

**Hubbard Trees and Kneading Sequences for Unicritical and Cubic
Polynomials**

by
Alexandra Kaffl

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Approved, Thesis Committee:

Prof. Dierk Schleicher

Prof. John W. Milnor

Prof. Mary Rees

Prof. Michael Stoll

Prof. Raymond O. Wells

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School of Engineering and Science

Abstract

We investigate two combinatorial models for polynomial dynamics: Hubbard trees and kneading sequences. We consider abstract versions of unicritical and cubic Hubbard trees and study their interdependence with kneading sequences. This gives structure to the set of Hubbard trees, from which we derive results on the structure of the unicritical and cubic connectedness loci.

The manuscript is divided into two parts. The first part is motivated by the dynamics of unicritical polynomials $p_c : \mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto z^d + c$, where $d \geq 2$ and $c \in \mathbb{C}$. We show that (unicritical) Hubbard trees and (unicritical) kneading sequences are two equivalent concepts. Indeed, we develop an algorithm to construct a Hubbard tree from a given \star - or preperiodic kneading sequence (a \star -periodic kneading sequence is a periodic sequence containing the symbol \star). This yields a bijection between the set of (equivalence classes of) Hubbard trees and the set Σ_d^\sharp of \star - and preperiodic kneading sequences. Furthermore, we give a characterization of admissible kneading sequences in Σ_d^\sharp , that is of kneading sequences that are generated by some unicritical polynomial. First we give a topological criterion for Hubbard trees to be admissible and then, using this result, we find a purely combinatorial criterion for the admissibility of elements in Σ_d^\sharp .

On the parameter level, we introduce a partial order “ $<$ ” on $\Sigma_d^\star := \{0, \dots, d-1\}^{\mathbb{N}} \cup \{\star\text{-periodic sequences}\} \supset \Sigma_d^\sharp$, which is based on comparing Hubbard trees. Roughly speaking, a Hubbard tree $\tilde{\mathcal{T}}$ is smaller than a Hubbard tree \mathcal{T} if it is represented by a periodic orbit in \mathcal{T} . Transitivity of “ $<$ ” follows from a forcing relation on periodic orbits between comparable Hubbard trees (indeed, we can force periodic orbits in a more general context). As our main result in parameter space, we give a complete description of the structure of Σ_d^\star by showing that this set is structured like a tree. We determine the locus of non-admissible kneading sequences in the space $(\Sigma_d^\sharp, <)$, answering a question of Kauko. Thus, we can apply our results to the Multibrot sets \mathcal{M}_d ; we provide a new proof for the Branch Theorem, which asserts that branching in \mathcal{M}_d can only happen at postcritically finite parameters.

The motor of our study of Σ_d^\star are results on the structure of periodic

and precritical points in Hubbard trees: a periodic orbit is represented by a single point, its characteristic point. Our results on the arrangements of characteristic and precritical points allow us to take advantage of the 1-to-1 correspondence between kneading sequences and Hubbard trees.

We conclude the unicritical part by discussing alternative approaches to define a partial order for the set of Hubbard trees which does not rely on kneading sequences. Finally, we compare the Σ_d^* for different degrees d and show that there are $\binom{d'-1}{d-1}$ ways to embed Σ_d^* into $\Sigma_{d'}^*$ for any $d' \geq d$.

The second part of this thesis focuses on cubic polynomials. We again define Hubbard trees and kneading sequences in an abstract way and investigate their interaction. The main difficulty lies in the existence of two critical points. As a consequence, the itinerary of a (characteristic) periodic or a critical point x does not always encode the mutual location of the points in $\text{orb}(x)$. Still, we show that there is an injection from the set of (equivalence classes of) Hubbard trees into the set of kneading sequences. We show by way of example that this map is not surjective.

We follow the successful strategy of the unicritical case to link Hubbard trees and kneading sequences in order to gain information about the parameter space. For this, we first explore the dynamical properties of a Hubbard tree. We put an emphasis on fixed and periodic points and show that every periodic orbit contains at least one characteristic point. This yields a classification of the behavior of arms at periodic points under their first return maps. Using these results, we give a topological criterion for a cubic Hubbard tree to be realizable by a cubic polynomial.

On the parameter level, we define a partial order “ $<$ ” on hyperbolic Hubbard trees. For the transitivity of “ $<$ ”, it is crucial to have a forcing relation on orbits between Hubbard trees. We first consider the set \mathcal{H}_μ consisting of hyperbolic Hubbard trees whose critical point c_2 has itinerary μ . We show that orbit forcing is always possible for combinatorially related Hubbard trees of disjoint type in \mathcal{H}_μ . Under one further assumption, this statement extends to all elements of \mathcal{H}_μ . As a consequence, “ $<$ ” is indeed a partial order on this set. Furthermore we show that in \mathcal{H}_μ , the set of Hubbard trees smaller than a given one is linearly ordered. These results can be applied to families of cubic polynomials.

We generalize the obtained results to arbitrary Hubbard trees of disjoint type: we give sufficient conditions that guarantee orbit forcing on these Hubbard trees. Without these conditions, orbit forcing might or might not be possible (we provide examples for both cases). Where orbit forcing holds, we can again define a partial order.

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Chapter 1

Introduction

The topic of our work is symbolic dynamics for complex polynomials. We investigate and compare properties of Hubbard trees and kneading sequences for unicritical polynomials (Part I) and cubic polynomials (Part II). We define them in an abstract way so that not all Hubbard trees and kneading sequences are generated by polynomials. In both settings, we give necessary and sufficient conditions under which a Hubbard tree is realizable by a (not necessarily unique) polynomial. In the unicritical case, we give a combinatorial criterion so that one can read off from a kneading sequence whether it is generated by a polynomial.

Our goal in both parts is to explore the parameter spaces, i.e., the set of all Hubbard trees. In the unicritical part, our main result on this level is the *Branch Theorem*, which provides a complete description of the space of all Hubbard trees, or equivalently of all kneading sequences. We determine the locus of non-admissible kneading sequences. Therefore, our Branch Theorem also describes how postcritically finite parameters are arranged in the connectedness loci of unicritical polynomials. Finally, we investigate the relation between parameter spaces of unicritical polynomials of different degrees in combinatorial terms. To gain information about the cubic parameter space, we discuss under what conditions a periodic orbit can be forced from one Hubbard tree into another. We define a partial order on the set of Hubbard tree for which orbit forcing holds.

Our work is set on a very abstract level and can be regarded more generally as a work in low-dimensional dynamics, or more precisely a work on tree maps (compare various articles by Baldwin and Alsedà et al., e.g. [A, Ba]). However, the motivation of our discussion is holomorphic dynamics. In Sections 1.1–1.3, we give a brief overview of important tools and results in polynomial dynamics. We summarize properties of the connectedness loci of unicritical and cubic polynomials. It might be helpful to keep them in mind, especially for our discussion of unicritical Hubbard trees and kneading sequences in Part I. Furthermore, we give a short introduction to

combinatorial methods in polynomial dynamics, focusing on Hubbard trees and kneading sequences. In their original definitions Hubbard trees and kneading sequences are always generated by postcritically finite polynomials. We use these original definitions as basis for our abstract definitions. Section 1.4 gives an overview of the structure of this manuscript. There, we also describe our results in more detail.

1.1 Dynamics of Complex Polynomials

We assume familiarity with holomorphic iteration theory, in particular with the following concepts: Fatou-, Julia- and filled-in Julia sets; (attracting, repelling and indifferent) periodic, preperiodic and critical points; basins of attraction, Fatou components and the multiplier of a periodic orbit. For an introduction to holomorphic dynamics, we refer to [M3].

Let us recall some terminology and results which are more specific to polynomial dynamics and which relate our work to holomorphic iteration theory. To determine the dynamical behavior of a polynomial, it is important to know the behavior of its critical points. For example, the filled-in Julia set $K(p)$ of a polynomial p is connected if and only if none of the critical points of p escape to infinity. Note that every polynomial of degree d has exactly $d - 1$ critical points, counting multiplicities. If there is just one critical point, which then has multiplicity $d - 1$, we call the polynomial *unicritical* following Milnor. Every unicritical polynomial of degree d is affinely conjugate to one of the form $z \mapsto z^d + c$ with $c \in \mathbb{C}$. Note that the conjugacy is only unique up to a $(d - 1)$ -st root of unity, which explains the $(d - 1)$ -fold rotation symmetry of the parameter space of this family. If all critical points are contained in attractive basins of the polynomial p , then p is called *hyperbolic*. *Postcritically finite* polynomials are polynomials p for which the postcritical set $P := \{p^{on}(c) : n \in \mathbb{N}, c \text{ is critical}\}$ is finite. Note that we denote by \mathbb{N} the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers. Every critical point c of a postcritically finite polynomial is either periodic or preperiodic. The dynamics of such polynomials is completely encoded in the behavior of the critical orbits [D2, DH, Po1].

Let us now concentrate on the case when $K(p)$ is connected for a polynomial p of degree d . There is a unique Riemann map, i.e. a conformal isomorphism $\varphi_p : \mathbb{C} \setminus K(p) \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $\varphi_p(z)/z \rightarrow 1$ as $z \rightarrow \infty$. Moreover, $\varphi_p(p(z)) = \varphi_p(z)^d$, that is, the Riemann map conjugates the dynamics of p on the complement of $K(p)$ to $z \mapsto z^d$ on the complement of the closed unit disk $\overline{\mathbb{D}}$. An *external ray at angle θ* , written as R_θ , is the preimage of the radial line $\{re^{2\pi i\theta} : r > 1\}$ under φ_p . External rays foliate the complement of $K(p)$ and $p(R_\theta) = R_{d\theta}$. We say that a ray R_θ *lands* if $\lim_{r \rightarrow 1} \varphi_p^{-1}(re^{2\pi i\theta}) \in J(p)$ exists. By work of Carathéodory, φ_p^{-1} extends to a continuous map to the boundary if and only if $\partial K(p) = J(p)$ is locally

connected. In this case, all external rays land and conversely, every point in $J(p)$ is the landing point of an external ray. Note that then, $\varphi_p^{-1}|_{\mathbb{S}^1}$ semi-conjugates multiplication of angles in $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by d to the dynamics of p on $J(p)$. It is known that rays at rational angles always land; the landing points are periodic or preperiodic points [M4, S1].

1.2 The Connectedness Loci of Unicritical and Cubic Polynomials

In this section, we give an overview of properties of the connectedness loci of unicritical and cubic polynomials.

Let us start with the family $\{z \mapsto z^d + c : c \in \mathbb{C}\}$ of unicritical polynomials of degree d . Its parameter space is the complex plane. Following [LS], we call its connectedness locus the *Multibrot set* of degree d and denote it by \mathcal{M}_d . For $d = 2$, \mathcal{M}_2 is the well known Mandelbrot set. The following results go back to [DH] for the quadratic case, for degree $d > 2$, see [E].

Let us start by looking at \mathcal{M}_d from the outside: by the Riemann mapping theorem, there is a conformal isomorphism $\Phi_{\mathcal{M}_d} : \mathbb{C} \setminus \mathcal{M}_d \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$. One defines external rays for the parameter space exactly the same way as it is done for the dynamical plane (see Section 1.1). These rays, which are usually called *parameter rays* to distinguish them from external rays in the dynamical plane, foliate the complement of \mathcal{M}_d . Looking at \mathcal{M}_d from the inside, we first encounter *hyperbolic components*. These are connected components of the set $\{c \in \mathcal{M}_d : z \mapsto z^d + c \text{ has an attracting periodic orbit}\}$. The period of the attracting orbit is constant throughout any hyperbolic component; thus one can speak of the *period* of a hyperbolic component. It is well known that for any hyperbolic component W , the *multiplier map* $\mu_W : W \longrightarrow \mathbb{D}$ is a degree $d - 1$ branched covering, ramified over 0, which extends continuously to the boundary. A *sector* of a hyperbolic component is the image of \mathbb{D} under a branch of μ_W^{-1} . The point $\mu_W^{-1}(0)$ is called the *center* of the hyperbolic component and is the unique parameter in W for which the critical point is periodic. There is a unique parameter $c_W \in \partial W$ where two parameter rays land and separate the hyperbolic component from the origin. These two rays are always periodic (i.e., their angles in \mathbb{S}^1 are periodic under multiplication by d) and $\mu_W(c_W) = 1$. This special parameter in ∂W is called the *root* of W . The other $d - 2$ preimages of 1 under μ_W are the *co-roots* of W . At each of them exactly one periodic parameter ray lands. Each co-root lies on the boundary of no other hyperbolic component than W . The root c_W is contained in the boundary of at most one further hyperbolic component W' . If it is not, then W is called *primitive*. If it is, then W is a *satellite* component of W' ; we say that W *bifurcates* from W' . Note that in this case, the period of W is a strict multiple of the period of W' . At every parameter $c \in \partial W$ with $\mu_W(c) = e^{2\pi ip/q}$, $p/q \in \mathbb{Q}/\mathbb{Z}$ in

lowest terms, exactly two parameter rays land. Moreover, c is the root of a satellite component of W , which sometimes is called the p/q -satellite of W . The *wake* of W is the open region in \mathbb{C} that is separated from the origin by c_W and the two parameter rays landing at c_W . A p/q -*subwake* of W is the wake of the p/q -satellite component.

Another set of parameters which play an important role for the structure of \mathcal{M}_d are the *Misiurewicz points*. A Misiurewicz point is a parameter $c \in \mathcal{M}_d$ for which the critical point of $z \mapsto z^d + c$ is preperiodic under iteration. Every Misiurewicz point c is the landing point of a finite, positive number of parameter rays with preperiodic angles. Together with c , they partition \mathbb{C} into a finite set of open regions. Each of them except the one which contains the origin is called a *subwake* of the Misiurewicz point c . The *wake* of c is the union of all its subwakes together with the parameter rays in between them.

Hyperbolic components and Misiurewicz points reveal a lot of the structure of the Multibrot sets: for any two hyperbolic components or Misiurewicz points A_1, A_2 , either one is contained in a subwake of the other, or there is another hyperbolic component or Misiurewicz point B such that A_1 and A_2 are contained in two different subwakes of B . This result, known as the Branch Theorem, is proven by Douady and Hubbard in [DH, XXII.3] for the Mandelbrot set. A proof for arbitrary degree can be found in [S2]. The importance of this theorem is not only that it asserts that all branch points in the Multibrot sets are postcritically finite (if one thinks of branching at a hyperbolic component as a branching that happens at its center); it also is a crucial step in proving that local connectivity implies density of hyperbolicity. In fact, Schleicher uses the Branch Theorem to show that local connectivity and triviality of fibers are equivalent [S2, Corollary 4.6].

Now let us take a look at the cubic parameter space. Since up to affine conjugacy, every cubic polynomial can be written in the Branner-Hubbard form as $p_{a,b}(z) = z^3 - 3a^2z + b$, this space is of complex dimension two. By the seminal work of Branner and Hubbard [BH1, BH2], the connectedness locus \mathcal{C}_3 is compact, connected and cellular, i.e. its complement in $\mathbb{R}^4 \cup \{\infty\}$ is an open topological cell. The complement $\mathbb{C}^2 \setminus \mathcal{C}_3$, the *escape locus*, comprises all polynomials such that at least one of the two critical points c_1, c_2 escapes to infinity. Let c_2 be the one of higher escape rate, i.e. the one that has larger potential h . Then the escape locus is partitioned by the sets $\mathcal{S}_\rho := \{p_{a,b} : h(c_2) = \log(\rho)\}$, which are homeomorphic to \mathbb{S}^3 . They in turn are foliated by so-called *turning curves*, which are either simple closed curves or infinite curves whose closure is a torus or a dyadic solenoid. Any two polynomials on a turning curve are quasi-conformally conjugate [Br]. Observe that turning curves are the analogous concept to equipotential lines in the unicritical case while *stretching rays* correspond to external rays. Both are used to study the cubic connectedness locus from the outside. Nakane and co-authors deal

with the landing properties of stretching rays in several papers, cf. [KN]. The escape locus can also be partitioned into the following two regions: the *shift locus* S_3 where both critical points escape, and the set of parameters where exactly one critical point escapes. The latter region has been studied intensely in [BH2]. For the shift locus, [BDK] shows that there is a surjective map from the fundamental group $\pi_1(S_3)$ onto the automorphism group Aut_3 of the one-sided shift acting on Σ_3 . Aut_3 is the set of all homeomorphisms on Σ_3 that commute with the shift and Σ_3 is the set of one-sided infinite sequences over a three letter alphabet. The Julia set of every polynomial $p_{a,b}$ in S_3 is a Cantor set and the action of $p_{a,b}$ on it is conjugate to the action of the left shift on Σ_3 ; thus the name shift locus.

So far, there is no description of \mathcal{C}_3 from the inside. There has been some effort to describe one-dimensional slices of \mathcal{C}_3 . This was started by Milnor in a draft circulated in 1991 [M5]. He suggests to consider slices \mathcal{S}_n , $n \in \mathbb{N}$, consisting of polynomials that have one periodic critical point of exact period n . Building on this work, the thesis of Faught [Fa] investigates the slice \mathcal{S}_1 . Using Yoccoz' puzzle technique, he gives a complete description of $\mathcal{C}_3 \cap \mathcal{S}_1$. Except for this result, not much is known about the slices \mathcal{S}_n . To our knowledge, it is still open whether the curves \mathcal{S}_n are always connected and what their genus is. It is known, however, that the genus tends to infinity as $n \rightarrow \infty$ [M5]. One noteworthy fact about the cubic connectedness locus is that it is not locally connected as shown by Lavaurs (unpublished). This also holds for the real slice of the cubic connectedness locus, cf. [M1] and [EY].

In [M2], it is shown that every hyperbolic component of \mathcal{C}_3 has a unique center, i.e., for each hyperbolic component there is a unique parameter pair (a, b) such that $p_{a,b}$ is postcritically finite. This paper builds on Rees' methods to study hyperbolic components of rational maps of degree two. We will discuss in the next section that every postcritically finite polynomial generates a Hubbard tree that characterizes it completely. So proving structural properties for the set of Hubbard trees gives structure to the cubic connectedness locus. This is the approach we pursue in Part II of this manuscript.

1.3 Combinatorics in Holomorphic Dynamics

The goal in holomorphic dynamics is to classify different dynamical behavior. In this process, combinatorial models are very important since they extract the essential features of a given dynamical system. Very often they are more convenient to study than the actual dynamical systems. So a possible working plan is the following: given a family of dynamical systems (in our case, unicritical or cubic polynomials), distinguish different dynamics in combinatorial terms. Then structure the parameter space according to the different combinatorics. This yields a partition into combinatorial

classes. To gain information about the actual structure of the parameter space, one has to investigate the different combinatorial classes. If they are all singletons, which is known as *combinatorial rigidity*, then the structure of the combinatorial classes corresponds to parameter space. If combinatorial classes are not trivial, one has to investigate what kind of dynamical systems are contained in a class.

The groundbreaking work of Douady and Hubbard [DH] shows that the dynamics of polynomials, including the geometry of their Julia sets, can be understood in terms of combinatorics. In particular, they show that the dynamics of any polynomial whose critical points are all preperiodic is characterized by the behavior of the critical orbits. This result was extended to all postcritically finite polynomials by Poirier [Po1, Po2]. So in this situation, combinatorial rigidity is guaranteed and our investigation of combinatorial models will give structure to the parameter space of (unicritical and cubic) polynomials. Note that Henriksen shows that in general, rigidity fails for cubic polynomials: there are two cubic polynomials which generate the same *rational lamination*, a combinatorial model described below, yet they are not topologically conjugate [H].

Let us give a brief overview of the various combinatorial concepts used to describe polynomial dynamics. Each of them is more natural than the others for certain types of polynomials. Recall from Section 1.1 that the complement of the filled-in Julia set is foliated by external rays if it is connected. Furthermore, if the Julia set is locally connected then all rays land; this induces a semi-conjugacy between the map $\vartheta \mapsto d\vartheta$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and the dynamics on the Julia set, where d is the degree of the polynomial. Let us define an equivalence relation on \mathbb{S}^1 by setting $\theta \sim \theta'$ if the rays at angle θ and θ' land together. This equivalence relation is called a *rational lamination* [McM] and gives rise to the *pinched disk model* of the filled-in Julia set [D2]. This concept was extended by Kiwi to all polynomials that do not have an indifferent periodic orbit; in particular, the Julia set is not required to be locally connected. This equivalence relation is called a *real lamination* [Ki2]. Closely related to it is the *quadratic invariant lamination* defined by Thurston as a model for the dynamics of quadratic polynomials. He pioneered the concept of laminations in [T], where he defines laminations on an abstract level and investigates their properties. In particular, he shows that the set of all laminations is a lamination again, the so-called *quadratic minor lamination QML*. Recently, using laminations, Blokh and Oversteegen showed that there are polynomials whose Julia sets contain branch points with infinite forward orbit, a question that had been open for about 20 years [BO].

There are two further combinatorial invariants that are defined via external rays: *orbit portraits* and *critical portraits*. An orbit portrait for an n -periodic point z is the set $\Theta = \{\Theta_1, \dots, \Theta_n\}$, where Θ_i is the collec-

tion of angles of external rays landing at $f^{\circ i}(z)$. Orbit portraits have been successfully used to prove structural statements about the dynamical and parameter planes of polynomials [M4, Ki1]. This approach was pioneered by Goldberg and Milnor in [GM] by their definition of *fixed point portraits*. They relate this concept to critical portraits, which were introduced and explored by Fisher for polynomials whose critical points are all preperiodic [Fi]. Similar to an orbit portrait, a critical portrait is a collection of sets Θ_i of angles; however, the sets Θ_i are defined via external rays landing at the critical points c_i . Building upon this work, it is proven in [BFH] that polynomials whose critical points are all preperiodic are characterized by their critical portraits. Poirier extended this result to the set of all postcritically finite polynomials [Po1].

The remainder of this section is devoted to the two concepts of Hubbard trees and kneading sequences. We discuss them in more detail because they are at the core of the work at hand. From now on, we only consider postcritically finite polynomials. Hubbard discovered in the early 80s that the filled-in Julia set of every postcritically finite polynomial contains a special topological tree, its *Hubbard tree*. According to [DH], the Hubbard tree of such a polynomial is the smallest forward invariant set that contains all critical orbits. Let us first consider polynomials whose critical points are all preperiodic. In this case, the filled-in Julia set is a dendrite and thus uniquely arcwise connected. The Hubbard tree is exactly the convex hull of the points on the critical orbits, taken within the filled-in Julia set. If however at least one of the critical points is periodic, we have to add an additional condition so that the Hubbard tree is uniquely defined: let us consider each periodic critical point and its Fatou component separately. Analogous to the construction of external rays, every such Fatou component U can be foliated by *internal rays* R_θ , starting at the critical point and landing at the boundary. This foliation can be chosen such that $f^{\circ n}(R_\theta) = R_{d\theta}$, where n is the exact period of U and d is the degree of $f^{\circ n}|_U$. Taking preimages induces a foliation on all Fatou components. If we now require that any arc connecting two points in the Hubbard tree may intersect at most two internal rays per Fatou component, then the Hubbard trees for all postcritically finite polynomials are uniquely defined.

Douady and Hubbard augment Hubbard trees with the so-called *secondary* or *angle information* so that they characterize polynomials with all critical points preperiodic [DH, Chapter VI]. These statements are extended to all postcritically finite polynomials in [Po2]: Poirier reduces the problem to finding a bijection between his *abstract* Hubbard trees and critical portraits. An essential step to show that critical portraits characterize postcritically finite polynomials is the introduction of *critically marked polynomials*; these are postcritically finite polynomials with associated sets of angles for each critical point: take any Fatou-component U which contains

a periodic critical value $p(c)$, say of exact period n . Pick a ray landing at some $z \in \partial U$ with $p^{\circ n}(z) = z$. For a preperiodic critical point c , choose a ray landing at $p(c)$. In either case, pull this ray back by p . The preimage rays landing at c form the angle set associated to the critical point c . If there are critical orbit relations, one has to be more careful when choosing the ray, see [Po1, Section I.2]. These sets of rays partition \mathbb{C} (if extended by internal rays where necessary). The rays also partition the Hubbard tree. We will use this partition to interpret a Hubbard tree in the sense of Douady and Hubbard as a Hubbard tree according to our definition.

Our definition extends Hubbard trees in the sense of Bruin and Schleicher who have lifted the concept of quadratic Hubbard trees to an abstract level:

Definition 1.3.1 (Hubbard trees). [BS, Definition 3.2] *A Hubbard tree (T, f) is a topological tree T together with a continuous and surjective map $f : T \rightarrow T$ and a distinguished point c_0 , the critical point, with the following properties:*

- (i) *f is at most 2-to-1 and f is locally 1-to-1 on $T \setminus \{c_0\}$,*
- (ii) *all endpoints of T are on the critical orbit,*
- (iii) *c_0 is periodic or preperiodic,*
- (iv) *if $x \neq y$ are branch points or in $\text{orb}(c_0)$, then $\exists n \geq 0$ such that $c_0 \in f^{\circ n}([x, y])$.*

Note in particular that these Hubbard trees are not embedded into the plane. In some sense, they are a purified version of Douady's and Hubbard's definition. This makes their investigation easier. Another advantage is that this is the perfect definition to match another combinatorial concept, namely *kneading sequences*.

Kneading sequences come from real dynamics and were introduced in [MT] to study iterated interval maps. They have the following extension to complex dynamics, cf. [LS]. Let us again restrict to the quadratic case. Fix any $\theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then the *itinerary* $\tau(\vartheta) = (\tau_i(\vartheta))_{i=1}^{\infty}$ of $\vartheta \in \mathbb{S}^1$ with respect to θ is given by

$$\tau_i(\vartheta) = \begin{cases} 0 & 2^{i-1}(\vartheta) \in]\frac{\theta}{2}, \frac{\theta+1}{2}[\\ 1 & \text{if } 2^{i-1}(\vartheta) \in]\frac{\theta+1}{2}, \frac{\theta}{2}[\\ \star & 2^{i-1}(\vartheta) \in \{\frac{\theta}{2}, \frac{\theta+1}{2}\} \end{cases} .$$

The *kneading sequence* of θ , usually denoted by $\nu(\theta)$, is the itinerary $\tau(\theta)$ with respect to θ .

Let us connect this definition to the dynamics of postcritically finite quadratic polynomials p_c . If the critical point is preperiodic, then the critical value c is the landing point of an external ray R_θ ; if it is periodic of exact period n and U denotes the critical-value Fatou component, then there is

an external ray R_θ that lands at the unique fixed point of $f^{\circ n}$ in ∂U . This immediately implies that the kneading sequence of θ encodes the dynamics of the critical value. In the preperiodic case, the corresponding parameter ray R_θ lands at the parameter c , in the periodic case it is part of the ray pair that lands at the root of the hyperbolic component whose center is c . (This statement is part of the correspondence principle for dynamic and parameter rays [L, E].) Moreover, all rays landing at the same Misiurewicz point (or at the root of the same hyperbolic component) define the same kneading sequence. So, we can speak of the kneading sequence associated to a Misiurewicz point or to a hyperbolic component. Since all rational rays land, there is a 1-to-1 correspondence between the set $\{\nu(\theta) : \theta \in \mathbb{Q}/\mathbb{Z}\}$ and the set of hyperbolic components and Misiurewicz points up to symmetries of \mathcal{M}_2 . Note that analogous statements also hold for the unicritical case of arbitrary degree $d \geq 2$; in particular, we get a bijection (again up to symmetries) between the set of hyperbolic components and Misiurewicz points of \mathcal{M}_d and the set of kneading sequences in $\{0, \dots, d-1, \star\}^{\mathbb{N}}$ that are generated by rational angles, where angle doubling is replaced by the map $\vartheta \mapsto d\vartheta$.

Again in [BS], one finds an abstract version of kneading sequences for the quadratic case: a kneading sequence is any element of the set

$$\Sigma_2^\star := \{\nu \in \{0, 1\}^{\mathbb{N}} : \nu_1 = 1\} \cup \{\nu \text{ is } \star\text{-periodic}\},$$

where a \star -periodic sequence is a sequence of the form $\overline{1\nu_2 \dots \nu_{n-1}\star}$ or equals $\overline{\star}$. The notation $\overline{\nu_1 \dots \nu_n}$ means that the word $\nu_1 \dots \nu_n$ is repeated periodically.

Penrose, who investigated topological models for quadratic Julia sets and the Mandelbrot set by gluing itineraries and kneading sequences together, noted that not all such kneading sequences come from (quadratic) polynomials [Pe]. This means that not all of them are generated by a rational angle according to the definition above. This problem was also mentioned in [Ke], where the two concepts of laminations and kneading sequences (for the quadratic setting) are linked. In [BS], one finds a complete characterization of kneading sequences that are generated by quadratic polynomials. We call such kneading sequences *admissible*. Similarly, Hubbard trees that are realizable by polynomials are called admissible.

In [Kau2], Kauko investigates the space of kneading sequences of degree d by studying the equivalent concept of *internal addresses* (cf. see Lemma 2.3.3.) She finds a class of kneading sequences which are not admissible, however leaves open if every non-admissible kneading sequence is contained in this class. We answer this in the affirmative in Proposition 3.3.14.

1.4 Outline of the Thesis

The present work is an extension of the manuscript [BS]. There, Bruin and Schleicher review various methods in symbolic dynamics for quadratic polynomials and also discuss (old and new) algorithms how to go from one concept to the other. A central part of their work is to lift the concepts of Hubbard trees and kneading sequences to an abstract level and to investigate their properties. In particular, they give a purely combinatorial characterization of all admissible (quadratic) kneading sequences and define a partial order on the set of kneading sequences, which turns this set into a model for the Mandelbrot set. Moreover, they investigate biaccessible points from a measure theoretical point of view and show that the set of admissible kneading sequences has positive $\frac{1}{2}$ - $\frac{1}{2}$ product measure.

There are two natural extensions to their work: the first one is to stick to the existence of exactly one critical point but to allow that it has arbitrary multiplicity; in a second direction of extension, one allows for two critical points (counted with their multiplicities). The first one yields combinatorial models for unicritical polynomials and the Multibrot sets, the second one for general cubics. We deal with both extensions, omitting the measure theoretical part of Bruin's and Schleicher's work. Part I is devoted to the unicritical case, Part II to the cubic case. Both parts are to a large extent independent from each other (some definitions of Part I are repeated in Part II).

The dynamics and the structure of the parameter spaces of unicritical polynomials are fairly well understood (see the discussion in Section 1.2). Although not all techniques of the quadratic setting (e.g. Yoccoz' methods to prove local connectivity) carry over in a 1-to-1 fashion to the general unicritical case, the relation to quadratic polynomials is very close. Thus, our main results in Part I are to find necessary and sufficient conditions for kneading sequences to be admissible and, secondly, to give structure to the set of all kneading sequences; in particular, we determine the locus of non-admissible sequences. As a corollary of this discussion, we derive structural properties of the Multibrot sets: our results provide a new combinatorial proof of the Branch Theorem [DH, XXII.3], see also page 4.

In contrast to the unicritical situation, there are still many open questions about the cubic connectedness locus \mathcal{C}_3 . In particular, there is no description of \mathcal{C}_3 from the inside. Our work with Hubbard trees provides a first step into this direction.

Summary

In more detail, our thesis is structured the following way. Each of the two parts is subdivided into a section about the dynamical plane, where properties of individual Hubbard trees and kneading sequences are discussed,

and one section about the parameter plane, where we compare different Hubbard trees (or kneading sequences). In the unicritical setting, we devote Section 3.5 to the comparison of Σ_d^* for different degrees d (recall that Σ_d^* is the set of all kneading sequences of degree d).

In Section 2.1, we introduce the concept of Hubbard trees and discuss their basic properties. In particular, we prove that every periodic orbit in a Hubbard tree has a unique distinguished point, the so-called *characteristic point*. Its existence is crucial for our definition of a partial order on Σ_d^* . Propositions 2.1.21 and 2.1.23 characterize the behavior of arms at periodic points under the first return map.

In Section 2.2, we define equivalence classes of Hubbard trees and discuss the concept of *minimal Hubbard trees* introduced in [Ka]. These are Hubbard trees with restricted possibilities for their dynamics: in a minimal Hubbard tree, any two (pre-)periodic points have distinct itineraries and the critical orbit is locally attracting if it is periodic. We show that every equivalence class of Hubbard trees contains a minimal representative. Since we are interested in classifying equivalence classes, the existence of a minimal representative with its purified dynamics is very convenient.

In Section 2.3, we introduce kneading sequences, our second combinatorial tool of interest, and link it with the concept of Hubbard trees. One of our main results is Corollary 2.3.22, which says that there is a bijection between the set of \star - and preperiodic kneading sequences and the set of equivalence classes of Hubbard trees. In particular, we give an algorithm how to construct a Hubbard tree from a given kneading sequence. Most work goes into verifying that the defined dynamics turns the obtained topological tree into a Hubbard tree.

Section 2.4 deals with the question of admissibility. More precisely, we give topological conditions on Hubbard trees and combinatorial conditions on kneading sequences that are necessary and sufficient for their admissibility.

We start the discussion of the parameter plane in Section 3.1 with introducing *lower* and *upper kneading sequences*. These are special elements of $\Sigma_d^0 = \{0, \dots, d-1\}^{\mathbb{N}}$ associated to a given \star -periodic kneading sequence. They can be used to distinguish different types of \star -periodic kneading sequences: *primitive*, *bifurcation* and *backward bifurcation* sequences. The first two objects correspond to the two different types of hyperbolic components, namely primitive and satellite components as introduced on page 3. While bifurcation and primitive kneading sequences might be admissible or not, backward bifurcation sequences never are: they correspond to a branching off into (a higher level of) non-admissibility.

In Section 3.2, we investigate the arrangements of characteristic points in a Hubbard tree (Theorem 3.2.1). Another important result in this section is that given two Hubbard trees $\mathcal{T}_1, \mathcal{T}_2$, under certain conditions, \mathcal{T}_1 contains a characteristic point of itinerary τ if and only if \mathcal{T}_2 contains a character-

istic point of itinerary τ (this is Proposition 3.2.3 on orbit forcing). These two results form a combinatorial analog to the correspondence principle of dynamic and parameter rays for Multibrot sets.

Building upon this work, we structure the set Σ_d^* in Section 3.3: we first define a partial order on \star - and preperiodic kneading sequences, which we extend in Section 3.3.4 to all elements of Σ_d^* . Using this order, we determine the locus of non-admissible kneading sequences. This answers a question posed by Kauko in [Kau2]. Our main result of the discussion on the parameter level completely describes the structure of Σ_d^* :

Theorem 3.3.20 (Branch Theorem for Σ_d^*). *Let $\nu \neq \tilde{\nu} \in \Sigma_d^*$. Then either $\nu < \tilde{\nu}$, or $\tilde{\nu} < \nu$, or there is a unique $\mu \in \Sigma_d^*$ such that $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \mu]$. Moreover, μ is either preperiodic or $\mu \in \{\mu^*, \mathcal{A}_i(\mu^*), \overline{\mathcal{A}}(\mu^*)\}$ for some \star -periodic kneading sequence μ^* .*

We conclude the section on parameter space by discussing alternative approaches to define a partial order on Hubbard trees. More precisely, we concentrate on finding a partial order that does not rely on kneading sequences. We know of regions in the cubic connectedness locus where the presented strategy to use itineraries of characteristic points and kneading sequences to define a partial order cannot be applied to. It turns out that possible approaches to go around this problem have their flaws as well and cause difficulties already in the unicritical case. This is addressed in Section 3.4.

The last section in the unicritical case is devoted to a comparison of Σ_d^* for different degrees d . In particular, we show that there is an embedding $\iota : \Sigma_d^* \longrightarrow \Sigma_{d'}^*$ for any $1 < d \leq d'$. We also show that a non-admissible kneading sequence cannot become admissible when interpreted as a kneading sequence of higher degree.

The cubic part starts in Section 4.1, where we define a (cubic) Hubbard tree to be a triple (T, f, \mathcal{P}) consisting of a topological tree T , a tree map $f : T \rightarrow T$ and a partition \mathcal{P} of T which meet certain conditions. The conditions on \mathcal{P} are motivated by properties of Hubbard trees in the sense of Douady and Hubbard, see Section 4.1.2. In Sections 4.1.3–4.1.5, we determine basic properties of (cubic) Hubbard trees and discuss four fundamentally different types. Again the investigation of periodic orbits reveals a lot about the dynamics of Hubbard trees: we show in Proposition 4.1.20 that each periodic orbit contains one or two characteristic points and exploit this result to characterize the behavior of arms at periodic points under the first return map in Section 4.1.7. Observe that every branch point of a Hubbard tree is preperiodic.

In Section 4.1.8, we address the question of admissibility: we provide necessary and sufficient conditions under which a Hubbard tree is admissible. In this section, we also introduce equivalence classes of Hubbard trees.

As a next step towards our investigation of the parameter space, we

associate to each Hubbard tree its *kneading sequence*, i.e. the tuple consisting of the itineraries of the two critical values, see Section 4.2. Theorem 4.2.7 proves that two Hubbard trees generate the same kneading sequence if and only if they are equivalent. We give an example of a kneading sequence that is not generated by a Hubbard tree. Recall that in the unicritical case, there was a bijection between equivalence classes of Hubbard trees and kneading sequences.

In Section 4.3, we introduce minimal Hubbard trees. We define them in a slightly more general way than in the unicritical case. As a basis for our discussion of a forcing relation between Hubbard trees, we determine the fixed-point sets of minimal Hubbard trees and show under which conditions which preimages of a characteristic point exist. We illustrate the various possibilities by examples.

In Chapter 5, we define a partial order “ $<$ ” on hyperbolic Hubbard trees, which is based on kneading sequences and itineraries. Transitivity of “ $<$ ” follows from a forcing relation between combinatorially related Hubbard trees. We first restrict ourselves to the set \mathcal{H}_μ comprising all hyperbolic Hubbard trees whose critical values c_2 have itinerary μ (Section 5.1.1). We show that forcing is always possible for Hubbard trees in \mathcal{H}_μ of disjoint type. Under one further assumption, the statement on orbit forcing extends to all Hubbard trees in \mathcal{H}_μ , which allows us to define a partial order for this set. We prove that in \mathcal{H}_μ , the set of all Hubbard trees smaller than a Hubbard tree \mathcal{T} is linearly ordered. We give examples of families of cubic polynomials which our results can be applied to, including the family studied by Faught [Fa].

Section 5.1.2 considers arbitrary Hubbard trees of disjoint type. We give sufficient conditions under which orbit forcing from a Hubbard tree into another one is always possible. By way of examples we show that without these conditions, forcing might or might not be possible. We define a partial order on the set of Hubbard trees where orbit forcing is possible. We give an example that in this situation, the set of smaller Hubbard trees is not linearly ordered.

In the last Section 5.2, we discuss whether the partial order defined so far can be extended to a larger set of Hubbard trees. We present examples which suggest that this is possible; however, this would require a different approach of defining a partial order.

Part I

The Unicritical Case

Chapter 2

The Dynamical Plane

2.1 Hubbard Trees

We start this section by recalling some definitions from topology. A connected and simply connected metric space T is called a *tree* if it can be written as the finite union of closed intervals. We call $x \in T$ an *endpoint*, *inner point* or *branch point* of T according as $T \setminus \{x\}$ consists of one, two or at least three connected components. The connected components of $T \setminus \{x\}$ are called *global arms* at x . The global arm at x containing $y \in T$ is denoted by $G_x(y)$. A *local arm* L at x is a suitable representative of the *germ* of a global arm G at x . In particular, given a map $f : T \rightarrow T$ on a tree T and given any $n \in \mathbb{N}$, the local arm L associated to a global arm G of x is a small enough interval $]x, p[\subset G$ such that $f^{on}|_L$ is a homeomorphism onto its image. We say that a local arm L at x is *pointing to* $y \in T$ if its global arm contains the point y . We denote this local arm by $L_x(y)$.

Let $x_1, \dots, x_n \in T$ be n pairwise distinct points of the tree T . We denote the *connected hull* of these n points in T by $[x_1, \dots, x_n]$. This set is a tree itself, indeed it is a *subtree* of T . For $n = 2$, $[x, y]$ equals the unique arc in T connecting x and y . The arc without its endpoints is denoted by $]x, y[$, and $[x, y[$ is the arc containing the endpoint x but not the endpoint y .

An *n-od*, $n \geq 3$, is a tree T with exactly one branch point b such that $T \setminus \{b\}$ falls into exactly n connected components. We allow that an *n-od* is lacking some or all of its endpoints. A 3-od is usually called a *triod*. For the purpose of our discussion, we weaken the definition of a triod: consider the connected hull of any three distinct points $x, y, z \in T$. The set $[x, y, z]$ is either homeomorphic to the letter “Y”, i.e., it is a triod in the above sense, or it is an interval. In both cases, we call the set $[x, y, z]$ a triod. More precisely, we say that $[x, y, z]$ is a *non-degenerate triod* if it is homeomorphic to the letter “Y”. It is a *degenerate triod with x in the middle* if $[x, y, z]$ is homeomorphic to an interval and x is in the interior of $[x, y, z]$.

2.1.1 Expanding Trees

The following definition of *expanding trees* is the straightforward adaption of the definition of quadratic Hubbard trees in [BS, Chapter 3] to arbitrary degree $d \geq 2$.

Definition 2.1.1 (Expanding trees). *An expanding tree of degree $d \geq 2$ is a tuple $(T, f)_d$ consisting of a topological tree T with one distinguished point c_0 , the critical point, and a continuous surjective map $f : T \rightarrow T$ such that the following are satisfied:*

- (i) *The critical point c_0 is preperiodic or periodic.*
- (ii) *All endpoints of T are contained in $\text{orb}(c_0)$.*
- (iii) *$f : T \rightarrow T$ is at most d -to-1.*
- (iv) *f is locally injective at any point $p \neq c_0$.*
- (v) (Expansivity) *If $V := \{v \in T : v \text{ is a branch point or } v \in \text{orb}(c_0)\}$, then for all $x \neq y \in V$, there is an $n \in \mathbb{N}_0$ such that $c_0 \in f^{on}([x, y])$.*

Following [BS], an element of the set V is called a *marked point*. This set corresponds to the set of vertices defined in [Po2]. Two marked points $v, \tilde{v} \in V$ are *adjacent* if $]v, \tilde{v}[\cap V = \emptyset$.

Let $(T, f)_d$ be an expanding tree. A point $p \in T$ is called *periodic of period n* if there is an $n \in \mathbb{N}$ such that $f^{on}(p) = p$. The number n is called *exact period* of p if n is the smallest positive integer with this property. We say that a point is *n -periodic* if it is periodic of exact period n . The point p is *preperiodic* if p is not periodic, yet there is a smallest integer $l > 0$ such $f^{ol}(p)$ is periodic. The integer l is called the *preperiod* of p . The orbit of a periodic point is usually called *cycle*. If a cycle \mathcal{C} contains the critical point, then we say that \mathcal{C} is *critical*. Any expanding tree contains at most one critical cycle. The image $f(c_0) =: c_1$ of the critical point is called the *critical value*. We extend this notation to the whole critical orbit: $f^{oi}(c_0) =: c_i$ for all $i \in \mathbb{N}_0$. Observe that $c_i = c_{i \bmod n}$ for all $i \in \mathbb{N}_0$ if c_0 is n -periodic, and for all $i \geq l > 1$ if c_0 is preperiodic of preperiod l and the periodic part has exact period n . We sometimes use this notation also for non-critical cycles (especially when labeling points in pictures).

A point ξ is called *precritical* if there is a $k > 0$ such that $f^{ok-1}(\xi) = c_0$. If k is the smallest number with this property, then we call k the *step* of the precritical point ξ and write $\text{STEP}(\xi) = k$, i.e., $\text{STEP}(\xi)$ indicates how many iteration steps it takes until ξ is mapped onto the critical value. Observe that according to this definition, the critical point c_0 is *precritical of step one* both for periodic and preperiodic c_0 . The critical point is the unique precritical point with step equal to one. We call a point $p \in T$ *(pre-)critical* if it is either critical or strictly precritical. Similarly, p is *non-(pre-)critical*

if p is neither critical nor strictly precritical. If ξ and ξ' are precritical with $\text{STEP}(\xi) = \text{STEP}(\xi')$, then there is precritical point $\xi'' \in]\xi, \xi'[$ with $\text{STEP}(\xi'') < \text{STEP}(\xi)$.

Finally, let us define the *image* of a local arm. If L_x is a local arm of x which is represented by the interval $]x, p[$, then its image $f(L_x)$ is the local arm at $f(x)$ which intersects $f(]x, p[)$. We say L_x is *periodic of exact period k* (or *k -periodic*) if x is n -periodic and k is the smallest number such that $f^{\circ kn}(L_x) = L_x$. The set $\{f^{\circ jn}(L_x) : j \in \mathbb{N}_0\}$ is called a *cycle of local arms*. The *length* of a cycle of local arms equals the number of its elements.

The following lemma describes some basic properties of expanding trees.

Lemma 2.1.2 (Properties of expanding trees). *For any expanding tree $(T, f)_d$, the following are true:*

- (i) *The critical value $f(c_0)$ is an endpoint of T and $f(c_0)$ is not a fixed point of f . Any local arm at c_0 is mapped to the unique local arm at $f(c_0)$ and the set $T \setminus \{c_0\}$ consists of at most d connected components.*
- (ii) *All branch points of T are periodic or preperiodic.*
- (iii) *For any non-(pre-)critical periodic point p , the number of arms at $f^{\circ i}(p)$ equals the number of arms at p for all $i \in \mathbb{N}_0$. If p is a non-(pre-)critical preperiodic point, then the number of arms at $f^{\circ i}(p)$ can be at most larger than the number of arms at p .*
- (iv) *$f|_{[x,y]}$ is injective if and only if $c_0 \notin]x, y[$. Thus, the expansivity condition implies that for any $x \neq y \in V$, $f^{\circ n}|_{[x,y]}$ is not injective for some $n \in \mathbb{N}$ unless at least one of the endpoints x, y is (pre-)critical and the critical point is periodic. \square*

Any unicritical postcritically finite polynomial generates a unique expanding tree (compare Proposition 2.2.8). However, not every expanding tree can be realized by a polynomial as Figure 2.1 shows. The expanding tree pictured there has a periodic non-(pre-)critical branch point x_1 with two cycles of local arms of different lengths (namely of length one and two). An expanding tree generated by a polynomial p comes from a Hubbard tree in the sense of Douady and Hubbard which lives in the complex plane so that the first return map preserves the cyclic order of local arms at periodic non-(pre-)critical branch points (because $p^{\circ i}$ is injective in a neighborhood $U \subset \mathbb{C}$ of such points). Consequently, for any periodic non-(pre-)critical branch point, we have that all cycles of local arms have equal length. We will give necessary and sufficient condition for expanding trees to be realizable by some postcritically finite unicritical polynomial in Section 2.4.

In general, there may be several polynomials which are not affinely conjugate yet generate the same expanding tree (cf. Figure 2.2). This is because

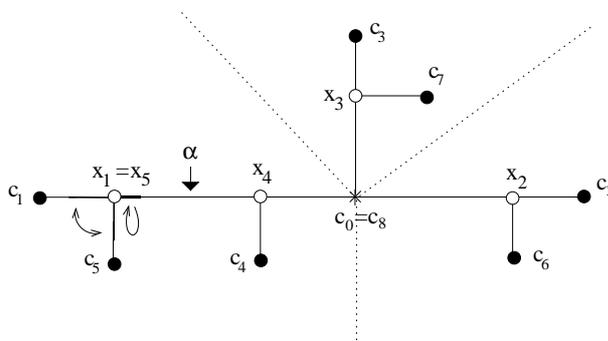


Figure 2.1: An expanding tree of degree three which cannot be generated by any polynomial. The periodic branch point x_1 has two cycles of local arms of different lengths. The different cycles are illustrated by differently thick lines. The dotted lines indicate a possible partition which turns the expanding tree into a Hubbard tree according to Definition 2.1.3. Hence, this figure is also an example for a non-admissible Hubbard tree.

expanding trees do not come with an embedding into the plane; in particular, they do neither specify a combinatorial rotation number for the cycles of local arms at non-(pre-)critical branch points nor do they specify what mutual location the global arms of the critical point have. However, if the Hubbard trees in the sense of [DH] of two unicritical polynomials have the same underlying topological tree but different combinatorial rotation numbers for some branch point, then these polynomials are not affinely conjugate; compare e.g. the Hubbard tree of the rabbit and of the anti-rabbit in the Mandelbrot set. The same is true for higher degree postcritically finite unicritical polynomials which generate the same topological tree but whose global arms at the critical point lie at different internal angles of their Fatou components. In parameter space, this means that the two polynomials in question lie in subwakes of different sectors of some hyperbolic component. Douady and Hubbard give necessary and sufficient conditions, the so-called *secondary* or *angle* information of a Hubbard tree, which allow to distinguish all postcritically finite polynomials by their Hubbard trees.

2.1.2 Basic Properties of Hubbard Trees

Definition 2.1.3 (Hubbard trees). *A unicritical Hubbard tree $(T, f, \mathcal{P})_d$ is an expanding tree $(T, f)_d$ together with a partition \mathcal{P} of T that has the following property:*

\mathcal{P} is the set of labeled connected components of $T \setminus \{c_0\}$ plus $\{c_0\}$. The non-trivial elements are denoted by T_i with $0 \leq i < d$ and are labeled in such

a way that $f(c_0) \in T_0$ and no index appears twice. The singleton $\{c_0\}$ gets assigned the label \star .

Observe that given an expanding tree $(T, f)_d$, two different labelings of its connected components $T \setminus \{c_0\}$ define two different partitions \mathcal{P} and thus, two different Hubbard trees, see Figure 2.2. When we picture a Hubbard tree, we order its subtrees T_i so that the indices are monotonously increasing in a counterclockwise way. We also indicate empty elements T_i of \mathcal{P} .

We call the triple $(T, f, \mathcal{P})_d$ a *unicritical* Hubbard tree because T contains a unique critical point with respect to f . In general, a Hubbard tree of degree d in the sense of [DH] has $d - 1$ critical points counting multiplicities, i.e., it can have up to $d - 1$ distinct critical points. Since we only deal with unicritical Hubbard trees in the first part of this manuscript, we skip the term “unicritical” and just speak of Hubbard trees in the following.

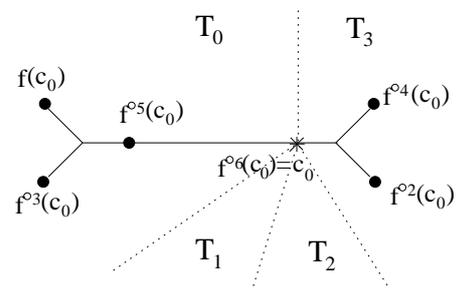
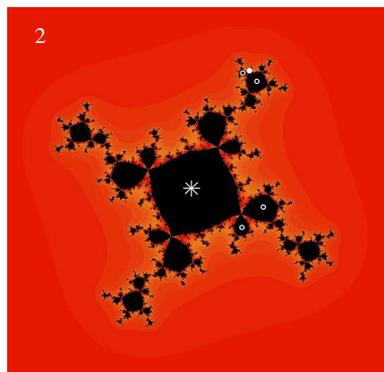
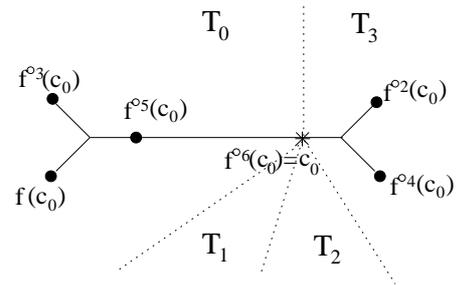
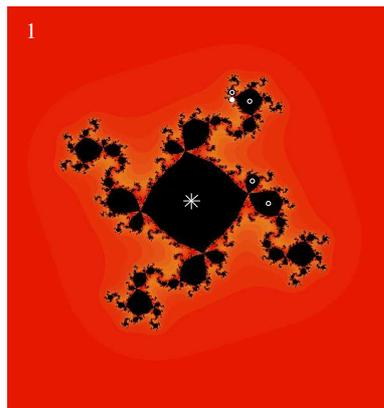
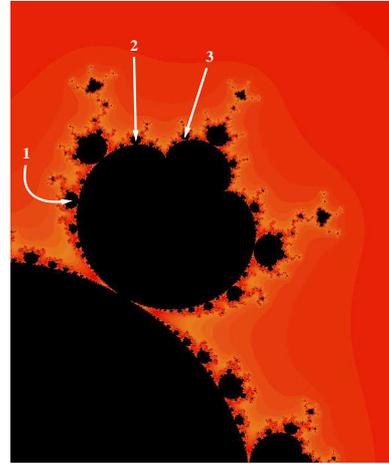
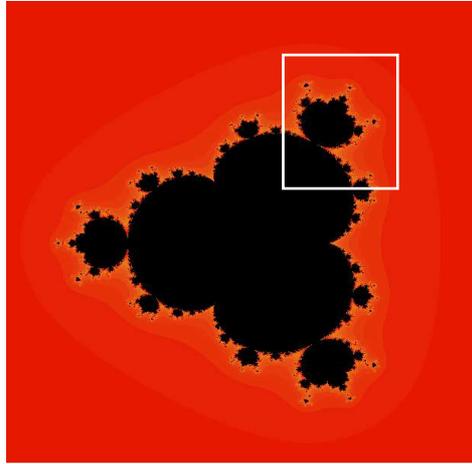
Hubbard trees in the sense of Definition 2.1.3 do not provide the full secondary information that is necessary to distinguish non-conjugate polynomials by their Hubbard trees [DH, Chapter VI]: we do not specify a cyclic order on the local arms of non-(pre-)critical branch points. So just as for expanding trees, there might be several postcritically finite polynomials of degree d that generate $(T, f, \mathcal{P})_d$. However, with Definition 2.1.3, we are now able to distinguish polynomials which lie in subwakes of different sectors of some hyperbolic component. This is illustrated in Figure 2.2. For Hubbard trees of degree d which are generated by some polynomial, the requirement that $f(c_0) \in T_0$ implies that we only regard one sector S of the main hyperbolic component of \mathcal{M}_d . That is, every such (non-trivial) Hubbard tree is generated by some postcritically finite polynomial contained in a subwake of the sector S .

Remark 2.1.4 (Trivial Hubbard tree). The triple $(\{c_0\}, \text{id}, \{\star\})_d$ is also a Hubbard tree according to Definition 2.1.3. It corresponds to the main hyperbolic component of \mathcal{M}_d ; so it makes sense to include this degenerate case in the definition of unicritical Hubbard trees. However sometimes, this very special situation gives rise to counterexamples to our statements. In such cases, we always assume that $(T, f, \mathcal{P})_d \neq (\{c_0\}, \text{id}, \{\star\})_d$ without explicitly stating this in the hypothesis. We call $(\{c_0\}, \text{id}, \{\star\})_d$ the *trivial* Hubbard tree.

Definition 2.1.5 (Itinerary). *The itinerary of a point $z \in T$ is the infinite sequence $\tau(z) = (\tau_i(z))_{i=1}^\infty \in \{0, \dots, d-1, \star\}^\mathbb{N}$ given by*

$$\tau_i(z) = \begin{cases} j & \text{if } f^{\circ i-1}(z) \in T_j, j \in \{0, \dots, d-1\} \\ \star & \text{if } f^{\circ i-1}(z) = c_0 \end{cases} .$$

If τ is periodic of period n then τ is composed of an infinite repetition of a finite word $\tau_1 \cdots \tau_n$ of length n . We write $\tau = \overline{\tau_1 \cdots \tau_n}$.



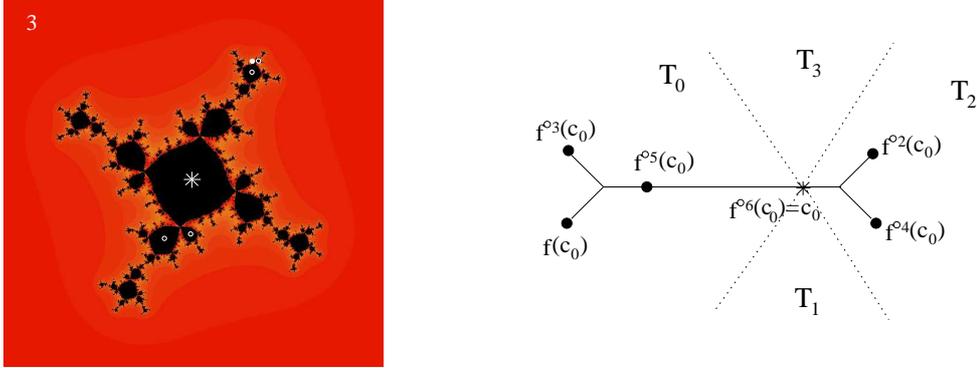


Figure 2.2: The first picture shows \mathcal{M}_4 , the second one is an enlargement of the framed region. The following pictures show the Julia sets of the centers of the hyperbolic components labeled by (1) to (3) (in this order). The critical point is label by \star , the critical value by \bullet and all other points on the critical orbit by \circ . Next to each Julia set is its Hubbard tree in the sense of Definition 2.1.3. The first two Hubbard trees are equal because Definition 2.1.3 does not capture the cyclic order at non-(pre-)critical branch points unlike Hubbard trees in the sense of Douady and Hubbard. The third Hubbard tree is distinct from the previous ones. This illustrates that our version of Hubbard trees contains some but not all secondary information.

Remark 2.1.6 (Combinatorial expansivity). It follows immediately that the action of f on z commutes with the action of the standard left shift σ on $\tau(z)$, i.e., $\tau(f(z)) = \sigma(\tau(z))$. Furthermore, for any $i \in \mathbb{N}_0$, $\tau_i(x) \neq \tau_i(y)$ if and only if $c_0 \in [f^{oi}(x), f^{oi}(y)]$. And if $f^{oi}(x) \neq c_0 \neq f^{oi}(y)$, then this is equivalent to $f|_{[f^{oi}(x), f^{oi}(y)]}$ is not injective. This allows us to formulate the expansivity condition of Definition 2.1.1 in the language of symbolic dynamics: if $x \neq y$ are marked points then $\tau(x) \neq \tau(y)$.

Lemma 2.1.7 (Itineraries of limit points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $x \in T$ be non-(pre-)critical. If $\{x_k\}_{k \in \mathbb{N}}$ is a sequence of points converging to x then $\tau(x_k) \rightarrow \tau(x)$ as $k \rightarrow \infty$.*

Proof. Since there are only finitely many precritical points of any fixed step, the sequence $\tau(x_k)$ does converge, say to τ . By possibly taking a subsequence, we can assume that for all $k \geq k_0$, the first k_0 entries of $\tau(x_k)$ and τ coincide. Fix any $m \in \mathbb{N}$ and pick a neighborhood U of x so that U contains no precritical point of step at most $m + 1$. Since x is not precritical, the itineraries of all points in U have the same first m entries. There is an $M > m$ such that $x_j \in U$ for all $j > M$ and therefore, $\tau_m(x_j) = \tau_m(x)$ for

all $j > M$. On the other hand, we have $\tau_m(x_j) = \tau_m$ for all $j > M$ and thus $\tau_m = \tau_m(x)$. \square

Definition 2.1.8 (Points of equal itinerary). *Let $\tau \in \{0, \dots, d-1, \star\}^{\mathbb{N}}$ and $(T, f, \mathcal{P})_d$ a Hubbard tree. Then we define the subset T_τ of T to be the set of all points with itinerary τ , i.e. $T_\tau := \{x \in T : \tau(x) = \tau\}$.*

Lemma 2.1.9 (Properties of T_τ). *Let $(T, f, \mathcal{P})_d$, τ and T_τ be as above. If $T_\tau \neq \emptyset$, then T_τ is either a point, an interval or an n -od (which might lack some of its endpoints).*

Moreover, if τ is n -periodic then the map $f^{\circ n}|_{T_\tau} : T_\tau \rightarrow T_\tau$ is a homeomorphism. For all $x \in \partial T_\tau$, the point x is periodic of period $kn > 0$ and either $\tau(x) = \tau$ or $x \in \text{orb}(c_0)$.

Proof. First observe that the symbol \star is contained in $\tau(x)$ if and only if x is eventually mapped on the critical point. In this case, there is at most one point with itinerary $\tau(x)$ by expansivity and because for any $0 \leq i < d$, $f|_{T_i}$ is a homeomorphism onto its image. Thus $T_{\tau(x)}$ is a point or empty. Now suppose that $\tau = j_1 j_2 \dots$ such that all $j_i \in \{0, \dots, d-1\}$ and that $T_\tau \neq \emptyset$. Let us first show that T_τ is connected. It is enough to show that for any two points $x, y \in T_\tau$, the connecting arc $[x, y]$ is contained in T_τ . Since T is uniquely arcwise connected, $x, y \in T_{j_1}$ implies that $[x, y] \in T_{j_1}$. Thus, $f|_{[x, y]}$ is injective and $f([x, y]) = [f(x), f(y)]$. By repeating the argument, $f([x, y]) \in T_{j_2}$. Inductively we get that $f^{\circ n}([x, y]) = [f^{\circ n}(x), f^{\circ n}(y)] \in T_{j_n}$ for all n , and by Remark 2.1.6, $\tau_n(z) = j_n$ for all $z \in [x, y]$ and $n \in \mathbb{N}$. By expansivity, T_τ contains at most one branch point, and hence is an n -od.

For the second statement we can restrict ourselves to the case that τ does not contain the symbol \star . Clearly, $f^{\circ n}|_{T_\tau}$ is continuous, injective and $f^{\circ n}(T_\tau) \subset T_\tau$. If $f^{\circ n}(T_\tau) \neq T_\tau$ then there is an $x \in \partial T_\tau$ such that $f^{\circ n}(x) \in \overset{\circ}{T}_\tau$. Thus there is an open interval I containing x such that $f^{\circ n}(I) \subset T_\tau$. If I is sufficiently small and x is not (pre-)critical then all points $p \in I$ have itinerary τ , in contradiction to the definition of T_τ . If however x is (pre-)critical then $I \setminus \{x\} \subset T_\tau$, contradicting that T_τ is connected. Hence $f^{\circ n}|_{T_\tau}$ is surjective. If $x \in \partial T_\tau$ is not (pre-)critical then $x \in T_\tau$. We have already seen that $f^{\circ n}(x) \in \partial T_\tau$. Since $f^{\circ n}|_{T_\tau}$ extends to a homeomorphism on $\overline{T_\tau}$ and since T_τ has only finitely many boundary points, x must be periodic of period kn . \square

2.1.3 Periodic Itineraries and Periodic Points

We start this section by investigating the set of fixed points of Hubbard trees.

Lemma 2.1.10 (Fixed points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree. Then T_0 contains a unique fixed point α and $\alpha \in]c_0, f(c_0)[$. If the critical value is*

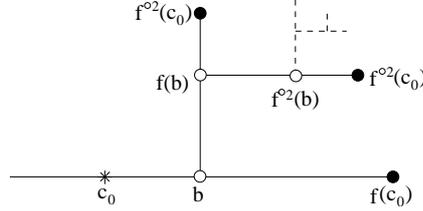


Figure 2.3: The pictured situation leads to $|\text{orb}(b)| = \infty$.

not eventually mapped to a fixed point, then α is the unique fixed point of T . Otherwise, there is a unique $0 < j_0 < d$ such that T_{j_0} contains further fixed points. In this case $T_{j_0} =]c_0, f^{\circ i}(c_0)[$ and $f^{\circ i}(c_0) = f^{\circ i+1}(c_0)$ for some $i > 1$.

The fixed point α is called the *alpha-fixed* point of $(T, f, \mathcal{P})_d$.

Proof. If the three points $c_0, f(c_0), f^{\circ 2}(c_0)$ form a degenerate triod, then $f(c_0)$ is an endpoint and the claim follows by the Intermediate Value theorem. Otherwise consider the behavior of the branch point b of the triod $[c_0, f(c_0), f^{\circ 2}(c_0)]$ under f : the point $f(b)$ is contained in $]f(c_0), f^{\circ 2}(c_0)[$. If $f(b) \in]b, f^{\circ 2}(c_0)[$ then $|\text{orb}(b)| = \infty$, and if $f(b) \in]b, f(c_0)[$ then $|\text{orb}(b)| = \infty$, too, unless $f^{\circ 2}(b) = b$. But this last possibility contradicts expansivity because then $c_0 \notin f^{\circ i}([b, f(b)])$ for all $i \in \mathbb{N}_0$. Thus b is fixed.

Now let $\alpha \in]c_0, f(c_0)[$ be fixed and suppose that there was another fixed point $p \in T_0$. Let C be the component of $T \setminus \{\alpha\}$ which contains p . Then $C \not\subset G_\alpha(c_0)$ and since $f|_{T_0}$ is a homeomorphism onto its image, $f(C) \subset C$. This again yields an immediate contradiction to expansivity unless C contains exactly one endpoint e and e is fixed by f . For degree reasons, $e \neq f(c_0)$ and thus, α is a branch point. But this means that for the two marked points α and e , $c_0 \notin f^{\circ n}([\alpha, e])$ for all $n \in \mathbb{N}$.

Now suppose that there is a $j_0 \neq 0$ such that $p_0 \in T_{j_0}$ is fixed. Then $L_{p_0}(c_0)$ is fixed under f . Let $i > 1$ such that $p_0 \in]f^{\circ i}(c_0), c_0[\subset T_{j_0}$. Since f is locally injective at p_0 we have that for all $n \geq 0$ the iterate $f^{\circ(i+n)}(c_0) \in T_{j_0}$. Now expansivity implies that $f^{\circ i}(c_0) = f^{\circ i+1}(c_0)$ and $T_{j_0} =]c_0, f^{\circ i}(c_0)[$. \square

The following two statements are immediate corollaries of Lemma 2.1.10.

Corollary 2.1.11 (Existence of further fixed points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree. Then there is an element $T_{j_0} \neq T_0$ of \mathcal{P} which contains a fixed point if and only if $\tau(c_0) = \star \tau_1 \dots \tau_l \overline{j_0}$ for some $j_0 \in \{1, 2, \dots, d-1\}$. \square*

Corollary 2.1.12 (Preimage in T_0). *Every point $p \in]\alpha, f(c_0)[$ has a preimage $p_0 \in]c_0, \alpha[$. \square*

Now, we turn to arbitrary periodic points. Unless stated otherwise, we always assume that the period of a periodic point is at least two. We start by comparing the exact period of periodic points to the exact period of their itineraries. In particular, we show that if a Hubbard tree contains a periodic point of itinerary τ then it always contains a periodic point of itinerary τ whose exact period coincides with the one of its itinerary. This fact guarantees that the *minimal Hubbard trees* introduced in Section 2.2 are well-defined.

Lemma 2.1.13 ((Pre-)periodic points and itineraries). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $p \in T$ be a periodic point of period n . Then the itinerary τ of p is periodic, too, and its period divides n .*

If a point $p \in T$ is preperiodic then so is its itinerary τ . The preperiod of p coincides with the one of τ .

Proof. Obviously, if p is n -periodic, then $\tau = (\tau_i)_{i \in \mathbb{N}}$ is periodic of period at most n . Let $m \leq n$ be the exact period of τ . If the greatest common divisor $(m, n) = 1$, then for any $0 < k \leq n$ there is an N such that $N \equiv 1 \pmod{m}$ and $N \equiv k \pmod{n}$ by the Chinese Remainder theorem. Thus, $\tau_1 = \tau_k$ for all $k \in \mathbb{N}$ and $m = 1$, which clearly divides n . If $(m, n) = q > 1$, then there are two integers \hat{m}, \hat{n} such that $\hat{m}q = m$, $\hat{n}q = n$. Now $(\hat{m}, \hat{n}) = 1$ and thus for any $i \in \mathbb{N}$, there is an \hat{N} such that $\hat{N} \equiv 0 \pmod{\hat{m}}$ and $\hat{N} \equiv i \pmod{\hat{n}}$. So,

$$\begin{aligned} \exists k, l : k\hat{m} = i + l\hat{n} &\iff \exists k, l : j + k(\hat{m}q) = j + iq + l(\hat{n}q) \\ &\iff \exists N : N \equiv j \pmod{m}, N \equiv (j + iq) \pmod{n}. \end{aligned}$$

But this implies that for all $i, j \in \mathbb{N}$, we have $\tau_j = \tau_{j+iq}$, i.e. τ is periodic of period q . This means that $m = q$ and thus, m divides n .

Now let p be preperiodic. By way of contradiction, suppose that τ and p have preperiods of different lengths. Clearly, the preperiod of τ cannot be longer than the one of p . By iterating p , we can assume that τ is periodic whereas p has preperiod of length $k > 0$. Let m be the exact period of τ and n the exact period of $f^{\circ k}(p)$. We are going to show that there is a j_0 such that $\tau_k = \tau_{k+j_0n}$. This implies that the two points $f^{\circ k-1}(p)$ and $f^{\circ k-1+j_0n}(p)$ are contained in some $T_i \subset T$. By the choice of n, k , $f^{\circ k}(p) = f^{\circ k+j_0n}(p)$, and thus $f|_{T_i}$ is not injective, a contradiction. Since τ is m -periodic, $\tau_k = \tau_{k+jm}$ for all $j \in \mathbb{N}$. So it suffices to show that there are $j_0, j_1 \in \mathbb{N}$ such that $j_0n + k = j_1m + k$. If $(m, n) = 1$, then this is true by the Chinese Remainder theorem. If $(m, n) = q$, then the reasoning above shows that the period of τ equals q , i.e. $m = q$. Hence $m|n$ and there is an $l > 0$ such that $lm = n$. Setting $j_0 = 1$ and $j_1 = l$ proves the claim. \square

Lemma 2.1.14 (Period of points and itineraries). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree. The exact period and preperiod of any marked point equals the exact period and preperiod of its itinerary.*

If $z \in T$ is periodic such that its period is greater than the exact period of $\tau(z)$, then there is a periodic point z' such that $\tau(z) = \tau(z')$ and the exact periods of z' and $\tau(z')$ are equal.

Proof. Let z be a marked point and $\tau(z)$ its itinerary. Suppose that z is m -periodic and m_τ is the exact period of $\tau(z)$. If $m = km_\tau$ for some $k > 1$ then the two distinct marked points $z, f^{om_\tau}(z)$ have the same itinerary, contradicting the expansivity condition. Now suppose that z is preperiodic with preperiod of length l and let l_τ be the length of the preperiod of $\tau(z)$. Then $l \geq l_\tau$ and if $l > l_\tau$, we get the same contradiction as for the periodic case using the two distinct points $f^{ol_\tau}(z), f^{ol_\tau+m_\tau}(z)$.

To prove the second part, suppose that z is a periodic, non-marked point and $m > m_\tau$. Consider the connected set $T_{\tau(z)}$ that consists of all points of T which have the same itinerary as z . If $T_{\tau(z)}$ contains a branch point, then this point is marked and hence must have period m_τ as we just have seen. Otherwise $T_{\tau(z)}$ is a (not necessarily closed) interval with endpoints z_1, z_2 . Since $T_{\tau(z)}$ contains only points that share the same itinerary and since it is maximal with respect to this property, we get that $f^{on}|_{[z_1, z_2]}$ is a homeomorphism onto its image for all n and $f^{om_\tau}([z_1, z_2]) \subset [z_1, z_2]$. If f^{om_τ} reverses orientation on $[z_1, z_2]$ or maps this interval strictly into itself, then there is a fixed point in $]z_1, z_2[$ of f^{om_τ} . So the only remaining case is that at least one of the endpoints, say z_1 , is fixed by f^{om_τ} . But since $f^{ojm_\tau}|_{[z_1, z_2]}$ is a homeomorphism (onto its image) for all j , this implies that the orbit of z is infinite. \square

One can actually say more about the relation between the period of a periodic point and the period of its itinerary.

Lemma 2.1.15 (Length of periodic orbits). *Let τ be a periodic itinerary of exact period n generated by a point of the Hubbard tree $(T, f, \mathcal{P})_d$. Then there exists exactly one $l \in \mathbb{N}$ such that the period of any periodic point in T with itinerary τ is either n or ln .*

We will see later in Proposition 2.1.23 that there are at most two disjoint cycles of local arms at any periodic point and if there are two, then (at least) one of the two cycles is trivial, i.e. consists of one fixed local arm. The first part of this proposition is needed to show uniqueness of l . So the logically correct place to put Lemma 2.1.15 would be after Proposition 2.1.23. We state the lemma already here because its content belongs to this section. (Note that Lemma 2.1.15 is not used to prove Proposition 2.1.23.)

Knowing Proposition 2.1.23, we can explain the number l : the subtree T_τ contains at most one periodic point $b \in T_\tau$ of period n such that not all

of its local arms are fixed under $f^{\circ n}$. If such a point b does not exist, then all periodic points of itinerary τ have period n . This can only happen if T_τ is an interval or a point. However, if such a point b exists, then $l > 1$ is the length of the non-trivial cycle C of local arms at b . If G is a global arm with local arm $L \in C$ and $p \in T_\tau \cap G$ is periodic, then the exact period of p is ln . If G' denotes the global arm with associated fixed local arm (if existing), then all periodic points in $T_\tau \cap G'$ have exact period n .

We will use the following easy observation to prove Lemma 2.1.15 .

Lemma 2.1.16 (Consecutive iterates). *Suppose that $f|_{[x,y]}$ is injective and there are three consecutive iterates $p, f(p), f^{\circ 2}(p) \in [x, y]$ such that $f(p) \in]p, f^{\circ 2}(p)[$. Then $f^{\circ 2}(p) \in]f(p), f^{\circ 3}(p)[$. \square*

Proof of Lemma 2.1.15. If τ contains the symbol \star , then T_τ is a single point and as marked point, this point has period n . Since for any $\tau \in \{0, \dots, d-1\}^{\mathbb{N}}$ the set T_τ contains at most one branch point, it is easy to see that there is at most one periodic point in T_τ of period n whose local arms are not all fixed. Lemma 2.1.16 implies that for any periodic point p with period ln ($l > 1$) there is an n -periodic point $z \in]p, f^{\circ n}(p)[$ (z is a branch point for $l > 2$, as otherwise the orbit of z would be infinite) and $f^{\circ n}$ does not fix $L_z(p)$ and $L_z(f^{\circ n}(p))$. Therefore, if for all n -periodic points in T_τ all local arms are fixed under $f^{\circ n}$, then all periodic points in T_τ have period n . Now suppose that there is an n -periodic point z that has a cycle C of local arms of length $l > 1$. By Proposition 2.1.23, there is at most one such cycle at z . If an arm of z whose respective local arm is in C contains a periodic point p of T_τ , then p must be fixed under the first return map $f^{\circ ln}$ of $L_z(p)$: if not, the reasoning above shows that $G_z(p) \cap T_\tau$ contains a kn -periodic (inner) point p' whose local arms are permuted. This contradicts that $f^{\circ ln}([z, p']) = [z, p']$. Thus, the period of any periodic point in T_τ is n or ln . \square

2.1.4 Characteristic Points

In this section we prove that every periodic orbit contains one distinguished point, the *characteristic* point. Characteristic points will become crucial when we investigate the set of kneading sequences. In fact, we will define a partial order on this set via characteristic points.

Definition 2.1.17 (Characteristic points). *Let x be a periodic point of period $n > 1$ of the Hubbard tree $(T, f, \mathcal{P})_d$. Suppose that there is a point $x_1 \in \text{orb}(x) \cap]c_0, f(c_0)[$ such that $\text{orb}(x) \subset \overline{G_{x_1}(c_0)}$. Then the point x_1 is called the characteristic point of $\text{orb}(x)$.*

This definition immediately implies that if a characteristic point $x_1 \in \text{orb}(x)$ exists, then it is unique for $\text{orb}(x)$. Furthermore, $f(c_0)$ is the characteristic point of the critical orbit if and only if c_0 is periodic. Whenever

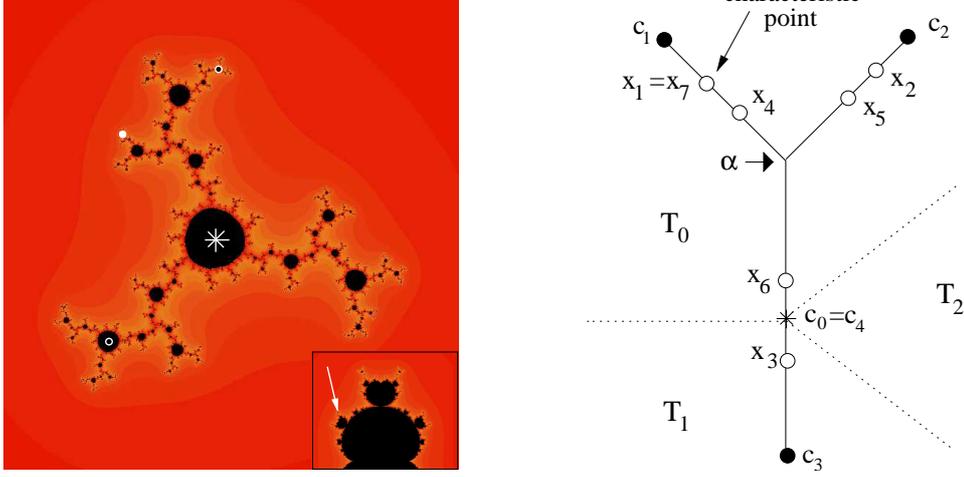


Figure 2.4: This figure illustrates the special location of characteristic points. The Hubbard tree is generated by the Julia set pictured on the left hand side.

we speak of a *characteristic point* p then we mean that p is contained in a periodic orbit O and that it is the characteristic point of O .

Proposition 2.1.18 (Existence of characteristic points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $x \in T$ be a periodic point of period $n > 1$ such that $\tau(x) \neq \tau(e)$ for all endpoints e of T . Then there is an $x_1 \in \text{orb}(x)$ that is characteristic.*

Proof. For any point $y \in \text{orb}(x)$ let \mathcal{X}_y be the union of the closures of global arms at y not containing c_0 . Observe that $f(\mathcal{X}_y) \subset \mathcal{X}_{f(y)}$ if \mathcal{X}_y contains no immediate preimage of c_0 . For $y \neq y'$, the sets $\mathcal{X}_y, \mathcal{X}_{y'}$ are either disjoint or one is strictly contained in the other. This implies that there is at least one \mathcal{X}_{y_0} which contains no element of $\text{orb}(x)$ besides y_0 . Moreover, since $\tau(x) \neq \tau(e)$ for all endpoints e of the Hubbard tree and \mathcal{X}_y clearly contains an endpoint, there is a precritical point in \mathcal{X}_y for all $y \in \text{orb}(x)$. Let ξ_y be the one of lowest step, say $\text{STEP}(\xi_y) = k > 1$. Then $f^{\circ k-2}(y) \in]c_0, f^{\circ k-2}(\xi_y)[$ and thus $f^{\circ k-1}(y) \in]f(c_0), c_0[$.

Let $x_1 \in]c_0, f(c_0)[$ be the point of $\text{orb}(x)$ such that $\text{orb}(x) \cap]x_1, f(c_0)[= \emptyset$ and let us assume that there is a $y \in \text{orb}(x) \cap \overset{\circ}{\mathcal{X}}_{x_1}$. Iterate \mathcal{X}_{x_1} until $f^{\circ k}(\mathcal{X}_{x_1})$ contains an immediate preimage of c_0 . We have that for all $i = 0, \dots, k-1$, $\overset{\circ}{\mathcal{X}}_{f^{\circ i}(x_1)}$ contains a point of $\text{orb}(x)$. Now $f^{\circ k-1}(x_1) \in]c_0, f(c_0)[$ and by the choice of x_1 , the interior of $\mathcal{X}_{f^{\circ k-1}(x_1)}$ contains a point of $\text{orb}(x)$. We continue this process until $f^{\circ n}(x_1) = x_1$. Summing up, we have shown that the assumption “ $\exists y \in \text{orb}(x) \cap \overset{\circ}{\mathcal{X}}_{x_1}$ ” implies that there is no $y \in \text{orb}(x)$

with $\text{orb}(x) \cap \overset{\circ}{\mathcal{X}}_y = \emptyset$. But such a point exists as we have seen before. \square

The proof shows that for any periodic point x , $\text{orb}(x) \cap [c_0, f(c_0)] \neq \emptyset$, and secondly, that the point $\underline{x_1} \in [c_0, f(c_0)]$ with $\text{orb}(x) \cap]x_1, f(c_0)[= \emptyset$ has the property that $\text{orb}(x) \subset \overline{G_{x_1}(c_0)}$. So, we could equally well have defined the characteristic point as the point in $\text{orb}(x) \cap [c_0, f(c_0)]$ that is closest to $f(c_0)$.

Note that the proposition is trivially true for the α -fixed point. For convenience, we say that the α -fixed point is the characteristic point of its orbit. Obviously, we have that any characteristic point $x_1 \neq \alpha$ is contained in $] \alpha, f(c_0) [$.

The next statement is an immediate consequence of expansivity because this condition guarantees that marked points have pairwise distinct itineraries.

Corollary 2.1.19 (Branch and characteristic points). *Let $b \in T$ be a periodic branch point. Then $\text{orb}(b)$ contains a characteristic point.* \square

Corollary 2.1.20 (Preimages of characteristic points). *Let $x_1 \in T$ be a characteristic point. Suppose there is an $i \in \{0, \dots, d-1\}$ such that $\text{orb}(x_1) \cap T_i \neq \emptyset$. Then T_i contains an immediate preimage x_0^i of x_1 , i.e. $f(x_0^i) = x_1$.*

Proof. Let $x' \in \text{orb}(x_1) \cap T_i$. Then $[x', c_0]$ is mapped homeomorphically onto $[f(x'), f(c_0)]$, and as x_1 is characteristic, $x_1 \in [f(x'), f(c_0)]$. Thus $[x', c_0] \subset \overline{T_i}$ contains a preimage of x_1 . \square

2.1.5 Global and Local Arms of Periodic Points

The existence of characteristic points is essential for our study of the behavior of global and local arms at periodic points under their first return maps.

Proposition 2.1.21 (Behavior of global arms). *Let x be an n -periodic, non-(pre-)critical point and let x_1 be the characteristic point of $\text{orb}(x)$. Then every global arm G_{x_1} at x_1 has exactly one of the following behaviors under $f^{\circ n}$:*

- $f^{\circ n}$ maps G_{x_1} homeomorphically onto its image so that $c_0 \notin f^{\circ j}(G_{x_1})$ for all $j = 0, \dots, n$;
- there is a $0 \leq j \leq n$ such that $c_0 \in f^{\circ j}(G_{x_1})$. Moreover, $f^{\circ n}$ maps the local arm L_{x_1} associated to G_{x_1} either to $L_{x_1}(c_0)$ or $L_{x_1}(f(c_0))$.

Proof. Assume that there is an iterate $f^{\circ j}(G_{x_1})$, $j \in \{0, \dots, n\}$, which contains the critical point. Let $\xi \in G_{x_1}$ be the unique precritical point with $\text{STEP}(\xi) =: l \leq n$ such that there is no other precritical point of step smaller than n in $]x_1, \xi]$. By definition, $f^{\circ n}|_{[x_1, \xi]}$ is injective and $f^{\circ l}(L_{x_1})$ points

towards c_0 . If $l = n$, then $f^{\circ n}(L_{x_1}) = L_{x_1}(c_0)$. If $l \leq n - 1$, then $f^{\circ l+1}(L_{x_1})$ points to $f(c_0)$. Hence for $l = n - 1$, $f^{\circ n}(L_{x_1}) = L_{x_0}(f(c_0))$, and for all $l < n - 1$, $f^{\circ l+1}(L_{x_1})$ points towards a point of $\text{orb}(x)$ and all further iterates will do so until we reach x_1 after n iterations. This implies that $f^{\circ n}(L_{x_1})$ points towards the critical point, i.e., equals $L_{x_1}(c_0)$. \square

Definition 2.1.22 (Hitting c_0). *Let G, G' be two global arms of an n -periodic point x . We say that $f^{\circ n}$ maps G into G' without hitting c_0 if $f^{\circ n}$ maps G homeomorphically into G' so that $c_0 \notin f^{\circ j}(G)$ for all $j = 0, \dots, n$.*

Proposition 2.1.23 (Behavior of local arms). *Let x be an n -periodic, non-(pre-)critical point. Then the first return map $f^{\circ n}$ either permutes the set of local arms of x transitively or it fixes one local arm and permutes the remaining ones transitively. In the latter case, the fixed local arm is the image of the local arm at the characteristic point of $\text{orb}(x)$ pointing towards c_0 .*

If x is the α -fixed point then all local arms are permuted transitively.

Proof. We are going to show the claim for the characteristic point x_1 of $\text{orb}(x)$. Since for all $i \in \mathbb{N}$ and all non-(pre-)critical points $p \in T$ the map $f^{\circ i}$ is locally injective at p , the claim carries over to any point on $\text{orb}(x)$.

Note first that f maps two distinct local arms to two distinct local arms. Thus the set of local arms at x_1 splits into a finite number of cycles and the elements of each cycle are permuted transitively. By expansivity, for any global arm G_{x_1} , there is a $0 \leq k \leq ln$ such that $f^{\circ k}(G_{x_1})$ contains the critical point, where l is the length of the cycle of local arms containing L_{x_1} . Hence Lemma 2.1.21 implies that for any L_{x_1} , $\text{orb}_{f^{\circ n}}(L_{x_1})$ contains a local arm which either points to the critical point or to the critical value. It follows that there are at most two disjoint cycles of local arms at x_1 , one of which consists of all local arms L with $L_{x_1}(c_0) \in \text{orb}_{f^{\circ n}}(L)$, the second one contains all L such that $L_{x_1}(f(c_0)) \in \text{orb}_{f^{\circ n}}(L)$.

Since $G_{x_1}(c_0)$ is not mapped homeomorphically by $f^{\circ n}$ without hitting c_0 , $f^{\circ n}(L_{x_1}(c_0))$ either points towards the critical point or the critical value. Now the discussion above yields that in the first case, $L_{x_1}(c_0)$ is fixed by $f^{\circ n}$ and the remaining arms are permuted transitively, and in the second one that there is only one cycle of local arms and the set of all local arms is permuted transitively.

To finish the proof, let us assume that x is the α -fixed point. As we have seen in Lemma 2.1.10, T_0 contains no further fixed point. Suppose that the local arm L_α is fixed and that it is not pointing to c_0 . If G_α denotes its global arm, then $f(G_\alpha) \subset G_\alpha$. But by expansivity, this is only possible if G_α is an interval with both endpoints fixed, a contradiction. If $L_\alpha(c_0)$ is fixed, then $f^{\circ i}(f(c_0)) \notin G_\alpha(c_0)$ because f is locally injective at α . This is again a contradiction to expansivity. \square

Definition 2.1.24 (Tame and evil points). *Let $z \neq f(c_0)$ be a characteristic point of exact period n . We say that z is tame if none of its local arms are fixed under f^{on} . If z is a branch point and one of its local arms is fixed under the first return map, then z is called evil.*

This terminology is motivated by the following fact: in Section 2.4.1 we will see that the only obstruction for a Hubbard tree to be realized by a polynomial is the existence of an evil branch point. Inner points whose local arms are fixed are no obstruction. So we use the term evil solely for branch points.

Remark 2.1.25 (Labeling of global and local arms). Going through the argument of the proof above, we get an even stronger statement about the behavior of global arms. Except for the case that c_0 is an endpoint of T , $G_{x_1}(c_0)$ is never mapped homeomorphically onto its image under f^{on} . We can label the q global arms at x_1 in such a way that G_0 contains the critical point, G_1 the critical value and for all $1 \leq i < q - 1$, $f^{on}(G_i) \subset G_{i+1}$. The local arm associated to G_i is denoted by L_i . Note that f^{on} maps G_i into G_{i+1} without hitting c_0 for all $1 \leq i \leq q - 1$. For the global arm G_{q-1} we have to distinguish whether x_1 is tame or not: suppose that L_0 is not fixed under f^{on} . Then it must be mapped to L_1 by Proposition 2.1.21. It follows that L_{q-1} is mapped onto L_0 . The global arm G_{q-1} may or may not be mapped homeomorphically into G_0 such that $c_0 \notin f^{oj}(G_{q-1})$ for all $j = 0, \dots, n$. Figure 2.5 gives an example for each case. If $L_{x_1}(c_0)$ is fixed under f^{on} , then by expansivity, the global arm G_{q-1} cannot be mapped homeomorphically into G_1 such that $c_0 \notin f^{oj}(G_{q-1})$ for all $0 < j < n$. However, it might be mapped homeomorphically (cf. the last picture in Figure 2.5).

Corollary 2.1.26 (Permutation of global arms). *Let x be a characteristic point of exact period n with global arms G_0, \dots, G_{q-1} (labeled as described above). Then for all $0 < i < q - 1$, we have $f^{on}(G_i) = G_{i+1}$ and c_0 is not hit. If x is tame then $f^{on}(L_{q-1}) = L_0$, otherwise $f^{on}(L_{q-1}) = L_1$. While in the first case, G_{q-1} might map to G_0 without hitting c_0 , in the second case, G_{q-1} is always mapped into G_1 so that c_0 is hit. \square*

Corollary 2.1.27 (Period and itinerary of branch points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $b \in]c_0, f(c_0)[$ be a characteristic branch point of exact period m which has q arms. Then $\tau_i(x) = \tau_i(c_1)$ for all $0 < i \leq (q - 2)m$.*

Furthermore, if c_0 is periodic of exact period n , then $n > m$. \square

2.2 Minimal Hubbard Trees

In the definition of Hubbard trees, we left some freedom for the dynamics on $T \setminus V$. (Recall that V is the set of marked points of T .) However,

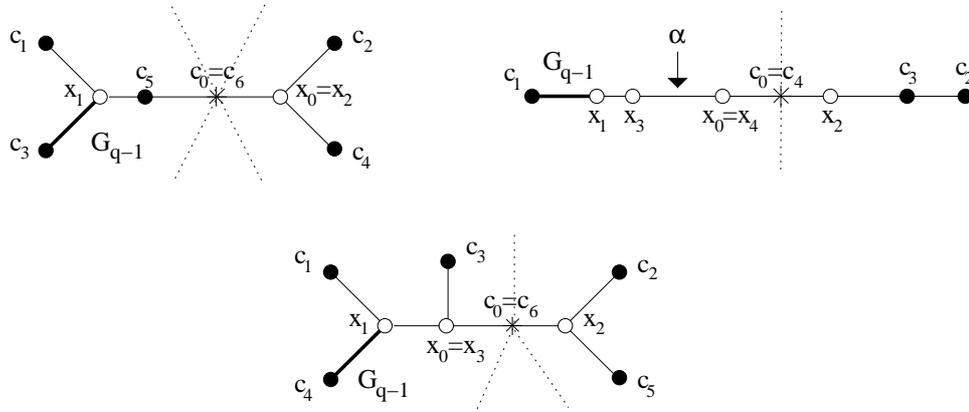


Figure 2.5: Different behaviors of G_{q-1} (marked by a thick line) under the first return map of x_1 . In the first two Hubbard trees, x_1 is tame; in the third one x_1 is an evil branch point. Partitions are indicated by dotted lines.

most of the possible choices cannot be realized by a polynomial. Besides describing the space of Hubbard trees we also want to get a meaningful model for the Multibrot sets. Therefore, we introduce an equivalence relation on the set of Hubbard trees so that every equivalence class is characterized by the dynamics on the set of marked points of any representative. We will investigate equivalence classes of Hubbard trees rather than individual Hubbard trees.

Definition 2.2.1 (Equivalent Hubbard trees). *We say that two Hubbard trees $(T, f, \mathcal{P})_d, (T', f', \mathcal{P}')_d$ of the same degree d are equivalent if there is a bijection $\phi : V \rightarrow V'$ between the sets of marked points of T and T' such that $\tau(v) = \tau(\phi(v))$ for all $v \in V$ and if $v, \tilde{v} \in V$ are adjacent marked points then so are $\phi(v), \phi(\tilde{v})$.*

This defines an equivalence relation on the set of Hubbard trees.

2.2.1 Hubbard Trees with Attracting Dynamics

For the investigation of the set of (equivalence classes of) Hubbard trees, we will work with special representatives, the so-called *minimal Hubbard trees*. They were introduced in [Ka] for quadratic Hubbard trees. Every minimal Hubbard tree carries all the dynamically relevant information which characterizes its equivalence class but it has rather plain dynamics compared to other elements of its class. This eases their investigation, in particular the comparison of different equivalence classes. Before giving a definition

of minimal Hubbard trees in the next section, we discuss what it means for a Hubbard tree to have *attracting dynamics*, a property minimal Hubbard trees are required to have.

Definition 2.2.2 (Attracting dynamics). *A Hubbard tree $(T, f, \mathcal{P})_d$ with n -periodic critical point is said to have attracting dynamics if for each $c_k \in \text{orb}(c_0)$, there is a neighborhood U_k of c_k such that for all $x \in U_k$, $f^{\circ j n}(x) \rightarrow c_k$ as $j \rightarrow \infty$.*

The next lemma summarizes some properties of Hubbard trees with attracting dynamics. Lemma 2.2.4 then shows that every equivalence class of Hubbard trees with periodic c_0 contains a representative which has attracting dynamics.

Proposition 2.2.3 (Trees with attracting dynamics). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with attracting dynamics. Then the following are true:*

- (i) *Let $p \in T$ be a non-(pre-)critical point. If p is a limit point of periodic points p_n which all have the same itinerary τ , then $\tau(p) = \tau$ and p is periodic.*
- (ii) *There is a neighborhood U of c_1 such that $f^{\circ j}|_U$ is injective for all $j \in \mathbb{N}_0$.*

Proof. By Lemma 2.1.15, there is an $m \in \mathbb{N}$ such that all p_n have (not necessarily exact) period m . By continuity, $f^{\circ m}(p) = p$, i.e., p is periodic and its period divides m . Moreover, p has either itinerary τ or is on the critical orbit. But since T has attracting dynamics, no point on the critical orbit can be the limit point of periodic points.

For the second claim, let n be the period of c_1 . There is a neighborhood U of c_1 such that U contains no precritical point unequal to c_1 of step at most n . Since $(T, f, \mathcal{P})_d$ has attracting dynamics, we can choose U so small that for all $p \in U$, $f^{\circ n}(p) \in]p, c_1[$. If there was a precritical point $\xi \in U$ with $\xi \neq c_1$, then there is a j_0 such that $f^{\circ j_0}(\xi)$ is precritical of step at most n and by the choice of U , we have $f^{\circ j_0}(\xi) \in U$, a contradiction. \square

Lemma 2.2.4 (Attracting dynamics exists). *Every equivalence class of Hubbard trees with periodic critical point contains a representative with attracting dynamics.*

Proof. For any given equivalence class pick a Hubbard tree $(T, f, \mathcal{P})_d$ and suppose that the critical orbit is not attracting. We are going to define a new dynamics \tilde{f} on the topological tree T such that any point on the critical orbit is locally attracting. To achieve this, it suffices to change f locally at c_0 . Let V be the set of marked points, n be the exact period of the critical point c_0 and let G be the unique global arm of c_0 whose associated local arm is fixed under $f^{\circ n}$. Choose $y_0 \in G$ such that the interval $I :=]c_0, y_0[$

has the following properties: $f^{\circ n}|_I$ is a homeomorphism onto its image, $I \cap V = \emptyset = f^{\circ n}(I) \cap V$ and $f^{\circ i}(I) \cap I = \emptyset$ for all $0 < i < n$. Without loss of generality, we can assume that there is a $z \in I$ such that $f^{\circ n}(z) = z$: if such a point does not exist then $p \in]c_0, f^{\circ n}(p)[$ for all $p \in I$ and we can pick $z, y, y' \in I$ with $c_0 < y < z < f^{\circ n}(z) < y'$ ($<$ denotes the natural order on \bar{I} with c_0 as the smallest element). There is a homeomorphism $h' : T \rightarrow T$ such that $h'|_{T \setminus [y, y']} \equiv \text{id}$ and $h'(f^{\circ n}(z)) = z$. Set $f' := h' \circ f$. Then $(f')^{\circ n}(z) = z$ and the two Hubbard trees $(T, f, \mathcal{P})_d$ and $(T, f', \mathcal{P})_d$ are equivalent.

So we can assume that there is a $z \in I$ which is fixed under $f^{\circ n}$. Pick any homeomorphism $\varphi : [c_0, z] \rightarrow [0, 1]$ with $\varphi(c_0) = 0$, $\varphi(z) = 1$, and consider the function $h : [0, 1] \rightarrow [0, 1]$, $x \mapsto x^2$. It induces a map

$$\tilde{h} : T \longrightarrow T, p \mapsto \begin{cases} (\varphi^{-1} \circ h \circ \varphi)(p) & \text{if } p \in [c_0, z] \\ p & \text{otherwise} \end{cases}.$$

Let f_{-n} be the inverse branch of $f^{\circ n}|_I$ that maps $[c_0, z]$ onto itself and define

$$g : T \longrightarrow T, p \mapsto \begin{cases} (\tilde{h} \circ f_{-n})(p) & \text{if } p \in [c_0, z] \\ p & \text{otherwise} \end{cases}.$$

The map $\tilde{f} := g \circ f : T \rightarrow T$ is a continuous surjection and the triple $(T, \tilde{f}, \mathcal{P})_d$ is a Hubbard tree. Moreover, it is equivalent to the given one and has attracting dynamics: for the first part, observe that $V \subset T \setminus f^{-1}(]c_0, z[) =: S$ and $\tilde{f}|_S = g \circ f|_S = f|_S$; to prove attracting dynamics, it suffices to show that $\tilde{f}^{\circ j n}(x) \rightarrow c_0$ as $j \rightarrow \infty$ for all $x \in]c_0, z[$. For any $x \in]c_0, z[$, we have

$$\tilde{f}^{\circ n}(x) = ((g \circ f) \circ (\text{id} \circ f)^{\circ n-1})(x) = (g \circ f^{\circ n})(x) = (\tilde{h} \circ f_{-n} \circ f^{\circ n})(x) = \tilde{h}(x)$$

and $\tilde{h}(x) \in]c_0, z[$. Thus, $\tilde{f}^{\circ j n}(x) = \tilde{h}^{\circ j}(x)$ for all $j \in \mathbb{N}$ and all $x \in]c_0, z[$, and by definition $\tilde{h}^{\circ j}(x) \rightarrow c_0$ as $j \rightarrow \infty$. \square

2.2.2 Existence and Properties of Minimal Hubbard Trees

Definition 2.2.5 (Minimal Hubbard trees). *A Hubbard tree $(T, f, \mathcal{P})_d$ is minimal if the following two conditions are satisfied: there are no two periodic points $p \neq p'$ in T with $\tau(p) = \tau(p')$. If the critical point c_0 is periodic, then $(T, f, \mathcal{P})_d$ has attracting dynamics.*

Remark 2.2.6. Observe that this definition implies immediately that minimal Hubbard trees do not contain two preperiodic points which have the same itinerary either. Recall that expansivity implies that there are no (pre-)periodic marked points with the same itinerary. Thus, one could consider minimal Hubbard trees to be Hubbard trees with all (pre-)periodic points marked. Minimal Hubbard trees have the property that the exact period and preperiod of any (pre-)periodic point p equals the exact period and preperiod of its itinerary $\tau(p)$ (cf. Lemma 2.1.14).

Proposition 2.2.7 (Minimal Hubbard trees exist). *Every equivalence class of Hubbard trees contains a minimal representative.*

Proof. Given any equivalence class of Hubbard trees, pick a representative $(T, f, \mathcal{P})_d$ such that if c_0 is periodic, $(T, f, \mathcal{P})_d$ has attracting dynamics. For any periodic itinerary τ , let $X_\tau \subset T$ be the smallest connected subset of T which contains all periodic points of itinerary τ . X_τ is a closed set: this is trivial if there are only finitely many periodic points in X_τ , and if there are infinitely many, then this follows from Proposition 2.2.3, case (i). By expansivity, X_τ contains at most one branch point, so it is a closed (possibly degenerate) n -od.

We define the following equivalence relation on T :

$$x \sim y \quad : \iff \quad \begin{array}{l} x = y \text{ or } \tau(x) = \tau(y) \text{ and there is an } n \in \mathbb{N} \text{ such that} \\ f^{\circ n}(x) \text{ and } f^{\circ n}(y) \in X_\tau \text{ for some periodic itinerary } \tau. \end{array}$$

Observe that an equivalence class is either a singleton, of the form X_τ or a non-periodic iterated preimage of some X_τ . Thus, all equivalence classes are closed subsets of T . Moreover, the equivalence class of any precritical point is trivial and if $x_0 \notin X_\tau$ has itinerary τ , then there is a neighborhood U of x_0 such that the equivalence class of any $x \in U$ is trivial as well.

Let \tilde{f} be the dynamics induced by f on the quotient $\tilde{T} := T/\sim$ and $\pi : T \rightarrow \tilde{T}$ the natural projection map. We show that $(\tilde{T}, \tilde{f}, \mathcal{P})_d$ is a Hubbard tree. It is minimal by construction and equivalent to $(T, f, \mathcal{P})_d$ since we have not changed the mutual location of marked points (marked points and in particular branch points are preserved under π by expansivity of $(T, f, \mathcal{P})_d$).

We first prove that any two points $x \neq y \in \tilde{T}$ can be separated by a third point. This implies that \tilde{T} is metrizable. In fact \tilde{T} is a tree, because all equivalence classes are connected [N, 9.42, 9.45]. There are endpoints p_x and p_y of $\pi^{-1}(x)$ and respectively $\pi^{-1}(y)$ such that $]p_x, p_y[$ does not intersect $\pi^{-1}(x) \cup \pi^{-1}(y)$. If the itineraries of p_x and p_y are different, then there is a precritical point $\xi \in]p_x, p_y[$. If they are equal then the equivalence class of at least one of the two points p_x, p_y is trivial and thus, there is a $\xi \in]p_x, p_y[$ whose equivalence class is trivial as well. Hence, the two components U, U' of $T \setminus \{\xi\}$ are disjoint open saturated sets, one of which contains $\pi^{-1}(x)$ and the other one $\pi^{-1}(y)$. Therefore, $\pi(U), \pi(U')$ are open in \tilde{T} , disjoint and contain x and y , respectively. Since $\tilde{T} \setminus \{\pi(\xi)\} = \pi(U) \cup \pi(U')$, x and y are separated by $\pi(\xi)$. Observe that \tilde{f} is continuous and locally injective on $\tilde{T} \setminus \{\pi(c_0)\}$, and since $f^{-1}(X_\tau)$ splits into at most d equivalence classes, \tilde{f} is at most d -to-1. Moreover, $(\tilde{T}, \tilde{f}, \mathcal{P})_d$ meets the expansivity condition. Putting everything together, $(\tilde{T}, \tilde{f}, \mathcal{P})_d$ is a Hubbard tree. \square

Proposition 2.2.8 (Hubbard trees of polynomials). *Let f be a postcritically finite unicritical polynomial of degree d and \mathbb{T} its Hubbard tree in the sense*

of Douady and Hubbard. Then, if T is its underlying topological tree and \mathcal{P} is the partition induced by external rays, then $(T, f, \mathcal{P})_d$ is a Hubbard tree in the sense of Definition 2.1.3. Moreover, it is minimal.

Proof. Suppose that \mathbb{T} is a Hubbard tree in the sense of Douady and Hubbard, generated by the polynomial f . From their definition it is straightforward to see that \mathbb{T} fulfills requirements (i) – (iv) of Definition 2.1.1 and that it has attracting dynamics. Moreover, since it is embedded into the plane it comes with a labeled partition: in the preperiodic case, pick any external ray R_θ that lands at the critical value. If the critical point $c_0 = 0$ is n -periodic, let U be the critical-value Fatou component and pick any ray R_θ that lands at the unique fixed point on ∂U of f^{on} . In the first case, $f^{-1}(R_\theta) \cup \{c_0\}$ partitions the plane into d regions so that f restricted to each region is injective. In the periodic case, one has to extend the d preimage rays R_i of R_θ by internal rays of the critical-point Fatou component. (Internal rays were introduced on page 7.) These extended rays together with the critical point define a partition of the complex plane. In both cases, label the obtained d regions counterclockwise in a consecutive way so that the region with label 0 contains the critical value. This induces a partition \mathcal{P} on the underlying topological tree T of \mathbb{T} . So it only remains to verify expansivity and minimality for the triple $(T, f, \mathcal{P})_d$.

Since the first requirement of minimality is a stronger condition than expansivity, it suffices to show that for any two distinct periodic points p, p' there is an $i \in \mathbb{N}_0$ such that the critical point c_0 is contained in $f^{oi}([p, p'])$. This condition is trivially true if c_0 is periodic and one of the points p, p' is on the critical cycle. Suppose that there were two periodic points $p \neq p' \in T$ which have the same itinerary in $\{0, \dots, d-1\}^{\mathbb{N}}$. Then the interval $[p, p']$ contains no precritical point. Let m be the smallest common period of p and p' and suppose that we have chosen p and p' in such a way that $]p, p'[$ contains no point of period m . This is possible since there is only a finite number of such points. Observe that the periodic points p, p' are repelling. Thus, the fact that f^{om} fixes both p and p' and $f^{om}|_{[p, p']} : [p, p'] \rightarrow [p, p']$ is a homeomorphism implies that there is a fixed point of f^{om} in $]p, p'[,$ a contradiction to the choice of p and p' . \square

We have already seen in Section 2.1.1 that the converse statement is not true.

We conclude this section with two consequences of the fact that minimal Hubbard trees with periodic critical point have attracting dynamics. The first one is a technical observation which will become very helpful when we investigate the set of Hubbard trees or equivalently the set of \star - and preperiodic kneading sequences in Section 3.3; indeed it is used to prove Theorem 3.3.17, our main result in parameter space. The second one is of interest by itself.

Lemma 2.2.9 (Limit of characteristic points). *Let $(T, f, \mathcal{P})_d$ be a minimal Hubbard tree and $z \in]c_0, c_1[\subset T$ be a (pre-)periodic point. Set $P_z := \{p \in]c_0, z[: p \text{ is characteristic}\}$ and $p_0 := \sup(P_z)$. If $p_0 \neq z$, then $p_0 \in P_z$.*

Proof. Since p_0 is the limit of characteristic points, it is enough to show that p_0 is periodic. Let us suppose that p_0 is not periodic. We show first that the arc $]p_0, z[$ contains a precritical point: by minimality of T , z cannot have the same itinerary as any point $p \in P_z$. Since p_0 is the limit of characteristic points, p_0 is not precritical. Since p_0 is not periodic, Lemma 2.1.9 implies that there is a neighborhood $I \subset]c_0, f(c_0)[$ of p_0 such that all points $p \in I$ have itinerary $\tau(p_0)$. Consequently, there is a point $p \in P_z \cap I$ that has itinerary $\tau(p_0) = \tau(z)$, a contradiction.

Let ξ be the precritical point in $]p_0, z[$ such that there is no precritical point in $]p_0, \xi[$ with lower step. Set $\text{STEP}(\xi) =: k$. Since p_0 is a limit point of characteristic points, there is no $l \in \mathbb{N}$ such that $f^{ol}(p_0) \in]p_0, c_1[$. Hence $f^{ok}([p_0, \xi])$ covers $[p_0, \xi]$ homeomorphically and there is a point $z' \in [p_0, \xi[$ with $z' = f^{ok}(z')$. If $z' = p_0$, we are done. Otherwise, let $z'' \in [p_0, \xi[$ be the periodic point with lowest period. If $z'' = p_0$, we are done again; otherwise we show that the point z'' is characteristic and thus $z'' \in P_z$, which contradicts that $p_0 = \sup(P_z)$: if z'' is not characteristic, then there is an $s < k$ such that $f^{os}([p_0, z''])$ covers $[p_0, z'']$ homeomorphically. But this yields a periodic point in $]p_0, \xi[$ of period smaller than the one of z'' , a contradiction to the choice of z'' . \square

Corollary 2.2.10 (Periodic points are repelling). *If $(T, f, \mathcal{P})_d$ is a minimal Hubbard tree and $z \in T$ is an n -periodic point disjoint from the critical orbit, then z is repelling. That is, there is a neighborhood U of z such that for all $p \in U$, $p \in]z, f^{\circ j_p n}(p)[$, where j_p is the period of the local arm at z pointing to p .*

Proof. Let k equal the period of a non-fixed local arm at z if such a local arm exists, and set $k = 1$ otherwise. Furthermore, let $U \subset T$ be a neighborhood of z . Pick U so small that $f^{\circ kn}|_U$ is a homeomorphism onto its image. For any global arm G of z set $L := G \cap U$. By minimality, either $f^{\circ j_n}(p) \in]z, p[$ for all $p \in L$ or $p \in]z, f^{\circ j_n}(p)[$ for all $p \in L$, where $j \in \{1, k\}$ is the period of the local arm associated to G . Suppose that the first case holds. Then $f^{\circ j_n}(p) \in]z, p[$ extends to all $p \in L' := \{x \in G : \tau(x) = \tau(z)\} \supset L$. By minimality, the set L' is an interval and its boundary point $p_0 \neq z$ is a precritical point by Lemma 2.1.9. By continuity, p_0 is periodic and thus on the critical orbit, yet all points in L' (except for z) are repelled from p_0 by $f^{\circ j_n}$. This contradicts that $(T, f, \mathcal{P})_d$ has attracting dynamics. \square

From now on, we only regard minimal Hubbard trees. Whenever we speak of Hubbard trees we mean minimal ones (unless stated otherwise). Since any equivalence class of Hubbard trees contains a minimal representative and we are interested in classifying equivalence classes of Hubbard

trees rather than individual ones, this restriction does not result in any loss of generality.

2.3 Kneading Sequences and Internal Addresses

2.3.1 Definition and Properties

In this section we define kneading sequences in a more abstract way than they were introduced on page 8: following [BS], we define kneading sequences to be elements of the set $\{0, \dots, d-1, \star\}^{\mathbb{N}}$; in particular, they are not necessarily generated by some external angle. We will see in Section 2.3.2 that there is a bijection between the set of \star - and preperiodic kneading sequences and the set of equivalence classes of Hubbard trees. We set

$$\begin{aligned}\Sigma_d^0 &:= \{\nu = (\nu_i)_{i=1}^{\infty} \in \{0, \dots, d-1\}^{\mathbb{N}} : \nu_1 = 0\}, \\ \Sigma_d^{\star} &:= \Sigma_d^0 \cup \{\bar{\star}\} \cup \{\nu = \overline{0\nu_2 \dots \nu_{n-1}\star} : \nu_i \in \{0, \dots, d-1\}\}.\end{aligned}$$

Definition 2.3.1 (Kneading sequences, internal addresses). *A kneading sequence of degree d is an element of the set Σ_d^{\star} . A kneading sequence ν is called \star -periodic of period n if $\nu = \overline{0\nu_2 \dots \nu_{n-1}\star}$ (or $\nu = \bar{\star}$ for $n = 1$). It is periodic if $\nu = \overline{\nu_1 \dots \nu_n} \in \Sigma_d^0$.*

An internal address is a finite or infinite sequence of tuples

$$(1, 0) \rightarrow (n_1, s_1) \rightarrow \dots \rightarrow (n_k, s_k) \rightarrow \dots,$$

where $1 < n_1 < \dots < n_k < \dots$ and $s_k \in \{1, \dots, d-1, \star\}$ such that $s_k = \star$ is only possible if the sequence of tuples is finite and (n_k, s_k) is the last element of the sequence. An entry of the internal address is an element (n_k, s_k) of the sequence of tuples.

Note that if ν is periodic, then $\nu_i \neq \star$ for all $i \in \mathbb{N}$. So a \star -periodic sequence is not periodic. We will mainly work with special kneading sequences, namely \star - and preperiodic ones. We set

$$\Sigma_d^{\sharp} := \{\nu \in \Sigma_d^{\star} : \nu \text{ is pre- or } \star\text{-periodic}\}.$$

Kneading sequences and internal addresses are actually analogous concepts: there is a bijection between the set of kneading sequences and the set of internal addresses of degree d . We are going to define maps $\rho, \tilde{\rho}$ that allow us to give algorithms to go from the one to the other. These algorithms together with the concept of internal addresses were introduced in [LS].

Definition 2.3.2 (ρ -map). *For any kneading sequence $\nu \in \Sigma_d^{\star}$ define*

$$\rho_{\nu} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}, \quad \rho(n) = \begin{cases} \inf\{k > n : \nu_k \neq \nu_{k-n}\} & \text{if existing} \\ \infty & \text{otherwise} \end{cases}.$$

The ρ -orbit $\text{orb}_\rho(\nu)$ of a kneading sequence ν is the set $\text{orb}_{\rho_\nu}(1) \setminus \{\infty\}$. Furthermore, we define

$$\begin{aligned} \tilde{\rho}_\nu : \text{orb}_\rho(\nu) &\rightarrow \{1, \dots, d-1, \star\}, \\ n_{k+1} &\mapsto \begin{cases} (\nu_{n_{k+1}} - \nu_{n_{k+1}-n_k}) \bmod d & \text{if } \nu_{n_{k+1}} \neq \star \\ \star & \text{otherwise} \end{cases}, \end{aligned}$$

where $n_0 = 1$ and $n_{k+1} = \rho_\nu(n_k)$.

The ρ_ν -function determines for any given $k \in \mathbb{N}$ how long the entries in ν and $\sigma^{\circ k}(\nu)$ coincide. At the time they disagree for the first time, the function $\tilde{\rho}_\nu$ measures the difference of the respective entries in ν and $\sigma^{\circ k}(\nu)$.

Lemma 2.3.3 (Kneading sequence vs. internal address). *For any kneading sequence $\nu \in \Sigma_d^\star \setminus \{\bar{\star}\}$, the associated internal address $(1, 0) \rightarrow (n_1, s_1) \rightarrow \dots \rightarrow (n_k, s_k) \rightarrow \dots$ is obtained inductively by the following algorithm: set $n_0 := 1, s_0 := 0$. If (n_k, s_k) has already been defined then*

$$(n_{k+1}, s_{k+1}) := (\rho_\nu(n_k), \tilde{\rho}_\nu(n_k)), \quad \text{if } \rho_\nu(n_k) \neq \infty.$$

If $\rho_\nu(n_k) = \infty$ then stop the algorithm. The resulting internal address is finite.

Conversely, if the internal address $(1, 0) \rightarrow (n_1, s_1) \rightarrow \dots \rightarrow (n_k, s_k) \rightarrow \dots$ is given, then the associated kneading sequence ν is obtained by induction on k : for $k = 0$, we set $\nu_{n_0} = \nu_1 := 0$. if ν_{n_k} has already been defined and (n_k, s_k) is not the last entry of the internal address, then let $\nu^{n_k} = (\nu_i^{n_k}) := \overline{\nu_1 \cdots \nu_{n_k}}$ and define

$$\begin{aligned} \nu_{n_k+i} &:= \nu_i^{n_k} \quad \text{for all } i = 1, \dots, n_{k+1} - n_k - 1 \text{ and} \\ \nu_{n_{k+1}} &:= \begin{cases} s_{k+1} + \nu_{n_{k+1}-n_k} & \text{if } s_{k+1} \neq \star \\ \star & \text{otherwise} \end{cases}. \end{aligned}$$

However, if the internal address is finite with last entry (n_k, s_k) , then stop the algorithm and set $\nu := \overline{\nu_1 \cdots \nu_{n_k}}$. \square

By the definition of ρ and $\tilde{\rho}$ the first algorithm gives an injective map from $\Sigma_d^\star \setminus \{\bar{\star}\}$ to the set of internal addresses. Since the second algorithm describes exactly the inverse process of the first one, it follows that there is a bijection between these two sets. We associate to $\bar{\star}$ the internal address $(1, \star)$, although strictly speaking this is not an internal address as defined above. This establishes a bijection between Σ_d^\star and the set of internal addresses.

Observe the following geometric interpretation of $\text{orb}_\rho(\nu)$, which we will take advantage of later on: given $n_k \in \text{orb}_\rho(\nu)$, the next point n_{k+1} on the ρ -orbit indicates the first position where $\sigma^{\circ n_k}(\nu)$ and $\overline{\nu_1 \cdots \nu_{n_k}}$ differ. But this means that n_{k+1} is the smallest number bigger than n_k such that $c_0 \in f^{\circ n_{k+1}-n_k}([f(c_0), f^{\circ n_k+1}(c_0)])$. Thus the ρ -map indicates the step of precritical points between c_0 and certain iterates of c_0 (cf. Lemma 2.4.5).

Lemma 2.3.4 (ρ -orbit and (pre-)periodic sequences). *Let $\nu \in \Sigma_d^*$ be a kneading sequence and $\text{orb}_\rho(\nu) = \{1, \dots, n_k, n_{k+1}, \dots\}$.*

- (i) *If ν is preperiodic, then $|\text{orb}_\rho(\nu)| = \infty$ and the sequence $(n_{k+1} - n_k)_{k \geq 0}$ is preperiodic.*
- (ii) *If ν is \star -periodic, then ν has exact period n if and only if $|\text{orb}_\rho(\nu)| < \infty$ and n is the largest entry of $\text{orb}_\rho(\nu)$.*
- (iii) *Let $\nu \in \Sigma_d^0$ be periodic. Then ν has exact period n such that $n \in \text{orb}_\rho(\nu)$ if and only if $\text{orb}_\rho(\nu)$ is finite and n is its largest element. However, if ν is n -periodic and $n \notin \text{orb}_\rho(\nu)$ then $|\text{orb}_\rho(\nu)| = \infty$.*

Proof. Let us first show the statement of the preperiodic case. If the ρ -orbit of ν was finite, then there would be a last entry, say n , and $\nu = \overline{\nu_1 \cdots \nu_n}$ would be periodic of period dividing n , a contradiction. Suppose that $\nu = \nu_1 \cdots \nu_{m_0} \overline{\nu_{m_0+1} \cdots \nu_{m_0+m}}$. Then $\rho(i + jm) = \rho(i) \pmod m$ for all $i \in \{m_0 + 1, \dots, m_0 + m\}$ and all $j \in \mathbb{N}_0$. This implies that $n_{k+1} - n_k$ (for $n_k > m_0$) can assume at most m different values, namely elements of $\{\rho(i) - i : m_0 < i \leq m_0 + m\}$. Moreover, there are $i \in \{m_0 + 1, \dots, m_0 + m\}$ and $j < j'$ such that $i + jm \in \text{orb}_\rho(\nu)$, $i + j'm \in \text{orb}_\rho(\nu)$. Hence from $\rho(i + jm) - (i + jm)$ on, the sequence of increments $n_{k+1} - n_k$ repeats itself.

The second statement follows directly from the definition of \star -periodic kneading sequences.

For the third statement, let us first show that if ν has exact period n and $n \notin \text{orb}_\rho(\nu)$, then $|\text{orb}_\rho(\nu)| = \infty$. Bruin and Schleicher show in [BS, Lemma 5.12] that for any $\nu \in \Sigma_2^0$, if $m \in \text{orb}_\rho(\nu)$ and $\rho(m) = \infty$ then the exact period of ν is m . Their proof can be modified as to hold for any $\nu \in \Sigma_d^0$. From this statement our claim follows easily: if $|\text{orb}_\rho(\nu)| \neq \infty$ then there is a last entry in $\text{orb}_\rho(\nu)$. This entry equals jn for some $j > 1$, because otherwise the exact period of ν would be strictly smaller than n . But now [BS, Lemma 5.12] implies that jn is the exact period of ν .

To finish the proof of the third claim suppose that $|\text{orb}_\rho(\nu)| < \infty$ and that n is the largest entry of $\text{orb}_\rho(\nu)$. Let n' be the exact period of ν . Then $\nu = \overline{\nu_1 \cdots \nu_{n'}}$. If $n' < n$ such that $n' \in \text{orb}_\rho(\nu)$ then the ρ -orbit would already stop at the entry n' , and in particular $n \notin \text{orb}_\rho(\nu)$. If $n' \notin \text{orb}_\rho(\nu)$ then $|\text{orb}_\rho(\nu)| = \infty$ as we just have seen. \square

2.3.2 Kneading Sequences for Hubbard Trees

In this section we are going to link the two concepts of kneading sequences and Hubbard trees.

Definition 2.3.5 (Kneading sequence of $(T, f, \mathcal{P})_d$). *For any Hubbard tree $(T, f, \mathcal{P})_d$, we define its associated kneading sequence ν to be the itinerary of the critical value, i.e., $\nu = \tau(f(c_0))$.*

It is an easy observation that the kneading sequence of a Hubbard tree is \star -periodic if and only if the critical point is periodic. It is preperiodic if and only if c_0 is preperiodic.

From Definition 2.3.5 it follows immediately that there is an injection from the set of equivalence classes of Hubbard trees into Σ_d^\sharp . In the remainder of this section we are going to show that this map is actually a bijection. Given any kneading sequence $\nu \in \Sigma_d^\sharp$, we are going to construct a topological tree and define a dynamics on this tree that turns it into a Hubbard tree with kneading sequence ν . For constructing the tree, we will use the *combinatorial triod map*: it allows us to determine the mutual location that vertices of the tree should have. This is also the key step for showing that two Hubbard trees generate the same kneading sequence if and only if they are contained in the same equivalence class.

Definition 2.3.6 (Combinatorial triod map). *Let ν be a \star -periodic or pre-periodic kneading sequence and let $\Sigma_d^\nu := \{0, \dots, d-1\}^\mathbb{N} \cup \text{orb}_\sigma(\star\nu)$. For any three pairwise distinct elements $\tau^1, \tau^2, \tau^3 \in \Sigma_d^\nu$ define $\varphi_\nu : (\Sigma_d^\nu)^3 \rightarrow (\Sigma_d^\nu)^3$ by*

$$(\tau^1, \tau^2, \tau^3) \mapsto \begin{cases} (\sigma(\tau^1), \sigma(\tau^2), \sigma(\tau^3)) & \tau_1^1 = \tau_1^2 = \tau_1^3 \\ (\nu, \sigma(\tau^2), \sigma(\tau^3)) & \tau_1^1 \neq \tau_1^2 = \tau_1^3 \\ (\sigma(\tau^1), \nu, \sigma(\tau^3)) & \tau_1^2 \neq \tau_1^1 = \tau_1^3 \\ (\sigma(\tau^1), \sigma(\tau^2), \nu) & \tau_1^3 \neq \tau_1^1 = \tau_1^2 \\ \text{STOP} & \tau_1^1, \tau_1^2, \tau_1^3 \text{ are pairwise distinct} \end{cases}.$$

Let $\mathcal{T} := (\tau^1, \tau^2, \tau^3)$ and for $j \in \{1, 2, 3\}$, let $\pi^j : (\Sigma_d^\nu)^3 \rightarrow \Sigma_d^\nu$ be the projection to the j -th coordinate. We set

$$\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) := \pi^j \circ \varphi_\nu^{\circ i}(\mathcal{T}) \text{ for all } i \in \mathbb{N}_0.$$

If $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) = \nu$, we say that $\phi_{\nu, \mathcal{T}}^{\circ i-1}(\tau^j)$ is chopped off, or sometimes also that τ^j is chopped off at time $i-1$. This includes the case that $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) = \sigma^{\circ i}(\tau^j) = \nu$.

Observe that $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ depends on the underlying triod $\mathcal{T} = (\tau^1, \tau^2, \tau^3)$.

We call the triple $(\tau^1, \tau^2, \tau^3) \in (\Sigma_d^\nu)^3$ a *combinatorial triod* if the three sequences τ^1, τ^2, τ^3 are pairwise distinct. Following [BS], we say a combinatorial triod \mathcal{T} can be *iterated indefinitely* if for all $i \in \mathbb{N}$, $\varphi_\nu^{\circ i}(\mathcal{T}) \neq \text{STOP}$ and the three sequences $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ are pairwise distinct.

A combinatorial triod is *non-degenerate* if exactly one of the following holds:

- (τ^1, τ^2, τ^3) can be iterated indefinitely under φ_ν so that all three sequences are eventually chopped off;

- the STOP case occurs so that the three sequences $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ start with pairwise distinct symbols none of which equal \star .

A combinatorial triod is *degenerate* if either

- (τ^1, τ^2, τ^3) can be iterated indefinitely under φ_ν so that exactly one sequence is never chopped off, or
- the STOP case occurs so that the three sequences $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ start with pairwise distinct symbols exactly one of which equals \star .

In the degenerate case, we say that the sequence which is never chopped off or whose iterate starts with \star in the STOP case is *in the middle* of the triod.

Definition 2.3.7 (Collapsing triods). *Let $\nu \in \Sigma_d^\#$ and let $\tau^1, \tau^2, \tau^3 \in \Sigma_d^\nu$. The combinatorial triod $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ collapses if there is an $i \in \mathbb{N}$ so that $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) = \phi_{\nu, \mathcal{T}}^{\circ i}(\tau^{j'})$ for $j \neq j'$. Any triod that collapses is called collapsing triod.*

If the triod \mathcal{T} collapses at time i , then φ_ν cannot be applied to $\varphi_\nu^{\circ i}(\mathcal{T})$. Therefore, the triod \mathcal{T} cannot be iterated indefinitely under φ_ν .

Lemma 2.3.8 (Types of triods). *Suppose that $(\tau^1, \tau^2, \tau^3) \in (\Sigma_d^\nu)^3$ is a non-collapsing combinatorial triod. Then it must exhibit exactly one of the four behaviors described after Definition 2.3.6 when iterated under φ_ν . Therefore, there are exactly four types of non-collapsing triods.*

Proof. Since $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ does not collapse, either $\varphi_\nu^{\circ i}(\mathcal{T})$ is defined for all i or the STOP case occurs. First suppose that STOP does not occur. Then it suffices to show that at least two sequences are chopped off. But this must hold because otherwise the two not chopped sequences would be equivalent. On the other hand, if $\varphi_\nu^{\circ i}(\mathcal{T}) = \text{STOP}$ for some i , then the $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ must be pairwise distinct. If one of these three sequences starts with the symbol \star we are in the degenerate case, otherwise we are in the non-degenerate case. \square

Lemma 2.3.9 (Requirement for collapsing). *Let $\tau^1, \tau^2, \tau^3 \in \Sigma_d^\nu$ such that $\tau_1^j \in \{t, \star\}$ for all $j = 1, 2, 3$, where $t \in \{0, \dots, d-1\}$. If the combinatorial triod $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ collapses, then there is an $i \in \mathbb{N}$ and a $j \in \{1, 2, 3\}$ such that $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) = t_0\nu$ for some $t_0 \neq \star$.*

Proof. Let us assume that $i = 1$ generates the collapsing and that $\phi_{\nu, \mathcal{T}}(\tau^1) = \phi_{\nu, \mathcal{T}}(\tau^2)$. Then $\varphi_\nu(\mathcal{T}) \neq \text{STOP}$ and if we were in the first case of the definition of φ_ν , then already τ^1 and τ^2 would have been equal. Thus, one of the sequences is chopped off and, as the τ^i are pairwise distinct, it must be τ^1 or τ^2 ; say it is τ^1 . Then $\phi_{\nu, \mathcal{T}}(\tau^1) = \nu = \sigma(\tau^2)$. So, $\tau^2 = t_0\nu$ for some $t_0 \in \{\star, 0, \dots, d-1\}$ and since $\tau_1^2 = \tau_1^3$, $t_0 \neq \star$. \square

We will mainly use the contrapositive statement of Lemma 2.3.9: if there are three pairwise distinct sequences starting with \star or t and $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) \neq t_0\nu$ for all i, j and $t_0 \neq \star$, then \mathcal{T} does not collapse.

Corollary 2.3.10 (Non-collapsing triod). *Let $\nu \in \Sigma_d^{\sharp}$. If $\sigma^{\circ k}(\nu)$, $\sigma^{\circ l}(\nu)$, $\sigma^{\circ m}(\nu)$ are pairwise distinct sequences such that their first entries are in $\{t, \star\}$, then the associated combinatorial triod does not collapse.*

Proof. If ν is \star -periodic then $\sigma^{\circ i}(\nu) \neq j\nu$ for all $j \in \{0, \dots, d-1\}$ and by Lemma 2.3.9, the given triod cannot collapse. Now suppose that ν is preperiodic and the triod collapses. Then there is an $l \in \mathbb{N}$ such that $\sigma^{\circ l-1}(\nu) = t_0\nu$ for some $t_0 \in \{0, \dots, d-1\}$. Let $k > 0$ be the preperiod of ν and $n > 0$ its period. We have that

$$t_0\nu_1 \cdots \nu_k \nu_{k+1} \cdots \nu_{k+n} \nu_{k+1} \cdots = \nu_l \nu_{l+1} \cdots \nu_{l+k+1} \cdots \nu_{l+k+n} \nu_{l+k+1} \cdots$$

This implies that $l = in$ for some $i > 0$. And if we take again a look at the equality above with l replaced by in , we see that $\nu_{1+j} = \nu_{in+j}$ for all $j \in \mathbb{N}_0$. Thus ν was periodic with exact period in , a contradiction. \square

Definition 2.3.11 (Branch itinerary). *Let $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ be a non-degenerate combinatorial triod. If \mathcal{T} maps eventually to STOP, let i_0 be such that $\varphi_{\nu}^{\circ i_0}(\mathcal{T}) = \text{STOP}$, otherwise set $i_0 = \infty$. We define the branch itinerary $\tau_b = ((\tau_b)_i)_{i=1}^{\infty}$ of \mathcal{T} by*

$$(\tau_b)_i = \begin{cases} t & i < i_0, t = (\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^{j_i}))_1 = (\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^{k_i}))_1, j_i \neq k_i \\ \star & \text{if } i = i_0 \\ (\tau_b)_{i \bmod i_0} & i > i_0 \end{cases},$$

where $(\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^{j_i}))_1$ denotes the first entry of the sequence $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^{j_i})$.

Lemma 2.3.12 (Arms of non-degenerate triod). *Let τ_b be the branch itinerary of the triod (τ^1, τ^2, τ^3) . Then for any $j \neq j' \in \{1, 2, 3\}$, the triod $(\tau^j, \tau_b, \tau^{j'})$ is degenerate with τ_b in the middle.*

Proof. We show the statement for $j = 1, j' = 2$. Let $\mathcal{T} := (\tau^1, \tau^2, \tau^3)$ and $\mathcal{T}_b := (\tau^1, \tau_b, \tau^2)$. Suppose that i_0 is the smallest integer such that $(\tau_b)_{i_0} = \star$ if existing, otherwise set $i_0 = \infty$. The definition of the branch itinerary implies that $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ ($j = 1$ or 2) is chopped off in \mathcal{T} if and only if $\phi_{\nu, \mathcal{T}_b}^{\circ i}(\tau^j)$ is chopped off in \mathcal{T}_b . Thus, $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j) = \phi_{\nu, \mathcal{T}_b}^{\circ i}(\tau^j)$ for all $i < i_0$. It follows that for all $i < i_0$, $\phi_{\nu, \mathcal{T}_b}^{\circ i}(\tau_b) = \phi_{\nu, \mathcal{T}}^{\circ i}(\tau^j)$ for either $j = 1$ or 2 , or both, and thus, for all $i < i_0$, $\phi_{\nu, \mathcal{T}_b}^{\circ i}(\tau_b)$ is not chopped off. At time i_0 , we have that $\phi_{\nu, \mathcal{T}_b}^{\circ i_0}(\tau_b) = \sigma^{\circ i_0}(\tau_b) = \star$ and $\phi_{\nu, \mathcal{T}_b}^{\circ i_0}(\tau^1) \neq \phi_{\nu, \mathcal{T}_b}^{\circ i_0}(\tau^2)$. Consequently, $\varphi_{\nu}^{\circ i}(\mathcal{T}_b) = \text{STOP}$ and \mathcal{T}_b is degenerate with τ_b in the middle. \square

Proposition 2.3.13 (ν is an endpoint). *Let $\nu \in \Sigma_d^\#$ and $\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu$ be pairwise distinct with first entries in $\{0, \star\}$. Then $(\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu)$ is not degenerate with ν in the middle.*

The proof of this proposition is based on [BS, Lemma 3.10]. Bruin and Schleicher only consider sequences $\nu \in \Sigma_2^\star$. However, their reasoning carries over to arbitrary degree d . The main tool in their proof is to change the considered sequence ν at specific entries and to investigate this modified sequence $\tilde{\nu}$. In degree 2, there is only one way to change entries of sequences that are not \star -periodic so that one obtains a periodic sequence that does not contain \star . For arbitrary degree one has several options. In all cases they consider for the proof of [BS, Lemma 3.10] except one it does not matter which symbol we choose (as long as it differs from the original one). The exceptional case is their **Case I** with $m = m_0$, where one has to choose $\tilde{\nu}_{m_0} = \tilde{\nu}_m := \nu_{\rho(k)-m_0}$ when transferring their proof to the general setting.

Lemma 2.3.14 ([BS, Lemma 3.10], arbitrary degree). *For any $\nu \in \Sigma_d^\star$ and $k \in \mathbb{N}$ with $\rho(k) - k < \infty$, there exists an i with $\rho^{\circ i}(\rho(k) - k) \leq \rho(k)$ such that $\rho^{\circ i}(\rho(k) - k) \in \text{orb}_\rho(\nu)$. \square*

Proof of Proposition 2.3.13. Note that $\min\{\text{orb}_\rho(\nu) \cap \text{orb}_\rho(\rho(k) - k)\} \leq \rho(k)$ by [BS, Lemma 3.10]. Let us first prove the following statement:

CLAIM: Let ν be a \star -periodic kneading sequences of exact period n , $\rho(k) = n$ and let $\min\{\text{orb}_\rho(\nu) \cap \text{orb}_\rho(\rho(k) - k)\} = n$. If n_0, k_0 denotes the largest element smaller than n of $\text{orb}_\rho(\nu)$ and of $\text{orb}_\rho(\rho(k) - k)$ respectively, then $\nu_{n-k_0} = \nu_{n-n_0}$.

Note first that n_0, k_0 exist. By way of contradiction, suppose that $\nu_{n-k_0} \neq \nu_{n-n_0}$. Let us consider the sequence $\tilde{\nu} = \overline{\nu_1 \cdots \nu_{n-1} \nu_{n-n_0}}$ and let $\tilde{\rho} := \rho_{\tilde{\nu}}$. Then $\tilde{\rho}(k_0) = \rho(k_0) = n$, $\tilde{\rho}(n_0) > n$ and $\tilde{\rho}(k) \geq n$. We are going to show that n is the largest element of $\text{orb}_{\tilde{\rho}}(\tilde{\rho}(k) - k)$. If $\tilde{\rho}(k) = n$ then this is clearly true as $\text{orb}_\rho(\rho(k) - k) = \text{orb}_{\tilde{\rho}}(\tilde{\rho}(k) - k)$. Otherwise observe that $\nu_{n-k} = \nu_{n-n_0}$ and $k_0 \geq \rho(k) - k = n - k$. If $k_0 = n - k$ then the following finite words are equal

$$\nu_{k+1} \cdots \nu_{n-n_0} \nu_1 \cdots \nu_{n-1-k_0} = \nu_1 \cdots \nu_{k_0} \nu_{k_0+1} \cdots \nu_{n-1},$$

and thus, as $\nu_{n-k_0} \neq \nu_{n-n_0}$ by hypothesis, the first n entries of the sequences $\tilde{\nu}$ and $\sigma^{\circ k}(\tilde{\nu})$ disagree at exactly one position. But this means that they contain one symbol that appears differently often within the respective first n entries, which is impossible, as one is an iterate of the other. On the other hand, if $k_0 > n - k$, then $\rho(\rho(k) - k) < n$ and

$$\begin{aligned} \nu_{k+1} \cdots \nu_{n-n_0} \nu_1 \cdots \nu_{\rho(\rho(k)-k)-(\rho(k)-k)-1} = \\ \nu_1 \cdots \nu_{n-k} \nu_{\rho(k)-k+1} \cdots \nu_{\rho(\rho(k)-k)-1} \end{aligned}$$

and $\nu_{\rho(\rho(k)-k)} \neq \nu_{\rho(\rho(k)-k)-(\rho(k)-k)}$. Consequently, $\tilde{\rho}(k) = \rho(\rho(k) - k) + k$ and $\text{orb}_{\tilde{\rho}}(\tilde{\rho}(k) - k) = \{\rho(\rho(k) - k), \dots, k_0, n\}$, i.e. n is the largest element of $\text{orb}_{\tilde{\rho}}(\tilde{\rho}(k) - k)$.

It follows that $\text{orb}_{\tilde{\rho}}(\tilde{\nu}) \cap \text{orb}_{\tilde{\rho}}(\tilde{\rho}(k) - k) = \emptyset$, in contradiction to 2.3.14. This proves the claim.

Now let us compare the two sequences $\nu, \sigma^{\circ k}(\nu)$ and their iterates under σ . The first time that $\sigma^{\circ i}(\nu)$ and $\sigma^{\circ i}(\sigma^{\circ k}(\nu))$ start with different symbols is at time $i = \rho(k) - k$. Replace $\sigma^{\circ \rho(k)-k}(\sigma^{\circ k}(\nu))$ by ν and continue iterating $\sigma^{\circ \rho(k)-k}(\nu)$ and ν . The first time their iterates start with different symbols equals $\rho(\rho(k) - k)$. Suppose that this number is finite. Replacing $\sigma^{\circ \rho(\rho(k)-k)-(\rho(k)-k)}(\nu)$ by ν and repeating the whole procedure yields $\rho^{\circ 2}(\rho(k) - k)$.

We will use this observation to show that the combinatorial triod $(\sigma^{\circ k}(\nu), \sigma^{\circ l}(\nu), \nu) =: Y$ is not degenerate with ν in the middle. We proceed by way of contradiction. Let us first assume that ν is never chopped off and the STOP does not occur. Then the observation implies that the elements of $\text{orb}_{\rho}(\rho(k) - k)$ and $\text{orb}_{\rho}(\rho(l) - l)$ are exactly the times when $\phi_{\nu, Y}^{\circ i}(\sigma^{\circ k}(\nu))$ and $\phi_{\nu, Y}^{\circ i}(\sigma^{\circ l}(\nu))$ are chopped off in Y . By 2.3.14, $\text{orb}_{\rho}(\rho(k) - k)$ and $\text{orb}_{\rho}(\rho(l) - l)$ meet at some time i_0 and thus either $\varphi_{\nu}^{\circ i_0}(Y) = \text{STOP}$ or $\phi_{\nu, Y}^{\circ i_0}(\nu)$ is chopped off. Both possibilities contradict our hypothesis. Now let us assume that $\varphi_{\nu}^{\circ i_0}(Y) = \text{STOP}$ and $\phi_{\nu, Y}^{\circ i_0}(\nu)$ starts with \star . This, of course, can only happen if ν is \star -periodic of, say, exact period n . By possibly considering an appropriate iterate of Y instead of Y itself, we can assume that $i_0 = n$ and $\phi_{\nu, Y}^{\circ i}(\nu)$ has not been chopped off for any $i < n$. Therefore, $\min\{\text{orb}_{\rho}(\rho(k) - k) \cap \text{orb}_{\rho}(\rho(l) - l)\} = n$, and consequently, $\min\{\text{orb}_{\rho}(\nu) \cap \text{orb}_{\rho}(\rho(j) - j)\} = n$ for $j = k$ or $j = l$. Without loss of generality let us assume that $j = k$ and let k_0, l_0 be as in the claim above. Then by this claim, $\nu_{n-k_0} = \nu_{n-n_0}$. If $\min\{\text{orb}_{\rho}(\nu) \cap \text{orb}_{\rho}(\rho(l) - l)\} = n$ as well, then $\nu_{n-k_0} = \nu_{n-n_0} = \nu_{n-l_0}$. Otherwise $\min\{\text{orb}_{\rho}(\nu) \cap \text{orb}_{\rho}(\rho(j) - j)\} < n$ and $n_0 = l_0$. So also in this case, $\nu_{n-k_0} = \nu_{n-n_0} = \nu_{n-l_0}$. As a consequence, $\phi_{\nu, Y}^{\circ n-1}(\sigma^{\circ k}(\nu)) = \phi_{\nu, Y}^{\circ n-1}(\sigma^{\circ l}(\nu))$, and $\varphi_{\nu}^{\circ n}(Y) \neq \text{STOP}$, in contradiction to our assumption. \square

Corollary 2.3.15 (Chopped-off points). *Let $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ be a combinatorial triod such that $\varphi_{\nu}^{\circ i}(\mathcal{T})$ exists for all $i \in \mathbb{N}$. Then, if $\phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^j)$ is chopped off for some i_1 , then there is an $i_2 > i_1$ such that $\phi_{\nu, \mathcal{T}}^{\circ i_2}(\tau^j)$ is chopped off, too.*

If \mathcal{T} is degenerate with τ^2 in the middle and there is an i_0 such that $\varphi_{\nu}^{\circ i_0}(\mathcal{T}) = \text{STOP}$ then $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^2)$ has not been chopped off for all $i < i_0$.

Proof. To prove the first statement let us assume that τ^1 is chopped off at time $i_1 - 1$ yet $\phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^1)$ is never chopped off. If \mathcal{T} is degenerate then τ^2 or τ^3 is in the middle; say it is τ^2 . Now the assumption implies that $\phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^2) = \phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^1)$ and the triod \mathcal{T} is collapsing. If \mathcal{T} is non-degenerate,

then the fact that $\phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^1)$ is not eventually chopped off implies that $\phi_{\nu}^{\circ i_1}(\mathcal{T})$ is degenerate and $\phi_{\nu, \mathcal{T}}^{\circ i_1}(\tau^1) = \nu$ is contained in the middle, in contradiction to Proposition 2.3.13.

The second claim follows immediately from Proposition 2.3.13: if $\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^2)$ was chopped off then $(\phi_{\nu, \mathcal{T}}^{\circ i}(\tau^1), \nu, \phi_{\nu, \mathcal{T}}^{\circ i}(\tau^3))$ would be degenerate with ν in the middle. \square

Corollary 2.3.16 (Type of image). *Suppose that (τ^1, τ^2, τ^3) is not collapsing. Then (τ^1, τ^2, τ^3) and $\varphi_{\nu}((\tau^1, \tau^2, \tau^3))$ are of the same type.* \square

Lemma 2.3.17 (Extending triods). *Consider the four combinatorial triods $Y_1 := (\tau^1, \tau^2, \tau^3)$, $Y_2 := (\tau^2, \tau^3, \tau^4)$, $Y_3 := (\tau^1, \tau^2, \tau^4)$ and $Y_4 := (\tau^1, \tau^3, \tau^4)$, all contained in Σ_d^{ν} , and suppose that none of them are collapsing. Then the following are true:*

- (i) *If Y_1 and Y_2 are degenerate with τ^2, τ^3 respectively in the middle, then Y_3 and Y_4 are degenerate with τ^2, τ^3 respectively in the middle.*
- (ii) *If Y_1 and Y_4 are degenerate with τ^2, τ^3 respectively in the middle, then Y_2 is degenerate with τ^3 in the middle.*
- (iii) *If Y_1 is non-degenerate with branch itinerary τ_b and (τ^4, τ^1, τ_b) is degenerate with τ^1 in the middle, then Y_2 is non-degenerate and its branch itinerary is τ_b .*

Proof. For the first item, we are going to show that Y_3 is degenerate with τ^2 in the middle. The statement for Y_4 follows analogously. Let $\tilde{\tau}_i^j(X)$ denote the first entry of $\phi_{\nu, X}^{\circ i-1}(\tau^j)$, where X is the appropriate triod Y_m , and let $k := \min \{i : \varphi_{\nu}^{\circ i-1}(Y_j) = \text{STOP for } j \in \{1, 2, 3\}\}$. Then $k < \infty$ if and only if STOP occurs in (at least) one of the three triods Y_1, Y_2, Y_3 .

We first show that for all $i \leq k$, $\tilde{\tau}_i^j(X)$ is the same for all triods $X \in \{Y_m : \tau^j \in Y_m, m = 1, 2, 3\}$. This is not straightforward as the maps φ and ϕ depend on the underlying triod Y_m . Let $l - 1 < k$ be the first time that a chopping occurs in one of the three triods. Then for all $i < l$, $\tilde{\tau}_i^j(Y_m)$ is the same for all Y_m ; let us denote this symbol by $\tilde{\tau}_i^j$. The point $\phi_{\nu, Y_3}^{\circ l-1}(\tau^2)$ cannot be chopped off in Y_3 : if it was, then $\tilde{\tau}_l^2 \neq \tilde{\tau}_l^1 = \tilde{\tau}_l^4$. On the other hand, if we regard Y_1, Y_2 we see that $\tilde{\tau}_l^2 = \tilde{\tau}_l^3 \neq \tilde{\tau}_l^4$. But this implies that $\varphi_{\nu}^{\circ l}(Y_1) = (\phi_{\nu, Y_1}^{\circ l}(\tau^2), \phi_{\nu, Y_1}^{\circ l}(\tau^3), \nu) = (\phi_{\nu, Y_2}^{\circ l}(\tau^2), \phi_{\nu, Y_2}^{\circ l}(\tau^3), \nu) = \varphi_{\nu}^{\circ l}(Y_2)$, and by Corollary 2.3.15, the triods Y_1, Y_2 have the same point (i.e. τ^2 or τ^3) in the middle, a contradiction. The sequence $\phi_{\nu, Y_1}^{\circ l-1}(\tau^1)$ is chopped off in Y_1 if and only if $\phi_{\nu, Y_3}^{\circ l-1}(\tau^1)$ is in Y_3 : if it is chopped off in Y_1 , then we have that $\tilde{\tau}_l^1 \neq \tilde{\tau}_l^2 = \tilde{\tau}_l^3$ in Y_1 . Since no other sequence has been chopped off in any of the triods before and since $\phi_{\nu, Y_3}^{\circ l-1}(\tau^2)$ cannot be chopped off as we have seen before, $\tilde{\tau}_l^1 \neq \tilde{\tau}_l^2 = \tilde{\tau}_l^4$ in Y_3 . On the other hand, if $\phi_{\nu, Y_3}^{\circ l-1}(\tau^1)$ is chopped off in Y_3 , then $\tilde{\tau}_l^1 \neq \tilde{\tau}_l^2$ in Y_3 and Y_1 . Thus, $\phi_{\nu, Y_1}^{\circ l-1}(\tau^1)$ is also chopped off.

By the same reasoning, $\phi_{\nu, Y_2}^{ol-1}(\tau^4)$ is chopped off if and only if $\phi_{\nu, Y_3}^{ol-1}(\tau^4)$ is. Furthermore, if $\phi_{\nu, Y_2}^{ol-1}(\tau^2)$ is chopped off in Y_2 or if $\phi_{\nu, Y_2}^{ol-1}(\tau^3)$ is chopped off in Y_1 , then $\tilde{\tau}_l^1 = \tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 = \tilde{\tau}_l^4$, and $\varphi_\nu^{ol}(Y_1) = \varphi_\nu^{ol}(Y_3)$. Hence, Y_3 is degenerate with τ^2 in the middle.

As a consequence of this discussion, we can iterate the three triods Y_1, Y_2, Y_3 so that $\tilde{\tau}_i^j$ is the same for all of them until $\phi_{\nu, Y_j}^{oi}(\tau^2)$ and $\phi_{\nu, Y_j}^{oi}(\tau^3)$ ($j = 1, 2$) are separated in the respective image of Y_1 and Y_2 , or until we reach STOP at time $k < \infty$ (whatever comes first). The first case settles the claim as we have already seen in the previous paragraph, in the second case we distinguish which triod STOP occurs in.

If $\varphi_\nu^{ok-1}(Y_1) = \text{STOP}$, then $\tilde{\tau}_k^2 = \star$ and $\tilde{\tau}_k^1 \neq \tilde{\tau}_k^3$. Looking at $\varphi_\nu^{ok-1}(Y_2)$, we see that $\tilde{\tau}_k^3 = \tilde{\tau}_k^4$ and thus, $\varphi_\nu^{ok-1}(Y_3) = \text{STOP}$, and Y_3 is degenerate with τ^2 in the middle. If $\varphi_\nu^{ok-1}(Y_2) = \text{STOP}$, then $\tilde{\tau}_k^3 = \star$ and $\tilde{\tau}_k^2 \neq \tilde{\tau}_k^4$ and, looking at $\varphi_\nu^{ok-1}(Y_1)$, $\tilde{\tau}_k^1 = \tilde{\tau}_k^2$. Thus, $\varphi_\nu^{ok}(Y_1) = (\phi_{\nu, Y_1}^{ok}(\tau^1), \phi_{\nu, Y_1}^{ok}(\tau^2), \nu) = (\phi_{\nu, Y_3}^{ok}(\tau^1), \phi_{\nu, Y_3}^{ok}(\tau^2), \nu) = \varphi_\nu^{ok}(Y_3)$ and Y_3 is as claimed. Now suppose that $\varphi_\nu^{ok-1}(Y_3) = \text{STOP}$. If $\tilde{\tau}_k^2 = \star$, then we are done. If $\tilde{\tau}_k^1 = \star$ then $\tilde{\tau}_k^2 \neq \tilde{\tau}_k^4$ and looking at the respective images of Y_1 and Y_2 we see that $\tilde{\tau}_k^2 = \tilde{\tau}_k^3 \neq \tilde{\tau}_k^4$. Thus $\varphi_\nu^{ok}(Y_1) = (\nu, \phi_{\nu, Y_i}^{ok}(\tau^2), \phi_{\nu, Y_i}^{ok}(\tau^3)) = \varphi_\nu^{ok}(Y_2)$ ($i = 1$ or 2) and the triods Y_1, Y_2 have the same sequence τ^2 or τ^3 in the middle, a contradiction. We derive the same contradiction if $\tilde{\tau}_k^4 = \star$ or $\tilde{\tau}_k^1 \neq \tilde{\tau}_k^2 \neq \tilde{\tau}_k^4 \neq \tilde{\tau}_k^1$.

For the second claim, let l, k be defined as above. We again have to show first that for all $i \leq k$, $\tilde{\tau}_i^j$ is the same for the triods Y_1, Y_2, Y_3 if they contain τ^j . If $\phi_{\nu, Y_1}^{ol-1}(\tau^1)$ is chopped off in Y_1 then $\tilde{\tau}_l^1 \neq \tilde{\tau}_l^2 = \tilde{\tau}_l^3$, and $\phi_{\nu, Y_4}^{ol-1}(\tau^1)$ is also chopped off in Y_4 . On the other hand, if $\phi_{\nu, Y_4}^{ol-1}(\tau^1)$ is chopped off in Y_4 then we have that either $\phi_{\nu, Y_1}^{ol-1}(\tau^1)$ is also chopped off in Y_1 or $\tilde{\tau}_l^1 = \tilde{\tau}_l^2 \neq \tilde{\tau}_l^3$. In the latter case, $\tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 = \tilde{\tau}_l^4$ and $\varphi_\nu^{ol}(Y_4) = (\nu, \phi_{\nu, Y_i}^{ol}(\tau^3), \phi_{\nu, Y_i}^{ol}(\tau^4)) = \varphi_\nu^{ol}(Y_2)$ ($i = 2$ or 4). Thus, the triod Y_2 is degenerate with τ^3 in the middle. If $\phi_{\nu, Y_1}^{ol-1}(\tau^3)$ is chopped off in Y_1 then $\tilde{\tau}_l^1 = \tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 = \tilde{\tau}_l^4$. Hence, $\varphi_\nu^{ol}(Y_4) = \varphi_\nu^{ol}(Y_2)$ and again we are done. If $\phi_{\nu, Y_4}^{ol-1}(\tau^4)$ or $\phi_{\nu, Y_2}^{ol-1}(\tau^4)$ is chopped off, then $\tilde{\tau}_l^1 = \tilde{\tau}_l^2 = \tilde{\tau}_l^3 \neq \tilde{\tau}_l^4$, and $\phi_{\nu, Y_4}^{ol-1}(\tau^4)$, $\phi_{\nu, Y_2}^{ol-1}(\tau^4)$ respectively, is also chopped off. Now if $\phi_{\nu, Y_2}^{ol-1}(\tau^2)$ is chopped off then $\tilde{\tau}_l^1 = \tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 = \tilde{\tau}_l^4$. Hence $\varphi_\nu^{ol}(Y_4) = \varphi_\nu^{ol}(Y_2)$ and we are done. The last case is that $\phi_{\nu, Y_2}^{ol-1}(\tau^3)$ is chopped off. Then $\tilde{\tau}_l^3 \neq \tilde{\tau}_l^2 = \tilde{\tau}_l^4$, and looking at Y_1 yields that $\tilde{\tau}_l^1 = \tilde{\tau}_l^2$ whereas in $\varphi_\nu^{ol-1}(Y_4)$, $\tilde{\tau}_l^1 = \tilde{\tau}_l^3$ and thus $\tilde{\tau}_l^2 = \tilde{\tau}_l^3$, a contradiction.

Since τ^3 is eventually chopped off in Y_1 , the above discussion proves the claim unless STOP appears earlier: If $\varphi_\nu^{ok-1}(Y_1) = \text{STOP}$ then $\tilde{\tau}_k^2 = \star$ and $\tilde{\tau}_k^1 \neq \tilde{\tau}_k^3 = \tilde{\tau}_k^4$. Again $\varphi_\nu^{ok}(Y_4) = \varphi_\nu^{ok}(Y_2)$. If $\varphi_\nu^{ok-1}(Y_4) = \text{STOP}$ then $\tilde{\tau}_k^3 = \star$ and $\tilde{\tau}_k^1 = \tilde{\tau}_k^2 \neq \tilde{\tau}_k^4$. Consequently, $\varphi_\nu^{ok-1}(Y_2) = \text{STOP}$ and again Y_2 is as claimed. If $\varphi_\nu^{ok-1}(Y_2) = \text{STOP}$ we can assume that $\tilde{\tau}_k^4 = \star$ because otherwise we are in one of the two previous cases. Hence, looking at Y_1, Y_4 yields $\tilde{\tau}_k^1 = \tilde{\tau}_k^2 = \tilde{\tau}_k^3$ and $\varphi_\nu^{ok-1}(Y_2) \neq \text{STOP}$, a contradiction. Now suppose

that $\tilde{\tau}_k^2 \neq \tilde{\tau}_k^3 \neq \tilde{\tau}_k^4 \neq \tilde{\tau}_k^2$. In $\varphi_\nu^{\circ k-1}(Y_4)$, either $\tilde{\tau}_k^1 = \tilde{\tau}_k^3$ or $\tilde{\tau}_k^1 = \tilde{\tau}_k^4$, and in $\varphi_\nu^{\circ k-1}(Y_1)$, $\tilde{\tau}_k^1 = \tilde{\tau}_k^2$. But this implies that either $\tilde{\tau}_k^2 = \tilde{\tau}_k^3$ or $\tilde{\tau}_k^2 = \tilde{\tau}_k^4$, which is both impossible.

Now let us prove the statement about the non-degenerate triods. By Lemma 2.3.12 and the two previous claims of the current lemma, the triods Y_3 and Y_4 are both degenerate with τ^1 in the middle. Let $k := \min\{i : \varphi_\nu^{\circ i-1}(Y_j) = \text{STOP for some } j \in \{1, 2, 3, 4\}\}$ and l the earliest time that any sequence is chopped off in one of the four triods. Again, we show first that for all $i \leq k$, $\tilde{\tau}_i^j$ is the same for all triods in $\{Y_1, Y_3, Y_3, Y_4\}$ that contain τ^j . Observe that this also guarantees that the branch itinerary of Y_2 equals τ_b if Y_2 is non-degenerate.

If $l < k$ then there is nothing to show. Otherwise suppose that $\phi_{\nu, Y_1}^{\circ l-1}(\tau^1)$ is chopped off in Y_1 . Then looking at Y_3 yields that $\tilde{\tau}_l^4 = \tilde{\tau}_l^1 \neq \tilde{\tau}_l^2 = \tilde{\tau}_l^3$, and $\varphi_\nu^{\circ l}(Y_1) = \varphi_\nu^{\circ l}(Y_2)$, that is, the claim is proven. If $\phi_{\nu, Y_2}^{\circ l-1}(\tau^2)$ is chopped off, then $\tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 = \tilde{\tau}_l^1 \stackrel{Y_3}{\cong} \tilde{\tau}_l^4$, and $\phi_{\nu, Y_3}^{\circ l-1}(\tau^2)$, $\phi_{\nu, Y_4}^{\circ l-1}(\tau^2)$ are also chopped off. Analogously, if $\phi_{\nu, Y_1}^{\circ l-1}(\tau^3)$ is chopped off then so are $\phi_{\nu, Y_2}^{\circ l-1}(\tau^3)$, $\phi_{\nu, Y_4}^{\circ l-1}(\tau^3)$. And if $\phi_{\nu, Y_3}^{\circ l-1}(\tau^4)$ or $\phi_{\nu, Y_4}^{\circ l-1}(\tau^4)$ is chopped off, then the respective sequence is chopped off in all triods that contain τ^4 . If $\phi_{\nu, Y_3}^{\circ l-1}(\tau^2)$ is chopped off then $\tilde{\tau}_l^1 = \tilde{\tau}_l^4 \neq \tilde{\tau}_l^2$. If $\tilde{\tau}_l^1 = \tilde{\tau}_l^3$ then $\phi_{\nu, Y_1}^{\circ l-1}(\tau^2)$ is also chopped off. If $\tilde{\tau}_l^2 = \tilde{\tau}_l^3$ then $\varphi_\nu^{\circ l}(Y_1) = \varphi_\nu^{\circ l}(Y_2)$, and Y_2 is non-degenerate. The case that $\phi_{\nu, Y_3}^{\circ l-1}(\tau^3)$ is chopped off works the same way. It remains to consider the cases where the first chopping occurs in Y_2 : if $\phi_{\nu, Y_2}^{\circ l-1}(\tau^2)$ is chopped off then $\tilde{\tau}_l^2 \neq \tilde{\tau}_l^4 = \tilde{\tau}_l^3 \stackrel{Y_4}{\cong} \tilde{\tau}_l^1$, and $\phi_{\nu, Y_1}^{\circ l-1}(\tau^2)$, $\phi_{\nu, Y_3}^{\circ l-1}(\tau^2)$ are chopped off. Analogously, if $\phi_{\nu, Y_2}^{\circ l-1}(\tau^3)$ is chopped off then the respective sequence is also chopped off in Y_1, Y_4 . And if $\phi_{\nu, Y_2}^{\circ l-1}(\tau^4)$ is chopped off then $\tilde{\tau}_l^4 \neq \tilde{\tau}_l^2 \neq \tilde{\tau}_l^3 \stackrel{Y_3}{\cong} \tilde{\tau}_l^1$. Thus $\varphi_\nu^{\circ l}(Y_1) = \varphi_\nu^{\circ l}(Y_2)$, and we are done. Since τ^1 is eventually chopped off in Y_1 this proves the third claim unless a STOP case happens before.

Now let us consider the various possibilities for the first STOP case: if $\varphi_\nu^{\circ k-1}(Y_3) = \text{STOP}$ then $\tilde{\tau}_k^1 = \star$, $\tilde{\tau}_k^2 \neq \tilde{\tau}_k^4$ and, looking at the image of Y_1 , $\tilde{\tau}_k^2 = \tilde{\tau}_k^3$. Thus, $\varphi_\nu^{\circ k}(Y_1) = \varphi_\nu^{\circ k}(Y_2)$ and Y_2 is non-degenerate. The same is true if $\varphi_\nu^{\circ k-1}(Y_4) = \text{STOP}$. If $\varphi_\nu^{\circ k-1}(Y_1) = \text{STOP}$ then $\tilde{\tau}_k^1 \neq \tilde{\tau}_k^2 \neq \tilde{\tau}_k^3 \neq \tilde{\tau}_k^1$ and by considering the image of Y_3 , $\tilde{\tau}_k^1 = \tilde{\tau}_k^4$. Consequently, $\varphi_\nu^{\circ k-1}(Y_2) = \text{STOP}$ so that no sequence starts with \star . But this means that Y_2 is non-degenerate. Now suppose that $\varphi_\nu^{\circ k-1}(Y_3) = \text{STOP}$. If $\tilde{\tau}_k^4 = \star$ then $\tilde{\tau}_k^2 \neq \tilde{\tau}_k^3$, and considering the respective images of Y_3, Y_4 yields that $\tilde{\tau}_k^1 = \tilde{\tau}_k^2$, $\tilde{\tau}_k^1 = \tilde{\tau}_k^3$. Thus $\tilde{\tau}_k^2 = \tilde{\tau}_k^3$, a contradiction. We get a similar contradiction for $\tilde{\tau}_k^2 = \star$ and $\tilde{\tau}_k^3 = \star$. This settles the last claim. \square

An Algorithm for Constructing Hubbard Trees from Pre- or \star -Periodic Kneading Sequences

The goal of the following paragraph is to construct from a given $\nu \in \Sigma_d^\sharp$ a Hubbard tree with associated kneading sequence ν . We give an algorithm that generates a topological tree T so that T is spanned by points in $\text{orb}_\sigma(\star\nu)$ and so that any combinatorial triod $(\sigma^{\circ j}(\star\nu), \sigma^{\circ k}(\star\nu), \sigma^{\circ l}(\star\nu))$ is of the same type as the triod generated by its associated points in T . We first describe the algorithm and then show that the obtained tree exhibits these properties. Finally, we equip the tree T with dynamics, which results in a Hubbard tree with associated kneading sequence ν .

THE ALGORITHM: Let $\mathcal{V}^\nu := \{\sigma^{\circ k}(\nu) : k \in \mathbb{N}_0\}$ and for each $i \in \{0, \dots, d-1\}$, let $\mathcal{V}_i := \{\mu \in \mathcal{V}^\nu : \mu_1 = i\}$. For each $i \in \{0, \dots, d-1\}$, we construct a graph $T'_i = (V_i, E_i)$, where V_i (E_i) is the set of vertices (edges) of T'_i . The (planar representations of the) T'_i are going to be the subtrees of the yet to be constructed Hubbard tree $(T, f, \mathcal{P})_d$ that contain all points whose itineraries start with i plus the critical point c_0 .

Let us first construct T'_0 . Note that $|\mathcal{V}_0| \geq 1$. If $\mathcal{V}_0 = \{\nu\}$, then $T'_0 = (\{\star\nu, \nu\}, \{(\star\nu, \nu)\})$. In all other cases, we pick two points $\mu^1, \mu^2 \in \mathcal{V}_0$. By Lemma 2.3.8 and Proposition 2.3.13 and Corollaries 2.3.10 and 2.3.16, the combinatorial triod $(\star\nu, \mu^1, \mu^2)$ does not collapse and it is either non-degenerate or degenerate such that $\star\nu$ is not in the middle. If it is degenerate with μ^j in the middle, we set $V_0 := \{\star\nu, \mu^1, \mu^2\}$ and $E_0 := \{(\star\nu, \mu^j), (\mu^j, \mu^{j'})\}$, where $\{j, j'\} = \{1, 2\}$. If $(\star\nu, \mu^1, \mu^2)$ is a non-degenerate triod, we define $V_0 := \{\star\nu, \mu^1, \mu^2, \tau_b\}$ and $E_0 := \{(\star\nu, \tau_b), (\mu^1, \tau_b), (\mu^2, \tau_b)\}$, where τ_b is the branch itinerary of $(\star\nu, \mu^1, \mu^2)$. We add iteratively each point $\mu \in \mathcal{V}_0$ to the set V_0 , and to E_0 appropriate edges that have μ as endpoint. To find the ‘‘appropriate edges’’ we iterate combinatorial triods. After having described the algorithm, we show that no combinatorial triod that has been considered in this process is collapsing. This guarantees that the resulting tree T is well-defined.

We set $V'_0 := \mathcal{V}_0 \setminus V_0$, i.e. V'_0 comprises all points of \mathcal{V}_0 that are not an endpoint of an edge in E_0 yet. At the end of each loop, we update the sets V_0 , V'_0 and E_0 , and thus the tree T_0 . Note that e.g. by $V := V \cup \{p\}$ we mean that the updated set V contains all elements of the old V plus the point p .

(Outer) loop: If $V'_0 = \emptyset$ then T'_0 has been completely constructed and we continue with the construction of T'_1 . Otherwise take any point $v \in V'_0$ and let w be the unique point of V_0 so that $(\star\nu, w) \in E_0$. As discussed above, there are exactly the following three possibilities:

- If $(\star\nu, w, v)$ is a non-degenerate triod with branch itinerary $\tau_{b'}$, remove $(\star\nu, w)$ from E_0 and add the tree edges $(\star\nu, \tau_{b'}), (w, \tau_{b'}), (v, \tau_{b'})$. Add $\tau_{b'}$ and v to V_0 and set $V'_0 := V'_0 \setminus V_0$. Repeat the loop.

- If $(\star\nu, w, v)$ is degenerate with v in the middle, remove $(\star\nu, w)$ from E_0 and add the two edges $(\star\nu, v)$, (v, w) , add v to V_0 and set $V'_0 := V_0 \setminus V_0$. Repeat the loop.
- If $(\star\nu, w, v)$ is degenerate with w in the middle, pick any edge in E_0 with endpoint w such that the second endpoint $w' \neq \star\nu$. Consider the combinatorial triod (w, w', v) .
 - (*) We distinguish four cases in the following “inner loop”:
 - if (w, w', v) is non-degenerate with branch itinerary $\tau_{b''}$, then remove (w, w') from E_0 and add the three edges $(w, \tau_{b''})$, $(w', \tau_{b''})$, $(v, \tau_{b''})$. Add $\tau_{b''}$ and v to V_0 and set $V'_0 := V_0 \setminus V_0$. Repeat the (outer) loop.
 - If (w, w', v) is degenerate with v in the middle, remove (w, w') from E_0 and add the two edges (w, v) , (v, w') , add v to V_0 and set $V'_0 := V_0 \setminus V_0$. Repeat the (outer) loop.
 - If w is in the middle of (w, w', v) then mark the edge (w, w') as “considered” and pick another edge $(w, w'') \in E_0$ such that $w'' \neq \star\nu$ and (w, w'') is not marked “considered”. If there is no such edge, add (w, v) to E_0 and v to V_0 . Set $V'_0 := V_0 \setminus V_0$ and repeat the (outer) loop. However, if such an edge exists, go to (*), replace w' by w'' and repeat the inner loop.
 - if w' is in the middle of (w, w', v) pick an edge $(w', w'') \in E_0$ where $w'' \neq w$. If there is no such edge, add (w', v) to E_0 and v to V_0 . Set $V'_0 := V_0 \setminus V_0$ and repeat the (outer) loop. However, if such an arm exists go to (*), replace (w, w', v) by the triod (w', w'', v) and repeat the inner loop.

Since $|\mathcal{V}_0| < \infty$, V'_0 is empty after finitely many steps. The resulting graph T'_0 is a tree. We apply the algorithm to each $0 \leq i < d$ (where i replaces 0) to get the trees T'_i . Define the tree $T' := (\bigcup_{i=0}^{d-1} V_i, \bigcup_{i=0}^{d-1} E_i)$, where V_i (E_i) is the union of all vertices (edges) of T'_i obtained after completing the algorithm for the set \mathcal{V}_i . This tree can be turned into a topological tree, which we denote by T (consider e.g. a planar representation of T' with the relative topology of \mathbb{R}^2). The point $c_0 \in T$ associated to the sequence $\star\nu$ is called *critical point*. The set T_i is the connected component of $T \setminus \{c_0\}$ that corresponds to the subgraph T'_i . Note that $c_0 \notin T_i$ while $\star\nu \in T'_i$.

Denote by V the set of vertices of the topological tree T , where each element of V corresponds to a unique element of $\bigcup_{i=0}^{d-1} V_i$. The *marked points* of T are exactly the elements of V .

It remains to show that we have not generated collapsing triods.

Lemma 2.3.18 (T is well-defined). *Let (w, w', w'') be a combinatorial triod that has been considered in the above algorithm. Then (w, w', w'') is not collapsing.*

Proof. By Lemma 2.3.9, it is enough to show that for each added branch itinerary τ_b , $\tau_b \neq t_0\nu$. Note the following two facts: first, by the way we constructed T , each subtree T_j can contain at most one point with itinerary $j\nu$. Secondly, if $\tau^1, \tau^2 \in \text{orb}_\sigma(\nu) \cap \mathcal{V}_j$ such that $(\tau^1, \tau^2, \star\nu) =: \mathcal{T}$ is non-degenerate with branch itinerary τ_b , then $\tau_b \neq j\nu$: by definition, there is an integer $i > 0$ such that $\phi_{\nu, \mathcal{T}}^{o_i}(\star\nu)$ is chopped off (if ν is \star -periodic of exact period n then $i \leq n$) and hence, $(\tau_b)_{i+1} \neq \nu_i$. Now consider any $w, w', w'' \in V_j \cup \{\star\nu\}$ which form a non-degenerate triod and let τ_b be its branch itinerary. The algorithm and Lemma 2.3.17 imply that in this case there are $\tau^1, \tau^2 \in \mathcal{V}_j \cap \text{orb}_\sigma(\nu)$ such that $(\star\nu, \tau^1, \tau^2)$ is non-degenerate with branch itinerary τ_b . As we just have seen, this implies that $\tau(b) \neq j\nu$. \square

Lemma 2.3.19 (Topological vs. combinatorial triods). *Let T be the topological tree constructed from a given $\nu \in \Sigma_d^\#$ via the algorithm above and let $T_i \subset T$ be the subtree comprising all elements $p \in T$ with itinerary $\tau(p) = i\dots$ for some $i \in \{0, \dots, d-1\}$. Pick any three pairwise distinct points $x_1, x_2, x_3 \in V \cap \overline{T_i} = T_i \cup \{c_0\}$. Then the topological triod (x_1, x_2, x_3) is non-degenerate (or degenerate) if and only if the combinatorial triod $(\tau(x_1), \tau(x_2), \tau(x_3))$ is. Moreover, the itinerary $\tau(b)$ of the branch point b of (x_1, x_2, x_3) equals the branch itinerary τ of $(\tau(x_1), \tau(x_2), \tau(x_3))$. In the degenerate case, x_i is in the middle of (x_1, x_2, x_3) if and only if $\tau(x_i)$ is in the middle of $(\tau(x_1), \tau(x_2), \tau(x_3))$.*

Proof. Observe first that for any combinatorial triod which has been considered in the construction and its associated topological triod in T the claim holds trivially. In the following we will call such (combinatorial or topological) triods “considered” triods. For any vertex $v' \in T_i$ adjacent to v (i.e. $]v, v'[\cap V = \emptyset$), there is a “considered” combinatorial triod \mathcal{Y}' such that $\tau(v), \tau(v')$ are generating points of \mathcal{Y}' . We set $Y := (x_1, x_2, x_3) \subset \overline{T_i}$ and $\mathcal{Y} := (\tau(x_1), \tau(x_2), \tau(x_3))$.

Using Lemma 2.3.12, we find for any degenerate Y a finite set of “considered” triods which we can apply Lemma 2.3.17 to (more precisely cases (i) and (ii), and these are enough) so that we get that \mathcal{Y} is also degenerate. Moreover, if x_i is in the middle of Y then $\tau(x_i)$ is in the middle of the combinatorial triod.

Now let Y be a non-degenerate triod with branch point b . The way we constructed T yields that either the itinerary $\tau(b)$ of b was generated as branch itinerary of some “considered” non-degenerate combinatorial triod \mathcal{Y}' or $\tau(b) \in \text{orb}(c_0)$. We deal with the first possibility first. Let $Y' \in T$ be the topological triod associated to \mathcal{Y}' . From what we have shown so far in

this proof, it follows that Y' is non-degenerate. We investigate the mutual location of Y and Y' , which both contain the point b .

If $Y' \subset Y$, then \mathcal{Y} is non-degenerate and has branch itinerary $\tau(b)$ by Lemma 2.3.17, case (iii), and by the just shown result about degenerate triods.

Next assume that $Y \subset Y'$. If the combinatorial triod \mathcal{Y} is non-degenerate then its branch itinerary equals $\tau(b)$ by repeating the argument of the previous case with the role of Y and Y' interchanged. If \mathcal{Y} is degenerate, let us first consider the situation that Y and Y' have two generating points in common, say x_1, x_2 . Then $x_3 \in]b, x_4[$, where x_4 is the third generating point of Y' . By Lemma 2.3.17, case (ii), it follows that $(\tau(x_2), \tau(b), \tau(x_3))$ is degenerate with $\tau(b)$ in the middle, and from this that $(\tau(x_2), \tau(x_3), \tau(x_4))$ is degenerate with $\tau(x_3)$ in the middle (case (i)). This in turn implies that \mathcal{Y}' is degenerate with $\tau(x_2)$ in the middle (again by case (i) because $\mathcal{Y} = (\tau(x_1), \tau(x_2), \tau(x_3))$), a contradiction. Now the claim for this case follows iteratively by working one's way out to Y' and applying this argument at every step.

The last possibility is that $Y \cup Y'$ is a (non-degenerate) 4-od with branch point b . Using the results obtained so far, we can assume that two generating points, say x_1, x_2 , of Y and Y' coincide. Let x_4 be the third generating point of Y' . Then b has been added to the tree at an earlier time than x_4 because otherwise b would not have been generated by Y' . Hence Lemma 2.3.12 and the definition of the algorithm yield that the combinatorial triods $(\tau(x_i), \tau(b), \tau(x_{i'}))$ are degenerate with $\tau(b)$ in the middle for all $i \neq i' \in \{1, 2, 3, 4\}$. Now, if \mathcal{Y} was degenerate, then applying the results about degenerate triods gives that for one of the degenerate triods $(\tau(x_i), \tau(b), \tau(x_{i'}))$, $\tau(b)$ is not in the middle, a contradiction. On the other hand, if \mathcal{Y} is non-degenerate with branch itinerary $\tau(a) \neq \tau(b)$, then there is a smallest integer i_0 such that $(\tau(a))_{i_0} \neq (\tau(b))_{i_0}$ (these are the i_0 -th entries of $\tau(a), \tau(b)$). In particular, for all $i < i_0$, $(\tau(a))_i \neq \star \neq (\tau(b))_i$, and for $i = 1, 2$, $\phi_{\nu, \mathcal{Y}'}^{\circ i}(\tau(x_i))$ is chopped off if and only if $\phi_{\nu, \mathcal{Y}}^{\circ i}(\tau(x_i))$ is chopped off. It follows that for $\mathcal{X} = \mathcal{Y}', \mathcal{Y}$, $\phi_{\nu, \mathcal{X}}^{\circ i_0}(\tau(x_1))$ and $\phi_{\nu, \mathcal{X}}^{\circ i_0}(\tau(x_2))$ start with different symbols. If $(\tau(b))_{i_0} = \star$, then for $j = 1$ or $j = 2$, $\left(\phi_{\nu, \mathcal{Y}}^{\circ i_0}(\tau(x_j))\right)_1 = \left(\phi_{\nu, \mathcal{Y}}^{\circ i_0}(\tau(x_3))\right)_1$. Consequently, $(\tau(x_j), \tau(b), \tau(x_3))$ is not degenerate with $\tau(b)$ in the middle ($\tau(b)$ is chopped off at time i_0). Observe that Lemma 2.3.12 allows us to transfer the result on the images of $\tau(x_j), \tau(x_3)$ from the triod \mathcal{Y} to $(\tau(x_j), \tau(b), \tau(x_3))$. If $(\tau(a))_{i_0} = \star$, then it follows that there is a $j \in \{1, 2\}$ such that $\left(\phi_{\nu, \mathcal{Y}'}^{\circ i_0}(\tau(x_j))\right)_1, \left(\phi_{\nu, \mathcal{Y}'}^{\circ i_0}(\tau(x_4))\right)_1$ and $(\sigma^{\circ i}(\tau(b)))_1 = (\tau(b))_{i_0}$ are pairwise distinct, which is only possible if $(\tau(b))_{i_0} = \star$, contradicting our assumption. Similarly, we see that in the case of $\varphi^{\circ i_0-1}(\mathcal{Y}) \neq \text{STOP} \neq \varphi^{\circ i_0-1}(\mathcal{Y}')$, the triod $(\tau(x_j), \tau(b), \tau(x_4))$ is not degenerate with $\tau(b)$ in the middle for some $j \in \{1, 2\}$.

If b is not a branch point of any “considered” triod, then there are three adjacent points y_1, y_2, y_3 such that b is the branch point of (y_1, y_2, y_3) . Let τ^i be the itinerary of y_i for all $i = 1, 2, 3$. It is enough to show that (τ^1, τ^2, τ^3) is non-degenerate with branch itinerary $\tau(b)$, as we then can apply the arguments of the previous paragraph, where (τ^1, τ^2, τ^3) takes over the role of \mathcal{Y}' . Since T was constructed via the above algorithm, we know that the combinatorial triods $(\tau^1, \tau(b), \tau^2), (\tau^1, \tau(b), \tau^3), (\tau^2, \tau(b), \tau^3)$ are degenerate with $\tau(b)$ in the middle. If (τ^1, τ^2, τ^3) was degenerate with, say τ^2 in the middle, then Lemma 2.3.17, case (i), implies that $(\tau(b), \tau^2, \tau^3)$ is degenerate with τ_2 in the middle, a contradiction. We get the same contradiction if τ^1 or τ^3 is in the middle. Thus (τ^1, τ^2, τ^3) is non-degenerate. If its branch itinerary $\tau(b')$ is not equal to $\tau(b)$ then let i_0 be defined as in the previous case. It follows that there are $j \neq j' \in \{1, 2, 3\}$ such that $(\tau^j, \tau^{j'}, \tau(b))$ is not degenerate with $\tau(b)$ in the middle, which is not true. \square

Lemma 2.3.20 (Marked points and itineraries). *Let $\nu \in \Sigma_d^\sharp$ and T be the topological tree spanned by $\text{orb}_\sigma(\nu)$ as constructed via the above algorithm. Then the following hold:*

- (i) *If $v_1 \neq v_2 \in V$, then $\tau(v_1) \neq \tau(v_2)$.*
- (ii) *If $\mathcal{V} = \{\tau(v) : v \in V\}$, then $\sigma(\mathcal{V}) \subset \mathcal{V}$.*

Proof. The first statement follows immediately by the way we constructed T . For the second statement, it is clearly true that if $\sigma^{oi}(\nu) \in \mathcal{V}$ then $\sigma^{oi+1}(\nu) \in \mathcal{V}$. So, it only remains to consider branch points which are not on the critical orbit. Pick such a point and let τ be its itinerary. By Lemma 2.3.19, there is a combinatorial triod $(\tau^1, \tau^2, \tau^3) =: \mathcal{T}$ such that all $\tau^i \in \{\star\nu\} \cup (\text{orb}_\sigma(\nu) \cap \mathcal{V}_j)$ for some j and that τ is the branch itinerary of this triod. We know that $\phi_{\nu, \mathcal{T}}(\tau^i) \in \mathcal{V}$, so there are three pairwise distinct points $x_i \in V$ with $\tau(x_i) = \phi_{\nu, \mathcal{T}}(\tau^i)$. By Corollary 2.3.16 and Definition 2.3.11, the triod $(\phi_{\nu, \mathcal{T}}(\tau^1), \phi_{\nu, \mathcal{T}}(\tau^2), \phi_{\nu, \mathcal{T}}(\tau^3))$ is non-degenerate and its branch itinerary is $\sigma(\tau)$. Now Lemma 2.3.19 implies that the triod $(x_1, x_2, x_3) \subset T$ is non-degenerate and that its branch point has itinerary $\sigma(\tau)$. This settles the second part of the claim. \square

Theorem 2.3.21 (Existence and uniqueness). *Let $\nu \in \Sigma_d^\sharp$. Then there is a Hubbard tree $(T, f, \mathcal{P})_d$ that generates the kneading sequence ν . Moreover, $(T, f, \mathcal{P})_d$ is unique up to equivalence.*

Proof. Let T be the topological tree constructed by the algorithm above and let V be the set of its vertices. Now let us specify a dynamics f on T . We define f on all points in V by $f(v) = v'$, where $v' \in V$ such that $\sigma(\tau(v)) = \tau(v')$. By Lemma 2.3.20, v' exists and is unique, thus $f|_V$ is well-defined. To extend f to the whole tree T pick any two adjacent points $v_1 \neq v_2$ in V and define f on the arc $[v_1, v_2]$ such that it maps $[v_1, v_2]$

homeomorphically onto $[f(v_1), f(v_2)]$. This defines a continuous map $f : T \rightarrow T$. The construction of T via the subtrees T_i gives rise to the following partition \mathcal{P} of T : $\mathcal{P} := \{T_0, \dots, T_{d-1}, \{c_0\}\}$ and we assign to T_i the symbol i and to $\{c_0\}$ the symbol \star .

We claim that $(T, f, \mathcal{P})_d$ is a Hubbard tree of degree d . The way we defined f on marked points guarantees that c_0 is periodic (preperiodic) if and only if ν is periodic (preperiodic), and that $(T, f, \mathcal{P})_d$ generates the kneading sequence ν . Moreover, by the construction of T , every endpoint of T is contained in the f -orbit of c_0 . By Proposition 2.3.13, $f(c_0)$ is an endpoint of the tree, and by Corollary 2.3.10, f is locally injective at any point unequal to c_0 . In particular, $f|_{T_i}$ is injective. Hence f is at most d -to-1. It is not hard to see that f is surjective: every endpoint of T has a preimage in T and f is continuous. By Lemma 2.3.20, item (i), $(T, f, \mathcal{P})_d$ meets the expansivity condition of Definition 2.1.1.

It remains to show that $(T, f, \mathcal{P})_d$ is unique up to equivalence: let $(T', f', \mathcal{P}')_d$ be another Hubbard tree that generates ν . We have seen that the mutual locations of the points associated to three (pairwise) distinct iterates of ν are uniquely determined by the combinatorial triod map (Lemma 2.3.8 and Corollary 2.3.10). Thus there is a bijection ψ between the sets V and V' of marked points of T and T' such that $\tau(v) = \tau(\psi(v))$ for all $v \in V$ and if $v_1, v_2 \in V$ are adjacent so are $\psi(v_1)$ and $\psi(v_2)$. Thus the two Hubbard trees are equivalent. \square

Combining Theorem 2.3.21 and Definition 2.3.5, we get the following result:

Corollary 2.3.22 (Equivalent concepts). *For all $d \geq 2$, there is a bijection between Σ_d^\sharp and the set of equivalence classes of Hubbard trees of degree d .* \square

2.4 Admissibility

Unlike Hubbard trees in the sense of Douady and Hubbard, Definition 2.1.3 includes Hubbard trees which are not generated by any postcritically finite polynomial (cf. Figure 2.1). We are going to investigate when a Hubbard tree of degree d can be generated by a postcritically finite unicritical polynomial of degree d . We first give a necessary and sufficient topological condition. The second part of this section deals with finding a combinatorial condition that allows us to read admissibility off from the kneading sequence.

Definition 2.4.1 (Admissible Hubbard trees). *A Hubbard tree $(T, f, \mathcal{P})_d$ is called admissible if its equivalence class contains a representative that is generated by a degree d postcritically finite unicritical polynomial. A kneading sequence is admissible if its associated Hubbard tree is admissible. Both are called non-admissible if they are not admissible.*

2.4.1 The Topological Admissibility Condition

Theorem 2.4.2 (Topological condition). *A Hubbard tree $(T, f, \mathcal{P})_d$ is admissible if and only if it contains no evil branch point.*

Proof. Let p_c be a postcritically finite polynomial of degree d that has a unique critical point c_0 and let \mathbb{T} be its Hubbard tree in the sense of Douady and Hubbard. We have shown in Proposition 2.2.8 that \mathbb{T} gives rise to a minimal Hubbard tree $(T, p_c, \mathcal{P})_d$. Since p_c is locally injective at any point in $\mathbb{C} \setminus \{c_0\}$, the cyclic order of local arms at any periodic point must be preserved under the first return map of this point. This together with Proposition 2.1.23 implies that there are no evil branch points.

For the other direction, let $(T, f, \mathcal{P})_d$ be a Hubbard tree in the sense of Definition 2.1.3 that does not contain an evil branch point. It is enough to show that it is an abstract Hubbard tree of degree d in the sense of Poirier. The main result of [Po2] is that any such abstract Hubbard tree is realizable by a (unique) postcritically finite polynomial. Let us be more precise: following Poirier, we distinguish two kinds of marked points. Recall that the set of marked points is denoted by V . We call an element $v \in V$ a *Fatou vertex* if v eventually maps on the critical cycle, i.e., if there is a $j \in \mathbb{N}$ such that $f^{oj}(v) = c_0$ and c_0 is periodic. Any other element of V is called a *Julia vertex*. An *edge* of T is the closure of a component of $T \setminus V$. For any $v \in V$, let E_v be the set of edges that have v as vertex. This means that E_v corresponds to the set of local arms at v . An *abstract Hubbard tree* in the sense of Poirier is a topological tree T with dynamics f and an *angle function* \angle that, for any $v \in V$, assigns to two elements $l, l' \in E_v$ a rational number modulo 1. The tree must meet an expansivity condition: for any two Julia vertices $v \neq v'$ there is an i such that $V \cap f^{oi}(]v, v'[) \neq \emptyset$. The function \angle is required to have the following properties (all equalities are modulo 1):

- $\angle(l, l') = -\angle(l', l) \quad \forall l, l' \in E_v.$
- $\angle(l, l') = 0 \iff l = l'.$
- $\angle(l, l'') = \angle(l, l') + \angle(l', l'') \quad \forall l, l', l'' \in E_v.$
- $\angle(f(l), f(l')) = \delta(v) \cdot \angle(l, l') \quad \forall v \in V$ and $l, l' \in E_v$, where $\delta(v)$ is the local degree of f at v .
- For any periodic Julia vertex v with q arms we have that for all $l, l' \in E_v$: $\angle(l, l') = \frac{i}{q}$ for some $i = 0, \dots, q - 1$.

Most work goes into defining an angle function for $(T, f, \mathcal{P})_d$ that exhibits the properties listed above. Observe that the action of f on T meets all conditions that the dynamics has to have to fit Poirier's definition of abstract Hubbard trees. Especially, E_v can be extended to a set \mathcal{E}_v such that $f|_{\mathcal{E}_v}$:

$\mathcal{E}_v \rightarrow E_{f(v)}$ is a degree d_v covering map, with $d_v = d$ for $v = c_0$ and $d_v = 1$ otherwise. Therefore, the abstract Hubbard tree that we are going to construct has degree d . Note also that our expansivity condition implies the one of Poirier.

For any vertex $v \in V$, we define a function $a_v : E_v \rightarrow \mathbb{Q}/\mathbb{Z}$ which associates an angle to each local arm of v . The function a_v defines a cyclic order on each E_v . We require that at any point $v \in V$, the function a_v is injective. Given this function, we set

$$\angle : \{(l, l') : l, l' \in E_v \text{ for some } v \in V\} \rightarrow \mathbb{Q}/\mathbb{Z}, (l, l') \mapsto a_v(l') - a_v(l)$$

From this definition, properties (i) – (iii) follow immediately, so that it only remains to verify property (iv) in the respective cases. It is no problem to define the angle function on periodic Julia vertices to meet (v).

Let us start with periodic Julia vertices. For each orbit of periodic Julia vertices we pick one element b . Suppose that n is the period of b , q the number of its arms and $k \in \{1, 2\}$ the number of cycles of local arms. Since there are no evil branch points it follows that if $k = 2$ then b is an inner point and $q = 2$. In this case, set for the two local arms L_0, L_1 of b , $a_b(L_i) := i/2$. If $k = 1$, pick one local arm L at b and define $a_b(f^{ojn}(L)) = j/q$ for $j = 0, \dots, q-1$. For the local arms of the image $f(b)$ of b set $a_{f(b)}(f(L)) := (a_b(L) + 1/q) \bmod 1$, and for all $1 < i < n$ set $a_{f^oi(b)}(f^oi(L)) := a_{f(b)}(f(L))$. This defines \angle for all E_v with $v \in \text{orb}(b)$. For preperiodic Julia vertices, we define a_v via (finite) induction on the number of iterations they need in order to be mapped onto a point of a periodic orbit. (It is very well possible that the number of local arms at a preperiodic point is smaller than at the periodic points of its orbit.) Suppose that b is a preperiodic branch point and $a_{f(b)}$ has been defined. If b is not critical, set for any local arm L , $a_b(L) := a_{f(b)}(f(L))$. If b is critical then it has degree at most d and all local arms $L_i \subset T_i$ ($i \in \{0, \dots, d-1\}$) collapse. We set $a_b(L_i) := i/d + a_{f(b)}(f(L_i))/d$ (if existing). It is not hard to see that this definition of \angle meets requirement (iv).

It remains to define a_v at Fatou vertices. We consider the critical point c_0 first. It has at most d local arms $L_i \subset T_i$, exactly one of which is fixed under the first return map. Let this arm have the label i_0 . We set $a_{c_0}(L_i) := (i - i_0)/d \bmod 1$. For any other element of $\text{orb}(c_0)$ and for any (iterated) preimage of c_0 , we assign to its local arms the same angles that the arms at c_0 have which they are mapped to. Since this definition is in accordance with property (iv), this concludes the proof. \square

2.4.2 Closest Precritical Points

Before we give a combinatorial criterion for the admissibility of kneading sequences, we return to the study of precritical points. In the beginning of

Section 2.1.1, we defined precritical points and their steps. Now we consider special precritical points, the so-called *closest precritical points* (cf. [BS, Chapter 5]). Just as in the quadratic setting, closest precritical points play an important role when linking combinatorial properties of kneading sequences and structural properties of Hubbard trees in the general unicritical case.

Definition 2.4.3 (Closest precritical point). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $p \in T$. A precritical point $\xi_k \in T$ with $\text{STEP}(\xi_k) = k$ is called closest to p if $c_0 \notin f^{oi}([p, \xi_k])$ for all $i = 0, \dots, k-1$. If $p = c_1$ then we say that ξ_k is a closest precritical point (i.e. we drop “to c_1 ”).*

Remark 2.4.4. Note that for any positive integer k there is a unique precritical point ξ with $\text{STEP}(\xi) = k$ that is closest to p because $f^{oi}([c_1, \xi])$ is injective for all $i = 0, \dots, k$. If ξ, ξ' are precritical points closest to p with $\xi' \in]p, \xi[$ then $\text{STEP}(\xi') > \text{STEP}(\xi)$. Moreover, if $] \xi', \xi[$ contains no precritical point closest to p then it contains no precritical point ζ with $\text{STEP}(\zeta) < \text{STEP}(\xi')$: if $] \xi', \xi[$ contained such a precritical point then pick the one with the lowest step. This precritical point is closest to p in contradiction to the hypothesis. The critical point c_0 is the closest precritical point of $\text{STEP}(c_0) = 1$. And if c_0 is n -periodic then c_1 is the closest precritical point of $\text{STEP}(c_1) = n$. In this case, there are no closest precritical points of step larger than n .

Lemma 2.4.5 (Closest precritical points and ρ_ν). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with kneading sequence ν and let $j \in \mathbb{N} \cup \{-1\}$ such that $j \not\equiv 0 \pmod{N}$ if ν is \star -periodic of exact period N . Then*

$$\exists \text{ a closest precritical point } \xi \in [c_1, c_{j+1}] \text{ with } \text{STEP}(\xi) = k \iff \begin{cases} k \in \text{orb}_\rho(\rho(j) - j) & \text{if } j \geq 1 \\ k \in \text{orb}_\rho(\nu) & \text{if } j = -1 \end{cases} .$$

Proof. If ν is \star -periodic of exact period N , then for all $j = 1, \dots, N-1$ and $i \in \mathbb{N}_0$, $\rho(j+iN) = \rho(j) + iN$ and $\rho(j+iN) - (j+iN) = \rho(j) - j$. Similarly, if $\nu = \nu_1 \cdots \nu_k \overline{\nu_{k+1} \cdots \nu_{k+n}}$ is preperiodic, then for all $j = 0, \dots, n$ and $i \in \mathbb{N}_0$, $\rho(k+j+in) = \rho(k+j) + in$ and $\rho(k+j+in) - (k+j+in) = \rho(k+j) - (k+j)$. Thus we can assume that $j < N$, where N is the smallest integer such that $c_i = c_{1+N}$ for $i = 1$ in the \star -periodic case and $i = k+1$ in the preperiodic case.

Consider first the case $j \geq 1$. By definition, $\rho(j) > j$ is the index of the first entry of $\sigma^{oj}(\nu) = \nu_{j+1}\nu_{j+2}\cdots$ with $\nu_{j+k} \neq \nu_k$. This means that $\rho(j) - j$ is the smallest number such that $c_0 \in f^{o\rho(j)-j-1}([c_1, c_{1+j}])$. If $k_1 := \rho(j) - j$, then the closest precritical point ξ_1 of lowest step in $[c_1, c_{1+j}]$ has $\text{STEP}(\xi_1) = k_1$. Let ξ_2 be the closest precritical point in $[c_1, c_{1+j}]$ of the next higher step, say $\text{STEP}(\xi_2) = k_2$. (If in the \star -periodic case $j = N$, then there is no such point; this is in accordance with $\rho(N) = \infty$.) Then $f^{ok_1}(\xi_{k_2})$ is the closest precritical point of lowest step in $[c_1, c_{k_1+1}]$. As we just have

seen, this one has step $\rho(k_1) - k_1$, and thus $k_2 = \rho(k_1) = \rho(\rho(j) - j)$. By induction, the elements of $\text{orb}_\rho(\rho(j) - j)$ encode exactly the steps of the closest precritical points in $[c_1, c_{j+1}]$ (with monotonously increasing steps). Note that if c_1 is n -periodic then $\rho(i) \leq n$ for all $i < n$. This is in accordance with the fact that there is no closest precritical point in $[c_1, c_{j+1}]$ of larger step than $\text{STEP}(c_1) = n$. Thus, the if-and-only-if statement is proven for $j \neq -1$.

For the case $j = -1$, recall that $c_0 = \xi_1$ is the closest precritical point of lowest step in $[c_1, c_0]$ and $\text{STEP}(c_0) = 1$. The precritical point in $[c_1, c_2]$ of lowest step is the f -image of the closest precritical point $\xi_2 \in [c_0, c_1]$ that has the next higher step after c_0 . As we just have seen, $\text{STEP}(f(\xi_2)) = \rho(1) - 1$, and hence $\text{STEP}(\xi_2) = \rho(1)$. The claim follows by the same inductive reasoning as above. \square

One can derive a similar statement for characteristic points:

Lemma 2.4.6 (Precritical and characteristic points). *Suppose that $p \in T$ is characteristic with itinerary τ , $f(p) \neq p$, and let $p_0^0 \in T_0$ be a preimage of p . Let $k > 1$ and $N := \sup\{n \in \mathbb{N} : n \in \text{orb}_\rho(\tau)\}$. Then $k \in \text{orb}_\rho(\tau)$ if and only if there is a precritical point $\xi \in [p_0^0, p]$ with $\text{STEP}(\xi) = k \leq N$ that is closest to p .*

Proof. Let n be the exact period of p and p_0^i the preimage of p in T_i (if existing). If k_1 is the smallest number such that $c_0 \in f^{\circ k_1 - 1}([p_0^0, p])$ then the precritical point $\xi_1 \in [p_0^0, p]$ of lowest step has $\text{STEP}(\xi_1) = k_1$. Thus, $\rho_\tau(1) = k_1$ (cf. Remark 2.1.6). If $\rho(k_1) \neq \infty$, then $f^{\circ k_1 - 1}(p) \neq f^{\circ n - 1}(p)$ and thus, there is an $0 \leq i < d$ such that the interval $[p_0^i, f^{\circ k_1 - 1}(p)] \subset T_i$ is not degenerate. If j denotes the step of the precritical point of lowest step in $[p_0^i, f^{\circ k_1 - 1}(p)]$, then $j = \rho_\tau(k_1) - k_1 + 1$. And if ξ_2 is the precritical point closest to p that has the next higher step after ξ_1 , then $k_2 := \text{STEP}(\xi_2) = k_1 - 1 + j = \rho_\tau(k_1) = \rho_\tau^{\circ 2}(1)$. Now, if $\rho(k_2) \neq \infty$ then again there is an i' such that the interval $[p_0^{i'}, f^{\circ k_2 - 1}(p)] \subset T_{i'}$ is not degenerate and we can repeat the above reasoning. This shows inductively that, under the hypothesis that $k \leq N$, $k \in \text{orb}_\rho(\tau)$ if and only if there is a precritical point $\xi \in [p_0^0, p]$ closest to p with $\text{STEP}(\xi) = k$. \square

Note that the statement of Lemma 2.4.6 is not true if we drop the assumption that $\text{STEP}(\xi) \leq \sup\{n \in \mathbb{N} : n \in \text{orb}_\rho(\tau)\}$. That is, there might be a precritical point $\xi \in]p_0^0, p[$ with $\text{STEP}(\xi) > N$ that is closest to p yet $\text{STEP}(\xi) \notin \text{orb}_\rho(\tau)$.

The next lemma is an easy but useful observation.

Lemma 2.4.7 (Local arms and precritical points). *Let $b \in T$ be a characteristic point of exact period n , L be any local arm of b and let G be the*

global arm associated to L . Assume that $\xi_k \in G$ is a precritical point such that $f^{\circ n}|_{[b, \xi_k]}$ is injective. If $\text{STEP}(\xi_k) = k \leq n$ then

$$\begin{aligned} f^{\circ n}(L) = L_b(c_0) &\iff k < n && \text{and} \\ f^{\circ n}(L) = L_b(c_1) &\iff k = n. \end{aligned}$$

Proof. If $k = n$, then $f^{\circ n}([b, \xi_k]) = [b, c_1]$ implies that $f^{\circ n}(L)$ points towards the critical value. If $k < n$, then $f^{\circ k}([b, \xi_k]) = [f^{\circ k}(b), c_1]$, and since n is the exact period of b , $b \in]f^{\circ k}(b), c_1[$. Thus $f^{\circ n-k}(b) \in]b, f^{\circ n-k}(c_1)[= f^{\circ n}([b, \xi_k])$, which implies $f^{\circ n}(L) = L_b(c_0)$. The other direction now follows in both cases easily as $L_b(c_0) \neq L_b(c_1)$. \square

2.4.3 The Combinatorial Admissibility Condition

The goal of this section is to show that a kneading sequence ν (or a Hubbard tree) is admissible if and only if ν does not fail the combinatorial admissibility condition. Let us state this condition:

Definition 2.4.8 (Combinatorial condition). *Let $\nu \in \Sigma_d^\sharp$. We say that ν fails the admissibility condition for $n \in \mathbb{N}$ if the following conditions are satisfied:*

- (A0) *if ν is \star -periodic, then n is smaller than the period of ν ,*
- (A1) *$\nexists j, n' > 1 : jn' = n$ and $\rho_\nu(n) = \rho_\nu(n')$,*
- (A2) *$\exists r \in \{1, \dots, n\}$ such that $r = \rho(n) \pmod n$ and $n \in \text{orb}_{\rho_\nu}(r)$*
- (A3) *$n \notin \text{orb}_\rho(\nu)$.*

We are going to show that for any ν , failing the admissibility condition for an integer n is equivalent to the existence of an n -periodic evil branch point in the Hubbard tree $(T, f, \mathcal{P})_d$ of ν . In fact, condition (A2) makes sure that T contains a periodic point b of period n and (A1) guarantees that n is the exact period of b . The last condition is needed to turn b into an evil branch point, the first one takes into account that for Hubbard trees with periodic critical point, the exact period of every periodic branch point is strictly less than the exact period of c_0 .

Lemma 2.4.9 (Itineraries of tame points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $z \in T$ be a characteristic point of exact period n and let τ be its itinerary. Then $n \in \text{orb}_\rho(\tau)$ if and only if z is tame.*

Proof. Recall from Lemma 2.4.6 that the elements of $\text{orb}_\rho(\tau) \setminus \{1\}$ smaller or equal to n correspond exactly to the steps of the precritical points in $[z_0^0, z]$ that are closest to z , where $z_0^0 \in T_0$ is a preimage of z . Among these precritical points, let ξ_k be the one that has largest step, say k . Note that ξ_k

exists: by minimality, there is an $i \leq n$ such that $c_0 \in f^{\circ i}([z_0^0, z])$. If n was the smallest integer with this property, then the itineraries of $f^{\circ n-1}(z)$ and z would differ in exactly one position, which is impossible. Hence, $k \leq n$ and $f^{\circ n}|_{[\xi_k, z]}$ is injective. By Lemma 2.4.7, $f^{\circ n}(L_b(c_0)) = L_b(c_1)$ if and only if $k = n$. Therefore, z is tame if and only if $\text{STEP}(\xi_k) = n$, i.e., if and only if $n \in \text{orb}_\rho(\tau)$. \square

Corollary 2.4.10 (Evil branch point and $\text{orb}_\rho(\nu)$). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and let $b \in T$ be a characteristic branch point of exact period n . If ν is the kneading sequence of $(T, f, \mathcal{P})_d$, then $n \notin \text{orb}_\rho(\nu)$ if and only if b is evil.*

Proof. By Lemma 2.4.9, $n \notin \text{orb}_\rho(\tau(b))$ if and only if b is evil. Now by Corollary 2.1.26, the first n entries in $\tau(b)$ and ν are equal, and thus the claim is proven. \square

Corollary 2.4.11 (Periods of branch points and $\text{orb}_\rho(\nu)$). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with kneading sequence ν and let $b \in T$ be a tame characteristic branch point of exact period N . If $(1, 0) \rightarrow \cdots \rightarrow (n_k, s_k) \rightarrow \cdots$ is the internal address of ν then there is a k such that $(n_k, s_k) = (N, s_k)$. In fact, if $(T, f, \mathcal{P})_d$ is admissible, then the period of every branch point of T generates an entry of the internal address of ν . \square*

Next we prove that certain combinatorial properties of ν imply the existence of an n -periodic point in the associated Hubbard tree. The main argument is taken from [BS, Lemma 5.11]. But first observe the following easy fact:

Lemma 2.4.12 (ρ for bifurcation sequence). *Let $\nu \in \Sigma_d^*$. Then*

$$\nu = (\nu_1 \cdots \nu_n)^q \nu_{qn+1} \cdots \quad \text{with } \nu_{nq+j} \neq \nu_j \text{ for some } 0 < j \leq n \iff \\ \rho(jn) = \rho(n) \text{ for all } j = 1, \dots, q \quad \text{and} \quad \rho(n) \leq (q+1)n.$$

Here $(\nu_1 \cdots \nu_n)^q$ means that the word $\nu_1 \cdots \nu_n$ is repeated q -times. \square

Lemma 2.4.13 (Existence of n -periodic point). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with kneading sequence ν and let $n \in \mathbb{N}$ be such that n is smaller than the period of ν if ν is \star -periodic. Set $r := (\rho(n) \bmod n) \in \{1, \dots, n\}$ and suppose that $n \in \text{orb}_\rho(r)$. Then T contains a periodic point b with itinerary $\tau(b) = \overline{\nu_1 \cdots \nu_n}$.*

Proof. Pick $q > 0$ such that $\rho(n) = qn + r$. By Lemma 2.4.12, $r = \rho(qn) - qn$ and since $n \in \text{orb}_\rho(r)$, Lemma 2.4.5 implies that there is a closest precritical point $\xi_n \in]c_1, c_{1+qn}[$ such that $\text{STEP}(\xi_n) = n$.

If $q = 1$, then $\xi_n \in [c_1, c_{1+n}]$ and $f^{\circ n}$ maps $[\xi_n, c_1]$ injectively over itself, reversing orientation. By the intermediate value theorem, $[\xi_n, c_1]$ contains a periodic point of (not necessarily exact) period n .

For $q > 1$, consider the convex hull $H := [\xi_n, c_1, c_{1+n} \dots, c_{1+(q-1)n}]$. Since $\rho(n) = (q+1)n$ and $\xi_n \in]c_1, c_{qn}[$ is the precritical point closest to c_1 , H is mapped homeomorphically under f^{on} onto its image $[c_1, c_{1+n} \dots, c_{1+qn}] =: H'$. The triod $[\xi_n, c_1, c_{1+n}]$ is non-degenerate: if it was degenerate, then $c_{1+n} \in [c_1, \xi_n]$ because $\rho(n) > 2n$. But this implies that the orbit of c_{1+n} is infinite. Hence, H contains at least one branch point $b \in]c_1, \xi_n[$. Among the points $c_1, c_{1+n} \dots, c_{1+qn}$ only c_{1+qn} might not be contained in H (this happens if and only if $\xi_n \neq c_{1+qn}$). Since $[\xi_n, c_{1+jn}, c_{1+j'n}]$ is not degenerate with ξ_n in the middle for all $j, j' < q$ ($j \neq j'$), the point ξ_n is an endpoint of H . This together with $\xi_n \in]c_1, c_{qn}[$ implies that H and H' contain exactly the same branch points of T , and since $f^{on}|_H$ is injective, f^{on} permutes them. Now expansivity implies that the point $b \in]c_1, \xi_n[$ is the only branch point in H, H' . Moreover, b is fixed under f^{on} and $\tau(b) = \overline{\nu_1 \dots \nu_n}$. \square

Remark 2.4.14. Observe that the point b that we just constructed is characteristic: if it was not then there is an i such that $f^{oi}(b) \in]c_1, b[$ is characteristic. Since $f^{oi}(b)$ and b do not have the same itinerary by minimality, there is a precritical point $\xi_k \in]c_1, \xi_n[$ with $\text{STEP}(\xi_k) < n$. But this contradicts the fact that ξ_n is the closest precritical point with step at most n .

The converse of Lemma 2.4.13 is not true in general, i.e., the existence of an n -periodic point does not imply that $n \in \text{orb}_\rho(r)$, where $r = \rho(n) \bmod n$: let b be a tame n -periodic branch point with q arms. Suppose that there is a precritical point ξ of $\text{STEP}(\xi) = k < n$ in $[b, c_{1+(q-2)n}] \subset G_{q-1}$ (which is possible because f^{on} does not need to map G_{q-1} homeomorphically into G_0). Then $\rho((q-2)n) - (q-2)n = k$, and if $n \in \text{orb}_\rho(k)$, Lemma 2.4.5 implies that there is a closest precritical point ξ_n of step n in $[c_1, c_{1+(q-2)n}]$, in fact, $\xi_n \in]b, \xi_k[$. But this means that $f^{on}(L_{q-1}) = L_1$, contradicting Corollary 2.1.26.

The following lemma is not only an important step for the combinatorial classification of admissible kneading sequences but will also become useful when we investigate Σ_d^\sharp . We use the labeling of (local, global) arms introduced on page 32. In some sense, Lemma 2.4.15 is the complementary Lemma to [BS, Lemma 5.13]. There Bruin and Schleicher show that the existence of a precritical point $\xi \in]c_0, c_1[$ with certain properties forces the existence of a characteristic point $z \in]\xi, c_1[$. We show that the existence of a characteristic point z forces the existence of a precritical point $\xi \in]z, c_1[$ with certain properties. Bruin's and Schleicher's statement is a special case of Theorem 3.2.1.

Lemma 2.4.15 (Existence of precritical point). *Let $z \in]c_1, c_0[$ be a characteristic point of exact period n with q arms and let Q be the period of the local arm $L_1 = L_z(c_1)$. Then there is a precritical point $\xi \in [c_1, z[$ closest to z with $\text{STEP}(\xi) = Qn$.*

Proof. Recall that $Q = q$ if $f^{\circ n}(L_0) = L_1$ and $Q = q - 1$ if $f^{\circ n}(L_0) = L_0$. For any precritical point $\xi \in G_1$, $\text{STEP}(\xi) > (q - 2)n$ and by Corollary 2.1.26, $\text{STEP}(\xi) \neq (q - 1)n$ if $f^{\circ n}(L_0) = L_1$. Set $N := Qn$ and, by way of contradiction, assume that there is no precritical point $\xi \in]z, c_1]$ with $(q - 2)n < \text{STEP}(\xi) \leq N$. Then in particular, $c_1 \neq c_{1+N}$ and consequently either $f^{\circ N}([z, c_1]) \subset [z, c_1[$ or $[z, c_1, c_{1+N}]$ is a non-degenerate triod with branch point b . The first possibility contradicts finiteness of $\text{orb}(c_0)$; in the second case, finiteness of $\text{orb}(b)$ and minimality imply that $b_{1+N} := f^{\circ N}(b) \in G_b(c_{1+N})$. Set $G' := [b, b_{1+N}] \cup (T \setminus G_{b_{1+N}}(b))$. Since $f^{\circ N}$ cannot map G' homeomorphically into itself, there is a precritical point $\xi' \in G'$ with $(q - 2)n < \text{STEP}(\xi') \leq N$. Let ξ' be such that $f^{\circ N}|_{[z, \xi']}$ is injective and let $k' := \text{STEP}(\xi')$. First let us assume that $f^{\circ N}(\xi') \notin G_b(c_{1+N})$. Then the connected hull $[b, b_{1+N}, \xi', f^{\circ N}(\xi')]$ is an interval and the points b, ξ' are mapped into different directions by $f^{\circ N}$. Consequently, $]b, \xi'[$ contains a periodic point p and $\tau(p) = \tau(z)$, a contradiction to minimality. If $f^{\circ N}(\xi') \in G_b(c_{1+N})$, then $k' \neq N$. And since $k' \neq jn$, we have that $z \in f^{\circ k'}(]z, \xi']$ and hence $f^{\circ N - k'}(z) \in f^{\circ N}(]z, \xi']) =]z, f^{\circ N}(\xi')[\subset G_z(c_1)$. But this contradicts that z is characteristic.

Among all precritical points in $]z, c_1]$ that are closest to z and have step at most N , let ζ be the one of largest step. Then $f^{\circ N}|_{[z, \zeta]}$ is injective. Suppose that $\text{STEP}(\zeta) = k < N$. For $j = q - 2$ or $j = q - 1$, we have that $f^{\circ jn}(\zeta) =: \zeta'$ is precritical with $1 \leq \text{STEP}(\zeta') =: k' < n$. Let L be the local arm at z pointing to ζ' . By Lemma 2.4.7, $f^{\circ n}(L) = L_0$. We know that if $Q = q - 1$, then $f^{\circ n}(L_{q-1}) = L_1$. On the other hand, in this case $j = q - 2$ and $L = L_{q-1}$, so that $f^{\circ n}(L_{q-1}) = L_0$, a contradiction. We derive a similar contradiction for $Q = q$ and $j = q - 1$. If $Q = q$ and $j = q - 2$, then $z \in f^{\circ k'}(]z', z])$. Since $f^{\circ N}|_{[z, \zeta]}$ is injective, we have that $f^{\circ 2n}|_{[\zeta', z]}$ is injective as well (by the definition of ζ'). It follows that $f^{\circ 2n - k'}(z) \in f^{\circ 2n}(]z', z]) \subset G_1$, a contradiction to z being characteristic. \square

Theorem 2.4.16 (Combinatorial admissibility). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and ν the associated kneading sequence. Then $(T, f, \mathcal{P})_d$ and ν are admissible if and only if there is no $n \in \mathbb{N}$ such that ν fails the admissibility condition for n .*

Proof. By Theorem 2.4.2, it is enough to show that $(T, f, \mathcal{P})_d$ contains an evil branch point of exact period n if and only if ν fails the admissibility condition for the integer n .

For the first direction, let b be the evil characteristic branch point of period n and let q be the number of its arms. By Corollary 2.1.26, $\tau(b) = \overline{\nu_1 \cdots \nu_n}$ and n is smaller than the period of ν if ν is \star -periodic. Since $(T, f, \mathcal{P})_d$ is minimal, condition (A1) is trivially satisfied and Corollary 2.4.10 says that ν meets requirement (A3). (A2) follows from Lemma 2.4.15: it guarantees the existence of a precritical point $\xi \in]z, c_1]$ with $\text{STEP}(\xi) =$

$(q-1)n$ that is closest to z . Therefore, $\xi' := f^{\circ(q-2)n}(\xi) \in [c_{1+(q-2)n}, z] \subset [c_{1+(q-2)n}, c_1]$ is a precritical point closest to z and thus, closest to c_1 . Moreover, $\text{STEP}(\xi') = n$ and by Lemma 2.4.12, $n \in \text{orb}(\rho((q-2)n) - (q-2)n)$. To finish this direction it suffices to show that $\rho((q-2)n) - (q-2)n =: r \leq n$. This follows easily because $\xi' \in]c_1, c_{1+(q-2)n}[$, and thus $\rho((q-2)n) \leq \text{STEP}(\xi') + (q-2)n = n + (q-2)n$.

For the other direction, we are going to construct an evil branch point b of period n . Most work has already been done in the previous lemmas: Lemma 2.4.13 provides the existence of a characteristic point b of period n and (A1) guarantees that n is the exact period of b (cf. Lemma 2.4.12). If we show that b is a branch point then Corollary 2.4.10 tells us that it must be evil. Let us suppose that b is an inner point. Then the proof of Lemma 2.4.13 shows that $q = 1$, where q is the number such that $\rho(n) = qn + r$ with $r = 1, \dots, n$. This proof also shows that there is a closest precritical point $\xi_n \in]c_1, c_{1+n}[$ with $\text{STEP}(\xi_n) = n$ such that $b \in [c_1, \xi_n]$. Hence $\xi_n \in G_b(c_0)$, and consequently, $f^{\circ n}(L_0) = L_1$. However, $n \notin \text{orb}_\rho(\nu)$ implies that $\xi_n \notin [c_0, c_1]$ and thus, there is a precritical point $\xi_k \in [c_0, b[$ with $\text{STEP}(\xi_k) < n$ ($\xi_k = c_0$ is allowed). It follows that $f^{\circ n}|_{[b, \xi_k]}$ is injective and thus by Lemma 2.4.7, $f^{\circ n}(L_0) = L_0$, an obvious contradiction. \square

Chapter 3

The Parameter Plane

3.1 Types of Kneading Sequences

Before we investigate the structural properties of the set Σ_d^\sharp , we take a closer look at \star -periodic kneading sequences. We distinguish three kinds of such sequences that are quite different from a dynamical point of view. As a first step into this direction, we introduce special periodic sequences associated to \star -periodic kneading sequences.

3.1.1 Upper and Lower Kneading Sequences

Let ν be a \star -periodic kneading sequence of period n and degree d , and let $\text{orb}_\rho(\nu) = \{1, \dots, n', n\}$ with $\rho_\nu(n') = n$. Choosing an element $i \in \{0, \dots, d-1\}$ and replacing each \star in ν by i yields a periodic kneading sequence. This way, we can generate d distinct periodic kneading sequences, $d-1$ of which have the property that $\nu_n \neq \nu_{n-n'}$, or equivalently that n is contained in their respective ρ -orbits.

Definition 3.1.1 (Upper and lower kneading sequences). *Let $\nu \in \Sigma_d^\sharp$ be \star -periodic of exact period n and let n' be the largest element of $\text{orb}_\rho(\nu)$ smaller than n . For all $i \in \{0, \dots, d-1\}$ such that $i \neq \nu_{n-n'}$, we set $\mathcal{A}_i(\nu) := \overline{\nu_1 \cdots \nu_{n-1} i}$. These sequences are called **upper kneading sequences** of ν . For the remaining symbol i_0 , the sequence $\overline{\nu_1 \cdots \nu_{n-1} i_0} =: \overline{\mathcal{A}}(\nu)$ is called the **lower kneading sequence** of ν .*

The lower kneading sequence is characterized by the property that $n \notin \text{orb}_\rho(\overline{\mathcal{A}}(\nu))$. The above definition provides a way to associate periodic kneading sequences to \star -periodic ones. Sometimes it is necessary to go the other way round: given an n -periodic itinerary $\tau \in \{0, \dots, d-1\}^\mathbb{N}$ and an integer $N = kn > 0$, \mathcal{A}_N^{-1} denotes the unique \star -periodic sequence of period N that coincides with τ at all places except for the jN -th entries, where it has the symbol \star .

Lemma 3.1.2 (Lower kneading sequence as limit). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with \star -periodic kneading sequence ν and let $(x_k)_{k=1}^\infty$ be a sequence of points converging to c_1 . Then $\tau(x_k) \rightarrow \overline{\mathcal{A}}(\nu)$.*

Proof. Let n be the exact period of c_1 . By Proposition 2.2.3, there is a K such that $\tau(x_k) = \tau(x_{k'})$ for all $k, k' > K$. Thus $\lim_{k \rightarrow \infty} \tau(x_k) =: \tau$ exists trivially. Moreover, $\tau_i = \nu_i$ for all $i \neq kn$ and $\tau_n = \tau_{kn}$, where $k \in \mathbb{N}$. Let $\xi \in [c_0, c_1]$ be the closest precritical point such that $\text{STEP}(\xi) = m < n$ and $f^{on}|_{[\xi, c_1]}$ is injective. Then $x_k \in]\xi, c_1[$ for all $k > K$, and consequently $f^{om}(x_k) \in]c_1, c_{1+m}[$. This implies that

$$\tau_1(x_k) \cdots \tau_{n-m}(x_k) = \nu_1 \cdots \nu_{n-m} = \tau_{1+m}(x_k) \cdots \tau_n(x_k). \quad (*)$$

By Lemma 2.4.5, $m \in \text{orb}_\rho(\nu)$, and since $\nu_1 \cdots \nu_m = \tau_1(x_k) \cdots \tau_m(x_k)$, m is also contained in $\text{orb}_\rho(\tau(x_k))$. Now $(*)$ yields that $n \notin \text{orb}_\rho(\tau(x_k))$, and the first n entries of $\overline{\mathcal{A}}(\nu)$ and $\tau(x_k)$ coincide. Hence, $\overline{\mathcal{A}}(\nu) = \tau(x_k) = \tau$ for some $k > K$. \square

Lemma 3.1.2 allows us to interpret lower and upper kneading sequences the following way: suppose ν is an admissible kneading sequence, generated by a unicritical polynomial p_c of degree d with periodic critical point. Let W_c be the hyperbolic component of \mathcal{M}_d containing c . It is well known that at each of the $d-2$ co-roots of W_c exactly one periodic parameter ray lands and that exactly two rays land at its root; say they are of angles $\theta^1 < \theta^2$. In the dynamical plane of p_c , the rays at angles θ^1, θ^2 land at the same point on the critical-value Fatou component, separating the critical value and the critical point (cf. the correspondence principle of dynamic and parameter rays for Multibrot sets [E]). Now let $\theta_n^1 < \theta^1$ ($\theta_n^2 > \theta^2$) be a sequence of angles converging to θ^1 (θ^2). The lower kneading sequence of ν corresponds to the limit of kneading sequences generated by θ_n^1 and θ_n^2 (these two limits are equal). The upper kneading sequences are the limits of kneading sequences generated by angles that converge in a monotone fashion to the angles of the rays landing at the co-roots. Note that lower and upper kneading sequences were defined this way in [LS].

Lemma 3.1.3 (Existence of dynamic root). *Suppose that $(T, f, \mathcal{P})_d$ is a Hubbard tree such that its kneading sequence ν is \star -periodic. Then there is a characteristic point z such that $\tau(z) = \overline{\mathcal{A}}(\nu)$. Moreover, $]z, c_1[$ contains no characteristic point.*

Proof. Let n be the period of ν and $I := \{z \in T : \tau(z) = \overline{\mathcal{A}}(\nu)\}$. By Lemma 3.1.2 and Proposition 2.2.3, $I \neq \emptyset$. It suffices to show that I contains a periodic point; by minimality, this point is characteristic.

If I contains a branch point, then this point is periodic by expansivity. If I contains no branch point, then either $I = [z, c_1[$ or $I =]z, c_1[$. In the first case, we have that z is periodic: if $f^{on}([z, c_1]) \supsetneq [z, c_1]$, then $f^{on}(z) \notin I$ has

itinerary $\overline{A}(\nu)$, contradicting maximality of I , and if $f^{\circ n}([z, c_1]) \subsetneq [z, c_1]$, we have that $f^{\circ n}(z) \in]z, c_1[$. Thus by continuity, there is a $y \notin I$ close to z such that $[y, z]$ contains no precritical point of step at most n and $f^{\circ n}(y) \in I$. But this means that $\tau(y) = \tau(z)$, in contradiction to maximality of I . If $I =]z, c_1[$, then z is precritical by continuity. If I contains no periodic point then $f^{\circ n}(p) \in]p, c_1[$ for all $p \in I$. Since z is not locally attracting, z cannot be on the critical orbit and hence, is not periodic. Thus we must have that $f^{\circ n}(z) \in I$, which contradicts that I contains no (pre-)critical point.

The second claim follows by minimality because $]z, c_1[$ contains no pre-critical point. \square

The following combinatorial statement about periodic itineraries is part of Lemma 2.3.4. However there, we referred to [BS] for a combinatorial proof. Here, we want to give an alternative proof that uses Hubbard trees.

Lemma 3.1.4 (ρ -orbit and periodic sequences). *Let $\tau \in \{0, \dots, d-1\}^{\mathbb{N}}$. The sequence τ is periodic of exact period n with $n \in \text{orb}_\rho(\tau)$ if and only if $\text{orb}_\rho(\tau)$ is finite and its largest element is n . However, if τ is n -periodic and $n \notin \text{orb}_\rho(\tau)$, then $|\text{orb}_\rho(\tau)| = \infty$.*

Proof. Obviously, if τ is n -periodic then $\rho_\tau(n) = \infty$ and $|\text{orb}_\rho(\tau)| < \infty$. For the other direction we only have to show that n equals the exact period \tilde{n} of τ . Consider a Hubbard tree that contains an \tilde{n} -periodic point p with itinerary τ (such a tree exists by Theorem 2.3.21 and Lemma 3.1.3). Note that n must be a multiple of \tilde{n} (cf. the proof of Corollary 2.1.13). Let us assume that $n = j_0 \tilde{n} > \tilde{n}$ and let p_0^0 be the preimage of p contained in T_0 . There is a precritical point $\xi \in]p, p_0^0[$ with $\text{STEP}(\xi) \leq \tilde{n}$ such that $f^{\circ \tilde{n}}|_{[p, \xi]}$ is injective. This precritical point has step strictly smaller than \tilde{n} because otherwise $\tilde{n} \in \text{orb}_\rho(\tau)$ and the largest entry of $\text{orb}_\rho(\tau)$ would be $\tilde{n} < n$. Hence, Lemma 2.4.7 yields that for all $k \in \mathbb{N}$, there is an interval $I_k \subset [p, c_0]$ such that $f^{\circ k \tilde{n}}(I_k) \subset [p, c_0]$. However by Lemma 2.4.6, $n \in \text{orb}_\rho(\tau)$ implies that there is a precritical point $\xi' \in]p, p_0^0[$ with $\text{STEP}(\xi') = n$ such that $f^{\circ n} = f^{\circ j_0 \tilde{n}}$ maps $[p, \xi']$ homeomorphically onto $[p, c_1]$. But this contradicts the existence of the interval I_{j_0} .

For the last statement, assume that τ is n -periodic with $n \notin \text{orb}_\rho(\tau)$ and $|\text{orb}_\rho(\tau)| < \infty$. Then $\text{orb}_\rho(\tau)$ contains a last entry N and $N = j_0 n$ for some $j_0 > 0$, because otherwise the exact period of ν would be smaller than n . But as we just have seen, this implies that ν has exact period N , a contradiction. \square

Corollary 3.1.5 (Exact period and ρ -orbit). *Let $\nu \in \Sigma_d^*$ be a periodic sequence of period n . If $n \in \text{orb}_\rho(\nu)$, then n is the exact period of ν . \square*

Corollary 3.1.6 (Period of upper and lower sequence). *Let ν be \star -periodic of exact period n . Then the exact period of its associated lower kneading*

sequence $\overline{\mathcal{A}}(\nu)$ divides n whereas the exact period of any upper kneading sequence equals n .

Proof. Let m be the exact period of $\overline{\mathcal{A}}(\nu)$. By definition, $\overline{\mathcal{A}}(\nu)$ is periodic of period n . Now we can argue exactly the same way as in Lemma 2.1.13 to show that $m|n$. The second statement follows by the definition of upper kneading sequences and Corollary 3.1.5. \square

3.1.2 Primitive, Bifurcation and Backward Bifurcation Sequences

Besides upper and lower kneading sequences, there is another type of sequences which is closely related to a given ν . Unlike upper and lower kneading sequences, we can associate such a sequence to any \star -periodic or periodic element $\nu \in \Sigma_d^*$.

Let $\mu = \overline{\mu_1 \dots \mu_n} \in \Sigma_d^*$ be periodic or \star -periodic. For any $q \geq 2$, we set, if $\mu_n \neq \star$,

$$B_i^q(\mu) := \overline{(\mu_1 \dots \mu_n)^{q-1} \mu_1 \dots \mu_{n-1} i} \quad \text{with } i \neq \mu_n,$$

and, if $\mu_n = \star$,

$$B_i^q(\mu) := \overline{(\mu_1 \dots \mu_{n-1} i)^{q-1} \mu_1 \dots \mu_{n-1} \star} \quad \text{with } \overline{\mu_1 \dots \mu_{n-1} i} = \mathcal{A}_i(\mu).$$

The expression $(\mu_1 \dots \mu_n)^{q-1}$ means that the finite word $\mu_1 \dots \mu_n$ is repeated $(q-1)$ times. Any periodic or \star -periodic element $\mu \in \Sigma_d^*$ defines $d-1$ pairwise distinct sequences B_i^q .

If $\mu \in \{0, \dots, d-1\}^{\mathbb{N}}$ is n -periodic and $n \notin \text{orb}_\rho(\mu)$, then we also allow that $q = 1$. Let us take a closer look at this special situation: if $\mu^\star = \overline{\mu_1 \dots \mu_{n-1} \star}$, then $\mu = \overline{\mathcal{A}(\mu^\star)}$ and $B_i^1(\mu) = \mathcal{A}_i(\mu)$. This means that each 1-bifurcation sequence of the lower kneading sequence of μ^\star yields one of the $d-1$ upper kneading sequences of μ .

If μ is \star -periodic such that μ and $\overline{\mathcal{A}}(\mu)$ have the same period, we additionally define for $q \geq 2$ the following sequence:

$$\overline{B}^q(\mu) := \overline{(\mu_1 \dots \mu_n)^{q-1} \mu_1 \dots \mu_{n-1} \star}, \quad \text{where } \overline{\mu_1 \dots \mu_n} = \overline{\mathcal{A}}(\mu).$$

Definition 3.1.7 (Types of kneading sequences). *Let ν be a \star -periodic kneading sequence with exact period n and let $m \in \mathbb{N}$ such that $\rho_\nu(m) = n$. We say that ν is*

- a bifurcation sequence if $m | n$,
- a primitive sequence if $m \nmid n$ and $\overline{\mathcal{A}}(\nu)$ has exact period n .
- a backward bifurcation sequence if $m \nmid n$ and $\overline{\mathcal{A}}(\nu)$ has period strictly dividing n .

It follows immediately from this definition that every \star -periodic kneading sequence ν is of exactly one of the three types. The type is completely encoded in the lower kneading sequence $\overline{\mathcal{A}}(\nu)$ as we can see in the subsequent lemma.

Lemma 3.1.8 (Lower sequence determines type). *Let ν be a \star -periodic kneading sequence. Then ν is either primitive or a bifurcation sequence or a backward bifurcation sequence. Moreover, if n denotes the exact period of ν , \overline{n} the exact period of $\overline{\mathcal{A}}(\nu)$ and if $m \in \text{orb}_\rho(\nu)$ such that $\rho_\nu(m) = n$, then*

$$(i) \ \nu \text{ is primitive} \iff \overline{n} = n.$$

$$(ii) \ \nu \text{ is a bifurcation sequence} \iff \overline{n} = m \iff \overline{\mathcal{A}}(\nu) = \mathcal{A}_i(\nu'), \text{ where } \nu' = \overline{\nu_1 \cdots \nu_{m-1} \star}.$$

$$(iii) \ \nu \text{ is a backward bifurcation sequence} \iff \overline{n} \notin \text{orb}_\rho(\nu) \iff \overline{\mathcal{A}}(\nu) = \overline{\mathcal{A}}(\nu'), \text{ where } \nu' = \overline{\nu_1 \cdots \nu_{\overline{n}-1} \star} \text{ is primitive.}$$

In the first and the last case $|\text{orb}_\rho(\overline{\mathcal{A}}(\nu))| = \infty$ whereas in the bifurcation case, $|\text{orb}_\rho(\overline{\mathcal{A}}(\nu))| < \infty$.

Proof. First observe that, since we have not changed the first $n-1$ entries of ν , the ρ -orbits of ν and $\overline{\mathcal{A}}(\nu)$ agree up to the entry m . For future reference, we denote this property by (*).

The statement about primitive sequences follows immediately from Definition 3.1.7.

If ν is a bifurcation sequence, then $n/m =: l \in \mathbb{N}$ and (*) yields that $\overline{\mathcal{A}}(\nu) = (\nu_1 \cdots \nu_m)^l$. Since $m \in \text{orb}_\rho(\overline{\mathcal{A}}(\nu))$, $m = \overline{n}$ by Lemma 3.1.4. This however implies that $\overline{\nu_1 \cdots \nu_m} = \mathcal{A}_i(\nu')$ for the \star -periodic kneading sequence $\nu' = \overline{\nu_1 \cdots \nu_{m-1} \star}$. The converse directions follow from Corollary 3.1.6 and the definition of bifurcation sequences.

Now suppose that ν is a backward bifurcation sequence. Then $\overline{n} < n$ by definition, and if $\overline{n} \in \text{orb}_\rho(\nu)$ then $\overline{n} = m$, and ν would be a bifurcation sequence. Consider the \overline{n} -periodic kneading sequence $\nu' = \overline{\nu_1 \cdots \nu_{\overline{n}-1} \star}$. We have that either $\overline{\mathcal{A}}(\nu) = \mathcal{A}_i(\nu')$ or $\overline{\mathcal{A}}(\nu) = \overline{\mathcal{A}}(\nu')$. The first option is not possible because then $\overline{n} \in \text{orb}_\rho(\overline{\mathcal{A}}(\nu))$ and thus $\overline{n} \in \text{orb}_\rho(\nu)$ by (*). So it only remains to prove that ν' is primitive. If it was not, then $\overline{\mathcal{A}}(\nu') = \mathcal{A}_i(\nu'')$ for some \star -periodic sequence ν'' of period smaller than \overline{n} and consequently, \overline{n} was not the exact period of $\overline{\mathcal{A}}(\nu)$. For the reverse direction, note that if $\overline{\mathcal{A}}(\nu) = \overline{\mathcal{A}}(\nu')$ and $\nu' = \overline{\nu_1 \cdots \nu_{\overline{n}-1} \star}$ is primitive then $\overline{n} \notin \text{orb}_\rho(\overline{\mathcal{A}}(\nu))$. Thus $\overline{n} \notin \text{orb}_\rho(\nu)$ by (*). The last implication holds because every \star -periodic kneading sequence is of exactly one of the three described types.

Finally, the last statement is an immediate consequence of Lemma 3.1.4. \square

Corollary 3.1.9 (B_i^q, \overline{B}^q are bifurcation sequences). *Let $\nu \in \Sigma_d^\sharp$ be \star -periodic. Then ν is a bifurcation sequence if and only if there is a \star -periodic*

μ such that $\nu = B_i^q(\mu)$ for some $i \in \{0, \dots, d-1\}$. If n is the exact period of ν , then the exact period m of μ equals the last entry of $\text{orb}_\rho(\nu)$ before n , and $q = n/m$.

Furthermore, ν is a backward bifurcation sequence if and only if there is a primitive kneading sequence μ such that $\nu = \overline{B}^q(\mu)$. If n, m are the exact periods of ν, μ , then again $q = n/m$. \square

Remark 3.1.10. With this result in mind, observe the following terminology: we call the sequences $B_i^q(\mu)$ the q -th bifurcation sequences of μ . The sequence $\overline{B}^q(\mu)$ is called the q -th backward bifurcation sequence of μ . In the first case, μ might be \star -periodic or periodic.

Let us give some motivation for these names. Assume the \star -periodic kneading sequence μ is admissible; more precisely, assume that μ is generated by the center of a hyperbolic component W_c in the Multibrot set \mathcal{M}_d and let W_c^i be the sector of W_c labeled by the symbol i (cf. the discussion on page 66). Then the kneading sequence $B_i^q(\mu)$ corresponds to all hyperbolic components that *bifurcate* from the sector W_c^i at an internal angle p/q (in lowest terms) for some p . Let us also give some justification for the term *backward bifurcation*: we have seen that any bifurcation sequence is of the form $B_i^q(\mu) = \mathcal{A}_{qn}^{-1}(\mathcal{A}_i(\mu))$ and any backward bifurcation sequence of the form $\overline{B}^q(\mu) = \mathcal{A}_{qn}^{-1}(\overline{\mathcal{A}}(\mu))$, where μ is \star -periodic of exact period n and in the second case, μ is primitive. So \overline{B}^q is also a bifurcation sequence, just that we use the lower kneading sequence to build it as opposed to the upper one for B_i^q . As above, let W_c be the hyperbolic component associated to μ . The lower kneading sequence of μ is the limit of sequences generated by angles outside of the wake of W_c that converge to the angle of a ray landing at the root of W_c . So to speak, it is the periodic sequence *just before* μ and thus from μ 's point of view it is the first sequence looking *backwards* to the main hyperbolic component.

The following statement is an immediate corollary of Lemma 2.4.9 and the definition of primitive sequences.

Corollary 3.1.11 ($\overline{\mathcal{A}}(\nu)$ and evil points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and z be the characteristic point of an n -periodic orbit. Then the local arm $L_z(c_0)$ at z is fixed under $f^{\circ n}$ if and only if $\tau(z) = \overline{\mathcal{A}}(\nu)$ for a primitive sequence ν .*

In particular, if z is a branch point, then z is evil if and only if its itinerary equals the lower kneading sequence of a primitive sequence. \square

3.2 A Forcing Relation

We start this section by investigating the arrangement of periodic orbits in a Hubbard tree $(T, f, \mathcal{P})_d$. More precisely, we will show that for any

characteristic point $z \in T$, there is a characteristic point $z' \in]z, c_1]$ such that the interval $]z, z'[$ contains no further characteristic points. Moreover, $\tau(z')$ is a bifurcation sequence of $\tau(z)$. In the second part of this section, we compare non-equivalent Hubbard trees of degree d both of which have a characteristic point with the same itinerary. The forcing relation that we obtain in this process will give rise to a partial order on the set Σ_d^\sharp . Recall that we only regard minimal Hubbard trees although we usually do not state this explicitly (see Section 2.2).

3.2.1 Arrangement of Characteristic Points

Theorem 3.2.1 (Bifurcation points in T). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with kneading sequence ν , let $z \in T$ be a characteristic n -periodic point with itinerary τ , $z \neq c_1$, and let Q be the period of $L_z(c_1)$. Then there is a characteristic point $z' \in]z, c_1]$ such that $]z, z'[$ contains no characteristic point. Moreover, either $z' \in]z, c_1[$ and $\tau(z') = B_i^{Qn}(\tau)$, or $z' = c_1$ and $\nu = \mathcal{A}_{Qn}^{-1}(\tau)$.*

Proof. By Lemma 2.4.15, there is a precritical point $\xi \in]z, c_1]$ closest to z with $\text{STEP}(\xi) = N$. If $\xi = c_1$ we are done; otherwise pick ξ' so that $[\xi, \xi']$ is the longest arc in $[\xi, c_1]$ such that $f^{\circ N}|_{[\xi, \xi']}$ is injective. It suffices to find an N -periodic characteristic point in $]\xi, \xi']$. If $f^{\circ N}(\xi') \notin G_\xi(c_1)$, then $]\xi, \xi'[$ contains a fixed point of $f^{\circ N}$ by the intermediate value theorem. If $f^{\circ N}(\xi') \in G_\xi(c_1)$, we distinguish two cases:

First suppose that $\xi' \neq c_1$. Then ξ' is precritical of $\text{STEP}(\xi') =: k' < N$ and $k' \neq jn$ for all j . It follows that $z \in f^{\circ k'}(]z, \xi']) =]f^{\circ k'}(z), c_1[$ and thus, $f^{\circ N-k'}(z) \in f^{\circ N}(]z, \xi']) =]z, c_1] \cup [c_1, f^{\circ N}(\xi')] \subset G_z(c_1)$, contradicting that z is characteristic. If $\xi' = c_1$ then $c_{1+N} \neq c_1$, and c_{1+N} must be contained in a subtree branching off at $b \in]\xi, c_1[$ by finiteness of $\text{orb}(c_1)$. If $b_{1+N} := f^{\circ N}(b) \notin G_b(c_{1+N})$, we find a periodic point of period N in $[b, c_1[$. Otherwise, consider the set $\hat{G}_b := [b, b_{1+n}] \cup (T \setminus G_{b_{1+N}}(b))$. \hat{G}_b cannot be mapped homeomorphically into itself by $f^{\circ N}$ and thus contains a precritical point ζ of $\text{STEP}(\zeta) =: l < N$. We can assume that $f^{\circ N}|_{[\xi, \zeta]}$ is injective. If $f^{\circ N}(\zeta) \notin G_b(c_{1+N})$ then there is a periodic point $p \in]b, \zeta[$. Let $p' \in]b, c_1[$ be the characteristic point of $\text{orb}(p)$. Since $f^{\circ N}|_{[\zeta, c_1]}$ is injective, $\tau(p) = \tau(p')$, a contradiction to minimality. If on the other hand $f^{\circ N}(\zeta) \in G_b(c_{1+N})$, then $z \in f^{\circ l}(]z, \zeta])$ implies that $f^{\circ N-l}(z) \in f^{\circ N}(]z, \zeta]) =]z, c_1] \cup [c_1, f^{\circ N}(\zeta)] \subset G_z(c_1)$, which is not possible as z is characteristic.

This shows the existence of a periodic point $z' \in [\xi, \xi']$ with itinerary $B_i^N(\tau)$ and exact period N . Before we prove that z' is characteristic, note that there is no characteristic point $z'' \in]z, z'[$: since $f^{\circ N}$ maps $[\xi, z']$ homeomorphically onto $[z', c_1]$, there is no characteristic point in $]\xi, z'[$ (ξ is the precritical point of step N from above). And if $]z, \xi[$ contained a characteristic point z'' then the interval $[z'', \xi]$ covers itself under $f^{\circ N}$. This

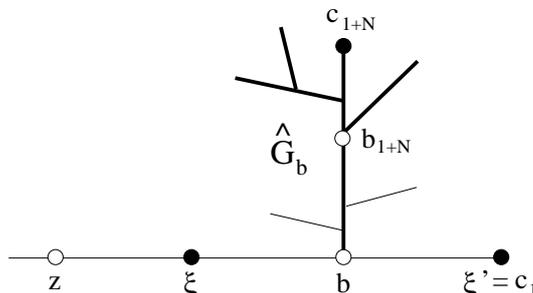


Figure 3.1: A sketch of the situation in Theorem 3.2.1 when $\xi' = c_1$ and $c_{1+N} \in G_\xi(c_1)$. The subset \hat{G}_b is drawn in thick lines.

yields a periodic point in $]z, \xi[$ of period dividing N . Hence $\tau(p) = \tau(z)$, in contradiction to minimality.

Now if z' is not characteristic then there is an $l < N$ such that $]z, z'[,$ covers itself under f^{ol} , which yields a periodic point of period dividing l in $]z, z'[,$. If z'' denotes the periodic point of lowest period in $]z, z'[,$ then z'' is characteristic because otherwise we find a periodic point of lower period in $]z, z'[,$. But we have already seen that there is no characteristic point in $]z, z'[,$. \square

3.2.2 Orbit Forcing

Many of the subsequent proofs will be based on iterating triods in T homeomorphically. Suppose a triod Y is given as the convex hull of the three pairwise distinct points $x, y, z \in T$; we call these three points the *generating points* of the triod. It follows that at least two of the generating points must be in the boundary of Y , and Y is non-degenerate (i.e. not an interval) if and only if $\partial Y = \{x, y, z\}$. Observe that a triod $[x, y, z]$ can be pushed forward homeomorphically if and only if c_0 is not contained in its interior. If it is and we want to push forward $[x, y, z]$ homeomorphically, we have to *chop* it first. The chopping must happen in such a way that the resulting triod does not longer contain c_0 in its interior and is topologically the same as the original one. More precisely, we require that it contains two points of $\{f(x), f(y), f(z)\}$ and the third one is *chopped off*, that is, replaced by a point p distinct from the two not chopped points. The mutual location of the new generating points must be the same as the one of x, y, z . Usually, it suffices to choose $p = c_0$. However, sometimes we want the endpoints to have specific itineraries and thus we have to replace the separated point in a more tricky way. Note that this procedure of chopping a triod is not possible if and only if either one of the generating points is mapped onto c_0 and the images of the two other generating points are in two distinct

global arms of c_0 or the generating points are mapped into three pairwise distinct global arms of c_0 . In the first case $[x, y, z]$ is degenerate, in the second case $[x, y, z]$ is a non-degenerate triod. The two described events are called *stop case*, denoted by STOP, because it prevents any further iteration. Note that the itineraries of x, y, z determine the type of the STOP uniquely: $[x, y, z]$ is degenerate if and only if $\star \in \{\tau_1(x), \tau_1(y), \tau_1(z)\}$. (Compare also the definition of the combinatorial triod map in Section 2.3.2.) Let us make this more precise.

Definition 3.2.2 (Chopping map). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree, x, y, z be three pairwise distinct points in T and let $p \in T$ be a characteristic point. By possibly extending T , we can assume that for all $T_i \neq \emptyset$, the preimage $p_0^i \in T_i$ of p exists. We define the formal chopping map φ_p for the triod $[x, y, z] =: Y$ by*

$$[x, y, z] \mapsto \begin{cases} [f(x), f(y), f(z)] & x, y, z \in T_i \text{ for some } i \\ [f(x), f(y), p] & x, y \in T_i, z \notin T_i \text{ for some } i \\ [f(x), p, f(z)] & \text{if } x, z \in T_i, y \notin T_i \text{ for some } i \\ [p, f(y), f(z)] & y, z \in T_i, x \notin T_i \text{ for some } i \\ \text{STOP} & \exists i_1 \neq i_2: x \in T_{i_1}, y \in T_{i_2}, z \notin T_{i_1} \cup T_{i_2} \end{cases}.$$

Whenever the image $f(a)$ of a generating point a is replaced by p , we say that the point a has been chopped off.

The label p of φ_p indicates that whenever one of the generating points of (the image of) Y is chopped off then it is replaced by a preimage of p . The point p is called the replacing point.

For any generating point a of Y , we define $\phi_{Y,p}^{oi}(a)$ to be the generating point of $\varphi_p^{oi}(Y)$ that a is mapped to.

We call the map φ_p “formal” chopping map because we only want to apply it if the respective generating points of $\varphi_p(Y)$ and Y have the same mutual location. In this case, we say that φ_p is *well-defined*. Whether this holds depends, of course, on the choice of p (and the triod in question). The choice $p = c_1$ always yields a well-defined chopping map (for any triod in T).

If for a given triod $[x, y, z] =: Y$, φ_p is well-defined for all iterated images of Y and $\varphi_p^{oi}(Y) \neq \text{STOP}$ for all $i \in \mathbb{N}$, we say that Y can be *iterated indefinitely*. In this case, either Y is non-degenerate and the branch point b is not (pre-)critical or it is degenerate and the point in the middle, say y , is never chopped off. In the latter situation, $\tau_i(y) \neq \star$ for all $i \in \mathbb{N}$. Observe that in the first case all three points must eventually get chopped off because otherwise the branch point b and one of the three points x, y, z (the one which is never chopped off) would have the same itinerary, contradicting minimality. The same is true for the two endpoints in the degenerate case.

The following proposition describes a forcing relation between different Hubbard trees. Together with the order on Σ_d^\sharp that we are going to introduce in Section 3.3.1, it is a combinatorial analogue of the correspondence of dynamical and parameter rays for polynomials (cf. [L], [M4] and [E]).

Proposition 3.2.3 (Orbit forcing). *Let $(T, f, \mathcal{P})_d$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ be two Hubbard trees with \star -periodic kneading sequences ν and $\tilde{\nu}$.*

- (a) *Let $p \in T$, $\tilde{p} \in \tilde{T}$ be two periodic characteristic points of exact period n such that $\tau_i(p) = \tau_i(\tilde{p})$ for all $i < n$ ($p = c_1$ or $\tilde{p} = \tilde{c}_1$ is allowed). If $z \in]c_0, p[$ is a characteristic point, then there is a characteristic point $\tilde{z} \in]\tilde{c}_0, \tilde{p}[\subset \tilde{T}$ such that z and \tilde{z} have the same itinerary, the same type and the same number of arms.*
- (b) *Suppose that $p \in T$ is a periodic non-precritical point of exact period n and itinerary $\mathcal{A}_i(\tilde{\nu})$. Let $b \in T$ be the point where the arc containing p branches off from $[0, c_1]$, i.e., $[0, b] = [0, c_1] \cap [0, p]$ ($b = p$ is allowed). Denote by \tilde{T} the Hubbard tree associated to $\tilde{\nu}$. If $z \in]0, b[$ is a characteristic point, then there is a characteristic point \tilde{z} in \tilde{T} such that z and \tilde{z} have the same itinerary and are of the same type. Moreover, if z is chosen so that $]z, b[$ contains a further characteristic point, then z and \tilde{z} also have the same number of arms.*

Proof. In order to tell points in the two Hubbard trees apart, all points in \tilde{T} are marked by \sim . In the following, we give a detailed proof of case (b) and note the differences for the various subcases of (a) as footnotes. Note however that most of the reasoning for case (a) is identical to the presented one. To read the stated proof for case (a), set $b := p$.

We can assume that T contains all d preimages p_0^i of p : if it does not contain a preimage p_0^i , then we attach an arc $[c_0, p_0^i]$ to the tree T such that $[c_0, p_0^i]$ is mapped homeomorphically onto $[c_1, p]$. Strictly speaking, the extended tree, also denoted by T , is not a Hubbard tree anymore as not all of its endpoints are contained in $\text{orb}(c_0)$. However, this tree inherits all other properties of (minimal) Hubbard trees. Denote by T_{p_0} the subtree of T which maps onto $T \setminus G_b(c_0)$ under f . T_{p_0} contains c_0 and all preimages of p . Note that if $p \in T_{p_0}$ then there is no characteristic point in $[c_0, b]$ and the statement is empty. So from now on, we assume that $p \notin T_{p_0}$. Figure 3.2 illustrates the location of the mentioned points in the tree T .

We are going to construct closed intervals $I_k \subset T \setminus T_{p_0}$ with endpoints in $P = \{p_0^0, \dots, p_0^{d-1}, p, f(p), \dots, f^{o(n-1)}(p)\}$ such that $f^{ok}(z) \in I_k$ and $f|_{I_k}$ is injective. Any point in T_{p_0} maps into the global arm of z which contains c_1 . Since z is characteristic, it follows that $\text{orb}(z) \cap T_{p_0} = \emptyset$. Hence $z \in [p_0^0, p]$ implies that for any $l \in \mathbb{N}$, $f^{ol}(z)$ is contained in the tree spanned by the orbit of p minus T_{p_0} .¹ We define the I_k iteratively:

¹This implies that in case sl (a), T contains all relevant preimages p_0^i of p (cf. Lemma 2.1.20), and thus does not have to be extended.

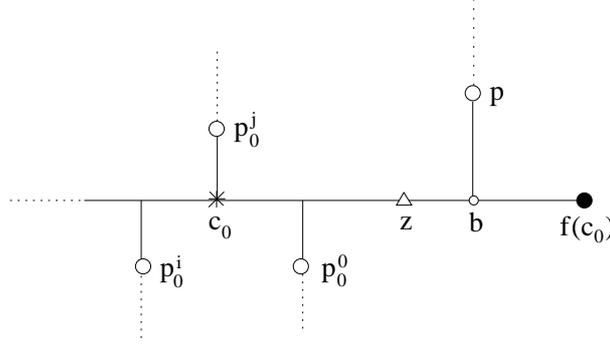


Figure 3.2: Possible location of points in the proof of Proposition 3.2.3, case (b).

For $k = 0$, we set $I_0 := [p_0, p]$. This interval contains z and $f|_{I_0}$ is injective. Now suppose that $I_k = [x, y]$ has already been defined. Then $f^{\circ k+1}(z) \in [f(x), f(y)]$. We set $I_{k+1} := [f(x), f(y)]$ if $c_0 \notin [f(x), f(y)]$. Otherwise, there are i_x, i_y such that both $[f(x), p_0^{i_x}]$ and $[f(x), p_0^{i_y}]$ do not contain c_0 , and exactly one of them contains $f^{\circ k+1}(z)$. Pick I_{k+1} to be this interval. Clearly, I_{k+1} has all the required properties.

Let m be the period of z . We repeat the whole construction until $I_{jm} = I_{j'm}$ for some $0 \leq j < j'$. Such j, j' exist because P is a finite set. For all $k > j'm$, we set $I_k := I_{k \bmod (j'-j)m}$. This defines the intervals I_k for all $k \in \mathbb{N}_0$.

Now we are going to define analogous closed intervals \tilde{I}_k in \tilde{T} . The critical value $\tilde{p} = \tilde{c}_1$ has itinerary $\tilde{\nu}$ and $\tilde{p}_0^i = \tilde{c}_0$ for all $i = 0, \dots, d-1$. For all $k \in \mathbb{N}_0$, set \tilde{I}_k to be the closed arc with endpoints in $\{\tilde{c}_0, \tilde{c}_1, \dots, f^{\circ(n-1)}(\tilde{c}_1)\}$ corresponding to the ones of I_{k+jm} . Since p and \tilde{c}_1 have the same itinerary except for the \star , \tilde{f} maps \tilde{I}_k homeomorphically to an arc containing \tilde{I}_{k+1} .²

Consider the set

$$\tilde{S}_j = \{x \in [\tilde{p}_0^0, \tilde{p}] : \tilde{f}^{\circ k}(x) \in \tilde{I}_k \text{ for all } k < j\}.$$

Since $\tilde{I}_{k+1} \subset f(\tilde{I}_k)$, we have that $\tilde{S}_{j+1} \subset \tilde{S}_j$, and for each $j \in \mathbb{N}$, the set \tilde{S}_j is a compact interval. Hence $\tilde{S} = \bigcap_{j=1}^{\infty} \tilde{S}_j$ is also non-empty, compact and connected, and for all x in the interior of \tilde{S} , we have $\tau(x) = \tau(z)$. By definition $\tilde{f}^{\circ lm}(\tilde{S}) = \tilde{S}$, where $l = j' - j$. If the interior of \tilde{S} contains a

²If $\tilde{p} \neq \tilde{c}_1$, then $\partial \tilde{I}_k \subset \{\tilde{p}_0^0, \dots, \tilde{p}_0^{d-1}, \tilde{p}, \dots, \tilde{f}^{\circ(n-1)}(\tilde{p})\}$. If $p = c_1 \in T$, $c_0 \in \partial I_k$ and $I_k \subset \tilde{T}_i$, then we choose the point \tilde{p}_0^i to be the endpoint of \tilde{I}_k that corresponds to the endpoint c_0 of I_k . Note that all preimages of \tilde{p} that are needed to define the intervals \tilde{I}_k are contained in \tilde{T} (even if $\tau_n(p) \neq \tau_n(\tilde{p})$): p_0^i (which might be equal to c_0) is only an endpoint of I_k if $(\text{orb}(p) \setminus f^{\circ(n-1)}(p)) \cap T_i \neq \emptyset$. Since $\tau_i(p) = \tau_i(\tilde{p})$ for all $i < n$, this implies that $(\text{orb}(\tilde{p}) \setminus \tilde{f}^{\circ(n-1)}(\tilde{p})) \cap \tilde{T}_i \neq \emptyset$ and hence, $\tilde{p}_0^i \in \tilde{T}$ by Lemma 2.1.20.

periodic point, we are done. Otherwise consider the boundary of \tilde{S} . Suppose first that it consists of two distinct points s_1, s_2 . Then $f^{olm}(s_1) = s_1$ and $f^{olm}(s_2) = s_2$ and we are done unless s_1 and s_2 are both on the critical orbit (which in this case must be periodic). Since the interior of \tilde{S} contains no precritical points and no branch point of T , there is a $k < lm$ such that $f^{ok}(s_1) = s_2$ and $f^{ok}(s_2) = s_1$. But this forces a periodic point in the interior of \tilde{S} , which does not exist. It remains to consider the case where $\tilde{S} = \{\tilde{s}\}$ and \tilde{s} is on the (again periodic) critical orbit. Now the way we constructed \tilde{S} implies that the critical point is the limit of precritical points. Consequently, $(T, \tilde{f}, \tilde{\mathcal{P}})_d$ does not have attracting dynamics, a property it must have as minimal Hubbard tree. This shows the existence of a periodic point $\tilde{z} \in \tilde{S}$ of period lm and itinerary $\tau(z)$. And since m was the exact period of $\tau(z)$, minimality implies that \tilde{z} is periodic and has exact period m .

The point \tilde{z} is characteristic: suppose it was not, then there is an $l \leq m$ such that $\tilde{z}_l := f^{o(l-1)}(\tilde{z}) \in]\tilde{z}, \tilde{p}[$. Hence the three points \tilde{z}_l, \tilde{z} and \tilde{c}_1 form a degenerate triod \tilde{Y} with \tilde{z}_l in the middle. In the Hubbard tree T , the points z_l, z, p form a degenerate triod Y with z in the middle, since z was characteristic. Note that $\varphi_p^{on}(Y)$ and $\varphi_{\tilde{c}_1}^{on}(\tilde{Y})$ are well-defined for all n , in particular, the STOP case does not eventually occur: in T , no image of z is contained in T_{p_0} , so the n -th image is well-defined unless an earlier one equaled STOP. But this is not possible because the orbits of the three generating points of Y are disjoint from $\text{orb}(c_0)$. In \tilde{T} , only \tilde{c}_1 could eventually be mapped onto \tilde{c}_0 . However, $\phi_{\tilde{Y}, \tilde{c}_1}^{oi}(\tilde{c}_1)$ is always an endpoint of the (iterated) $\varphi_{\tilde{c}_1}$ -image of \tilde{Y} . Observe that the way we chose the replacing points for Y and \tilde{Y} guarantees that respective generating points of $\varphi_p^{oi}(Y)$ and $\varphi_{\tilde{c}_1}^{oi}(\tilde{Y})$ have the same itinerary (up to \star). By minimality, $\tau(z) \neq \tau(z_l)$ and there is a first time k ($k < m$) such that $c_0 \in f^{ok}([z, z_l])$. Consider the triods $\varphi_p^{ok}(Y)$ and $\varphi_{\tilde{c}_1}^{ok}(\tilde{Y})$. We have that $c_0 \notin](\varphi_{Y,p})^{ok}(z), (\varphi_{Y,p})^{ok}(p)[$ but $\tilde{c}_0 \in](\varphi_{\tilde{Y}, \tilde{c}_1})^{ok}(\tilde{z}), (\varphi_{\tilde{Y}, \tilde{c}_1})^{ok}(\tilde{p})[$. This contradicts that corresponding endpoints have the same itinerary (up to \star).

After proving the existence of the characteristic point \tilde{z} , we show that z and \tilde{z} are of the same type and have the same number of arms except for the case that $]z, b[$ contains no characteristic point. (An example for such a situation is given in Remark 3.2.4). The first part of the claim is immediate because the type is completely encoded into the internal address by Corollary 3.1.11. For the second part, let us assume that we are not in the exceptional case. Let q, \tilde{q} the number of arms at z, \tilde{z} , respectively. By Theorem 3.2.1, there is a characteristic point $z^q \in]z, b[\subset T$ with itinerary $B_i^q(\tau(z))$ and a characteristic point $\tilde{z}^{\tilde{q}} \in \tilde{T}$ with $B_i^{\tilde{q}}(\tau(z))$ or $\mathcal{A}_{\tilde{q}n}^{-1}(\tau(z))$. If $q \neq \tilde{q}$, then there is a characteristic point $\tilde{z}^q \in \tilde{T}$ with itinerary $B_i^q(\tau(z))$ and $\tilde{z}^q \in]\tilde{z}, \tilde{c}_1[$, as we just have proven. Let us assume that $\tilde{z}^q \in]\tilde{z}, \tilde{z}^{\tilde{q}}[$

(the case that $\tilde{z}^q, \tilde{z}^{\tilde{q}}$ are arranged the other way round works exactly the same way). Then the precritical point ξ^q of smallest step between $[\tilde{z}, \tilde{z}^q]$ has step qn and the one between $[\tilde{z}, \tilde{z}^{\tilde{q}}]$, denoted by $\xi^{\tilde{q}}$, is of step $\tilde{q}n$. Therefore we must have that $q > \tilde{q}$. Since $\xi^{\tilde{q}} \in]\tilde{z}^q, \tilde{z}^{\tilde{q}}[$ and \tilde{z}^q is characteristic, the interval $] \xi^{\tilde{q}}, \tilde{z}^q[$ covers itself under $f^{\circ \tilde{q}n}$ and thus contains a periodic point of (not necessarily exact) period $\tilde{q}n$ and itinerary $(\nu_1 \cdots \nu_n)^{\tilde{q}} = \tau(\tilde{z})$. But this contradicts minimality. If $\tilde{z}^q = \tilde{c}_1$, then $q > \tilde{q}$, and $f^{\circ \tilde{q}n}$ maps the interval $[\tilde{z}, \tilde{c}_1]$ homeomorphically onto itself without fixing \tilde{z}^q . This yields an infinite orbit for this point, a contradiction.³ \square

Remark 3.2.4 (Statement for extended Hubbard trees). Suppose that T contains no n -periodic point p with itinerary $\mathcal{A}_i(\tilde{\nu})$. We can extend $(T, f, \mathcal{P})_d$ such as to contain the orbit of the periodic point p . One can check that this extended tree $(T', f', \mathcal{P}')_d$ satisfies all conditions of a Hubbard tree with the exception that not all endpoints are on the critical orbit. We claim that in this case, the statement of Proposition 3.2.3, item (b), also holds true. There are several possibilities for the location of p in T' :

If $p \in T'_i$ for some $i \neq 0$, then the statement is empty. If p is contained in a subtree branching off from $[c_0, c_1]$ then we have exactly the same situation as in Proposition 3.2.3. The last possibility is that $c_1 \in [c_0, p]$. In this case, every preimage p_i^0 of p defines a new arm at c_0 . We set $T_{p_0} := \bigcup_{i=0}^{d-1} [p_0^i, c_0]$. For all characteristic points $z \in]c_0, c_1[$, we can define intervals I_k so that for all $k \in \mathbb{N}_0$, $f^{\circ k}(z) \in I_k$, $f|_{I_k}$ is a homeomorphism and $\partial I_k \in \text{orb}(p) \cup \{-p_0, p_0\}$. However, it is not longer true that $f(I_k) \supset I_{k+1}$. We construct intervals \tilde{I}_k in \tilde{T} just the same way as we did in the proof of Proposition 3.2.3. Observe that in \tilde{T} , $\tilde{f}(\tilde{I}_k) \supset \tilde{I}_{k+1}$ for all $k \in \mathbb{N}_0$. Thus, the remainder of the above proof carries over to this case, and the statement of case (b) in Proposition 3.2.3 extends to the case that $c_1 \in]c_0, p[$.

Remark 3.2.5 (Attracting dynamics is crucial). For the claim of Proposition 3.2.3 to hold, it is crucial that we assumed that Hubbard trees with periodic critical point have attracting dynamics. If one ignores this requirement then the statement of Proposition 3.2.3, case (a), is false in exactly the following situation: suppose that $(T, f, \mathcal{P})_d, (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ are two Hubbard trees with \star -periodic kneading sequences ν and $\tilde{\nu}$ such that T contains characteristic periodic points $p \neq p'$ with itineraries $\mathcal{A}_i(\nu)$ and $\overline{\mathcal{A}}(\tilde{\nu})$. Then $p' \in [c_0, p] \subset T$. Suppose furthermore that \tilde{T} contains no characteristic point with itinerary $\overline{\mathcal{A}}(\tilde{\nu})$. This is only possible if \tilde{T} has no attracting dynamics (cf. Lemma 3.1.3) and $\tilde{\nu}$ is primitive. Although p and the critical point \tilde{c}_0 meet the requirements of Proposition 3.2.3, the characteristic point p' is not forced in \tilde{T} .

³In case (a), the hypothesis that $]z, b[$ contains a further characteristic point is not necessary: either $z^q \in]z, c_1[$ and the hypothesis is trivially satisfied or $z^q = c_1$. But then \tilde{T} contains a periodic point \tilde{z}^q with itinerary $\mathcal{A}_i(\nu)$ by assumption, and we derive the same contradiction as before.

As an example for this situation consider in the Mandelbrot set the center of any primitive component W (unequal to the main cardioid) and the center of one of its satellite components. Denote by $p_{\tilde{c}}$ the unique postcritically finite polynomial corresponding to the center of W and let $\tilde{\nu}, \mathbb{T}$ be its associated kneading sequence and Hubbard tree respectively. We define a Hubbard tree $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_2$ by

$$\tilde{T} := \mathbb{T}/\sim, \text{ where } x \sim y : \iff x = y \quad \text{or} \quad \begin{cases} x, y \in \overline{F} \cap \mathbb{T}, \text{ for some} \\ \text{Fatou component } F \end{cases} .$$

Moreover, define $\tilde{f} := p_{\tilde{c}}|_{\tilde{\mathcal{T}}}$ and let $\tilde{\mathcal{P}}$ be the partition induced by the partition of $\mathbb{T} \subset \mathbb{C}$ generated by external rays (cf. page 8). We claim that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_2$ is a minimal Hubbard tree except that it does not have attracting dynamics. It is well known that if $F_1 \neq F_2$ are two Fatou components, then $\overline{F_1} \cap \overline{F_2} = \emptyset$. For the convenience of the reader we give a sketch of a proof using Hubbard trees.

Since all Fatou components are preperiodic (Sullivan's classification of Fatou components, cf. [M3]), it is enough to consider the case that F_1 and F_2 are periodic. Then $F_i \cap \text{orb}(\tilde{c}_0) = \{\tilde{c}^i\}$ and the legal arc $[\tilde{c}^1, \tilde{c}^2]$ is contained in the Hubbard tree of the polynomial. Without loss of generality, let us assume that \tilde{c}^1 equals the critical value. If now $\overline{F_2} \cap \overline{F_1} \neq \emptyset$ then $f^{oi}|_{[\tilde{c}^1, \tilde{c}^2]}$ is injective for all i . Let k be the smallest number such that $f^{ok}(\tilde{c}^i) = \tilde{c}^j$ for $i \neq j \in \{1, 2\}$. Then either $[\tilde{c}^1, \tilde{c}^2]$ covers itself under f^{ok} and contains a fixed point of f^{ok} , or $[\tilde{c}^1, \tilde{c}^2] \cup f^{ok}([\tilde{c}^1, \tilde{c}^2])$ is a non-degenerate triod and its branch point is fixed (by finiteness of $\text{orb}(\tilde{c}_0)$ and minimality). Thus in both cases, $[\tilde{c}^1, \tilde{c}^2]$ contains a periodic point \tilde{p} of exact period $k'|k$. If $\tau(\tilde{p}) = \tilde{\nu}_1 \cdots \tilde{\nu}_{k'}$ then the critical value has itinerary $(\tilde{\nu}_1 \cdots \tilde{\nu}_{k'})^j \tilde{\nu}_1 \cdots \tilde{\nu}_{k'-1} \star = \tilde{\nu}$ for some $j > 0$, and $\tilde{\nu}$ was not primitive.

Now it follows that the equivalence relation \sim is closed and \tilde{T} is Hausdorff. Since the quotient map is monotone, \tilde{T} is a tree. It is straightforward to verify the remaining conditions for Hubbard trees. (Note that we did not change the mutual location of points of $\text{orb}(\tilde{c}_0)$.)

Remark 3.2.6 (Number of arms at z and \tilde{z}). In case (b) of Proposition 3.2.3, we claimed that z and \tilde{z} have the same number of arms if $]z, b[$ contains a characteristic point. In the following we are going to show that this also holds if b is characteristic. We state the proof for the weakened hypothesis separately because of its rather technical and lengthy proof. Furthermore, the statement with this weaker hypothesis is not needed for our further discussions. Observe that we cannot make any statement in the case that $]z, b[$ contains no characteristic point as Figures 3.3 and 3.4 illustrate. In the chosen example $b = p$ (this is not necessary).

CLAIM. In the situation of Proposition 3.2.3, let $z \in [c_0, b[$ be characteristic and $\tilde{z} \in \tilde{T}$ the forced characteristic point with itinerary $\tau(z)$. If $]z, b[$ contains a characteristic point then the number of arms at z and \tilde{z} are equal.

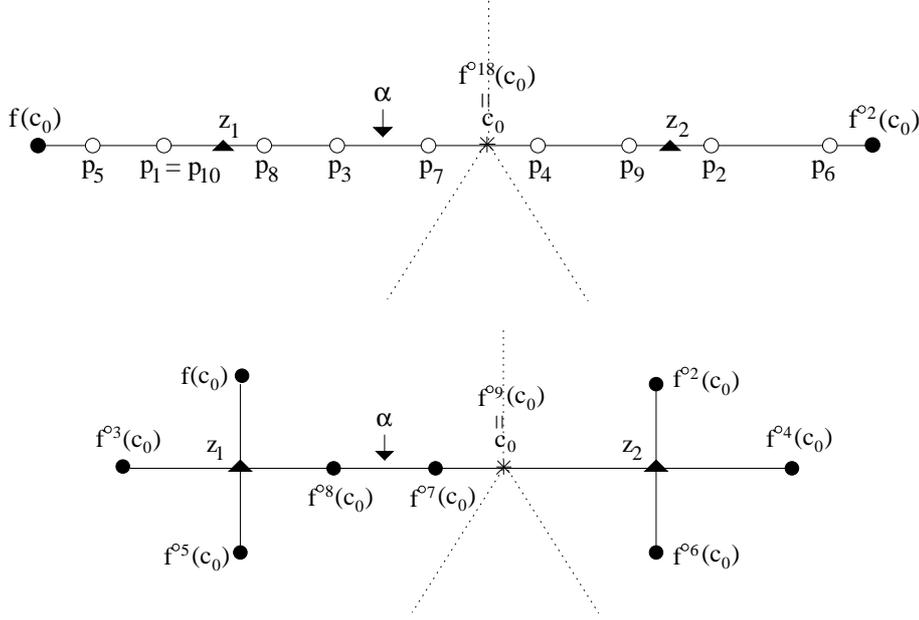


Figure 3.3: The two Hubbard trees illustrate that in the situation of Remark 3.2.6, the number of arms at $z = z_1$ might be different for $(T, f, \mathcal{P})_d$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$. At the top, the Hubbard tree associated to the kneading sequence $B_1^2(\overline{01001010\star})$ (only the first three iterates of c_0 are drawn; $\text{orb}(c_0)$ is the 2-bifurcation of $\text{orb}(p_5)$); the point p_1 is 9-periodic and $\tau(p_1) = \mathcal{A}_1(\overline{01010100\star})$. At the bottom, the Hubbard tree associated to $\overline{01010100\star}$. In both trees, $f^{\circ 2}(z_1) = z_1$.

Proof. Let q, \tilde{q} be the number of arms at z, \tilde{z} respectively. By Proposition 3.2.3 it only remains to show the statement if b is the only characteristic point in $]z, b]$.

Let us assume first that $q > \tilde{q}$. Then, using Corollary 2.1.26, there are iterates $f^{\circ jn}(p) =: p^j, f^{\circ j'n}(p) =: p^{j'}, j \neq j'$, so that $[p_0^0, p^j, p^{j'}] =: Y$ is a non-degenerate triod with branch point z whereas the respective triod $[\tilde{c}_0, \tilde{c}^j, \tilde{c}^{j'}] =: \tilde{Y}$ in \tilde{T} is either degenerate with \tilde{c}^j in the middle, or it is non-degenerate with branch point $\tilde{y} \neq \tilde{z}$. Iterate Y under the map φ_p and \tilde{Y} under the map $\varphi_{\tilde{c}_1}$. The images $\varphi_p^{\circ n}(Y)$ are all well-defined because $f^{\circ i}(z) \notin T_{p_0}$ for all i , and $\varphi_p^{\circ n}(Y) \neq \text{STOP}$. In particular, chopping causes no problems. Let $k < \infty$ be the first time that one of the following events happens: $\varphi_{\tilde{c}_1}^{\circ k+1}(\tilde{Y}) = \text{STOP}$, or $\tilde{c}_0 \in \tilde{f}^{\circ k}(] \tilde{y}, \tilde{z}])$.

In the first case, we have that $\phi_{Y,p}^{\circ k}(p^j) = p_0^i$ for some i in T . Since the k -th images of p_0^0 and $p^{j'}$ are on different sides of c_0 , $f^{\circ k}(z) \in]p_0^i, c_0[$ and thus $f^{\circ k+1}(z) \in]p, c_1[$, contradicting that z is characteristic. The second possibil-

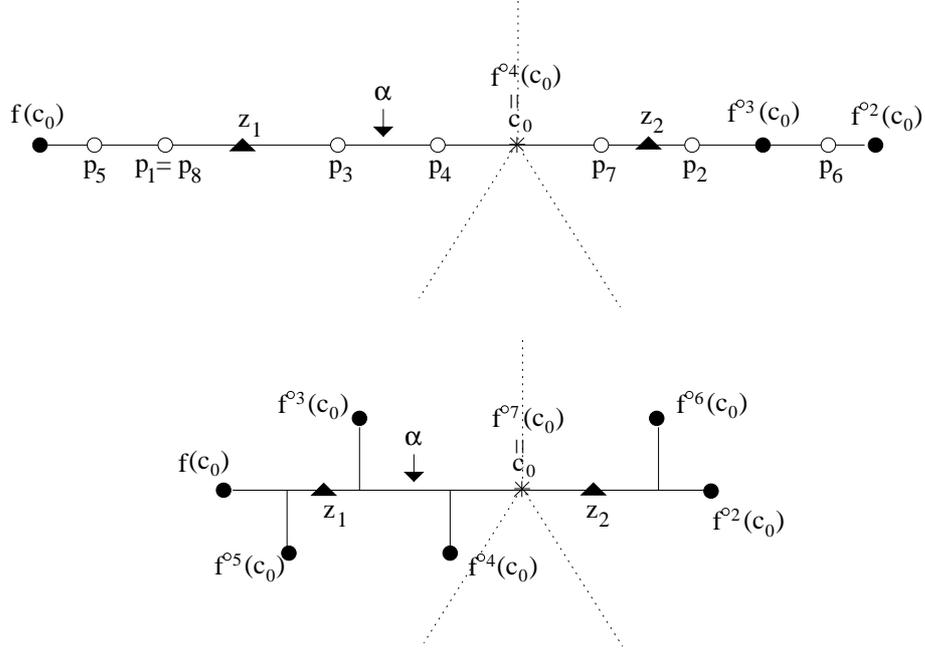


Figure 3.4: An example where the number of arms at $z = z_1$ are equal for two Hubbard trees as considered in Remark 3.2.6. The Hubbard tree at the top with kneading sequence $\overline{011}\star$ contains a 7-periodic point p_1 with $\tau(p_1) = \overline{0100011} = \mathcal{A}_1(\overline{010001}\star)$. On the bottom, the Hubbard tree associated to $\overline{010001}\star$.

ity implies that there is exactly one point in $\{\phi_{\tilde{Y}, \tilde{c}_1}^{\circ k}(\tilde{c}_0), \phi_{\tilde{Y}, \tilde{c}_1}^{\circ k}(\tilde{c}^j), \phi_{\tilde{Y}, \tilde{c}_1}^{\circ k}(\tilde{c}^{j'})\}$ from which $\tilde{f}^{\circ k}(\tilde{z})$ is not separated by \tilde{c}_0 . But since z is the branch point of Y this is clearly not possible in $\varphi_p^{\circ k}(Y)$. This contradicts that respective endpoints in $\varphi_p^{\circ k}(Y)$ and $\varphi_{\tilde{c}_1}^{\circ k}(\tilde{Y})$ have equal itinerary (up to \star).

Now let us assume that $\tilde{q} > q$. There are iterates $\tilde{c}^j, \tilde{c}^{j'} \in \tilde{T}$ and $p^j, p^{j'} \in T$, which are defined as in the previous case, such that $[\tilde{c}_0, \tilde{c}^j, \tilde{c}^{j'}] =: \tilde{Y}$ is non-degenerate with branch point \tilde{z} whereas the triod $Y := [p_0^0, p^j, p^{j'}]$ is either degenerate with p^j in the middle, or it is non-degenerate with branch point $y \neq z$.

Let us deal with the latter case first. There is a first time k_0 such that $c_0 \in f^{\circ k_0}([y, z])$. Suppose that for all $i \leq k_0$, $\varphi_p^{\circ i}(Y)$ is well-defined. Then $f^{\circ k_0-1}(z)$ is chopped off. This implies that $\tilde{f}^{\circ k_0-1}(\tilde{z}) \in \tilde{T}$ is also chopped off, but this is impossible. Unfortunately, $\varphi_p^{\circ i}(Y)$ might not be well-defined for some $i \leq k_0$, namely if one of the endpoints is chopped off. Let k be the smallest number such that an endpoint is chopped off and that replacing this point according to our requirements is not possible. Let Y_k be the

triod generated by the two points in $\{\phi_{Y,p}^{\circ k}(p_0^0), \phi_{Y,p}^{\circ k}(p_j), \phi_{Y,p}^{\circ k}(p)\}$ that are not chopped off and the appropriate preimage $p_0^{i_k}$. If Y_k is degenerate then $f^{\circ k}(y) \in]p_0^{i_k}, f^{\circ k}(z)[$ and $f^{\circ k+1}(y) = b$. We iterate the triod Y_k under φ_p until $c_0 \in f^{\circ k_1}(]b, f^{\circ k}(z)[)$ for the first time. At this time we get the same contradiction as described above. Again it might happen that an iterate $\varphi_p^{\circ i}(Y_k)$ is not well-defined. We define a new triod $Y_{k'}$ analogously to the way we defined Y_k ($Y_{k'}$ might be degenerate or non-degenerate). Note however, that in any case the interval $[b, z]$ is contained in the closure of a component of $Y_{k'} \setminus \{\text{generating points}\}$, as b and z are characteristic. This ensures that the respective images of b and z are always contained in the triods under consideration. We continue until after k_1 push forwards the images of b and z are separated by c_0 .

If the triod Y_k is non-degenerate then the branch point of $\varphi_p(Y_k)$ equals b . Now, since b is characteristic, all $\varphi_p^{\circ i}(Y_k)$ are well-defined, and we get the same contradiction as before at time k_0 .

The remaining case is that Y is degenerate. Independent from which of the three points is the inner point, the interval $[z, b]$ is contained in the interior of Y . By the same reasoning as in the previous paragraph for the case “ Y_k is degenerate”, this is impossible. \square

Lemma 3.2.7 (Not forced characteristic points). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree, $p \in T$ be an n -periodic non-precritical point with itinerary $\mathcal{A}_i(\tilde{\nu})$ and $b \in T$ such that $[0, b] = [0, c_1] \cap [0, p]$. Moreover, denote by \tilde{T} the Hubbard tree associated to $\tilde{\nu}$. Suppose that $z \in]b, c_1[$ is a characteristic point unequal to the α -fixed point and that either there is a characteristic point in $[b, z[$ or b is the limit of characteristic periodic points. Then there is no characteristic point \tilde{z} with itinerary $\tau(z)$ in the Hubbard tree \tilde{T} of $\tilde{\nu}$.*

Proof. We state the proof for the case that there is a characteristic point $y \in [b, z[$. For the case that such a point does not exist but b is a limit of characteristic points, we refer to the footnotes. By way of contradiction, we assume that there is a characteristic point $\tilde{z} \in \tilde{T}$ such that $\tau(z) = \tau(\tilde{z})$.

Consider the triod $Y := [p, y, z] \subset T$, which is degenerate and has y as an inner point. The point $\tilde{z} \in \tilde{T}$ forces a characteristic point $\tilde{y} \in \tilde{T}$ with itinerary $\tau(\tilde{y}) = \tau(y)$ by Proposition 3.2.3, item (a). The triod $\tilde{Y} := [\tilde{y}, \tilde{z}, \tilde{p}]$ is degenerate with \tilde{z} in the middle.⁴ There is a smallest number k such that $c_0 \in f^{\circ k}([y, z])$. Consider the iterates of Y, \tilde{Y} under the functions φ_z and $\varphi_{\tilde{z}}$. The iterates are well-defined for all $n \leq k$: the STOP case cannot occur because in T none of the generating points of (the images) of Y are

⁴In this case, $p \neq b$ and we set $y = b$. Any characteristic point in $[0, b[\subset T$ is forced in \tilde{T} by Proposition 3.2.3. Hence there is a limit point $\tilde{b} \in \tilde{T}$, which is either the critical value or has the same itinerary as b (cf. Lemma 2.1.7 and observe that \tilde{b} cannot be precritical as limit of characteristic points). Because of the existence of \tilde{z} , the second case must hold. We consider the degenerate triod $[\tilde{b}, \tilde{z}, \tilde{p}]$. (The proof does not require that \tilde{b} is periodic.)

on the critical orbit, and in \tilde{T} only the endpoint \tilde{p} is. If we have to chop a point while iterating Y it must be p . And since no image of y is in $[z_0^j, c_0]$ for all $j \in \{0, \dots, d-1\}$ ⁵, we can replace the respective iterate of p by some z_0^j . The chopping for \tilde{Y} can only occur at the same time as in Y (as corresponding generating points of $\varphi_z^{\circ i}(Y)$ and $\varphi_{\tilde{z}}^{\circ i}(\tilde{Y})$ have the same itineraries). If $\phi_{Y,z}^{\circ i}(p)$ is chopped off then $\phi_{Y,z}^{\circ i}(z) \neq z_0^j$ for all j . Thus, $\phi_{\tilde{Y},\tilde{z}}^{\circ i}(\tilde{z}) \neq \tilde{z}_0^j$ for all j , and we can replace $\phi_{\tilde{Y},\tilde{z}}^{\circ i}(\tilde{p})$ by some preimage of \tilde{z} . Now at time k , $c_0 \in [\phi_{Y,z}^{\circ k}(z), \phi_{Y,z}^{\circ k}(p)]$ whereas $\tilde{c}_0 \notin [\phi_{\tilde{Y},\tilde{z}}^{\circ k}(\tilde{z}), \phi_{\tilde{Y},\tilde{z}}^{\circ k}(\tilde{p})]$, which contradicts that corresponding generating points have equal itineraries. \square

Part (b) of Proposition 3.2.3 and Lemma 3.2.7 provide a classification of all characteristic points $z \neq b$ that can be forced in the Hubbard tree $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$. In the proof of Theorem 3.3.17, it will turn out that for the case that b is a limit of characteristic points or characteristic itself, b is also forced in \tilde{T} . However, we cannot make any statement about the number of arms.

3.3 Structure of the Parameter Space

3.3.1 A Partial Order on Kneading Sequences

In the following section, we introduce a partial order “ $<$ ” on the set Σ_d^\sharp . This order gives structure to the set Σ_d^\sharp . Moreover, it allows us to determine the non-admissible locus in Σ_d^\sharp , see Proposition 3.3.14.

Definition 3.3.1 (Order). *Let ν and $\tilde{\nu}$ be two \star -periodic kneading sequences. Then $\nu < \tilde{\nu} : \iff$ the Hubbard tree of $\tilde{\nu}$ contains a characteristic point with itinerary $\mathcal{A}_i(\nu)$ for some i .*

Proposition 3.2.3 guarantees that this relation is transitive. From the next lemma, it follows that it is also non-reflexive, and thus a partial order.

Lemma 3.3.2 (Unique characteristic point for ν). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree and $\nu \in \Sigma_d^\sharp$ be \star -periodic. Then T does not contain two characteristic points y, z , $z \in]y, c_1]$, with itineraries $\tau(y) = \mathcal{A}_i(\nu)$ and $\tau(z) \in \{\mathcal{A}_j(\nu), \nu\}$ for any j .*

Proof. Let n be the exact period of ν . If $i = j$ and $z \neq c_1$, then the statement follows by minimality. Otherwise, the precritical point $\xi \in]y, z]$ with smallest step has $\text{STEP}(\xi) = n$ (if $z = c_1$, then $\xi = c_1$). Hence, $f^{\circ n}(L_y(c_1)) = L_y(c_1)$. This implies that the local arm at z pointing towards c_0 must also be fixed under $f^{\circ n}$. So by Corollary 3.1.11, $\tau(y) = \overline{\mathcal{A}}(\nu)$, a contradiction. \square

⁵Since b is the limit point of characteristic points no iterate of b is in $[z_0^j, c_0]$.

Definition 3.3.3 (Truncated sequence). *Let $\nu \in \Sigma_d^\sharp$ be a kneading sequence. For all $n_k \in \text{orb}_\rho(\nu)$, we define $\nu^k := \overline{\nu_1 \cdots \nu_{n_k-1} \star}$. The sequence ν^k is called the truncated sequence of length n_k of ν .*

With this definition in mind, let us extend the partial order defined on \star -periodic kneading sequences to Σ_d^\sharp : let ν be a preperiodic kneading sequence. We set $\nu > \nu^k$ for all $k \in \mathbb{N}$. Furthermore, for any \star -periodic $\tilde{\nu}$, we set $\tilde{\nu} > \nu : \iff \tilde{\nu} > \nu^k$ for all $k \in \mathbb{N}$. From this, a partial order on Σ_d^\sharp is obtained by taking the transitive hull of the relation defined so far.

Let μ be \star -periodic or preperiodic and ν be preperiodic. It follows from the definition that $\mu < \nu$ if and only if there is a $k \in \mathbb{N}$ such that $\mu < \nu^k$. This observation implies immediately that $\nu \not< \nu$ for any preperiodic sequence ν . So “ $<$ ” on Σ_d^\sharp is non-reflexive and hence a partial order. We call the tuple $(\Sigma_d^\sharp, <)$ the *parameter tree* of degree d .

The next lemma proves that for any $\nu \in \Sigma_d^\sharp$, all its truncated sequences ν^k are represented in the Hubbard tree of ν by a characteristic point each. So, defining the order on preperiodic sequences via characteristic points just as in the periodic case would have yielded the same result. We chose the approach via truncated sequences because it allows for extending the order to all of Σ_d^\star whereas the approach via characteristic points fails there: elements in Σ_d^\star do not have an associated Hubbard trees in general. In Section 3.3.4, we discuss in more detail how one can extend “ $<$ ” to the set Σ_d^\star .

Lemma 3.3.4 (Internal address and characteristic points). *Let $\nu \in \Sigma_d^\sharp$, let ν^k be its truncated kneading sequence of length n_k and let $(T, f, \mathcal{P})_d$ be the Hubbard tree associated to ν . Then for any $k \in \mathbb{N}_0$ such that n_k is not the largest element of $\text{orb}_\rho(\nu)$, there is a characteristic point $z^k \in]c_0, c_1[\subset T$ with itinerary $\tau = \overline{\nu_1 \cdots \nu_{n_k}} = \mathcal{A}_{\nu_{n_k}}(\nu^k)$.*

Proof. By Lemma 2.4.5, there is a sequence of closest precritical points $\xi_i \in [c_0, c_1]$ with $c_0 < \xi_1 < \cdots < \xi_k < \cdots < c_1$ and $\text{STEP}(\xi_i) = n_i$. We proceed by induction: for $n_0 = 1$, note that the underlying T of every non-degenerate Hubbard tree contains the α -fixed point, which has itinerary $\overline{0} = \overline{\nu_1}$. Now let us assume that there is a characteristic point $p \in]c_0, c_1[$ with itinerary $\tau(p) = \overline{\nu_1 \cdots \nu_{n_k}}$. Then $c_0 < \xi_k < p < \xi_{k+1}$. First suppose that $n_{k+1} = jn_k$. By Theorem 3.2.1, the next characteristic point $p' \in]p, c_1[$ has itinerary $B_i^{j'}(\tau(p))$. We claim that $j' = j$; then clearly, $\tau(p') = \overline{\nu_1 \cdots \nu_{n_{k+1}}}$. If $j' \neq j$ then, since ξ_{k+1} is closest, $j' > j$, and $\xi_{k+1} \in]p', c_1[$. It follows that $f^{\circ jn}(p') \in]p, c_1[$, and since p' is characteristic, $f^{\circ jn}(p') \in]p, p'[$. That is, $f^{\circ jn}$ maps $]p, p'[$ homeomorphically into itself, which contradicts the existence of a precritical point $\xi \in]p, p'[$ with $\text{STEP}(\xi) = j'n_k$.

If $n_{k+1} \neq jn_k$, then consider the interval $]p, \xi_{k+1}[$. This interval covers itself homeomorphically under $f^{\circ n_{k+1}}$ so that there is a periodic point $x \in]p, \xi_{k+1}[$ with exact period m and $m|n_{k+1}$. Moreover, $\tau(x) = \overline{\mathcal{A}(\nu^{k+1})}$

because $\tau_{n_{k+1}}(x) = \nu_{n_{k+1}-n_k}$. We are going to show that x is not tame and that the number of its arms equals $(n_{k+1}/m + 1)$. Then Theorem 3.2.1 implies that there is a characteristic periodic point $p' \in]\xi_{k+1}, c_1[$ with itinerary $\overline{\nu_1 \cdots \nu_{n_{k+1}}}$. Note that the point x is characteristic: if it was not, then there is an image x_i of x contained in $]x, c_1[$. By minimality, $x \in]\xi_{k+1}, c_1[$, and thus the itineraries of x and x_i differ in exactly one entry. This means that there is one symbol in $\{0, \dots, d-1\}$ which appears differently often in the itineraries of x and x_i . But this is not possible, as x and x_i are iterates of each other. Now suppose that $m = n_{k+1}$. Then $f^{\circ n_{k+1}}$ maps $[x, \xi_{k+1}]$ homeomorphically onto $[x, c_1]$, i.e., $f^{\circ n_{k+1}}$ fixes the local arm of x pointing towards the critical value. It follows that the arm towards c_0 is also fixed under $f^{\circ n_{k+1}}$ and that x is an inner point. If $Qm = n_{k+1}$ for some $Q > 1$, then x is an evil branch point with $Q + 1$ arms: since p is characteristic, $f^{\circ m}(L_x(c_0)) = L_x(c_0)$ and x is not tame. Now let us consider the possible location of $f^{\circ m}(\xi_{k+1})$. Since $x \in]f^{\circ m}(\xi_{k+1}), f^{\circ m}(p)[$, it follows that $f^{\circ m}(\xi_{k+1}) \notin G_x(c_0)$. By finiteness of $\text{orb}(\xi_{k+1})$, $f^{\circ m}(\xi_{k+1}) \notin]\xi_{k+1}, x]$, and since ξ_{k+1} is a closest precritical point, $f^{\circ m}(\xi_{k+1}) \notin]c_1, \xi_{k+1}[$. If $f^{\circ m}(\xi_{k+1})$ was contained in a subtree B branching off in $]c_1, x[$, then $f^{\circ jm}(\xi_{k+1}) \in B$ for all $j \leq Q$, as $f^{\circ Qm}|_{]c_1, x]}$ is injective. (In order to get a contradiction otherwise, use minimality if the branching happens in $]x, \xi_{k+1}[$ and Lemma 2.1.16 if it happens in $]\xi_{k+1}, c_1[$.) In particular, $f^{\circ n_{k+1}}(\xi_{k+1}) = c_1 \in B$, which is clearly false. Therefore the only remaining possibility is that $f^{\circ m}(\xi_{k+1})$ is contained in an arm of x not pointing to the critical point or value. Since $f^{\circ i}$ is locally injective at x for all i , we have that all $f^{\circ jm}(\xi_{k+1})$ ($0 < j \leq Q$) are contained in different arms of x not pointing to c_0 . Putting everything together, x is an evil branch point with $Q + 1$ arms. \square

As a consequence of Lemma 3.3.4, we get our first statement about the structure of the parameter tree.

Corollary 3.3.5 ($B_i^q(\nu)$ and “ $<$ ”). *Let $\nu \in \Sigma_d^\sharp$ be any \star -periodic kneading sequence. Then*

- (i) $B_i^q(\nu) > \nu$ for all $q > 1$ and $i \in \{0, \dots, d-1\}$ for which $B_i^q(\nu)$ is defined.
- (ii) $B_i^q(\nu) \not> B_j^q(\nu)$ for all $i \neq j$.

Proof. The first claim follows immediately by the definition of bifurcation sequences. For the second claim, suppose that there are $i \neq j$ such that $B_i^q(\nu) > B_j^q(\nu)$. Let T_i, T_j be the underlying topological trees associated to the two bifurcation sequences. Then by Lemma 3.3.4 and the definition of bifurcation sequences, T_i (T_j) contains a characteristic point with itinerary $\mathcal{A}_i(\nu)$ ($\mathcal{A}_j(\nu)$). Now, $B_i^q(\nu) > B_j^q(\nu)$ and Proposition 3.2.3 imply that T_i also contains a characteristic point with itinerary $\mathcal{A}_j(\nu)$, a contradiction to Lemma 3.3.2. \square

Let us take a closer look at the subset of $(\Sigma_d^\#, <)$ that consists of all kneading sequences smaller than a given one.

Lemma 3.3.6 (Linear order for smaller sequences). *For any \star - or pre-periodic kneading sequence ν , the set $\{\mu \in \Sigma_d^\# : \mu < \nu\}$ is linearly ordered.*

Proof. Let us assume that $\mu \neq \mu'$ are two \star -periodic kneading sequences smaller than ν . The proof for preperiodic sequences works the same way by using the truncated sequences converging to the given preperiodic one. Let $(T, f, \mathcal{P})_d$ be the Hubbard tree associated to ν . By the definition of “ $<$ ”, there are two characteristic points $z, z' \in]c_0, c_1[\subset T$ such that $\tau(z) = \mathcal{A}_i(\mu)$ and $\tau(z') = \mathcal{A}_j(\mu')$. We have that either $z \in]z', c_1[$ or $z' \in]z, c_1[$. Theorem 3.2.3 yields in the first case that the Hubbard tree associated to μ' contains a characteristic point with itinerary $\mathcal{A}_i(\mu)$ and thus $\mu < \mu'$, and in the second case that $\mu' < \mu$. \square

Lemma 3.3.7 (Primitive sequence as limit). *Every primitive sequence ν is the limit of \star -periodic kneading sequences $\mu^n < \nu$. One can choose the μ^n to be primitive.*

Proof. Let ν be a primitive sequence and $(T, f, \mathcal{P})_d$ be its associated Hubbard tree. By Lemma 3.1.3, T contains a characteristic point z with $\tau(z) = \overline{\mathcal{A}}(\nu)$. We have to show that there is a sequence of characteristic points z^n converging to z . Consider the set $P := \{p \in]c_0, z[: p \text{ is characteristic}\}$. To prove the claim it suffices to show that $\sup(P) = z$. But this follows immediately by Lemma 2.2.9 and the following observation: there is no characteristic point $z' \in]c_0, z[$ such that $]z', z[$ contains no further characteristic point. If there was such a z' then $\tau(z) = B_i^q(\tau(z'))$ by Proposition 3.2.3, and $\tau(z)$ would not be the lower kneading sequence of a primitive sequence.

For the second part of the claim suppose that there is a one-sided neighborhood $]p, z[\subset]c_0, z[$ of z such that for all characteristic points $y \in]p, z[$, $\mathcal{A}_{n_y}^{-1}(\tau(y))$ is not primitive, where n_y denotes the exact period of y . We can choose $]p, z[$ so small that it contains no branch point of T . Now Proposition 3.2.3 implies that there is a periodic sequence $\tau \in \Sigma_d^\star$ such that the set of characteristic points in $]p, z[$ equals $\{z^n \in]p, z[\text{ characteristic} : \tau(z^{n+1}) = B_i^2(\tau(z^n))\}$, where $\tau(z^1) = \tau$ and i depends on n . Thus if N denotes the period of τ , we have that the ρ -orbit of z^n equals $1 \mapsto \dots \mapsto N \mapsto 2N \mapsto \dots \mapsto 2^n N$. Observe that there is an $M \in \mathbb{N}$ such that $\rho_z(k) - k \leq M$ for all $k \in \mathbb{N}$. On the other hand, there is an n_0 such that $2^{n_0+1}N - 2^{n_0}N = 2^{n_0}N > M$. Set $m := \rho_{\tau(z)}(2^{n_0}N)$. Then there is an $m' \leq m$ such that $\tau_{m'}(z) \neq \tau_{m'}(z^n)$ for all $n > n_0$. In contradiction to this, there is an n_1 such that for all $n > n_1$, $]z^n, z[$ contains no precritical point of step at most $m+1$ and thus, the first m entries of $\tau(z)$ and $\tau(z^n)$ are the same. \square

The proof above shows that the sequence $(\mu^n)_{n=1}^\infty$ cannot exclusively

consist of consecutive 2-bifurcation sequences, i.e., it is not possible that $\nu^{n+1} = B_{i(n)}^2(\nu^n)$ for all $n > n_0$.

3.3.2 Location of Non-Admissible Kneading Sequences

Let us start our investigation of the set of non-admissible kneading sequences by some immediate consequences of the results we have shown so far:

Lemma 3.3.8 (Evil branch points and “<”). *Let $(T, f, \mathcal{P})_d$ be a Hubbard tree with kneading sequence ν and let $z \in T$ be a characteristic n -periodic point that is not tame. Furthermore, suppose that μ is \star -periodic such that $\tau(z) = \overline{\mathcal{A}}(\mu)$. If z is an inner point, then $\mu \leq \nu$. If z is a branch point, then $\mu \not\leq \nu$ but $\tilde{\mu} < \nu$ for all \star -periodic $\tilde{\mu} < \mu$.*

Proof. If z is an inner point then by Theorem 3.2.1 either there is a characteristic point $z' \in]z, c_1[$ with itinerary $B_i^1(\overline{\mathcal{A}}(\mu)) = \mathcal{A}_i(\mu)$ or $z' = c_1$ and $\nu = \mathcal{A}_n^{-1}(\mathcal{A}(\mu)) = \mu$. Thus, in this case $\mu \leq \nu$ by definition. If z is a branch point we have to show that there is no characteristic n -periodic point with itinerary $\mathcal{A}(\mu)$. By way of contradiction, let us assume that there is such a characteristic point $p \in [c_0, c_1]$. By Corollary 2.1.26, there are no n -periodic points in $]z, c_1]$. On the other hand, if $p \in]c_0, z[$ then $f^{on}(L_z(c_0)) = L_z(c_1)$, because $c_0 \in f^{o(n-1)}([p, z])$. But this contradicts that z is evil. The last part of the claim is an immediate consequence of Proposition 3.2.3. \square

Corollary 3.3.9 (Backward bifurcation not larger). *Let $\nu \in \Sigma_d^\sharp$ be a \star -periodic primitive sequence. Then for all $q > 1$, $\overline{B}^q(\nu) \not\leq \nu$. For all $\tilde{\nu} < \nu$, $\overline{B}^q(\nu) > \tilde{\nu}$.*

Proof. Consider the Hubbard tree $(T, f, \mathcal{P})_d$ of $\overline{B}^q(\nu)$. By Lemma 3.1.3, T contains a characteristic point z with itinerary $\overline{\mathcal{A}}(\overline{B}^q(\nu)) = \overline{\mathcal{A}}(\nu)$. By Lemma 2.4.9 and Corollary 2.1.26 it follows that z is an evil branch point. Now the statement follows from Lemma 3.3.8. \square

The proof of the previous corollary also yields the following result:

Corollary 3.3.10 (Backward bifurcation not admissible). *For every primitive $\nu \in \Sigma_d^\sharp$ and every $q > 1$, the backward bifurcation sequence $\overline{B}^q(\nu)$ is non-admissible.* \square

Lemma 3.3.11 (Forcing of (non-)admissibility). *Let ν be a \star - or preperiodic kneading sequence. If ν is admissible then $\tilde{\nu}$ is admissible for all $\tilde{\nu} < \nu$, or equivalently, if ν is non-admissible then $\tilde{\nu}$ is non-admissible for all $\tilde{\nu} > \nu$.*

Proof. The statement follows immediately by Proposition 3.2.3: if $\tilde{\mu} < \mu$ and $\tilde{\mu}$ is non-admissible, then the Hubbard tree of $\tilde{\mu}$ contains an evil branch point. By orbit forcing this evil branch point can also be found in the Hubbard tree of μ , and thus μ is non-admissible. \square

Definition 3.3.12 (Combinatorial arc, wake). *Let $\nu, \nu' \in \Sigma_d^\sharp$ with $\nu \leq \nu'$. The combinatorial arc between ν and ν' is the set $[\nu, \nu'] := \{\mu \in \Sigma_d^\sharp : \nu \leq \mu \leq \nu'\}$.*

If ν is \star -periodic, we define for any $i \in \{0, \dots, d-1\}$ and $q \geq 2$

- *its (q, i) -subwake $\mathcal{W}_i^q(\nu) := \{\mu \in \Sigma_d^\sharp : \mu \geq B_i^q(\nu)\}$,*
- *its non-admissible q -subwake $\mathcal{W}_{non}^q(\nu) := \{\mu \in \Sigma_d^\sharp : \mu \geq \overline{B}^q(\nu)\}$,*
- *its admissible wake $\mathcal{W}_{ad}(\nu) := \{\mu \in \Sigma_d^\sharp : \mu \geq \nu\}$ and*
- *its wake $\mathcal{W}(\nu) := \bigcup_{q>1} \mathcal{W}_{non}^q(\nu) \cup \mathcal{W}_{ad}(\nu)$.*

The subwake $\mathcal{W}_i^q(\nu)$ is empty for the one symbol in $\{0, \dots, d-1\}$ for which $B_i^q(\nu)$ does not exist. For any $q > 1$, $\mathcal{W}_{non}^q \neq \emptyset$ if and only if $\nu \neq \bar{\kappa}$ is primitive (for non-primitive sequences, $\overline{B}^q(\nu)$ was not defined), and for any $\mu \in \mathcal{W}_{non}^q(\nu)$, $\nu \not\leq \mu$. An immediate corollary of this definition and Theorem 3.2.1 is the following statement.

Corollary 3.3.13 (Wakes and subwakes). *Let $\nu \in \Sigma_d^\sharp$ be \star -periodic. Then*

$$\mathcal{W}_{ad}(\nu) = \bigcup_{q=2}^{\infty} \bigcup_{i=0}^{d-1} \mathcal{W}_i^q(\nu) \cup \{\nu\}.$$

□

We get a description of the location of all non-admissible kneading sequences as a further corollary of our previous work. This positively answers the question posed in [Kau2] whether all non-admissible kneading sequence correspond to *shadow components*. Using the notation of wakes, our result reads as follows.

Proposition 3.3.14 (Location non-admissible sequences). *Let ν be a non-admissible kneading sequence. Then there is a unique admissible primitive sequence μ and an integer $q > 1$ such that ν is contained in $\mathcal{W}_{non}^q(\mu)$.*

Proof. Take any non-admissible kneading sequence ν and let b be the unique evil branch point such that $[c_0, b[$ contains no further evil branch points. Since T contains only finitely many branch points, such a point b exists. Let τ be the itinerary of b , n be its exact period and let q be the number of global arms at b . If $\mu := \mathcal{A}_n^{-1}(\tau)$, then Theorem 3.2.1 implies that $\nu \in \mathcal{W}_{non}^{q-1}$. The primitive sequence μ is admissible: if it was not then the Hubbard tree T' associated to μ contains an evil branch point $z' \in]c'_0, c'_1[$. By Orbit Forcing 3.2.1, there is a characteristic point $z \in [c_0, b[$ which has the same number of arms and is of the same type as z' . But this means that z is an evil branch point closer to c_0 than b , contradicting the choice of b .

It only remains to show that μ is unique. If ν is contained in a (non-admissible or admissible) subwake of $\tilde{\mu}$ then there is a characteristic periodic point z with itinerary $\mathcal{A}_i(\tilde{\mu})$ or $\overline{\mathcal{A}}(\tilde{\mu})$. If z was in $]b, c_1[$ then the Hubbard tree associated to $\tilde{\mu}$ would contain an evil branch point with itinerary $\tau(b)$ and $\tilde{\mu}$ would not be admissible. On the other hand, if $z \in]c_0, b[$ then z is not an evil branch point and $\tilde{\mu} < \nu$ by Lemma 3.3.8. In both cases, $\tilde{\mu}$ does not have the claimed properties. \square

An immediate consequence of Propositions 3.3.14 and 3.2.3 is the following statement:

Corollary 3.3.15 (Branching into non-admissibility). *Let $\nu, \tilde{\nu} \in \Sigma_d^\sharp$. Suppose that the Hubbard trees of ν and $\tilde{\nu}$ are non-admissible and they contain an evil periodic branch point b, \tilde{b} , respectively, with $\tau(b) = \tau(\tilde{b})$. Then there is a unique admissible primitive sequence μ such that $\nu \in \mathcal{W}_{non}^q(\mu)$, $\tilde{\nu} \in \mathcal{W}_{non}^{\tilde{q}}(\mu)$ (q, \tilde{q} are not necessarily distinct). \square*

3.3.3 The Branch Theorem

Now we come to the main result of our investigation of Σ_d^\sharp , the so-called *Branch Theorem*. In spirit, it says that either $\nu, \tilde{\nu} \in \Sigma_d^\sharp$ can be compared or there is a unique maximal sequence which ν and $\tilde{\nu}$ can be compared with. The latter means that there is a unique point in Σ_d^\sharp where the linearly ordered sets $[\bar{x}, \nu]$, $[\bar{x}, \tilde{\nu}]$ branch off. For postcritically finite parameters in the Mandelbrot set, this result was proven by Douday and Hubbard in [DH, Proposition XXII.3] as the main statement of their study of “nervures”. This theorem is also true for the Multibrot sets \mathcal{M}_d [S2]. Our proof is inspired by the techniques used there. The Branch Theorem for Σ_2^\sharp has been proven in [Ka]. We will work in the Hubbard tree of one of the two given sequences, say ν . This Hubbard tree contains, or can be extended to contain, a representative p of $\tilde{\nu}$. The branch point of $[c_0, p] \cap [c_0, c_1]$ will give rise to the branch point in $(\Sigma_d^\sharp, <)$. That this setup makes sense is the content of the following lemma.

Lemma 3.3.16 (Periodic points behind c_1). *Let $(T, f, \mathcal{P})_d$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ be two Hubbard trees with associated kneading sequences $\nu, \tilde{\nu}$ and suppose that $\nu \neq \tilde{\nu}$. If necessary, extend T so that it contains a periodic point $p_{\tilde{\nu}}$ with itinerary $\tau(p_{\tilde{\nu}}) = \mathcal{A}(\tilde{\nu})$ and extend \tilde{T} so that it contains a periodic point \tilde{p}_ν with $\tau(\tilde{p}_\nu) = \mathcal{A}(\nu)$. Then it is not possible that $c_1 \in]c_0, p_{\tilde{\nu}}[$ and $\tilde{c}_1 \in]\tilde{c}_0, \tilde{p}_\nu[$.*

Proof. By way of contradiction assume that $c_1 \in]c_0, p_{\tilde{\nu}}[$ and $\tilde{c}_1 \in]\tilde{c}_0, \tilde{p}_\nu[$. By Lemma 3.1.3, there is a characteristic point $z \in]c_0, c_1[\subset T$ with itinerary $\overline{\mathcal{A}}(\nu)$ and since $c_1 \in]c_0, p_{\tilde{\nu}}[$, this point is forced in \tilde{T} by Remark 3.2.4. More precisely, there is a characteristic point $\tilde{z} \in]\tilde{c}_0, \tilde{c}_1[\subset \tilde{T}$ with $\tau(\tilde{z}) = \overline{\mathcal{A}}(\nu)$. Since $\tilde{c}_1 \in]\tilde{c}_0, \tilde{p}_\nu[$, we have that $\tilde{c}_1 \in]\tilde{z}, \tilde{p}_\nu[$. The precritical point ξ of lowest

step in $]\tilde{z}, \tilde{p}_\nu[$ has step n , if n equals the period of ν . Because of finiteness of $\text{orb}(c_1)$, it follows that $\xi = c_1$, and thus $\tilde{\nu} = \tau(\tilde{c}_1) = \nu$, a contradiction. \square

Theorem 3.3.17 (Branch Theorem). *Let $\nu \neq \tilde{\nu}$ be \star -periodic or preperiodic kneading sequences. Then there is a kneading sequence μ such that exactly one of the following cases holds:*

- (i) $[\bar{\kappa}, \nu] \cap [\bar{\kappa}, \tilde{\nu}] = [\bar{\kappa}, \mu]$, where μ is either \star -periodic or preperiodic.
- (ii) $[\bar{\kappa}, \nu] \cap [\bar{\kappa}, \tilde{\nu}] = [\bar{\kappa}, \mu] \setminus \{\mu\}$, where μ is a \star -periodic primitive sequence.

If $\mu \notin \{\nu, \tilde{\nu}\}$, we say that μ is the *branch point* of the kneading sequences ν and $\tilde{\nu}$. In the second case, we say that ν or $\tilde{\nu}$ *branches off into non-admissibility* at μ according as $\mu \not\prec \nu$ or $\mu \not\prec \tilde{\nu}$.

Proof. We have that $[\bar{\kappa}, \nu] \cap [\bar{\kappa}, \tilde{\nu}] = \{\mu' \in \Sigma_d^\sharp : \mu' \leq \nu \text{ and } \mu' \leq \tilde{\nu}\}$. This together with the way the order on Σ_d^\sharp was defined yields that finding the branch point μ is equivalent to finding the supremum of the set $\{\mu' \text{ is } \star\text{-periodic} : \mu' \leq \nu \text{ and } \mu' \leq \tilde{\nu}\}$. By the definition of “ $<$ ” on preperiodic kneading sequences, we can assume that both ν and $\tilde{\nu}$ are \star -periodic. Let T be the Hubbard tree of ν and \tilde{T} the one of $\tilde{\nu}$. If T (or \tilde{T}) contains a characteristic point with itinerary $\mathcal{A}(\tilde{\nu})$ (or $\mathcal{A}(\nu)$), then $\nu > \tilde{\nu}$ (or $\tilde{\nu} > \nu$). This is in particular true if one of the given kneading sequences equals $\bar{\kappa}$. Suppose that there is no $n > 1$ such that T, \tilde{T} contain an n -periodic characteristic point with itinerary $\overline{0 \cdots 0i}, \overline{0 \cdots 0\tilde{i}}$, respectively, where $i, \tilde{i} \in \{1, \dots, d-1, \star\}$. Then $\bar{\kappa}$ is the branch point of ν and $\tilde{\nu}$.

For the remaining cases note that, by possibly enlarging T , the tree T contains a periodic point p with itinerary $\mathcal{A}(\tilde{\nu})$ and by Lemma 3.3.16, $c_1 \notin [c_0, p]$ (interchange T and \tilde{T} if necessary). Let $b \in T$ such that $[c_0, p] \cap [c_0, c_1] = [c_0, b]$. The point b is either a branch point or equals p . Consider the set

$$P := \{z \in [c_0, b] : z \text{ is a characteristic point}\}.$$

In all the cases that we still have to consider, P contains the α -fixed point and hence is not empty. Set $a := \sup P$ (where the supremum is taken with respect to the natural order on $[c_0, b]$ with c_0 as smallest element). We distinguish whether b equals the supremum of P or not.

- (i) If $a = b$ then $b \neq p$: the local arm of b pointing towards c_0 is fixed because b is the limit of characteristic points. Hence the itinerary of b cannot be the upper kneading sequence $\mathcal{A}_i(\tilde{\mu})$, and b is a preperiodic branch point or an evil periodic one.

We first consider the case that b is preperiodic. Let $\mu = \tau(b)$ be the associated preperiodic kneading sequence. We claim that μ is the branch point of ν and $\tilde{\nu}$.

Let $(1, 0) \rightarrow (n_1, s_1) \rightarrow \dots \rightarrow (n_k, s_k) \rightarrow \dots$ be the infinite internal address of μ and let μ^k be the unique \star -periodic sequence associated to $(1, 0) \rightarrow (n_1, s_1) \rightarrow \dots \rightarrow (n_k, \star)$. We first show that b is the limit point of characteristic points $p_k \in T$ with itinerary $\tau(p_k) = \mathcal{A}_i(\mu^k)$, where $i = \mu_{n_k}$: by definition, b is the limit point of a sequence of characteristic points z_l of period m_l . By taking a subsequence, we can assume that for all $l \geq l_0$ the itineraries of z_l have the same first m_{l_0} entries as μ . Fix an entry n_k of $\text{orb}_\rho(\mu)$. Then there is an $m_{l(k)} \in \mathbb{N}$ such that $m_{l(k)-1} < n_k \leq m_{l(k)}$. Since the itinerary $\tau(z_{l(k)})$ coincides with μ for the first $m_{l(k)}$ entries, n_k is contained in $\text{orb}_\rho(\mathcal{A}_{m_{l(k)}}^{-1}(\tau(z_{l(k)})))$. Then by Lemma 3.3.4, there is a characteristic point p'_{n_k} with itinerary $\mathcal{A}_i(\mu^k)$ in the Hubbard tree of $\mathcal{A}_{m_{l(k)}}^{-1}(\tau(z_{l(k)}))$. This forces a characteristic periodic point $p_k \in]z_{l(k)-1}, z_{l(k)}] \subset T$ with itinerary $\mathcal{A}_i(\mu^k)$. By induction on k , we get a sequence of characteristic points $p_k \in T$, which have to converge to b because $p_k \in [z_{l(k)-1}, z_{l(k)})$ for all k .

We have seen that T contains a characteristic point with itinerary $\mathcal{A}(\mu^k)$ for all $k \in \mathbb{N}$. Since these points are contained in the arc $[c_0, b[$, $\nu, \tilde{\nu} > \mu^k$ for all $k \in \mathbb{N}$, and thus $\nu, \tilde{\nu} > \mu$. Assume that $\tilde{\mu}$ is \star -periodic with $\nu, \tilde{\nu} > \tilde{\mu}$ and let $\tilde{q} \neq b$ be the characteristic periodic point with itinerary $\mathcal{A}(\tilde{\mu})$. Since b is the limit of characteristic points, Lemma 3.2.7 implies that $\tilde{q} \in [c_0, b[$, and since $p_{n_k} \rightarrow b$, there is a k_0 such that $\tilde{q} \in]c_0, p_{n_{k_0}}[$. By Proposition 3.2.3, the Hubbard tree of μ^{k_0} contains a characteristic periodic point corresponding to \tilde{q} and hence $\tilde{\mu} < \mu^{k_0} < \mu$.

Now suppose that b is an evil periodic branch point: as limit of characteristic points b itself is characteristic. Let $\{z_n\} \subset P$ be a sequence of characteristic points converging to b such that $c_0 < z_n < z_{n+1} < b$. Hence, there is a sequence of corresponding characteristic points $\tilde{z}_n \in \tilde{T}$. By compactness of the Hubbard tree and since $\tilde{c}_0 < \tilde{z}_n < \tilde{z}_{n+1} < \tilde{c}_1$, this sequence has to converge to a point $\tilde{b} \in [\tilde{c}_0, \tilde{c}_1] \subset \tilde{T}$. We know that $\tau(\tilde{z}_n) \rightarrow \overline{\mathcal{A}}(\mu)$. Hence by Lemma 2.1.7, the itinerary of \tilde{b} is either $\overline{\mathcal{A}}(\mu)$ or μ , the latter if and only if $\tilde{b} = \tilde{c}_1$. In the second case $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \tilde{\nu}]$. In the first case we claim that μ is the branch point for ν and $\tilde{\nu}$. Note that at least ν is contained in some non-admissible subwake $\mathcal{W}_{\text{non}}^q(\mu)$ by Lemma 3.3.8. Thus $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \tilde{\mu}]$. The point \tilde{b} is a characteristic point: since \tilde{b} is the limit of characteristic points, $f^{\circ i}(\tilde{b}) \in \overline{G_{\tilde{b}}(\tilde{c}_0)}$ for all i . Let m be the exact period of $\overline{\mathcal{A}}(\mu)$ and suppose that \tilde{b} is not m -periodic. Then the interval $]f^{\circ m}(\tilde{b}), \tilde{b}[$ contains a characteristic point \tilde{z}_n which has itinerary unequal to $\overline{\mathcal{A}}(\mu)$. But according to Lemma 2.1.9, we must have that $\tau(\tilde{z}_n) = \overline{\mathcal{A}}(\mu)$, a contradiction. Therefore, $\tau(\tilde{b}) = \overline{\mathcal{A}}(\mu)$ implies that $\mu < \tilde{\nu}$ if and only

if \tilde{b} is an inner point. If \tilde{b} is a branch point then $\tilde{\nu} \in \mathcal{W}_{\text{non}}^{\tilde{q}}(\mu)$ for some $\tilde{q} > 1$. There is no \star -periodic kneading sequence $\mu' \not\prec \mu$ such that $\mu' < \nu, \tilde{\nu}$: for any $\mu' < \nu$ with $\mu' \not\prec \mu$, there is a characteristic point $z \in]b, c_1] \subset T$ with itinerary $\mathcal{A}_i(\mu')$ by Proposition 3.2.3. Since b is characteristic, Lemma 3.2.7 implies that $\mu' \not\prec \nu$.

- (ii) The second possibility is that $a \in]c_0, b[$. By Lemma 2.2.9, $a \in P$, i.e. a is a characteristic point. Let τ be the itinerary of a , n be its period and let μ be the \star -periodic kneading sequence so that $\tau = \mathcal{A}_i(\mu)$ or $\tau = \overline{\mathcal{A}}(\mu)$, according as a is tame or not. Let q be the number of arms at a . By Proposition 3.2.3, there is a periodic characteristic point $\tilde{a} \in \tilde{T}$ that has the same itinerary τ and is of the same type as a . Let \tilde{q} be the number of its arms. We claim that if $q \neq \tilde{q}$ then the μ is the branch point of ν and $\tilde{\nu}$, and if $q = \tilde{q}$ then this is $B_i^Q(\mu)$. Here $Q = q$ if a is tame and $Q = q - 1$ otherwise.

Theorem 3.2.1 implies that T contains a characteristic point x with itinerary $B_i^Q(\tau)$, and \tilde{T} contains one, say \tilde{x} , with itinerary $B_i^{\tilde{Q}}(\tau)$ or $B_i^{\tilde{Q}}(\mu)$ (\tilde{Q} is defined analogously to Q). Consider the case $q \neq \tilde{q}$ first. In the proof of Proposition 3.2.3, we have seen that a Hubbard tree cannot contain two different bifurcation sequences of a given sequence. This together with Lemma 3.2.7 yields that if μ' is a \star -periodic kneading sequence smaller than ν and $\tilde{\nu}$ then $\mu' \leq \mu$. If $q = \tilde{q}$ then we have that either $\tilde{x} = \tilde{c}_1$ and $\tilde{\nu} = B_i^Q(\mu) < \nu$, or $B_i^Q(\mu)$ is the largest sequence that is smaller than ν and $\tilde{\nu}$: let μ' have this property too. Then T contains a characteristic point y with itinerary $\mathcal{A}(\mu')$, and since $y \in [c_0, x[$ by Lemma 3.2.7, we have $\tilde{\mu} \leq B_i^Q(\mu)$. \square

Since by Lemma 3.3.11 the non-admissible locus is separated from the admissible one, Theorem 3.3.17 implies the Branch Theorem for the Multi-brot sets [S2].

Corollary 3.3.18 (Branch Theorem for \mathcal{M}_d). *Let A_1, A_2 be any two hyperbolic components or Misiurewicz points of \mathcal{M}_d . Then either one is contained in a subwake of the other, or there is another hyperbolic component or Misiurewicz point B such that A_1 and A_2 are contained in two different subwakes of B .* \square

3.3.4 The Space of All Kneading Sequences

So far, we only considered an order “ $<$ ” on the set Σ_d^\sharp . Now, we extend this order to the set Σ_d^\star . The basis was already laid in Section 3.3.1 when we defined “ $<$ ” for preperiodic sequences. Recall the definition of truncated sequences ν^k (which was only given for elements in Σ_d^\sharp). Given any $\nu \in \Sigma_d^\star$, we define for each element $n_k \in \text{orb}_\rho(\nu)$, $\nu^k := \overline{\nu_1 \cdots \nu_{n_k-1} \star}$.

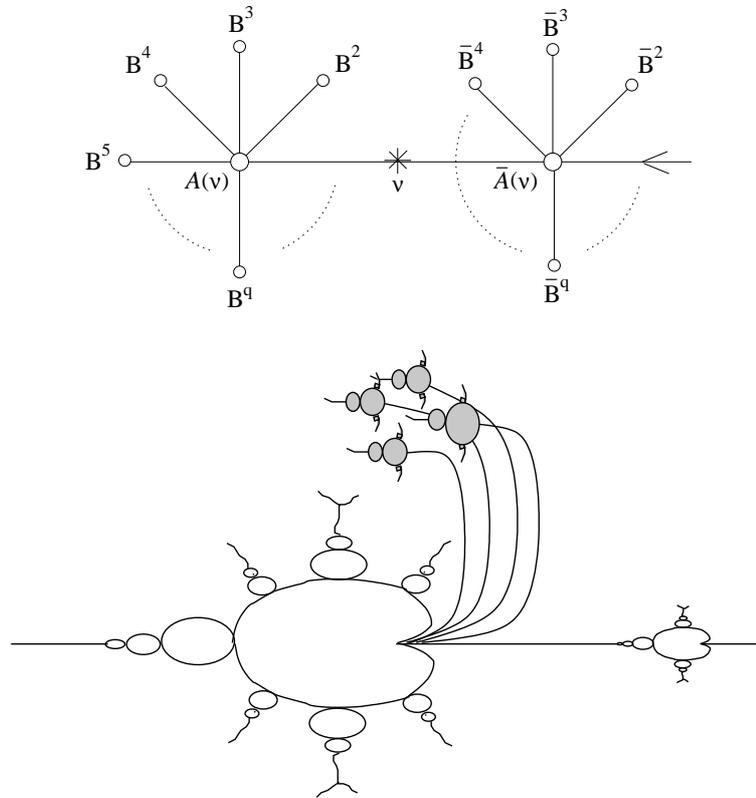


Figure 3.5: An illustration of the structure of Σ_d^\sharp as given by Theorem 3.3.17. At the top, a schematic picture of the structure of Σ_2^\sharp close to a primitive sequence ν . At $A(\nu)$ the bifurcation sequences B^q branch off, at $\bar{A}(\nu)$, the backward bifurcation sequences \bar{B}^q , where $q \in \mathbb{N}$. For more details see Section 3.3.4. At the bottom, the location of non-admissible backward bifurcation sequences (in gray) with respect to hyperbolic components in the Mandelbrot set \mathcal{M} (in white) is sketched. The figure shows the situation at one primitive hyperbolic component of \mathcal{M} . To obtain this picture, we associate to non-admissible \star -periodic kneading sequences “non-existing hyperbolic components”. These correspond to *shadow components* in [Kau2]. Every backward bifurcation corresponds to the main cardioid of a (non-admissible) copy of \mathcal{M} . At every primitive component of such a copy we have in turn backward bifurcation sequences. That is, they give rise to a higher level of non-admissibility. This event of course repeats itself, yielding for any $n \in \mathbb{N}$ an n -th level of non-admissibility. Hubbard trees corresponding to kneading sequences in this n -th level of non-admissibility contain exactly n distinct cycles of evil branch points.

Definition 3.3.19 (Order on Σ_d^*). Let $\nu \in \Sigma_d^* \setminus \Sigma_d^\sharp$.

- (i) If ν is not periodic, we set $\nu > \nu^k$ for all $k \in \mathbb{N}$. For any \star -periodic μ , we set $\mu > \nu : \iff \mu > \nu^k$ for all $k \in \mathbb{N}$.
- (ii) If ν is n -periodic such that $\nu = \mathcal{A}_i(\nu^*)$ for some \star -periodic kneading sequence ν^* , then set $\nu > \nu^*$. Moreover, for any \star -periodic μ , we set $\mu > \nu : \iff \mu \geq B_i^q(\nu^*)$.
- (iii) If ν is n -periodic such that $\nu = \overline{\mathcal{A}}(\nu^*)$ for some \star -periodic kneading sequence ν^* , then we set $\nu^* > \nu$. For any \star -periodic μ , we set $\nu > \mu : \iff \nu^* > \mu$. Moreover, if ν^* is primitive, we define that $\overline{B}^q > \nu$ for all $q \geq 2$.

The partial order on Σ_d^* is obtained by taking the transitive hull of the relation defined so far.

There are periodic sequences ν such that $\mathcal{A}(\mu) = \nu = \overline{\mathcal{A}}(\tilde{\mu})$ for some \star -periodic kneading sequences $\mu, \tilde{\mu}$. In such cases, using item (ii) or (iii) yields the same result (cf. Theorem 3.2.1). So the relation is well-defined and, by definition, transitive and non-reflexive. Note also that by Theorem 3.2.1, for all \star -periodic ν , there is no $\mu \in \Sigma_d^*$ such that $\mathcal{A}_i(\nu) > \mu > \nu$ or $\nu > \mu > \overline{\mathcal{A}}(\nu)$, nor is there a μ such that $B_i^q(\nu) > \mu > \mathcal{A}_i(\nu)$ or $\overline{B}^q(\nu) > \mu > \overline{\mathcal{A}}(\nu)$.

For any $\nu < \tilde{\nu}$ in Σ_d^* , we define the *combinatorial arc* connecting $\nu, \tilde{\nu}$ to be $[\nu, \tilde{\nu}] := \{\mu \in \Sigma_d^* : \nu \leq \mu \leq \tilde{\nu}\}$. Now we can formulate the Branch Theorem for the set $(\Sigma_d^*, <)$.

Theorem 3.3.20 (Branch Theorem for Σ_d^*). Let $\nu \neq \tilde{\nu} \in \Sigma_d^*$. Then either $\nu < \tilde{\nu}$, or $\tilde{\nu} < \nu$, or there is a unique $\mu \in \Sigma_d^*$ such that $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \mu]$. Moreover, μ is either preperiodic or $\mu \in \{\mu^*, \mathcal{A}_i(\mu^*), \overline{\mathcal{A}}(\mu^*)\}$ for some \star -periodic kneading sequence μ^* .

Proof. Let us first assume that $\nu, \tilde{\nu}$ are \star -periodic. Then by the Branch Theorem 3.3.17 for Σ_d^\sharp , $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \mu]$ or $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \mu[$. In the first case, $\mu \in \Sigma_d^\sharp$, and there is nothing to prove. In the second case, μ is a primitive \star -periodic sequence. We claim that in this situation, $[\overline{\mathfrak{x}}, \nu] \cap [\overline{\mathfrak{x}}, \tilde{\nu}] = [\overline{\mathfrak{x}}, \overline{\mathcal{A}}(\mu)]$. It suffices to prove that $\overline{\mathcal{A}}(\mu) < \nu$ and $\overline{\mathcal{A}}(\mu) < \tilde{\nu}$. The proof of Theorem 3.3.17 shows that either $\mu \in [\overline{\mathfrak{x}}, \nu]$ or ν branches off into non-admissibility at μ . In the first case, $\nu > \overline{\mathcal{A}}(\mu)$ by the definition of “ $<$ ”. In the second case, the Hubbard tree of ν contains an evil branch point b with itinerary $\overline{\mathcal{A}}(\mu)$. By Theorem 3.2.1, we get that $\nu \geq \overline{B}^q$, if $q+1$ is the number of arms at b . Now it follows by definition that $\nu > \overline{\mathcal{A}}(\mu)$. By symmetry, the same argument holds for $\tilde{\nu}$.

The way “ $<$ ” was defined for elements of Σ_d^* that are not periodic, the statement easily extends to these sequences. It remains to consider the situation where at least one of $\nu, \tilde{\nu}$ is periodic. Without loss of generality, we can

assume that the respective other sequence is either periodic or \star -periodic. Suppose that under these assumptions, ν and $\tilde{\nu}$ cannot be compared. Let n, \tilde{n} be the exact period of $\nu, \tilde{\nu}$ and set $\nu^* := \mathcal{A}_n^{-1}(\nu), \tilde{\nu}^* := \mathcal{A}_{\tilde{n}}^{-1}(\tilde{\nu})$. If $\nu^* = \tilde{\nu}^*$, then $\nu, \tilde{\nu} \in \{\mathcal{A}_i(\nu^*)\}$. (If one equaled the lower kneading sequence or ν^* itself then they would have been comparable to begin with.) Therefore, $[\bar{x}, \nu] \cap [\bar{x}, \tilde{\nu}] = [\bar{x}, \nu^*]$. If $\nu^* \neq \tilde{\nu}^*$ are comparable, then so are ν and $\tilde{\nu}$ by the definition of “ $<$ ”. If they are not comparable, let μ be their branch point and let μ^* be the associated \star -periodic kneading sequence ($\mu^* = \mu$ if μ is \star -periodic itself). If $\mu^* \notin \{\nu^*, \tilde{\nu}^*\}$ then $[\bar{x}, \nu] \cap [\bar{x}, \tilde{\nu}] = [\bar{x}, \mu]$. Otherwise assume that $\nu^* = \mu^*$. It follows that $\tilde{\nu}^*$, and thus $\tilde{\nu}$, lies in a non-admissible subwake of ν^* . Consequently, $[\bar{x}, \nu] \cap [\bar{x}, \tilde{\nu}] = [\bar{x}, \bar{\mathcal{A}}(\nu^*)]$. \square

3.4 Alternative Approaches for an Order on Hubbard Trees

Our discussion in the uncritical case is built on combining the two combinatorial concepts kneading sequences and Hubbard trees. This led to the Branch Theorem 3.3.17 which provides important information on the structure of the set of Hubbard trees in the sense of Definition 2.1.3. In Part II of this manuscript, we extend this approach to general cubic polynomials. Our goal is to get a hand on the structure of the set of cubic Hubbard trees because this would imply structural properties of the set of hyperbolic components in the cubic connectedness locus \mathcal{C}_3 . In general, cubic Hubbard trees have two critical points, which might interact in various ways (cf. the four fundamentally different types of hyperbolic components in Definition 4.1.8). In this environment of very diverse dynamics, kneading sequences and itineraries seem to be too inflexible to describe the global structure of the set of (admissible) Hubbard trees. For example, we know regions in \mathcal{C}_3 which contain Hubbard trees such that their critical values are not characteristic with respect to themselves. For our discussion of the partial order “ $<$ ” however, it is very important that the critical value is characteristic with respect to itself (this is automatically true in the unicritical case) because only then it makes sense that a smaller Hubbard tree is represented by a characteristic point in the larger one. Nevertheless, we think that in these regions one might be able to define a partial order: we can find a topological subtree in T whose vertices exhibit precisely the dynamics of the smaller tree \tilde{T} . So there is some dynamical relation between these two Hubbard trees though it cannot be observed by a partial order which is analogously defined as “ $<$ ” (see Section 5.1). Therefore, in this last section of Chapter 3, we want to discuss two alternative approaches for a partial order on the set of Hubbard trees which do not rely on kneading sequences. They are based on conversations with John Milnor and Mary Rees. We do not give a full discussion of the presented concepts. We want to point out their basic

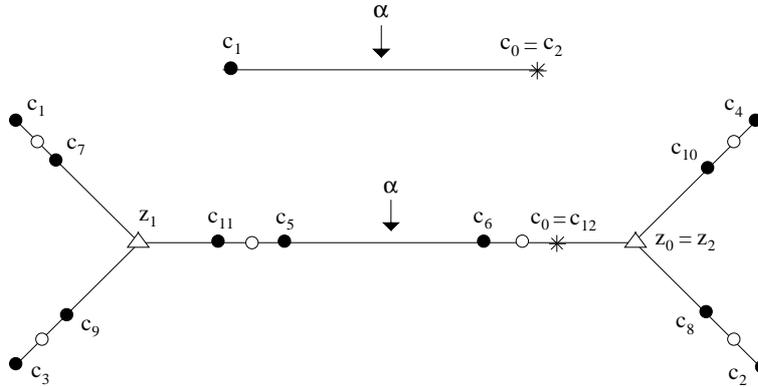


Figure 3.6: At the bottom, the Hubbard tree generated by the polynomial $z \mapsto z^2 + c$ with $c \approx -1.152 + 0.270i$. The subtree spanned by $\text{orb}(z_1)$ is an embedding of the underlying topological tree of the Hubbard tree of the basilica, that is pictured at the top. The embedding does not commute with the dynamics. (“o” marks another periodic orbit which represents a smaller Hubbard tree.)

ideas and the difficulties involved. This might serve as a basis for further research.

3.4.1 A Semi-Conjugacy Between Hubbard Trees

Let us start with some heuristics. Looking at two quadratic Hubbard trees $(T, f, \mathcal{P})_2 > (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_2$ as pictured in Figure 3.6, one sees that T contains a homeomorphic copy of \tilde{T} such that the action of f on the embedding of \tilde{V} is exactly the same as the action of \tilde{f} on \tilde{V} in \tilde{T} .

This suggests a semi-conjugacy φ between $f : T \rightarrow T$ and $\tilde{f} : \tilde{T} \rightarrow \tilde{T}$. A first difficulty for finding a semi-conjugacy is that we only regard minimal Hubbard trees. Hence, if the critical orbit is periodic, then it is locally attracting. Any other periodic orbit is locally repelling. In particular, the periodic orbit $\text{orb}(z_1)$ in T that spans the embedded copy of the smaller tree \tilde{T} is repelling while the corresponding orbit in \tilde{T} , the critical cycle, is attracting. Recall that we restricted ourselves to compare equivalence classes of Hubbard trees. So it suffices to find a semi-conjugacy between $(T, f, \mathcal{P})_d$ and any representative of the class $[(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d]$. In other words, we are looking for an f -invariant equivalence relation \sim on T such that $(T/\sim, f|_{T/\sim}, \mathcal{P}_\sim)_d \in [(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d]$. Here \mathcal{P}_\sim denotes the induced partition on the quotient. Since \sim is f -invariant, the quotient map is a semi-conjugacy.

Let us take a closer look at the two Hubbard tree of Figure 3.6. Let $z_1 \in T$ be the characteristic point that represents the smaller Hubbard tree. In order to get a semi-conjugacy, we have to identify every decoration of

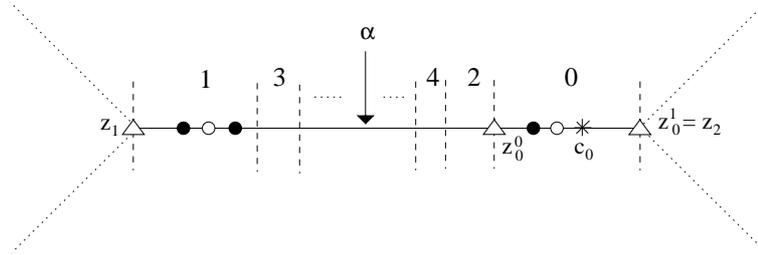


Figure 3.7: An attempt to find an f -invariant equivalence relation on T so that $(T/\sim, f_{T/\sim}, \mathcal{P}_\sim)_2$ is a Hubbard tree that is equivalent to the basilica. The interval $[z_0^0, z_0^1]$ has to be collapsed to a point. Therefore, the iterated preimages of $[z_0^0, z_0^1]$, which are the intervals labeled by $1, 2, \dots$, also have to be collapsed. However, they pairwise intersect so that $T/\sim = \{p\}$.

the subtree $[\text{orb}(z_1)]$ with the point where it is attached. Since we want f -invariant equivalence classes, we also have to identify (iterated) images and connected components of (iterated) preimages of such decorations. Note also that we also want to obtain a closed equivalence relation to make sure that the quotient is Hausdorff. Applying all this to the tree T yields that we have to identify all points of T with each other, so that we just get one single equivalence class (cf. Figure 3.7). Consequently, T/\sim is a point and not homeomorphic to \tilde{T} .

This phenomenon is not specific for the two considered Hubbard trees. In fact, it can be observed for any two admissible quadratic Hubbard trees $\mathcal{T}_1 < \mathcal{T}_2$ which have the property that, if W_1, W_2 are the corresponding hyperbolic components of $\mathcal{T}_1, \mathcal{T}_2$, then the combinatorial arc $]W_1, W_2]$ consists only of satellite components. The reason for this is that there is a (finite) sequence of tuning operation which turns \mathcal{T}_1 into \mathcal{T}_2 . Or from the opposite point of view, there is a sequence of renormalizations of β -type that maps \mathcal{T}_2 to \mathcal{T}_1 . The crucial point is that only β -type renormalizations occur and hence, in every renormalization step some images of the little filled-in Julia set intersect. To get an f -invariant equivalence relation on T_2 , we have to collapse the interval in T_2 around the critical point that lies in the critical-point Fatou component of $K(p_2)$, where p_2 is the polynomial generating \mathcal{T}_2 . Therefore, we also have to collapse all connected components of $f^{-n}(U) \cap T$. But this means that the part of the Hubbard tree that is contained in the little filled-in Julia set of the innermost quadratic-like map of $K(T)$ is collapsed to a point and thus all its images and preimages. Repeating this argument yields that at the end, we remain with the Hubbard tree associated to the primitive component W_0 , where W_0 is the hyperbolic component such that $[W_0, W_1]$ contains no further primitive component. For the Hubbard trees of Figure

3.6, W_0 is the main cardioid of \mathcal{M}_2 and we end up with a single point.

A way to go around this problem is to only consider an embedding of \tilde{T} into T that conjugates the dynamics on the marked points of \tilde{T} . This approach is discussed in the next section.

3.4.2 Embeddings of Hubbard Trees

Let us focus on Hubbard trees with periodic critical point. In Section 3.3.1, we defined a partial order “ $<$ ” on the set of \star -periodic kneading sequences. By Theorem 2.3.21, this induces an order on the set of Hubbard trees with periodic critical point. In fact, $[(T, f, \mathcal{P})_d] > [(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d]$ if and only if T contains a characteristic point z with itinerary $\tau(z) = \mathcal{A}_i(\tilde{\nu})$, where $\tilde{\nu}$ is the kneading sequence generated by $[(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d]$. Note that this definition is independent of the chosen representative and thus, the partial order “ $<$ ” is well-defined.

Our aim is to replace “ $<$ ” by a partial order which does not rely on kneading sequences and itineraries. This motivates the following definition. Recall that V denotes the set of marked points of a Hubbard tree $(T, f, \mathcal{P})_d$, i.e. the union of its branch points and of points in $\text{orb}(c_0)$.

Definition 3.4.1 (Dynamical embedding). *Let $(T, f, \mathcal{P})_d, (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ be two Hubbard trees with periodic critical point and let n be the exact period of \tilde{c}_0 . We say that $(T, f, \mathcal{P})_d$ dynamically contains the Hubbard tree $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ if the following holds: there is an embedding $\iota : \tilde{T} \rightarrow T$, an n -periodic characteristic point $z \in T$ and a bijection $\varphi : \tilde{V} \rightarrow \text{orb}(z) \cup \iota(\tilde{V} \setminus \text{orb}(\tilde{c}_0)) =: V_z$ such that*

- (i) $\varphi(\tilde{f}(\tilde{c}_0)) = z$;
- (ii) $\varphi \circ \tilde{f}|_{\text{orb}(\tilde{c}_0)} \equiv f \circ \varphi|_{\text{orb}(\tilde{c}_0)}$;
- (iii) $\iota(\tilde{T}) = [\text{orb}(z)]$, where $[\text{orb}(z)]$ is the convex hull of $\text{orb}(z)$ in T ;
- (iv) for any endpoint $f^{oi}(z) \in \iota(\tilde{T})$, $f^{oi}(z) \in T_j$ if and only if $\tilde{f}^{oi}(\tilde{c}_0) \in \tilde{T}_j$;
- (v) If $\tilde{v} \neq \tilde{w} \in \tilde{V}$ are adjacent in \tilde{T} then so are $\varphi(\tilde{v}), \varphi(\tilde{w}) \in V_z$ as vertices of the subtree $[\text{orb}(z)]$ of T .

On the other hand, if $\varphi(\tilde{v})$ and $\varphi(\tilde{w})$ are adjacent as vertices of $[\text{orb}(z)]$, then \tilde{v}, \tilde{w} are either adjacent or there is a unique branch point $\tilde{c}_i \in \text{orb}(\tilde{c}_0)$ of \tilde{T} such that $]\tilde{v}, \tilde{w}[\cap \tilde{V} = \{\tilde{c}_i\}$.

Note that in general, $V_z \not\subset V \subset T$. When we say that $\varphi(\tilde{v}), \varphi(\tilde{w}) \in V_z$ are adjacent as vertices of $[\text{orb}(z)]$ in item (iv), we mean that $]\varphi(\tilde{v}), \varphi(\tilde{w})[\cap V_z = \emptyset$. It is very well possible that $]\varphi(\tilde{v}), \varphi(\tilde{w})[\cap V \neq \emptyset$. In fact, the arc in T between two vertices that are adjacent in T_z might contain several elements of V .

Remark 3.4.2 (Images of adjacent vertices). Suppose that $\varphi(\tilde{v})$ and $\varphi(\tilde{w})$ are adjacent as vertices of T_z but $\tilde{v}, \tilde{w} \in \tilde{T}$ are not. Then item (iv) implies that $[\varphi(\tilde{v}), \varphi(\tilde{w}), \varphi(\tilde{c}_i)]$ is a non-degenerate triod and its interior does not intersect V_z : since \tilde{c}_i is the unique point of \tilde{V} which is contained in $] \tilde{v}, \tilde{w}[, \tilde{v}, \tilde{c}_i$ and \tilde{w}, \tilde{c}_i are adjacent. Therefore $\varphi(\tilde{v}), \varphi(\tilde{c}_i)$ and $\varphi(\tilde{w}), \varphi(\tilde{c}_i)$ and, by hypothesis, $\varphi(\tilde{v}), \varphi(\tilde{w})$ are adjacent, which proves the claim.

The requirements in Definition 3.4.1 might seem a bit artificial at the first glance. However, looking at some Hubbard trees generated by unicritical polynomials, we see that one has to ask for such conditions: of course, it is not enough to only consider a topological embedding of \tilde{T} without taking into account the dynamics on marked points: if we did so we could embed every Hubbard tree whose topological tree is an interval into any other Hubbard tree. Also, if we only considered $\iota(\tilde{V})$ instead of the set V_z , we would not get a desired result: if the critical point $\tilde{c}_0 \in \tilde{T}$ is a branch point, then any Hubbard tree larger than \tilde{T} (with respect to “ $<$ ”) will also have a critical branch point. It follows that $\iota(\tilde{c}_0) = c_0$. But c_0 has nothing to do with $\text{orb}(z)$ that spans the embedded tree $\iota(\tilde{T})$ (cf. Figures 3.9 and 3.11). What we really want is to associate to $\tilde{f}^{oi}(\tilde{c}_0)$ the point $f^{oi}(z)$. This is done by the bijection φ . Note that we cannot expect that $\varphi \circ \tilde{f}|_{\tilde{V}} \equiv f \circ \varphi|_{\tilde{V}}$: suppose that \tilde{T} contains a precritical branch point \tilde{b} so that $\tilde{b} \notin \text{orb}(\tilde{c}_0)$. Then $\varphi(\tilde{b})$ might also be a precritical branch point of the same step as \tilde{b} . As an example for such a situation, consider the Hubbard trees associated to $\tilde{\nu} = \overline{0112\star}$ and $B_2^2(\tilde{\nu})$. There $\tilde{f}(\tilde{b}) = \tilde{c}_0$ and $(f \circ \varphi)(\tilde{b}) = c_0$ while $(\varphi \circ \tilde{f})(\tilde{b}) = z_0^2$. As a further condition, we required for each endpoint e of $\iota(\tilde{T})$ that it is contained in the element of \mathcal{P} labeled by the same symbol as the element in $\tilde{\mathcal{P}}$ which contains the endpoint $\varphi^{-1}(e)$ of \tilde{T} . This is necessary, since we do not want to compare Hubbard trees that are contained in wakes of different sectors of some hyperbolic component. Finally, the exceptional case in item (v) is necessary because if $\tilde{c}_i \in \text{orb}(\tilde{c}_0) \subset \tilde{T}$ is a branch point, then in general $\varphi(\tilde{c}_i)$ is not on the critical orbit of T so that the degenerate triod $[\tilde{v}, \tilde{c}_0, \tilde{w}]$ with \tilde{c}_0 in the middle might become a non-degenerate triod $[\varphi(\tilde{v}), z_0^i, \varphi(\tilde{w})]$ in T . An example is given in Figure 3.11.

Definition 3.4.1 seems promising to find a partial order on Hubbard trees that does not rely on kneading sequences: we find the potentially smaller Hubbard tree in the potentially larger one as topological subtree and in addition, the relevant dynamics of the smaller Hubbard tree, namely the one on the marked points, is preserved. So let us define a relation “ \prec ” on the set of (equivalence classes of) Hubbard trees of degree d . Suppose that $\mathcal{T}, \tilde{\mathcal{T}}$ are minimal Hubbard trees of degree d . Then

$$\tilde{\mathcal{T}} \prec \mathcal{T} \quad : \iff \quad \mathcal{T} \text{ dynamically contains } \tilde{\mathcal{T}} \text{ and } \mathcal{T}, \tilde{\mathcal{T}} \text{ are not contained in the same equivalence class.}$$

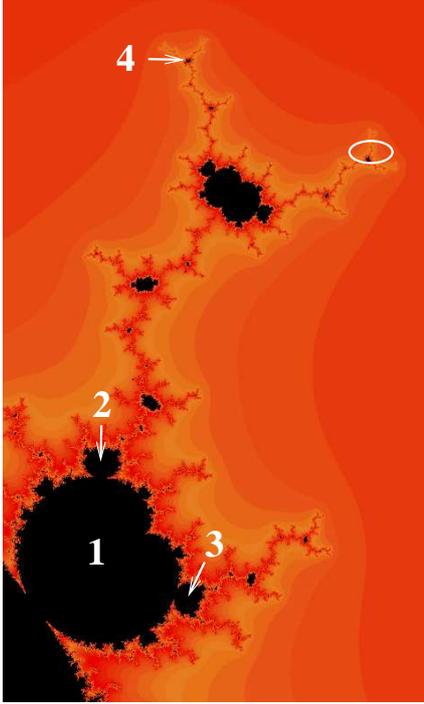


Figure 3.8: Location of the polynomials that generate the Julia sets considered in Section 3.4.2. The hyperbolic components W_1, W_2, W_3 , labeled by 1, 2, 3, show that an additional arm at the critical point of a Hubbard tree might be generated when bifurcating from a hyperbolic component. The corresponding Julia sets and Hubbard trees are pictured in Figure 3.10. The region surrounded by the white circle indicates the location of the two polynomials of Figure 3.11 which show that $\mathcal{T} > \tilde{\mathcal{T}} \not\asymp \mathcal{T} \succ \tilde{\mathcal{T}}$. The Hubbard tree of the hyperbolic component W_4 is used to show that $\mathcal{T} \succ \tilde{\mathcal{T}} \not\asymp \mathcal{T} > \tilde{\mathcal{T}}$, compare Figure 3.9.

Suppose that $[\mathcal{T}], [\tilde{\mathcal{T}}]$ are two distinct equivalence classes of Hubbard trees of degree d . Then we set $[\tilde{\mathcal{T}}] \prec [\mathcal{T}]$ if there are minimal representatives $(T, f, \mathcal{P})_d, (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ with $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d \prec (T, f, \mathcal{P})_d$.

The relation “ \prec ” is non-reflexive by definition. In order to check transitivity, one has to prove a result about *orbit forcing* similar to Proposition 3.2.3. We do not want to go into details here. We rather want to compare the relation “ \prec ” with the partial order “ $<$ ”. Under the assumption that “ \prec ” is a partial order, is it equivalent to the partial order “ $<$ ” defined via kneading sequences?

The answer, unfortunately, is no. Indeed, we are going to show that none implies the other. First, let us give an example of two cubic Hubbard trees $\mathcal{T} := (T, f, \mathcal{P})_3 \succ (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_3 =: \tilde{\mathcal{T}}$ such that $\mathcal{T} \not\asymp \tilde{\mathcal{T}}$. For $\tilde{\mathcal{T}}$, we choose the Hubbard tree associated to the kneading sequence $\overline{012\star}$. Its topological tree is a non-degenerate triod where the branch point is the critical point \tilde{c}_0 (in \mathcal{M}_3 , one possibility for a parameter generating this tree is $c \approx 0.2440 + 1.322i$). For \mathcal{T} , we choose the non-admissible Hubbard tree associated to $\overline{0120012\star}$. It is not hard to see that $\mathcal{T} \succ \tilde{\mathcal{T}}$, indeed $\iota(\tilde{T}) = [\text{orb}(x_1)]$. However, $\mathcal{T} \not\asymp \tilde{\mathcal{T}}$: this follows from Corollary 3.3.9 because $\overline{0120012\star} = \overline{B^2(012\star)}$. The situation is pictured in Figure 3.9.

For admissible Hubbard trees the described obstruction cannot occur; in fact, we have the following result.

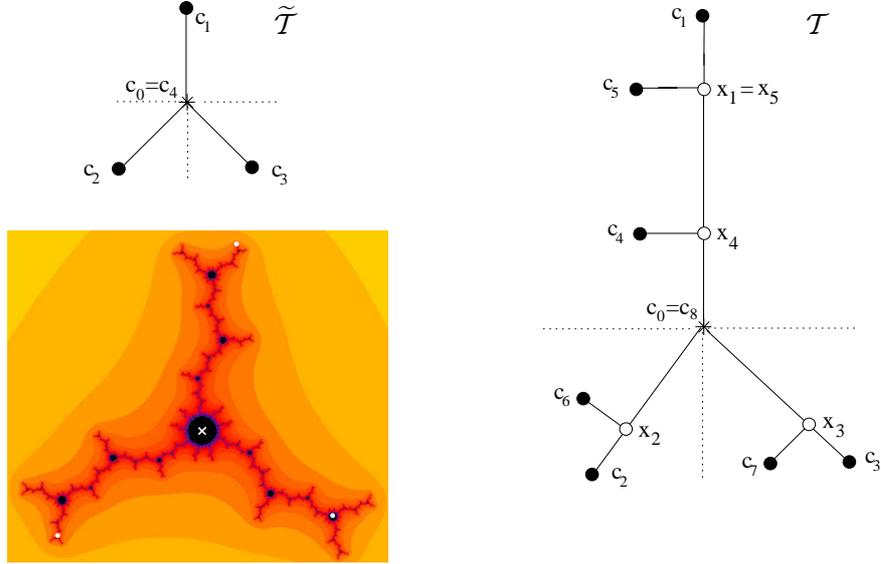


Figure 3.9: The Hubbard tree \tilde{T} can be dynamically embedded in to the non-admissible Hubbard tree T . Therefore $T \succ \tilde{T}$ yet $T \not\asymp \tilde{T}$. (\tilde{T} is generated by the hyperbolic component W_4 of Figure 3.8.)

Theorem 3.4.3 (Embedding and order). *If $(T, f, \mathcal{P})_d, (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ are two admissible Hubbard trees, then*

$$(T, f, \mathcal{P})_d \succ (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d \implies (T, f, \mathcal{P})_d > (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d.$$

Proof. Let $\tilde{\nu}$ be the kneading sequence generated by $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$. By Definition 3.4.1, the Hubbard tree T contains a characteristic point z of exact period n . So, we only have to show that $\tau(z) = \overline{\tilde{\nu}_1 \cdots \tilde{\nu}_{n-1} j}$ for some $j \in \{0, \dots, d-1\}$. If $\tau(z) = \overline{\mathcal{A}(\tilde{\nu})}$, then z is an inner point because $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})_d$ is admissible. Thus in this case, the claim follows by Theorem 3.2.1.

By way of contradiction, let us suppose that there is an i such that $\tau_i(z) \neq \tilde{\nu}_i$ for some $1 < i < n$. Then $f^{oi-1}(z) =: z_i$ and \tilde{c}_i are contained in elements of $\mathcal{P}, \tilde{\mathcal{P}}$ that have different labels. Since z_i is not an endpoint, there is an endpoint z_k of $[\text{orb}(z)]$ such that $z_i \in]z_k, c_0[$. Let $y \in]z_k, z_i] \cap V_z$ such that y and $\varphi^{-1}(y) \in \tilde{T}$ are contained in subtrees with different labels and such that there is no point in $]z_k, y[\cap V_z$ which has this property, too. Let $x \in V_z \cap]z_k, y[$ be adjacent to y . By the choice of x, y , we have that $\tilde{c}_0 \in]\varphi^{-1}(x), \varphi^{-1}(y)[$. Now Remark 3.4.2 implies that $[x, y, \varphi(\tilde{c}_0)]$ is a non-degenerate triod, where $\varphi(\tilde{c}_0)$ is a preimage of z . Therefore, $z_i \in]\varphi(\tilde{c}_0), c_0[$ and thus $f(z_i) \in]z, c_1[$, contradicting that z is characteristic. \square

Now let us take a closer look at the converse direction. Let $T > \tilde{T}$ be

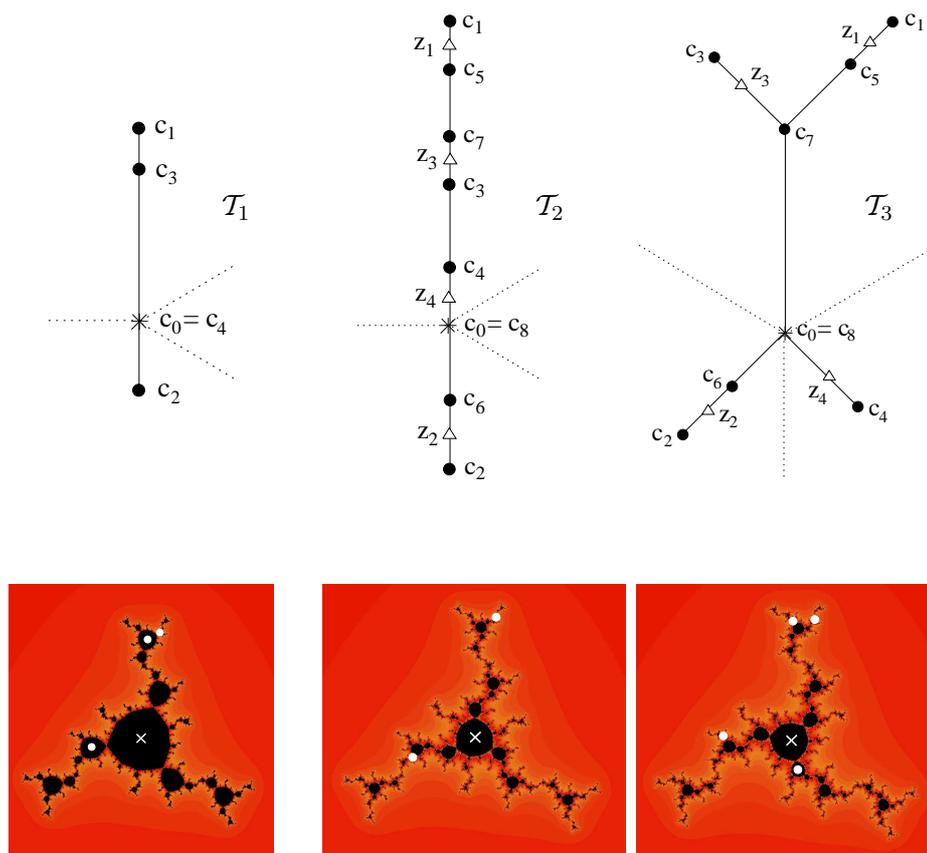


Figure 3.10: The Julia sets and Hubbard trees associated to the hyperbolic components W_1, W_2, W_3 from Figure 3.8. W_2 and W_3 are the satellite components at internal angle $\frac{1}{2}$ of the two sectors of W_1 . In terms of kneading sequences, this means the following: $\nu(W_1) = \overline{010\star}$, $\nu(W_2) = \overline{010010\star} = B_0^2(W_1)$ and $\nu(W_3) = \overline{010210\star} = B_2^2(W_1)$. We have that $\mathcal{T}_2 > \mathcal{T}_1$ and $\mathcal{T}_2 \succ \mathcal{T}_1$. However, $\mathcal{T}_1 \not\asymp \mathcal{T}_3$ although $\mathcal{T}_3 > \mathcal{T}_1$.

two admissible Hubbard trees such that, if $W, \widetilde{W} \subset \mathcal{M}_d$ are the hyperbolic components of $\mathcal{T}, \widetilde{\mathcal{T}}$, then W is a satellite component of \widetilde{W} . Let T, \widetilde{T} be the underlying topological trees and let $z \in T$ be the characteristic point with itinerary $\tau(z) = \mathcal{A}_j(\tilde{\nu})$, where $\tilde{\nu}$ is the kneading sequence generated by $\widetilde{\mathcal{T}}$. If now at this bifurcation a further arm at the critical point of T is generated, then $[\text{orb}(z)]$ is not homeomorphic to \widetilde{T} . In fact, $[\text{orb}(z)]$ will have one further arm compared to \widetilde{T} . In this situation, it is in general not possible to find an embedding of T into \widetilde{T} that meets all requirements of Definition 3.4.1. Figure 3.10 illustrates this problem.

This obstacle is of very special kind. It happens only in regions of \mathcal{M}_d where an additional arm at the critical point in the Hubbard tree is generated. Recall that the number of arms of Hubbard trees is at most d , and if the critical point c_0 of the Hubbard tree \mathcal{T} has d' arms then the critical point of any Hubbard tree $\mathcal{T}' > \mathcal{T}$ has at least d' arms. In other words, the discussed problem only happens if we branch off from an embedded copy of a Multibrot set of lower degree (see Section 3.5). However, this is not the only possibility for “ $> \Rightarrow \succ$ ” to fail, as illustrated in Figure 3.11. For simplicity, let us identify the Hubbard trees with their underlying topological trees. The Hubbard tree \mathcal{T}_0 generates the itinerary $\nu^0 = \overline{011201\star}$ and the Hubbard tree \mathcal{T}_2 generates $\nu^2 = \overline{011201201\star}$. Both Hubbard trees have a critical branch point with three arms. Nevertheless, the characteristic point $z_1 \in \mathcal{T}_2$ with itinerary $\tau(z_1) = \mathcal{A}_2(\nu^0)$ has the property that $[\text{orb}(z_1)]$ is not homeomorphic to \mathcal{T}_0 . We should mention that \mathcal{T}_2 contains also a characteristic point y with itinerary $\tau(y) = \overline{\mathcal{A}(\nu^0)}$ and $[\text{orb}(y)]$ is homeomorphic to \mathcal{T}_0 . Moreover, there is a bijection between V^0 , the set of marked points of \mathcal{T}_0 , and V_y that meets all conditions of Definition 3.4.1, so that $\mathcal{T}_2 \succ \mathcal{T}_0$. However this solution does not work if we consider \mathcal{T}_1 and \mathcal{T}_2 . The Hubbard tree \mathcal{T}_1 has kneading sequence $B_2^2(\nu^0)$. \mathcal{T} contains no characteristic point z whose itinerary starts with $0112012 \cdots$ such that $[\text{orb}(z)]$ is homeomorphic to \mathcal{T}_1 . But if $\mathcal{T}_2 \succ \mathcal{T}_1$ then such a point exists by (the proof of) Theorem 3.4.3. As a consequence, we have that $\mathcal{T}_2 > \mathcal{T}_1$ yet $\mathcal{T}_2 \not\succeq \mathcal{T}_1$.

3.5 Relations Between Different Degrees

3.5.1 Embeddings into Higher Degree

Every kneading sequence in Σ_d^* can be considered an element of $\Sigma_{d'}^*$ for all $d' \geq d$. So for any $2 \leq d < d'$, there is a natural injection $\iota : \Sigma_d^* \hookrightarrow \Sigma_{d'}^*$. We show that this map is an embedding with respect to an appropriate topology.

Lemma 3.5.1 (Sequences omitting a symbol). *Let $\nu \in \Sigma_d^*$ and suppose that the symbol $0 < s < d$ is not contained in ν . Then no kneading sequence $\tilde{\nu} \in \Sigma^*$ with $\tilde{\nu} < \nu$ contains s .*

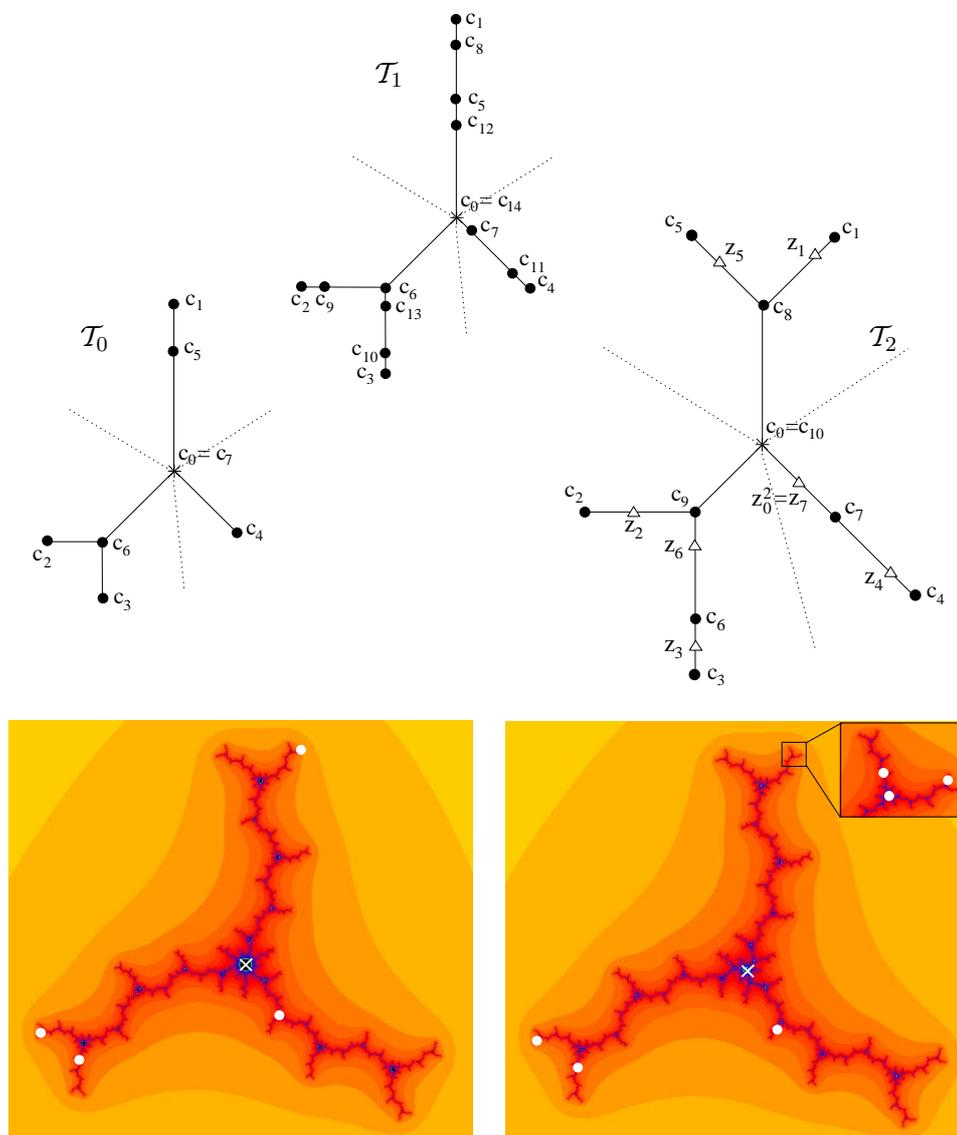


Figure 3.11: The Hubbard trees $\mathcal{T}_1, \mathcal{T}_2$ show that $\mathcal{T}_2 > \mathcal{T}_1 \not\asymp \mathcal{T}_2 \succ \mathcal{T}_1$, even if the number of arms at the critical point does not change. If W_0, W_1 are the hyperbolic components associated to $\mathcal{T}_0, \mathcal{T}_1$, then W_1 is the $1/2$ -satellite component of W_0 . The Hubbard tree \mathcal{T}_0 is represented by $\text{orb}(z_1)$ in \mathcal{T}_2 , the periodic orbit corresponding to \mathcal{T}_1 is not pictured. The parameters for the polynomials generating \mathcal{T}_0 and \mathcal{T}_2 are $\tilde{c} = 0.3288 + 1.281i$ and $c \approx 0.3293 + 1.280i$. Their Julia sets are pictured below.

Proof. Let us first show the claim for \star -periodic kneading sequences $\tilde{\nu}, \nu$. Since $\tilde{\nu} < \nu$, the Hubbard tree associated to ν contains a characteristic point z with itinerary $\mathcal{A}_i(\tilde{\nu})$. Thus, if $\tilde{\nu}$ contains the symbol t then so does $\mathcal{A}_i(\tilde{\nu})$, and there is a global arm G at c_0 that is labeled with the symbol t . Since T is spanned by $\text{orb}(c_0)$, it follows that there is an image $f^{\circ i}(c_0)$ ($i > 0$) of the critical point that is contained in G . Consequently, $\tau(f^{\circ i}(c_0)) = \sigma^{\circ i-1}(\nu) = t \cdots$ and the symbol t is contained in ν .

If $\mu \in \Sigma_d^0$ is not periodic and it contains the symbol t , then there is a truncated sequence ν^k which also contains the symbol t . Now the definition of the order on non-periodic sequences yields the claim for the set $\Sigma_d^* \setminus \{\nu \text{ is periodic}\}$.

So it only remains to consider the case when at least one of the sequences is periodic. By way of contradiction, suppose that $\tilde{\nu}$ contains the symbol s but ν does not. We first consider the case when ν is \star -periodic and $\tilde{\nu}$ is periodic. Let \tilde{n} be the period of $\tilde{\nu}$ and set $\tilde{\nu}^* := \mathcal{A}_{\tilde{n}}^{-1}(\tilde{\nu})$. Note that if s is contained in $\tilde{\nu}$ but not in $\tilde{\nu}^*$, then $\tilde{\nu}_{k\tilde{n}} = s$ for all $k \in \mathbb{N}$. Thus, $\tilde{n} \in \text{orb}_\rho(\tilde{\nu})$ and $\tilde{\nu} = \mathcal{A}_s(\tilde{\nu}^*) > \tilde{\nu}^*$. As a consequence, $\tilde{\nu}^* \neq \nu$ in this case. Suppose that $\tilde{\nu} = \mathcal{A}_i(\tilde{\nu}^*)$. Then the \star -periodic kneading sequence $B_i^q(\tilde{\nu}^*)$ also contains the symbol s for all $q \geq 2$. By the definition of “ $<$ ” on Σ_d^* , $\nu \geq B_i^q(\tilde{\nu}^*)$ for some q and s is contained in ν , as we have seen in the first paragraph. If $\tilde{\nu} = \overline{\mathcal{A}}(\tilde{\nu}^*)$, then s is contained in $\tilde{\nu}^*$ and $\overline{B}^q(\tilde{\nu}^*)$ for all $q \geq 2$. Since either $\nu > \tilde{\nu}^*$ or $\nu > \overline{B}^q$ for some q , we get the same contradiction as before.

Now suppose that ν is periodic and let n be its exact period. Then the \star -periodic kneading sequence $\mathcal{A}_n^{-1}(\nu) =: \nu^*$ does not contain the symbol s either. We have that either $\nu^* > \nu$ or $\nu > \nu^*$ such that there is no $\mu \in \Sigma_d^*$ with $\nu > \mu > \nu^*$. Thus, the claim follows by one of the cases that have been treated so far. \square

Now we define a topology on Σ_d^* that respects the partial order “ $<$ ”.

Lemma 3.5.2 (Topology on Σ_d^*). *For any $\nu \in \Sigma_d^*$, define*

$$L_\nu := \{\mu \in \Sigma_d^* : \mu > \nu\} \quad \text{and} \quad S_\nu := \{\mu \in \Sigma_d^* : \mu \not\geq \nu\}.$$

Then the set $\{L_\nu, S_\nu : \nu \in \Sigma_d^\}$ is a subbasis for a topology.* \square

Clearly, the sets L_ν, S_ν cover the whole space Σ_d^* . We call the obtained topology the *topology induced by “ $<$ ”*.

Proposition 3.5.3 (Embedding). *Let $2 \leq d < d'$ be two integers. Then there are $\binom{d'-1}{d-1}$ distinct ways to embed Σ_d^* into $\Sigma_{d'}^*$ (with respect to the topology induced by “ $<$ ”).*

Proof. Let $\iota : \Sigma_d^* \longrightarrow \Sigma_{d'}^*$ be the natural injection. In order to show that this map is continuous, it suffices to show that $\nu_1 < \nu_2$ in Σ_d^* if and only if $\iota(\nu_1) < \iota(\nu_2)$ in $\Sigma_{d'}^*$. But this follows immediately from the definition of the partial order via Hubbard trees and from Lemma 3.5.1.

Now consider any injective map $h : \{\star, 0, \dots, d-1\} \rightarrow \{\star, 0, \dots, d'-1\}$, which fixed 0 and \star . It induces a map $\tilde{h} : \Sigma_d^\star \rightarrow \Sigma_{d'}^\star$ via $\tilde{h}(\nu) = (h(\nu_i))_{i=1}^\infty$. Instead of picking the natural injection map, we can also consider the map $\tilde{h} : \Sigma_d^\star \rightarrow \Sigma_{d'}^\star$. By the same argument as before, \tilde{h} is an embedding. Since there are $\binom{d'-1}{d-1}$ choices for the map h , the claim follows. \square

Remark 3.5.4. If $\tilde{h}_i : \Sigma_d^\star \rightarrow \Sigma_{d'}^\star$ for $i = 1, 2$ such that $h_1(\{\star, 0, \dots, d-1\}) \cap h_2(\{\star, 0, \dots, d-1\})$ contains exactly δ elements and $\delta > 2$, then $\tilde{h}_1(\Sigma_d^\star) \cap \tilde{h}_2(\Sigma_d^\star)$ is an embedded image of $\Sigma_{\delta-1}^\star$.

3.5.2 Admissibility in Different Degrees

In [Kau2], Kauko studies the space of internal addresses as they were defined in Definition 2.3.1. She focuses on constructing and comparing *visible trees* in this set. With the methods developed there, she finds a class of non-admissible internal addresses that she calls *shadow components* [Kau2, —Section 5.19]. She leaves as an open question whether this class of non-admissible internal addresses comprises *all* non-admissible internal addresses. Our manuscript answers this question in the affirmative: any shadow component corresponds to a kneading sequence which is contained in a non-admissible subwake of an admissible kneading sequence. (The notion of subwakes was introduced in Section 3.3.2.) Proposition 3.3.14 says that any non-admissible kneading sequence has this property.

Kauko also gives sufficient conditions for internal addresses (and hence kneading sequences) to exist. Under the assumption that all non-admissible kneading sequences in Σ_d^\sharp correspond to shadow components, she derives (among others) the following results, which hold true by our classification of non-admissible kneading sequences:

- A kneading sequence with largest symbol $d-1$ is either realized by a postcritically finite polynomial in each connectedness locus $\mathcal{M}_{d'}$ of unicritical polynomials of degree $d' \geq d$ or in none.
- Every rough internal address (see definition below) is realized by some set of *sector numbers*.

Definition 3.5.5 (Rough internal addresses). *A strictly increasing sequence of integers that starts with 1 is called a rough internal address.*

Every rough internal address can be made into an internal address of degree d by replacing each entry n_i of the rough internal address by the tuple (n_i, s_i) , where $s_i \in \{1, \dots, d-1\}$. The s_i are exactly what Kauko calls sector numbers. Kauko's proofs are based on estimates on the width of *narrow sectors*.

Observe that the first statement follows immediately from our results. Recall that a \star - or preperiodic kneading sequence ν is non-admissible if

it violates the admissibility condition for some integer m (see Definition 2.4.8). Using the ρ -map, this can be read off directly from ν . Here, it does not matter if $\nu \in \Sigma_d^*$ is interpreted as a kneading sequence of degree larger than d . This is also in accordance with what we expect looking at Hubbard trees. Any Hubbard tree of degree d can be considered a Hubbard tree of degree $d' > d$. If the Hubbard tree is non-admissible, then it contains a branch point such that one of its local arm is fixed under the first return map. But this destroys admissibility for degree d' , too.

Proposition 3.5.6. *Let ν be a pre- or \star -periodic kneading sequence that is non-admissible. Then there is no complex polynomial p (of arbitrary high degree) such that p generates the kneading sequence ν . \square*

Part II

The Cubic Case

Chapter 4

The Dynamical Plane

4.1 Hubbard Trees

As in the unicritical case, we define cubic Hubbard trees in an abstract way so that not all of them are realizable by cubic polynomials. In this section, we discuss the basic dynamical properties of Hubbard trees and give necessary and sufficient conditions for their realizability by cubic polynomials.

4.1.1 Dynamic Trees

Let us first fix some notation. Most of the terminology was already introduced in Section 2.1. To keep Part I and Part II independent from each other, we repeat the basic definitions presented there and adapt them to the cubic setting where necessary.

A connected metric space T is a topological *tree* if it can be written as the finite union of closed intervals. We call $x \in T$ an *endpoint*, *inner point* or *branch point* if $T \setminus \{x\}$ consists of one, two or at least three connected components, respectively. Denote by $[x_1, \dots, x_n] \subset T$ the *convex hull* of $x_1, \dots, x_n \in T$. In the special case of $n = 2$, $[x, y]$ is the unique arc in T connecting x and y . The arc without its endpoints x, y is denoted by $]x, y[$, and $[x, y[$ is the arc containing the endpoint x but not the endpoint y . An *n-od* is a tree T with exactly one branch point b such that $T \setminus \{b\}$ falls into n connected components. For the special case of $n = 3$, we weaken this condition and require only that $T \setminus \{b\}$ falls into *at most* three connected components: given an tree T , we define a *triod* in T to be the connected hull of three distinct points $x, y, z \in T$. A triod is *non-degenerate* if it is homeomorphic to the letter “Y” and *degenerate* with x in the middle if $[x, y, z]$ is homeomorphic to an interval and x is contained in the interior of $[x, y, z]$. Observe that an *n-od* may lack one or several of its endpoints; when we speak of an *n-od* in a tree T , we do not imply that this is a closed set.

Definition 4.1.1 (Dynamic trees). *A dynamic tree is a tuple (T, f) consisting of a topological tree T with one or two critical points c_1, c_2 , and a continuous surjective map $f : T \rightarrow T$ such that the following are true:*

- (i) *The critical points are preperiodic or periodic.*
- (ii) *All endpoints of T are contained in $\text{orb}(c_1) \cup \text{orb}(c_2)$.*
- (iii) *f is a local homeomorphism on $T \setminus \{c_1, c_2\}$ and at most $3 - 1$. If $c_1 \neq c_2$, then f is at most $2 - 1$ in a neighborhood of any critical point.*

We refer to the set of points on the critical orbits as \mathcal{O} , i.e. $\mathcal{O} := \text{orb}(c_1) \cup \text{orb}(c_2)$. The set $[c_1, c_2] \subset T$ is called the *critical interval*, the image $v_i := f(c_i)$ of a critical point c_i is called a *critical value*. We usually denote the critical value by $f(c_i)$ to emphasize that it is the image of the critical point c_i . However, sometimes it is more convenient to use v_i . A point $p \in T$ is called *periodic of period n* if there is an $n \in \mathbb{N}$ such that $f^{\circ n}(p) = p$. The number n is called *exact period* of p if n is the smallest positive integer with this property. We say that a point is *n -periodic* if it is periodic of exact period n . The point p is *preperiodic* if $f^{\circ n}(p) \neq p$ for all n , however there is a smallest integer $l > 0$ such $f^{\circ l}(p)$ is periodic. The integer l is called *preperiod* of p . The orbit of a periodic point is usually called a *cycle*. If a cycle \mathcal{C} contains a critical point, then we say that \mathcal{C} is a *critical cycle*. A point ξ is called *precritical* if there is a $k > 0$ such that $f^{\circ k-1}(\xi) \in \{c_1, c_2\}$. If k is the smallest number with this property, then we call k the *step* of the precritical point ξ and write $\text{STEP}(\xi) = k$. That is, $\text{STEP}(\xi)$ indicates how many iteration steps it takes until ξ is mapped onto one of the critical values. Observe that according to this definition, any critical point is *precritical* of step 1 (both for periodic and preperiodic critical points). If a point p is critical or precritical of step at least two, we say that p is *(pre-)critical*.

Definition 4.1.2 (Global, local and regular arms). *Let (T, f) be a dynamic tree and $x \neq y \in T$. The connected components of $T \setminus \{x\}$ are called global arms of x . The global arm of x containing the point y is denoted by $G_x(y)$. A local arm L_x at x is a suitable representative of the germ of the respective global arm, i.e. L_x is a small enough interval $]x, p[\subset G_x(y)$. In particular, for $n \in \mathbb{N}$, choose the representative in such a way $f^{\circ n}|_{]x, p]}$ is injective. The local arm associated to the global arm $G_x(y)$ is denoted by $L_x(y)$. The image $f(L)$ of a local arm $L =]x, p[$ at x is the local arm at $f(x)$ that is represented by $]f(x), f(p)[$.*

Furthermore, we define the set of regular arms \mathcal{X}_x at x to be the union of all global arms of x which do not contain any critical point.

From the definition of dynamic trees, it follows quite easily that any branch point is periodic or preperiodic, no matter if it is (pre-)critical or not. For a non-(pre-)critical branch point b , the number of arms at any

point of the periodic part of $\text{orb}(b)$ is constant, say q , and at any point on the preperiodic part the number of arms is at most q .

At least one of the critical values must be an endpoint. It follows that at most one critical point can be a branch point, the second one must be an endpoint or an inner point such that its local arms collapse when mapped forward by f . If the two critical points c_1, c_2 are distinct and c_1 maps eventually onto c_2 such that $f(c_2)$ is an endpoint of the tree, then c_1 has at most four arms and $f(c_1)$ has at most two. If in addition c_1 is n -periodic and the local arms of c_1 are denoted by L_1, \dots, L_q for $1 \leq q \leq 4$, then there is one local arm, say L_1 , such that $f^{on}(L_i) = L_1$ for all $i = 1, \dots, q$.

A degree argument yields that $f(c_1) \neq f(c_2)$ if and only if $c_1 \neq c_2$. Since any critical orbit has finitely many elements, $\{f(c_1), f(c_2)\} \not\subset]c_1, c_2[$, and if $\{f(c_1), f(c_2)\} \subset [c_1, c_2]$, then either both critical points are fixed by f or $f(c_1) = c_2$ and $f(c_2) = c_1$.

4.1.2 From Dynamic Trees to Hubbard Trees

We will define *Hubbard trees* to be dynamic trees that meet an expansivity condition and are equipped with a partition \mathcal{P} . For quadratic Hubbard trees, a preferred partition is intrinsically contained in Definition 1.3.1; for unicritical Hubbard trees of arbitrary degree, we had to specify a partition, compare Definition 2.1.3. In the cubic case, we additionally have to impose certain requirements on the partition \mathcal{P} . These restrictions are motivated by the behavior of internal and external rays for cubic polynomials. Let us take a closer look at the relation between dynamic rays of a postcritically finite cubic polynomial $p_{a,b}$ and its Hubbard tree \mathbb{T} in the sense of Douady and Hubbard. Suppose the critical point c_1 is n -periodic and let F be the Fatou component containing c_1 . Then each local arm in \mathbb{T} at c_1 is contained in an internal ray of F . Therefore, investigating the behavior of local arms at c_1 under the first return map $p_{a,b}^{on}$ is equivalent to investigating the behavior of radial lines in \mathbb{D} under $z \mapsto z^d$, where d is the degree of the cycle containing c_1 , i.e. d is the product of the local degrees of points in $\text{orb}(c_1)$.

Suppose that c_1 is a periodic critical point and the second critical point c_2 is not contained in $\text{orb}(c_1)$. Then the Fatou component F_{c_1} containing c_1 is divided into two parts F_0, F_1 (recall the partition of the dynamical plane \mathbb{C} by internal and external rays described on page 8). This partition corresponds to the partition of the unit disk via the two radial lines at angle 0 and $\frac{1}{2}$ into the two semicircles $[0, \frac{1}{2}[$, $[\frac{1}{2}, 1[$ for the dynamical system (\mathbb{D}, z^2) .

Lemma 4.1.3 (Angle doubling map). *Let $S_0 = [0, \frac{1}{2}[$, $S_1 = [\frac{1}{2}, 1[\subset \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then there are no angles $\theta \neq \theta'$ such that for all i there is a $j_i \in \{0, 1\}$ with $2^i\theta, 2^i\theta' \in S_{j_i}$.*

Proof. If $\alpha, \beta \in S_j$ with $\alpha < \beta$, then $2\alpha \bmod 1 < 2\beta \bmod 1$. If $d := \beta - \alpha < 1/2$ then $2\beta - 2\alpha = 2d$. Using this, the claim follows immediately: if there

are two angles θ, θ' with $2^i\theta, 2^i\theta' \in \mathbb{S}_{j_i}$ for all i , then $2^i\theta' - 2^i\theta = 2^i d < 1/2$ for all i . Therefore $\theta = \theta'$. \square

Observe that the lemma also implies that there is no cycle of periodic angles of length at least two which is completely contained in some S_j . By the correspondence of inner rays and local arms, we can reformulate the previous lemma to get the following statement about Hubbard trees:

Corollary 4.1.4 (Local arms of periodic critical points). *Let \mathbb{T} be a Hubbard tree in the sense of Douady and Hubbard that is generated by the cubic polynomial $p_{a,b}$ and suppose that \mathbb{T} contains a periodic critical point c of degree two that is not eventually mapped onto the second critical point. Let F be the Fatou component of the critical point c and let F_0, F_1 be the two elements of the partition induced by internal rays. Then for any two local arms $L_c \neq L'_c$ at c there is an i_0 such that $f^{o_{i_0}}(L) \in F_0$ and $f^{o_{i_0}}(L'_c) \in F_1$, or vice versa. \square*

Definition 4.1.5 (Hubbard trees). *A Hubbard tree is a triple (T, f, \mathcal{P}) , where (T, f) is a dynamic tree and \mathcal{P} is a partition of T such that the following are true:*

- (P1) \mathcal{P} has five elements $\{c_1\}, \{c_2\}, T_0, T_1, T_2$ (several might be empty) such that $f|_{\overline{T_i}}$ is a homeomorphism for all i . If $c_1 \neq c_2$, then $c_1, c_2 \in \overline{T_0}$ and for $i = 1, 2$, $c_i \in \overline{T_i}$ unless $T_i = \emptyset$.
- (P2) If $c \in T$ is a periodic critical point such that c is disjoint from the orbit of the second critical point, then for any two distinct local arms L, L' at c there is an $n \in \mathbb{N}$ such that $f^{on}(L) \in T_i$ and $f^{on}(L') \in T_{i'}$ for some $i \neq i'$.
- (P3) (Expansivity) If $V := \{v \in T : v \text{ is a branch point or } v \in \mathcal{O}\}$, then for all $x \neq y \in V$, there is an $n \in \mathbb{N}$ such that $f^{on}(x) \in T_i$ and $f^{on}(y) \in T_{i'}$ for some $i \neq i'$.

An element of V is called a *marked point*. Two marked points $v, \tilde{v} \in V$ are *adjacent* if $]v, \tilde{v}[\cap V = \emptyset$.

There are Hubbard trees such that one or two of the subtrees T_i are empty. This can happen both if the two critical points coincide and if the two critical points are distinct. In the latter case, T_0 is never empty because it contains the critical interval. As an example, consider the Hubbard tree of the shape of an n -od such that $c_1 = c_2$, the non-(pre-)critical branch point is fixed by f and all points on the critical orbit are endpoints of T . In this case, $T = \overline{T_i}$ for some $i \in \{0, 1, 2\}$. For the Hubbard tree $([c_1, c_2], f, \{c_1, c_2,]c_1, c_2[\})$ with $f(c_1) = c_2$ and $f(c_2) = c_1$, we have that $T_1 = \emptyset = T_2$.

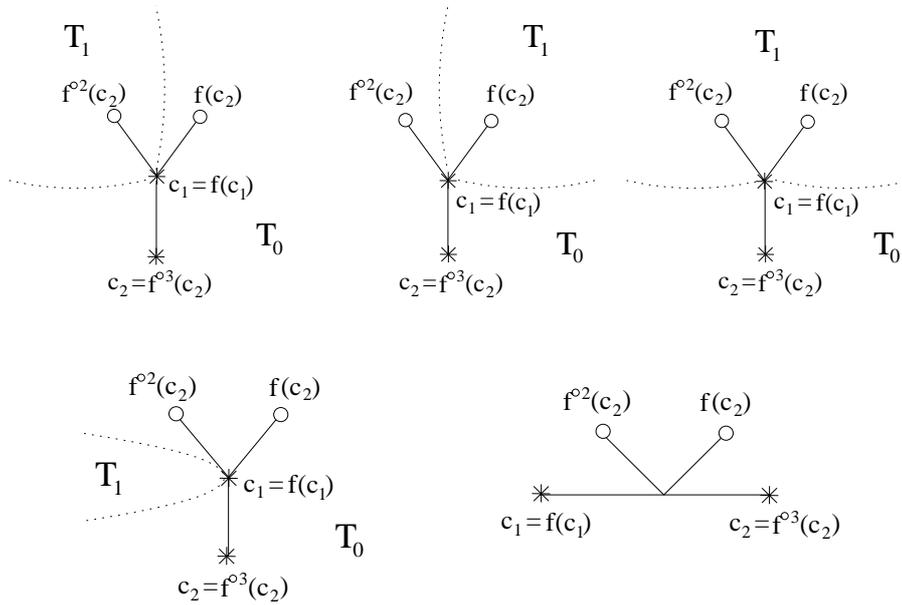


Figure 4.1: Top: three different possibilities to turn the given dynamic tree into a Hubbard tree. Bottom: the partition of the first tree violates (P2) of Definition 4.1.5, so it is not a Hubbard tree. The Hubbard tree on the right hand side with kneading sequence $(\overline{\star_1}, \overline{00\star_2})$ is not generated by a cubic polynomial: the two cycles of arms at its branch points have different length (compare Section 4.1.8).

Note that given a dynamic tree, there is not a unique choice for the partition \mathcal{P} to turn it into a Hubbard tree. For Hubbard trees that are generated by polynomials, different choices of the partition correspond to non-conjugate polynomials in general. This is always true if different partitions correspond to different choices for the secondary information of Hubbard trees in the sense of Douady and Hubbard. Figure 4.1 shows a very simple dynamic tree with different partitions.

Recall that Hubbard trees \mathbb{T} in the sense of Douady and Hubbard are only defined for postcritically finite polynomials. Their underlying topological trees equal the convex hulls of the points on the critical orbits, where connecting arcs within Fatou components correspond to inner rays. Therefore, a Hubbard tree in the sense of Douady and Hubbard can be interpreted as a Hubbard tree in the sense of Definition 4.1.5, where the secondary information at the critical points defines a partition \mathcal{P} . We ignore the secondary information for non-critical points. Thus there might be several Hubbard trees in the sense of Douady and Hubbard that generate the same Hubbard tree (T, f, \mathcal{P}) in the sense of Definition 4.1.5.

Proposition 4.1.6 (Hubbard trees of polynomials). *Every cubic Hubbard tree \mathbb{T} in the sense of Douady and Hubbard gives rise to a Hubbard tree (T, f, \mathcal{P}) .*

Proof. Consider the tuple $(\mathbb{T}, p_{a,b}|_{\mathbb{T}})$. By the definition of \mathbb{T} , it meets requirements (i)–(iii) of Definition 4.1.1. Let P_0, P_1, P_2, c_1, c_2 be the partition of \mathbb{C} by external and internal rays and the two critical points as described on page 8. This partition induces a partition \mathcal{P} on \mathbb{T} which meets condition (P1) of Definition 4.1.5. Corollary 4.1.4 yields condition (P2), so that it only remains to check expansivity. Let x, y be either branch points of \mathbb{T} or on the critical orbits. Expansivity trivially holds if one of the points x, y is on a critical cycle. If none of them are, then x and y are contained in the Julia set J . Without loss of generality we can assume that x and y are periodic. Let n be the smallest number such that x and y are fixed by $p_{a,b}^{on}$. Since they are contained in J , x and y are repelling and the interval $[p_{a,b}^{on}(x), p_{a,b}^{on}(y)] \subset \mathbb{T}$ must contain an attracting periodic point. Since any attracting periodic point in \mathbb{T} is an element of \mathcal{O} , there is a j such that c_1 or $c_2 \in [p_{a,b}^{oj}(x), p_{a,b}^{oj}(y)]$. \square

In the situation of Proposition 4.1.6, we say that (T, f, \mathcal{P}) is *generated by a polynomial*. The converse direction of the above statement is not true, i.e., there are Hubbard trees which are not generated by any postcritically finite cubic polynomial: we do not impose restrictions on the dynamical behavior of local arms of periodic points. We will see in Section 4.1.8 that there are exactly two obstructions for a Hubbard tree (T, f, \mathcal{P}) to be realizable by a cubic polynomial:

- T contains a non-(pre-)critical periodic branch point which has two cycles of local arms of different lengths.
- T contains a periodic critical point c which is disjoint from the second critical orbit and c has two cycles of local arms that have either both length one or both length two.

The partition \mathcal{P} allows us to associate to any point $z \in T$ a symbol sequence, its *itinerary*.

Definition 4.1.7 (Itinerary, kneading sequence). *Let (T, f, \mathcal{P}) be a Hubbard tree. Then for any $z \in T$, we define its itinerary $\tau(z) = (\tau_n(z))_{n=1}^{\infty}$ to be the sequence given by*

$$\tau_n(z) = \begin{cases} i & \text{if } f^{\circ(n-1)}(z) \in T_i \text{ for some } i \in \{0, 1, 2\}, \\ \star_i & \text{if } f^{\circ(n-1)}(z) = c_i \text{ for some } i \in \{1, 2\}. \end{cases}$$

The kneading sequence associated to (T, f, \mathcal{P}) is the tuple $(\tau(c_1), \tau(c_2))$, usually denoted by (ν^1, ν^2) .

We call a sequence $\tau \in \{0, 1, 2, \star_1, \star_2\}^{\mathbb{N}}$ \star_i -periodic if it is of the form $\overline{\tau_1 \cdots \tau_{n-1} \star_i}$, where $\tau_j \in \{0, 1, 2\}$ for all $0 < i < n$. Observe that a \star_1 -periodic sequence does not contain the symbol \star_2 (and vice versa). The itinerary of the critical point c_i is \star_i -periodic if and only if c_i is periodic and the second critical point is not contained in $\text{orb}(c_i)$.

4.1.3 Different Types of Hubbard Trees

There are various possibilities for the two critical points of a Hubbard tree to interact. In [M5], Milnor points out that there are four types of hyperbolic cubic polynomials that are fundamentally different with respect to this interaction. Following Rees [R], he calls these four cases *adjacent*, *bitransitive*, *capture* and *disjoint*. We transfer these notions to Hubbard trees.

Definition 4.1.8 (Types of Hubbard trees). *A Hubbard tree (T, f, \mathcal{P}) is called hyperbolic if each critical point eventually lies on a critical cycle.*

Suppose that (T, f, \mathcal{P}) is a hyperbolic Hubbard tree such that c_1 is periodic.

- *If $c_1 = c_2$, then (T, f, \mathcal{P}) is of adjacent type. We call the Hubbard tree (T, f, \mathcal{P}) unicritical.*
- *If $c_1 \neq c_2$ but there are $n_1, n_2 > 0$ such that $f^{n_1}(c_1) = c_2$ and $f^{n_2}(c_2) = c_1$, then (T, f, \mathcal{P}) is of bitransitive type.*
- *If there is an $n \in \mathbb{N}$ such that $f^{n_1}(c_2) = c_1$ and c_1 is never mapped onto c_2 , then (T, f, \mathcal{P}) is of capture type.*
- *If $\text{orb}(c_1) \cap \text{orb}(c_2) = \emptyset$ and c_2 is periodic, then (T, f, \mathcal{P}) is of disjoint type.*

Any hyperbolic Hubbard tree is of one of the four types above. If the itinerary $\tau(c_i)$ of the critical point c_i of (T, f, \mathcal{P}) is \star_i -periodic, then (T, f, \mathcal{P}) is neither adjacent nor bitransitive. Non-hyperbolic Hubbard trees show three distinct types of behavior: the two critical points coincide, the two critical orbits are disjoint or there are $n_1 \neq n_2$ such that $f^{n_1}(c_1) = f^{n_2}(c_2)$. In the first and third case, both critical point are preperiodic. Note that whenever we speak of a Hubbard tree that is of one of the four types of Definition 4.1.8, we assume implicitly that it is hyperbolic.

Remark 4.1.9 (Basic observations). *If (T, f, \mathcal{P}) is a unicritical Hubbard tree then the unique critical point c is of degree three and the critical value $f(c)$ is an endpoint of the tree. Thus, there are at most three local arms at c , exactly one of which is fixed under the first return map, the other two local arms are preperiodic. The elements T_i of \mathcal{P} are exactly the connected components of $T \setminus \{c\}$.*

If (T, f, \mathcal{P}) has two critical points $c_1 \neq c_2$, then both are of degree two. As an immediate consequence of requirements (P1) to (P3) on \mathcal{P} we get the following: suppose the critical point c_i is periodic and the second critical point is not contained in $\text{orb}(c_i)$, then the two subtrees T_0, T_i are not empty if and only if c_i is not an endpoint: let us assume that c_i is an inner or branch point and that $T_i = \emptyset$. Since $f|_{T_0}$ is injective, we have for any two local arms L, L' of c_i that $f^{\circ k}(L), f^{\circ k}(L') \subset T_0$ for all k , a contradiction to condition (P2). It follows that there are Hubbard trees (T, f, \mathcal{P}) such that f is locally injective at the critical point c_i yet T_i, T_0 are both non-empty, cf. Figure 4.2. If $T_i \neq \emptyset \neq T_0$, we say that c_i generates two non-trivial elements of \mathcal{P} .

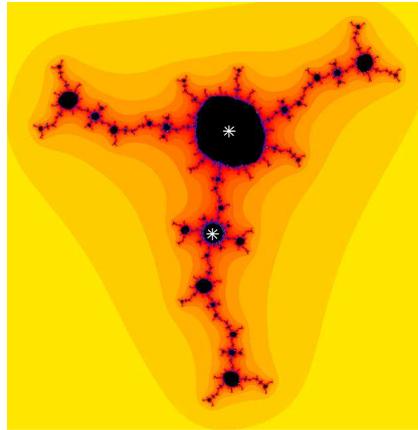
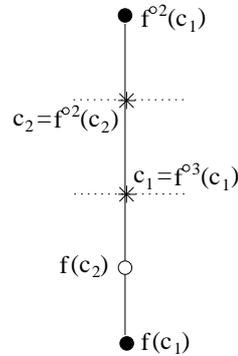


Figure 4.2: A Hubbard tree such that f is locally injective at the critical point c_2 . The critical points are marked by \star , black dots represent points on $\text{orb}(c_1)$, white dots represent points on $\text{orb}(c_2)$. We will stick to this convention in all subsequent figures. The Hubbard tree is generated by the cubic polynomial with parameters $a \approx -0.107 + 0.321i$, $b \approx 0.188 - 1.306i$ of the Branner–Hubbard form. Its Julia set is pictured on the right hand side.

If (T, f, \mathcal{P}) is a bitransitive tree that contains a critical branch point c_i , then c_i must generate two non-trivial elements of \mathcal{P} because two local arms at c_i have to collapse. However, if c_i is an inner point, then the two local arms may or may not collapse. In the latter case, c_i may or may not generate two non-trivial elements. The same is true for preperiodic critical points of non-hyperbolic Hubbard trees or of Hubbard trees of capture type. Figure 4.3 shows the (unique) Hubbard trees that generate the kneading sequences $(\overline{i2\star_1}, \overline{\star_1 i2})$ for $i = 0, 1$. As an example for the bitransitive case, consider the (unique) Hubbard trees generating the kneading sequences $(\overline{1i\star_2 0\star_1}, \overline{0\star_1 1i\star_2})$, where $i = 0, 2$.

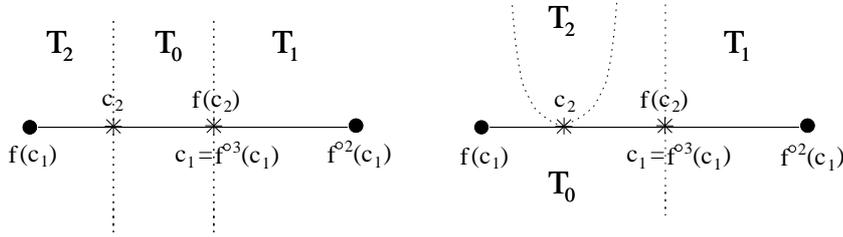


Figure 4.3: Possible choices to turn a dynamic tree into a Hubbard tree. The sketch illustrates that for the capture case an inner critical point might generate only T_0 . These two trees are the unique Hubbard trees that generate the kneading sequences $(\overline{i2\star_1}, \overline{\star_1 i2})$ for $i = 0, 1$ (up to changing the dynamics outside the set of marked points).

4.1.4 Points with Equal Itinerary

We start our study of the dynamical properties of (T, f, \mathcal{P}) by determining the structure of the subset in T of points that have the same itinerary. Let us set $\Sigma^\star := \{0, 1, 2, \star_1, \star_2\}^\mathbb{N}$.

Definition 4.1.10 (The set T_τ). *For any $\tau \in \Sigma^\star$, set*

$$T_\tau := \{p \in T : \tau(p) = \tau\}.$$

If $\sigma : \Sigma^\star \rightarrow \Sigma^\star$, $(\tau_i)_{i=1}^\infty \mapsto (\tau_i)_{i=2}^\infty$ be the standard shift map, then the action of f on T is semi-conjugate to the action of σ on Σ^\star . In particular, $f(T_\tau) \subset T_{\sigma(\tau)}$.

Lemma 4.1.11 (Properties of T_τ). *The set T_τ is either a (possibly degenerate) n -od, or there is (pre-)critical point b such that $T_\tau \cup \{b\}$ is an n -od. Moreover, b is preperiodic and disjoint from any critical cycle. If τ contains the symbol \star_1 or \star_2 , then T_τ is a singleton.*

If τ is periodic then T_τ is connected. If m_τ denotes the period of τ then $f^{om_\tau}|_{T_\tau} : T_\tau \rightarrow T_\tau$ is a homeomorphism. Moreover, the endpoints of T_τ are periodic, even if they do not belong to T_τ .

Proof. By expansivity, T_τ is a singleton for all itineraries τ containing the symbol \star_1 or \star_2 . For $\tau \in \{0, 1, 2\}^\mathbb{N}$, pick any $x, y \in T_\tau$. For all i , there is a j_i such that $f^{oj_i}(x), f^{oj_i}(y) \in T_{j_i}$ because $\tau_i(x) = \tau_i(y)$. Since T is simply connected and $f_{T_i}^{oj_i}$ is injective, we get inductively that $f^{oi}([x, y]) \in \overline{T_{j_i}}$ for all i . Consequently, for any $z \in \overline{T_\tau}$, we have either $\tau(z) = \tau$ or z is (pre-)critical. This argument also shows that $f^{oi}|_{\overline{T_\tau}}$ is injective for all i and thus any critical value which has a preimage in $\overline{T_\tau}$ is not an endpoint. This implies immediately that there is at most one critical point, say c_1 , such that

$\bigcup f^{-n}(c_1) \cap \overset{\circ}{T}_\tau \neq \emptyset$, and $c_1 \notin \text{orb}(c_2)$. Moreover, c_1 is not periodic: suppose it was. Then there are $x, y \in T_\tau$ and an i_0 such that $c_1 \in f^{\circ i_0}([x, y])$. We have that for all i , there is a T_{j_i} such that $f^{\circ i}(L_{c_1}(f^{\circ i_0}(x))), f^{\circ i}(L_{c_1}(f^{\circ i_0}(y))) \subset T_{j_i}$, in contradiction to requirement (P2). Now suppose that $\overset{\circ}{T}_\tau$ contains two iterated preimages ξ_1, ξ_2 of c_1 . They are of different step and without loss of generality, we can assume that $i_1 := \text{STEP}(\xi_1) < \text{STEP}(\xi_2) =: i_2$. The interval $f^{\circ i_2}([\xi_1, \xi_2]) = [f^{\circ(i_2-i_1)}(c_1), c_1]$ is mapped homeomorphically for all $k \in \mathbb{N}$. Since $f^{\circ j}(c_1) \notin \{c_1, c_2\}$ for all j , it follows that $c_1, f^{\circ(i_2-i_1)}(c_1)$ have the same itinerary, contradicting expansivity. Consequently, $\overset{\circ}{T}_\tau$ contains at most one (pre-)critical point. Moreover, by expansivity, T_τ contains at most one branch point of T and if a preimage of c_1 is in $\overset{\circ}{T}_\tau$, then T_τ contains no branch point at all, because otherwise this branch point and the precritical point would eventually have the same itinerary.

It remains to show the claimed properties for periodic $\tau \in \{0, 1, 2\}^{\mathbb{N}}$. First suppose that T_τ is not connected. Then $\overset{\circ}{T}_\tau$ is an n -od with branch point b such that $f^{\circ i_0}(b) = c_i$ and c_i is a preperiodic critical point that is not eventually mapped onto the second critical point. So $f^{\circ i_0+m_\tau}(c_i) \in T_{\sigma^{\circ i_0}(\tau)}$ and by expansivity, $f^{\circ i_0+2m_\tau}(c_i) = f^{\circ i_0+m_\tau}(c_i)$. However, since $f^{\circ i}|_{\overset{\circ}{T}_\tau}$ is injective for all $i \in \mathbb{N}_0$, $\overset{\circ}{T}_\tau$ contains no preperiodic point in its interior, a contradiction. Now let us show that $f^{\circ m_\tau}|_{T_\tau}$ is a homeomorphism. Clearly, $f^{\circ m_\tau}|_{T_\tau}$ is continuous, injective and $f^{\circ m_\tau}(T_\tau) \subset T_\tau$. If $f^{\circ m_\tau}(T_\tau) \neq T_\tau$ then there is an $x \in \partial T_\tau$ such that $f^{\circ m_\tau}(x) \in T_\tau$. This yields an open interval I containing x such that $f^{\circ m_\tau}(I) \subset T_\tau$. If I is sufficiently small and x is not (pre-)critical then all points $p \in I$ have itinerary τ , in contradiction to the definition of T_τ . If however x is precritical then $|\text{orb}(x)| < \infty$. Consequently, it must be preperiodic because $\overset{\circ}{T}_\tau$ contains no point of a critical cycle. But this is impossible as we have already seen; so $f^{\circ m_\tau}|_{T_\tau}$ is surjective. Since $f^{\circ m_\tau}|_{T_\tau}$ extends to a homeomorphism on $\overset{\circ}{T}_\tau$, it follows that for any $x \in \partial T_\tau$, $f^{\circ m_\tau}(x) \in \partial T_\tau$. And since T_τ has only finitely many boundary points, x must be periodic of period km_τ for some k . \square

The following lemma is an easy but useful observation.

Lemma 4.1.12 (No turning). *Suppose $f|_{[x,y]}$ is injective and there are three consecutive iterates $p, f(p), f^{\circ 2}(p) \in [x, y]$ such that $f(p) \in]p, f^{\circ 2}(p)[$. Then $f^{\circ 2}(p) \in]f(p), f^{\circ 3}(p)[$. \square*

Lemma 4.1.13 (Orbit points in $\overset{\circ}{T}_\tau$). *Let (T, f, \mathcal{P}) be a Hubbard tree, τ be periodic with exact period m and $x \in T_\tau$. Then for each critical point c_i there is a global arm G_{c_i} such that $\text{orb}_{f^{\circ m}}(x) \subset G_{c_i}$.*

Moreover, the point x is either (nm) -periodic such that the points on $\text{orb}_{f^{\circ m}}(x)$ are exactly the endpoints of a non-degenerate n -od, or $\text{orb}(x)$ is infinite and the iterates of x under $f^{\circ m}$ accumulate in a monotone fashion at an (nm) -periodic point y with $\text{orb}(y) \in \overset{\circ}{T}_\tau$.

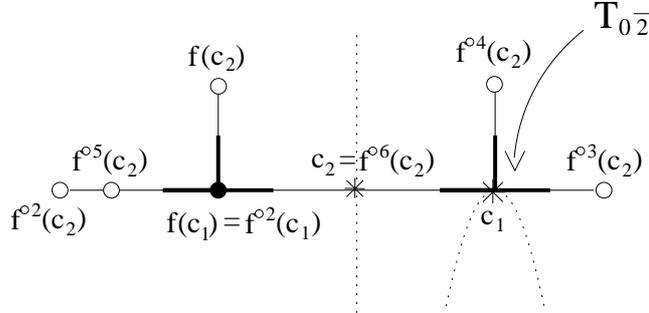


Figure 4.4: A sketch of a Hubbard tree that contains a disconnected set T_τ . In the picture $\tau = 0\bar{2}$.

Proof. The first statement is an immediate consequence of Lemma 4.1.11. Observe that x is not preperiodic because $f|_{T_i}$ is injective. If x is a periodic point, then by Lemma 4.1.12 either $f^{om}(x) = x$ or the points on $\text{orb}(x)$ are exactly the endpoints of an n -od for some $n > 1$. By expansivity, the branch point is fixed under f^{om} . Since $f_{T_\tau}^{om}$ is a homeomorphism, the claim for non-periodic points x follows (use again Lemma 4.1.12). \square

Corollary 4.1.14 (Orbits in the critical interval). *If $x \in T$ is periodic such that $\text{orb}(x) \in [c_1, c_2]$, then the exact period of x is at most two. Consequently, for any periodic point x of exact period at least three, there is an $x' \in \text{orb}(x)$ such that $x' \notin [c_1, c_2]$.* \square

In the case that $\text{orb}(c_1) \subset [c_1, c_2]$, c_2 might or might not be contained in $\text{orb}(c_1)$.

4.1.5 Periodic Points and Their Itineraries

In this section, we study the relation between periodic points and periodic itineraries. In particular, we are interested in comparing the exact periods of a point and of its itinerary. We do not present proofs of our statements because the proofs of the unicritical setting in Section 2.1.3 carry over literally to the cubic case.

Lemma 4.1.15 ((Pre-)periodic points and their itineraries). *If p is periodic of period n , then $\tau(p)$ is periodic of period n_τ , where $n_\tau|n$. A point $p \in T$ is preperiodic if and only if its itinerary $\tau(p)$ is preperiodic. The length of the preperiods of p and $\tau(p)$ coincide.* \square

Note that if $\tau(p)$ is periodic then it does not follow that p is periodic as well: consider for example a Hubbard tree \mathbb{T} in the sense of Douady

and Hubbard with a superattracting cycle \mathcal{C} . If $p \in \mathbb{T}$ is a point which is contained in one of the Fatou components associated to \mathcal{C} but not on the critical cycle itself then p has a periodic itinerary but its orbit is infinite, converging to the critical cycle.

Lemma 4.1.16 (Periods of itineraries and points). *Let (T, f, \mathcal{P}) be a Hubbard tree. The exact period and preperiod of any marked point equals the exact period and preperiod of its itinerary.*

If $z \in T$ is a periodic point with itinerary τ such that the period of τ is smaller than the period of z , then T_τ contains a periodic point z' that has the same period as τ . More precisely, z' is contained in the convex hull of $\text{orb}(z) \cap T_\tau$.

Proof. If in the second part of the statement $\overline{T_\tau}$ contains a branch point then it is not (pre-)critical by Lemma 4.1.11. Therefore the proof of Lemma 2.1.14 applies. \square

Lemma 4.1.17 (Length of periodic orbits). *Let τ be a periodic itinerary of exact period n . Then there exist $k_1, k_2 \in \mathbb{N}$ such that the period of any periodic point with itinerary τ is either n , $k_1 n$ or $k_2 n$.* \square

The proof of Lemma 4.1.17 works analogous to the proof of Lemma 2.1.15. Note that in the unicritical setting, we had only two possibilities for the periods of points in T_τ . This is because the number of possible periods depends on the maximal number of disjoint cycles of local arms at periodic points. This number is two for unicritical and three for cubic Hubbard trees. Let us explain the connection in more detail. There is at most one periodic point $b \in T_\tau$ of exact period n such that not all of its local arms are fixed under f^{on} . We will see later in Proposition 4.1.25 that there are at most three disjoint cycles of local arms at a periodic point and if there are exactly three then there is one distinguished arm that is fixed under f^{on} . (This distinguished arm is the image of the arm at (one of) the characteristic point(s) of $\text{orb}(b)$ pointing towards the critical points.) The numbers $k_1, k_2 > 1$ are the length of the non-trivial cycles C_1, C_2 of local arms at b (if existing). If G is the global arm associated to a local arm $L \in C_i$, then all periodic points contained in $T_\tau \cap G$ have exact period $k_i n$. If for all n -periodic points in T_τ all local arms are fixed, then all periodic points in T_τ have period n .

4.1.6 Characteristic Points

In this section, we investigate the mutual location of points in a periodic orbit.

For any n -periodic point $x \notin \mathcal{O}$, we set $x_j := f^{oj}(x)$ and $\mathcal{X}_j := \mathcal{X}_{x_j}$ and for each $0 \leq j < n$. Observe that for any $j \neq k$, we have either $\mathcal{X}_j \cap \mathcal{X}_k = \emptyset$, $\mathcal{X}_j \subsetneq \mathcal{X}_k$ or $\mathcal{X}_k \subsetneq \mathcal{X}_j$. Moreover, $x_j \notin [c_1, c_2]$ if and only if c_1, c_2 are in

the same global arm of x_j . Then $\mathcal{X}_j \neq \emptyset$ because no point in $\text{orb}(x)$ is an endpoint. If $\text{orb}(x) \not\subset [c_1, c_2]$, then there is an $l \in \mathbb{N}_0$ such that $x_l \notin [c_1, c_2]$ and $\text{orb}(x) \cap \mathcal{X}_l = \emptyset$.

Lemma 4.1.18 (Orbit intersects $[c_i, f(c_i)]$). *Let x be a periodic point such that $\text{orb}(x) \not\subset [c_1, c_2]$ and suppose that $\tau(x) \neq \tau(e)$ for all non-(pre-)critical endpoints e . Then there is an $\hat{x} \in \text{orb}(x)$ and an $i \in \{1, 2\}$ such that $\hat{x} \in [c_i, f(c_i)]$ and c_1, c_2 are in the same global arm of \hat{x} . In particular, if $x = c_1$ and $f(c_1) \in]c_1, c_2[$, then there is an iterate $f^{\circ j}(c_1) \in [c_2, f(c_2)]$. (The same holds if we interchange the role of 1 and 2.)*

Proof. First consider the case that there is an $x_j \in \text{orb}(x) \cap [c_1, c_2]$. Let us first assume that x is disjoint from any critical cycle. There is a first iterate \hat{x} of x_j which is mapped outside $[c_1, c_2]$. Since $f|_{[c_1, c_2]}$ is injective, $\hat{x} \in]c_i, f(c_i)[$ for some $i \in \{1, 2\}$. Now suppose that $x \in \text{orb}(c_1)$. If $f(c_1) \notin [c_1, c_2]$, then $\hat{x} := f(c_1)$ does the job. Otherwise there is a first iterate \hat{x} of c_1 such that $\hat{x} \notin [c_1, c_2]$ and hence $\hat{x} \in [f(c_2), c_2] \setminus [c_1, c_2]$.

If $\text{orb}(x) \cap [c_1, c_2] = \emptyset$, then for all k , the critical points c_1, c_2 are contained in the same global arm of x_k . Pick any \mathcal{X}_k . By expansivity, there is a first time k_0 such that $f^{\circ k_0}(\mathcal{X}_k)$ contains an immediate preimage of a critical point, say c_1 , unless $\tau(x)$ equals the itinerary of a non-(pre-)critical endpoint. Let $-c_1$ be this preimage. If $c_2 \notin [-c_1, c_1]$, then $\hat{x} := f^{\circ k_0+1}(x_k) \in [c_1, f(c_1)]$. Otherwise $\hat{x} \in [c_1, f(c_2)]$, and since both critical points are contained in the same global arm of \hat{x} , we have $\hat{x} \in [c_2, f(c_2)]$. \square

Note that in the statement of the above lemma also includes unicritical Hubbard trees.

For the next definition, recall that the critical value $f(c_i)$ is also denoted by v_i .

Definition 4.1.19 (Characteristic points). *Let x be a periodic point. If there is an $i \in \{1, 2\}$ and an $\hat{x} \in \text{orb}(x) \cap]c_i, f(c_i)[\setminus [c_1, c_2]$ such that $\text{orb}(x) \in \overline{G_x(c_i)}$, then the point \hat{x} is called the v_i -characteristic point of $\text{orb}(x)$. A point z is characteristic if it is periodic and if it is the v_i -characteristic point of $\text{orb}(z)$ for $i = 1$ or $i = 2$.*

Proposition 4.1.20 (Existence of characteristic points). *Let x be a periodic point such that $\text{orb}(x) \not\subset [c_1, c_2]$ and such that $\tau(x) \neq \tau(e)$ for all non-(pre-)critical endpoints e . Then $\text{orb}(x)$ contains a characteristic point \hat{x} . If $\text{orb}(x)$ is a critical cycle, then $\hat{x} = f(c_i)$ might be an endpoint.*

Proof. If c is a critical point such that $\text{orb}(c)$ contains an endpoint, then $\text{orb}(c)$ also contains at least one critical value which is an endpoint, and we are done. For all other cases, let us assume by way of contradiction that $\mathcal{X}_j \cap \text{orb}(x) \neq \emptyset$ for all points x_j for which there is a critical point c_i with $x_j \in [c_i, f(c_i)] \setminus [c_1, c_2]$. By Lemma 4.1.18, there is an iterate x_{j_0} and a

critical point, say c_1 , such that $x_{j_0} \in [c_1, f(c_1)] \setminus [c_1, c_2]$. Denote by \hat{x} the point in $[c_1, f(c_1)]$ closest to $f(c)$ and by $\hat{\mathcal{X}}$ its associated set of regular arms. By assumption, $\hat{\mathcal{X}} \cap \text{orb}(x) \neq \emptyset$. We will show that there is no $x_j \in \text{orb}(x)$ such that $x_j \notin [c_1, c_2]$ and $\mathcal{X}_j \cap \text{orb}(x) = \emptyset$, which clearly is impossible.

Push \hat{x} and $\hat{\mathcal{X}}$ forward until $f^{\circ k-1}(\hat{\mathcal{X}})$ contains an immediate preimage of one of the critical points, which must happen by our assumptions. For $j = 0, \dots, k-1$, the set $f^{\circ j}(\hat{\mathcal{X}})$ is a homeomorphic image of $\hat{\mathcal{X}}$ and hence contains at least one point of $\text{orb}(x)$. Since $f^{\circ j}(\hat{\mathcal{X}}) \subset \mathcal{X}_j$, \mathcal{X}_j also contains at least one point of $\text{orb}(x)$. If $f^{\circ k}(\hat{x}) \in [c_1, c_2]$, we consider the first time this point is mapped out of the critical interval. Let this image be $\hat{\hat{x}}$. Since $f|_{[c_1, c_2]}$ is injective, $\hat{\hat{x}} \in [c_i, f(c_i)]$ for some $i \in \{1, 2\}$ and hence its set of regular arms contains a point of $\text{orb}(x)$ by assumption. Now replace \hat{x} by $\hat{\hat{x}}$ and start all over again. If $f^{\circ k}(\hat{x}) \notin [c_1, c_2]$, recall that $f^{\circ k-1}(\hat{x}) \in]-c, c_i[$ for some critical point c_i and some immediate preimage $-c$ of the critical point c . Thus, $f^{\circ k}(\hat{x}) \in [c_i, f(c_i)]$, and again \mathcal{X}_k contains at least one point of the orbit. Replace \hat{x} by $f^{\circ k}(\hat{x})$ and repeat the argument. After a finite number of iterations we are back at \hat{x} . So we showed that there is no point in $\text{orb}(\hat{x}) \setminus [c_1, c_2]$ whose union of regular arms contains no point of $\text{orb}(x)$. \square

Remark 4.1.21. Note that unlike in the unicritical case, it is not true that x is v_i -characteristic if $x \in]c_i, f(c_i)[$ is periodic such that $\text{orb}(x) \cap]x, f(c_i)[= \emptyset$. A counterexample can be found in Figure 4.6.

A periodic orbit meeting the requirements of Proposition 4.1.20 might or might not contain a characteristic point with respect to each critical value. There are also examples where a point is v_1 - and v_2 -characteristic. For a v_1 -characteristic point \hat{x} , it is very well possible that the second critical point $c_2 \in [c_1, \hat{x}]$. Also, $f(c_1)$ need not be an endpoint of the tree T . This happens for example when T contains a subtree that is spanned by $\text{orb}(c_1)$ which is conjugate to a quadratic Hubbard tree. Some examples are shown in Figure 4.5.

4.1.7 Behavior of Local and Global Arms

We will take advantage of the existence of characteristic points to describe the behavior of global and local arms at periodic points under the action of the first return map. The basic ideas are very similar to the unicritical case.

Definition 4.1.22 (Hitting a critical point, cycle of local arms). *Let G be a global arm of a point x and $n \in \mathbb{N}$. If no critical point is contained in $f^{\circ j}(G)$ for all $j = 0, \dots, n$, then we say $f^{\circ n}$ maps G homeomorphically onto its image without hitting a critical point. If there is an $0 \leq j \leq n$ for which $c_i \in f^{\circ j}(G)$ we say that G is mapped over the critical point c_i .*

Let y be an n -periodic point disjoint from any critical cycle. For any local arm L of y the set $\mathcal{C}(L) := \{f^{\circ j n}(L) : j \in \mathbb{N}_0\}$ is called a cycle of local

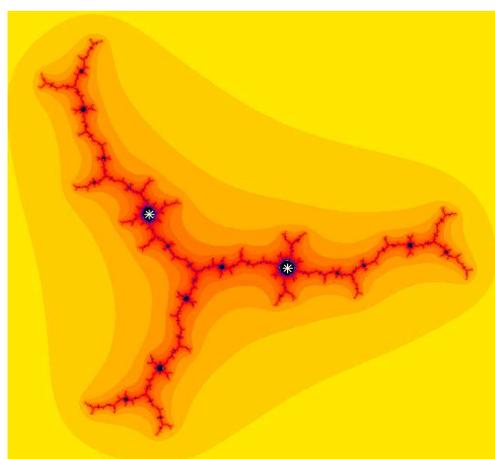
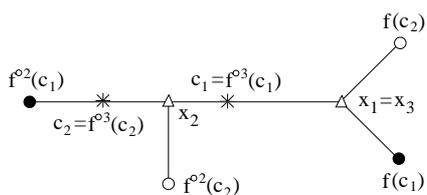
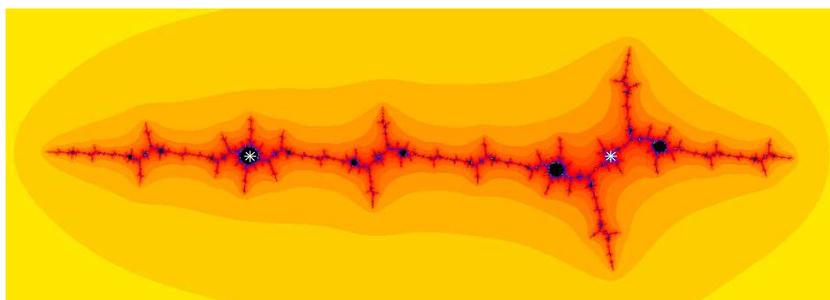
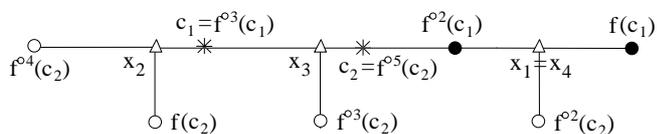


Figure 4.5: The upper figure shows a Julia set whose Hubbard tree contains a periodic orbit $\text{orb}(x_1)$ with two characteristic points: x_1 is v_1 -characteristic and x_2 is v_2 -characteristic. At the bottom, a Julia set whose Hubbard tree has a periodic orbit $\text{orb}(x_1)$ such that x_1 is v_1 - and v_2 -characteristic at the same time.

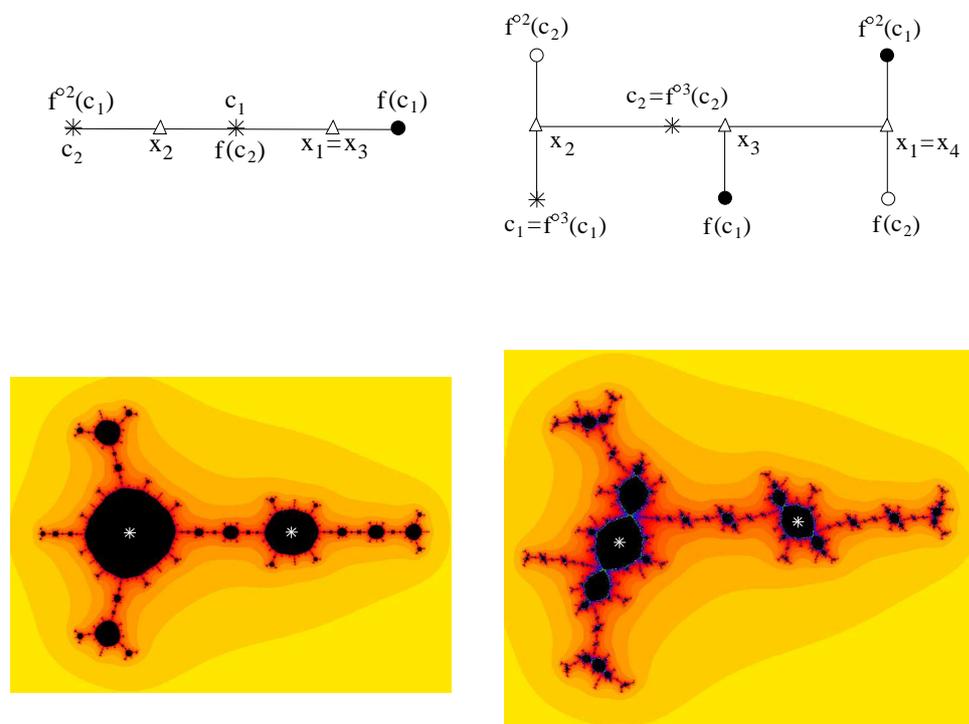


Figure 4.6: The right Hubbard tree shows that a periodic point in $]c_i, f(c_i)[$ which is closest to the critical value need not be characteristic: here the periodic point $x_3 \in]c_1, f(c_1)[$ is such that $\text{orb}(x_1) \cap]x_3, f(c_1)[= \emptyset$ yet x_3 is not characteristic. It also provides an example for a periodic branch point x that has three local arms L_i which are all fixed under the first return map (which is $f^{\circ 3}$ in the picture). The Hubbard tree belongs to a satellite component of the hyperbolic component associated to the left Hubbard tree. The left Hubbard tree shows an example for a periodic orbit that has just one characteristic point that is characteristic with respect to exactly one critical value. The Julia sets that generate the two Hubbard trees are pictured below.

arms. The length of such a cycle equals the number of its elements. We call a cycle trivial if it has only one element.

Proposition 4.1.23 (Global arms under first return map). *Let x be an n -periodic point such that $\text{orb}(x) \not\subset [c_1, c_2]$ and $\tau(x) \neq \tau(e)$ for all endpoints e with $\text{orb}(e) \cap \{c_1, c_2\} = \emptyset$. Suppose that \hat{x} is the v_1 -characteristic point of $\text{orb}(x)$. Then there is a $0 < \kappa \leq n$ such that any global arm G at \hat{x} exhibits exactly one of the following behaviors under $f^{\circ n}$:*

- $f^{\circ n}$ maps G homeomorphically onto its image without hitting a critical point.
- There is an $0 \leq j \leq n$ and a critical point c with $c \in f^{\circ j}(G)$ and $f^{\circ n}$ maps the local arm of G either
 - to the local arm of \hat{x} that points to the critical points, or
 - to the local arm of \hat{x} that points to $f(c_1)$, or
 - to the local arm of \hat{x} that points to the iterate $f^{\circ \kappa}(c_2)$ of the critical point c_2 and $f^{\circ \kappa}(c_2)$ is not contained in the same global arms as the critical points or as $f(c_1)$.

Note that the statement also holds if $\text{orb}(x)$ is a critical cycle and also if \hat{x} is v_2 -characteristic (interchange then the symbols 1 and 2 in the statement).

Proof. Assume that there is an iterate $f^{\circ j}(G)$, $j \in \{0, \dots, n\}$ which contains a critical point. Let L be the local arm associated to G and let G' be the maximal subset of G with $\hat{x} \in \partial G'$ that is mapped homeomorphically by $f^{\circ n}$.

First suppose that G' contains a precritical point ξ of $\text{STEP}(\xi) = n + 1$. Then there is a critical point $c \in f^{\circ n}(G')$, and hence $f^{\circ n}(L)$ points towards the critical points. So from now on, let us assume that for all precritical points $\xi \in G'$, $\text{STEP}(\xi) \neq n + 1$. If G' contains a precritical point ξ such that $f^{\circ j_0}(\xi) = c_1$ for some $j_0 < n$, then $f^{\circ j_0+1}(G')$ contains the critical value $f(c_1)$ and consequently a point of $\text{orb}(x)$ unless $j_0 + 1 = n$; in the latter case $f^{\circ j_0+1}(L) = f^{\circ n}(L)$ points to $f(c_1)$. Otherwise, any iterate $f^{\circ j}(G')$ with $j_0 < j \leq n$ also contains to a point of $\text{orb}(x)$, and thus $f^{\circ n}(L)$ points to the critical points.

The remaining case is that no precritical points in G' with step at most n is mapped onto c_1 by $f^{\circ j}$ for $j \leq n$. Then $f^{\circ n}(L)$ points towards $f^{\circ k}(c_2)$ for some $k \leq n$. We will show that among all global arms of x that have this property there is at most one such that $f^{\circ n}(L)$ is neither pointing towards the critical points nor to the critical value $f(c_1)$. As a consequence, there is a unique κ such that the statement of the Lemma holds true. By way of contradiction, suppose there were two global arms G_1, G_2 of x with this property (and for case that we deal with the critical orbit, suppose that the

associated local arms do not collapse under $f^{\circ i}$ for some $0 < i \leq n$). Then $\text{orb}(\hat{x}) \cap f^{\circ j}(G'_i) = \emptyset$ for all $j \leq n, i = 1, 2$, because otherwise $f^{\circ n}(G'_i)$ would also contain a point of $\text{orb}(\hat{x})$ and hence point towards the critical points. Let ξ_1, ξ_2 be the precritical points in G'_1, G'_2 that have the largest step smaller than n , and let $l_i := \text{STEP}(\xi_i)$. Then $l_1 \neq l_2$ because $f^{\circ n}|_{G'_1 \cup G'_2}$ is injective. Without loss of generality, let $l_1 < l_2$. The interval $[\xi_1, \xi_2] \ni \hat{x}$ maps homeomorphically onto $[f^{\circ l_2 - l_1}(c_2), f(c_2)] \ni f^{\circ l_2}(\hat{x})$. Since $]f^{\circ l_2}(\hat{x}), f(c_2)[$ does not contain a point of $\text{orb}(x)$ by hypothesis, the three points $f^{\circ l_1}(\hat{x}), f^{\circ l_2}(\hat{x}), f(c_2)$ are either the endpoints of a non-degenerate triod or form a degenerate triod such that $f^{\circ l_2}(\hat{x})$ is in the middle. Let Y be this triod. Since $f^{\circ n - l_2}|_Y$ is injective, $f^{\circ n - l_2}(Y)$ is contained in a global arm G of \hat{x} . Since $f^{\circ n}(G'_2) \subset G$, G does not contain the critical points. On the other hand, G contains $f^{\circ n - l_2 + l_1}(\hat{x})$ and thus, since \hat{x} is characteristic, it must also contain c_1, c_2 , a contradiction. \square

Remark 4.1.24. Observe that there are orbits of branch points where the last possibility of case (ii) does not occur (this is trivial for inner points). For example take any cubic Hubbard tree that is obtained by intertwining two quadratic Hubbard trees at their α - or β -fixed points. (For the definition of *intertwining* we refer to [EY].) If \hat{x} is v_1 -characteristic and one of its global arms hits a critical point, then the associated local arm points either to the critical points or to $f(c_1)$. There are also examples for other types of Hubbard trees.

Suppose that \hat{x} is characteristic with respect to both critical values. Then we get a slightly stronger statement: $f^{\circ n}$ maps each global arm of \hat{x} either homeomorphically onto its image without hitting a critical point or the image of the corresponding local arm points towards the critical points, to $f(c_1)$ or to $f(c_2)$, i.e. $\kappa = 1$ in the statement of the above proposition. This is because if ξ is precritical of step $k < n - 1$, then $f^{\circ k}(L)$ points to the characteristic point both if $f^{\circ k - 1}(\xi) = c_1$ and $f^{\circ k - 1}(\xi) = c_2$.

The next proposition is an analog to Kiwi's statement about cycles of external rays that land at periodic points of polynomial Julia sets [Kil, Theorem 3.1].

Proposition 4.1.25 (Local arms of periodic points). *Let x be a non-(pre)-critical, n -periodic point such that $\text{orb}(x) \not\subset [c_1, c_2]$ and such that $\tau(x) \neq \tau(e)$ for all endpoints e with $\text{orb}(e) \cap \{c_1, c_2\} = \emptyset$. Then there are at most three cycles of local arms at x . If there are exactly three cycles, then at least one of them is trivial. More precisely, in this case, if \hat{x} is a characteristic point of $\text{orb}(x)$ and L_0 is its local arm pointing to the critical points, then the local arm at x that is the image of L_0 is fixed under $f^{\circ n}$.*

Proof. Let $\hat{x} \in \text{orb}(x)$ be v_1 -characteristic. It suffices to show the statement for \hat{x} because $f^{\circ n}$ is locally injective at points of $\text{orb}(x)$. This property also

implies that the orbit of any local arm L is periodic and all elements in $\mathcal{C}(L)$ are permuted transitively by $f^{\circ n}$. By expansivity, for any global arm G , there is a $f^{\circ l}(G)$ with $0 \leq l \leq |\text{orb}(L)|$ that contains one of the critical points. Hence for any L , $\mathcal{C}(L)$ contains a local arm that points either to the critical points, or to $f(c_1)$, or to $f^{\circ k}(c_2)$ by Proposition 4.1.23. Consequently, there are at most three distinct elements in $\{\mathcal{C}(L) : L \text{ is a local arm at } \hat{x}\}$.

Now, let us pick L_0 such that it points towards the two critical points. If $f^{\circ n}(L_0) = L_0$, then one cycle of local arms at \hat{x} is trivial and the remaining local arms split into at most two disjoint cycles $\mathcal{C}_1, \mathcal{C}_2$. If \mathcal{C}_1 contains the local arm pointing to $f(c_1)$, then the set \mathcal{C}_2 might be empty. If $f^{\circ n}(L_0) \neq L_0$, then the set of all local arms split into the two sets $\mathcal{C}_1, \mathcal{C}_2$ described above, where again \mathcal{C}_2 might be empty. \square

Observe that it is very well possible that there is a branch point with three arms that are all fixed under the first return map. In particular, if (T, f, \mathcal{P}) comes from a cubic polynomial and T contains a characteristic n -periodic branch b such that the local arm pointing to the critical points is fixed under $f^{\circ n}$, then b has exactly three arms and all of them are fixed by $f^{\circ n}$. Figure 4.6 gives an example for such a tree.

Lemma 4.1.26 (Local arms of periodic critical points). *Let c_1 be an n -periodic critical point and let \mathcal{L} be the set of local arms at c_1 that are periodic under $f^{\circ n}$. The set \mathcal{L} splits into at most three disjoint sets \mathcal{C}_i and the arms contained in each \mathcal{C}_i are permuted cyclically.*

More precisely, if $f(c_1)$ is an endpoint then \mathcal{L} contains exactly one element. If $f(c_1) = c_1$ then there is only one cycle of local arms at c_1 . If c_1 is periodic of exact period two and $f(c_1) \in]c_1, c_2[$, then there exist at most two cycles and if there are exactly two cycles then one cycle \mathcal{C}_1 is trivial, more precisely, $\mathcal{C}_1 = \{L_{c_1}(c_2)\}$.

Proof. If the critical point c_2 is also contained in $\text{orb}(c_1)$, then there is exactly one local arm which is the image of a local arm at c_1 under $f^{\circ n}$ and the statement is trivial. If $c_2 \notin \text{orb}(c_1)$ and the exact period of c_1 is at least three then $\text{orb}(c_1)$ contains a characteristic point and we can apply the arguments of Proposition 4.1.25.

If c_1 is fixed then all global arms $G \neq G_{c_1}(c_2)$ are mapped homeomorphically into $f(G)$ without hitting c_2 unless $f(G) \subset G_{c_1}(c_2)$. It follows that there is exactly one cycle of local arms. Now let us consider the cases that c_1 has period two and $f(c_1) \in]c_1, c_2[$. Then $f^{\circ 2}(L_{c_1}(c_2)) = L_{c_1}(c_2)$. Consider any global arm $G \neq G_{c_1}(c_2)$ of c_1 that is mapped over a critical point c_i under $f^{\circ 2}$ and let L be its local arm. If $c_i = c_1$ then $f^{\circ 2}(L) = L_{c_1}(c_2)$. If $c_i = c_2$ then $c_2 \in f(G)$, and $f^{\circ 2}(L) = L_{c_1}(f(c_2))$. Now the claim follows easily. \square

Remark 4.1.27 (Labeling of arms). Going through the argument of the proof of Proposition 4.1.25, we can actually say more about the behavior of global arms. We can label the elements $L_1^k, \dots, L_{m_k}^k$ of \mathcal{C}_k in such a way that L_1^k points towards $f(c_1)$, L_1^k to $f^{\circ\kappa}(c_2)$, $f^{\circ n}(L_j^k) = L_{j+1}^k$ for $j < m_k$ and $f^{\circ n}(L_{m_k}^k) = L_1^k$. Denote by G_j^k the global arm associated to L_j^k . We have to distinguish whether the local arm pointing to the critical points is fixed under the first return map $f^{\circ n}$ or not.

Let us first suppose that it is, i.e., $f^{\circ n}(L_{\hat{x}}(c_1)) = L_{\hat{x}}(c_1)$. If a global arm G_j^k that does not contain the critical points hits a critical point when pushed forward by $f^{\circ n}$, then $f^{\circ n}(L_j^k)$ points either towards $f(c_1)$ or $f^{\circ\kappa}(c_2)$. If $\mathcal{C}_2 \neq \emptyset$, then exactly one global arm per cycle hits a critical point, namely the one whose local arm maps to $L_{\hat{x}}(f(c_1))$ or to $L_{\hat{x}}(f^{\circ\kappa}(c_2))$. Thus, using the above labeling, the global arms $G_1^k, \dots, G_{m_k-1}^k$ are mapped homeomorphically into $G_2^k, \dots, G_{m_k}^k$ without hitting any critical point but $G_{m_k}^k$ is mapped over some critical point. If $\mathcal{C}_2 = \emptyset$, then again $G_{m_1}^1$ is mapped over the critical point c_1 . However now, there might be a second global arm G_j^1 that is mapped over the second critical point c_2 because \mathcal{C}_1 might also contain the local arm pointing towards $f^{\circ\kappa}(c_2)$. That means G_j^1 is the global arm whose local arm maps to the one pointing to $f^{\circ\kappa}(c_2)$.

If the local arm pointing to the critical points is not fixed, then its image under $f^{\circ n}$ is pointing either towards $f(c_1)$ or $f^{\circ\kappa}(c_2)$. By the same arguments as above, if $\mathcal{C}_2 \neq \emptyset$, then the set \mathcal{C}_k that contains the local arm pointing to the critical points might have one or two arms which hit a critical point under $f^{\circ n}$, the other set contains exactly one. If $\mathcal{C}_2 = \emptyset$, then \mathcal{C}_1 might contain one, two or three global arms that are mapped over a critical point under $f^{\circ n}$. We give some examples for the different possibilities in Figure 4.7. Observe that it is possible that a global arm G is mapped homeomorphically onto its image under the first return map and, at the same time, hits a critical point.

Summing up the above discussion we get the following result:

Corollary 4.1.28 (Number of arms mapped over c_i). *Let x, \hat{x} be as in Proposition 4.1.23 and let $l \in \{0, 1, 2\}$ be the number of disjoint cycles of local arms at \hat{x} minus one. Pick any cycle \mathcal{C} and let \mathcal{G} be the set of associated global arms. Then \mathcal{G} contains at most $3 - l$ arms that are mapped over some critical point under the first return map of \hat{x} . \square*

4.1.8 Admissible Hubbard Trees

In this section, we discuss which Hubbard trees are generated by cubic polynomials. Recall that Definition 4.1.5 imposed very little restriction on the action of f on $T \setminus V$. In fact, most of the choices for the dynamics f on $T \setminus V$ will not be realizable by a polynomial. The essential feature of a quadratic

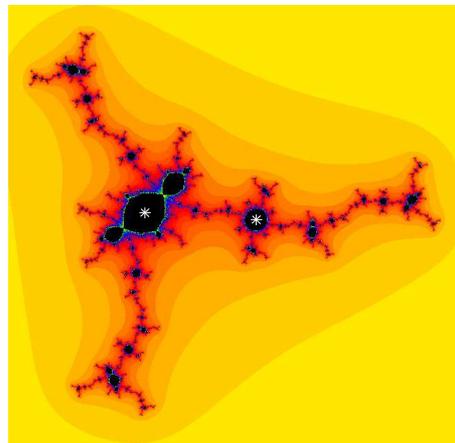
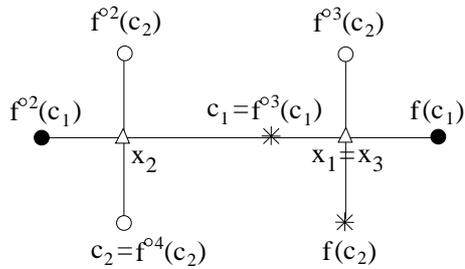
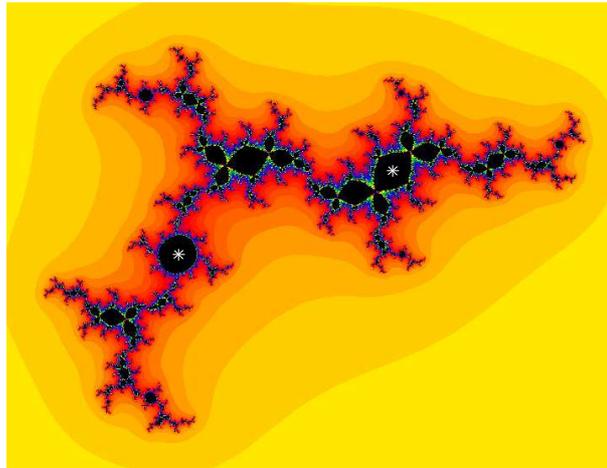
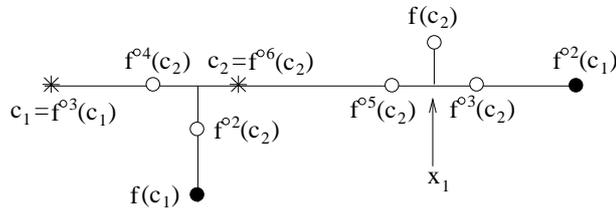


Figure 4.7: The upper Hubbard tree contains a periodic branch point x_1 such that the global arm $G_{x_1}(f(c_1))$ is mapped homeomorphically into itself under f^{o6} . However, $G_{x_1}(f(c_1))$ hits a critical point (by expansivity); more precisely, $c_1 \in f^{o5}([x_1, c_1])$. The lower Hubbard tree gives an example for a periodic orbit that has exactly two cycles of local arms. Again we also show the Julia sets generating the Hubbard trees. Their location in parameter space is indicated in Figure 4.8.

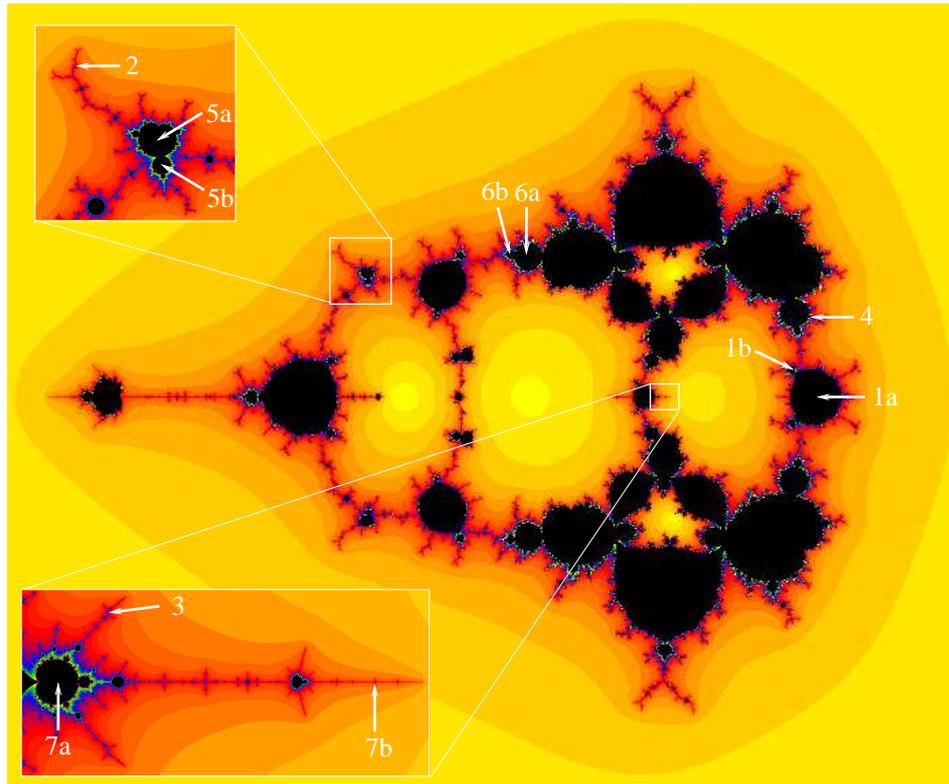


Figure 4.8: The one-dimensional slice $\mathcal{S}_3/\mathcal{I}$ of the cubic parameter space. It is characterized by the existence of one critical point of exact period 3. \mathcal{I} is the involution $(a, b) \mapsto (-a, -b)$, where a, b are the parameters of the Branner-Hubbard form; so the two critical points are not distinguishable in the pictured slice. We indicate the location of polynomials that we use for examples throughout this section and in Section 5.1. In detail, $(1a)$ and $(1b)$ are pictured in Figure 4.6, (2) is pictured at the bottom and (3) at the top of Figure 4.5, (4) at the top and $(5b)$ at the bottom of Figure 4.7 and finally, $(5a)$ is pictured in Figure 4.2. In Section 5.1, we find pictures of $(6a)$ and $(6b)$ in Figure 5.1 and of $(7a)$ and $(7b)$ in Figure 5.7.

Hubbard tree in the sense of Douady and Hubbard is the dynamics on the set of marked points. (For higher degrees we also have to consider the so-called secondary information.) With this information, the complete filled-in Julia set can be reconstructed [D2]. This motivates to regard two Hubbard trees as equivalent if they only differ in their dynamics on the complement of the set of marked points. Let us make this more precise.

Definition 4.1.29 (Equivalent trees). *Let $\tau(x)$ be the itinerary of a point x of a Hubbard tree. We denote by $\overleftrightarrow{\tau}(x)$ the itinerary obtained by interchanging the symbols 1 and 2.*

Two Hubbard trees (T, f, \mathcal{P}) and (T', f', \mathcal{P}') are equivalent if there is a bijection $\varphi : V \rightarrow V'$ between the sets of marked points such that the following are true: φ conjugates $f|_V$ and $f'|_{V'}$, two points $v_1, v_2 \in V$ are adjacent if and only if $\varphi(v_1), \varphi(v_2) \in V'$ are adjacent, and finally either $\tau(v) = \tau(\varphi(v))$ for all $v \in V$ or $\tau(v) = \overleftrightarrow{\tau}(\varphi(v))$ for all $v \in V$.

Passing from τ to $\overleftrightarrow{\tau}(x)$ corresponds to interchanging the labels of the two critical points c_1, c_2 in T .

Note in particular, that if two Hubbard trees are equivalent, then their underlying topological trees are homeomorphic. Definition 4.1.29 gives an equivalence relation on the set of all Hubbard trees (T, f, \mathcal{P}) . We are interested in classifying equivalence classes of Hubbard trees rather than individual Hubbard trees. The following definitions emphasizes this goal.

Definition 4.1.30 (Admissible Hubbard trees). *We call a Hubbard tree (T, f, \mathcal{P}) admissible if its equivalence class contains a representative that is generated by a postcritically finite cubic polynomial (in the sense of Proposition 4.1.6). A Hubbard tree is non-admissible if it is not admissible.*

In Theorem 4.1.34, we give necessary and sufficient conditions for a Hubbard tree to be admissible. An obvious obstruction for admissibility is the existence of so-called evil branch points and evil critical points. In the remainder of this section, we are going to show that these are the only obstructions.

Definition 4.1.31 (Evil branch and critical points). *A periodic non-(pre)-critical branch point b is called evil if there are two cycles of local arms at b which have different lengths.*

A periodic critical point c is called evil if there are (at least) two cycles of local arms at c of length l for some $l \in \{1, 2\}$.

By definition, a bitransitive Hubbard tree or a Hubbard tree with two preperiodic critical points cannot have an evil critical point. In general, the two obstructions are independent from each other. Figure 4.9 gives some examples of non-admissible Hubbard trees.

For the proof that evil branch and critical points are the only obstructions for admissibility, we need the following two lemmas.

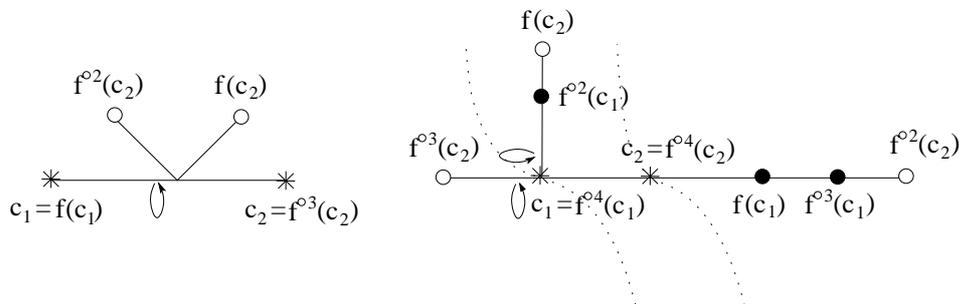


Figure 4.9: Two non-admissible Hubbard trees. The first one contains an evil branch point, the second one an evil critical point. The arrows indicate local arms that are fixed under the first return map. The kneading sequences generated by the two trees are $(\overline{\star_1}, \overline{00\star_2})$ and $(\overline{202\star_1}, \overline{021\star_2})$.

Lemma 4.1.32 (Number of periodic angles under angle doubling). *For any $n \geq 3$, there are at least $2n$ angles in $[0, 1[$ which have exact period n under the angle doubling map on \mathbb{S}^1 .*

Proof. An angle $\theta \in [0, 1[$ is periodic of period n if and only if $2^n\theta = \theta \pmod{1}$, or equivalently, if $\theta = \frac{k}{2^n-1}$ for some $k \in \{0, \dots, 2^n-2\}$. Moreover, $n > 1$ is the exact period of θ if and only if n is minimal such that $\theta = \frac{k}{2^n-1}$. For any $n \geq 3$, we set

$$K_n := \left\{ k < 2^n - 1 : \frac{k}{2^n - 1} \neq \frac{\tilde{k}}{2^{\tilde{n}} - 1} \quad \forall \tilde{n} < n, \forall \tilde{k} \in \mathbb{N} \right\}.$$

To prove the lemma it is enough to show that $|K_n| \geq 2n$. Observe that $\frac{k}{2^n-1} \neq \frac{\tilde{k}}{2^{\tilde{n}}-1}$ for all $k, \tilde{k} \in \mathbb{N}$ because $2^n - 1 = 2(2^{n-1} - 1) + 1$ and $2^n - 1 < 3(2^{n-1} - 1)$ for all $n \geq 3$. Moreover, $\sum_{i=1}^n 2^i = 2^{n+1} - 2$. Therefore, we get

$$\begin{aligned} |K_n| &\geq |\{k \in \mathbb{N} : 0 < k < 2^n - 1\}| - \sum_{i=1}^{n-2} (2^i - 2) = \\ &= (2^n - 2) - (2^{n-1} - 2) + (n - 2)2 = 2^{n-1} - 4 + 2n \geq 2n. \quad \square \end{aligned}$$

Lemma 4.1.33 (Evil points as invariants). *Let (T, f, \mathcal{P}) and (T', f', \mathcal{P}') be two equivalent Hubbard trees. A point $b \in V$ is an evil branch point of T if and only if $\varphi(b) \in V'$ is an evil branch point of T' . The same is true for evil critical points.*

Proof. We are going to show that if $b \in T$ is an evil branch point, then so is $b' := \varphi(b) \in T'$. The backward direction follows by symmetry. Let n be the exact period of b . The motor of the proof is the fact that the bijection φ between the set of marked points preserves adjacency. For further reference, we abbreviate this property by (A). First note that (A) implies that the number of local arms at b and b' are the same. Let us denote the ones at b by L_i and the ones at b' by L'_i , where $L_i \subset [b, v]$ for some $v \in V$ if and only if $L'_i \subset [b', \varphi(v)]$. It suffices to show that $f^{on}(L_i) = L_j$ if and only if $(f')^{on}(L'_i) = L'_j$, because this implies that the dynamics on the sets of local arms at b and b' are the same. Let $v \in V$ such that $L_i \subset [b, v]$ and let $f^{on}(L_i) = L_j$ ($i = j$ is possible). Iterate the interval $[b, v]$ under f . If f^{on} maps $[b, v]$ homeomorphically onto $[b, f^{on}(v)]$ without hitting a critical point at time $k < n$, then (A) implies that $[b', \varphi(v)]$ has the same property under f' . Thus $(f')^{on}([b', \varphi(v)]) = [b', \varphi(f^{on}(v))]$, and again using (A), we get that $(f')^{on}(L'_i) = L'_j$. If however a critical point, say c_1 , is contained in $f^{ok}([b, v])$ for some $k < n$, then $c'_1 \in f'^{ok}([b', \varphi(v)]) = [\varphi(f^{ok}(b)), \varphi(f^{ok}(v))]$. We pick as new intervals $[f^{ok}(b), c_1]$ and $[\varphi(f^{ok}(b)), c'_1]$, and continue the iteration until we reach time n . This proves the claim. Note that the same arguments hold for evil critical points. \square

Theorem 4.1.34 (Admissibility). *A Hubbard tree is admissible if and only if it contains no evil critical and no evil branch point.*

We have to show that any Hubbard tree \mathbb{T} in the sense of Douady and Hubbard gives rise to a Hubbard tree (T, f, \mathcal{P}) in the sense of Definition 4.1.5 so that T contains no evil branch or critical points. For the converse direction, we show that an admissible Hubbard tree defines a (not necessarily unique) *abstract Hubbard tree in the sense of Poirier*. For this, we first have to show that (T, f, \mathcal{P}) give rise to an *angled tree* in the sense of [Po2] and then that this angled tree is expanding. By the main result in [Po2] any abstract Hubbard tree is realizable by a postcritically finite polynomial. We want to note that the proof is very similar to the one in the unicritical case in Section 2.4.1.

Recall that V is the set of marked points of (T, f, \mathcal{P}) . An *edge* of T is the closure of a component of $T \setminus V$. For any $v \in V$, let E_v be the set of edges which have v as vertex, i.e., E_v corresponds to the set of local arms at v . We define for any point in V a degree map

$$\deg(v) = \begin{cases} 1 & v \notin \{c_1, c_2\} \\ 2 & \text{if } v \in \{c_1, c_2\} \text{ and } c_1 \neq c_2 \\ 3 & v = c_1 = c_2 \end{cases} .$$

Most work in proving the theorem goes into showing that (T, f, \mathcal{P}) gives rise to an *angled tree*: this is a topological tree with dynamics and an *angle function* \angle , that, for any $v \in V$, assigns a rational number (modulo 1) to

two elements $l, l' \in E_v$. The function \angle is required to have the following properties (\star) (all equalities are modulo 1):

- $\angle(l, l') = -\angle(l', l) \quad \forall l, l' \in E_v$
- $\angle(l, l') = 0 \iff l = l'$
- $\angle(l, l'') = \angle(l, l') + \angle(l', l'') \quad \forall l, l', l'' \in E_v$
- $\angle(f_v(l), f_v(l')) = \deg(v) \cdot \angle(l, l') \quad \forall v \in V$.

Proof. “ \implies ” Let \mathbb{T} be a Hubbard tree in the sense of Douady and Hubbard and let $p_{a,b}$ be the postcritically finite cubic polynomial generating it. We have already seen in Proposition 4.1.6 that every Hubbard tree in the sense of Douady and Hubbard is a Hubbard tree. So it only remains to show that T contains no evil branch and critical points. Pick any periodic non-(pre)-critical point x . Then x is contained in the Julia set $J(p_{a,b})$ and any two local arms of T are separated by two periodic external rays landing at x . Since the first return map of x is a local homeomorphism, the cyclic order on the set of external rays landing at x is preserved. This implies that all cycles of arms at x have the same length. Recall from Section 4.1.2 that locally at a periodic critical point whose orbit is disjoint from the second critical point, the action of $p_{a,b}$ is conjugate to $z \mapsto z^2$ in \mathbb{D} . Moreover, this conjugacy respects the foliation by internal rays. As a consequence, there is at most one cycle of local arms at $c \in \mathbb{T}$ of length one and at most one cycle of length two for every periodic critical point c that is not eventually mapped onto the second critical point.

“ \impliedby ” Let us call an element v of V a *Fatou vertex* if v is eventually mapped to a point on a critical cycle, i.e., if there is a $j \in \mathbb{N}_0$ and a periodic critical point c such that $f^{\circ j}(v) = c$. Any other element of V is called a *Julia vertex*.

Let (T, f, \mathcal{P}) be a Hubbard tree that contains no evil branch and no evil critical points. (T, f, \mathcal{P}) is an angled tree of degree three: let us define the function \angle . For this, we define for any vertex $v \in V$ a function $a_v : E_v \rightarrow \mathbb{Q}/\mathbb{Z}$ which associates to each local arm an angle. This induces a cyclic order on any E_v . We require that at any point $v \in V$, the function a_v is injective. Given this function, we set

$$\angle : \{(l, l') : l, l' \in E_v \text{ for some } v \in V\} \rightarrow \mathbb{Q}/\mathbb{Z}, (l, l') \mapsto a_v(l') - a_v(l).$$

From this definition, properties (i) – (iii) stated on page 134 follow immediately, so that it only remains to verify property (iv) in the respective cases.

Let O be an n -periodic orbit of a Julia vertex v and q be the number of local arms at v . Then for all i , the number of local arms at $f^{\circ i}(v)$ also equals q . Since all cycles of local arms at v have the same length by assumption, we get $q = k\tilde{q}$, where $0 < k \leq 3$ is the number of disjoint cycles and \tilde{q} the

period of the local arms. For each of these cycles choose one local arm $L_{\tilde{k}}$, $0 \leq \tilde{k} < k$, and set $a_v(f^{\circ jn}(L_{\tilde{k}})) := (jk + \tilde{k})/q$ for $j = 0, \dots, \tilde{q} - 1$. For the local arms of $f(v)$, set $a_{f(v)}(f(L)) := (a_v(L) + k/q) \pmod 1$, and for all $1 < i < n$ set $a_{f^{\circ i}(v)}(f^{\circ i}(L)) = a_{f(v)}(f(L))$. This defines \angle for all $E_{v'}$ with $v' \in O$. For preperiodic Julia vertices, we define a_v via induction on the number of iterations they need to be mapped onto a point of a periodic orbit. It is very well possible that the number of local arms at a preperiodic point is smaller or larger than the number at the periodic points of its orbit. Suppose v is a preperiodic Julia vertex and $a_{f(v)}$ has been defined. If v is not critical, set for any local arm L , $a_v(L) := a_{f(v)}(f(L))$. Suppose v is critical. For any local arm L at $f(v)$, let $L_0^{-1}, \dots, L_d^{-1}$ be the set of preimages of L at v , where $0 \leq d < \deg(v)$. We set $a_v(L_i^{-1}) := i/\deg(v) + a_{f(v)}(L)/\deg(v)$. It is not hard to see that this definition of \angle meets requirement (iv).

It remains to define a_v for Fatou vertices. Let C be a critical orbit of exact period n and degree d . Note that $2 \leq d \leq 4$ and $d = \prod_{v \in C} \deg(v)$. Pick any point $v \in C$ and let L be a periodic local arm of v and \tilde{q} the period of L . We set for all $j \in \{1, \dots, \tilde{q} - 1\}$, $a_v(f^{\circ jn}(L)) := \frac{d^j p}{d^{\tilde{q}} - 1}$, where p is chosen such that p and $d^{\tilde{q}} - 1$ are coprime. For $0 < i < n$, let $a_{f^{\circ i}(v)}(f(L)) := \deg(v) \cdot a_{f^{\circ i-1}(v)}(L)$. Now consider the preperiodic arms at v . Let $L_1^{-1}, \dots, L_{\deg(v)-1}^{-1}$ be preperiodic with $f(L_j^{-1}) = L$ and assume that $a_v(L)$ has been defined. We set $a_v(L_j^{-1}) := (j/\deg(v) + a_v(L)) \pmod 1$. By induction on the number of iterations it takes for v to map onto a periodic point, this procedure defines an angle $a_v(L)$ for any $v \in V$ and $L \in E_v$, and hence the function \angle .

Next we prove that the angled tree we constructed is an *expanding* angled tree. Let

$$N(v, v') := |\{w \in V \cap]v, v'[\}|$$

be the number of vertices contained in the open arc $]v, v'[$. Observe that an angled tree is not expanding in the sense of Poirier if and only if there are periodic $v, v' \in V$ such that $\text{orb}(v), \text{orb}(v')$ are disjoint from any periodic critical cycle and $N(f^{\circ m}(v), f^{\circ m}(v')) = 0 \quad \forall m \geq 0$ [Po2]. Let v, v' be the two periodic endpoints of an edge whose orbits do not contain any the critical point. Now our expansivity condition says that there is a critical point c and an n such that $c \in f^{\circ n}(]v, v'[)$. If we pick n to be the smallest number with this property, then $f^{\circ n}(]v, v'[) =]f^{\circ n}(v), f^{\circ n}(v')[\ni c$, and hence $N(f^{\circ n}(v), f^{\circ n}(v')) > 0$. Thus, (T, f, \mathcal{P}) is expanding.

We have chosen the angles at any periodic Julia vertex with q local arms to be i/q . This is the only requirement for an expanding angled tree to be an abstract Hubbard tree in the sense of Poirier. \square

4.2 Kneading Sequences

In the unicritical case, the combination of kneading sequences and Hubbard trees led to valuable insight into the structure of the Multibrot sets as we have seen in Part I. We want to further pursue this strategy. In order to determine the dynamics of a postcritically finite cubic polynomial one has to consider both critical points. So, instead of one sequence as in the unicritical case, we will have to work with two symbol sequences, one for each critical value. We already prepared the ground by our definition of kneading sequences in Definition 4.1.7. Note, however, that the approach to use kneading sequences to structure the set of Hubbard trees will only generate meaningful results if every kneading sequence is generated by at most one equivalence class of Hubbard trees. To show that this holds is the content of this section.

4.2.1 Iteration of Triods

Recall that if two Hubbard trees (T, f, \mathcal{P}) , (T', f', \mathcal{P}') are equivalent, then there is a bijection φ between the marked points of T and T' such that either $\tau(v) = \tau(\varphi(v))$ for all $v \in V$, or $\tau(v) = \overleftarrow{\tau}(\varphi(v))$ for all $v \in V$, and such that φ preserves adjacency of marked points. This means that two Hubbard trees can only be equivalent if they have the same kneading sequence ($\text{mod } \leftrightarrow$), where ($\text{mod } \leftrightarrow$) means “modulo interchanging ν^1, ν^2 and the symbols 1 and 2”. In the following, we are going to show that the converse also holds, i.e., that two Hubbard trees that generate the same kneading sequence ($\text{mod } \leftrightarrow$) are in fact equivalent. To prove this, it suffices to study the location of marked points in the two Hubbard trees: two Hubbard trees (T, f, \mathcal{P}) , (T', f', \mathcal{P}') with the same kneading sequence are equivalent if and only if any three distinct points $x, y, z \in \mathcal{O}$ form a triod that is *combinatorially equivalent* to the triod $[x', y', z'] \subset T'$, where $\tau(s) = \tau(s')$ for all $s \in \{x, y, z\}$.

Definition 4.2.1 (Combinatorial equivalence). *Given two Hubbard trees with topological trees T, T' , pick any three distinct points $x, y, z \in T$ and $x', y', z' \in T'$ and let Y, Y' be the connected hulls of the three points in T, T' . The triods Y and Y' are combinatorially equivalent if the following two requirements are satisfied:*

- $\tau(s) = \tau(s')$ (modulo \star_j) for all $s \in \{x, y, z\}$
- the triods Y, Y' are topologically equivalent and, in the degenerate case, $s \in \{x, y, z\}$ is an inner point of Y if and only if s' is an inner point of Y' .

Here, “modulo \star_j ” means that it is allowed that $\tau_i(s) \in \{j, 0\}$ whereas $\tau_i(s') = \star_j$ (or vice versa).

In order to study the mutual location of the three points x, y, z , we study the behavior of the triod $Y = [x, y, z]$ under iteration. Expansivity implies that, in general Y is not mapped forward homeomorphically by f^{oi} for all $i \in \mathbb{N}_0$. If $f^{oi}|_Y$ is not a homeomorphism onto its image, then $f^{oi}(Y)$ and Y might or might not be topologically the same. To distinguish such different behaviors we introduce the *triod map* φ .

Definition 4.2.2 (Triod map). *Let (T, f, \mathcal{P}) be a Hubbard tree and x, y, z three distinct points of \mathcal{O} . We define the triod map $\varphi : T^3 \rightarrow T^3$ by $(x, y, z) \mapsto$*

$$\left\{ \begin{array}{ll} (f(x), f(y), f(z)) & x, y, z \in T_i \\ (f(x), f(y), f(c)) & x, y \in T_i, z \notin T_i \text{ and } c \text{ critical with } c \notin [x, y] \\ & c \in [x, y, z] \\ (f(c_1), f(y), f(c_2)) & \text{if } y \in T_0, x \in \overline{T_1}, z \in \overline{T_2} \\ (f(c_2), f(y), f(c_1)) & y \in T_0, x \in \overline{T_2}, z \in \overline{T_1} \\ \text{STOP} & [x, y] \cap [y, z] \cap [z, x] = \{c\}, c \text{ critical and} \\ & [x, y, z] \not\subset \overline{T_i} \text{ for all } i \in \{0, 1, 2\} \end{array} \right. .$$

Of course, in the second, third and fourth case, the role of z and y can be played by any of the three points x, y, z . This follows by simple interchanging the order of the three points. If the image of a point, say $f(x)$, is replaced by a critical value $f(c_i)$ and $x \neq c_i$, then we say that x has been chopped off at c_i .

Observe also that we often identify the triple (x, y, z) with its convex hull $[x, y, z]$, which is either a degenerate or non-degenerate triod.

Note that we can extend the triod map φ to any set of three points of T , i.e., x, y, z do not necessarily have to be elements of \mathcal{O} . If x is not on the critical orbit and chopped off at the critical point c , then we sometimes replace x not by c but by a non-critical point of T so that the resulting triod has the same shape as $[x, y, z]$ and can be mapped forward homeomorphically by f . For example, if a periodic point x is chopped off at the critical point c , then we might substitute $f(x)$ by the point of $\text{orb}(x)$ which is characteristic with respect to $f(c)$. The advantage of this choice is that the resulting triod is still spanned by points on the orbits of x, y, z .

We defined an analogous triod map in the unicritical case, see Definition 3.2.2. There, the behavior of Y under the iteration by φ determines uniquely whether Y is degenerate or not, and in the degenerate case which point is in the middle of Y . This builds mainly on the fact that for any two points x, y in a unicritical Hubbard tree with itineraries $\tau(x), \tau(y)$ the following two statements are equivalent:

- There is an i such that $\tau_i(x) \neq \tau_i(y)$.
- The interval $[f^{oi}(x), f^{oi}(y)]$ contains the unique critical point c_0 .

And if $f^{\circ i}(x) \neq c_0 \neq f^{\circ i}(y)$, there is a third equivalent statement, namely that $f|_{[f^{\circ i}(x), f^{\circ i}(y)]}$ is not injective.

All of this is generally no longer true for cubic Hubbard trees: whenever the simple critical point c is a branch point then there are two global arms which are in different elements of the partition, e.g. T_0 and T_1 , yet they are mapped homeomorphically in a neighborhood of c . This is because one of T_0, T_1 contains at least two arms of c , say L_1, L_2 . Pick an arm L_3 at c which is contained in another element of \mathcal{P} than L_1 . Then for at most one arm L_1 or L_2 , $f(L_i) = f(L_3)$, because the local degree of f at c equals two. Thus, it is no longer possible to read off from the itinerary of two points whether the arc connecting them is mapped homeomorphically or not. In particular, one has to expect that the itinerary of a characteristic point x does not encode the mutual location of three given points of $\text{orb}(x)$ anymore (compare the Hubbard trees of Figure 5.2). As a consequence, it is not straightforward that a given kneading sequence determines a Hubbard tree uniquely up to equivalence.

Despite these difficulties, we will successfully use the iteration of triods by φ to determine whether two triods, contained in different Hubbard trees, are equivalent. Therefore, let us take a closer look at the triod map. From the definition, we see that we can reach the STOP in the following cases:

- Y is degenerate, the middle point is a critical point c_i and exactly one of the remaining points is contained in T_i .
- Y is non-degenerate, the branch point b is critical, i.e. $b = c$, and $\{x, y, z\} \not\subset \overline{T_j}$ for all $j = 0, 1, 2$.

Now suppose the triod Y can be iterated *indefinitely*, i.e., $\varphi^{\circ i}(x, y, z) \neq \text{STOP}$ for all $i \in \mathbb{N}$. Then the following behaviors are possible:

- Y is non-degenerate and its branch point b is not (pre-)critical.
- Y is non-degenerate, b is (pre-)critical and whenever b is mapped to the critical point, say at time n , then $\varphi^{\circ n}((x, y, z))$ is contained in $\overline{T_{j_n}}$ for some j_n .
- Y is degenerate and the generating point contained in the middle is not (pre-)critical.
- Y is degenerate and whenever the middle point is mapped to a critical point c_i either each or none of the two endpoints are contained in T_i .

Remark 4.2.3. The discussion above shows that reaching the STOP does not imply that $f|_Y$ is not injective. Thus we might stop the iteration of Y although there is no topological necessity for it. If Y is degenerate and the STOP case occurs, then the mutual location of the three points defining Y

is uniquely determined by their itineraries. Conversely, if $x, y, z \in \mathcal{O}$ such that $\{\tau_1(x), \tau_1(y), \tau_1(z)\} = \{i, \star_i, j\}$ and $i \neq j$, then $\varphi(Y) = \text{STOP}$ and Y is degenerate with c_i in the middle.

Definition 4.2.4 (Chopping map). *Let (T, f, \mathcal{P}) be a Hubbard tree, x_1, x_2, x_3 be three distinct points of $\mathcal{O} \subset T$ and set $(x_1, x_2, x_3) =: Y$. For every j such that $\varphi^{\circ j}(Y) \neq \text{STOP}$ for all $j' \leq j$, we define the iterated chopping map $\phi_Y^{\circ j}$ by*

$$\phi_Y^{\circ j} : \{x_1, x_2, x_3\} \longrightarrow T, x_i \mapsto \pi_i(\varphi^{\circ j}(Y)),$$

where $i = 1, 2, 3$ and π_i is the projection onto the i -th coordinate.

Let $N := \min\{N \in \mathbb{N} : \varphi^{\circ N+1}(Y) = \text{STOP}\}$. The chopping itinerary $\tau_Y(x_i) = ((\tau_Y(x_i))_j)_{j=1}^N$ of a vertex $x_i \in Y$ is the itinerary obtained by noting for all $i \leq N$ which elements of \mathcal{P} the point $\phi_Y^{\circ j}(x_i)$ is contained in.

We say that two points are *on the same side* of c_1 if $x, y \in T_1$ or $x, y \in T_0 \cup T_2 \cup \{c_2\}$. If two points are not on the same side of c_1 , we say that they are *on different sides* of c_1 . Interchanging 1 and 2 yields the analogous definition for c_2 . Note that if x, y are on different sides of c_i , then $\tau_1(x) \neq \tau_1(y)$. Conversely, $\tau_1(x) \neq \tau_1(y)$ implies that x, y are on different sides of a critical point but we cannot specify of which one unless we explicitly know the first entries of $\tau(x)$ and $\tau(y)$.

Lemma 4.2.5 (Triod iteration). *Let $(T, f, \mathcal{P}), (T', f', \mathcal{P}')$ be two Hubbard trees generating the same kneading sequence. Moreover, let $x, y, z \in \mathcal{O}, x', y', z' \in \mathcal{O}', Y := (x, y, z)$ and $Y' := (x', y', z')$ such that $\varphi_f(Y) \neq \text{STOP} \neq \varphi_{f'}(Y')$. If the triods spanned by Y and Y' are combinatorially equivalent, then so are the triods spanned by $\varphi_f(Y)$ and $\varphi_{f'}(Y')$.*

If Y can be iterated indefinitely, then at most one of the three points x, y, z is never chopped off. In particular, a point $y \in Y$ is never chopped off if and only if the triod spanned by Y is degenerate and y is contained in the middle. In this case, $\phi_Y^{\circ i}(y) = f^{\circ i}(y)$ for all $i \in \mathbb{N}_0$.

Proof. First observe that the action of φ is completely encoded in the itineraries of the three points x, y, z . Thus, since $(T, f, \mathcal{P}), (T', f', \mathcal{P}')$ generate the same kneading sequence, $\tau(s) = \tau(s')$ implies $\tau(f(s)) = \tau(f'(s'))$. In the following, we identify the triples Y, Y' with the respective triod they span. We can interpret the action of the triod map the following way: φ cuts the arms of the triod Y such that this chopped triod is contained in the closure of an element of \mathcal{P} , and f restricted to it is injective. The cutting preserves the topological type of the triod and in the degenerate case, the image of the point in the middle is in the middle again. All this put together implies the first claim.

Now assume that Y can be iterated indefinitely by φ , i.e., does not eventually reach STOP. Let us first consider the case that Y is degenerate and $y \in Y$ is not (pre-)critical. If exactly two points out of x, y, z are contained

in some T_i , then one of these two points must be y . If every point of x, y, z is contained in a different element of \mathcal{P} (including the two trivial ones), then $y \in T_0$. Since we can argue the same way for all $\phi_Y^{oi}(x), \phi_Y^{oi}(y), \phi_Y^{oi}(z)$, it follows that y is never chopped off. Now if y is (pre-)critical then at any time when $f^{oi}(y) = c$ for some periodic critical point $c \in T$, we have that $\phi_Y^{oi}(x)$ and $\phi_Y^{oi}(z)$ are on the same side of c . If c is periodic then this behavior is not allowed by the condition (P2) imposed on \mathcal{P} in Definition 4.1.5. Thus y is preperiodic and the periodic part of its orbit contains no critical point. So after finitely many steps we are in the above situation. Now if there was a second point, say x , which is not chopped off, then x has the same properties as just described for y . In particular, after finitely many steps, the iterates of x and y would have the same itinerary, contradicting expansivity.

Last we claim that if Y is non-degenerate then x, y and z must eventually be chopped off. The reasoning is very similar: expansivity implies that at least two of the three points x, y, z must eventually be chopped off. If one endpoint is not chopped off then the branch point b of Y is eventually mapped onto a periodic critical point c whose orbit is disjoint from the second critical point. Again $\varphi^{oi}(Y) \neq \text{STOP}$ implies that c has two local arms L_1, L_2 with $f^{oi}(L_1), f^{oi}(L_2) \in T_{j_i}$ for all $i \in \mathbb{N}_0$, which is prohibited. \square

Observe that in Lemma 4.2.5 we could equivalently have stated that under the assumption that $\varphi^{oi}(Y) \neq \text{STOP}$ for all $i \in \mathbb{N}_0$, the triod spanned by Y is non-degenerate if and only if all three points are eventually chopped off.

4.2.2 Kneading Sequences and Hubbard Trees

In the following we are going to prove that two Hubbard trees are equivalent if and only if they generate the same kneading sequence (mod \leftrightarrow). While it follows by definition that equivalent Hubbard trees have the same kneading sequence, we have to do some work for the other direction. We show that two non-equivalent Hubbard trees $(T, f, \mathcal{P}), (T', f', \mathcal{P}')$ can only generate the same kneading sequence if three marked points in T, T' are arranged in a very special way. Lemma 4.2.6 shows that such an arrangement violates the properties that any partition of a Hubbard tree has.

Lemma 4.2.6 (Ambiguities are impossible). *Let $(T, f, \mathcal{P}), (T', f', \mathcal{P}')$ be two Hubbard trees that have the same kneading sequence (ν^1, ν^2) . Then there are no three pairwise distinct points $x, y, c \in \mathcal{O}$ and $x', y', c' \in \mathcal{O}'$ (c, c' are critical points) which have the following properties: $\tau(x) = \tau(x')$, $\tau(y) = \tau(y')$, $\tau(c) = \tau(c')$, the triples $(x, y, z), (x', y', z')$ can be iterated indefinitely and one of the following holds:*

(A1) $c \in [x, y] \subset T$ whereas $x' \in [c', y'] \subset T'$ (or vice versa);

(A2) $c \in [x, y] \subset T$ whereas $[c', x', y'] \subset T'$ is a non-degenerate triod with non-(pre-)critical branch point b' (or vice versa).

Proof. Let us assume that $c = c_1$ and set $Y := (c_1, x, y)$, $Y' := (c'_1, x', y')$. Observe that we identify the triples Y, Y' with the topological triods they span. Since (T, f, \mathcal{P}) and (T', f', \mathcal{P}') have the same kneading sequence, either both trees are bitransitive or none of them are. Suppose first that they are bitransitive. Then we iterate Y under φ until $f^{\circ i_0}(c_1) = c_2$. Since Y is degenerate with the critical point c_1 in the middle, $f^{\circ i}(L_{c_1}(x)) \neq f^{\circ i}(L_{c_1}(y))$ for all $0 \leq i \leq i_0$. Thus, c_2 is an inner point and its local arms must collapse under f . But this is only possible if $\phi_Y^{\circ i_0+1}(x)$ and $\phi_Y^{\circ i_0+1}(y)$ are on different sides of c_2 , yet in T' , $\phi_{Y'}^{\circ i_0+1}(x')$ and $\phi_{Y'}^{\circ i_0+1}(y')$ are on the same side of c'_2 . This contradicts that $\tau_Y(x) = \tau_{Y'}(x')$ and $\tau_Y(y) = \tau_{Y'}(y')$.

Now suppose that the given Hubbard trees are not bitransitive. Let us first consider item (A1). Since $\tau(s) = \tau(s')$ for $s = x, y, c$, $\phi_Y^{\circ i}(c_1)$ and $\phi_Y^{\circ i}(x)$ are on the same side of c_1 and c_2 for all $i \in \mathbb{N}_0$. This implies by expansivity that at most one critical point is preperiodic. First suppose that c is periodic of, say, period n . Since $\varphi^{\circ i}(Y) \neq \text{STOP}$ for all $i \in \mathbb{N}_0$, $f^{\circ i}(L_{c_1}(x)) \neq f^{\circ i}(L_{c_1}(y))$ are contained in some T_{j_i} for all i , a contradiction to (P2) of Definition 4.1.5. If c_1 is preperiodic, we must have that $x \in \text{orb}(c_2)$. Let k_0 be the smallest number such that $\phi_Y^{\circ k_0}(x) = c_2$. In $\varphi^{\circ k_0}(Y')$, the critical point c'_2 is an inner point whereas in $\varphi^{\circ k_0}(Y)$, c_2 is an endpoint. Now by the same reasoning as above we see that $c'_2 \in T'$ has two local arms L_1, L_2 with $f^{\circ i}(L_1), f^{\circ i}(L_2) \in T_{j_i}$ for all i , contradicting (P2) again.

It remains to investigate item (A2), in which case Y' is non-degenerate. Since the branch point b' of Y' is not (pre-)critical and $\phi_{Y'}^{\circ i}(c')$, $\phi_{Y'}^{\circ i}(b')$ are on the same side of c'_1 and of c'_2 for all $i \in \mathbb{N}_0$ (otherwise we get an immediate contradiction to T and T' having the same kneading sequence), the critical point c'_1 is periodic of say period n . The points $\phi_{Y'}^{\circ kn}(x') \neq \phi_{Y'}^{\circ kn}(y') \in T'$ are on the same side of c' for all i . Thus we derive the same contradiction again: in T , for all $k \in \mathbb{N}_0$, there is a non-trivial element $T_{j_k} \in \mathcal{P}$ such that the local arms $f^{\circ k}(L_{c_1}(x)) \neq f^{\circ k}(L_{c_1}(y))$ are in T_{j_k} . \square

Theorem 4.2.7 (Uniqueness). *Two Hubbard trees (T, f, \mathcal{P}) , (T', f', \mathcal{P}') have the same kneading sequence $(\nu^1, \nu^2) \pmod{\leftrightarrow}$ if and only if they are equivalent.*

Proof. We only have to show the reverse direction, because two equivalent Hubbard trees have the same kneading sequence $\pmod{\leftrightarrow}$ by definition. By possibly relabeling the critical points in of one of the two given Hubbard trees, we can assume that $(T, f, \mathcal{P}), (T', f', \mathcal{P}')$ generate the same kneading sequence. We study the mutual locations of any three distinct points $x, y, z \in \mathcal{O}$ and $x', y', z' \in \mathcal{O}'$, where points in T and T' are denoted by the same letter if and only if they have equal itineraries. We again identify the triples $Y := (x, y, z)$, $Y' := (x', y', z')$ with their connected hull. The two Hubbard

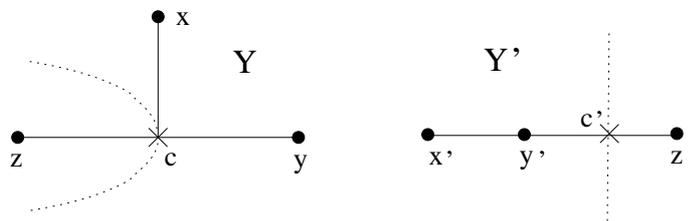


Figure 4.10: A sketch of the forbidden situation (A1) of Lemma 4.2.6 which would arise if two non-equivalent Hubbard trees generate the same itinerary. Note that the letter s in the figure ($s = x, y, z$) corresponds to $\phi_Y^{oi_0}(s)$ in the proof of Theorem 4.2.7.

trees are equivalent if for all pairs of triods Y, Y' obtained as described above, Y and Y' are combinatorially equivalent. Let us pick any such pair $Y \subset T$ and $Y' \subset T'$. We show that they are either equivalent or we are in one of the two situations (A1), (A2) described in Lemma 4.2.6, which is impossible.

Assume first that there is an i_0 such that $f^{oi_0}(Y) = \text{STOP} = (f')^{oi_0}(Y')$. By the definition of the triod map φ , it is not hard to see that this is only possible if Y, Y' are either both degenerate or both non-degenerate. In either case, $\varphi^{oi_0}(Y)$ and $\varphi^{oi_0}(Y')$ are equivalent, and so are Y, Y' .

If Y reaches the STOP at time i_0 but Y' does not, then by Remark 4.2.3 this is only possible if $\varphi^{oi_0}(Y)$, and thus Y , is non-degenerate. Let us assume that c is the critical branch point of $\varphi^{oi_0}(Y)$ and that $\phi_Y^{oi_0}(z)$ is not on the same side of c as $\phi_Y^{oi_0}(x)$ and $\phi_Y^{oi_0}(y)$ are. If Y' is not equivalent to Y , then Y' is degenerate and hence, we get situation (A1) for the three points $\phi_Y^{oi_0}(x), c, \phi_Y^{oi_0}(y)$. Figure 4.10 illustrates this situation.

By symmetry, it only remains to investigate the situation when both triods can be iterated indefinitely. If they are both degenerate and, say, $y \in Y$ is an inner point, it might happen that there is an i_0 such that $\phi_Y^{oi_0}(y) = c$ and the critical point c is an inner point while c' is an endpoint of $\varphi^{oi_0}(Y')$. But this is exactly situation (A1) for the points $\phi_Y^{oi_0}(x), \phi_Y^{oi_0}(y), \phi_Y^{oi_0}(z)$. If Y and Y' are both non-degenerate, then they are combinatorially equivalent by definition. The last case is that Y is degenerate with inner point say y and Y' is non-degenerate with branch point b' .

First assume that b' is eventually mapped onto a critical point. In this case, we eventually get situation (A1): let i_0 be the first time such that $f^{oi_0}(b') = c'$ for some critical point $c' \in T'$. Then $c' \in]\phi_{Y'}^{oi_0}(x'), \phi_{Y'}^{oi_0}(y')[$ and $c' \in]\phi_{Y'}^{oi_0}(x'), \phi_{Y'}^{oi_0}(z')[$, however in T , the critical point c is either an endpoint of the triod $[c, \phi_Y^{oi_1}(y), \phi_Y^{oi_1}(x)]$ or of the triod $[c, \phi_Y^{oi_1}(y), \phi_Y^{oi_1}(z)]$. Both possibilities are versions of (A1).

If b' is not (pre-)critical, we have to distinguish whether or not y is (pre-)critical. If there is an i_0 and a critical point $c \in T$ such that $f^{\circ i_0}(y) = c$, then we are in situation (A2) for the $\phi_Y^{\circ i_0}$ -images of x, y, z . (In this case, (A1) cannot occur.) If y is not (pre-)critical, then we consider the time i_1 , where $f^{\circ i_1}(y')$ and $f^{\circ i_1}(b')$ are on different sides of some critical point $c' \in T'$. Since $\varphi^{\circ i_1}(Y')$ is non-degenerate, $\phi_{Y'}^{\circ i_1}(x')$ and $\phi_{Y'}^{\circ i_1}(z')$ are on the same side of c' but on a different side than $f^{\circ i_1}(y')$. Thus in T , the points $\phi_Y^{\circ i_1}(x)$ and $\phi_Y^{\circ i_1}(z)$ must be on the same side of c , too. But this implies that also $f^{\circ i_1}(y)$ is on this side of c , contradicting that $\tau_Y(y) = \tau_{Y'}(y')$.

It remains to show that in situation (A1) or (A2), Y, Y' can be iterated forever: if a degenerate triod reaches the STOP, then the itineraries of the three points determine uniquely the inner point of the triod. This means if Y or Y' in (A1) or Y in (A2) are mapped to STOP, then Y and Y' are combinatorially equivalent, which they are not by assumption. Finally, for Y' in (A2), the STOP case can never occur by definition. \square

Observe that this result implies in particular that not all kneading sequences are generated by admissible Hubbard trees (compare Example 4.9 and Lemma 4.1.33).

Theorem 4.2.7 implies that there is an injection from the set of equivalence classes of Hubbard trees into the set of kneading sequences (ν^1, ν^2) , where ν^i is \star_i -periodic or preperiodic. In the unicritical case, we showed that there is a bijection between the set of equivalence classes of unicritical Hubbard trees of degree d and the set Σ_d^\sharp of \star - and preperiodic kneading sequences (Corollary 2.3.22). In the cubic case, there is no bijection as the next proposition illustrates.

Proposition 4.2.8 (No 1-to-1 correspondence). *There is no (cubic) Hubbard tree that generates the tuple $(\overline{\star_1 1}, \overline{\star_2 1 1})$.*

Proof. Let us try to construct a Hubbard tree (T, f, \mathcal{P}) that would generate the kneading sequence $(\overline{\star_1 1}, \overline{\star_2 1 1})$. For any such tree the critical point c_2 must be an endpoint, and consequently $f(c_2), f^{\circ 2}(c_2)$ are endpoints as well. Moreover, no images of c_1 and c_2 are in $T_0 \cup T_2$. If $f(c_1)$ is an endpoint, then c_1 is an inner point and $f(c_1), f(c_2), c_1$ form a non-degenerate triod with branch point b . By expansivity, b is a fixed point of f . Furthermore, $f(L_b(f(c_1))) = L_b(c_1)$. Since f must be locally injective at b , $f^{\circ i}(c_2) \notin G_b(c_1)$ for all $i \in \mathbb{N}$; in particular, $f^{\circ 3}(c_2) \neq c_2$, a contradiction. Therefore, $f(c_1) \in]f(c_2), c_1[$. Since $f^{\circ 2}(c_2)$ is an endpoint of T_1 , it must branch off from $]f(c_2), c_1[$ at a fixed branch point $b \neq f(c_1)$. If $b \in]f(c_2), f(c_1)[$, then $f(L_b(f(c_1))) = L_b(f(c_1))$ and we get the same contradiction as above. If however $b \in]f(c_1), c_1[$, then $f(L_b(f(c_1)))$ and $L_b(c_1)$ form a cycle of length two. Consequently, $L_b(f^{\circ 2}(c_2))$ must be fixed by f and $f^{\circ 3}(c_2) \in G_b(f^{\circ 2}(c_2))$. So c_2 cannot be 3-periodic. The last possibility is that c_1 is a branch point such that the two arms $L_{c_1}(f(c_2)), L_{c_1}(f^{\circ 2}(c_2))$

are contained in T_1 . Thus, they do not collapse if we push them forward by f . But this means that $f^{\circ 3}(c_2) \in G_{f(c_1)}(f(c_2))$, and again we get the contradiction that $f^{\circ 3}(c_2) \neq c_2$. \square

4.3 Minimal Hubbard Trees

We continue our study of Hubbard trees (T, f, \mathcal{P}) by taking a closer look at fixed points of f . In general, statements on fixed points might not be meaningful since Definition 4.1.5 allows for whole subtrees in T which are pointwise fixed. Therefore, we restrict our investigation to special types of Hubbard trees, the so-called minimal Hubbard trees. Fixed points then reveal the basic dynamics of the map f .

Definition 4.3.1 (Minimal Hubbard tree). *A minimal Hubbard tree is a Hubbard tree (T, f, \mathcal{P}) such that for each itinerary $\tau \in \{0, 1, 2\}^{\mathbb{N}}$, there is at most one periodic point $p \in T$ with $\tau(p) = \tau$.*

A minimal Hubbard tree is attracting if the following holds: if \mathcal{C} is a critical cycle of length n then for each $p \in \mathcal{C}$, there is a neighborhood $U_p \subset T$ such that for all $x \in U_p$, $f^{\circ jn}$ converges to p as j goes to ∞ .

The arguments of Lemmas 2.2.4 and 2.2.7 of the unicritical part carry over to prove the following statement.

Proposition 4.3.2 (Minimal and tame representatives). *Every equivalence class of Hubbard trees contains a attracting minimal Hubbard tree.* \square

4.3.1 Fixed Points

In the following, we are going to investigate how many fixed point a minimal Hubbard tree (T, f, \mathcal{P}) has.

Lemma 4.3.3 (Fixed points in T). *Let (T, f, \mathcal{P}) be a minimal Hubbard tree. Then any subtree $T_i \subset T$, $i = 0, 1, 2$, contains at most one fixed point. The tree T itself contains at least one and at most four fixed points. If (T, f, \mathcal{P}) is admissible, then there are at most three fixed points in T .*

Proof. Since a tree has the fixed point property (compare e.g. [N]), T contains at least one fixed point. If both critical points are fixed, then $T = [c_1, c_2]$ and T might contain an additional fixed point in $]c_1, c_2[$. In the tame case, it actually has to by attracting dynamics. Any subtree T_i can contain at most one fixed point by minimality. So, if none of the critical points are fixed, then T contains at most three fixed points. If the critical point c_i is fixed and there are fixed points in T_0 and T_i , then c_i has two local arms that are fixed by f . Thus, c_i is an evil critical point. So we get a maximal number of four fixed points for minimal Hubbard trees and of three fixed points for admissible minimal Hubbard trees. \square

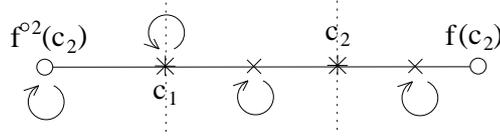


Figure 4.11: A minimal Hubbard tree with four fixed points.

Figure 4.11 gives an example for a minimal Hubbard tree that contains four fixed points.

Proposition 4.3.4 (Fixed points in T_1). *Let (T, f, \mathcal{P}) be a minimal Hubbard tree with kneading pair (ν^1, ν^2) . Then T_1 does not contain a fixed point if and only if for all $n > 0$ and $k = 1, 2$, $\nu_1^1 \neq 1$ and $\sigma^{on}(\nu^k) \neq \bar{1} = 111\dots$.*

If $\alpha \in T_1$ is a fixed inner point, then $\alpha \in]c_1, f(c_1)[$.

There is an analogous statement for T_2 .

Note that the proof actually shows a slightly stronger statement: T_1 contains no fixed point if and only if $f(c_1) \notin T_1$ and T_1 contains no fixed endpoint.

Proof. Let us first assume that $f(c_1) \in T_1$. We are going to show that there is a fixed point in $]c_1, f(c_1)[$. By Lemma 4.1.12, $f(c_1) \notin]c_1, f^{\circ 2}(c_1)[$. If $f^{\circ 2}(c_1)$ is not contained in a subtree branching off at $b \in]c_1, f(c_1)[$, the interval $[c_1, f(c_1)]$ contains a fixed point by the intermediate value theorem. If it is, then $f(b) = b$: if $f(b) \in]b, f^{\circ 2}(c_1)[$, then the fact that $f|_{[c_1, f(c_1), f^{\circ 2}(c_1)]}$ is injective implies that the two marked points b, c_1 have an infinite orbit. If $f(b) \in]b, f(c_1)[$, then $f^{\circ 2}(b) \in]f(b), b[$ or $f^{\circ 2}(c_1) \in]b, f^{\circ 2}(c_1)[$ by expansivity. Both times, we get $|\text{orb}(b)| = \infty$ because $f|_{T_1}$ is injective.

Next suppose that $f(c_1) \notin T_1$ and there is no $n > 0$ such that $\sigma^{on}(\nu^j) = \bar{1}$ for any $j = 1, 2$. Suppose there is a fixed point $\alpha \in T_1$. Then $L_\alpha(c_1)$ is fixed. By hypothesis, α is not an endpoint of T . Therefore, there is an $e \in \mathcal{O}$ such that $\alpha \in]c_1, e[$. Since f is locally injective at α , $f^{\circ i}(e) \in \mathcal{X}_\alpha \subset T_1$ for all $i \in \mathbb{N}_0$, where \mathcal{X}_α is the set of regular arms at α . This contradicts expansivity because $f(e) \neq e$. \square

Remark 4.3.5. For unicritical cubic Hubbard trees the statement also holds for $i = 0$. As a consequence, a unicritical cubic Hubbard tree contains a unique fixed point $\alpha \in]c, f(c)[$ unless the critical point c is eventually mapped on a fixed endpoint unequal to α . Then, the Hubbard tree has exactly two fixed points.

To determine whether T_0 contains a fixed point or not, we distinguish two cases: in Proposition 4.3.6, we assume that none of the critical points are fixed under f ; in Proposition 4.3.7, we assume that at least one c_i is fixed.

Proposition 4.3.6 (Fixed point in T_0). *Let (T, f, \mathcal{P}) be a minimal Hubbard tree with kneading sequence (ν^1, ν^2) such that none of the critical points are fixed. Then T_0 does not contain a fixed point if and only if for all $n > 0$ and $j = 1, 2$, $\sigma^{on}(\nu^j) \neq \bar{0}$ and one of the following holds:*

- (i) $\nu_1^1 = 1$ and $\nu_1^2 \in \{1, \star_1\}$,
- (i') $\nu_1^2 = 2$ and $\nu_1^1 \in \{2, \star_2\}$,
- (ii) $\nu_1^1 \in \{0, \star_2\}$, $\nu_1^2 = 2$ and there is no fixed evil branch point in $]c_1, c_2[$,
- (ii') $\nu_1^2 \in \{0, \star_1\}$, $\nu_1^1 = 1$ and there is no fixed evil branch point in $]c_1, c_2[$.

This means that T_0 contains no fixed point if and only if T_0 contains no fixed endpoint and either

- (i) $f(c_1), f(c_2) \in T_i$ for $i = 1$ or 2 , or
- (ii) $f(c_1) \in \overline{T_0}$, $f(c_2) \in T_2$, and there is no fixed evil branch point in $]c_1, c_2[$, or
- (ii') $f(c_2) \in \overline{T_0}$, $f(c_1) \in T_1$, and there is no fixed evil branch point in $]c_1, c_2[$.

The left Hubbard tree in Figure 4.9 illustrates that in case (ii), it is necessary to require that there is no fixed evil branch point in $]c_1, c_2[$. An example for option (ii) is provided by the upper Hubbard tree in Figure 4.7.

Proof. If $\nu_1^1, \nu_1^2 \in \{1, \star_1\}$, then $f([c_1, c_2]) \in \overline{T_1}$. Thus, only an arm that branches off at $b \in [c_1, c_2]$ can contain a fixed point. Suppose there is such an arm and let α be the fixed point in this arm. Since f is a locally injective at α it follows that for all $p \in \mathcal{X}_\alpha$, $f^{oi}(p) \in \mathcal{X}_\alpha$ for all $i \in \mathbb{N}$. Thus by minimality, α must be an endpoint and consequently $\alpha \in \mathcal{O}$. The same reasoning holds for (i').

Now suppose that $f(c_1), f(c_2) \in \overline{T_0}$. If $f(c_1) \in [c_1, c_2]$ and $f(c_2) \in [c_1, c_2]$ or $f(c_2)$ branches off at c_1 from $[c_1, c_2]$, then there is a fixed point in $]c_1, c_2[$. If $f(c_1) \in [c_1, c_2]$ but $f(c_2)$ branches off from $]c_1, c_2[$ at b and the branch point b is not fixed, then $b \neq c_2$ (since c_2 is not a branch point) and $f(b) \in]c_1, c_2[\cap]f(c_1), f(c_2)[$ by finiteness of $\text{orb}(b)$. Depending on whether $f(c_1) \in]c_1, b[$ or $f(b) \in]b, c_2[$, there is a fixed point in $]c_1, b[$ or $]b, c_2[$. By symmetry, the last case is that $f(c_1), f(c_2) \notin [c_1, c_2]$. Let $b_1, b_2 \in [c_1, c_2]$ be the points where the critical values branch off. Suppose first that $\{b_1, b_2\} \not\subset]c_1, c_2[$. Then by finiteness of \mathcal{O} , this is only possible if $f(c_1)$ is branching off at c_2 or if $f(c_2)$ is branching off at c_1 (compare Lemma 4.1.12). Independently from where the second critical value branches off, there is a fixed point in T_0 . (It is contained in $]c_1, c_2[$ unless both critical values branch off at the same c_i .) Now let us consider the cases for $b_1, b_2 \in]c_1, c_2[$. If $b_1 = b_2$ and

$f(c_1), f(c_2)$ are contained in the same global arm of b , then $b, f(c_1), f(c_2)$ form a non-degenerate triod with branch point \hat{b} because otherwise $|\mathcal{O}| = \infty$. Then $f(b) = \hat{b}$, and this in turn implies (together with finiteness of \mathcal{O}) that $f(\hat{b}) \in G_{\hat{b}}(b)$. So there is a fixed point in $]b, \hat{b}[$. In all other possible combinations, $f(b_i) \notin G_{b_i}(f(c_i))$ for $i = 1, 2$ again by finiteness of \mathcal{O} . This yields that $f(b_1) = b_1$ if $b_1 = b_2$; otherwise there is a fixed point in $]b_1, b_2[$.

If $\nu_1^1 = 1, \nu_1^2 = 2$ or $\nu_1^1 \in \{2, \star_2\}, \nu_1^2 \in \{1, \star_1\}$, then $]c_1, c_2[$ contains a fixed point by the intermediate value theorem.

Now suppose that $\nu_1^1 = 0, \nu_1^2 \in \{1, \star_1\}$. The case for $\nu_1^1 \in \{2, \star_2\}, \nu_1^2 = 0$ works the same way. If $f(c_1)$ branches off at $b \in]c_1, c_2[$, then $f(b) \in [c_1, b]$ and there is a fixed point in $]c_1, b[$ by the intermediate value theorem. The same holds if $f(c_1) \in]c_1, c_2]$ or $f(c_1)$ branches off at c_2 . By finiteness of $\text{orb}(c_1)$, $f(c_1)$ cannot branch off at c_1 .

The last case is that $\nu_1^1 \in \{0, \star_2\}$ and $\nu_1^2 = 2$ (or $\nu_1^1 = 1$ and $\nu_1^2 \in \{0, \star_1\}$). We claim that there is no fixed point in T_0 unless one of the endpoints of T_0 is fixed. Suppose first that $f(c_1) \in]c_1, c_2[$. Then there is an $i_0 > 1$ such that $f^{o_{i_0}}(c_1) \in T_2$ and $f^{o_i}(c_1) \in]c_1, c_2[$ for all $i < i_0$. It follows that $G_{c_1}(c_2) \cap T_0$ contains no fixed point: such a fixed point could only be contained in an arm branching off at $b \in]c_1, c_2[$, but then two local arms at b collapse. An arm of T_0 branching off at c_1 might contain a fixed point, which then must be an endpoint of the tree. Since $f(c_1) \notin \mathcal{X}_{c_1} \cap T_0$ by finiteness of $\text{orb}(c_1)$, it remains to consider that $f(c_1)$ is contained in an arm branching off at $b \in]c_1, c_2]$. If the branch point b is fixed, then it is evil. Otherwise finiteness of $\text{orb}(c_1)$ implies that $f(b) \in G_b(c_2)$. Similarly to the previous case, the fact that f is injective on $\overline{T_0}$ implies that there can only be a fixed point β in an arm branching off in $[c_1, b[$. And together with minimality, it follows that β is an endpoint of T . \square

Proposition 4.3.7 (Fixed point in T_0 when c_1 fixed). *Let (T, f, \mathcal{P}) be a minimal Hubbard tree with kneading pair $(\overline{\star^1}, \nu^2)$ so that c_1 is fixed. Then T_0 does not contain a fixed point if for all $n > 0$ and $j = 1, 2, \sigma^{on}(\nu^j) \neq \overline{0}$ and one of the following holds:*

- (i) $\nu_1^2 = 1$
- (ii) $\nu_1^2 = 0$ and $f(c_2) \notin G_{c_1}(c_2)$.

If $\nu_1^2 \in \{2, \star_2\}$, then T_0 might or might not have a fixed point.

If (T, f, \mathcal{P}) is moreover attracting and admissible, then T_0 does not contain a fixed point if and only if for all $n > 0$ and $j = 1, 2, \sigma^{on}(\nu^j) \neq \overline{0}$ and $\nu_1^2 \in \{0, 1\}$.

Therefore in the situation of Proposition 4.3.7, the subtree T_0 of an attracting minimal Hubbard tree which is admissible does not contain a fixed point if and only if T_0 contains no fixed endpoint and $f(c_2) \notin \overline{T_2}$.

Of course, the proposition is also true for the role of ν^1 and ν^2 interchanged.

Proof. If $\nu_1^2 \in \{2, \star_2\}$, then $]c_1, c_2[$ contains a fixed point if (T, f, \mathcal{P}) has attracting dynamics. If it does not, then we cannot make any statement.

If $\nu_1^2 = 0$, then $f(c_2) \notin [c_1, c_2]$. Since c_2 is not a branch point, $f(c_2)$ branches off at $b \in [c_1, c_2]$. If $b = c_1$ then there is no fixed point in $G_{c_1}(c_2) \cap T_0$. However, it is possible that there is a arm at c_1 that is contained in T_0 and has a fixed point β and by minimality, β must be an endpoint of T . If $b \in]c_1, c_2]$, then $f(b) = b$ by finiteness of $\text{orb}(b)$. In this case, b is an evil branch point .

If $\nu_1^2 = 1$, then $f(]c_1, c_2]) \subset T_1$. Thus, if T_0 contains a fixed point α , then α is contained in an arm of c_1 unequal to $G_{c_1}(c_2)$. Since $[c_1, \alpha]$ is fixed by f , it follows that α is an endpoint and consequently $\alpha \in \mathcal{O}$. Since $\nu^2 \neq \star_1$ this finishes the proof. \square

Note that if a Hubbard does not have to have attracting dynamics then T_0 might not contain a fixed point.

As a corollary of Propositions 4.3.4 and 4.3.6, we get the following surprising statement:

Corollary 4.3.8 (Fixed points for admissible trees). *Suppose (T, f, \mathcal{P}) is an attracting minimal Hubbard tree which is admissible. If none of its critical points are fixed or eventually fixed, then T contains either one or three fixed points.*

4.3.2 Preimages of Characteristic Points

Let $T_i \in \mathcal{P}$ a subtree of the Hubbard tree (T, f, \mathcal{P}) . In this section, we are going to determine under what conditions T_i contains a preimage of a given characteristic point. The existence of preimages of characteristic points is essential for our proof of a forcing relation between (combinatorially related) Hubbard trees, see Section 5.1. Note that the presented statements hold for arbitrary Hubbard trees, i.e. minimality or the like is not required.

For any characteristic point p we denote the the preimage of p contained in T_i by p_0^i (if existing).

Lemma 4.3.9 (Periodic orbit of unicritical trees). *Let (T, f, \mathcal{P}) be a unicritical Hubbard tree with critical point c and let $f(c) \in T_j$ for $j = 1$ or 2 . Furthermore suppose that there is a smallest number k such that $f^{\circ k}(c) \notin \overline{T_j}$. If p is a periodic point and j' is such that $f^{\circ k}(c) \in T_{j'}$, then $\text{orb}(p) \cap T_{j'} \neq \emptyset$.*

Proof. Let us assume by way of contradiction that $\text{orb}(p) \cap T_{j'} = \emptyset$. Since $f^{\circ i}(c) \in T_j$ for all $i < k$, since $p \in]c, f(c)[$ and since $f|_{T_i}$ is injective, we have that $f^{\circ j}(p) \in T_j$ for all $j < k - 1$. Moreover, $f^{\circ k-1}(p) \in]f^{\circ k-1}(c), f^{\circ k}(c)[\subset T_j \cup T_{j'}$, and hence by assumption $f^{\circ k-1}(p) \in]c, f^{\circ k-1}(c)[$. This implies that

$f^{\circ k}(p) \in]f(c), f^{\circ k}(c)[\subset T_j \cup T_{j'}$, and again by our assumption, $f^{\circ k}(p) \in]c, f(c)[$. Now we can repeat the argument, where p is replaced by $f^{\circ k}(p)$. Since p is periodic, we get after finitely many steps that $\text{orb}(p) \subset T_j$, contradicting minimality. \square

In the following, we always assume that the regarded point p is v_1 -characteristic. The analog statements hold, of course, for p being v_2 -characteristic. We just have to interchange the role of the symbols “1” and “2”.

Lemma 4.3.10 (Existence of p_0^0, p_0^1). *Let (T, f, \mathcal{P}) be a Hubbard tree and let $p \in T$ be v_1 -characteristic. If $i \in \{0, 1\}$ and $T_i \cap \text{orb}(p) \neq \emptyset$, then $p_0^i \in T_i$ exists.*

If (T, f, \mathcal{P}) is unicritical, then this also holds for $i = 2$.

Proof. Suppose that $f^{\circ l}(p) \in T_k$ for some $k \in \{0, 1\}$. Then $[f^{\circ l}(p), c_1]$ maps homeomorphically onto $[f^{\circ l+1}(p), f(c_1)]$, which contains p because p is v_1 -characteristic. And thus, T_k contains a preimage p_0^k of p . \square

Lemma 4.3.11 (Existence of p_0^2). *Suppose that p is v_1 -characteristic and $f(c_1) \in T_2$. Then at least one of the two preimages p_0^0 or p_0^2 exists.*

Proof. Note that $p \in]c_2, f(c_1)[$. Thus, if $f(c_2) \in G_p(c_2)$, then the preimage $p_0^0 \in T_0$ of p exists. If $f(c_2) \notin G_p(c_2)$, then p is also characteristic with respect to $f(c_2)$. By Lemma 4.3.10, p_0^2 exists. \square

As a summary of Lemmas 4.3.9, 4.3.10 and 4.3.11, we get the following proposition. Keep in mind that for all $j = 0, 1, 2$, $\text{orb}(p) \not\subset T_j$ by minimality.

Proposition 4.3.12 (Existence of preimages). *Let (T, f, \mathcal{P}) be a Hubbard tree and let $p \in T$ be v_1 -characteristic. Then the following are true:*

- (a) *Suppose that (T, f, \mathcal{P}) is unicritical. For any $i \in \{0, 1, 2\}$, if $\text{orb}(p) \cap T_i \neq \emptyset$ then $p_0^i \in T_i$. In particular, if T_j is the subtree that contains the first iterate of c_1 that is not contained in the same subtree as $f(c_1)$, then p_0^j exists.*
- (b) *If (T, f, \mathcal{P}) is not unicritical, then at least one of the two preimages p_0^0 and p_0^1 exists. More precisely, for $i \in \{0, 1\}$, if $\text{orb}(p) \cap T_i \neq \emptyset$ then $p_0^i \in T_i$ exists. If p_0^0 does not exist, then p is also v_2 -characteristic and the two preimages p_0^1, p_0^2 exist.*

The preimage p_0^2 might or might not exist regardless in which T_i the critical value $f(c_1)$ is contained and whether $\text{orb}(p) \cap T_2 = \emptyset$ or not.

Corollary 4.3.13 (Unique preimage). *If p is a periodic point which is v_1 -characteristic and T contains exactly one preimage of p , then this preimage is $p_0^0 \in T_0$.* \square

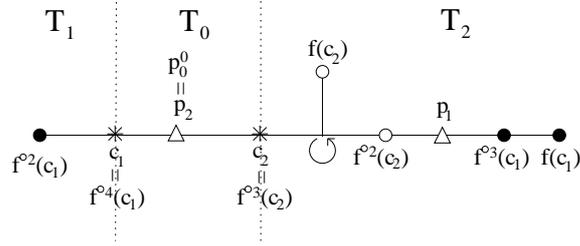


Figure 4.12: A Hubbard tree that contains exactly one preimage of p , which must be p_0^0 .

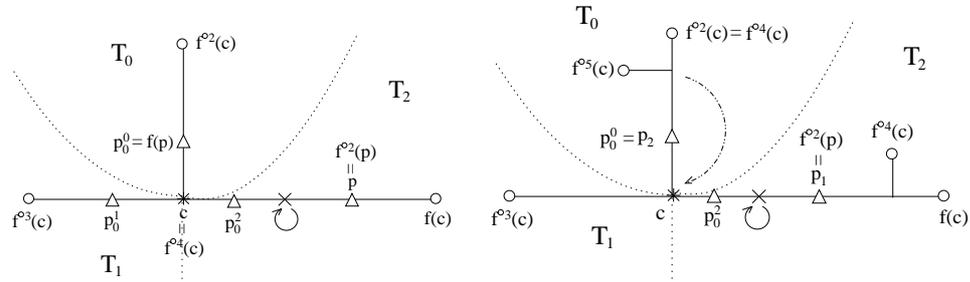


Figure 4.13: Two unicritical Hubbard trees. In the left example all three preimages p_0^i of the characteristic point p exist, in the right example only two exist.

Figures 4.13 to 4.12 illustrate several options described in Proposition 4.3.12. In particular, they show that one cannot make any statement about the existence of the preimage p_0^2 . Observe that for any cubic polynomial which is obtained by intertwining two quadratic polynomials at their β -fixed points (cf. [EY]), the preimage p_0^2 is never contained in T .

Figure 4.12 shows that there are indeed Hubbard trees that contain exactly one preimage of a characteristic point p .

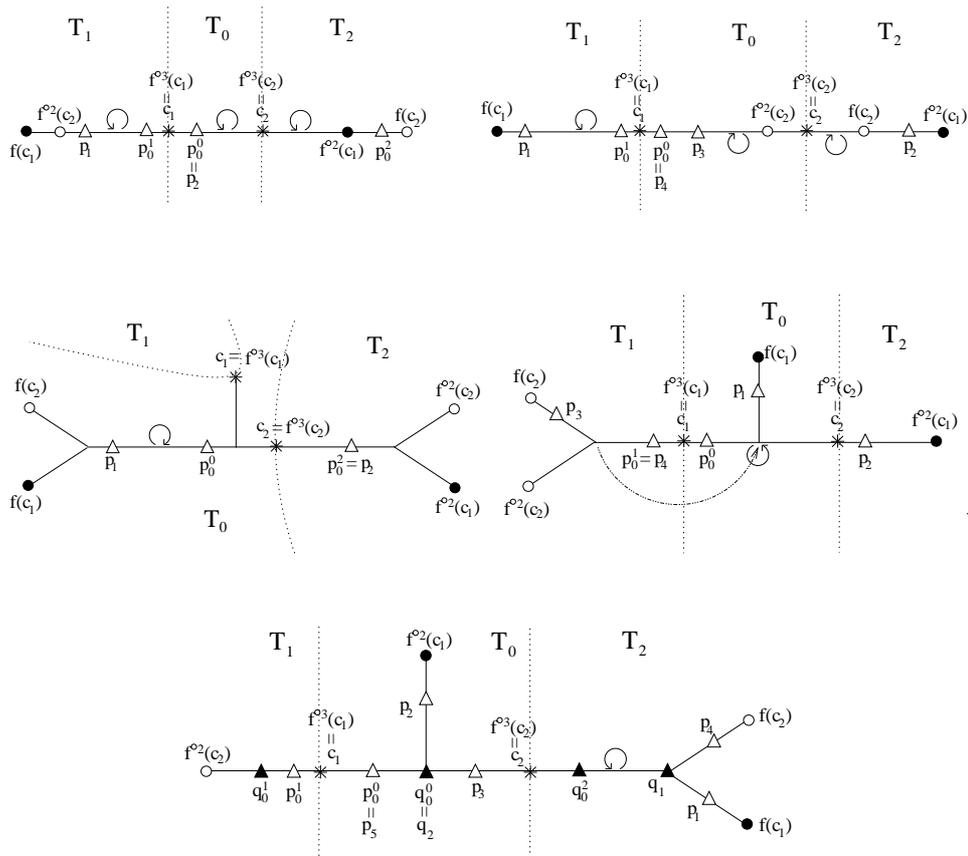


Figure 4.14: The critical value $f(c_1)$ is contained in T_1, T_0 and T_2 (from top to bottom). The left Hubbard trees of the first two lines contain the preimage p_0^2 , the ones on the right hand side do not. The last Hubbard tree contains v_1 -characteristic points p, q such that p_0^2 does not exist but q_0^2 does exist.

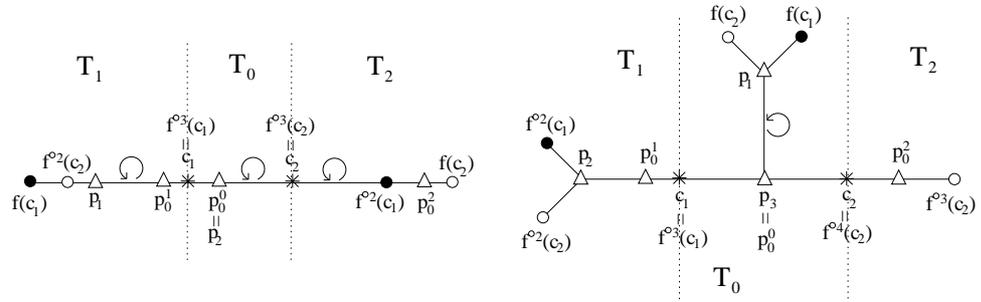


Figure 4.15: Examples for $\text{orb}(p) \cap T_2 = \emptyset$ and p_0^2 exists.

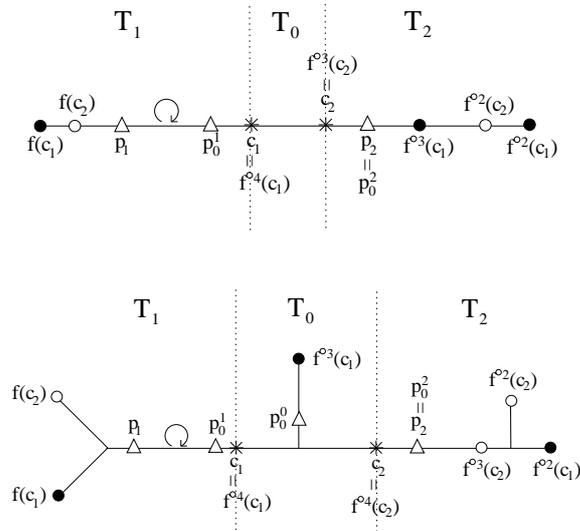


Figure 4.16: At the top, a Hubbard tree such that $\text{orb}(p) \cap T_0 = \emptyset$ and $p_0^0 \notin T_0$. Whenever we have a situation like this, then $p \notin [f(c_1), f(c_2)]$. The Hubbard tree on the bottom shows that it is also possible that $\text{orb}(p) \cap T_0 = \emptyset$ and $p_0^0 \in T_0$ exists.

Chapter 5

The Parameter Plane

5.1 Orbit Forcing

Before we start the study of a forcing relation on periodic orbits of Hubbard trees, let us fix some notation. Suppose (T, f, \mathcal{P}) is a Hubbard tree of capture or disjoint type and let (ν^1, ν^2) be its kneading sequence. If ν^1 is \star_1 -periodic, then we associate to ν^1 the sequences $\mathcal{A}_k(\nu^1)$ for $k = 0, 1$ by replacing each symbol \star_1 by k . With $\mathcal{A}(\nu^1)$ we mean any element of $\{\mathcal{A}_0(\nu^1), \mathcal{A}_1(\nu^1), \nu^1\}$. If ν^1 is preperiodic, then we set $\mathcal{A}(\nu^1) := \nu^1$. We define $\mathcal{A}_k(\nu^2)$ for $k = 0, 2$ and $\mathcal{A}(\nu^2)$ analogously. Note that in the unicritical case, we distinguished whether $n \in \text{orb}_\rho(\mathcal{A}_k(\nu))$ or not, which led to the notion of upper and lower kneading sequences. If $n \notin \text{orb}_\rho(\mathcal{A}_k(\nu))$ then we denoted $\mathcal{A}_k(\nu)$ by $\overline{\mathcal{A}}(\nu)$. Such a distinction is not useful in the general cubic setting: the combinatorial information whether $n \in \text{orb}_\rho(\mathcal{A}_k(\nu^i))$ or not has no geometric interpretation with respect to the location of points $p \in T$ with $\tau(p) = \mathcal{A}_k(\nu^i)$.

At this moment, we do not define $\mathcal{A}_k(\nu^i)$ for adjacent and bitransitive Hubbard trees.

In this section, we only consider hyperbolic Hubbard trees, i.e. Hubbard trees such that each of their critical points eventually maps onto a critical cycle. Suppose that (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ are hyperbolic Hubbard trees such that T contains two characteristic points with itineraries $\mathcal{A}(\tilde{\nu}^1), \mathcal{A}(\tilde{\nu}^2)$, where $(\tilde{\nu}^1, \tilde{\nu}^2)$ is the kneading sequence of $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$. We compare the characteristic points of the two Hubbard trees. In particular, we give sufficient conditions under which a characteristic point $z \in \tilde{T}$ is also found in T , and present examples that these conditions are not necessary. We use the obtained result on orbit forcing to introduce a partial order on equivalence classes of Hubbard trees.

5.1.1 Orbit Forcing for Hubbard Trees with One Free Critical Point

We start out with the set of Hubbard trees whose critical points c_2 have a fixed kneading sequence μ . Let $p \in T$ be a point “representing” the Hubbard tree $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ in T (this is made more precise below). We first compare the arrangements of points in $\text{orb}(p) \cup \text{orb}(c_2) \subset T$ and $\tilde{\mathcal{O}} \subset \tilde{T}$.

Definition 5.1.1 (Characteristic precritical, corresponding points). *Let (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be two Hubbard trees with kneading sequences (ν^1, ν^2) , $(\tilde{\nu}^1, \tilde{\nu}^2)$.*

Suppose that $c_2 \in T$ is periodic and $p \in T$ such that $f^{\circ i}(p) = c_2$ for some $i \in \mathbb{N}$. Let i_0 the smallest integer such that $f^{\circ i_0}(p) \in \text{orb}(c_2)$. We call p a v_1 -characteristic precritical point if $p \in [c_1, f(c_1)] \setminus [c_1, c_2]$ and $\{p, \dots, f^{\circ i_0}(p)\} \subset \overline{G_p(c_1)}$. If we say that a point p is v_1 -characteristic (precritical), then p is either v_1 -characteristic or v_1 -characteristic precritical.

v_2 -characteristic precritical points are defined in an analogous way.

For any $p \in T$ which is v_i -characteristic (precritical), we set $\text{orb}^i(p) := \text{orb}(p) \cup \{p_0^0, p_0^i\} \subset T$, where $p_0^0 \in T_0$, $p_0^i \in T_i$ are preimages of p (if existing).

Suppose that T contains a v_i -characteristic (precritical) point p with $\tau(p) = \mathcal{A}(\tilde{\nu})$. Then for all $j \in \mathbb{N}$, we call the points $f^{\circ j}(p) \in T$ and $\tilde{f}^{\circ j}(\tilde{\nu}_i) \in \tilde{T}$ corresponding. Both preimages p_0^0, p_0^i of p correspond to \tilde{c}_i .

Definition 5.1.2 (Combinatorially related Hubbard trees). *Let (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be two Hubbard trees of capture or disjoint type and let (ν^1, ν^2) , $(\tilde{\nu}^1, \tilde{\nu}^2)$ be their kneading sequences.*

Suppose that $\tilde{\nu}_1, \tilde{\nu}_2 \in \tilde{T}$ are characteristic (precritical) with respect to themselves and that $p, q \in T$ are v_1 -, v_2 -characteristic (precritical) such that $\tau(p) = \mathcal{A}(\tilde{\nu}^1)$ and $\tau(q) = \mathcal{A}(\tilde{\nu}^2)$. Then we say that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) . The two Hubbard trees (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ are called combinatorially related.

Finally, let us define

$$\mathcal{H}_\mu := \{(T, f, \mathcal{P}) \text{ minimal and attracting such that } \tau(c_2) = \mu\}.$$

Remark 5.1.3. The definition of \mathcal{H}_μ is only meaningful for \star_2 -periodic μ . Recall that μ is \star_2 -periodic if and only if c_2 is periodic such that $c_1 \notin \text{orb}(c_2)$. The set \mathcal{H}_μ is a single equivalence class of Hubbard trees if μ is not \star_2 -periodic: in this case, the Hubbard tree in question is either of adjacent or bitransitive type or c_2 is captured by the critical cycle $\text{orb}(c_1)$. For each possibility, we have that $\text{orb}(c_1) \subset \text{orb}(c_2)$. So the itinerary of c_2 uniquely determines the itinerary of the critical point c_1 . Therefore, when we fix the itinerary of c_2 , we automatically fix the itinerary of c_1 as well. Hence by Theorem 4.2.7, any two Hubbard trees in \mathcal{H}_μ are equivalent if μ is not \star_2 -periodic.

Lemma 5.1.4 (Location of corresponding points). *Let (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$ be two Hubbard trees. Suppose that μ is \star_2 -periodic that $\tilde{v}_1 \in \tilde{T}$ is \tilde{v}_1 -characteristic with exact period n . Let $k \in \{0, 1\}$ such that $k \neq \tau_n(\tilde{x})$ and $\tilde{x} \in]\tilde{c}_1, \tilde{v}_1]$ such that $[\tilde{x}, \tilde{v}_1[$ contains no precritical point of step at most $n+2$. Suppose that there is a v_1 -characteristic point $p \in T$ such that $\tau(p) = \mathcal{A}_k(\tilde{v})$.*

If $\tilde{x} \in \tilde{\mathcal{O}}$, $\tilde{x} \neq \tilde{v}_1$, and $x \in \text{orb}(p) \cup \text{orb}(c_2)$ is the corresponding point in T , then $\tilde{x} \in G_{\tilde{v}_1}(\tilde{c}_1)$ if and only if $x \in G_p(c_1)$.

Proof. The statement is clearly true if $\tilde{x} \in \{\tilde{c}_1, \tilde{c}_2\}$. In all other cases, we proceed by contradiction.

“ \implies ” Let us first assume that $\tilde{x} \in G_{\tilde{v}_1}(\tilde{c}_1)$ whereas $x \notin G_p(c_1)$. Since p is characteristic, $x \in \text{orb}(c_2)$ and thus $\tilde{x} \in \text{orb}(\tilde{c}_2)$. Let a denote the point in $\{p_0^0, p_0^1, c_2\}$ such that $Y := [a, p, x] \subset \overline{T_j}$ for some j and let $\tilde{a} \in \{\tilde{c}_1, \tilde{c}_2\}$ be its corresponding point. Then Y is degenerate with p in the middle while $[\tilde{a}, \tilde{v}_1, \tilde{x}] =: \tilde{Y}$ is either non-degenerate or degenerate with \tilde{x} in the middle. We are going to show that both possibilities yield a contradiction.

Let us first assume that both triods are degenerate. Then by condition (P2) of Definition 4.1.5, there is an i_0 such that $\varphi^{\circ i_0}(\tilde{Y}) = \text{STOP}$ (with \tilde{c}_2 in the middle). The iterates $\phi_{\tilde{Y}}^{\circ i_0}(\tilde{a}), \phi_{\tilde{Y}}^{\circ i_0}(\tilde{v}_1)$ are on different sides of \tilde{c}_2 while $\phi_Y^{\circ i_0}(a), \phi_Y^{\circ i_0}(p)$ are on the same of c_2 , contradicting that corresponding points have the same itinerary (modulo \star_1). It can of course happen that we have to chop the triod before reaching time i_0 (see Definition 4.2.2). No iterate of \tilde{v}_1 can be chopped off in \tilde{Y} and no iterate of x in Y ; so we only have to consider the case that a, \tilde{a} are chopped off. If the chopping happens at c_2, \tilde{c}_2 , then we replace the images of a, \tilde{a} by the critical points c_2, \tilde{c}_2 . If the chopping happens at c_1, \tilde{c}_1 , we replace $\phi_{\tilde{Y}}^{\circ j}(\tilde{a})$ by \tilde{c}_1 , and $\phi_Y^{\circ j}(a)$ by p_0^i if $\phi_Y^{\circ j}(p) \in]p_0^i, \phi_Y^{\circ j}(x)[$ ($i = 0$ or 1 such that the chopped triod is contained in $\overline{T_i}$). We continue the iteration. If however $\phi_Y^{\circ j}(p) \notin]p_0^i, \phi_Y^{\circ j}(x)[$, then $[c_1, \phi_Y^{\circ j}(p), p_0^i]$ is degenerate with $\phi_Y^{\circ j}(p)$ in the middle. Consequently $\phi_Y^{\circ j+1}(p) \in]v_1, p[$, contradicting the hypothesis that p is v_1 -characteristic.

Now suppose that \tilde{Y} is non-degenerate. As in the previous case, one can show that chopping does not cause any problems, so that all iterates $\varphi^{\circ i}(Y), \varphi^{\circ i}(\tilde{Y})$ are well-defined unless a STOP case occurs for some iterate of \tilde{Y} . Suppose that this happens at time i_0 and that \tilde{c} is the branch point of $\varphi^{\circ i_0}(\tilde{Y})$. Then either $[\phi_Y^{\circ i_0}(a), \phi_Y^{\circ i_0}(p), c]$ or $[\phi_Y^{\circ i_0}(x), \phi_Y^{\circ i_0}(p), c]$ is degenerate with $\phi_Y^{\circ i_0}(p)$ in the middle. If $c = c_1$, then for $q = x$ or a , the triod $[\phi_Y^{\circ i_0}(q), \phi_Y^{\circ i_0}(p), p_0^i]$ is degenerate with $\phi_Y^{\circ i_0}(p)$ in the middle, because p is v_1 -characteristic. (Again i is chosen such that the obtained triod lies in $\overline{T_i}$.) According as $\tilde{c} = \tilde{c}_1$ or $\tilde{c} = \tilde{c}_2$, we replace the chopped off point in $\varphi^{\circ i_0}(Y)$ by p_0^i or c_2 . In both cases, we iterate the newly obtained triod in T and the corresponding triod in \tilde{T} , which is degenerate with \tilde{c} in the middle. Following the reasoning of the previous case, we can iterate both triods until we eventually get a STOP for the triod in \tilde{T} . Then the images of p and q

lie on the same side of the image of c whereas the images of \tilde{q} and \tilde{v}_1 lie on different sides of \tilde{c} , a contradiction.

If $\varphi^{oi}(\tilde{Y}) \neq \text{STOP}$ for all $i \in \mathbb{N}$, then by condition (P2) of Definition 4.1.5, the branch point \tilde{b} of \tilde{Y} is not (pre-)critical; and since \tilde{v}_1 is never chopped off, $\tau(\tilde{b}) = \tilde{v}^1$ (modulo \star_1). Furthermore, $\tau_i(p) = \tau(\tilde{b}_i)$ for all $i < n$, where n is the period of \tilde{v}^1 . At time n , in T at most one point of $\phi_Y^{on}(a), \phi_Y^{on}(x)$ and in \tilde{T} at most one of $\phi_{\tilde{Y}}^{on}(\tilde{a}), \phi_{\tilde{Y}}^{on}(\tilde{x})$ is chopped off. So one of these points determines the n -th entry in $\tau(p)$ and in $\tau(\tilde{b})$. Therefore $\tau_n(p) = \tau_n(\tilde{b})$, contradicting our hypothesis on $\tau(p)$.

“ \Leftarrow ” To prove the converse direction, suppose that $\tilde{x} \notin G_{\tilde{v}_1}(\tilde{c}_1)$ and $x \in G_p(c_1)$. Then again $\tilde{x} \in \text{orb}(\tilde{c}_2)$. Let a, \tilde{a} be defined as above. By assumption, $\tilde{Y} := [\tilde{a}, \tilde{v}_1, \tilde{x}]$ is degenerate with \tilde{v}_1 in the middle while $Y := [a, p, x]$ is either non-degenerate or degenerate with x in the middle.

Let us first consider the case that Y is degenerate. Again by condition (P2) of Definition 4.1.5, we eventually reach the STOP case in Y or \tilde{Y} when we iterate the triods under φ . In both cases, the location of points contradicts that corresponding points have equal itineraries (modulo \star_1). If we have to chop the triods before we reach a STOP, then this can only happen at the images of a, \tilde{a} . (It must happen in both triods at the same time.) If the chopping occurs at c_2, \tilde{c}_2 , we replace the chopped off points by c_2, \tilde{c}_2 and continue the iteration. We do the same if the chopping occurs at c_1, \tilde{c}_1 and the appropriate preimage p_0^i of p is such that $\phi_Y^{oi}(x) \in [p_0^i, \phi_Y^{oi}(p)]$. Otherwise $\phi_Y^{oi+1}(x) = f^{oi+1}(x) \notin G_p(c_1)$. We have already shown that this implies that $\phi_{\tilde{Y}}^{oi+1}(\tilde{x}) = \tilde{f}^{oi+1}(\tilde{x}) \notin G_{\tilde{v}_1}(\tilde{c}_1)$ either. But this forces that $\phi_{\tilde{Y}}^{oi+1}(\tilde{v}_1) = \tilde{f}^{oi+1}(\tilde{v}_1) \notin G_{\tilde{v}_1}(\tilde{c}_1)$, contradicting the hypothesis that \tilde{v}_1 is \tilde{v}_1 -characteristic.

If Y is non-degenerate with branch point b , then expansivity and condition (P2) on \mathcal{P} imply that eventually p is chopped off or $\varphi^{oi_0}(Y) = \text{STOP}$. We can iterate the triods Y, \tilde{Y} until we reach one of the two events: again we only have to worry if the chopping occurs at c_1, \tilde{c}_1 . Only iterates of \tilde{a}, a or \tilde{x}, x can be chopped off. Without loss of generality, let us assume that a, \tilde{a} is chopped off. If $p_0^i \in G_b(c_1)$, then we replace the image of a by p_0^i and continue the iteration. Otherwise we get that $p_0^i \in [\phi_Y^{oi}(b), \phi_Y^{oi}(p)]$, and it follows that $\phi_Y^{oi+1}(x) \in G_p(v_1)$. Since \tilde{v}_1 is characteristic with respect to itself, we have that in \tilde{T} , $\phi_{\tilde{Y}}^{oi+1}(\tilde{x}) \in G_{\tilde{v}_1}(\tilde{c}_1)$. But we have already seen that this is not possible (this was the first direction we considered). So we can assume that Y, \tilde{Y} can be iterated until one of the two events mentioned above happens.

The image of p cannot be chopped off because this immediately contradicts that corresponding points have equal itineraries. If $\varphi^{oi_0}(Y) = \text{STOP}$ and $f^{oi_0}(b) = c_2$, then $[\tilde{c}_2, \phi_{\tilde{Y}}^{oi_0}(\tilde{v}_1), \phi_{\tilde{Y}}^{oi_0}(\tilde{q})] \subset \overline{\tilde{T}_j}$ is degenerate with $\phi_{\tilde{Y}}^{oi_0}(\tilde{v}_1)$ in the middle and $[\phi_Y^{oi_0}(p), c_2, \phi_Y^{oi_0}(q)] \subset \overline{T_j}$ is degenerate with c_2 in the

middle, where \tilde{q}, q equals \tilde{a}, a or \tilde{x}, x . Now we can apply the reasoning of the previous case to get the same contradiction as there. If $f^{\circ i_0}(b) = c_1$, then $[\phi_Y^{\circ i_0}(q), p_0^i, \phi_Y^{\circ i_0}(p)]$ is degenerate with p_0^i in the middle because p is v_1 -characteristic (q as above). In \tilde{T} , $[\tilde{c}_1, \phi_{\tilde{Y}}^{\circ i_0}(\tilde{v}_1), \phi_{\tilde{Y}}^{\circ i_0}(\tilde{q})]$ is degenerate with $\phi_{\tilde{Y}}^{\circ i_0}(\tilde{v}_1)$ in the middle. Now iterate these two triods. Since p is v_1 -characteristic, chopping does not cause any problems. By minimality, there is an iterate when $\phi_Y^{\circ i_0}(p)$ is chopped off, which cannot happen for $\phi_Y^{\circ i_0}(p) \in \tilde{T}$. \square

Lemma 5.1.5 (Orbit forcing in \mathcal{H}_μ , disjoint case). *Let μ be a \star_2 -periodic kneading sequence and $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$ be two Hubbard trees with kneading sequences (ν, μ) and $(\tilde{\nu}, \mu)$. Furthermore, suppose that \tilde{v}_1 is \tilde{v}_1 -characteristic and that T contains a periodic point p such that p is v_1 -characteristic and $\tau(p) = \mathcal{A}_k(\tilde{\nu})$, where k is chosen as in Lemma 5.1.4.*

Then for all \tilde{v}_1 -characteristic points $\tilde{z} \in \tilde{T}$, there is a v_1 -characteristic point $z \in]c_1, p[\subset T$ such that $\tau(z) = \tau(\tilde{z})$.

Proof. We pursue the same strategy as in the unicritical case. We start by iteratively constructing intervals $\tilde{I}_k \in \tilde{T}$ such that for all $k \in \mathbb{N}$, $\tilde{f}^{\circ k}(\tilde{z}) \in \tilde{I}_k$, $\partial \tilde{I}_k \in \tilde{\mathcal{O}}$ and such that $\tilde{f}|_{\tilde{I}_k}$ is a homeomorphism. For $k = 0$, we set

$$\tilde{I}_0 = \begin{cases} [\tilde{c}_1, \tilde{v}_1] & \text{if } \tilde{v}_1 \in \tilde{T}_1 \cup \tilde{T}_0 \\ [\tilde{c}_2, \tilde{v}_1] & \text{if } \tilde{v}_1 \in \tilde{T}_2 \end{cases}.$$

Now suppose that $\tilde{I}_k = [\tilde{x}, \tilde{y}]$ has been defined. Let $\mathcal{X}_{\tilde{v}_1}$ be the set of regular arms at \tilde{v}_1 . We have to distinguish the following three cases:

- (i) If $\tilde{f}(\tilde{I}_k) \subset \overline{\tilde{T}_i}$ for some $i \in \{0, 1, 2\}$, set $\tilde{I}'_{k+1} := \tilde{f}(\tilde{I}_k)$.
- (ii) If $\tilde{f}(\tilde{I}_k) \subset \overline{\tilde{T}_0} \cup \overline{\tilde{T}_i}$ for some $i \in \{1, 2\}$ and $\tilde{f}(\tilde{I}_k)$ intersects both subtrees, then let \tilde{I}'_{k+1} be the one interval of $[\tilde{f}(\tilde{x}), \tilde{c}_i]$ and $[\tilde{f}(\tilde{y}), \tilde{c}_i]$ that contains $\tilde{f}^{\circ k+1}(\tilde{z})$.
- (iii) If $\tilde{f}(\tilde{I}_k) \cap T_i \neq \emptyset$ for all $i \in \{0, 1, 2\}$, then $\tilde{f}(\tilde{x}) \in T_1$ and $\tilde{f}(\tilde{y})$ or vice versa. Let us suppose the first possibility holds (for the second one, we define \tilde{I}_{k+1} in an analogous way). We define \tilde{I}'_{k+1} to be the interval of $[\tilde{f}(\tilde{x}), \tilde{c}_1], [\tilde{c}_1, \tilde{c}_2]$ and $[\tilde{c}_2, \tilde{f}(\tilde{y})]$ that contains $\tilde{f}^{\circ k+1}(\tilde{z})$.

For all three cases, we set $\tilde{I}_{k+1} := \tilde{I}'_{k+1} \setminus \mathcal{X}_{\tilde{v}_1}$. This defines \tilde{I}_k for all $k \in \mathbb{N}$. Since $\partial \tilde{I}_k \in \tilde{\mathcal{O}}$ for all k and $\tilde{\mathcal{O}}$ is finite, the sequence of the \tilde{I}_k is preperiodic. In particular, if n is the period of \tilde{z} , then there are $0 \leq k_0 < k_1$ such that $\tilde{I}_{k_0 n} = \tilde{I}_{k_1 n}$. We set $\tilde{J}_k := \tilde{I}_{k+k_0 n}$ for all $k \in \mathbb{N}$.

Now, we define analogous intervals in T : for all $k \in \mathbb{N}$, we define J_k to be the interval bounded by two points in $\text{orb}'(p) \cup \text{orb}(c_2)$ which correspond

to the two endpoints of \tilde{J}_k . If one endpoint of \tilde{J}_k is \tilde{c}_1 and the second one lies in $\overline{\tilde{T}_i}$ for $i = 0$ or 1 , we pick in T the preimage p_0^i of p as the endpoint that corresponds to \tilde{c}_1 . This preimage exists by the following claim:

CLAIM: If $\tilde{f}^{\circ k}(\tilde{z}) \in [\tilde{x}, \tilde{c}_1] = \tilde{J}_k \subset \overline{\tilde{T}_i}$ for $i = 0$ or 1 and some $k \in \mathbb{N}$, then $p_0^i \in T$ exists and $p_0^i \in [x, c_1]$, where x is the corresponding points of \tilde{x} .

For the proof of this claim, observe that \tilde{z} being \tilde{v}_1 -characteristic implies that $\tilde{f}(\tilde{x}) \in G_{\tilde{v}_1}(\tilde{c}_1)$. Therefore $f(x) \in G_p(c_1)$ by Lemma 5.1.4. This is possible if and only if $p_0^i \in [x, c_1]$.

Next we show that $J_{k+1} \subset f(J_k)$ for all $k \geq 0$: it suffices to consider the cases where $J_{k+1} \neq f(J_k)$. This is only possible if we had to cut either the interval $\tilde{f}(\tilde{I}_{k+k_0n})$ at a critical point to get \tilde{I}'_{k+k_0n+1} or the interval \tilde{I}'_{k+k_0n+1} at \tilde{v}_1 to get \tilde{I}_{k+k_0n+1} . The above claim immediately implies that $J_{k+1} \subset f(J_k)$ in the first case. In the second case, this follows from Lemma 5.1.4, which says that $\tilde{x} \notin G_{\tilde{v}_1}(\tilde{c}_1)$ if and only if $x \notin G_p(c_1)$.

Since the itineraries of p and \tilde{c}_1 (modulo \star_1), and of c_2 and \tilde{c}_2 coincide, the construction of the intervals J_k yields that $f|_{J_k}$ is injective for all k . Let $S_k := \{x \in T : f^{\circ i}(x) \in J_i \text{ for all } 0 \leq i \leq k\}$. Then $S := \bigcap_{k=0}^{\infty} S_k$ is the nested intersection of non-empty, compact, connected sets and thus non-empty, compact and connected itself. That is, S is a possibly degenerate interval. We claim that S contains a periodic point z with itinerary $\tau(\tilde{z})$. Since $f^{\circ(k_1-k_0)n}(S) = S$, there is a periodic point $z \in S$, and if $z \in \overset{\circ}{S}$, then $\tau(z) = \tau(\tilde{z})$ by Lemma 4.1.11. Since c_2 might have been used to define the J_k (or c_1 in the case that $p = v_1$), the endpoints of S might not have itinerary $\tau(\tilde{z})$. Let us suppose that S contains no periodic point with itinerary $\tau(\tilde{z})$. If $S = \{z\}$ is a singleton then $z = f^{\circ i}(c_i)$. The construction implies that z is the limit of precritical points. As z is on a critical cycle, this contradicts the fact that (T, f, \mathcal{P}) has attracting dynamics. If S is non-degenerate, then it must be an interval because $\overset{\circ}{S}$ contains no branch point of T : this would be a periodic point with itinerary $\tau(\tilde{z})$. The two endpoints e_1, e_2 must be (pre-)critical and 1- or 2-periodic under $f^{\circ(k_1-k_0)n}$. Now attracting dynamics implies that there is a point $z \in \overset{\circ}{S}$ which is fixed by $f^{\circ 2n(k_1-k_0)}$. This shows the existence of a periodic point $z \in T$ that has itinerary $\tau(\tilde{z})$. Observe that the way the intervals J_k were defined, $\text{orb}(z) \subset G_p(c_1)$.

It remains to show that z is v_1 -characteristic. If it is not then either $z \notin]c_i, v_1[$, where c_i so that $]c_i, v_1[$ does not contain the second critical point, or $z \in]c_i, v_1[$ but there is an iterate z_l of z such that $z_l \notin G_z(c_i)$.

Let us start with the first possibility. We consider the degenerate triod $[\tilde{a}, \tilde{z}, \tilde{v}_1] =: \tilde{Y} \subset \overline{\tilde{T}}$ and the non-degenerate triod $[a, z, p] =: Y \subset T$ with branch point b , where $\tilde{a} \in \{\tilde{c}_1, \tilde{c}_2\}$ and $a \in \{p_0^0, p_0^1, c_2\}$ such that $\tilde{Y} \subset \overline{\tilde{T}_j}$, $Y \subset \overline{T_j}$ for some j . We iterate Y, \tilde{Y} under φ . In T , we replace points that are chopped off at c_1 (at c_2) by the appropriate preimage of p (by c_2). Let us suppose that at time k either z is chopped off or $\varphi^{\circ k}(Y) = \text{STOP}$. Such a time

exists because $\tau(z) \neq \tau(b)$ and by condition (P2) of Definition 4.1.5. Observe that p, a might have been chopped off before time k . However, it is possible to chop the triods Y, \tilde{Y} in such a way that corresponding generating points of the new triods have the same itineraries: if $\phi_Y^{\circ j}(p)$ or $\phi_Y^{\circ j}(a)$ is chopped off for some $j < k$, then let j_0 be the smallest number with this property and let r be the point that replaces the chopped one. Let us suppose the image of a is chopped off. The argument is similar for the case that p is chopped off (just interchange the roles of a and p). We claim that $r \in G_{f^{\circ j_0}(b)}(c_i)$ and thus the triod $[r, f^{\circ j_0}(z), f^{\circ j_0}(p)]$ is non-degenerate with branch point $f^{\circ j_0}(b)$. The claim is trivially true if $r = c_2$. If $r = p_0^0$ or $r = p_0^1$, then $r \in]c_1, f^{\circ j_0}(b)[$ because otherwise $f^{\circ j_0}(z) \notin G_p(c_1)$, which is not possible as we have seen. We continue the iteration until we reach time k . This might include further choppings of the just described kind (where f might have to be replaced by the chopping map ϕ_Y).

Now let us consider the situation at time k . Since the defining points of $\varphi^{\circ k}(Y)$ and $\varphi^{\circ k}(\tilde{Y})$ have the same itineraries (modulo \star_1) and \tilde{z} is contained in the middle of \tilde{Y} , we must have that $f^{\circ k}(Y) = \text{STOP}$. Then either $\phi_Y^{\circ k}(a)$ and $\phi_Y^{\circ k}(z)$ or of $\phi_Y^{\circ k}(p)$ and $\phi_Y^{\circ k}(z)$ are contained on the same side of the critical point $c = f^{\circ k}(b)$. We show that the first possibility yields a contradiction. One derives a contradiction in the second case by the same reasoning.

If $c = c_2$, then the triod $[\phi_Y^{\circ k}(\tilde{a}), \phi_Y^{\circ k}(\tilde{z}), \tilde{c}_2] =: \tilde{Y}_k \subset \tilde{T}$ is degenerate with $\phi_Y^{\circ k}(\tilde{z})$ in the middle, whereas the triod $[\phi_Y^{\circ k}(a), \phi_Y^{\circ k}(z), c_2] =: Y_k \subset T$ is degenerate with c_2 in the middle. We iterate both triods until we reach STOP in some image of Y_k (this must eventually happen by requirement (P2) imposed on \mathcal{P}). At this moment, the images of $\phi_Y^{\circ k}(a), \phi_Y^{\circ k}(z)$ are on different sides of c_2 while in \tilde{T} , the corresponding points are on the same side of \tilde{c}_2 , a contradiction. It might happen that we have to chop the triods Y_k, \tilde{Y}_k when we push them forward. By the same argument as in the previous paragraph, this causes no problems.

If $c = c_1$, then $\text{orb}(z) \subset G_p(c_1)$ implies that there is a preimage $p_0^i \in [\phi_Y^{\circ k}(z), \phi_Y^{\circ k}(a)]$. Now we iterate the triods $[\phi_Y^{\circ k}(\tilde{a}), \phi_Y^{\circ k}(\tilde{z}), \tilde{c}_1] =: \tilde{Y}_k$ and $[\phi_Y^{\circ k}(a), \phi_Y^{\circ k}(z), p_0^i] =: Y_k \subset T$ until in Y_k , the image of $\phi_Y^{\circ k}(z)$ is chopped off, which is a contradiction as this can never occur in \tilde{Y}_k . Also here, we might have to chop the triods before we reach this contradiction. In Y_k , only an iterate of $\phi_Y^{\circ k}(a)$ might be chopped off earlier than the iterate of $f^{\circ k}(z)$. Say this happens at time $l < k$. Again chopping at c_2 yields no problem and at c_1 , we have that $\phi_{Y_k}^{\circ l}(\phi_Y^{\circ k}(z)), \phi_{Y_k}^{\circ l}(p_0^i)$ and p_0^j form a degenerate triod with $\phi_{Y_k}^{\circ l}(p_0^i)$ in the middle because p is v_1 -characteristic.

If $z \in]c_1, v_1[$ but there is an iterate $z_l \notin G_z(c_1)$, then the triod $[a, z_l, z]$ is degenerate with z in the middle whereas $[\tilde{c}_i, \tilde{z}_l, \tilde{z}]$ is either non-degenerate or degenerate with \tilde{z}_l in the middle. Both possibilities yield similar contra-

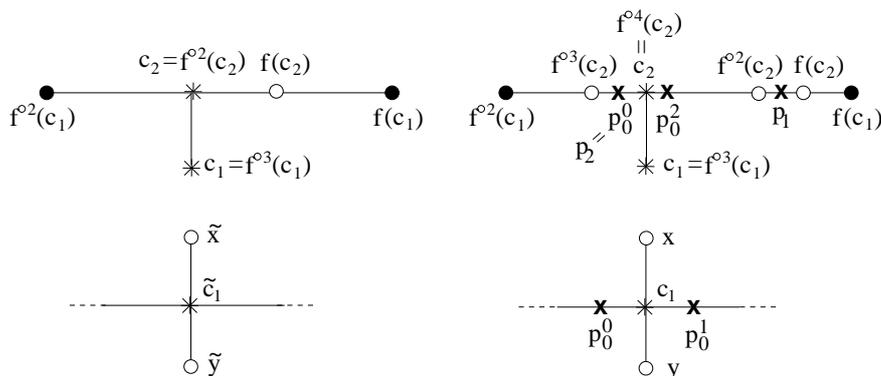


Figure 5.1: The above Hubbard trees illustrate the exceptional case of the orbit condition. The critical orbit c_1 of the left tree is represented in the right one by $\text{orb}(p_1)$. Both trees are admissible, the polynomials generating them are marked by (6a), (6b) in Figure 4.8. Note that for any degenerate triod $[x, c_1, y]$ with c_1 in the middle of the left tree, we can choose one preimage of p_0^i in the right tree so that $[x, p_0^i, y]$ is degenerate with p_0^i in the middle. In the situation at the bottom, this is not possible anymore. The two sketches can be extended to admissible Hubbard trees.

dictions as the cases discussed above. This finishes the proof. □

Remark 5.1.6. Let $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$ be two equivalent Hubbard trees and suppose that their critical values v_1, \tilde{v}_1 are v_1 -, \tilde{v}_1 -characteristic. Then Lemma 5.1.5 shows in particular that T contains a v_1 -characteristic point with itinerary τ if and only if \tilde{T} does. If the two Hubbard trees do not have attracting dynamics and the itinerary of the v_1 -characteristic point equals the itinerary of v_1 (modulo \star_1), then this is not true in general.

Definition 5.1.7 (Orbit condition for \mathcal{H}_μ). *Let $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$ such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) . We say that $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the orbit condition if any three points $x^1, x^2, x^3 \in \text{orb}'(p) \cup \text{orb}(c_2) \subset T$ and their corresponding points $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \tilde{\mathcal{O}} \subset \tilde{T}$ form combinatorially equivalent triods unless the following holds: the triod $[\tilde{x}, \tilde{c}_1, \tilde{y}] \subset \tilde{T}$ is degenerate with \tilde{c}_1 in the middle whereas $[x, p_0^i, y] \subset T$ is non-degenerate and c_1 is contained in its interior, where $i = 0$ or 1 .*

We allow for the exceptional case because if $\tilde{c}_1 \in \tilde{T}$ and $c_1 \in T$ are branch points, then it might very well happen that the degenerate triod $[\tilde{x}, \tilde{c}_1, \tilde{y}]$ becomes a non-degenerate triod $[x, p_0^i, y]$ in T . This is illustrated in Figure 5.1.

Remark 5.1.8. By Definition 4.1.29, any two representatives of an equivalence class meet the orbit condition.

Proposition 5.1.9 (Orbit forcing in \mathcal{H}_μ). *Let μ be a \star_2 -periodic kneading sequence and (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$. Suppose that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) such that these two Hubbard trees meet the orbit condition.*

Then for any \tilde{v}_1 -characteristic (precritical) point $\tilde{z} \in \tilde{T}$, there is a v_1 -characteristic point $z \in T$ such that $\tau(z) = \tau(\tilde{z})$.

Proof. We show first that under the assumption of the current lemma, the statement of Lemma 5.1.4 holds. Therefore, we do not need the assumption on $\tau(p)$ anymore for the statement of the orbit forcing lemma.

CLAIM: If $\tilde{x} \in \tilde{\mathcal{O}}$, $\tilde{x} \neq \tilde{v}_1$, and $x \in \text{orb}'(p) \cup \text{orb}(c_2)$ is the corresponding point in T , then $\tilde{x} \in G_{\tilde{v}_1}(\tilde{c}_1)$ if and only if $x \in G_p(c_1)$.

Let $\tilde{x} \notin \{\tilde{c}_1, \tilde{c}_2\}$ (for these two points, the claim is clearly true). If $\tilde{x} \in G_{\tilde{v}_1}(\tilde{c}_1)$ and $x \notin G_p(c_1)$, then the triod $[c_2, p, x]$ is degenerate with p in the middle whereas $[\tilde{c}_2, \tilde{x}, \tilde{v}_1]$ is either non-degenerate or degenerate with \tilde{x} in the middle. Both possibilities contradict that (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the orbit condition. Similarly, if $\tilde{x} \notin G_{\tilde{v}_1}(\tilde{c}_1)$ then $[\tilde{c}_2, \tilde{v}_1, \tilde{x}]$ is degenerate with \tilde{v}_1 in the middle. Now the orbit condition implies that $[c_2, p, x]$ is degenerate with p in the middle and thus, $x \notin G_p(c_1)$, and chopping causes no problems.

To prove the statement of this lemma, suppose first that $\tilde{z} \in \tilde{T}$ is \tilde{v}_1 -characteristic. The proof of Lemma 5.1.5 carries over to the current situation. We want to verify the crucial steps in that proof: first we have to make sure that for the intervals J_k in T , $J_{k+1} \subset f(J_k)$ for all $k \in \mathbb{N}_0$. This holds by the above claim and because \tilde{z} is \tilde{v}_1 -characteristic. So we find a periodic point $z \in T$ with $\tau(z) = \tau(\tilde{z})$. The above claim also guarantees that $\text{orb}(z) \subset G_p(c_1)$. So it only remains to show that z is v_1 -characteristic. We can argue literally the same way as in the proof of Lemma 5.1.5, only in its second to last paragraph, we have to use a different argument that chopping is no problem: we iterate the degenerate triods $[\phi_Y^{\circ k}(\tilde{a}), \phi_Y^{\circ k}(\tilde{z}), \tilde{c}_1] =: \tilde{Y}_k$ ($\phi_Y^{\circ k}(\tilde{z})$ is in the middle) and $[\phi_Y^{\circ k}(a), \phi_Y^{\circ k}(z), p_0^i] =: Y_k$ (p_0^i is in the middle). It might happen that $\phi_Y^{\circ k}(\tilde{a}), \phi_Y^{\circ k}(a)$ are chopped off at \tilde{c}_1, c_1 . Now the orbit condition implies that $[\phi_{Y_k}^{\circ l}(\phi_Y^{\circ k}(a)), p_0^j, \phi_{Y_k}^{\circ l}(p_0^i)]$ is either non-degenerate or degenerate with p_0^j in the middle, and thus, $[\phi_{Y_k}^{\circ l}(\phi_Y^{\circ k}(z)), p_0^j, \phi_{Y_k}^{\circ l}(p_0^i)]$ is degenerate with $\phi_{Y_k}^{\circ l}(p_0^i)$ in the middle.

Now let us prove the claim for the case that \tilde{z} is \tilde{v}_1 -characteristic precritical. Let i_0 be the smallest integer such that $\tilde{f}^{\circ i_0}(\tilde{z}) \in \text{orb}(\tilde{c}_2)$, say $\tilde{f}^{\circ i_0}(\tilde{z}) = \tilde{f}^{\circ i_1}(\tilde{c}_2)$. We define closed intervals $\tilde{I}_0, \dots, \tilde{I}_{i_0-1} \subset \tilde{T}$ as in the proof of Lemma 5.1.5; for i_0 , we set $\tilde{I}_{i_0} := \tilde{f}(\tilde{I}_{i_0-1})$. Let $I_k \subset T$ be the interval obtained by taking the convex hull of the two points corresponding to the endpoints of \tilde{I}_k . As above, one can show that $f(I_k) \supset I_{k+1}$ and $I_k \in \overline{G_p(c_1)}$ for all $0 \leq k < i_0$. (Recall that by definition, $\{\tilde{z}, \dots, \tilde{f}^{\circ i_0}(\tilde{z})\} \subset \overline{G_{\tilde{v}}(\tilde{c}_1)}$.) Then there is a point $z \in T$ such that $f^{\circ k}(z) \in I_k$ for all $0 \leq k < i_0$ and

$f^{\circ i_0}(z) = f^{\circ i_1}(c_2)$ and $\tau(z) = \tau(\tilde{z})$.

It only remains to show that z is v_1 -characteristic precritical. If z does not have this property, then there is an iterate $f^{\circ l-1}(z) =: z_l$ with $l \leq i_0 + 1$ such that $[a, z, z_l]$ is degenerate with z in the middle while $[\tilde{a}, \tilde{z}, \tilde{z}_l]$ is either degenerate with \tilde{z}_l in the middle or non-degenerate. The point $a \in \{p_0^0, p_0^1, c_2\}$ is again chosen so that $[a, z, z_l] \in \overline{T_j}$ for some j . Note that $z \in]c_1, v_1[$ by construction. Let us first assume that both triods are degenerate. There is a time where the images of z, z_l are on different sides of some critical point c . At this moment, the images of z, a are on the same side of c while the images of \tilde{z}, \tilde{a} are on different sides of \tilde{c} , a contradiction. Suppose the images of a, \tilde{a} are chopped off before this event happens. If this happens at c_2, \tilde{c}_2 , we replace the chopped off points by c_2, \tilde{c}_2 and continue the iteration. Suppose a chopping happens at c_1 at time k . If $p_0^i \notin G_{f^{\circ k}(z)}(c_1)$, then $f^{\circ k+1}(z) \notin G_p(c_1)$ and thus $f^{\circ k+1}(z) \in \text{orb}(c_2)$ and $f^{\circ k}(z_l) \in \text{orb}(c_2)$. Now by the above claim, $\tilde{f}^{\circ k+1}(\tilde{z}), \tilde{f}^{\circ k+1}(\tilde{z}_l) \notin G_{\tilde{v}_1}(\tilde{c}_1)$ and therefore $p_0^i \notin G_{f^{\circ k}(z_l)}(c_1)$. But this means that $[\phi_Y^{\circ k}(\tilde{a}), \tilde{c}_1, \tilde{f}^{\circ k}(\tilde{z}_l)]$ is a degenerate triod with \tilde{c}_1 in the middle whereas $[\phi_Y^{\circ k}(\tilde{a}), p_0^i, f^{\circ k}(z_l)]$ is degenerate with $f^{\circ k}(z_l)$ in the middle, contradicting the orbit condition.

Now if $[\tilde{a}, \tilde{z}, \tilde{z}_l] =: \tilde{Y}$ is non-degenerate, then either the branch point \tilde{b} is precritical and there is a k such that $\varphi^{\circ k}(\tilde{Y}) = \text{STOP}$, or the image of \tilde{z} is chopped off. The latter one is not possible because this cannot happen in $[a, z, z_l]$, which is a degenerate triod with z in the middle. By similar arguments as above, chopping before time k is no problem. Note that in $\varphi^{\circ k}(\tilde{Y})$, the image of \tilde{z} cannot be chopped off. Let us suppose that at time k , the image of \tilde{a} is chopped off (the reasoning for the case that the image of \tilde{z}_l is chopped off is similar). We first consider the possibility that $\tilde{f}^{\circ k}(\tilde{b}) = c_2$. The triod $[\phi_Y^{\circ k}(\tilde{z}_l), \tilde{c}_2, \phi_Y^{\circ k}(\tilde{z})]$ is degenerate with \tilde{c}_2 in the middle while $[c_2, \phi_Y^{\circ k}(z), \phi_Y^{\circ k}(z_l)]$ is degenerate with $\phi_Y^{\circ k}(z)$ in the middle. Eventually the first triod is mapped to STOP , which yields a contradiction to the fact that corresponding points have equal itineraries. Again chopping off the image of $\phi_Y^{\circ k}(\tilde{z}_l)$ before we reach STOP is no problem (no other point can be chopped off).

It remains to consider the case when $\tilde{f}^{\circ k}(\tilde{b}) = c_1$. Then $p_0^i \in G_{f^{\circ k}(z)}(c_1)$: suppose it was not. Then $f^{\circ k+1}(z) \notin G_p(c_1)$, and $f^{\circ k}(z) \in \text{orb}(c_2)$. Thus, the mutual location of $f^{\circ k}(a), f^{\circ k}(z)$ and p_0^i in T and the mutual location of the corresponding points in \tilde{T} contradict that $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the orbit condition. Now we can apply the same arguments as in the case $\tilde{f}^{\circ k}(\tilde{b}) = c_2$. This finishes the proof. \square

Lemma 5.1.10 (Forcing from larger to smaller Hubbard tree). *Let $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}) \in \mathcal{H}_\mu$ such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) and that the two Hubbard trees meet the orbit condition. Let $z \in]c_1, p[$ be a v_1 -characteristic (precritical) point. Then there is a \tilde{v}_1 -characteristic*

(precritical) point \tilde{z} in \tilde{T} such that $\tau(z) = \tau(\tilde{z})$.

Proof. For the proof in the case that z is periodic, we proceed as in Lemma 5.1.5. We first pick intervals $J_k \in T$ such that $f^{\circ k}(z)$ and $\partial J_k \in \text{orb}'(p) \cup \text{orb}(c_2)$. Then we define analogous intervals in \tilde{T} and find a periodic point \tilde{z} with $\tau(z) = \tau(\tilde{z})$. To show that \tilde{z} is characteristic, we proceed by contradiction and compare the triods $[a, z, p]$, $[a, z_l, z]$ (the first one is degenerate with z in the middle, the second one is non-degenerate or degenerate with z_l in the middle) with the non-degenerate triod $[\tilde{a}, \tilde{z}, \tilde{v}_1]$ and the degenerate triod $[\tilde{a}, \tilde{z}, \tilde{z}_l]$ with \tilde{z} in the middle. Note that $\text{orb}(z) \cap (G_{p_0^1})(c_1) \cap G(p_0^0)(c_1) = \emptyset$ because $z \in]c_1, p[$ is characteristic. So chopping does not cause any problems. We derive the same contradictions as in Lemma 5.1.5.

For the proof in the case that z is v_1 -characteristic precritical, we refer to Proposition 5.1.9. Note that one has to interchange the role of the triod $[a, z, z_l]$ and $[\tilde{a}, \tilde{z}, \tilde{z}_l]$. Since z is v_1 -characteristic and since the orbit condition holds, chopping causes no problems. \square

Now we are ready to define a partial order on the set of equivalence classes in

$$\mathcal{H}'_\mu := \{(T, f, \mathcal{P}) \in \mathcal{H}_\mu : v_1 \text{ is } v_1\text{-characteristic (precritical)}\}$$

for \star_2 -periodic μ . Recall that for such kneading sequences μ , the set \mathcal{H}_μ consists of Hubbard trees of disjoint or capture type. So we only ignore Hubbard trees whose critical values v_1 are not characteristic or characteristic precritical with respect to themselves.

Definition 5.1.11 (Order “ $<$ ” on \mathcal{H}_μ). *Let μ be a \star_2 -periodic sequence and let $\mathcal{T}, \tilde{\mathcal{T}} \in \mathcal{H}'_\mu$ be two non-equivalent Hubbard trees.*

Then $[\mathcal{T}] > [\tilde{\mathcal{T}}] : \iff \tilde{\mathcal{T}}$ is combinatorially represented in \mathcal{T} such that the two Hubbard trees meet the orbit condition.

Theorem 5.1.12 (Partial order). *The relation “ $<$ ” is a partial order on the set \mathcal{H}'_μ .*

Proof. We have to show that

- (i) the definition is independent of the choice of the representatives, that
- (ii) $(T, f, \mathcal{P}) \not> (T, f, \mathcal{P})$ and that
- (iii) transitivity holds, i.e., that for any three Hubbard trees $\mathcal{T}, \tilde{\mathcal{T}}, \tilde{\tilde{\mathcal{T}}}$ such that $\mathcal{T} > \tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}} > \tilde{\tilde{\mathcal{T}}}$, we have that $\mathcal{T} > \tilde{\tilde{\mathcal{T}}}$.

Item (i) follows by Remarks 5.1.6 and 5.1.8. For item (ii), we have to show that (T, f, \mathcal{P}) contains no v_1 -characteristic point with itinerary $\mathcal{A}(\tau(v_1))$

if c_1 is periodic: if there was such a point p , then $f^{oi}([p, v_1])$ would be injective for all i ; in particular, for any sequence $x_n \in]p, v_1[$ converging to v_1 , $\lim_{n \rightarrow \infty} \tau(x_n) = \mathcal{A}(\tau(v_1))$ contradicting the definition of upper and lower kneading sequences. (Note however, that T might very well contain a periodic point in $T \setminus G_{v_1}(c_1)$ with itinerary $\mathcal{A}(\tau(v_1))$.) If v_1 is preperiodic, then there is an i_0 such that $f^{oi_0}(v_1) = c_2$. It follows that there is no other point than v_1 which has itinerary $\tau(v_1)$.

It remains to show transitivity. Suppose $\tilde{p} \in \tilde{T}$ is \tilde{v}_1 -characteristic (pre-critical) with $\tau(\tilde{p}) = \mathcal{A}(\tau(\tilde{v}_1))$ and $q \in T$ is v_1 -characteristic (precritical) with $\tau(q) = \mathcal{A}(\tau(\tilde{v}_1))$. By Proposition 5.1.9, there is a v_1 -characteristic (precritical) point $p \in T$ with itinerary $\tau(p) = \tau(\tilde{p}) = \mathcal{A}(\tau(\tilde{v}_1))$. So to finish the proof, we have to show that \tilde{T} and T meet the orbit condition. For this, it is enough to show that any three points $a_1, a_2, a_3 \in \text{orb}(p) \cup \text{orb}(c_2) \subset T$ and their corresponding points $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in \text{orb}(\tilde{p}) \cup \text{orb}(\tilde{c}_2) \subset \tilde{T}$ form combinatorially equivalent triods Y, \tilde{Y} . We again determine their type via iteration under φ . Whenever we have to chop at c_2, \tilde{c}_2 , we replace the chopped off point by c_2, \tilde{c}_2 . If we have to chop at c_1, \tilde{c}_1 , we replace the chopped off point by the appropriate preimage q_0^i of q and by \tilde{c}_1 ($i = 0$ or 1). Let us first assume that chopping does not cause any problems; we verify this at the end of the proof. Let $\mathcal{O}_q := \text{orb}(q) \cup \text{orb}(c_2) \subset T$.

Let us first assume that we can iterate Y, \tilde{Y} forever. If one of them is non-degenerate, then its branch point is not (pre-)critical and all three end-points must eventually be chopped off or lie on a critical cycle. Recall that we only consider hyperbolic Hubbard trees and thus each critical point must eventually land on a periodic critical point. If Y is non-degenerate and \tilde{Y} is degenerate with, say, \tilde{a}_2 in the middle, then \tilde{a}_2 must be on a critical cycle and there is a time i_0 such that $\varphi^{oi_0}(\tilde{Y}) = \text{STOP}$. But this is impossible because corresponding points have equal itineraries. We obtain the same contradiction if \tilde{Y} is non-degenerate and Y is degenerate. If both are degenerate, then the middle point is not (pre-)critical. Since corresponding points have equal itineraries, it follows that \tilde{a}_i is contained in the middle if and only if a_i is contained in the middle, and the claim is proven for the case that no STOP occurs.

If STOP occurs at the same time in \tilde{Y}, Y , then they are combinatorially equivalent because we can read off from the itineraries whether in the STOP case a triod is degenerate or non-degenerate. If the STOP case happens in exactly one triod, then this triod must be non-degenerate (cf. Remark 4.2.3). Let us assume that the second triod is degenerate with a_1 or \tilde{a}_1 in the middle.

We consider first the case that $\varphi^{oi_0}(Y) = \text{STOP}$. Set $\phi_Y^{oi_0}(a_i) =: b_i$ and let \tilde{b}_i be the corresponding point in \tilde{T} . Without loss of generality, let us assume that b_2 is chopped off. If the critical branch point is c_1 such that $q_0^i \notin [b_1, b_3]$ (q represents \tilde{v}_1 in T), then $f(b_i) \notin G_q(c_1)$ for $i = 1, 3$ and

$b_1, b_3 \in \text{orb}(c_2)$. Now the triod $[\tilde{b}_1, \tilde{b}_3, \tilde{c}_1]$ is degenerate with \tilde{b}_1 . Since \mathcal{T} and $\tilde{\mathcal{T}}$ meet the orbit condition, $[b_1, b_3, q_0^i]$ is degenerate with b_1 in the middle. There is a point $x \in \mathcal{O}_q \cap G_{c_1}(b_2)$ (by the proof of Proposition 5.1.9). If \tilde{x} is the corresponding point in \tilde{T} , then $[q_0^i, b_1, x]$ is degenerate with b_1 in the middle while $[\tilde{c}_1, \tilde{b}_1, \tilde{x}]$ is degenerate with \tilde{c}_1 in the middle, contradicting that \mathcal{T} and $\tilde{\mathcal{T}}$ satisfy the orbit condition. On the other hand, if $q_0^i \in [b_1, b_3]$, then for either $i = 1$ or $i = 3$, $f(b_i) \notin G_q(c_1)$, and thus $b_i \in \text{orb}(c_2)$. Looking at $\varphi^{\circ i_0+1}(\tilde{Y})$, we see that either both $\tilde{f}(\tilde{b}_1)$ and $\tilde{f}(\tilde{b}_3)$ or none of them are contained in $G_{\tilde{v}_1}(\tilde{c}_1)$. By Lemma 5.1.4, it is only possible that none of them are in $G_{\tilde{v}_1}(\tilde{c}_1)$ and thus b_1 and b_3 are elements of $\text{orb}(c_2)$. But now the location of $f(b_1), f(b_3)$ contradicts Lemma 5.1.4.

Now suppose that c_2 is the branch point in $\varphi^{\circ i_0}(Y)$. Consider the degenerate triods $[\tilde{b}_1, \tilde{b}_3, \tilde{c}_2] =: \tilde{Y}_{i_0}$ with \tilde{b}_1 in the middle and $[b_1, b_3, c_2] =: Y_{i_0}$ with c_2 in the middle. In Y_{i_0} eventually STOP occurs by condition (P2) of Definition 4.1.5. At this time we get a contradiction to the fact that corresponding points have equal itineraries. Note that the images of b_1, \tilde{b}_1 and \tilde{c}_2, c_2 cannot be chopped off. If b_3, \tilde{b}_3 are chopped off at c_2, \tilde{c}_2 , replace them by c_2, \tilde{c}_2 and continue the iteration. If the chopping occurs at c_1, \tilde{c}_1 , then pick an $x \in \mathcal{O}_q \cap G_{c_1}(\phi_{Y_{i_0}}^{\circ k}(b_3))$ and let \tilde{x} the corresponding point in \tilde{T} . The orbit condition on \mathcal{T} and $\tilde{\mathcal{T}}$ implies that $[x, q_0^i, \phi_{Y_{i_0}}^{\circ k}(c_2)]$ is either degenerate with q_0^i in the middle or non-degenerate. In both cases, $[\phi_{Y_{i_0}}^{\circ k}(b_1), \phi_{Y_{i_0}}^{\circ k}(c_2), q_0^i]$ is degenerate with $\phi_{Y_{i_0}}^{\circ k}(c_2)$. So we replace $\phi_{Y_{i_0}}^{\circ k}(b_3)$ by q_1^0 and $\phi_{Y_{i_0}}^{\circ k}(\tilde{b}_3)$ by \tilde{c}_1 , and continue the iteration.

Now let us consider the case that Y is degenerate with a_1 in the middle and \tilde{Y} is non-degenerate such that $\varphi^{\circ i_0}(\tilde{Y}) = \text{STOP}$ and \tilde{c} is its critical branch point. We again set $\phi_Y^{\circ i_0}(a_i) =: b_i$ and $\phi_{\tilde{Y}}^{\circ i_0}(\tilde{a}_i) =: \tilde{b}_i$. Suppose that $\tilde{c} = \tilde{c}_1$ and that b_2 is chopped off before we reach the STOP. In this case, if $q_0^i \notin]c_1, b_1[$ then $b_1 \in \text{orb}(c_2)$. Because of the existence of a point $x \in \mathcal{O}_q \cap G_{c_1}(b_2)$ and because of the orbit condition for \mathcal{T} and $\tilde{\mathcal{T}}$ as above, this is not possible. Thus, $[b_1, b_3, q_0^i]$ is degenerate with b_1 in the middle whereas $[\tilde{b}_1, \tilde{b}_3, \tilde{c}_1]$ is degenerate with \tilde{c}_1 in the middle. We iterate both triods. By the just presented reasoning chopping causes no problems. Eventually we reach STOP in $[\tilde{b}_1, \tilde{b}_3, \tilde{c}_1]$ and, coming with it, a contradiction to the fact that corresponding points have the same itineraries. If $\tilde{c} = \tilde{c}_2$, we iterate the triod $[b_3, b_1, c_2]$, which is degenerate with b_1 in the middle, and the triod $[\tilde{b}_1, \tilde{b}_3, \tilde{c}_2]$, which is degenerate with \tilde{c}_2 in the middle. There is a k_0 such that $\varphi^{\circ k_0}([\tilde{b}_1, \tilde{b}_3, \tilde{c}_2]) = \text{STOP}$ so that the images of \tilde{b}_1 and \tilde{b}_3 are on different sides of \tilde{c}_2 while in $\varphi^{\circ k_0}(Y_{i_0})$, the images of b_1 and b_3 are on the same side of c_2 . Since corresponding points have equal itineraries only b_3, \tilde{b}_3 might be chopped off before we reach time k . Again by the above reasoning, this does not cause any problems.

We have seen that under the assumption that the iterates of the triods

Y, \tilde{Y} are well-defined until we reach the STOP, Y, \tilde{Y} must be combinatorially equivalent. Now suppose we have to chop the triods before the STOP case occurs. Again, we only have to worry if the chopping occurs at c_1 : let Y be non-degenerate with branch point b , \tilde{Y} be degenerate with \tilde{a}_1 in the middle and say, a_3, \tilde{a}_3 are chopped off at time i . Then if $q_0^i \in]c_1, \phi_Y^{oi}(b)[$, we can replace $\phi_Y^{oi}(a_3)$ by q_0^i and $\phi_{\tilde{Y}}^{oi}(\tilde{a}_3)$ by \tilde{c}_1 . If $q_0^i \notin]c_1, \phi_Y^{oi}(b)[$, then by the same argument as before, $\phi_Y^{oi}(a_1), \phi_Y^{oi}(a_2) \in \text{orb}(c_2)$. By the orbit condition, $[q_0^i, \phi_Y^{oi}(a_1), \phi_Y^{oi}(a_2)]$ is degenerate with $\phi_Y^{oi}(a_1)$ in the middle. Furthermore, there is an $x \in \mathcal{O}_q \cap G_{c_1}(\phi_Y^{oi}(a_3))$ and $[x, q_0^i, \phi_Y^{oi}(a_1)]$ is degenerate with $\phi_Y^{oi}(a_1)$ in the middle whereas $[\tilde{x}, \tilde{c}_1, \phi_{\tilde{Y}}^{oi}(a_1)]$ is degenerate with \tilde{c}_1 in the middle, contradicting that \mathcal{T} and $\tilde{\mathcal{T}}$ meet the orbit condition. By symmetry, the same argument holds if a_2 is chopped off.

By a similar reasoning, one shows that also in the case that Y is degenerate and \tilde{Y} is non-degenerate, or that both are degenerate with not corresponding points in the middle, chopping causes no problems. This finishes the proof. \square

Remark 5.1.13. For the disjoint case, orbit forcing is true even if the Hubbard trees in question do not meet the orbit condition of Definition 5.1.7. Nevertheless we require for the order that two Hubbard trees have to meet the orbit condition in order to be comparable. As Figure 5.2 shows, fixing the itinerary μ of the second critical point does not imply that we also fix the mutual location of the points in $\text{orb}(c_2)$. We do not know how the pictured Hubbard trees are arranged in parameter space. In particular, we do not know if these two trees should be comparable with respect to “ $<$ ”. However, it looks like we have to pass through a bitransitive tree (also pictured in Figure 5.2) when we transform $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ into (T, f, \mathcal{P}) . Note that in the pictured situation, the mutual location of points in $\text{orb}(c_2)$ necessarily has to change: in \tilde{T} , the location of points in $\text{orb}(\tilde{c}_2)$ forces \tilde{c}_1 to be a branch point. Since in T , v_1 is an endpoint, c_1 cannot be a branch point any more. So the mutual location of points in $\text{orb}(c_2)$ is bound to change. This discussion leads to the following question.

Question 5.1.14. Suppose that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}), (T, f, \mathcal{P}) \in \mathcal{H}_\mu$ are combinatorially related Hubbard trees of disjoint type. If $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}}), (T, f, \mathcal{P})$ do not meet the orbit condition, should they nevertheless be comparable with respect to the partial order “ $<$ ”?

Let us conclude this section with a statement about the structure of \mathcal{H}'_μ .

Proposition 5.1.15 (Smaller trees linearly ordered). *Let μ be \star_2 -periodic and let $\mathcal{T}, \mathcal{T}', \mathcal{T}'' \in \mathcal{H}'_\mu$ be three pairwise non-equivalent Hubbard trees. If $\mathcal{T}' < \mathcal{T}$ and $\mathcal{T}'' < \mathcal{T}$ then either $\mathcal{T}' < \mathcal{T}''$ or $\mathcal{T}'' < \mathcal{T}'$.*

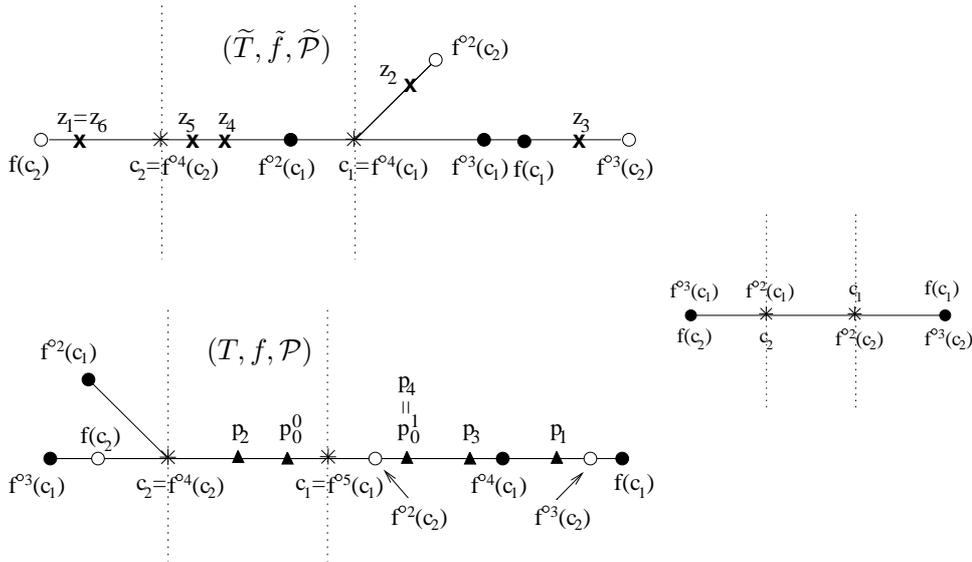


Figure 5.2: Two (admissible) Hubbard trees in \mathcal{H}_μ with $\mu = \overline{211\star_2}$ such that the mutual locations of points in $\text{orb}(\tilde{c}_2) \subset \tilde{T}$ and in $\text{orb}(c_2) \subset T$ are different. If there is a “combinatorial arc” between them, then presumably the bitransitive tree on the right must be contained in it. The two trees also show that orbit forcing is not always possible for Hubbard trees of disjoint types without any further assumptions: $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is represented in the tree T by the v_1 -characteristic point p_1 yet T contains no v_2 -characteristic point which has the same itinerary as $z_1 \in \tilde{T}$. For better readability, we dropped the “ \sim ” in the sketch of $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$. We stick to this convention in all further figures.

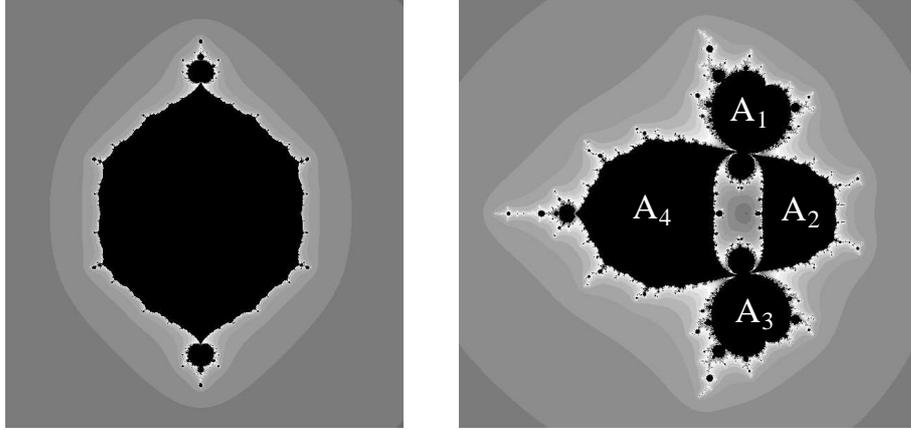


Figure 5.3: The cubic parameter slices \mathcal{S}_1 and $\mathcal{S}_2/\mathcal{I}$.

Proof. Let T, T', T'' be the underlying topological trees of the given Hubbard trees. By definition T contains v_1 -characteristic points p, q with itineraries $\tau(p) = \mathcal{A}(\tau(c'_1))$, $\tau(q) = \mathcal{A}(\tau(c''_1))$. We have that either $p \in]c_1, q[$ or $q \in]c_1, p[$. Let us suppose the latter holds (the reasoning for the first case is analogous). By Remark 5.1.10, T' contains a v'_1 -characteristic point q' with $\tau(q') = \tau(q)$. So it only remains to show that T' and T'' meet the orbit condition. For this, it suffices to show that any three point $x_1, x_2, x_3 \in \text{orb}(q) \cup \text{orb}(c_2) \in T$ and their corresponding points $x'_1, x'_2, x'_3 \in \text{orb}(q') \cup \text{orb}(c'_2) \in T$ form combinatorially equivalent triods. We are in a similar situation as in the proof of transitivity of “ $<$ ” in Theorem 5.1.12 and can apply the same arguments as there. This settles the claim. \square

Example 5.1.16. Recall that \mathcal{S}_i is the complex one-dimensional slice in the cubic parameter space consisting of all polynomials which have one critical point of exact period i . There is a bijection between the set \mathcal{H}'_μ with $\mu = \overline{\sqrt{2}}$ and the set of postcritically finite hyperbolic parameters in $\mathcal{S}_1/\mathcal{I}$ modulo symmetries, which can be interpreted as the different options to choose a combinatorial rotation number at branch points of Hubbard trees in the sense of Douady and Hubbard. So our results can be applied to this family of cubic polynomials.

We think the same holds true if in the slice $\mathcal{S}_2/\mathcal{I}$, we restrict ourselves to one hyperbolic component labeled by A_i in Figure 5.3 and its decorations. Also our results hold for various regions in the slice $\mathcal{S}_3/\mathcal{I}$.

5.1.2 Generalizations for Orbit Forcing

Lemma 5.1.5 shows that orbit forcing holds for Hubbard trees of disjoint type in \mathcal{H}_μ without any further assumption like the orbit condition of Definition 5.1.7. For our proof in the capture case, we already need that (T, f, \mathcal{P}) and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the orbit condition. Now we turn to the set of all Hubbard trees of disjoint types. We show that orbit forcing works under the assumption of an orbit condition which is more restrictive than Definition 5.1.7. If one weakens this condition, then forcing is not always possible.

Definition 5.1.17 (Orbit condition). *Let (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be two Hubbard trees of disjoint type such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) .*

- (i) (Weak version). *(T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the weak orbit condition if any three points $x^1, x^2, x^3 \in \text{orb}'(p) \cup \text{orb}'(q)$ and their corresponding points $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \tilde{\mathcal{O}}$ form combinatorially equivalent triods unless the following holds: for some $i \in \{1, 2\}$, the triod $[\tilde{x}, \tilde{c}_i, \tilde{y}] \subset \tilde{T}$ is degenerate with \tilde{c}_i in the middle whereas $[x, a, y] \subset T$ is non-degenerate such that c_i is contained in its interior. Here $a \in \{p_0^0, p_0^1\}$ if $i = 1$ and $a \in \{q_0^0, q_0^2\}$ if $i = 2$.*
- (ii) (Strong version). *(T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ meet the (strong) orbit condition if the following is true: let T' denote the tree which is the extension of T by the preimages p_0^0, p_0^1 of p and the preimages q_0^0, q_0^2 of q . Then any three points $x^1, x^2, x^3 \in \text{orb}'(p) \cup \text{orb}'(q) \subset T'$ and their corresponding points $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \tilde{\mathcal{O}}$ form combinatorially equivalent triods.*

Given a Hubbard tree (T, f, \mathcal{P}) , we obtain the extended tree T' as follows: if T contains the four points $p_0^0, p_0^1, q_0^0, q_0^2$, then we set $T' := T$. If a point $a = p_0^i$ or q_0^j is missing, then attach an arc with endpoint a at the critical point c_1 or c_2 according as a is a preimage of p or q .

The weak orbit condition is the straightforward generalization of the orbit condition for Hubbard trees in \mathcal{H}_μ . However, it is not sufficient for orbit forcing in the general setting. This is not even true for Hubbard trees in \mathcal{H}_μ if we want to force \tilde{v}_2 -characteristic points of \tilde{T} into T as Figure 5.4 shows.

Observe that it is not enough to require that any three points $x^1, x^2, x^3 \in \text{orb}(p) \cup \text{orb}(q) \subset T$ and their corresponding points $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \tilde{\mathcal{O}}$ form combinatorially equivalent triods, as illustrated in Figure 5.5. It does not even suffice to consider points in $\text{orb}'(p) \cup \text{orb}'(q) \subset T$ and $\tilde{\mathcal{O}}$: one of the preimages of p or q might not exist in T so that it is impossible to define appropriate intervals in T that are needed to force a point \tilde{z} with the techniques of the proof of Lemma 5.1.5. Indeed, the orbit condition is only sufficient for orbit forcing if we consider the extended tree T' instead of T , compare Figure 5.6.

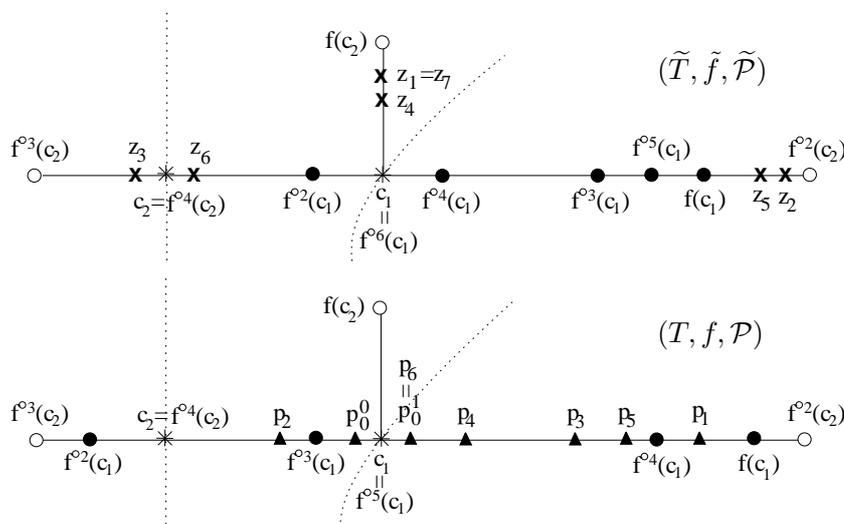


Figure 5.4: Two (admissible) Hubbard trees which are contained in \mathcal{H}_μ with $\mu = \overline{012^*2}$ and which meet the weak orbit condition. (The critical orbit $\text{orb}(\tilde{c}_1) \in \tilde{T}$ is represented in T by $\text{orb}(p_1)$.) Yet there is no v_2 -characteristic point in T which has the same itinerary as the \tilde{v}_2 -characteristic point $\tilde{z}_1 \in \tilde{T}$.

After pointing out several obstacles for orbit forcing, the next lemma shows that the strong orbit condition is sufficient to guarantee orbit forcing for Hubbard trees of disjoint type.

Proposition 5.1.18 (Orbit forcing, general version). *Let (T, f, \mathcal{P}) , $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be two attracting minimal Hubbard trees of disjoint type such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) and such that the two Hubbard trees satisfy the strong orbit condition.*

Then for each \tilde{v}_i -characteristic point $\tilde{z} \in \tilde{T}$, there is a v_i -characteristic point $z \in T$ such that $\tau(z) = \tau(\tilde{z})$.

Proof. The proof of Lemma 5.1.5 carries over to this situation. The orbit condition guarantees that all those preimages of p, q exist which are necessary to define the intervals J_k . Moreover, $J_k \subset f(J_{k-1})$ for all $k \in \mathbb{N}$. So we find a periodic point $z \in T$ with itinerary $\tau(z) = \tau(\tilde{z})$. For the proof that z is v_1 -characteristic observe the following: if we have to chop a triod in T at the critical point c_1 , then we replace the chopped off point by a suitable preimage p_0^0, p_0^i of p . If the chopping occurs at c_2 then we replace the chopped off point by q_0^0 or q_0^j , the preimages of q in T_0, T_2 . The orbit condition implies that $\text{orb}(z) \cap [c_1, p_0^i] = \emptyset$ and $\text{orb}(z) \cap [c_2, q_0^j] = \emptyset$: otherwise there would be $a, y \in \text{orb}(p) \cup \text{orb}(q)$ such that $[a, y, p_0^i] \subset \overline{T_i}$ is not degenerate with p_0^i in the middle, where $a \notin \overline{T_i}$. In \tilde{T} , $[\tilde{a}, \tilde{y}, \tilde{c}_1]$ is degenerate

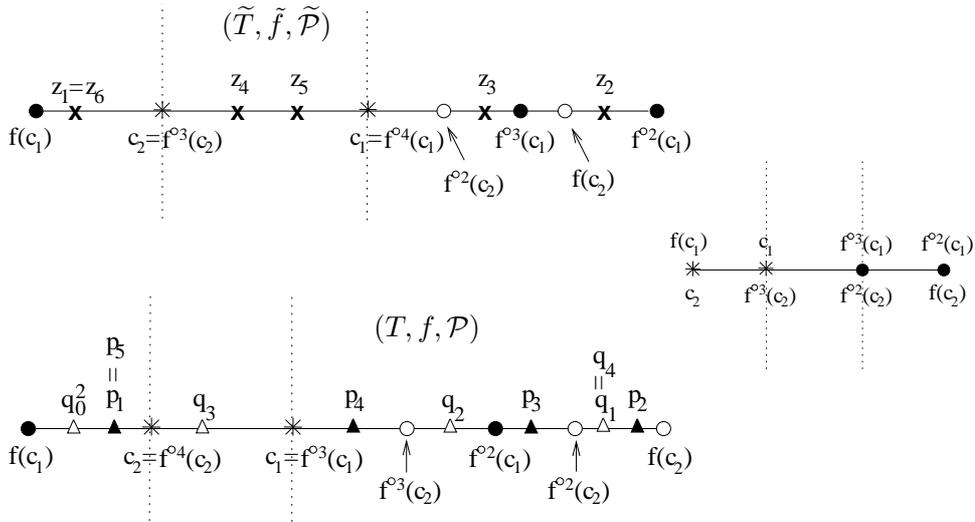


Figure 5.5: The Hubbard tree $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ generates the kneading sequence $(\overline{211\star_1}, \overline{11\star_2})$ and (T, f, \mathcal{P}) generates the kneading sequence $(\overline{21\star_1}, \overline{111\star_2})$ (both Hubbard trees are admissible). T contains a v_1 -characteristic point p_1 and a v_2 -characteristic point q_1 with $\tau(p_1) = \mathcal{A}(\tilde{\nu}^1)$, $\tau(q_1) = \mathcal{A}(\tilde{\nu}^2)$. The characteristic point $z \in \tilde{T}$ is not forced in T . This example shows that we have to consider both preimages of the characteristic points for the orbit condition: all triods formed by corresponding point $\text{orb}(p) \cup \text{orb}(q)$ and $\tilde{\mathcal{O}}$ are combinatorially equivalent yet $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ and (T, f, \mathcal{P}) do not meet the triod condition: e.g. the triods $[\tilde{v}_1, \tilde{c}_2, \tilde{c}_1] \subset \tilde{T}$ and $[q_0^2, p_1, p_4] \subset T$ are not combinatorially equivalent.

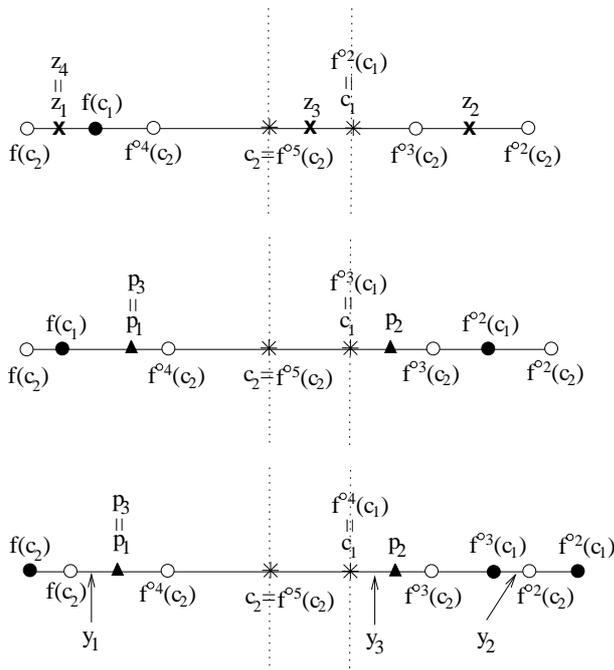


Figure 5.6: Three (admissible) Hubbard trees where the critical orbit $\text{orb}(c_2)$ is fixed. The second and third tree contain a v_1 -characteristic point p_1 which represents the critical value v_1 of the first tree. Neither of them contains the preimage p_0^0 of p_1 or a v_2 -characteristic point with itinerary $\overline{210}$, which is the itinerary of the v_2 -characteristic point z_1 of the first tree. However, the last tree contains a v_2 -characteristic point with itinerary $\overline{211}$.

with \tilde{c}_1 in the middle, in contradiction to the orbit condition.

We can apply the same arguments as in Lemma 5.1.5 to show that z is v_1 -characteristic. Let us only point out that chopping is not a problem: In the situation of the proof of Lemma 5.1.5 when $Y := [a, z, p]$ is non-degenerate, suppose that we have to chop an image of a or p at time k . Without loss of generality, let us assume that $\phi_Y^{\circ k}(p)$ is chopped off. Then, since $\phi_Y^{\circ k}(a), \phi_Y^{\circ k}(p)$ and the point $r \in \{p_0^0, p_0^1, q_0^0, q_0^2\}$ form a triod which is combinatorially equivalent to the one generated by the corresponding points in \tilde{T} , we get that $r \in [c_i, \phi_Y^{\circ k}(a)]$. Because of the existence of a point $y \in \text{orb}(p) \cup \text{orb}(q)$ such that $f^{\circ k}(z) \in]y, c_1[$, we have that $r \in]c_i, f^{\circ k}(b)[$, where b is the branch point of $[a, z, p]$. If $f^{\circ k}(b) \neq r$, we replace the chopped off point and continue the iteration; otherwise we consider the degenerate triod $[\phi_Y^{\circ k}(a_1), r, \phi_Y^{\circ k}(z)]$ with r in the middle and the degenerate triod $[\phi_Y^{\circ k}(\tilde{a}), \phi_Y^{\circ k}(\tilde{z}), \tilde{c}_i]$ with $\phi_Y^{\circ k}(\tilde{z})$ in the middle. Iterating them under φ yields a contradiction to the fact that corresponding points have equal itinerary. In the situation where $[a, z, z_l]$ is degenerate with z in the middle and the

k -th image of z_l (or a) is chopped off, the orbit condition implies that $r \in G_{c_1}(f^{\circ k}(z))$. Thus, $r \in]c_i, f^{\circ k}(z)[$. So replacing the chopped off points causes no problems. \square

Definition 5.1.19 (“ $<$ ” for disjoint type). *Let $\mathcal{T}, \tilde{\mathcal{T}}$ be two attracting minimal Hubbard trees of disjoint type. Then $[\mathcal{T}] > [\tilde{\mathcal{T}}] : \iff \tilde{\mathcal{T}}$ is combinatorially represented in \mathcal{T} and the two Hubbard trees satisfy the strong orbit condition.*

Proposition 5.1.20 (“ $<$ ” is a partial order). *The relation “ $<$ ” defined above is a partial order on the set of Hubbard trees of disjoint type whose critical values are characteristic with respect to themselves.*

Proof. The reasoning is very similar as for the set \mathcal{H}'_μ . We give an outline for the proof of transitivity. Let $\tilde{\tilde{\mathcal{T}}} < \tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}} < \mathcal{T}$. By the definition of “ $<$ ” and by the orbit forcing lemma, there are v_1 -, v_2 -characteristic points \tilde{p}, \tilde{q} , and p, q that represent the Hubbard tree $\tilde{\mathcal{T}}$ in $\tilde{\tilde{\mathcal{T}}}, \mathcal{T}$. To show that \mathcal{T} and $\tilde{\tilde{\mathcal{T}}}$ meet the orbit condition, it suffices to show that any three points in $\text{orb}'(p) \cup \text{orb}'(q)$ and their corresponding points in $\text{orb}'(\tilde{p}) \cup \text{orb}'(\tilde{q})$ form combinatorially equivalent triods. To verify this we iterate the triods under φ , where we replace chopped off points in $\tilde{\mathcal{T}}$ by \tilde{c}_1, \tilde{c}_2 and in \mathcal{T} by $r \in \{a_0^0, a_0^i, b_0^0, b_0^2\}$, where a, b are characteristic points representing the tree $\tilde{\mathcal{T}}$ in \mathcal{T} . We argue the same way as in Theorem 5.1.12 to derive a contradiction if the triods were not combinatorially equivalent. By the same arguments as in Proposition 5.1.18, chopping is not a problem. \square

Remark 5.1.21 (Smaller Hubbard trees not linearly ordered). Suppose $(\mathcal{T}, f, \mathcal{P}) > (\tilde{\mathcal{T}}, \tilde{f}, \tilde{\mathcal{P}})$ and let $p \in \mathcal{T}$ be the v_1 -characteristic point representing the critical value \tilde{v}_1 of $\tilde{\mathcal{T}}$. Then it is not true that for any v_1 -characteristic point z in $]c_1, p[$, there is a corresponding \tilde{v}_1 -characteristic point $\tilde{z} \in \tilde{\mathcal{T}}$ such that $\tau(z) = \tau(\tilde{z})$. (The same is true if we replace the label 1 by 2.) Figure 5.7 provides an example. It follows that the set of all Hubbard trees smaller than a given one is not linearly ordered in general.

5.2 Outlook

The examples presented in Figures 5.2, 5.4, 5.5 and 5.6 show that orbit forcing is not always possible without assuming the strong orbit condition. However, this condition is not necessary as Figure 5.8 illustrates. In fact, if we drop the orbit condition then we cannot make any statement whether forcing of a characteristic point with itinerary τ is possible and, in the case that it is, whether the forced point is characteristic.

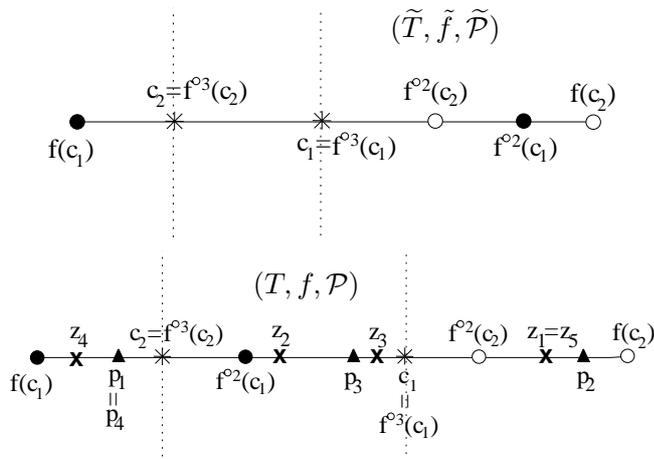


Figure 5.7: Two (admissible) Hubbard trees $(T, f, \mathcal{P}) > (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ in $\mathcal{H}_{11\star 2}$, where $\tilde{v}_1 \in \tilde{T}$ is represented by $p_1 \in T$ and $\tilde{v}_2 \in \tilde{T}$ by $q_1 := v_2 \in T$ (the parameters generating them are marked by (7a), (7b) in Figure 4.8). They illustrate that not all v_2 -characteristic points of a larger tree contained in $]c_2, q_1[$ are forced in a smaller tree. Here, there is no \tilde{v}_2 -characteristic point in \tilde{T} which has the same itinerary as the v_2 -characteristic point $z_1 \in]c_2, q_1[\in T$.

We have seen that for disjoint Hubbard trees in \mathcal{H}_μ , orbit forcing is always possible without imposing the orbit condition. Since also in the general case this condition is not necessary, we pose the following question:

Question 5.2.1. Which combinatorially related Hubbard trees of disjoint types satisfy the orbit condition naturally? For which combinatorially related Hubbard trees of disjoint type is orbit forcing possible without assuming any additional conditions?

Let $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be combinatorially represented in (T, f, \mathcal{P}) . Recall that there are two obstacles for orbit forcing. The first one is that the arrangement of points in $\tilde{O} \subset \tilde{T}$ and of points in $\text{orb}'(p) \cup \text{orb}'(q) \in T$ is not the same (cf. Figure 5.5). Secondly, it is possible that the arrangement of points is the same but T contains not all preimages of p, q that are needed for forcing a characteristic point (cf. Figure 5.6). The strong orbit condition takes both possibilities into account (it is a condition on the extended tree T').

Let us investigate under what circumstances not all preimages of p exist that are necessary in order to force a \tilde{v}_1 -characteristic point in T according to the construction in the proof of Lemma 5.1.5.

Lemma 5.2.2 (Non-existence of preimages). *Let $(T, f, \mathcal{P}), (\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ be two minimal Hubbard trees of disjoint type such that their critical values are*

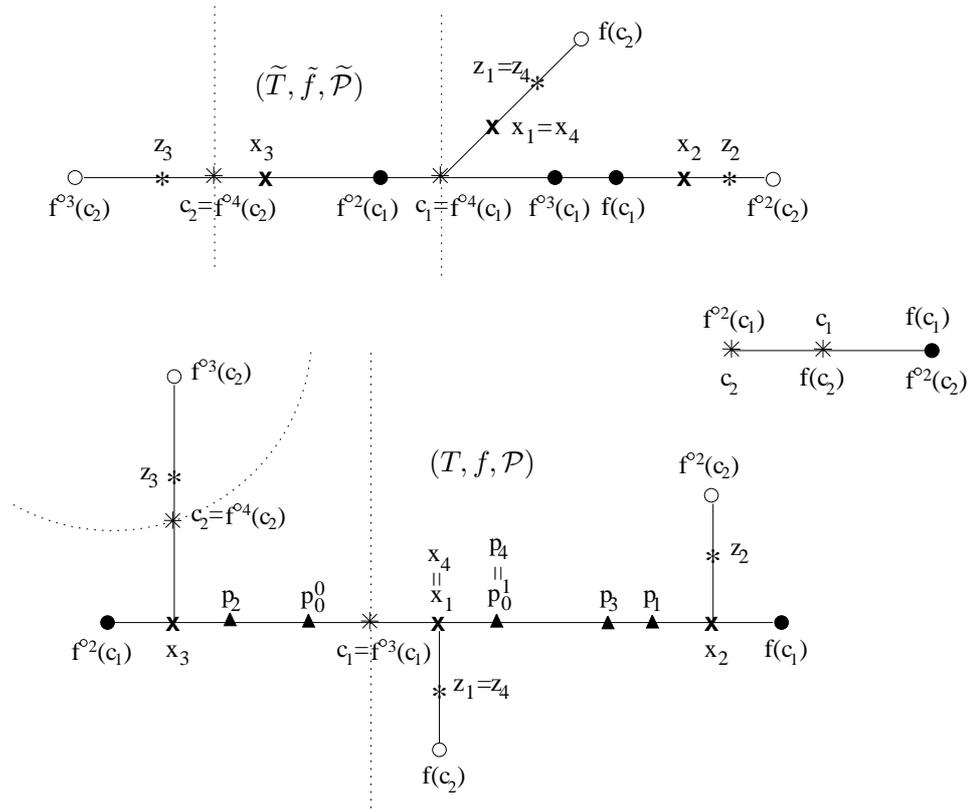


Figure 5.8: No statement about forcing is possible if the orbit condition is violated, as e.g. by the triads $[\tilde{c}_1, \tilde{c}_2, \tilde{f}(\tilde{c}_2)] \subset \tilde{T}$ and $[c_2, p_4, f(c_2)] \subset T$: the \tilde{v}_2 -characteristic point \tilde{z}_1 is forced in T as v_2 -characteristic point z although $J_{k+1} \not\subset f(J_k)$ for some k . The J_k are the intervals defined in the proof of Lemma 5.1.5 to find the forced point. Here, the sequence of intervals is $[f(c_2), p_4], [f^{\circ 2}(c_2), p_1], [f^{\circ 3}(c_2), c_2], [f(c_2), p_4], \dots; f([f^{\circ 3}(c_2), c_2]) \not\subset [f(c_2), p_4]$. On the other hand, if we consider the intervals J_k used to force the \tilde{v}_2 -characteristic point \tilde{x}_1 , then we see that $J_{k+1} \subset f(J_k)$ for all k yet the forced point x_1 is not v_2 -characteristic. (Here the sequence is $[f(c_2), p_4], [f^{\circ 2}(c_2), p_1], [c_2, p_2], [f(c_2), p_4], \dots$)

characteristic with respect to themselves. Suppose that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{P}})$ is combinatorially represented in (T, f, \mathcal{P}) . Suppose furthermore that any three points $x^1, x^2, x^3 \in \text{orb}'(p) \cup \text{orb}'(q) \subset T$ and their corresponding points $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \tilde{\mathcal{O}}$ form combinatorially equivalent triods. If \tilde{z} is a \tilde{v}_1 -characteristic point, then the following are true:

- (i) If $\text{orb}(\tilde{z}) \cap \tilde{T}_1 \neq \emptyset$, then p_0^1 exists.
- (ii) If $\text{orb}(\tilde{z}) \cap \tilde{T}_0 \neq \emptyset$ and p_0^0 does not exist then $\text{orb}(\tilde{z}) \cap \tilde{T}_0 \subset [\tilde{c}_1, \tilde{c}_2]$ and q_0^0 does not exist either. Furthermore, if q_0^0 does not exist then $\tilde{v}_2 \in]\tilde{v}_1, \tilde{c}_1[\subset \tilde{T}_i$, $q \in]p, c_1[\subset T_i$ for some $i = 1$ or 2 .

Proof. By way of contradiction, assume that $\tilde{f}^{\circ i_0}(\tilde{z}) \in \tilde{T}_1$ and that p_0^1 does not exist. Then by Proposition 4.3.12, $\text{orb}(p) \cap T_1 = \emptyset$ and thus by hypothesis, $\text{orb}(\tilde{c}_1) \cap \tilde{T}_1 = \emptyset$. Therefore, there is a $\tilde{q}_j \in \text{orb}(\tilde{c}_2) \subset \tilde{T}$ such that $\tilde{f}^{\circ i_0}(\tilde{z}) \in]\tilde{c}_1, \tilde{q}_j[$. Since \tilde{z} is \tilde{v}_1 -characteristic, $f(\tilde{q}_j) \in G_{\tilde{v}_1}(\tilde{c}_1)$. It follows that $p \in]f(c_1), f(q_j)[$, where q_j is the point corresponding to \tilde{q}_j . Thus $]c_1, q_j[\subset T_1$ contains a preimage of p .

For item two, observe first that p_0^0, q_0^0 exist if p, q are contained in different subtrees T_i or if they are both contained in T_0 (this follows again from Proposition 4.3.12). So let us suppose that $p, q \in T_i$ for $i = 1$ or 2 . Observe that p_0^0 exists if there is an i_0 such that $\tilde{f}^{\circ i_0}(\tilde{z}) \in \tilde{T}_0 \setminus [\tilde{c}_1, \tilde{c}_2]$: we can argue as in the previous paragraph that there is a $\tilde{q}_j \in \text{orb}(\tilde{c}_2)$ such that $\tilde{f}^{\circ i_0}(\tilde{z}) \in]\tilde{c}_1, \tilde{q}_j[\subset \tilde{T}_0$ and that, as a consequence, $]q_j, c_1[\subset T$ contains a preimage of p .

If $\text{orb}(z) \cap T_0 \subset [c_1, c_2]$ then $\tilde{v}_2 \in G_{\tilde{z}}(\tilde{c}_1)$, and thus $p \in [q, f(c_1)]$. Now assume that p_0^0 does not exist. If q_0^0 existed then $[c_1, q_0^0]$ would contain a preimage p_0^0 of p , a contradiction. So it only remains to show the implications when q_0^0 does not exist. Observe first that in this case, $q \notin [f(c_1), f(c_2)]$. It follows that $[p, q, c_1]$ is degenerate, and since $p \in [q, f(c_1)]$, $q \in [p, c_1]$ and by the hypothesis, the claim follows. \square

The strong version of the orbit condition is very restrictive, see Definition 5.1.17. Basically, it does not allow for critical branch points: the exceptional case pictured in Figure 5.1 is something one has to expect if the critical points \tilde{c}_1, c_1 of the Hubbard trees are both branch points. Also we require that T contains both preimages of the characteristic points p, q which represent the critical orbits of \tilde{T} . So we ignore a substantial set of Hubbard trees of disjoint type when we define the partial order “ $<$ ”. Furthermore, in our discussion of orbit forcing, we only consider Hubbard trees whose critical values are characteristic with respect to themselves and thus, we require that a Hubbard tree that is smaller than another Hubbard tree has this property, too. This assumption is essential for our proof of orbit forcing in Lemma 5.1.5. Yet many Hubbard trees do not have this property,

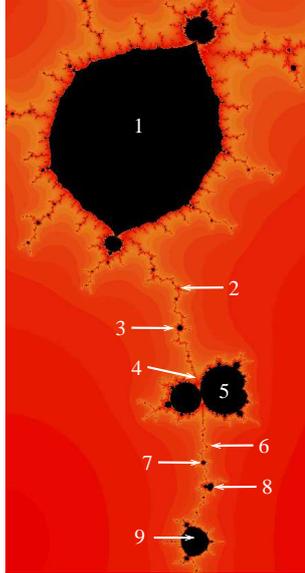


Figure 5.9: Part of the parameter slice $\mathcal{S}_3/\mathcal{I}$. The picture shows part of the third vertical connection from the right of Figure 4.8. The numbers indicate the location of the polynomials considered in Figure 5.10 (in this order).

while there might be a dynamical relation between them as illustrated in Figure 5.10. These pictures suggest that the partial order can be extended to Hubbard trees with critical values that are not characteristic with respect to themselves.

Another interesting dynamical relation is illustrated by the three Hubbard trees in $\mathcal{H}_{\overline{2112\star_2}}$ pictured in Figure 5.6. The top most Hubbard tree \mathcal{T}_1 contains a v_2 -characteristic point z_1 with itinerary $\overline{210}$. Now move $f(c_1)$ towards $f(c_2)$ and thus towards z_1 . The v_2 -characteristic point z_1 with itinerary $\overline{210}$ is preserved until $f(c_2)$ and z_1 merge to one point. At this moment, the v_2 -characteristic point with itinerary $\overline{210}$ ceases to exist. The situation is shown by the Hubbard tree \mathcal{T}_2 in the middle. If we now continue moving $f(c_1)$ towards $f(c_2)$, then the obtained Hubbard trees contain a v_2 -characteristic point y_1 with itinerary $\overline{211}$ as illustrated by the third Hubbard tree \mathcal{T}_3 . Note that y_1 is also v_1 -characteristic. A possible explanation for this behavior is the following: \mathcal{T}_1 is larger than the Hubbard tree \mathcal{T}_0 associated to the kneading sequence $(\overline{2\star_1}, \overline{21\star_2})$ with respect to “ $<$ ” defined in Definition 5.1.19. The critical value v_2 of \mathcal{T}_0 is represented in \mathcal{T}_1 by the v_2 -characteristic point z_1 . After z_1 collided with $f(c_1)$, the newly generated v_2 -characteristic point with itinerary $\overline{211}$ should rather be considered a v_1 -characteristic point, which represents the critical value v_1 of \mathcal{T}_2 . For this tree, $\tau(v_1) = \overline{21\star_1}$.

The dynamical relation between the trees suggests that \mathcal{T}_1 should be viewed as smaller than \mathcal{T}_2 , which in turn should be viewed as smaller than \mathcal{T}_3 . Let $\nu := \overline{21\star_2}$. We have seen that \mathcal{T}_1 contains a v_2 -characteristic point with itinerary $\mathcal{A}_0(\nu)$, yet \mathcal{T}_2 and \mathcal{T}_3 do not. However all three Hubbard trees

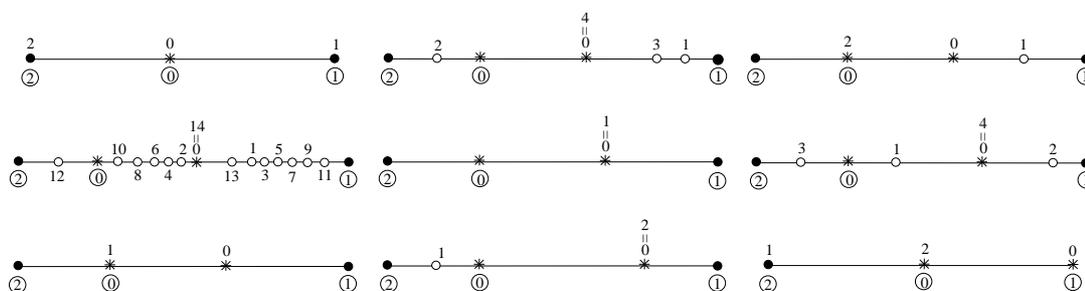
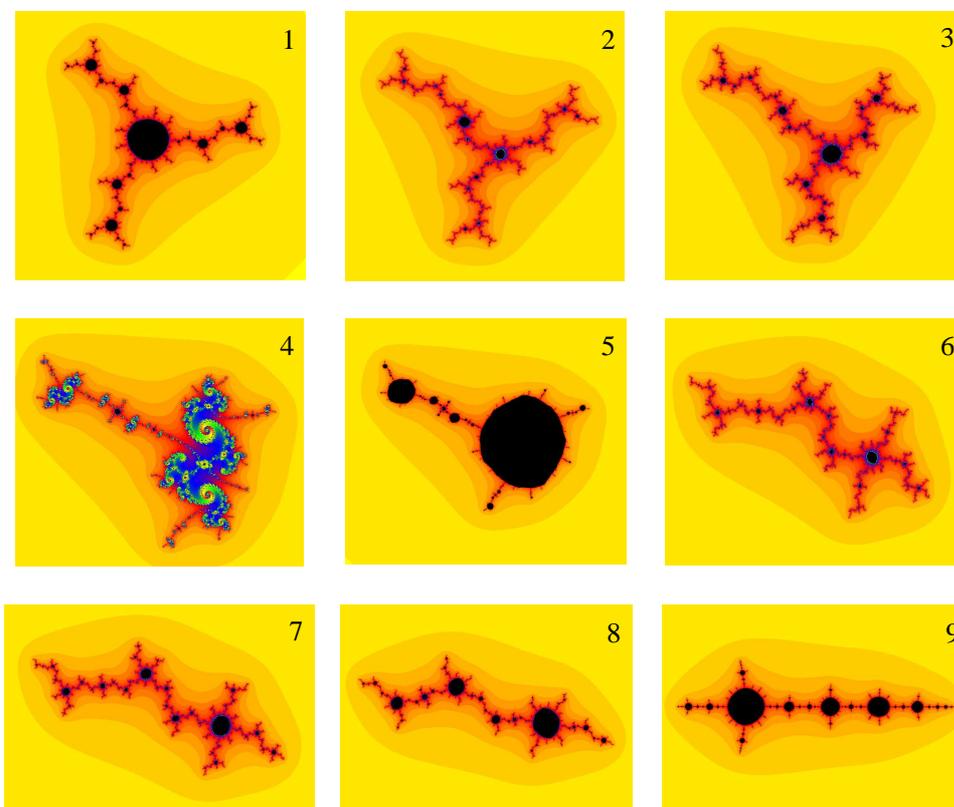


Figure 5.10: The picture shows how the location of v_1 with respect to c_1, c_2 might change when the parameters are varied (the parameters are marked in Figure 5.9). More precisely, v_1 is wandering from the right most tip of the Hubbard tree to the left one. In this process, Hubbard trees whose critical value v_1 is not v_1 -characteristic appear. The i -th Julia set generates the i -th Hubbard tree with the exception of the 4th Hubbard tree: it belongs to a Julia set which lies “between” the Julia sets no. 4 and 5. The Hubbard tree associated to Julia set no. 4 has the same structure as the 4th Hubbard tree but its critical point c_1 has exact period 18. Note that only points on the critical cycles are marked; the ones on the fixed orbit $\text{orb}(c_2)$ are encircled. f maps the point i to $i + 1$.

contain a v_2 -characteristic point with itinerary $\mathcal{A}_2(\nu)$. So if we ignore the orbit condition for these four Hubbard trees, we get that $\mathcal{T}_0 < \mathcal{T}_1 < \mathcal{T}_2 < \mathcal{T}_3$ are linearly ordered. (Recall that a smaller Hubbard tree $\tilde{\mathcal{T}}$ is represented by v_1 - and v_2 -characteristic points p, q with itinerary $\mathcal{A}_k(\tau(\tilde{c}_1)), \mathcal{A}_l(\tau(\tilde{c}_2))$, where $k \in \{0, 1\}$ and $l \in \{0, 2\}$.)

Note that one can construct similar examples by taking two Hubbard trees of real quadratic polynomials and glue them together to a cubic Hubbard tree as shown in the pictures. The two critical values and any of the critical points should form a degenerate triod such that the critical point is an endpoint. Not all possible choices will generate Hubbard trees and not all generated Hubbard trees will be admissible (they might contain an evil critical point). But for suitable quadratic Hubbard trees and a suitable way of gluing them, one can observe a similar behavior when moving the “inner” critical value towards the second critical value, which is an endpoint.

This discussion leads to the question whether one can define an order relation on Hubbard trees without being able to force all periodic orbits of a Hubbard tree $\tilde{\mathcal{T}}$ into \mathcal{T} , where $\tilde{\mathcal{T}}$ is combinatorially represented in \mathcal{T} .

So far, when we were changing critical orbits, we focused on moving a critical value further away or closer to the critical points. Both movements are so to speak “within the Hubbard tree”. One can also move the critical value to the “outside” of a tree (generating a new branch point). Such an action explains the loop in the parameter slice $\mathcal{S}_2/\mathcal{I}$ (cf. Figure 5.3). Figure 5.11 shows the Hubbard trees associated to this loop.

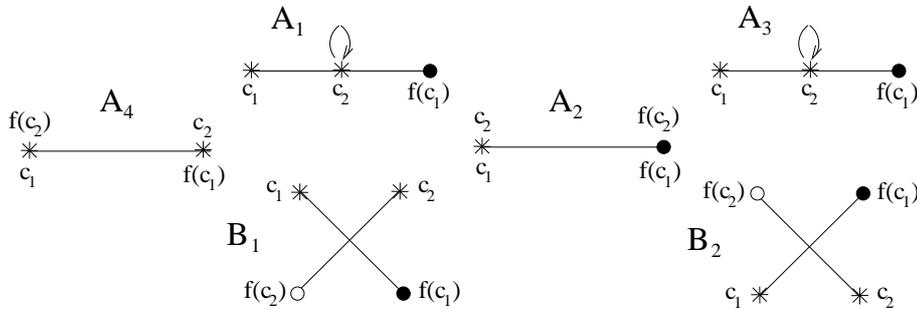


Figure 5.11: The Hubbard trees associated to the hyperbolic components forming the loop in $\mathcal{S}_2/\mathcal{I}$. The Hubbard tree A_i corresponds to the hyperbolic component A_i of Figure 5.3. The Hubbard trees B_i correspond to the $1/2$ -satellites of A_1, A_3 . Running through the loop A_4, B_1, A_2, B_2, A_4 corresponds to rotating the interval $[c_2, f(c_2)]$ around its fixed point while keeping $[c_1, f(c_1)]$ fixed. The fixed point represents the critical point c_2 of the Hubbard trees A_1, A_3 .

An analogous behavior can be observed when going around the holes in

$\mathcal{S}_3/\mathcal{I}$ on the top or bottom of Figure 4.8 which are formed by three hyperbolic components that touch each other. Call these hyperbolic components A_1, A_2, A_3 . Between any two of them, a Mandelbrot set is squeezed in. Consider its largest hyperbolic component in the interior of this hole. Let us call this component D_i if it is squeezed in between A_i and A_{i+1} (indices are modulo 3). The Julia set of its center consists of two rabbits intertwined at their α -fixed points. The corresponding Hubbard tree consists of two (non-degenerate) triods glued together at their branch points to form a 6-od. Now running through the loop $D_1, A_2, D_2, A_3, D_3, A_1, D_1$ corresponds on the Hubbard tree level to keeping one triod fixed and rotating the second one around the branch point. In all components D_i , we obtain a 6-od, in all components A_i , the two critical orbit collide to one orbit and the Hubbard tree is a triod (two of them are of bitransitive and one is of adjacent type).

Finally we want to point out that looking at Hubbard trees of adjacent or bitransitive hyperbolic components, e.g. in $\mathcal{S}_3/\mathcal{I}$, and at the hyperbolic components bifurcating from them, it is not clear what the right partitions for adjacent or bitransitive Hubbard trees are. In particular, it might be possible that for a given Hubbard tree, different choices of the partition belong to the same hyperbolic component. That is, there seems to be no canonical way to associate a unique kneading sequence to an adjacent or bitransitive hyperbolic component. A solution to these problems might be to define an order which does not rely on kneading sequences and itineraries. Two approaches in this direction have been discussed in Section 3.4 for the unicritical setting.

Let us summarize the discussion of this section by the following questions:

Question 5.2.3. Is it possible to extend the partial order “ $<$ ” on Hubbard trees of disjoint type to include Hubbard trees that do not meet the orbit condition or that contain a critical value v_i that is not v_i -characteristic? In particular, can one define a partial order on Hubbard trees such that not all comparable Hubbard trees allow an orbit forcing in full generality?

Is it possible to define a partial order so to include the other three types of hyperbolic Hubbard trees? Can such a partial order be based on kneading sequences and itineraries as the partial order “ $<$ ” we defined? Or should one rather work with a partial order which is based on comparing periodic orbits of Hubbard trees together with the arrangement of points on any such orbit?

Finally, in order to exploit a partial order on Hubbard trees to get a meaningful description of the connectedness locus of cubic polynomials, is it necessary to consider “angled” Hubbard trees, that is Hubbard trees as they were originally defined by Douday and Hubbard?

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