i\mathbb{R}^+$ acting on $\overline{i\mathbb{R}}$ are monotone.*

*Proof.* Given $a, b \in i\mathbb{R}^+$ such that $|a| < |b|$. Let $\mathcal{R}_{\mathcal{C}}(\xi_a)$ and $\mathcal{R}_{\mathcal{C}}(\frac{1}{2} - \xi_a)$ be the two external rays landing on $a$ (if $a \in \partial\mathcal{C}$) or root point of hyperbolic component which contains $a$ (if $a \in int(\mathcal{C})$). Similarly we can define $\mathcal{R}_{\mathcal{C}}(\xi_b)$ and $\mathcal{R}_{\mathcal{C}}(\frac{1}{2} - \xi_b)$.

Since $|a| < |b|$, it's easy to see that

$$\xi_a < \xi_b < \frac{1}{2} - \xi_b < \frac{1}{2} - \xi_a. \tag{4.5}$$

Also by the relations of parameter external ray and dynamical ray, on the dynamical plane of $f_a$, $R_a(\xi_a)$ and $R_a(\frac{1}{2} - \xi_a)$ land on $2a$ (or the root point of Fatou component which contain $2a$) and then $R_a(\theta_a)$ and $R_a(\frac{1}{2} - \theta_a)$ land on $f_a(2a)$ (or the root point of Fatou component which contain $f_a(2a)$). Where $\theta_a = 3\xi_a$.

Similarly, on the dynamical plane of $f_b$, $R_b(\xi_b)$ and $R_b(\frac{1}{2} - \xi_b)$ land on $2b$ (or the root point of Fatou component which contain $2b$) and then $R_b(\theta_b)$ and $R_b(\frac{1}{2} - \theta_b)$ land on $f_b(2b)$ (or the root point of Fatou component which contain $f_b(2b)$). Where $\theta_b = 3\xi_a$.

Now by inequality (4.5), $(\xi_a, \frac{1}{2} - \xi_a) \supset (\xi_b, \frac{1}{2} - \xi_b)$, then we have $(\theta_a, \frac{1}{2} - \theta_a) \supset (\theta_b, \frac{1}{2} - \theta_b)$ and $X_{\theta_a} \subset X_{\theta_b}$. Finally we get $h(X_{\theta_a}, p) \leq h(X_{\theta_b}, p)$ and then by theorem 4.20, $h(\overline{i\mathbb{R}}, f_a) \leq h(\overline{i\mathbb{R}}, f_b)$. $\square$

**Remark 4.22.** Since $f_a$ conjugate to $f_{-a}$ via involution $\mathcal{I} : z \mapsto -z$, we can see that for the parameter on the negative imaginary axis $i\mathbb{R}^-$, the entropy of $f_a$ acting on $\overline{i\mathbb{R}}$ is still monotone.

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