

Newton's Method as a Dynamical System

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Abstract

We study transcendental and rational mappings that arise as Newton maps of entire functions. If N_f is the Newton map of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$, then the roots of f are exactly the finite fixed points of N_f , all of which are attracting. It is well known that every finite fixed point ξ of N_f has an unbounded, connected and simply connected neighborhood U_ξ in which the dynamics converges to ξ , its *immediate basin*. Note that the existence of immediate basins suffices to explain why Newton's method can locally be used for root-finding.

We give a necessary and sufficient criterion under which a given transcendental or rational function $N : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ is a Newton map and study its behavior at infinity. If N is a rational Newton map, it extends to a function $N : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ and ∞ is a repelling or parabolic fixed point. In the latter case, N is necessarily the Newton map of a transcendental entire function and there exists an attracting parabolic petal at ∞ , that is an invariant, simply connected and unbounded component V of the Fatou set in which the dynamics converges to ∞ . We call V a *virtual immediate basin*, because it has many properties of an immediate basin but contains no fixed point. If ∞ is repelling, then N is necessarily the Newton map of a polynomial and has no virtual immediate basins.

We extend the study of Newton maps near ∞ to general (rational or transcendental) Newton maps N . Our first main result is that “in between” any two accesses to ∞ of an immediate basin, N exhibits either another immediate basin, a virtual immediate basin or a sequence $z_n \rightarrow \infty$ such that $N(z_n)$ is constant. This result is joint work with Dierk Schleicher and allows to locate virtual immediate basins. It also gives structure to the dynamical plane beyond the structure provided by the immediate basins. An important corollary is a proof of the folklore result that for Newton maps of polynomials, every complementary component of an immediate basin

contains another immediate basin. For the proof of this first main result we develop a fixed point inequality that we also use to show that transcendental Newton maps have no fixed Herman rings.

Let N_f be the Newton map of the transcendental entire function $f : \mathbb{C} \rightarrow \mathbb{C}$. The second main result of this thesis, which is joint work with Xavier Buff, shows an interesting connection between virtual immediate basins of N_f and asymptotic values of f , answering a 2003 question of Douady. More precisely, let V be a virtual immediate basin of N_f . Then, V can be of *parabolic* or *hyperbolic* type, where the parabolic type splits into two further cases. If V is parabolic, then 0 is an asymptotic value for f with asymptotic path in V . If V is hyperbolic, this is only true under an additional assumption. Conversely, if f has an asymptotic value of logarithmic type at 0, we prove that N_f has a virtual immediate basin. We show by way of counterexamples that this is not true for other types of asymptotic values.

With our third main result, we contribute to the study of parameter space of Newton maps by giving a combinatorial classification of a large sub-class of such maps. More precisely, let N_p be the Newton map of a polynomial p with simple roots such that all critical points of N_p land on a fixed point after finitely many iterations. In this case, we construct a graph that characterizes N_p uniquely up to Möbius conjugation. Conversely, we show that every graph with an associated map that satisfies several natural conditions is realized by a unique Newton map. This is a first step towards a classification of all postcritically finite Newton maps of polynomials, which in turn might be an important step towards a classification of arbitrary rational functions.

In an appendix, we introduce a class of bounded type transcendental entire functions with the property that its set of escaping points is organized in the form of unbounded rays. This fourth main result is joint work with Günter Rottenfuß, Lasse Rempe and Dierk Schleicher, and is part of an answer to a long-standing conjecture of Fatou and Eremenko. Since it is not immediately part of this thesis, it is presented in the appendix.

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Chapter 1

Introduction

1.1 Overview

We study holomorphic dynamical systems that arise as Newton maps of entire functions, and present several results concerning the structure of their dynamical planes. We then carry some of these results over to parameter space and give a classification of a large class of Newton maps of polynomials. In an appendix, we present a result about the set of escaping points for a transcendental entire function of bounded type.

We assume that the reader is familiar with the fundamental concepts of holomorphic dynamics, as laid out in e.g. [Be1, Mi]. All further definitions and prerequisites are treated in the chapters where they are needed. Each chapter is mathematically self-contained, so some concepts may be introduced more than once.

Let f be an entire function—polynomial or transcendental—and N_f its Newton map. It is well known that the roots of f are exactly the finite fixed points of N_f , and that all finite fixed points of N_f are attracting. The immediate basins of the attracting fixed points of N_f are unbounded domains and thus provide an elementary structure to the dynamical plane. In Chapter 2, we prove a necessary condition on the possible mutual locations of immediate basins. If an immediate basin U of an attracting fixed point of N_f separates the plane, then every component of $\mathbb{C} \setminus U$ contains another immediate basin of N_f or a *virtual immediate basin* (see Section 2.3.2 for the definition), provided that some technical assumptions are satisfied. For the proof, we develop a fixed point inequality that turns out to be useful in its

own right; as an example, it implies that transcendental Newton maps do not have fixed Herman rings.

Our result is related to a question of Smale, who posed the problem of characterizing all possible combinatorics of the basins of attraction for the Newton *flow* of a complex polynomial [Sm1]. This problem was solved in 1988 by Shub, Tischler and Williams [STW]. An easy corollary of our result shows that if f is a polynomial, then every complementary component of an immediate basin of N_f must contain another immediate basin. Thus, it gives a necessary condition on the combinatorics of immediate basins, providing a partial answer to Smale's question in the context of the *discrete* Newton method. Our condition is also used in the combinatorial classification of Newton maps as outlined below. The results of this chapter are being published jointly with Dierk Schleicher and are available on the arXiv [RuS].

In 2003, Douady asked whether the existence of a virtual immediate basin V for N_f implied that $f(z) \rightarrow 0$ as $z \rightarrow \infty$ within V . In Chapter 3, we present a condition on V under which this implication holds true; Bergweiler, Drasin and Langley recently showed that our condition is necessary [BDL]. Conversely, we show that if f has a logarithmic singularity over 0, then this singularity gives rise to a virtual immediate basin for N_f . We construct functions f with a non-logarithmic direct singularity over 0 that does not generate a virtual immediate basin for N_f . This shows that our result cannot be extended to larger classes of singularities. The results in Chapter 3 were developed in collaboration with Xavier Buff and have been published [BR].

Chapter 4 is concerned with rational mappings $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that arise as Newton maps of polynomials. If f satisfies the additional assumption that each critical point maps onto a fixed point after finitely many iterations, we construct a forward-invariant connected graph $\Gamma \subset \mathbb{S}^2$ that characterizes f . That is, $f|_\Gamma$ allows to reconstruct f uniquely (up to Möbius conjugation). Conversely, we show that a map $f' : \Gamma' \rightarrow \Gamma'$ on an abstract graph $\Gamma' \subset \mathbb{S}^2$ gives rise to a unique Newton map if it satisfies some natural conditions. The necessity of these conditions follows in part from results in Chapter 2. Thus, we give a classification of a large class of Newton maps.

Unrelated to the investigation of Newton maps, Eremenko asked in 1989 whether every (path-)component of the set of escaping points for a transcendental entire function of bounded type was necessarily unbounded [Er2]. In his 2005 thesis [Ro], Rottenfußer constructed an unbounded domain $T \subset \mathbb{H}$ in the right half-plane and a Riemann map $F : T \rightarrow \mathbb{H}$, whose set of es-

caping points has no unbounded path-components. The construction can be extended to a bounded type entire function f with the same property, thus answering Eremenko's famous question in the negative. Rottenfußer's thesis also gives a condition under which the answer to this question is positive. Some of these results have been obtained by Günter Rottenfußer in collaboration with Dierk Schleicher, Lasse Rempe and the author of the thesis on hand, and they are being published as a joint paper of the four authors [RRRS]. We include an excerpt of these results in Appendix A.

1.2 Newton's Method

Newton's root-finding method—named after Sir Isaac Newton (1643–1727)—is one of the oldest, most used and most heavily studied iterative procedures to approximate the roots of a differentiable function.

Newton's method was first used on the real line, where it is motivated as follows. Given a $\mathcal{C}^1$ -function $f : \mathbb{R} \rightarrow \mathbb{R}$, we want to find a point $\xi \in \mathbb{R}$ such that $f(\xi) = 0$ (a *root* of f). Starting with an initial guess x_1 , we calculate the root $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ of the linear map tangent to f at x_1 (see Figure 1.1). This tangent line approximates f well near x_1 , and it is reasonable to assume that x_2 will be a better approximation of ξ than x_1 . Indeed, it is well known that the Newton sequence obtained iteratively by setting $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges to ξ , if x_1 was sufficiently close to some root ξ of f . If this happens, we say that the starting value x_1 *finds* the root ξ .

However, in many cases there exist open sets of starting values for Newton's method that do not find any roots. Consider $f(x) = \frac{1}{2}x^3 - x + 1$. For any starting value $x_1 \in \mathbb{R}$, Newton's method gives the sequence

$$x_{n+1} = \frac{x_n^3 - 1}{\frac{3}{2}x_n^2 - 1}.$$

It is easy to see that for the choice $x_1 = 0$, we get $x_2 = 1$ and $x_3 = 0 = x_1$, so that the sequence will not converge. Furthermore, neither 0 nor 1 is a root of f . This Newton's method has a periodic critical point at 0, whose basin of attraction is an open set of starting values for which Newton's method does not find a root of f , compare Figure 1.2.

Nevertheless, Newton's method has been successfully used in many applications and in practice, “bad” starting values can often be avoided. Indeed, Smale [Sm2] has shown that for polynomials $f : \mathbb{C} \rightarrow \mathbb{C}$, the number of

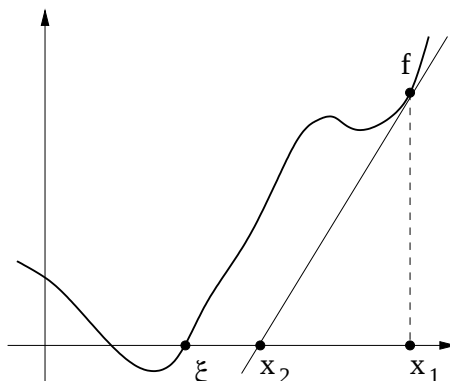


Figure 1.1: For a starting value x_1 sufficiently close to the root ξ of f , the root x_2 of the tangent to f at x_1 is a better approximation of ξ .

randomly chosen starting values needed to find a root of f is very small on average. But while Newton's method has practical uses in much more general contexts, e.g. for differentiable mappings between Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ (for a survey, we also refer to [Sm2]), Newton's method has not even been fully understood for a polynomial $f : \mathbb{C} \rightarrow \mathbb{C}$ in one complex variable.

1.3 A Dynamical Systems Approach

In the following, we restrict to an investigation of Newton's method for holomorphic functions $f : \mathbb{C} \rightarrow \mathbb{C}$. This is the easiest case in that it has only one variable and assumes analyticity of f rather than only differentiability. On the other hand, it contains the class of complex polynomials which are important in many applications. Moreover, the methods of holomorphic dynamics provide powerful tools to gain insight into the structure of Newton's method that are not present in more general settings. But still, this case is not yet completely understood and far from trivial.

While our results are not immediately of a numerical nature, they contribute to a better understanding of such starting points that do *not* find a root of f . Furthermore, our results show similarities and differences between the dynamics of certain classes of rational functions and certain classes of transcendental functions. Studying these similarities and differences is an important and active field of research in holomorphic dynamics.

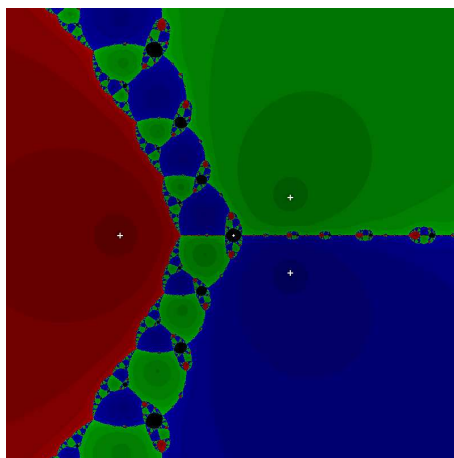


Figure 1.2: Newton's method for a cubic polynomial over $\mathbb{C}$. The roots of the polynomial are indicated by white crosses. Each point is colored according to the root to which Newton's method converges for this starting value. The open black regions indicate starting values that do not converge to any root. Such points converge to the 2-cycle $0 \mapsto 1 \mapsto 0 \mapsto \dots$. The white dot marks the periodic critical point 0.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. We assign to f the meromorphic function

$$N_f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}, z \mapsto z - \frac{f(z)}{f'(z)},$$

its *Newton map*. It is easy to see that for a starting value $z_1 \in \mathbb{C}$, the sequence of iterates $z_n = N_f^{\circ(n-1)}(z_1)$ is exactly the Newton sequence defined in Section 1.2. Observe also that N_f may be rational or transcendental; in the first case, it extends to a rational map $N_f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with a fixed point at ∞ . In Chapter 2, we will show for which f the Newton map N_f is rational.

An easy calculation shows that the roots of f are exactly the finite fixed points of N_f and all of them are attracting, see also Proposition 2.2.8. Therefore, each fixed point $\xi \in \mathbb{C}$ of N_f has a neighborhood U_ξ such that for $z_1 \in U_\xi$, the sequence $N_f^{\circ n}(z_1)$ converges to ξ . If ξ is a simple root of f , Böttcher's theorem [Mi, Theorem 9.1] immediately gives that this convergence is at least quadratic; if ξ is a multiple root, linear convergence follows from Koenigs' theorem [Mi, Theorem 8.2]. If N_f has an attracting cycle of higher period, the basin of attraction of this cycle is an open set of start-

ing values for Newton's method that do not find a root of f . This explains the black regions in Figure 1.2. In the transcendental case, virtual immediate basins are another class of open sets where Newton's method does not converge to a root of f . We investigate those in Chapters 2 and 3.

According to Milnor, “the problem of understanding Newton's method has been a primary inspiration for the study of iterated rational functions” [Mi, p. 51]. Besides the probabilistic approach of Smale in the 1980s, Manning [Ma] constructed in 1992 explicit finite sets $\mathcal{S}_d$ for $d \geq 10$ such that for each properly normalized polynomial p of degree d , the Newton sequence of at least one point in $\mathcal{S}_d$ converges to a root of p . Hubbard, Schleicher and Sutherland extended this result in 2001 and constructed near-optimally small finite sets $\mathcal{S}_d$ for $d \geq 2$ such that for each polynomial p of degree d —normalized such that all its roots are in $\mathbb{D}$ —*every* root of p is approached by the Newton sequence of at least one point in $\mathcal{S}_d$. In 2002, Schleicher [Sch] published an *a-priori* upper bound on the number of Newton steps necessary for points in $\mathcal{S}_d$ to approach all roots up to a distance of $\varepsilon > 0$. While this upper bound of steps grows exponentially in d , he recently announced a sharper bound with polynomial growth and an *a-posteriori* criterion to decide which points in $\mathcal{S}_d$ actually are close to a root after the prescribed number of Newton iterations. These results combined may make Newton's method into a true *algorithm* for the first time, i.e. into an iterative procedure that is guaranteed to terminate after an a-priori determined time.

1.4 The Dynamics of Rational Newton Maps

Besides their application to root-finding, Newton maps form a class of functions that is interesting to study in its own right. One of the main goals in complex dynamics is to gain an understanding of the space of rational functions (of a given degree $d \geq 2$). While the special case of *polynomials* is quite well understood, a classification of *all* rational functions seems still far away. The space of Newton maps of polynomials is large enough to form an interesting sub-case of this problem, while it has enough structure to make a classification seem feasible. Hence, a classification of Newton maps might provide an important intermediate step towards the longterm goal of a classification of all rational functions.

In this section, let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a rational map of degree $d \geq 2$ that is the Newton map of a polynomial p of degree $d \geq 2$. Observe that f and p

have the same degree if and only if all roots of p are simple. If $d = 2$, the dynamics of f is very easy: up to Möbius conjugation, we may assume that the roots of p are at $-i$ and i . Setting $J = \mathbb{R} \cup \{\infty\}$, an easy calculation shows that J is the Julia set of f and $f|_J$ is conformally conjugate to $\sigma|_{\mathbb{S}^1}$, where $\sigma(z) = z^2$. For any point $z \in \widehat{\mathbb{C}} \setminus J$, we get $f^{\circ n}(z) \rightarrow -i$ if $\text{Im}(z) < 0$ and $f^{\circ n}(z) \rightarrow i$ if $\text{Im}(z) > 0$. Hence, there is only one quadratic Newton map up to Möbius conjugation and the space of such maps reduces to a point.

The situation is very different for $d = 3$. A complete description of the space of cubic Newton maps was given by Tan Lei in 1997 [TL]. She constructed an isomorphism between the space of cubic Newton maps (up to conformal conjugacy) and $\mathbb{C}$. Figure 4.1 shows this parameter plane. Building upon the thesis of Janet Head [He], Tan Lei gave a classification of all hyperbolic components in this space and showed that every postcritically finite cubic Newton map (up to conformal conjugacy) can be represented as a mating of two cubic polynomials, a capture, or both. Here, a capture is a Thurston map that is constructed from the combinatorics of a single cubic polynomial, compare [TL, Section 5]. Moreover, Tan Lei gave another combinatorial classification of cubic Newton maps: she shows that every postcritically finite cubic Newton map gives rise to a forward-invariant finite connected graph, which contains the orbits of all critical points. Conversely, every abstract graph that satisfies certain properties is realized, i.e. there exists a unique postcritically finite cubic Newton map whose graph is homeomorphic to the given abstract one. The result of Tan Lei also describes when two given graphs give rise to the same Newton map.

However, very little is known in the case $d > 3$. A result of Jiaqi Luo [Lu] extends part of Tan Lei's work to "unicritical" Newton maps of arbitrary degree. Here, we say that a Newton map is unicritical if it has only one non-fixed critical value. For such maps, Luo constructs a forward-invariant, finite, connected *Newton graph*, which contains the forward orbits of all critical values. Conversely, he considers *topological Newton maps*, i.e. branched covering maps of $\mathbb{S}^2$ that imitate the behavior of Newton maps and have only one free critical value. A topological Newton map g also gives rise to a Newton graph, and Luo proves that if the unique free critical value is either periodic or lands on a fixed critical point after finitely many iterations, then g is equivalent (in the sense of Thurston) to a unique Newton map.

In Chapter 4, we extend the construction of Newton graphs to *postcritically fixed* Newton maps, that is Newton maps whose critical points land on

fixed points after finitely many iterations. The main difficulty in the proof is that the natural candidate for the Newton graph need not be connected (connectedness is clear in the “unicritical” cases above). However, we identify one connected component of this graph that carries all the relevant information, and restrict our attention to that component. Conversely, we show that an abstract graph that satisfies several natural conditions gives rise to a unique postcritically fixed Newton map. In this sense, the Newton graphs classify all postcritically fixed Newton maps. This result lays the groundwork for a possible classification of *all* postcritically finite Newton maps, in particular the hyperbolic ones, see also Section 5.3.

1.5 From Rational to Transcendental Dynamics

The dynamics of *transcendental meromorphic* functions exhibits interesting new phenomena that do not occur for rational maps. As an example, the Fatou set of a transcendental meromorphic function may contain wandering domains or Baker domains. For a detailed introduction to the dynamics of meromorphic functions we refer the reader to the survey article [Be1]. In the special case of Newton maps however, we mention the following facts.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function—polynomial or transcendental—and let N_f be its Newton map. Let U be the immediate basin of an attracting fixed point of N_f . Then, U is simply connected and unbounded. If f is a polynomial, this follows from [Pr], the transcendental case was proved in [MS]. More generally, Shishikura [Sh] showed that the Julia set of N_f is connected if N_f is a rational function. The question if the Julia set is also connected for any transcendental Newton map is still open, but Jordi Taixes has recently announced progress towards an affirmative answer.

If f is a polynomial, the number of accesses to ∞ of the unbounded immediate basin U is determined by the number of critical points of N_f in U (see Theorem 2.2.12). If f is transcendental, these two numbers seem unrelated [My, Theorem 4.1] and may well be both infinite. Moreover, in the transcendental case it is not even clear that ∞ is always an accessible boundary point of U .

An invariant Fatou component of N_f is necessarily an immediate basin if f is a polynomial. For transcendental f , the Fatou set of N_f may contain

invariant components which do not have any interior fixed point, such as fixed Baker domains. In many cases, we are able to explain why they appear, see Chapter 3. By a result of Bergweiler [Be2], wandering domains do not exist for N_f in the special case $f(z) = \int_0^z p(t)e^{q(t)}dt + c$, where p and q are polynomials. We show in Section 3.5 that in general, transcendental Newton maps may have wandering domains.

The main reason for the dynamical differences between rational and transcendental functions is the behavior at infinity: a transcendental function has an essential singularity at ∞ . Note that for a transcendental meromorphic function, there is a countable set of points that can only be iterated finitely many times: the backward orbit of ∞ . By definition, it belongs to the Julia set.

An essential singularity is not accessible to direct investigation, but studying rays that converge to ∞ allows us to draw conclusions about the dynamics near ∞ . Rays to ∞ have often proved useful in other contexts, for example in the classification of escaping points of polynomials [DH1] or of exponential functions [SZ].

Unbounded rays play a crucial role in each of our results: as invariant curves within immediate basins in Chapter 2, as asymptotic paths in Chapter 3, and as edges of the channel diagram in Chapter 4. Of course, the dynamic rays we construct in Appendix A are also unbounded curves. Moreover, while the results of Chapters 2 and 3 are mainly formulated for transcendental maps, they can be applied to the rational case as well. In this sense, the unifying theme of our work are curves to infinity. They organize the dynamics of Newton's method, both in the rational and the transcendental case.

Chapter 2

Newton's Method for Entire Functions

2.1 Introduction

Newton's method is a classical way to approximate roots of entire functions by an iterative procedure. Trying to understand this method may very well be called the founding problem of holomorphic dynamics [Mi, p. 51].

Newton's method for a complex polynomial p is the iteration of a rational function N_p on the Riemann sphere. Such dynamical systems have been extensively studied in recent years. Tan Lei [TL] gave a complete classification of Newton maps of cubic polynomials. In 1992, Manning [Ma] constructed a finite set of starting values for N_p that depends only on the degree of p , such that for any appropriately normalized polynomial with degree $d \geq 10$, the set contains at least one point that converges to a root of p under iteration of N_p . Hubbard, Schleicher and Sutherland [HSS] extended this by constructing a small set of starting values that depends only on the degree $d \geq 2$ and trivial normalizations and finds all roots of p .

If f is a transcendental entire function, the associated Newton map N_f will generally be transcendental meromorphic, except in the special case $f = pe^q$ with polynomials p and q (see Proposition 2.2.11) which was studied by Haruta [Ha]. Bergweiler [Be2] proved a no-wandering-domains theorem for transcendental Newton maps that satisfy several finiteness assumptions. Mayer and Schleicher [MS] have shown that immediate basins for Newton maps of entire functions are simply connected and unbounded, extending a

result of Przytycki [Pr] in the polynomial case. They have also shown that Newton maps of transcendental functions may exhibit a type of Fatou component that does not appear for Newton maps of polynomials, so called *virtual immediate basins* (Definition 2.3.7) in which the dynamics converges to ∞ . The thesis [My] investigates the Newton map of the transcendental function $z \mapsto ze^{e^z}$ and shows that it exhibits virtual immediate basins; see Figure 2.1 for an illustration. While immediate basins of roots are by definition related to zeroes of f (compare Definition 2.2.1), under mild technical assumptions a virtual immediate basin leads to an *asymptotic* zero of f ; in other words, a virtual immediate basin often contains an asymptotic path of an asymptotic value at 0 for f , see Chapter 3.

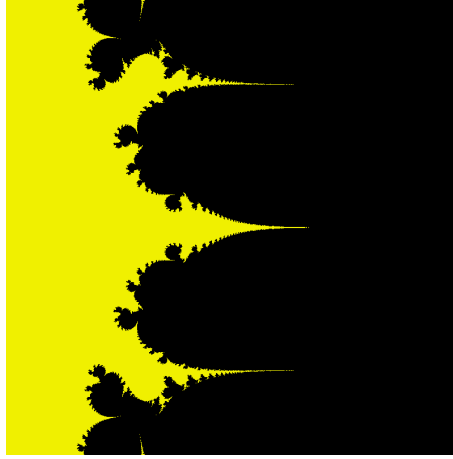


Figure 2.1: The Newton map for $z \mapsto ze^{e^z}$. The immediate basin of 0 has infinitely many accesses to the right. Any two of them surround a virtual immediate basin. More precisely, all curves of the form $(2k+1)\pi i + [2, \infty]$ are contained in a virtual immediate basin; the virtual basins for $k_1 \neq k_2$ are disjoint and separated by an access to ∞ of the immediate basin of 0. The visible area is from $-8 - 10i$ to $12 + 10i$.

In this chapter, we continue the work of [MS] and investigate the behavior of Newton maps in the complement of an immediate basin. Our main result (Theorem 2.5.1) is that if a complementary component can be surrounded by an invariant curve through ∞ , then it contains another immediate basin or virtual immediate basin, unless it maps infinite-to-one onto at least one

point of $\widehat{\mathbb{C}}$. We believe that the last “unless”-condition is unnecessary, but our methods do not allow us to show this.

An immediate corollary for Newton maps of polynomials is that between any two “channels” of any root, there is always another root. This is folklore, but we do not know of a published reference. This result can be viewed as a first step towards a classification of polynomial Newton maps.

This chapter is structured in the following way: In Section 2.2, we give an introduction to some general properties of Newton maps. In Section 2.3, we investigate homotopy classes of curves to ∞ in immediate basins and prove some auxiliary results. In Section 2.4, we prove a fixed point estimate which we will need and which might be interesting in its own right. In Section 2.5, we state and prove our main result.

2.2 Newton's Method as a Dynamical System

2.2.1 Immediate Basins

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant entire function and N_f its associated (meromorphic) Newton map

$$N_f = \text{id} - \frac{f}{f'}.$$

If f is a polynomial, then N_f extends to a rational map $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. If ξ is a root of f with multiplicity $m \geq 1$, then it is an attracting fixed point of N_f with multiplier $\frac{m-1}{m}$. Conversely, every fixed point $\xi \in \mathbb{C}$ of N_f is attracting and a root of f .

Definition 2.2.1 (Immediate Basin). *Let ξ be an attracting fixed point of N_f . The basin of ξ is $\{z \in \mathbb{C} : \lim_{n \rightarrow \infty} N_f^{\circ n}(z) = \xi\}$, the open set of points which converge to ξ under iteration. The connected component U of the basin that contains ξ is called its immediate basin.*

Immediate basins are N_f -invariant because they are Fatou components and contain a fixed point. The following theorem is the main result (Theorem 2.7) of [MS].

Theorem 2.2.2 (Immediate Basins Simply Connected). *If ξ is an attracting fixed point of the Newton map N_f , then its immediate basin U is simply connected and unbounded.* $\square$

We will use the following notation throughout the chapter:

If γ is a curve, the symbol γ denotes the mapping $\gamma : I \rightarrow \mathbb{C}$ from an interval into the plane as well as its image $\gamma(I) \subset \mathbb{C}$. By a *tail* of an unbounded curve we mean any unbounded connected part of its image.

For $r > 0$ and $z \in \mathbb{C}$, the symbol $B_r(z)$ designates the disk of radius r centered at z .

The *full preimage* of a point $z \in \widehat{\mathbb{C}}$ is the set $N_f^{-1}(\{z\})$. Its only accumulation point can be ∞ by the identity theorem. Any point $z' \in N_f^{-1}(\{z\})$ is called a *preimage* of z .

Unless stated otherwise, the boundary and the closure of a set are considered in $\mathbb{C}$.

2.2.2 Singular Values

Since the concept of singular values is crucial for the study of dynamical systems, we give a brief reminder of the most important types. In particular, we state some properties of *asymptotic values*; these appear only for transcendental maps.

Definition 2.2.3 (Singular Value). *Let $h : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function. We call a point $p \in \mathbb{C}$ a regular point of h if p has a neighborhood on which h is injective. Otherwise, we call p a critical point. A point $v \in \widehat{\mathbb{C}}$ is called a regular value if there exists a neighborhood V of v such that for every component W of $h^{-1}(V)$, $h^{-1}|_V : V \rightarrow W$ is a single-valued meromorphic function. Otherwise, v is called a singular value.*

The image of a critical point is a singular value and is called a critical value.

Critical points in $\mathbb{C}$ are exactly the zeroes of the first derivative. For a rational map, all singular values are critical values.

Definition 2.2.4 (Asymptotic Value). *Let $h : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a transcendental meromorphic function. A point $a \in \widehat{\mathbb{C}}$ is called an asymptotic value of h if there exists a curve $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{C}$ with $\lim_{t \rightarrow \infty} \Gamma(t) = \infty$ such that $\lim_{t \rightarrow \infty} h(\Gamma(t)) = a$. We call Γ an asymptotic path of a .*

In general, an asymptotic value is defined by having an asymptotic path towards any essential singularity. Note that in our definition, the set of singular values is the closure of the set of critical and asymptotic values.

We follow [BE] in the classification of asymptotic values.

Definition 2.2.5 (Direct and Indirect Singularity). *Let $h : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function and $a \in \mathbb{C}$ be a finite asymptotic value with asymptotic path Γ . For each $r > 0$, let U_r be the unbounded component of $h^{-1}(B_r(a))$ that contains an unbounded end of Γ .*

We say that a is a direct singularity (with respect to Γ) if there is an $r > 0$ such that $h(z) \neq a$ for all $z \in U_r$. We call a an indirect singularity if for all $r > 0$, there is a $z \in U_r$ such that $h(z) = a$ (then there are infinitely many such z in U_r).

Theorem 2.2.6 (Direct Singularities). [Hs, Theorem 5]. *The set of direct singularities of a meromorphic function is always countable.* $\square$

It is possible however that the set of (direct and indirect) singularities is the entire extended plane: Eremenko [Er1] constructed meromorphic functions of prescribed finite order whose set of asymptotic values is all of $\widehat{\mathbb{C}}$.

Lemma 2.2.7 (Unbounded Preimage). *Let $h : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function and $B \subset \mathbb{C}$ a bounded topological disk whose boundary is a simple closed curve β . Suppose that β contains no critical values and that $\widetilde{B}$ is an unbounded preimage component of B . Then $\partial\widetilde{B}$ contains an unbounded curve $\widetilde{\beta}$ with $h(\widetilde{\beta}) \subset \beta$ such that either $h|_{\widetilde{\beta}} : \widetilde{\beta} \rightarrow \beta$ is a universal covering map or $h(\widetilde{\beta})$ lands at an asymptotic value on β .*

Proof. Let $w \in \partial\widetilde{B}$. Clearly, $h(w) \in \beta$ and by assumption, h is a local homeomorphism in a neighborhood of w . It follows that the closed and unbounded set $\partial\widetilde{B}$ is locally an arc everywhere; therefore it cannot accumulate in any compact subset of $\mathbb{C}$ and must contain an arc $\widetilde{\beta}$ that converges to ∞ . The curve β contains no critical points. If $h|_{\widetilde{\beta}} : \widetilde{\beta} \rightarrow \beta$ is not a universal covering map, then it must land at an asymptotic value. $\square$

2.2.3 Newton Maps

We show that there is only one class of entire functions that have rational Newton maps. This class contains all polynomials. We give a classification of the dynamics within immediate basins for Newton maps of polynomials.

First, we investigate under which conditions a meromorphic function is the Newton map of an entire function. The following proposition uses ideas of Matthias Görner and extends a similar result for rational maps (see below) and certain transcendental functions [Be2, page 3]. We do not know if the proposition is new; however, we certainly do not know of a published reference.

Proposition 2.2.8 (Newton Maps). *Let $N : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function. It is the Newton map of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ if and only if for each fixed point $N(\xi) = \xi \in \mathbb{C}$, there is a natural number $m \in \mathbb{N}$ such that $N'(\xi) = \frac{m-1}{m}$. In this case, there exists $c \in \mathbb{C} \setminus \{0\}$ such that*

$$f = c \cdot \exp \left(\int \frac{d\zeta}{\zeta - N(\zeta)} \right).$$

Two entire functions f, g have the same Newton maps if and only if $f = c \cdot g$ for a constant $c \in \mathbb{C} \setminus \{0\}$.

Proof. We start with the last claim: f and cf have the same Newton map $\text{id} - f/f' = \text{id} - 1/(\ln f)'$. Conversely, if f and g have the same Newton maps, then $(\ln f)' = (\ln g)'$, and the claim follows.

It is easy to check that every Newton map satisfies the criterion on derivatives at fixed points.

For the other direction, we construct a map f such that $N_f = N$. Let $z_0 \in \mathbb{C}$ be any base point and define $\tilde{f}(z) = \int_{\gamma} \frac{d\zeta}{\zeta - N(\zeta)}$, where $\gamma : [0; 1] \rightarrow \mathbb{C}$ is any integration path from z_0 to z that avoids the fixed points of N . This defines $\tilde{f}$ up to $2\pi ik$: if γ' is another choice of integration path, the residue theorem shows that

$$\frac{1}{2\pi i} \int_{\gamma' \circ \gamma^{-1}} \frac{d\zeta}{\zeta - N(\zeta)} = \sum_{N(\xi)=\xi} \text{Res}_{\xi} \left(\frac{1}{\zeta - N(\zeta)} \right),$$

where the sum is taken over the finitely many fixed points of N that are contained in the compact regions bounded by the closed path $\gamma' \circ \gamma^{-1}$. Near a fixed point ξ , it is easy to show that $z - N(z) = \frac{1}{m}(z - \xi) + o(z - \xi)$. Hence we get $\text{Res}_{\xi} \left(\frac{1}{z - N(z)} \right) = m \in \mathbb{N}$.

It follows that the map $f = \exp(\tilde{f})$ is well defined and holomorphic outside the fixed points of N . Near such a fixed point ξ , $\tilde{f}$ has the form

$m \log(z - \xi) + O(1)$. Clearly, the real part of this converges to $-\infty$ for $z \rightarrow \xi$, hence setting $f(\xi) = 0$ makes f an entire function as desired. An easy calculation then shows that $N_f = N$. A different choice of base point z_0 will change f by a multiplicative constant and lead to the same Newton map N_f . $\square$

The following corollary is essentially due to Janet Head ([He, Proposition 2.1.2], [TL, Lemma 2.2]).

Corollary 2.2.9 (Rational Newton Maps). *A rational map $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is the Newton map of a polynomial of degree at least two if and only if $f(\infty) = \infty$ and for all other fixed points $a_1, \dots, a_d \in \mathbb{C}$ there exists a number $m_j \in \mathbb{N}$ such that $f'(a_j) = \frac{m_j-1}{m_j} < 1$. Then, f is the Newton map of the polynomial*

$$p(z) = a \prod_{j=1}^d (z - a_j)^{m_j}$$

for any complex $a \neq 0$.

Sketch of Proof. Let $a \in \mathbb{C} \setminus \{0\}$. Since N_p and f have the same fixed points with identical multiplicities, the residuals of the maps $\tilde{f} := (f - \text{id})^{-1}$ and $\tilde{N} := (N_p - \text{id})^{-1}$ at their common simple poles $a_1, \dots, a_d \in \mathbb{C}$ agree, and thus also those at ∞ . Hence, $\tilde{f} - \tilde{N}$ is a polynomial with $\lim_{z \rightarrow \infty} (\tilde{f} - \tilde{N})(z) = 0$. Hence $\tilde{f} = \tilde{N}$ and the claim follows. $\square$

We want to exclude the trivial case of Newton maps with degree one.

Lemma 2.2.10 (One Root). *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function such that its Newton map N_f has an attracting fixed point $\xi \in \mathbb{C}$ with immediate basin $U = \mathbb{C}$. Then, there exist $d > 0$ and $a \in \mathbb{C}$ such that $f(z) = a(z - \xi)^d$.*

Proof. Since N_f has no periodic points of minimal period at least 2, it cannot be transcendental [Be1, Theorem 2]. Hence N_f is rational and its fixed points can only be ξ and ∞ , both of which must be simple. It follows that N_f has degree at most one and since it has no poles in $\mathbb{C}$, it is a polynomial. The claim now follows from Proposition 2.2.8. $\square$

In the rest of this chapter, we will assume that N_f is not a Möbius transformation. Theorem 2.2.2 implies then that for each immediate basin U of N_f , there exists a Riemann map $\varphi : \mathbb{D} \rightarrow U$ with $\varphi(0) = \xi$.

The following simple proposition classifies rational Newton maps of entire functions. Its first half is stated without proof in [Be1].

Proposition 2.2.11 (Rational Newton Map). *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Its Newton map N_f is rational if and only if there are polynomials p and q such that f has the form $f = pe^q$. In this case, ∞ is a repelling or parabolic fixed point.*

More precisely, let $m, n \geq 0$ be the degrees of p and q , respectively. If $n = 0$ and $m \geq 2$, then ∞ is repelling with multiplier $\frac{m}{m-1}$. If $n = 0$ and $m = 1$, then N_f is constant. If $n > 0$, then ∞ is parabolic with multiplier $+1$ and multiplicity $n + 1 \geq 2$.

Proof. By [Mi, Corollary 12.7], every rational function of degree at least 2 has a repelling or parabolic fixed point. Since N_f is a Newton map, this non-attracting fixed point is unique and must be at ∞ . In addition to this, there are finitely many attracting fixed points $a_1, \dots, a_n \in \mathbb{C}$ with associated natural numbers $m_1, \dots, m_n \in \mathbb{N}$ such that the multipliers satisfy $N'_f(a_i) = \frac{m_i-1}{m_i}$. Let $p(z) = \prod_{i=1}^n (z - a_i)^{m_i}$.

Since attracting fixed points of N_f correspond exactly to the roots of f , f has the form $f = pe^h$ for an entire function h . If h was transcendental, so would be

$$N_f = \text{id} - \frac{pe^h}{p'e^h + h'pe^h} = \text{id} - \frac{p}{p' + h'p},$$

a contradiction. The other direction follows by direct calculation and the rest of the proof is left to the reader. $\square$

For Newton maps of $f = pe^q$, the area of every immediate basin is finite if $\deg q \geq 3$ [Ha] and infinite if $p(z) = z$ and $\deg q \in \{0, 1\}$ [Ci].

The dynamics within immediate basins of Newton maps of polynomials has an easy classification, because all singular values are critical values.

Theorem 2.2.12 (Polynomial Newton Maps). [HSS] *Let p be a polynomial of degree $d > 1$, normalized so that its roots are contained in the unit disk $\mathbb{D}$. Let ξ be a root of p and U its immediate basin for N_p . Then, U contains $k > 0$ critical points of N_p and $N_p|_U$ is a proper self-map of degree $k + 1$. Outside the disk $B_2(0)$, N_f is conformally conjugate to multiplication by $\frac{d-1}{d}$. Finally, $U \setminus B_2(0)$ has exactly k unbounded components, so called channels, each of which maps over itself under N_f .* $\square$

Figure 2.2 illustrates this theorem.

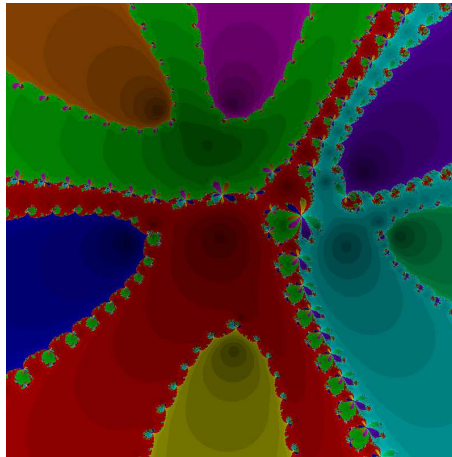


Figure 2.2: The Newton map for a polynomial of degree 9. The channels are clearly visible.

2.3 Accesses in Immediate Basins

2.3.1 Invariant Accesses

We investigate the immediate basins of attraction for the attracting fixed points of N_f . If f is a polynomial, we have seen in Theorem 2.2.12 that immediate basins have an easy geometric structure. In the general case, N_f has an essential singularity at ∞ and immediate basins may well have infinitely many accesses to ∞ . We use prime end theory to distinguish them.

Under a finiteness assumption, we have some control over the image of a sequence that converges to ∞ through an immediate basin.

Lemma 2.3.1 (Invariant Boundary). *Let U be an immediate basin of the Newton map N_f and U_R an unbounded component of $U \setminus B_R(0)$ with the property that no point has infinitely many preimages in U_R . Then for any sequence $(z_n) \subset U_R$ with $z_n \rightarrow \infty$, all limit points of $N_f(z_n)$ are contained in $\partial U \cup \{\infty\}$.*

The condition is necessary, because if there exists a point $p \in U$ with infinitely many preimages $p_1, p_2, \dots \in U_R$, we have $p_n \rightarrow \infty$ and $N_f(p_n) = p \in U$ for all $n \in \mathbb{N}$.

Proof. Assume there exists a sequence $(z_n) \subset U_R$ that converges to ∞ with $N_f(z_n) \rightarrow p \in U$. Let $B \subset U$ be a closed neighborhood of p inside U such that its boundary ∂B is a simple closed curve β that contains no direct singularities (this is possible by Theorem 2.2.6) nor critical values.

Suppose first that $p \notin N_f(\partial B_R(0))$. Then we may choose B small enough such that $B \cap N_f(\partial B_R(0)) = \emptyset$. The image of the first finitely many z_n need not be in B ; ignoring those, each z_n is contained in a component W_n of $N_f^{-1}(B) \cap U$. If a W_n is bounded, it maps surjectively onto B under N_f . Therefore, by the finiteness assumption, there can be only finitely many bounded W_n . Each bounded W_n contains finitely many z_n ; hence there must be an n such that W_n is unbounded. By Lemma 2.2.7 and again because of the finiteness assumption, ∂W_n contains an asymptotic path of an asymptotic value on β . But this asymptotic value must be an indirect singularity, which also contradicts the finiteness assumption.

If $p \in N_f(\partial B_R(0))$, a small homotopy of the curve $\partial B_R(0)$ in a neighborhood of p solves the problem. $\square$

Figure 2.1 suggests that immediate basins can reach out to infinity in several different directions. We make this precise in the following definitions that generalize the concept of a channel in the polynomial case.

Definition 2.3.2 (Invariant Access). *Let ξ be an attracting fixed point of N_f and U its immediate basin. An access to ∞ of U is a homotopy class of curves within U that begin at ξ , land at ∞ and are homotopic with fixed endpoints.*

An invariant access to ∞ is an access with the additional property that for each representative γ , its image $N_f(\gamma)$ belongs to the access as well.

Lemma 2.3.3 (Access Induces Prime End). *Let $[\gamma]$ be an access to ∞ in U . Then $[\gamma]$ induces a prime end $\mathcal{P}$ in U with impression $\{\infty\}$. If $[\gamma]$ is invariant, then $N_f(\mathcal{P}) = \mathcal{P}$.*

Proof. Let $\gamma \subset U$ be a curve representing $[\gamma]$ that starts at the fixed point ξ and lands at ∞ . For $n \in \mathbb{N}$, let W_n be the component of $U \setminus B_n(0)$ that contains a tail of γ . The W_n represent a prime end $\mathcal{P}$ with impression ∞ . Now a curve $\gamma' \subset U$ that starts at ξ and lands at ∞ is homotopic to γ if and only if a tail of it is contained in W_n for n large enough. Hence the prime end $\mathcal{P}$ of $[\gamma]$ is well-defined. The last claim follows immediately from the definition. $\square$

It is clear that different accesses induce different prime ends. We state one more well-known topological fact about the boundary behavior of Riemann maps before using prime ends to characterize invariant accesses.

Lemma 2.3.4 (Accesses Separate Disk). *Let $U \subsetneq \mathbb{C}$ be a simply connected unbounded domain and $\gamma_1, \gamma_2 : \mathbb{R}_0^+ \rightarrow U \cup \{\infty\}$ two non-homotopic curves that land at ∞ and are disjoint except for their common base point $z_0 = \gamma_1(0) = \gamma_2(0) \in U$. Let C be a component of $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$ and $\varphi : \mathbb{D} \rightarrow U$ a Riemann map with $\varphi(0) = z_0$.*

Then $\varphi^{-1}(\gamma_1)$ and $\varphi^{-1}(\gamma_2)$ land at distinct points ζ_1 and ζ_2 of $\partial\mathbb{D}$. Furthermore, $\overline{\partial U \cap C} \subset \widehat{\mathbb{C}}$ corresponds under φ^{-1} to a closed interval on $\partial\mathbb{D}$ that is bounded by ζ_1 and ζ_2 .

This follows immediately because φ extends to a homeomorphism from $\overline{\mathbb{D}}$ to the Carathéodory compactification of U , see [Mi, Theorem 17.12].

If f is a polynomial, it follows from Theorem 2.2.12 that every immediate basin contains a curve that lands at ∞ , is homotopic to its image and induces an invariant access. In the general case, it is a priori not even clear that a curve that lands at ∞ and is homotopic within U to its image induces an invariant access. The following proposition deals with this issue.

Proposition 2.3.5 (Curve Induces Invariant Access). *Let $\gamma \subset U \cup \{\infty\}$ be a curve connecting the fixed point ξ to ∞ such that $N_f(\gamma)$ is homotopic to γ in U with endpoints fixed. Let W_n be a sequence of fundamental neighborhoods representing the prime end $\mathcal{P}$ induced by $[\gamma]$. Then γ defines an invariant access to ∞ if and only if there is no $z \in \widehat{\mathbb{C}}$ that has infinitely many preimages in all W_n .*

Proof. Suppose that γ defines an invariant access, i.e. if γ' is homotopic in U to γ , then $N_f(\gamma')$ is homotopic to $N_f(\gamma)$. Assume there is a point $z_0 \in \widehat{\mathbb{C}}$ with the property that $N_f^{-1}(\{z_0\}) \cap W_n$ is an infinite set for all W_n . Without loss of generality, we may assume that for all $n \in \mathbb{N}$, $W_n \setminus W_{n+1}$ contains one preimage of z_0 . Then we can find a curve γ' with a tail contained in each W_n that goes through a preimage of z_0 in each $W_n \setminus W_{n+1}$. Clearly, γ' is homotopic to γ , while its image does not land at ∞ and can therefore not be homotopic to $N_f(\gamma)$ with endpoints fixed, a contradiction.

Now suppose that no point has infinitely many preimages in all W_n . Since the W_n are nested, no point can have infinitely many preimages in any W_n for n sufficiently large. We uniformize U to the unit disk via a Riemann

map $\varphi : \mathbb{D} \rightarrow U$ such that $\varphi(0) = \xi$ and consider the induced dynamics $g = \varphi^{-1} \circ N_f \circ \varphi : \mathbb{D} \rightarrow \mathbb{D}$.

By [Mi, Corollary 17.10], $\varphi^{-1}(\gamma)$ and $\varphi^{-1}(N_f(\gamma))$ land on $\partial\mathbb{D}$. Since the curves are homotopic, they even land at the same point $\zeta \in \partial\mathbb{D}$. Now by assumption, there exists an $\varepsilon > 0$ such that within $B_\varepsilon(\zeta)$, no g -preimage of any point in $\mathbb{D}$ accumulates. By Lemma 2.3.1 it follows that the g -image of any sequence converging to $\partial\mathbb{D}$ inside $B_\varepsilon(\zeta) \cap \mathbb{D}$ will also converge to $\partial\mathbb{D}$. Hence we can use the Schwarz Reflection Principle [Ru, Theorem 11.14] to extend g holomorphically to a neighborhood of ζ in $\mathbb{C}$. It follows that for the extended map, ζ is a repelling fixed point with positive real multiplier: if the multiplier was not positive real, g would map points in $B_\varepsilon(\zeta) \cap \mathbb{D}$ out of $\mathbb{D}$. Also, ζ cannot be attracting or parabolic, because in this case it would attract points in $\mathbb{D}$, which all converge to 0 under iteration.

Since $\mathbb{D}$ is simply connected, all curves in $\mathbb{D}$ from 0 to ζ will be homotopic to each other and their g -images. A curve in $\mathbb{D}$ that starts at 0 lands at ζ if and only if its φ -image in U is homotopic to γ with endpoints fixed, because $\varphi^{-1}(\mathcal{P})$ is a prime end in $\mathbb{D}$ with impression ζ . $\square$

REMARK. We have shown that each invariant access defines a boundary fixed point in the conjugated dynamics on the unit disk, and the dynamics can be extended to a neighborhood of this boundary fixed point, necessarily yielding a repelling fixed point. By [Mi, Corollary 17.10] it follows that different invariant accesses induce distinct boundary fixed points.

If f is a polynomial, there exists a one-to-one correspondence between accesses to ∞ of U and boundary fixed points of the induced map g [HSS, Proposition 6].

Corollary 2.3.6 (Invariant Curve). *Each invariant access has an invariant representative, i.e. a curve $\gamma : \mathbb{R}_0^+ \rightarrow U$ that lands at ∞ with $\gamma(0) = \xi$ and $N_f(\gamma) = \gamma$.*

Proof. For the extension of g to a neighborhood of $\zeta \in \partial\mathbb{D}$, the multiplier of ζ is positive real. A short piece of straight line in linearizing coordinates around ζ maps over itself under g . Its forward orbit lands at the fixed point. $\square$

Since there are uncountably many choices of such invariant curves, we can always find one that contains no critical or direct asymptotic values outside a sufficiently large disk.

2.3.2 Virtual Basins

If f is a polynomial and $U \subset \mathbb{C}$ an invariant Fatou component of N_f , then U is the immediate basin of a root of f , because the Julia set of N_f is connected [Sh], all finite fixed points are attracting and the fixed point at ∞ is repelling. If f is transcendental entire, N_f may possess invariant unbounded Fatou domains in which the dynamics converges to ∞ . Such components are Baker domains or attracting petals of an indifferent fixed point at infinity. In many cases, such components contain an asymptotic path of an asymptotic value at 0 for f , see Chapter 3.

Definition 2.3.7 (Virtual Basin). *An unbounded domain $V \subset \mathbb{C}$ is called virtual immediate basin of N_f if it is maximal (among domains in $\mathbb{C}$) with respect to the following properties:*

1. $\lim_{n \rightarrow \infty} N_f^n(z) = \infty$ for all $z \in V$;
2. *there is a connected and simply connected subdomain $S_0 \subset V$ such that $N_f(\overline{S_0}) \subset S_0$ and for all $z \in V$ there is an $m \in \mathbb{N}$ such that $N_f^m(z) \in S_0$.*

We call the domain S_0 an absorbing set for V .

Clearly, virtual immediate basins are forward invariant.

Theorem 2.3.8 (Virtual Basin Simply Connected). [MS, Theorem 3.4] *Virtual immediate basins are simply connected.* $\square$

It might be possible to extend Shishikura's theorem [Sh] to show that for Newton maps of entire functions, all Fatou components are simply connected. Taixes has announced partial results in this direction, in particular he rules out the existence of cycles of Herman rings (see also Corollary 2.4.7 below). If it were also known that Baker domains are always simply connected, then a result of Cowen [Co, Theorem 3.2] would imply that every invariant Fatou component of a Newton map is an immediate basin or a virtual immediate basin (see [MS, Remark 3.5]).

2.4 A Fixed Point Estimate

Let X be a compact, connected and triangulable real n -manifold and let $f : X \rightarrow X$ be continuous with finitely many fixed points. Each fixed point

of f has a well-defined *Lefschetz index*, and f has a global *Lefschetz number*. The classical Lefschetz fixed point formula says that the sum of the Lefschetz indices is equal to the Lefschetz number of f , up to a factor of $(-1)^n$ [Le, Br].

In [GM, Lemma 3.7], Goldberg and Milnor give a version of this theorem for weakly polynomial-like mappings $f : \overline{\mathbb{D}} \rightarrow \mathbb{C}$. We prove a similar result for a class of maps $f : \Delta \rightarrow \widehat{\mathbb{C}}$, where $\Delta \subset \widehat{\mathbb{C}}$ is a closed topological disk. By extending the range of f to $\widehat{\mathbb{C}}$, i.e. allowing poles, we lose equality and get an inequality between the Lefschetz number and the sum of the Lefschetz indices that we will need in the proof of our main theorem.

Definition 2.4.1 (Lefschetz Map). *Let $\Delta \subset \widehat{\mathbb{C}}$ be a closed topological disk with boundary curve $\partial\Delta$ and $f : \Delta \rightarrow \widehat{\mathbb{C}}$ an orientation preserving open mapping with isolated fixed points. We call f a Lefschetz map if it satisfies the following additional conditions: for any $z \in \widehat{\mathbb{C}}$, the full preimage $f^{-1}(\{z\}) \subset \Delta$ is a finite set. Furthermore, $f(\partial\Delta)$ is a simple closed curve so that $f|_{\partial\Delta} : \partial\Delta \rightarrow f(\partial\Delta)$ is a covering map of finite degree. Assume also that*

$$f(\partial\Delta) \cap \overset{\circ}{\Delta} = \emptyset$$

and that if $\xi \in \partial\Delta$ is a fixed point of f , then ξ has a neighborhood U such that $f(\partial\Delta \cap U) \subset \partial\Delta$, and f is expanding on $\partial\Delta \cap U$.

REMARK. The definition of “expanding” is with respect to the local parametrization of $\partial\Delta$ near ξ so that $f|_{\partial\Delta \cap U}$ is topologically conjugate to $x \mapsto 2x$ in a neighborhood of 0.

In this case, the map f can be extended continuously to $U \setminus \Delta$ so that f on $U \setminus \overset{\circ}{\Delta}$ is topologically conjugate to $z \mapsto 2z$ on the half disk $\{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im}(z) \geq 0\}$ (possibly after shrinking U). Such an extension will be called the *simple extension outside of Δ* .

Definition 2.4.2 (Lefschetz Index). *Let $W \subset \mathbb{C}$ be a closed topological disk and $f : W \rightarrow \mathbb{C}$ be continuous with an isolated fixed point at $\xi \in W$. With $g(z) = f(z) - z$, we assign to ξ its Lefschetz index*

$$\iota(\xi, f) := \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \oint_{g(\partial B_\varepsilon(\xi))} \frac{d\zeta}{\zeta}.$$

This is the number of full turns that the vector $f(z) - z$ makes when z goes once around ξ in a sufficiently small neighborhood.

If $\xi \in \partial W \cap \mathbb{C}$ is an isolated boundary fixed point which has a simple extension outside of W , then we define its Lefschetz index as above for this simple extension.

For an interior fixed point, it is easy to see that the limit exists and is invariant under homotopies of f that avoid additional fixed points. Strictly speaking, the curve $g(\partial B_\varepsilon(\xi))$ need not be an admissible integration path (i.e. rectifiable), but because of homotopy invariance, we may ignore this problem, and we will often do so in what follows.

The Lefschetz index is clearly a local topological invariant; for boundary fixed points, it does not depend on the details of the extension. Therefore, the index is also defined if $\xi = \infty$, using local topological coordinates. Note that for boundary points, the simple extension as defined above generates the least possible Lefschetz index for all extensions of f to a neighborhood of ξ .

If f is holomorphic in a neighborhood of a fixed point ξ , then $\iota(\xi, f)$ is the multiplicity of ξ as a fixed point.

Definition 2.4.3 (Lefschetz Number). *The Lefschetz number of f is the degree of the covering map $f|_{\partial\Delta}: \partial\Delta \rightarrow f(\partial\Delta)$.*

In this definition, we need to find compatible orientations for $\partial\Delta$ and its image. The set $\widehat{\mathbb{C}} \setminus f(\partial\Delta)$ consists of two connected components; let V be the one which intersects Δ . Then $\Delta \subset \overline{V}$ are two closed topological disks, and they can be oriented in a compatible way.

With this convention, the Lefschetz number is invariant under topological conjugacies. We may thus choose coordinates so that $\overline{V} \subset \mathbb{C}$.

Lemma 2.4.4 (Winding Number). *Suppose $f: \Delta \rightarrow \widehat{\mathbb{C}}$ is a Lefschetz map with $f(\partial\Delta) \subset \mathbb{C}$ such that $\mathring{\Delta}$ is contained in the bounded component of $\mathbb{C} \setminus f(\partial\Delta)$. If $\partial\Delta$ contains no fixed points of f , then the Lefschetz number of f equals*

$$\frac{1}{2\pi i} \oint_{g(\partial\Delta)} \frac{d\zeta}{\zeta}, \quad (2.1)$$

where $g(z) = f(z) - z$.

Proof. Let $w_0 \in \mathring{\Delta}$ be any base point. Since Δ is contractible to w_0 within $\mathring{\Delta}$, $g|_{\partial\Delta} = (f - \text{id})|_{\partial\Delta}$ is homotopic to $(f - w_0)|_{\partial\Delta}$ in $\mathbb{C} \setminus \{0\}$.

The integral (2.1) counts the number of full turns of $f(z) - z$ as z runs around $\partial\Delta$. By homotopy invariance, this is equal to the number of full turns $f(\partial\Delta)$ makes around w_0 , and this equals the Lefschetz number of f . $\square$

Lemma 2.4.5 (Equality in $\mathbb{C}$). *Let $V \subset \mathbb{C}$ be a simply connected and bounded domain with piecewise $\mathcal{C}^1$ boundary and let $f: \overline{V} \rightarrow f(\overline{V}) \subset \mathbb{C}$ be a continuous map with finitely many fixed points, none of which are on ∂V . Then*

$$\sum_{f(\xi)=\xi} \iota(\xi, f) = \frac{1}{2\pi i} \oint_{g(\partial V)} \frac{d\zeta}{\zeta},$$

where again $g = f - \text{id}$.

Proof. Break up V into finitely many disjoint simply connected open pieces V_i with piecewise $\mathcal{C}^1$ boundaries so that each V_i either contains a single fixed point of f or $f(\overline{V_i}) \cap \overline{V_i} = \emptyset$, and each fixed point of f is contained in some V_i . This can be done by first choosing disjoint neighborhoods for all fixed points and then partitioning their compact complement in $\overline{V}$ into pieces of diameter less than θ , where θ is chosen in such a way that $|f - \text{id}| > \theta$ in this complement. Set

$$c_i := \frac{1}{2\pi i} \oint_{g(\partial V_i)} \frac{d\zeta}{\zeta}.$$

Then

$$\frac{1}{2\pi i} \oint_{g(\partial V)} \frac{d\zeta}{\zeta} = \sum_i \frac{1}{2\pi i} \oint_{g(\partial V_i)} \frac{d\zeta}{\zeta} = \sum_i c_i.$$

On the pieces with $f(\overline{V_i}) \cap \overline{V_i} = \emptyset$, we have $c_i = 0$, and on a piece V_i with fixed point ξ_i , we have $\iota(\xi_i, f) = c_i$ by definition. The claim follows. $\square$

Theorem 2.4.6 (Fixed Point Inequality). *Let $f: \Delta \rightarrow \hat{\mathbb{C}}$ be a Lefschetz map with Lefschetz number $L \in \mathbb{Z}$. Then,*

$$L \leq \sum_{f(\xi)=\xi} \iota(\xi, f).$$

Proof. Let V be the component of $\hat{\mathbb{C}} \setminus f(\partial\Delta)$ containing $\mathring{\Delta}$, and choose coordinates of $\hat{\mathbb{C}}$ such that $\overline{V}$ is bounded.

Suppose first that f has no fixed points on $\partial\Delta$. Let $\{U_i\}$ be the collection of components of $f^{-1}(\hat{\mathbb{C}} \setminus \overline{V})$ and let $\{V_j\}$ be the collection of components of $f^{-1}(V)$. Since f is open, each U_i maps onto $\hat{\mathbb{C}} \setminus \overline{V}$ and each V_j maps onto V as a proper map. It follows that there are only finitely many U_i and V_j , and they satisfy $f(\partial U_i) = f(\partial V_j) = f(\partial\Delta)$.

Subdivide the U_i and V_j into finitely many simply connected pieces so that no poles or fixed points of f are on the boundaries; call these subdivided

domains $U'_{i'}$ and $V'_{j'}$. The orientation of $\mathbb{C}$ induces a boundary orientation on $U'_{i'}$ and $V'_{j'}$.

Set again $g := f - \text{id}$. Then, applying Lemma 2.4.5 to $V'_j \subset \Delta \subset \overline{V} \subset \mathbb{C}$ yields

$$\sum_{f(\xi)=\xi} \iota(\xi, f) = \sum_{j'} \frac{1}{2\pi i} \oint_{g(\partial V'_{j'})} \frac{d\zeta}{\zeta} = \sum_j \frac{1}{2\pi i} \oint_{g(\partial V_j)} \frac{d\zeta}{\zeta}.$$

Note that for every $U'_{i'}$, we have $\sum_i \frac{1}{2\pi i} \oint_{g(\partial U'_{i'})} \frac{d\zeta}{\zeta} \leq 0$ (this counts indices of poles).

The covering map $f: \partial\Delta \rightarrow f(\partial\Delta)$ has mapping degree L . Since the contributions from the boundaries within Δ cancel, we get

$$L = \sum_j \frac{1}{2\pi i} \oint_{g(\partial V_j)} \frac{d\zeta}{\zeta} + \sum_{i'} \frac{1}{2\pi i} \oint_{g(\partial U'_{i'})} \frac{d\zeta}{\zeta} \leq \sum_{f(\xi)=\xi} \iota(\xi, f).$$

If f has boundary fixed points, we employ a simple extension outside of Δ in a small neighborhood of each such fixed point. In order for the extended map to be a Lefschetz map, the preimages need to be extended as well. If the extended neighborhoods are sufficiently small, this does not change the Lefschetz number of f . $\square$

As an immediate corollary of this theorem, we observe that Newton maps do not have fixed Herman rings. Note that Taixes has announced a more general result: he uses quasiconformal surgery to rule out any periodic cycles of Herman rings for Newton maps.

Corollary 2.4.7 (No Fixed Herman Rings). *Newton maps of entire functions have no fixed Herman rings.*

Proof. By [Sh], we may assume that $N: \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ is a transcendental meromorphic Newton map. Suppose it has a fixed Herman ring, i.e. an invariant Fatou component H such that $N|_H$ is conjugate to an irrational rotation of an annulus of finite modulus. Then, H contains an invariant and essential simple closed curve γ . Clearly, $\deg(N: \gamma \rightarrow \gamma) = +1$. Let Δ be the bounded component of $\mathbb{C} \setminus \gamma$. Then, $N|_{\overline{\Delta}}$ is a Lefschetz map and by Theorem 2.4.6, $\overline{\Delta}$ contains a fixed point. This is a contradiction, because all fixed points of N have an unbounded immediate basin (Theorem 2.2.2). $\square$

2.5 Between Accesses of an Immediate Basin

In this section, we state and prove our main result. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function and N_f its Newton map. Let $\xi \in \mathbb{C}$ be a fixed point of N_f and U its immediate basin. Suppose that U has two distinct invariant accesses, represented by N_f -invariant curves Γ_1 and Γ_2 . Consider an unbounded component $\tilde{V}$ of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. We keep this notation for the entire section.

Theorem 2.5.1 (Main Theorem). *If no point in $\hat{\mathbb{C}}$ has infinitely many preimages within $\tilde{V}$, then the set $V := \tilde{V} \setminus U$ contains an immediate basin or a virtual immediate basin of N_f .*

Note that we do not assume that V is connected.

Corollary 2.5.2 (Polynomial Case). *If N_f is the Newton map of a polynomial f , then each component of $\mathbb{C} \setminus U$ contains the immediate basin of another root of f .*

Proof of Corollary 2.5.2. If f is a polynomial, N_f is a rational map. It has finite mapping degree and there exists $R > 0$ such that all components of $U \setminus B_R(0)$ contain exactly one invariant access. Furthermore, all accesses are invariant [HSS, Proposition 6]. Since ∞ is a repelling fixed point of N_f , there are no virtual immediate basins. $\square$

The rest of this section will be devoted to the proof of Theorem 2.5.1. This proof will be based on the fixed point estimate in Theorem 2.4.6. In order to be able to use it in our setting, we will need some preliminary statements.

Proposition 2.5.3 (Pole on Boundary). *If no point $z \in U$ has infinitely many preimages within $\tilde{V} \cap U$, then $\partial V = \partial U \cap \tilde{V}$ contains at least one pole of N_f .*

In particular, if ∂V is connected or $N_f|_U$ has finite degree, then ∂V contains a pole of N_f . Every pole on ∂V is arcwise accessible from within U .

Proof. Let $\varphi : \mathbb{D} \rightarrow U$ be a Riemann map for the immediate basin U with $\varphi(0) = \xi$. It conjugates the dynamics of N_f on U to the induced map $g = \varphi^{-1} \circ N_f \circ \varphi : \mathbb{D} \rightarrow \mathbb{D}$. By Lemma 2.3.4, the Carathéodory extension $\overline{\varphi}^{-1}$ maps $\overline{\partial V} \subset \partial U \cup \{\infty\}$ to a closed interval $I \subset \partial \mathbb{D}$ that is bounded by the landing points ζ_1 and ζ_2 of $\varphi^{-1}(\Gamma_1)$ and $\varphi^{-1}(\Gamma_2)$. By assumption, there is an

open neighborhood of $\overset{\circ}{I}$ in $\mathbb{D}$ which contains only finitely many g -preimages of every $z \in \mathbb{D}$. By Proposition 2.3.5, there is a neighborhood W' of I in $\mathbb{D}$ with the same property. Consider a sequence $(z_n) \subset \mathbb{D}$ whose accumulation set is in $W' \cap \partial\mathbb{D}$. By Lemma 2.3.1, all limit points of $(g(z_n))$ are in $\partial\mathbb{D}$. Hence there is a neighborhood W of I in $\mathbb{C}$ such that we can extend g by Schwarz reflection to a holomorphic map $\tilde{g} : W \rightarrow \mathbb{C}$ that coincides with g on $W \cap \mathbb{D}$. The endpoints ζ_1 and ζ_2 of I are fixed under this map, because each is the landing point of an invariant curve. They are repelling, because otherwise they would attract points from within $\mathbb{D}$.

Clearly, $\tilde{g}(I) \subset \partial\mathbb{D}$. If $\tilde{g}(I) = I$, then $\tilde{g}$ has to have an additional fixed point on I which is necessarily parabolic and thus attracts points in $\mathbb{D}$. This is a contradiction because all points in $\mathbb{D}$ converge to 0 under iteration of g .

If I contained a critical point c of $\tilde{g}$, points in $\mathbb{D}$ arbitrarily close to c would be mapped out of $\mathbb{D}$ by $\tilde{g}$, again a contradiction. Hence $\tilde{g} : I \rightarrow \partial\mathbb{D}$ is surjective and there are points $z_1, z_2 \in \overset{\circ}{I}$ such that $\tilde{g}(z_1) = \zeta_1$, $\tilde{g}(z_2) = \zeta_2$.

For $i = 1, 2$, let $\beta_i : [0; 1) \rightarrow \mathbb{D}$ be the radial line from 0 to z_i . Then, $\varphi(\beta_i)$ accumulates at a continuum $X_i \subset \partial V$ while $N_f(\varphi(\beta_i)) = \varphi(g(\beta_i))$ lands at ∞ in the access of Γ_i . By continuity, $N_f(X_i) = \{\infty\}$; the identity theorem shows that $X_i = \{p_i\}$ is a pole and $\varphi(\beta_i)$ lands at p_i . $\square$

We use the following general lemma to show that $N_f|_{\tilde{V}}$ can be continuously extended to ∞ .

Lemma 2.5.4 (Extension Lemma). *Let $h : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a meromorphic function and $G \subset \mathbb{C}$ an unbounded domain. Suppose that ∂G can be parametrized by two asymptotic paths of the asymptotic value ∞ and that no point has infinitely many preimages within G . Then, $h|_{\overline{G}}$ can be continuously extended to ∞ .*

Proof. Since $h(\partial G)$ is unbounded, the only possible continuous extension is to set $h(\infty) = \infty$.

If h cannot be continuously extended to ∞ , there exists a sequence $z_n \rightarrow \infty$ in G such that $h(z_n) \rightarrow p \in \mathbb{C}$. Let $S > |p|$ and pick $R > 0$ such that $|h(z)| \geq S$ for all $z \in \partial G$ with $|z| \geq R$, and $p \notin h(\partial B_R(0))$. We may suppose that all $|z_n| > R$. Then we can choose a closed neighborhood $B \subset B_S(0)$ of p whose boundary is a simple closed curve that contains no critical values or direct singularities and so that B is disjoint from $h(\partial B_R(0))$. Now let W_n be the component of $h^{-1}(B)$ that contains z_n . Then $W_n \subset \mathbb{C} \setminus \overline{B_R(0)}$. Since $z_n \in G$, it follows that all $W_n \subset G \setminus \overline{B_R(0)}$.

If all W_n are bounded, each can contain only finitely many z_k and there must be infinitely many such components. Since bounded W_n map onto B , this would contradict the finiteness assumption. Hence there is an unbounded preimage component W_0 . By Lemma 2.2.7, $G \setminus \overline{B_R(0)}$ then contains an asymptotic path of an indirect singularity on ∂B , which also contradicts the finiteness assumption. $\square$

In the next proposition, we show that $N_f|_{\tilde{V}}$ is injective near ∞ . For the proof, we use an extremal length argument in the half-strip

$$Y := [0, \infty) \times [0, 1] ,$$

in which we measure the modulus of a quadrilateral by curves connecting the left boundary arc to the right. For $x \in \mathbb{R}$, define

$$\mathbb{H}_x := \{z \in \mathbb{C} : \operatorname{Re}(z) \geq x\} .$$

First, we prove a technical lemma.

Lemma 2.5.5 (Bound on Modulus). *Let $0 < t \leq s$, let $\beta \subset Y$ an injective curve from $(t, 1)$ to $(s, 0)$ and let Q the bounded component of $Y \setminus \beta$. Let $(0, 0)$, $(s, 0)$, $(t, 1)$ and $(0, 1)$ be the vertices of the quadrilateral Q . Then, $\operatorname{mod}(Q) \leq t + 1$.*

Proof. Let $R \subset Y$ be the rectangle with vertices $(0, 0)$, $(0, 1)$, $(t + 1, 1)$, $(t + 1, 0)$. Its area and modulus are both equal to $t + 1$. In particular, $\operatorname{area}(Q \cap R) \leq \operatorname{area}(R) = t + 1$. Using the admissible density $\rho(x) = \frac{1}{\sqrt{\operatorname{area}(Q \cap R)}}$. $\chi_{Q \cap R}(x)$, we get the estimate

$$\frac{1}{\operatorname{mod}(Q)} \geq \frac{1}{\operatorname{area}(Q \cap R)} \geq \frac{1}{\operatorname{area}(R)} = \frac{1}{t + 1} ,$$

because $\int_{\gamma} \rho d\gamma \geq \frac{1}{\sqrt{\operatorname{area}(Q \cap R)}}$ for this density and all rectifiable curves γ that connect the upper to the lower boundaries. $\square$

Proposition 2.5.6 (Invariance Near ∞). *Suppose that every $z \in \hat{\mathbb{C}}$ has only finitely many N_f -preimages in $\tilde{V}$. Then there exists $R_0 > 0$ such that for all $R > R_0$, the map N_f is injective on $\tilde{V} \setminus B_R(0)$. Moreover, there exists $S > 0$ with the property that*

$$N_f(\tilde{V} \setminus B_R(0)) \setminus B_S(0) = \tilde{V} \setminus B_S(0) .$$

Proof. Choose $R_0 > \max\{|z| : z \in N_f^{-1}(\infty) \cap \overline{\tilde{V}}\}$. It follows from the open mapping principle and invariance of $\partial\tilde{V}$ that there exists $S_0 > 0$ such that $\partial N_f(\tilde{V} \setminus B_{R_0}(0)) \setminus B_{S_0}(0) \subset \partial\tilde{V}$. Since there are points $z \in \tilde{V}$ with arbitrarily large $|z|$ such that $N_f(z) \in \tilde{V}$, it follows that either $N_f(\tilde{V} \setminus B_R(0)) \subset \tilde{V}$ or $N_f(\tilde{V} \setminus B_R(0))$ contains a punctured neighborhood of ∞ within $\hat{\mathbb{C}}$.

In the first case, the claims follow easily. By way of contradiction, we may thus assume that we are in the second case.

We consider the situation in logarithmic coordinates: with an arbitrary but fixed choice of branch, let $C \subset \mathbb{H}_{\log(R)}$ be the unique unbounded component of $\log(\tilde{V} \setminus B_R(0))$. This is a closed set whose boundary consists of two analytic curves γ_1 and γ_2 and a subset of the vertical line at real part $\log(R)$. Define a holomorphic map $g : C \rightarrow \mathbb{C}$ by $g(z) = \log(N_f(e^z))$, choosing the branch such that $\gamma_1 \subset g(\gamma_1)$. This is possible because $\Gamma_1 = e^{\gamma_1}$ is N_f -invariant. Since e^{γ_2} is also N_f -invariant, there exists $k \in \mathbb{Z}$ such that with $\gamma_4 := g(\gamma_2)$, $\gamma_4 = \gamma_2 + 2\pi ik$. Define also $\gamma_3 := \gamma_1 + 2\pi ik$. Since $N_f(\tilde{V} \setminus B_R(0))$ contains a neighborhood of ∞ , we get $k \neq 0$. See Figure 2.3 for an illustration of the notations and note that $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are pairwise disjoint. These four curves have a natural vertical order induced by the observation that each curve separates sufficiently far right half planes into two unbounded components. To fix ideas, suppose that γ_2 is below γ_1 . Then γ_4 is below γ_3 . The construction implies that γ_4 is below γ_1 , and no curve is between γ_1 and γ_2 . Then, the vertical order is $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

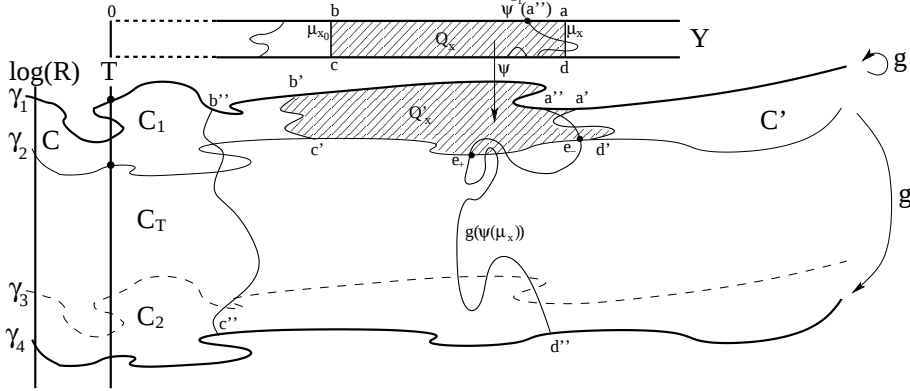


Figure 2.3: Illustrating the notations for Proposition 2.5.6.

By Lemma 2.5.4, g is continuous at $+\infty$. By the open mapping principle, there exists $T > \log(R)$ such that $\partial g(C) \cap \mathbb{H}_T \subset \gamma_1 \cup \gamma_4$. Let C_1 be the unique unbounded component of $C \cap \overline{\mathbb{H}}_T$, $C_2 = C_1 + 2\pi ik$ and $C_T := g(C) \cap \overline{\mathbb{H}}_T$. Note that $C_1 \cup C_2 \subset C_T$. We may choose T in such a way that $\partial \mathbb{H}_T$ does not contain any critical values of g . Define $C' = g^{-1}(C_T) \subset C$. Then, $g : C' \rightarrow C_T$ is a proper map and therefore has well-defined degree. Since g is injective on $\partial C'$ and has no pole, this degree is one and g is univalent.

The idea of the proof is as follows: the curves γ_2 and γ_3 subdivide C_T into three parts which are unbounded to the right. With an appropriate bound to the right, we obtain a large bounded quadrilateral consisting of three sub-quadrilaterals. Two of these sub-quadrilaterals, the upper and the lower ones, have moduli comparable to the modulus of the entire quadrilateral. This is a contradiction to the Grötzsch inequality if the right boundaries are sufficiently far out.

Define a homeomorphism $\psi : Y \rightarrow C_1$ that is biholomorphic on the interior and normalized so that it preserves the boundary vertex ∞ and the other two boundary vertices. We denote by $\mu_x \subset Y$ the vertical line segment at real part x . There exists an x_0 such that for $x \geq x_0$, $\psi(\mu_x) \subset C'$. For $x > x_0$, we denote by Q_x the rectangle in Y that is bounded by μ_{x_0} and μ_x . With vertices $a = (x, 1)$, $b = (x_0, 1)$, $c = (x_0, 0)$ and $d = (x, 0)$, its modulus is equal to $x - x_0$. We denote the vertices of its image $Q'_x := \psi(Q_x)$ by a', b', c', d' , respectively. Let $a'' = g(a')$ and $d'' = g(d')$. Since g and ψ are univalent, $\text{mod}(g(Q'_x)) = x - x_0$.

The curve $g(\psi(\mu_x))$ is a boundary curve of $g(Q'_x)$; it connects a'' and d'' within C_T . Let e_- be the intersection point $g(\psi(\mu_x)) \cap \gamma_2$ closest to a'' along $g(\varphi(\mu_x))$, and let e_+ be the intersection point furthest to the left along γ_2 . Let C'_1 be the bounded subdomain of C_1 bounded by $g(\psi(\mu_x))$ between a'' and e_- , viewed as a quadrilateral with vertices a'' and e_- and two more vertices on $\partial \mathbb{H}_T$. Similarly, let C_1'''' be the bounded subdomain of $\mathbb{C}$ bounded by $\partial \mathbb{H}_T$, γ_1 , the part of γ_2 to the left of e_+ , and the part of $g(\varphi(\mu_x))$ between a'' and e_+ , with right vertices a'' and e_+ . Finally, let $C_1'' := C'_1 \cup C_1''''$ with right vertices a'' and e_- , and let $C_1''' := C_1''$ but with right vertices a'' and e_+ (instead of a'' and e_-).

If $g(\psi(\mu_x))$ intersects γ_2 only once, then $e_- = e_+$ and $C_1' = C_1'' = C_1''' = C_1''''$. In general, the three domains C_1', C_1'', C_1''' may be different. However, we have $\text{mod}(C_1') \geq \text{mod}(C_1'') \geq \text{mod}(C_1''') \geq \text{mod}(C_1''')$: the first inequality holds because $C_1' \subset C_1''$, the second describes identical domains but with one boundary vertex moved, and the third follows again from the inclusion

$C_1'''' \subset C_1'''$, but this time the domain is extended on the “right” side of the domain, rather than on the “lower” side because the boundary vertex has moved.

Pulling back under ψ , we find that $\operatorname{Re}(\psi^{-1}(a'')) \leq \operatorname{Re}(a) = x$, because the map $\psi^{-1} \circ g \circ \psi$ repels points away from ∞ . By Lemma 2.5.5, it follows that $\operatorname{mod}(C_1''') \leq \operatorname{mod}(C_1'') \leq x + 1$.

Similar considerations on the left end of C_1 , as well as for C_2 , allow to subdivide $g(Q'_x)$ by a single curve segment of γ_2 and γ_3 into three sub-quadrilaterals, two of which have modulus at most $x + 1$. But the Grötzsch inequality implies that

$$\frac{1}{x - x_0} = \frac{1}{\operatorname{mod}(g(Q'_x))} \geq \frac{1}{x + 1} + \frac{1}{x + 1},$$

hence $x \leq 2x_0 + 1$ which is a contradiction for large x . $\square$

Proof of Theorem 2.5.1. In order to use Theorem 2.4.6, we construct an injective curve that surrounds an unbounded domain in $\tilde{V}$ such that the image of the curve does not intersect this domain. Consider a Riemann map $\varphi : \mathbb{D} \rightarrow U$ with $\varphi(0) = \xi$ and the induced dynamics $g = \varphi^{-1} \circ N_f \circ \varphi$ on $\mathbb{D}$. By the Remark after Proposition 2.3.5, the curves $\varphi^{-1}(\Gamma_1)$ and $\varphi^{-1}(\Gamma_2)$ land at points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$, and g extends to a neighborhood of ζ_1 and ζ_2 so that ζ_1 and ζ_2 become repelling fixed points. These fixed points have linearizing neighborhoods in which the curves $\varphi^{-1}(\Gamma_1)$, respectively $\varphi^{-1}(\Gamma_2)$, are straight lines in linearizing coordinates. If $0 < r < 1$ is large enough, these two curves intersect the circle at radius r only once and we can join them by a circle segment at radius r to an injective curve $\Gamma' \subset \mathbb{D}$ in such a way that $\Gamma := \varphi(\Gamma')$ separates V from ξ . Let W be the closure in $\hat{\mathbb{C}}$ of the connected component of $\mathbb{C} \setminus \Gamma$ that contains V (Figure 2.4). Note that no component of $N_f^{-1}(\Gamma)$ that intersects $\mathring{W}$ can leave W : in $\mathbb{D}$, any such component would have to intersect Γ' . But by the Schwarz Lemma, $g^{-1}(\Gamma')$ has greater absolute value than r everywhere and Γ' has only one g -preimage within the linearizing neighborhood of ζ_1 ; this preimage is contained in Γ' . The same is true at ζ_2 .

By Proposition 2.5.6, W contains an unbounded preimage component W' of itself such that the boundary $\partial W'$ is contained in $\Gamma_1 \cup \Gamma_2$ outside a sufficiently large disk. Make W' simply connected by filling in all bounded complementary components.

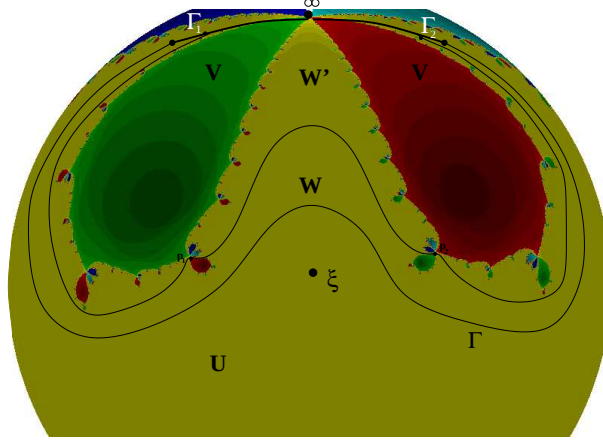


Figure 2.4: A schematic illustration of some notations in the proof of Theorem 2.5.1.

We claim that $\partial W'$ contains at least one finite pole on ∂U : if it did not, then $\partial W' \subset U$ and $N_f|_{\partial W'} : \partial W' \rightarrow \Gamma$ would be injective for all choices of r above. In the limit for $r \rightarrow 1$, this would imply that $N_f|_{\partial V}$ was injective, contradicting Proposition 2.5.3.

Therefore, $\partial W'$ maps onto Γ with covering degree at least $+2$. If ∞ is not an isolated fixed point in W' , we are done. Otherwise it is easy to see that $N_f|_{W'}$ is a Lefschetz map: there is a single boundary fixed point ∞ ; the conditions on this boundary fixed point are satisfied because $\Gamma_1, \Gamma_2 \subset U$, where the dynamics is expanding away from ∞ . Now Theorem 2.4.6 implies that W' contains fixed points of combined Lefschetz indices at least 2, because $\partial W'$ contains a pole. If W' contains a finite fixed point, we are done. If not, it follows that the fixed point at ∞ has Lefschetz index at least 2. Consider a Riemann map $\psi : W' \rightarrow \mathbb{H}^+$ that uniformizes W' to the upper half plane and maps ∞ to 0; this map preserves the Lefschetz index. By Proposition 2.5.6, the map $g = \psi \circ N_f \circ \psi^{-1}$ is defined in a relative neighborhood of 0 in $\mathbb{H}^+$. If a sequence converges to $\mathbb{R}$ in this neighborhood, then so will the image of this sequence. Hence we can extend g to a neighborhood of 0 in $\mathbb{C}$ by reflection. This extension does not reduce the Lefschetz index of 0: for a boundary fixed point, the index is defined by extending g to the lower half-plane in the way which generates the least possible fixed point index (compare Definition 2.4.2). Reflection however may increase the Lefschetz

index. Therefore, 0 is a parabolic (since multiple) fixed point of the extended map, and it is easily seen that $\partial\mathbb{H}^+$ is in the repelling direction. By the Fatou flower theorem [Mi, Theorem 10.5], 0 has an attracting petal in $\mathbb{H}^+$ that induces a virtual immediate basin inside V . $\square$

Chapter 3

Virtual Immediate Basins and Asymptotic Values

3.1 Introduction

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Newton's root finding method for f is implemented by iterating the associated *Newton map*

$$N_f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}, \quad z \mapsto z - \frac{f(z)}{f'(z)}.$$

It is well known that $\xi \in \mathbb{C}$ is a fixed point of N_f if and only if $f(\xi) = 0$. Furthermore, every finite fixed point ξ of N_f is attracting, so it has an invariant neighborhood on which N_f -orbits converge locally uniformly to ξ . In 2003, Douady raised the following question: if there exists a *virtual immediate basin* (an invariant, unbounded domain on which N_f -orbits converge locally uniformly to ∞), does this imply that ∞ is a 'virtual root' of f , in other words, does this imply that 0 is an *asymptotic value* of f ? In this chapter, we give a condition under which this is true. A recent result of Bergweiler, Drasin and Langley [BDL] implies that the condition is sharp when the Julia set of Newton maps is connected. Conversely, we show that if f has a singularity of logarithmic type over 0, then this singularity is contained in a virtual immediate basin of N_f ; if it is not of logarithmic type, then we provide counterexamples.

The dynamics of N_f partitions the Riemann sphere $\widehat{\mathbb{C}}$ into two completely invariant parts: the open *Fatou set* of all points at which the iterates

$\{N_f^{\circ n}\}_{n=0}^{\infty}$ are defined and form a normal family in the sense of Montel, and its complementary *Julia set* that contains the backward orbit of ∞ ; see [Be1, Mi] for an introduction to these concepts. Note that starting values in the Julia set will never converge to an attracting fixed point of N_f .

A component of the Fatou set of N_f for which no point converges to a root of f under iteration is either *wandering* or will eventually land on a cycle of *Böttcher domains*, *Leau domains*, *Siegel disks*, *Herman rings* or *Baker domains* (compare [Be1, Theorem 6]).

The possibilities become much more restricted when considering an *invariant* component U of the Fatou set, so that $N_f(U) \subset U$. In this case, it follows from Proposition 3.2.1 that U either contains a root of f , or is an invariant Herman ring or Baker domain.

Shishikura [Sh] has shown that if N_f is rational, then its Julia set is connected (see Proposition 3.2.5 for a characterization of rational Newton maps). Hence rational Newton maps have no Herman rings. In Corollary 2.4.7, we have shown that even transcendental Newton maps have no invariant Herman rings; hence, an invariant Fatou component of N_f either contains a root of f or is a virtual immediate basin (see Section 3.2 for the precise definition).

In this chapter, we continue the analysis of virtual immediate basins in [MS] and [RuS] (see also Chapter 2). We prove that if f has a logarithmic singularity over 0, then N_f has a virtual immediate basin (in 1994, Bergweiler, von Haeseler, Kriete, Meier and Terglane investigated a class of functions f that tend to 0 in a sector and showed that a right end of this sector is contained in a Baker domain of N_f [BHK, Theorem 3.3]).

For non-logarithmic singularities over 0, we give examples of functions whose Newton maps do not have a virtual immediate basin associated to these singularities.

Furthermore, we show that there are three classes of virtual immediate basins for N_f , two of which induce an asymptotic value at 0 for f . For the third class, this statement requires an additional assumption, without which it is false. Every such virtual immediate basin even has an open subset of starting values z_0 such that as $z_n = N_f^{\circ n}(z_0) \rightarrow \infty$, $f(z_n) \rightarrow 0$.

This chapter is structured as follows: in Section 3.2, we give a precise definition of virtual immediate basins and state several of their properties. In Section 3.3, we recall some fundamental notions concerning singular values. In Section 3.4, we prove that a logarithmic singularity over 0 for f induces a

virtual immediate basin for N_f , while the counterexamples for direct singularities are treated in Section 3.5. The converse theorem is stated and proved in Section 3.6. The underlying idea of the proof is to compare iterates of the Newton map $N_f = \text{id} - \frac{f}{f'}$ to the time 1 flow of $\dot{z} = -\frac{f(z)}{f'(z)}$.

3.2 Virtual Immediate Basins

The concept of a *virtual immediate basin* was introduced in [MS] to explain the behavior of Newton maps between different accesses to ∞ of an immediate basin. Examples of Newton maps having virtual immediate basins can be found in [MS, RuS]; these example are discussed in detail in [My]. The name was chosen to suggest that these domains behave in many ways similar to immediate basins.

The following proposition characterizes Newton maps of entire functions, see Proposition 2.2.8 for a proof.

Proposition 3.2.1 (Newton Maps). *Let $N : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function. It is the Newton map of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ if and only if for each fixed point $N(\xi) = \xi \in \mathbb{C}$, there exists a natural number $m > 0$ such that $N'(\xi) = \frac{m-1}{m} < 1$. In this case, there exists $c \neq 0$ such that*

$$f = c \cdot \exp \left(\int \frac{d\zeta}{\zeta - N(\zeta)} \right).$$

□

Note that while all definitions in this section are written in terms of Newton maps, they make sense for arbitrary meromorphic functions.

Definition 3.2.2 (Immediate Basin). *Let N_f be a Newton map. If ξ is an attracting fixed point of N_f , we call the open set*

$$\{z \in \mathbb{C} : \lim_{n \rightarrow \infty} N_f^{\circ n}(z) = \xi\}$$

its basin (of attraction). The component of the basin that contains ξ is called its immediate basin and denoted U_ξ .

For the definition of virtual immediate basins, we need the following concept.

Definition 3.2.3 (Absorbing Set). Let V be an N_f -invariant domain. A connected and simply connected open set $A \subset V$ is called a weakly absorbing set for V if $N_f(A) \subset A$ and for each compact $K \subset V$, there exists $k \in \mathbb{N}$ such that $N_f^{\circ k}(K) \subset A$.

We call A an absorbing set if it is weakly absorbing and additionally satisfies $N_f(\overline{A}) \subset A$, where the closure is taken in $\mathbb{C}$.

Definition 3.2.4 (Virtual Immediate Basin). A domain $V \subset \mathbb{C}$ is called a virtual immediate basin for N_f if it is maximal (among domains in $\mathbb{C}$) with respect to the following conditions:

1. for every $z \in V$, $\lim_{n \rightarrow \infty} N_f^{\circ n}(z) = \infty$;
2. V contains an absorbing set.

Every virtual immediate basin is unbounded, invariant and simply connected [MS, Theorem 3.4]. Since Newton maps of polynomials have a repelling fixed point at ∞ , virtual immediate basins can appear only for Newton maps of transcendental functions.

Proposition 3.2.5 (Rational Newton Map). (see Proposition 2.2.11.) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Its Newton map N_f is rational if and only if there exist polynomials p, q such that $f = p \cdot e^q$. In this case, ∞ is a repelling or parabolic fixed point.

More precisely, let $m := \deg p$ and $n := \deg q$. If $n = 0$ and $m \geq 2$, then ∞ is repelling with multiplier $\frac{m}{m-1}$. If $n > 0$, then ∞ is parabolic with multiplier $+1$ and multiplicity $n + 1 \geq 2$. $\square$

In the following, let f be a transcendental entire function. If N_f is rational, then it has virtual immediate basins which are the attracting petals of the parabolic fixed point at ∞ (see [Mi, Theorem 10.5]). If N_f is transcendental meromorphic, then any virtual immediate basin is (contained in) an invariant Baker domain.

Definition 3.2.6 (Baker Domain). Let B be an invariant component of the Fatou set of N_f . If $\lim_{n \rightarrow \infty} N_f^{\circ n}(z) = \infty \in \partial B$ for all $z \in B$ and N_f has an essential singularity at ∞ , then we call B a Baker domain of N_f .

If B is a simply connected Baker domain, it contains a weakly absorbing set A by a result of Cowen [Co, Theorem 3.2]. Using Cowen's work, it is easy to find an absorbing subset of A , hence B is a virtual immediate basin.

Moreover, Cowen's result implies that there are three dynamically defined classes of virtual immediate basins. The following notations are based on [Ko] and [BF].

Definition 3.2.7 (Conformal Conjugacy). *Let V be a virtual immediate basin of N_f and define $T(z) = z + 1$. If there exists a weakly absorbing set A for V , a T -invariant domain $\Omega \subset \mathbb{C}$ and a holomorphic map $\varphi : V \rightarrow \Omega$ such that*

$$\varphi \circ N_f(z) = T \circ \varphi(z)$$

for all $z \in V$, φ is univalent on A and $\varphi(A) \subset \Omega$ is a weakly absorbing set for $T|_{\Omega}$, then we call the triple (Ω, φ, T) a conformal conjugacy for N_f on V .

Definition 3.2.8 (Types of Virtual Immediate Basins). *Let V be a virtual immediate basin of N_f . We say that V is parabolic of type I if it has a conformal conjugacy (Ω, φ, T) such that $\Omega = \mathbb{C}$. It is parabolic of type II if there exists a conjugacy such that Ω is an upper or lower half-plane and hyperbolic with constant h if there exists $h > 0$ such that Ω is the strip*

$$S_h := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < h\}.$$

Theorem 3.2.9 (Classification of Virtual Immediate Basins). [Co, Theorem 3.2]. *Every virtual immediate basin V has a conformal conjugacy and is of exactly one of the three types defined above. If V is hyperbolic, the constant h is uniquely defined.* $\square$

REMARK. We believe that any Baker domain of a Newton map is simply connected; if this were proved, the notion of a virtual immediate basin would simply stand for either an attracting petal or a Baker domain, depending on whether the map under consideration is rational or not.

3.3 Asymptotic Values

We recall several important definitions concerning the singular values of a meromorphic map. Singular values play an important role in iteration theory, because their orbits determine the dynamics of a map in many ways.

We denote by $B_r(z)$ the open disk of radius $r > 0$ around $z \in \mathbb{C}$. In this section, let $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a meromorphic function.

Definition 3.3.1 (Regular and Singular Value). Let $a \in \mathbb{C}$ and assume that for $r > 0$, U_r is a connected component of $g^{-1}(B_r(a))$ such that $U_{r_1} \subset U_{r_2}$ if $r_1 < r_2$.¹ We have the following two cases:

1. If $\bigcap_{r>0} U_r = \{z\}$ for some $z \in \mathbb{C}$, then $g(z) = a$. If $g'(z) \neq 0$, then we call z a regular point of g . If $g'(z) = 0$, then z is called a critical point and a a critical value. In this case, we say that the critical point z lies over a .
2. If $\bigcap_{r>0} U_r = \emptyset$, then we say that $U : r \mapsto U_r$ defines a singularity of f^{-1} and we call a an asymptotic value. For simplicity, we call U a singularity and say it lies over a .

A singular value is an asymptotic or critical value. If no singularities or critical points lie over a point, we call it a regular value.

Note that there can be many different singularities as well as regular or critical points over any given point $a \in \mathbb{C}$.

For a rational map, all singular values are critical values. Asymptotic values of transcendental maps have a well-known characterization via paths.

Lemma 3.3.2 (Asymptotic Path). A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value of g if and only if there exists a path $\Gamma : (0, \infty) \rightarrow \mathbb{C}$ with $\lim_{t \rightarrow \infty} \Gamma(t) = \infty$ such that $\lim_{t \rightarrow \infty} g(\Gamma(t)) = a$. $\square$

We call Γ an *asymptotic path* of a . We follow [BE] in the classification of asymptotic values.

Definition 3.3.3 (Direct, Indirect and Logarithmic Singularity). Let U be a singularity of g^{-1} lying over $a \in \mathbb{C}$.

If $a \notin g(U_r)$ for some $r > 0$, then we call U a *direct singularity*. Otherwise, U is called an *indirect singularity*.

A direct singularity U over a is called *logarithmic* if $g : U_r \rightarrow B_r(a) \setminus \{a\}$ is a universal covering map for all sufficiently small r .

As an example, the positive real axis is an asymptotic path of 0 for the map $z \mapsto \sin(z)/z$. Since its image assumes this value infinitely many times, it is contained in an indirect singularity over 0. For $z \mapsto \exp z$, any left half plane is a logarithmic singularity over 0.

¹The function $U : r \mapsto U_r$ is completely determined by its germ at 0. Since $\bigcap_{r>0} U_r$ is connected, the intersection contains at most one point.

3.4 A Criterion for Virtual Immediate Basins

Our first result is the following.

Theorem 3.4.1 (Logarithmic Singularity Implies Virtual Immediate Basin). *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with a logarithmic singularity U over 0. Then there exists $r_0 > 0$ such that U_{r_0} is an absorbing set for a parabolic virtual immediate basin of type I for N_f .*

Note that if U is an indirect singularity, each U_r contains infinitely many roots of f and hence infinitely many attracting fixed points of N_f . Therefore, U_r cannot be part of a virtual immediate basin. In Section 3.5, we show that there exist functions $f : \mathbb{C} \rightarrow \mathbb{C}$ with a direct singularity U over 0 which does not induce a virtual immediate basin for N_f .

Proof. The idea is to compare the iterates of N_f to the time 1 flow of the differential equation $\dot{z} = -\frac{f(z)}{f'(z)}$. If r is small enough, this flow sends U_r isomorphically to $U_{r/e}$. We will see that for r small enough, N_f maps U_r univalently into itself and is an absorbing set for a virtual immediate basin of N_f .

First, let $r > 0$ be small enough so that $f : U_r \rightarrow B_r(0) \setminus \{0\}$ is a universal covering. Set $\eta := -\log r$ and $\mathbb{H}_\eta := \{w \in \mathbb{C} : \operatorname{Re}(w) > \eta\}$. Since $e^{-\operatorname{id}} : \mathbb{H}_\eta \rightarrow B_r(0) \setminus \{0\}$ is also a universal covering, the map $-\log(f) : U_r \rightarrow \mathbb{H}_\eta$ is biholomorphic with inverse $\psi : \mathbb{H}_\eta \rightarrow U_r$ (see Figure 3.1). With this, we get $\log(f(\psi(w))) = -w$ for $w \in \mathbb{H}_\eta$.

Taking derivatives yields

$$\frac{f'(\psi(w))}{f(\psi(w))} \cdot \psi'(w) = -1; \quad \text{hence} \quad \psi'(w) = -\frac{f(\psi(w))}{f'(\psi(w))}.$$

In other words, ψ is a solution of $\dot{z} = -\frac{f(z)}{f'(z)}$ and following the flow during time 1 maps $U_r = \psi(\mathbb{H}_\eta)$ to $U_{r/e} = \psi(\mathbb{H}_{\eta+1})$.

We now want to compare N_f to the time 1 flow of $\dot{z} = -\frac{f(z)}{f'(z)}$. We will do the comparison in time space: we will show that if $z = \psi(w)$ with $\operatorname{Re}(w)$ large enough, then $N_f(z) = \psi(w')$ with w' close to $w + 1$. More precisely, we have the following lemma.

Lemma 3.4.2. *There exists $\eta_0 > \eta$ and a holomorphic map $G : \mathbb{H}_{\eta_0} \rightarrow \mathbb{H}_{\eta_0+1/2}$ such that for all $w \in \mathbb{H}_{\eta_0}$, we have*

$$N_f \circ \psi(w) = \psi \circ G(w) \quad \text{and} \quad |G(w) - (w + 1)| < \frac{1}{2}.$$

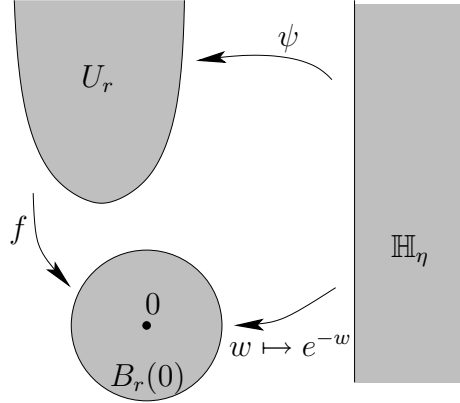


Figure 3.1: If $f : U_r \rightarrow B_r(0) \setminus \{0\}$ is a universal covering, there exists a biholomorphic map $\psi : \mathbb{H}_\eta \rightarrow U_r$.

The proof of Theorem 3.4.1 is then easily completed. Indeed, set $V_0 := \psi(\mathbb{H}_{\eta_0}) = U_{r_0}$ with $r_0 = e^{-\eta_0}$ and let V_{n+1} be the component of $N_f^{-1}(V_n)$ that contains V_0 . Since all points in $\mathbb{H}_{\eta_0}$ converge to ∞ under iteration of G (the real part increases by at least $1/2$ in each step), we conclude that $V := \bigcup_{n \in \mathbb{N}} V_n$ is a virtual immediate basin of N_f with absorbing set V_0 .

Let us now prove Lemma 3.4.2. Note that

$$N_f(\psi(w)) = \psi(w) - \frac{f(\psi(w))}{f'(\psi(w))} = \psi(w) + \psi'(w) .$$

Thus, it is equivalent to prove that there exists $\eta_0 > \eta$ and a holomorphic map $G : \mathbb{H}_{\eta_0} \rightarrow \mathbb{H}_{\eta_0+1/2}$ such that for all $w \in \mathbb{H}_{\eta_0}$, we have

$$\psi(w) + \psi'(w) = \psi(G(w)) \quad \text{and} \quad |G(w) - (w + 1)| < \frac{1}{2} . \quad (3.1)$$

Given $w \in \mathbb{H}_{\eta+2}$, define functions $g, h : B_2(w) \rightarrow \mathbb{C}$ by

$$g : \zeta \mapsto \frac{\psi(\zeta) - \psi(w) - \psi'(w)}{\psi'(w)} \quad \text{and} \quad h : \zeta \mapsto \zeta - (w + 1) .$$

Since g and h satisfy $g(w) = h(w) = -1$, $g'(w) = h'(w) = 1$ and can both be extended to all of $\mathbb{H}_\eta$ as univalent maps, by Koebe's distortion theorem there exists $\eta_0 > \eta + 2$ such that for every $w \in \mathbb{H}_{\eta_0}$ and every $\zeta \in B_2(w)$, $|g(\zeta) - h(\zeta)| < 1/4$.

Clearly, $h(w+1) = 0$. Note that $|h(\zeta)| = 1/2 > |g(\zeta) - h(\zeta)|$ when ζ belongs to the circle $\partial B_{1/2}(w+1)$. By Rouché's theorem, the map g has a (unique) root $\xi_w \in B_{1/2}(w+1)$. It is now easy to see that the map $G : \mathbb{H}_{\eta_0} \rightarrow \mathbb{C}$ defined by $G(w) = \xi_w$ satisfies equations (3.1). $\square$

3.5 A Direct Singularity Counterexample

In this section, we will exhibit examples of entire functions with direct singularities over 0 that do not induce Baker domains of the associated Newton maps. This shows that Theorem 3.4.1 cannot be improved much further; 0 is an omitted value in all examples, so that a generalization is not even possible to this case. We will only treat the first example in full detail.

For $\alpha \in]0, +\infty[$, consider the entire function f_α defined by

$$f_\alpha(Z) = \exp \left(-\frac{1}{\alpha} \left(Z + \frac{1}{2i\pi} e^{2i\pi Z} \right) \right) .$$

The function f_α has infinitely many singularities over 0 which are necessarily direct since f_α does not vanish. We have two kinds of asymptotic paths:

1. for $k \in \mathbb{Z}$, as $t \in \mathbb{R} \rightarrow +\infty$, $f_\alpha(k + \frac{1}{4} - it) \rightarrow 0$;
2. as $t \in \mathbb{R} \rightarrow +\infty$, $f_\alpha(t) \rightarrow 0$.

The singularities of the first kind are of logarithmic type. Thus, each one induces a Baker domain of parabolic type I for the Newton map

$$N_\alpha(Z) = Z + \frac{\alpha}{1 + e^{2i\pi Z}} .$$

The singularity of the second kind is not of logarithmic type and contains infinitely many critical points of f . We will see that for some values of α , it does not induce a Baker domain for N_α .

More precisely, observe that $N_\alpha(Z+1) = N_\alpha(Z) + 1$. It follows that we can study the dynamics of N_α modulo 1. In other words, we have

$$e^{2i\pi N_\alpha(Z)} = g_\alpha(e^{2i\pi Z}) \quad \text{with} \quad g_\alpha(z) = ze^{2i\pi\alpha/(1+z)} .$$

The map g_α has a fixed point with multiplier $e^{2i\pi\alpha}$ at $z = 0$, a fixed point with multiplier 1 at $z = \infty$ and an essential singularity at $z = -1$.

Let $\mathcal{F}(N_\alpha)$ and $\mathcal{F}(g_\alpha)$ be the Fatou sets of N_α and g_α and let $\pi : \mathbb{C} \rightarrow \mathbb{C}^*$ be the universal covering $\pi : Z \mapsto z = e^{2i\pi Z}$. We claim that

$$\mathcal{F}(N_\alpha) = \pi^{-1}(\mathcal{F}(g_\alpha)) .$$

It is easy to see that $\pi^{-1}(\mathcal{F}(g_\alpha)) \subset \mathcal{F}(N_\alpha)$ (see for example [Be3]). The inclusion $\mathcal{F}(N_\alpha) \subset \pi^{-1}(\mathcal{F}(g_\alpha))$ is less immediate. One may argue as follows. Assume $z_0 = \pi(Z_0) \notin \mathcal{F}(g_\alpha)$. Then, z_0 lies in the closure of the set of iterated g_α -preimages of -1 (otherwise, the family of iterates of g_α would be well defined near z_0 and avoid the infinite set $g_\alpha^{-1}(\{-1\})$, thus it would be normal). It follows that any neighborhood of Z_0 contains a preimage of a pole of N_α . Thus, $Z_0 \notin \mathcal{F}(N_\alpha)$.

As $z \rightarrow \infty$, we have

$$g_\alpha(z) = z + 1 + \frac{2i\pi\alpha}{z} + o(1/z) .$$

Thus, the parabolic fixed point at ∞ has multiplicity 2. It has a single attracting direction along the positive real axis. The full preimage of its parabolic basin under the map $e^{2i\pi Z}$ is the union of the Baker domains of N_α induced by the singularities of f_α of the first kind. The map g_α has exactly two critical points: the solutions to $(1+z)^2 - 2i\pi\alpha z = 0$.

Conjugating with $z \mapsto w = 1/(z+1)$, we may put the singularity at ∞ and the fixed points at 0 and 1. The map g_α is thus conjugate to the meromorphic function

$$h_\alpha(w) = \frac{w}{w + (1-w)e^{2i\pi\alpha w}} .$$

The map h_α has growth order 1 and two critical points. Thus, it has at most 2 asymptotic values by [BE, Corollary 3]. But as $t \in \mathbb{R} \rightarrow +\infty$, $h_\alpha(it) \rightarrow 0$ and $h_\alpha(-it) \rightarrow 1$. Thus, h_α has exactly 2 (fixed) asymptotic values and 2 critical values and is therefore a finite type map. It is well known that finite type meromorphic functions have neither wandering domains nor Baker domains [BKL, RS1].

The map h_α has a fully invariant parabolic point at 0 and for suitably chosen α , the fixed point at 1 is Cremer (in analogy to [Mi, Theorem 11.13]). We want to prove that in this case, the Fatou set of h_α consists of the parabolic basin at 0 and its preimage components. We deduce that then, the Fatou set of g_α is equal to the parabolic basin of ∞ and its preimage

components. Thus, every Fatou component of N_α maps after finitely many iterations into one of the invariant Baker domains induced by the first kind of singularities of f_α . There is no Fatou component associated to the second kind of singularity of f_α .

So it remains to show that h_α has no additional non-repelling periodic points nor Herman rings. While both claims follow directly from Epstein's version of the Fatou-Shishikura inequality for finite type maps [Ep1, Ep2, Ep3], we provide a version of Epstein's proof that is sufficient for our purposes; we treat Herman rings separately in Lemma 3.5.2.

Lemma 3.5.1 (Epstein). *There cannot be any additional non-repelling periodic points.*

Proof. Suppose that h_α has an additional non-repelling cycle

$$\{z_1 \mapsto z_2 \mapsto \dots, \mapsto z_k \mapsto z_1\} .$$

Let v_1 and v_2 be the two critical values of h_α , set

$$X = \{0, 1, z_1, \dots, z_k\}, \quad X' = X \cup \{v_1, v_2\} .$$

Let $\mathcal{Q}^1(X)$ (resp. $\mathcal{Q}^1(X')$) be the set of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ which are holomorphic outside X (resp. X') and have at most simple poles in X (resp. X'). Let $\mathcal{Q}^2(X)$ be the set of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ which are holomorphic outside X , have at most double poles in X and whose polar part of order 2 along X is of the form

$$A \frac{dz^2}{z^2} + B \frac{dz^2}{(z-1)^2} + C \sum_{i=1}^k \frac{dz^2}{(z-z_i)^2} \quad \text{with } A, B, C \in \mathbb{C} .$$

The sets $\mathcal{Q}^1(X)$, $\mathcal{Q}^1(X')$ and $\mathcal{Q}^2(X)$ are vector spaces of respective dimensions $k-3$, $k-1$ and k . We can define a linear map $\nabla : \mathcal{Q}^2(X) \rightarrow \mathcal{Q}^1(X')$ as follows. If U is a simply connected subset of $\widehat{\mathbb{C}} \setminus X'$, then $h_\alpha : h_\alpha^{-1}(U) \rightarrow U$ is a (trivial) covering map. We let $(g_i : U \rightarrow \widehat{\mathbb{C}})_{i \in I}$ be the countably many inverse branches and we set

$$(h_\alpha)_* q|_U = \left(\sum_{i \in I} g_i^* q \right) .$$

The sum is convergent because

$$\sum_{i \in I} \int_U |g_i^* q| = \int_{h_\alpha^{-1}(U)} |q| < \infty .$$

We can define in such a way a quadratic differential $(h_\alpha)_* q$ which is holomorphic outside X' . A local analysis shows that

$$\nabla q := (h_\alpha)_* q - q$$

has at most simple poles at points of X' and thus, belongs to $\mathcal{Q}^1(X')$.

Since the dimension of $\mathcal{Q}^1(X')$ is less than the dimension of $\mathcal{Q}^2(X)$, the linear map ∇ is not injective and there is a $q \in \mathcal{Q}^2(X)$ such that $\nabla q = 0$, i.e., $(h_\alpha)_* q = q$. To see that this is not possible, set

$$U_\varepsilon := D(0, \varepsilon) \cup D(1, \varepsilon) \cup \bigcup_{i=1}^k h_\alpha^{-i}(D(z_1, \varepsilon)) , \quad V_\varepsilon := h_\alpha^{-1}(U_\varepsilon) ,$$

let $W_\varepsilon \subset \widehat{\mathbb{C}} \setminus (U_\varepsilon \cup \{v_1, v_2\})$ be a simply connected subset of full measure and let $g_i : W_\varepsilon \rightarrow \widehat{\mathbb{C}}$ be the countably many inverse branches of h_α . Then, for ε sufficiently small, we have

$$\int_{\widehat{\mathbb{C}} \setminus U_\varepsilon} |(h_\alpha)_* q| = \int_{W_\varepsilon} \left| \sum_i g_i^* q \right| \leq \sum_i \int_{W_\varepsilon} |g_i^* q| = \int_{\mathbb{C} \setminus V_\varepsilon} |q|$$

with equality if and only if each $g_i^* q$ is a (real positive) multiple of $(h_\alpha)_* q = q$. In particular $q = h_\alpha^*(g_i^* q)$ has to be locally, and thus globally, a constant multiple of $h_\alpha^* q$, i.e. $q = c \cdot h_\alpha^* q$ for some constant $c > 0$. But in that case $g_i^* q = c \cdot q$ and the sum $\sum_i \int_{W_\varepsilon} |g_i^* q|$ will be diverging which is not the case.

Thus,

$$\int_{\widehat{\mathbb{C}} \setminus U_\varepsilon} |q| \leq \int_{\widehat{\mathbb{C}} \setminus V_\varepsilon} |q| - C_\varepsilon \quad \text{with } C_\varepsilon > 0 .$$

Note that for $\delta < \varepsilon$, we have

$$\begin{aligned} \int_{\widehat{\mathbb{C}} \setminus U_\delta} |q| &= \int_{\widehat{\mathbb{C}} \setminus U_\varepsilon} |q| + \int_{U_\varepsilon \setminus U_\delta} |q| \\ &\leq \int_{\widehat{\mathbb{C}} \setminus V_\varepsilon} |q| - C_\varepsilon + \int_{V_\varepsilon \setminus V_\delta} |q| \\ &= \int_{\widehat{\mathbb{C}} \setminus V_\delta} |q| - C_\varepsilon , \end{aligned}$$

thus

$$\int_{\widehat{\mathbb{C}} \setminus V_\delta} |q| - \int_{\widehat{\mathbb{C}} \setminus U_\delta} |q| \geq C_\varepsilon > 0.$$

We will obtain a contradiction by proving

$$\liminf_{\delta \rightarrow 0} \left(\int_{\widehat{\mathbb{C}} \setminus V_\delta} |q| - \int_{\widehat{\mathbb{C}} \setminus U_\delta} |q| \right) \leq 0.$$

This is the place where we use the fact that the cycle is non-repelling. As $\delta \rightarrow 0$, we can find a radius $r_\delta = \delta + o(\delta)$ such that

$$D(0, r_\delta) \cup D(1, r_\delta) \cup D(z_1, r_\delta) \cup \bigcup_{i=2}^k h_\alpha^{-i}(D(z_1, \delta)) \subset V_\delta.$$

Then, $U_\delta \setminus V_\delta$ is contained within the union of three annuli

$$\{z; r_\delta \leq |z| < \delta\} \cup \{z; r_\delta \leq |z-1| < \delta\} \cup \{z; r_\delta \leq |z-z_1| < \delta\}.$$

Since q has at most double poles at 0, 1 and z_1 , the integral of $|q|$ on those annuli tends to 0 as δ tends to 0 and we have

$$\int_{\widehat{\mathbb{C}} \setminus V_\delta} |q| - \int_{\widehat{\mathbb{C}} \setminus U_\delta} |q| = \int_{U_\delta \setminus V_\delta} |q| - \int_{V_\delta \setminus U_\delta} |q| \leq \int_{U_\delta \setminus V_\delta} |q| \xrightarrow{\delta \rightarrow 0} 0. \quad \square$$

Lemma 3.5.2. *There cannot be any cycle of Herman rings.*

Proof. Recall that 0 is a multiple fixed point and its immediate basin of attraction must contain a critical point ω_0 and the critical value $v_0 = h_\alpha(\omega_0)$. Also, 1 is a Cremer point. It must be accumulated by the orbit of the second critical point ω_1 with critical value $v_1 = h_\alpha(\omega_1)$.

Assume there is a cycle of Herman rings $H_1 \mapsto H_2 \mapsto \dots \mapsto H_k \mapsto H_1$. Let Γ be the union of the equators of the Herman rings H_i (Γ is the union of a cycle of Jordan curves). Choose a connected component W of $\widehat{\mathbb{C}} \setminus \Gamma$ which does not contain 1. Then, there are infinitely many iterates of v_1 contained in W (accumulating a boundary component of some Herman ring). In particular, there is an integer $m > 2$ such that $h_\alpha^{om}(v_1) \in W$. Let D be a disk around 1 avoiding Γ , the forward orbit of v_0 and the m first iterates of v_1 . Let D_{-1} be the connected component of $h_\alpha^{-1}(D)$ containing 1. Since $D \setminus \{1\}$ does not contain any singular value of h_α , $h_\alpha : D_{-1} \rightarrow D$ has to be an isomorphism. Since D_{-1} contains 1 and avoids Γ , it does not contain $h_\alpha^{om}(v_1)$. So, D_{-1}

is a disk avoiding Γ , the forward orbit of v_0 and the m first iterates of v_1 . We can therefore construct inductively a sequence of disks D_{-k} containing 1 such that $h_\alpha^{\circ k} : D_{-k} \rightarrow D$ is an isomorphism. Since $|(h_\alpha^{\circ k})'(1)| = 1$ for all $k \in \mathbb{N}$, by Koebe's one quarter theorem the disks D_{-k} contain a common neighborhood of 1 on which the iterates of h_α form a normal family. This contradicts the fact that 1 is a Cremer point contained in the Julia set of h_α . $\square$

Note that if we choose $\alpha \in \mathbb{Q}$, N_α will have a wandering domain that projects to a parabolic basin of a parabolic fixed point. If α is a Brjuno number, N_α will have a univalent Baker domain of parabolic type II which projects to a Siegel disk of g_α .

We can construct other examples in a similar way. The maps we will present do not have fixed points. It follows from Proposition 3.2.1 that they are Newton maps of non-vanishing entire functions, whose singularities over 0 are therefore direct.

Assume

$$N(Z) = Z + \frac{\alpha}{1 + \varepsilon \sin(2\pi Z)}$$

with

$$0 < \varepsilon < 1 \quad \text{and} \quad 0 < \alpha < m_\varepsilon = \left\lfloor \frac{(1 - \varepsilon)^2}{2\pi\varepsilon} \right\rfloor.$$

Then, N is the Newton map of an entire function f such that $f(t) \rightarrow 0$ as $t \in \mathbb{R} \rightarrow +\infty$. The restriction of N to $\mathbb{R}$ is an increasing homeomorphism which commutes with translation by 1. Indeed,

$$N'(Z) = 1 - \frac{2\pi\varepsilon\alpha \cos(2\pi Z)}{(1 + \varepsilon \sin(2\pi Z))^2} \geq 1 - \frac{2\pi\varepsilon\alpha}{(1 - \varepsilon)^2} > 0.$$

Thus, it has a well defined rotation number $\text{Rot}(N)$. This rotation number is positive since $N(Z) > Z$. Note that for $\alpha = m_\varepsilon$, $N(0) = m_\varepsilon$ and thus, $\text{Rot}(N) = m_\varepsilon$. For each fixed $\varepsilon \in (0, 1)$, the rotation number increases continuously from 0 to m_ε as α increases from 0 to m_ε . If $\text{Rot}(N)$ is rational, then N has a chain of wandering domains along the real axis. If $\text{Rot}(N)$ is a Brjuno number, N has a univalent Baker domain of hyperbolic type centered on the real axis. For suitably chosen parameters α , $\text{Rot}(N)$ is irrational and the induced map $N : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is topologically but not analytically conjugate to the rotation $Z \mapsto Z + \text{Rot}(N) : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$. It should follow

that N does not have any Baker domain associated to the singularity of f containing the large positive real numbers. The proof should be similar to the one we presented above: study the dynamics modulo 1.

In the previous examples, f had a direct singularity containing critical points of f . One may wonder whether it is the presence of critical points that prevents N_f from having a Baker domain associated to the singularity. The following example shows that this is not the case. We still assume $\alpha > 0$ and set

$$N_\alpha(Z) = Z + \alpha e^{2i\pi Z}.$$

Then, N_α does not have any fixed points. So, it is the Newton map of the non-vanishing entire function

$$f_\alpha(Z) = \exp\left(-\frac{1}{\alpha} \int_0^Z e^{-e^{2i\pi W}} dW\right).$$

Note that when $W \in \mathbb{R}$, the real part of $e^{-e^{2i\pi W}}$ is greater than $1/e$. Thus, for $\alpha > 0$ and for $t \in [0, +\infty)$, we have

$$|f_\alpha(t)| \leq e^{-t/(e\alpha)} \xrightarrow[t \rightarrow +\infty]{} 0.$$

The entire map f_α has a singularity over 0 containing large real numbers. This is a direct singularity since f_α does not vanish. In addition, N_α does not have poles and so, f_α does not have critical points.

Again, $N_\alpha(Z+1) = N_\alpha(Z) + 1$ and N_α projects via $Z \mapsto z = e^{2i\pi Z}$ to an entire map g_α fixing 0 with multiplier $e^{2i\pi\alpha}$:

$$g_\alpha(z) = ze^{2i\pi\alpha e^z}.$$

By a result of Bergweiler [Be3], the Fatou sets of N_α and g_α correspond under the map $Z \mapsto e^{2i\pi Z}$. If g_α has a Siegel disk around 0, the map N_α has a Baker domain of parabolic type II which corresponds to the singularity of f_α described above. But if g_α has a Cremer point at 0, there can be no Baker domain for N_α associated to the singularity of f_α described above.

3.6 Asymptotic Paths in Virtual Basins

Theorem 3.6.1 (Asymptotic Paths in Virtual Basins). *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function such that its Newton map N_f has a virtual immediate*

basin V . If V is parabolic of type I or type II, then 0 is an asymptotic value of f with asymptotic path in V . There exists $H > 0$ such that the same is true if V is hyperbolic with constant $h \geq H$.

Bergweiler, Drasin and Langley have constructed an entire function for which 0 is not an asymptotic value and whose Newton map has a virtual immediate basin of hyperbolic type [BDL]. Thus, the statement of Theorem 3.6.1 cannot be extended to all hyperbolic virtual immediate basins.

Using Theorem 3.6.1, we can give the following formulation of Theorem 2.5.1.

Corollary 3.6.2 (Outside Immediate Basins). *Let N_f be the Newton map of an entire function f and U_ξ the immediate basin of the attracting fixed point $\xi \in \mathbb{C}$ for N_f . Let $\Gamma_1, \Gamma_2 \subset U_\xi$ be two N_f -invariant curves connecting ξ to ∞ such that Γ_1 and Γ_2 are non-homotopic in U_ξ and let $\tilde{V}$ be an unbounded component of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. If the set $N_f^{-1}(\{z\}) \cap \tilde{V}$ is finite for all $z \in \hat{\mathbb{C}}$, then $f|_{\tilde{V}}$ assumes the value 0 or has 0 as an asymptotic value.*

Proof. If $0 \notin f(\tilde{V})$, then the virtual immediate basin constructed in the proof of Theorem 2.5.1 is parabolic of type I. $\square$

For the proof of Theorem 3.6.1, we will need the following corollary to the Koebe distortion theorem.

Lemma 3.6.3 (Bounded Non-Linearity). *Let $R > 0$, $g : B_R(0) \rightarrow \mathbb{C}$ be univalent and $\varepsilon > 0$. If r/R is sufficiently small, then*

$$\left| \frac{g(z) - g(w)}{g'(z)(z - w)} - 1 \right| < \varepsilon$$

for all $w, z \in B_r(0)$.

Proof. By possibly conjugating g with $z \mapsto Rz$, multiplying g with a constant or adding a constant to g , we may assume that $R = 1$, $g(0) = 0$ and $g'(0) = 1$. Fix $0 < r < 1$. By the Koebe distortion theorem, there is an $\alpha > 0$ independent of g such that

$$|g(z) - g(w) - (z - w)g'(z)| < \alpha|z - w|^2$$

for all $z, w \in B_r(0)$ (Taylor expansion around z). Moreover, there is a $\beta > 0$ so that $|g'(z)| > \beta$ for all $z \in B_r(0)$. This yields

$$\left| \frac{g(z) - g(w)}{g'(z)(z - w)} - 1 \right| < \alpha \left| \frac{z - w}{g'(z)} \right| < \frac{2\alpha r}{\beta}.$$

It follows from the Koebe distortion theorem that $\alpha \rightarrow 0$ and $\beta \rightarrow 1$ as $r \rightarrow 0$. The claim follows. $\square$

Proof of Theorem 3.6.1. Suppose first that V is parabolic of type I. Then, there exists a weakly absorbing set A of V and a conformal conjugacy $(\mathbb{C}, \varphi, T)$ such that $F := \varphi(A)$ is an absorbing set for $T : z \mapsto z + 1$ in $\mathbb{C}$. Since $\varphi|_A$ is univalent, it has a univalent inverse $\psi : F \rightarrow A$. With this, we get for $z \in F$ that $N_f(\psi(z)) = \psi(z + 1)$, and hence

$$\psi(z) - \frac{f(\psi(z))}{f'(\psi(z))} = \psi(z + 1).$$

It follows that

$$\frac{f'(\psi(z))}{f(\psi(z))} \cdot (\psi(z + 1) - \psi(z)) = -1 \quad (3.2)$$

(note that since V is a virtual immediate basin, f has no roots in $\psi(F)$). Let $0 < \varepsilon < 1$. By Lemma 3.6.3, there exists $R > 2$ such that if $B_R(z) \subset F$, then

$$\left| \frac{\psi'(z)}{\psi(z + 1) - \psi(z)} - 1 \right| < \varepsilon, \quad (3.3)$$

and by equation (3.2) and inequality (3.3) we get

$$\left| \frac{f'(\psi(z))}{f(\psi(z))} \cdot \psi'(z) + 1 \right| = \left| \frac{f'(\psi(z))}{f(\psi(z))} \cdot \psi'(z) \cdot \frac{\psi(z + 1) - \psi(z)}{\psi(z + 1) - \psi(z)} + 1 \right| < \varepsilon. \quad (3.4)$$

Since F contains all sufficiently far right translates of the disk $B_R(z_0)$, for every $z_0 \in F$ there exists $S_{z_0} \geq 0$ such that (3.4) holds for all $z_0 + t$ with real $t \geq S_{z_0}$.

Let $z_0 \in F$ such that $S_{z_0} = 0$. Then, for $t \geq 0$ and $z = z_0 + t \in F$, we use

a standard estimate in complex variables and inequality (3.4) to get

$$\begin{aligned}
|\log(f(\psi(z))) + z| &\leq \left| \int_{z_0}^z ((\log \circ f \circ \psi)'(\zeta) + 1) d\zeta \right| + |\log(f(\psi(z_0))) + z_0| \\
&\leq \sup_{w \in [z_0, z]} \left\{ \left| \frac{f'(\psi(w))}{f(\psi(w))} \cdot \psi'(w) + 1 \right| \right\} \cdot |z - z_0| + C' \\
&\leq \varepsilon \cdot |z - z_0| + C' \\
&\leq \varepsilon \cdot |z| + C,
\end{aligned}$$

where $C' = |\log(f(\psi(z_0))) + z_0|$ and $C > 0$ depend only on z_0 ; $[z_0, z]$ denotes the straight line segment in F connecting z_0 to z . It follows that $\log(f(\psi(z))) \in B_{\varepsilon|z|+C}(-z)$ and

$$\operatorname{Re}(\log(f(\psi(z)))) < -\operatorname{Re}(z) + \varepsilon|z| + C. \quad (3.5)$$

Since $\operatorname{Im}(z)$ does not depend on t , we have that $|z|/\operatorname{Re}(z) \rightarrow 1$ as $t \rightarrow \infty$ and the right hand side of inequality (3.5) converges to $-\infty$. Hence, exponentiating (3.5) yields $\lim_{t \rightarrow +\infty} f(\psi(z)) = 0$.

Analogous estimates hold for sufficiently large imaginary parts if V is parabolic of type II. If V is hyperbolic, sufficiently large h will permit a construction as above. This finishes the proof. $\square$

REMARK. In fact, we not only show the existence of an asymptotic path to 0 for f in V , but even that V has an N_f -invariant open subset in which f converges to 0 along N_f -orbits. This is another similarity between immediate basins and their virtual counterparts.

Chapter 4

A Combinatorial Classification of Postcritically Fixed Newton Maps of Polynomials

4.1 Introduction

In this chapter, we consider rational functions $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that appear as Newton's method of a complex polynomial. Newton's method of a linear or quadratic polynomial is easy to understand and we will exclude these cases from our investigation.

Definition 4.1.1 (Newton Map). *We call a rational function $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 3$ a Newton map if there exists a polynomial p of degree d with d distinct roots such that in $\mathbb{C}$, f has the form*

$$f(z) = z - \frac{p(z)}{p'(z)}.$$

Observe that f and p have the same degree if and only if p has d distinct roots, which is the only case that we will treat here. For a characterization of Newton maps of polynomials with possible multiple roots, see Corollary 2.2.9.

One of the most important open problems in rational dynamics is understanding the structure of the space of rational functions of a fixed degree $d \geq 2$. This problem is today wide open.

Aside from being a useful tool for numerical root-finding (see for example [HSS, Sch]), Newton maps form an interesting subset of the space of rational maps that is more accessible for studying than the full space of rational maps. So a partial goal in the classification of *all* rational maps is to gain an understanding of the space of Newton maps (of a given degree). Newton maps have a number of properties that help to study them.

Proposition 4.1.2 (Head’s Theorem). *A rational map $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 3$ is a Newton map if and only if $f(\infty) = \infty$ is a repelling fixed point and all other fixed points $\xi_1, \dots, \xi_d \in \mathbb{C}$ are superattracting. Then, for any complex $a \neq 0$, f is the Newton map of the polynomial*

$$p(z) = a \prod_{j=1}^d (z - \xi_j) .$$

A proof of this proposition can be found in [He, Proposition 2.1.2], see also Corollary 2.2.9.

Since ∞ is the only non-attracting fixed point of a Newton map, a result of Shishikura [Sh] implies the following.

Proposition 4.1.3 (Julia Set Connected). *The Julia set of a Newton map is always connected.* $\square$

A number of people have studied Newton maps and used combinatorial models to structure the parameter spaces of some Newton maps. Janet Head [He] introduced the so-called *Newton tree* to characterize many postcritically finite cubic Newton maps. Tan Lei [TL] built upon this thesis and gave a classification of all postcritically finite cubic Newton maps, see Figure 4.1. Jiaqi Luo [Lu] extended some of these results and gave a combinatorial classification of postcritically finite Newton maps of any degree with the property that they have only one free critical value.

In this chapter, we extend these results to postcritically finite Newton maps with the only restriction that all critical points map onto a fixed point after finitely many iterations. We call such maps *postcritically fixed*. More precisely, we introduce a combinatorial object that we call an *abstract Newton graph* and show that every abstract Newton graph is realized by a postcritically fixed Newton map (which is unique up to Möbius conjugation) and that vice versa, every postcritically fixed Newton map generates an abstract

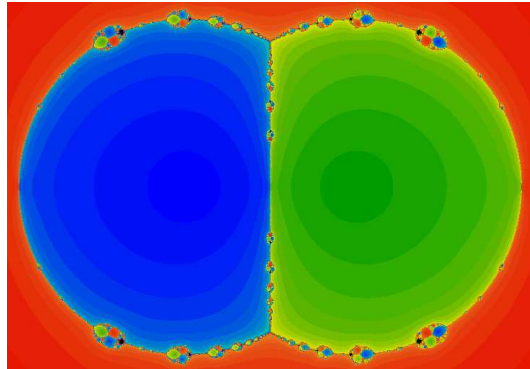


Figure 4.1: The parameter space of cubic Newton maps up to Möbius conjugation. The regions are colored depending on the fixed point to which the unique free critical value converges; every component of the red, green or blue set is a *hyperbolic component* and contains a unique postcritically finite parameter. The black regions are little Mandelbrot sets in which the free critical point converges to a nontrivial attracting cycle.

Newton graph. Hence we may say that the graphs classify these Newton maps.

It might be possible to extend our construction to a classification of *all* postcritically finite Newton maps by adding combinatorial information about those critical points that are not eventually fixed, see also Section 5.3. This would be an important step towards a classification of all rational maps.

Structure of the Chapter

This chapter is structured as follows. In Section 4.2, we introduce some aspects of Thurston theory, in particular Thurston equivalence, Thurston obstructions and Thurston's characterization of postcritically finite rational maps. We also give an introduction to the combinatorics of *arc systems* and state a result by Kevin Pilgrim and Tan Lei that restricts the possibilities of how arc systems and Thurston obstructions can intersect (Theorem 4.2.10). We will use this theorem in the proof of our main result.

In Section 4.3, we define abstract Newton graphs and prove that every such graph is realized by a postcritically fixed Newton map. In Section 4.4, we show how a postcritically fixed Newton map gives rise to an abstract

Newton graph.

Some Basic Notations

Let f be a Newton map of degree $d \geq 3$. We call a point $z \in \mathbb{C}$ *critical* if $f'(z) = 0$ (in this case, we do not need to make special conventions for ∞ because it can never be a critical point). It follows from the Riemann-Hurwitz formula [Mi, Theorem 7.2] that f has exactly $2d - 2$ critical points, counting multiplicities.

Definition 4.1.4 (Postcritically Fixed). *Let f be a Newton map of degree $d \geq 3$ with fixed points $\xi_1, \dots, \xi_d, \infty$ and critical points $c_1, \dots, c_{2d-2}$. Then, f is called postcritically finite if the set*

$$P_f := \bigcup_{i=1}^{2d-2} \bigcup_{n>0} \{f^{\circ n}(c_i)\}$$

is finite. We say that f is postcritically fixed if there exists $N \in \mathbb{N}$ such that for each $i \in \{1, \dots, 2d - 2\}$, $f^{\circ N}(c_i) \in \{\xi_1, \dots, \xi_d, \infty\}$.

Definition 4.1.5 (Immediate Basin). *Let f be a Newton map and $\xi \in \mathbb{C}$ a fixed point. Let $B_\xi = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} f^{\circ n}(z) = \xi\}$ the basin (of attraction) of ξ . The component of B_ξ containing ξ is called the immediate basin of ξ and denoted U_ξ .*

Clearly, B_ξ is open and by a theorem of Przytycki [Pr], U_ξ is simply connected and unbounded (in fact, by Proposition 4.1.3 every component of the Fatou set is simply connected). Moreover, $\infty \in \partial U_\xi$ is an accessible boundary point.

Definition 4.1.6 (Access to ∞). *Let U_ξ be the immediate basin of the fixed point $\xi \in \mathbb{C}$. Consider a curve $\Gamma : [0, \infty) \rightarrow U_\xi$ with $\Gamma(0) = \xi$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \infty$. Its homotopy class within U_ξ defines an access to ∞ for U_ξ , i.e., a curve Γ' with the same properties lies in the same access as Γ if they are homotopic in U_ξ with endpoints fixed.*

Proposition 4.1.7 (Accesses). (c.f. [HSS]) *Let f be a Newton map of degree $d \geq 3$ and U_ξ an immediate basin for f . Then, U_ξ contains $0 < k \leq d - 1$ critical points of f (counting multiplicities) and $f : U_\xi \rightarrow U_\xi$ is a covering map of degree $k + 1$. Furthermore, U_ξ has exactly k accesses to ∞ . $\square$*

4.2 Thurston Theory

In this section, we recall some fundamental notions of Thurston's characterization of rational maps. Thurston's theorem is a very deep and powerful method to prove the existence of a rational map with given combinatorics: it provides a necessary and sufficient condition on the existence of rational maps with a certain combinatorial behavior in terms of a collection of linear maps that are generated from a set of simple closed curves.

The notations and results in this section are based on [DH3] and [PT].

4.2.1 Thurston's Criterion For Marked Branched Coverings

Before we can state Thurston's criterion, we need several definitions. Recall that by the Riemann-Hurwitz formula [Mi, Theorem 7.2], a degree- d branched covering of $\mathbb{S}^2$ has $2d - 2$ branch points (counting multiplicities).

Definition 4.2.1 (Marked Branched Covering). *Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a branched covering map of degree $d \geq 2$ with branch points $c_1, \dots, c_{2d-2}$. If its postcritical set P_f is finite, then all branch points are (eventually) periodic and we call f postcritically finite.*

A marked branched covering is a pair (f, X) , where $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a postcritically finite branched covering map and X is a finite set containing P_f such that $f(X) \subset X$.

Definition 4.2.2 (Thurston Equivalence). *Let (f, X) and (g, Y) be two marked branched coverings. We say that they are (Thurston) equivalent and write $f \simeq g$, if there are two homeomorphisms $\varphi_0, \varphi_1 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that*

$$\varphi_0 \circ f = g \circ \varphi_1$$

and there exists an isotopy $\Phi : [0, 1] \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with $\Phi(0, \cdot) = \varphi_0$ and $\Phi(1, \cdot) = \varphi_1$ such that $\Phi(t, \cdot)|_X$ is constant in $t \in [0, 1]$ with $\Phi(t, X) = Y$.

If (f, X) is a marked branched covering and γ a simple closed curve in $\mathbb{S}^2 \setminus X$, then the set $f^{-1}(\gamma)$ is a disjoint union of simple closed curves.

Definition 4.2.3 (Multicurve). *Let (f, X) be a marked branched covering. We say that a simple closed curve $\gamma \subset \mathbb{S}^2$ is a simple closed curve in $(\mathbb{S}^2, X)$*

if $\gamma \subset \mathbb{S}^2 \setminus X$. It is called *peripheral* if there exists a component of $\mathbb{S}^2 \setminus \gamma$ that intersects X in at most one point, and *non-peripheral* otherwise.

Two simple closed curves γ_1, γ_2 in $(\mathbb{S}^2, X)$ are called *isotopic* (relative X) (write $\gamma_1 \simeq \gamma_2$) if there exists a continuous one-parameter family γ_t , $t \in [1, 2]$, of such curves joining γ_1 to γ_2 . We denote the isotopy class of γ_1 by $[\gamma_1]$.

A finite set $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of disjoint, non-peripheral and pairwise non-isotopic simple closed curves in $(\mathbb{S}^2, X)$ is called a *multicurve*.

Definition 4.2.4 (Irreducible Thurston Obstruction). Let (f, X) be a marked branched covering and Γ a multicurve. Denote by $\mathbb{R}^\Gamma$ the real vector space spanned by the isotopy classes of the curves in Γ . Then, we associate to Γ its Thurston transformation $f_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$ by specifying its action on representatives $\gamma \in \Gamma$ of basis elements:

$$f_\Gamma(\gamma) := \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\deg(f|_{\gamma'} : \gamma' \rightarrow \gamma)} [\gamma']. \quad (4.1)$$

The sum is taken to be zero if there are no preimage components isotopic to a curve in Γ .

The linear map given by equation (4.1) is represented by a square matrix with non-negative entries and thus its largest eigenvalue $\lambda(\Gamma)$ is non-negative and real by the Perron-Frobenius theorem.

A square matrix $A_{i,j} \in \mathbb{R}^{n \times n}$ is called *irreducible* if for each (i, j) , there exists $k \geq 0$ such that $(A^k)_{i,j} > 0$. We say that the multicurve Γ is *irreducible* if the matrix representing f_Γ is.

An irreducible multicurve Γ is called an *irreducible* (Thurston) *obstruction* if $\lambda(\Gamma) \geq 1$.

Definition 4.2.5 (Hyperbolic Orbifold). Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a postcritically finite branched covering map with postcritical set P_f . There exists a minimal function $\nu_f : \mathbb{S}^2 \rightarrow \mathbb{N} \cup \{\infty\}$ with the properties

1. $\nu_f(x) = 1$ if $x \notin P_f$;
2. $\nu_f(y) \cdot \deg_y(f) \mid \nu_f(x)$ for all $y \in f^{-1}(\{x\})$. Here, $\deg_y(f)$ denotes the local degree of f at y .

We say that f has *hyperbolic orbifold* if

$$2 - \sum_{x \in P_f} \left(1 - \frac{1}{\nu_f(x)} \right) < 0.$$

It is easy to see that a postcritically finite branched covering $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with at least three fixed branch points will have hyperbolic orbifold: the minimal orbifold weight on a fixed branch point is ∞ and hence three such points suffice to make its Euler characteristic negative. In general, f has hyperbolic orbifold if $\#P_f \geq 5$.

Now we are ready to state Thurston's theorem for marked branched coverings as given in [PT, Theorem 3.1] and proved in [DH3].

Theorem 4.2.6 (Marked Thurston Theorem). *Let (f, X) be a marked branched covering with hyperbolic orbifold. It is Thurston equivalent to a marked rational map (R, Y) if and only if it has no irreducible Thurston obstruction, i.e. for each irreducible multicurve Γ , $\lambda(\Gamma) < 1$. In this case, the rational map R is unique up to automorphism of $\widehat{\mathbb{C}}$. $\square$*

REMARK. Note that a marked rational map is in particular a rational map, “forgetting” the marked set Y .

4.2.2 Arcs Intersecting Obstructions

We present a theorem of Kevin Pilgrim and Tan Lei that is useful to show that certain marked branched coverings are equivalent to rational maps. Again, we first need to introduce some notation.

Let (f, X) be a marked branched covering of degree $d \geq 3$.

Definition 4.2.7 (Arc System). *An arc in $(\mathbb{S}^2, X)$ is a map $\alpha : [0, 1] \rightarrow \mathbb{S}^2$ such that $\alpha(\{0, 1\}) \subset X$, $\alpha((0, 1)) \cap X = \emptyset$ and $\alpha|_{(0, 1)}$ is injective. The notion of isotopy relative X extends to arcs and is also denoted by $\simeq$.*

A set of pairwise non-isotopic arcs in $(\mathbb{S}^2, X)$ is called an arc system. Two arc systems Λ, Λ' are isotopic if each curve in Λ is isotopic relative X to a unique element of Λ' and vice versa.

Note that arcs connect marked points (note that we do not require the endpoints to be distinct) while simple closed curves run around them. We will see that this leads to intersection properties that will give us some control over the location of possible Thurston obstructions. Since arcs and curves are only defined up to isotopy, we make precise what we mean by arcs and curves intersecting.

Definition 4.2.8 (Intersection Number). Let α and β each be an arc or a simple closed curve in $(\mathbb{S}^2, X)$. Their intersection number is

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \# \{(\alpha' \cap \beta') \setminus X\}.$$

This intersection number extends bilinearly to arc systems and multicurves, compare [ST, Section 3D].

If λ is an arc in $(\mathbb{S}^2, X)$, then the closure of a component of $f^{-1}(\lambda \setminus X)$ is called a *lift* of λ . Each arc clearly has d distinct lifts. If Λ is an arc system, an arc system $\tilde{\Lambda}$ is called a *lift* of Λ if each $\tilde{\lambda} \in \tilde{\Lambda}$ is a lift of some $\lambda \in \Lambda$.

An arc system Λ is (*forward*) *invariant* (up to isotopy relative X) if it contains a sub-system $\Lambda_0 \subset \Lambda$ and a lift $\tilde{\Lambda}_0$ such that $\tilde{\Lambda}_0 \simeq \Lambda$.

For a multicurve Γ , we introduce its *unweighted Thurston transformation* $f_{\#, \Gamma} : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$ by setting

$$f_{\#, \Gamma}(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} [\gamma'] ,$$

analogous to the definition of f_Γ . As for the usual Thurston linear transformation, this map is independent of the choice of representatives and commutes with iteration. Since $(f_\Gamma)_{i,j} \leq (f_{\#, \Gamma})_{i,j}$ and one is zero if and only if the other is, f_Γ is irreducible if and only if $f_{\#, \Gamma}$ is.

For an arc system Λ , introduce a linear map $f_{\#, \Lambda}$ in an analogous way and say that Λ is *irreducible* if $f_{\#, \Lambda}$ is. Denote by $\tilde{\Lambda}(f^{on})$ the union of those components of $f^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X , and define $\tilde{\Gamma}(f^{on})$ in an analogous way. Note that if Λ is irreducible, each element of Λ is isotopic to an element of $\tilde{\Lambda}(f^{on})$. We omit the easy proof of the following lemma.

Lemma 4.2.9. Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ a branched covering and $B \subset \mathbb{S}^2$ such that $f|_B : B \rightarrow f(B)$ is a k -to-1 mapping. Let A be any subset of $\mathbb{S}^2$. Then, $\#(f^{-1}(A) \cap B) = k \cdot \#(A \cap f(B))$. $\square$

The following theorem is Theorem 3.2 of [PT]. It says that up to isotopy relative X , a Thurston obstruction cannot intersect any preimage components of an irreducible arc system, except possibly for the arc system itself. For the sake of completeness, we provide a proof here.

Theorem 4.2.10 (Arcs Intersecting Obstructions). *Let (f, X) be a marked branched covering, Γ an irreducible Thurston obstruction and Λ an irreducible arc system. Suppose furthermore that $\#(\Gamma \cap \Lambda) = \Gamma \cdot \Lambda$. Then, exactly one of the following is true:*

1. $\Gamma \cdot \Lambda = 0$ and $\Gamma \cdot f^{-n}(\Lambda) = 0$ for all $n \geq 1$.
2. $\Gamma \cdot \Lambda \neq 0$ and for $n \geq 1$, each component of Γ is isotopic to a unique component of $\tilde{\Gamma}(f^{on})$. The mapping $f^{on} : \tilde{\Gamma}(f^{on}) \rightarrow \Gamma$ is a homeomorphism and $\tilde{\Gamma}(f^{on}) \cap (f^{-n}(\Lambda) \setminus \tilde{\Lambda}(f^{on})) = \emptyset$. The same is true when interchanging the roles of Γ and Λ .

Proof. We give the proof for $n = 1$ and write $\tilde{\Lambda}$ for $\tilde{\Lambda}(f^{o1})$. The proof generalizes easily to $n > 1$. Let $\tilde{\Lambda}' \subset \tilde{\Lambda}$ such that $\tilde{\Lambda}' \simeq \Lambda$. Then, $f : \tilde{\Lambda}' \setminus X \rightarrow \Lambda \setminus X$ is a homeomorphism, because $\Lambda \cap P_f = \emptyset$. From Definition 4.2.8, we immediately get for $\gamma_j \in \Gamma$

$$f_{\#, \Gamma}(\gamma_j) \cdot \Lambda \leq \#(f^{-1}(\gamma_j) \cap \tilde{\Lambda}') = \#(\gamma_j \cap \Lambda) = \gamma_j \cdot \Lambda, \quad (4.2)$$

where the last two steps follow by Lemma 4.2.9 and by hypothesis. Define coefficients b_{ij} by setting $f_{\#, \Gamma}(\gamma_j) = \sum_i b_{ij} \gamma_i$. By inequality (4.2), we get $(\sum_j \sum_i b_{ij} \gamma_i) \cdot \Lambda \leq \Gamma \cdot \Lambda$. On the other hand,

$$\left(\sum_j \sum_i b_{ij} \gamma_i \right) \cdot \Lambda = \left(\sum_i \left(\sum_j b_{ij} \right) \gamma_i \right) \cdot \Lambda \geq \sum_i \gamma_i \cdot \Lambda = \Gamma \cdot \Lambda, \quad (4.3)$$

because $b_{ij} \geq 1$ whenever $b_{ij} \neq 0$ and $f_{\#, \Gamma}$ has neither zero rows nor columns by irreducibility. Therefore, inequality (4.3) is actually an equality. If inequality (4.2) was strict, then there exists $\gamma' \subset f^{-1}(\gamma_j)$ such that $\#(\gamma' \cap \tilde{\Lambda}') > \#(\gamma_j \cap \Lambda)$, which contradicts the fact that $f : \tilde{\Lambda}' \setminus X \rightarrow \Lambda \setminus X$ is a homeomorphism. Hence, equation (4.2) is also an equality.

If $\Gamma \cdot \Lambda = 0$, let $\gamma_i \in \Gamma$. Then, there is $\gamma_j \in \Gamma$ and $\gamma' \subset f^{-1}(\gamma_j)$ such that $\gamma_i \simeq \gamma'$. By Definition 4.2.8 and Lemma 4.2.9, $\gamma_i \cdot f^{-1}(\Lambda) \leq \#(\gamma' \cap f^{-1}(\Lambda)) = k \cdot \#(\gamma_j \cap \Lambda) = 0$. This implies the first case of the claim.

Now consider the case $\Gamma \cdot \Lambda \neq 0$. By equation (4.3), for each i there exists a unique j such that $b_{ij} \neq 0$ and $b_{ij} = 1$. Thus for $\gamma_i \in \Gamma$, there is a unique j and a unique $\gamma' \subset f^{-1}(\gamma_j)$ such that $\gamma_i \simeq \gamma'$. Also, γ' is the unique preimage component of γ_j isotopic to an element of Γ . Equation (4.2) says that $f_{\#, \Gamma}(\gamma_j) \cdot \Lambda = \gamma_j \cdot \Lambda$. Since $f_{\#, \Gamma}(\gamma_j) = \gamma_i$, we get $\gamma_i \cdot \Lambda = \gamma_j \cdot \Lambda = \gamma' \cdot \Lambda$.

Clearly, $\gamma' \cdot \Lambda \leq \#(\gamma' \cap \tilde{\Lambda}')$, while $\gamma' \cap \tilde{\Lambda}' \subset f^{-1}(\gamma_j) \cap \tilde{\Lambda}'$. Hence, $\gamma' \cap \tilde{\Lambda}' = f^{-1}(\gamma_j) \cap \tilde{\Lambda}'$ and γ' is the unique curve in $f^{-1}(\gamma_j)$ intersecting $\tilde{\Lambda}'$. Since this holds for all j , $f^{-1}(\Gamma) \cap \tilde{\Lambda}' = \tilde{\Gamma} \cap \tilde{\Lambda}'$.

We have seen that the matrix f_Γ has exactly one non-zero entry of the form $1/k_j$, $k_j \in \mathbb{N}$, in each row and column and no non-zero entries on the diagonal. It is easy to check that the leading eigenvalue of f_Γ is $(k_1 \cdot \dots \cdot k_n)^{-\frac{1}{n}} \geq 1$ by hypothesis. It follows that $k_j = 1$ for all j and $f_\Gamma = f_{\#, \Gamma}$.

Repeating the above arguments with Γ and Λ interchanged, we find that $\tilde{\Lambda}' = \tilde{\Lambda}$. Hence, $\tilde{\Gamma} \cap f^{-1}(\Lambda) = \tilde{\Gamma} \cap \tilde{\Lambda} = f^{-1}(\Gamma) \cap \tilde{\Lambda}$ and $\#(\tilde{\Gamma} \cap \tilde{\Lambda}) = \#(\Gamma \cap \Lambda) = \Gamma \cdot \Lambda$. It follows that $\tilde{\Gamma} \cap (f^{-1}(\Lambda) \setminus \tilde{\Lambda}) = \emptyset$. $\square$

4.3 Combinatorial Models

4.3.1 The Channel Diagram

In the following, by a (*finite*) *graph* we mean a connected topological space Γ homeomorphic to the quotient of a finite disjoint union of closed arcs by an equivalence relation on the set of their endpoints. The arcs are called *edges* of the graph, an equivalence class of endpoints a *vertex*.

We usually consider imbedded graphs on $\mathbb{S}^2$, i.e. the homeomorphic image of a graph in $\mathbb{S}^2$. In the following, the closure and boundary operators will be understood with respect to the topology of $\hat{\mathbb{C}}$, unless otherwise stated. Also, we will say that a set $X \subset \hat{\mathbb{C}}$ is *bounded* if $\infty \notin \overline{X}$.

Lemma 4.3.1 (Only Critical Point). *Let f be a postcritically finite Newton map, $\xi \in \mathbb{C}$ a fixed point of f and U_ξ its immediate basin. Then, there is no critical point in U_ξ except ξ .*

Proof. Let $k \geq 2$ be the multiplicity of ξ as a critical point and suppose there is a critical point other than ξ in U_ξ . Then, by [Mi, Theorem 9.3] there exists a maximal $0 < r < 1$ and an open set $\xi \in V \subset \overline{V} \subset U_\xi$ such that the Böttcher map near ξ extends to a conformal isomorphism $\varphi : B_r(0) \rightarrow V$. Then, $\varphi \circ f(z) = \varphi(z)^k$ for $z \in V$ and there is a critical point $\zeta \in \partial V$. Let $z_l \in V$ be a sequence satisfying $z_l \rightarrow \zeta$. By assumption, there exists $n \in \mathbb{N}$ such that $f^{on}(\zeta) = \xi$. By continuity, we also get $\lim_{l \rightarrow \infty} f^{on}(z_l) = \xi$, a contradiction. $\square$

Hence, each immediate basin U_ξ has a global Böttcher map $\varphi_\xi : (\mathbb{D}, 0) \rightarrow (U_\xi, \xi)$ with the property that $f(\varphi_\xi(z)) = \varphi_\xi(z^{k_\xi})$ for each $z \in \mathbb{D}$ [Mi, Theorem 9.1 & 9.3], where $k_\xi \geq 2$ is the multiplicity of ξ as a critical point of f . The $k_\xi - 1$ radial lines (or *internal rays*) in $\mathbb{D}$ which are fixed under $z \mapsto z^{k_\xi}$ map under φ to $k_\xi - 1$ pairwise disjoint, non-homotopic injective curves $\Gamma_\xi^1, \dots, \Gamma_\xi^{k_\xi}$ in U_ξ that connect ξ to ∞ and are invariant under f . They represent all accesses to ∞ of U_ξ . The union

$$\Delta := \bigcup_{\xi} \bigcup_{i=1}^{k_\xi} \overline{\Gamma_\xi^i}$$

of these invariant curves over all immediate basins forms a connected graph in $\widehat{\mathbb{C}}$ that we call the *channel diagram* of f . The channel diagram records the mutual locations of the immediate basins of f , providing an elementary combinatorial structure to the dynamical plane. Figure 4.2 shows a Newton map and its channel diagram. The following definition is an axiomatization of the channel diagram.

Definition 4.3.2 (Abstract Channel Diagram). *An abstract channel diagram of degree d is a graph $\Delta \subset \mathbb{S}^2$ with vertices $v_0, \dots, v_d$ and edges $e_1, \dots, e_l$ that satisfies the following properties:*

1. $l \leq 2d - 2$;
2. each edge joins v_0 to a v_i , $i > 0$;
3. each v_i is connected to v_0 by at least one edge;
4. if e_i and e_j both join v_0 to v_k , then each connected component of $\mathbb{S}^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex.

We say that an abstract channel diagram Δ is *realized* if there exists a Newton map whose channel diagram is isotopic to Δ .

It is clear from the definition that a channel diagram has at most $2d - 2$ edges. If U_ξ is an immediate basin, it is shown in Corollary 2.5.2 that every component of $\mathbb{C} \setminus U_\xi$ contains at least one fixed point (see also Theorem 4.3.4). Hence every channel diagram is an abstract channel diagram.

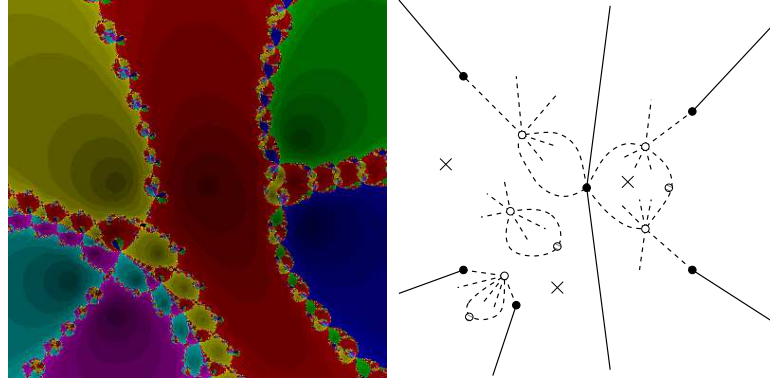


Figure 4.2: A Newton map of degree 6 with its channel diagram. The solid lines represent the fixed rays of the immediate basins, the black dots correspond to the fixed points. The dashed lines show the first preimage of the channel diagram: white circles represent poles, a cross is a free critical point. Clearly visible are the free pole, and that the right boundary component of the central immediate basin contains two poles.

4.3.2 Basic Properties of Channel Diagrams

The following useful observation is a direct consequence of Theorem 2.4.6. As always, f denotes a postcritically fixed Newton map (although the lemma is true in much more general situations).

Lemma 4.3.3 (Fixed Points). *Let $D \subset \widehat{\mathbb{C}}$ be a closed topological disk such that $\gamma := f(\partial D)$ is a simple closed curve with the property that $\gamma \cap \mathring{D} = \emptyset$. Let V be the unique component of $\widehat{\mathbb{C}} \setminus \gamma$ intersecting D and let $\{\gamma'_i\}_{i \in I}$ be the boundary components of $f^{-1}(V) \cap D$. Then, the number of fixed points in D equals*

$$\sum_{i \in I} \deg(f : \gamma'_i \rightarrow \gamma) ,$$

where the sum equals 0 if $I = \emptyset$. □

REMARK. Since f has no parabolic fixed points, we do not need to take multiplicities into account.

The following theorem shows a relation between poles and fixed points outside immediate basins. It considerably sharpens Corollary 2.5.2.

Theorem 4.3.4 (Fixed Points and Poles). *Let U_ξ be an immediate basin of f and V a component of $\mathbb{C} \setminus U_\xi$. Then, the number of fixed points in V equals the number of poles in V , counting multiplicities.*

Proof. If U_ξ does not separate the plane, i.e. it has only one access to ∞ , then the claim follows trivially. So suppose in the following that there is a Böttcher map $\varphi : (\mathbb{D}, 0) \rightarrow (U_\xi, \xi)$ with $f(\varphi(z)) = \varphi(z^k)$ for $z \in \mathbb{D}$ such that $k \geq 3$.

We are going to construct a simple closed curve Γ through ∞ that surrounds V and has a preimage component Γ'_1 with the same properties such that Γ does not intersect the component of $\widehat{\mathbb{C}} \setminus \Gamma'_1$ containing V . Then, we will use Lemma 4.3.3.

Let p be the number of poles in V and choose $0 < \rho < 1$. Let

$$\gamma := \{re^{2\pi i \frac{d}{k-1}} : r \geq \rho\} \cup \{re^{2\pi i \frac{d+1}{k-1}} : r \geq \rho\} \cup \{\rho e^{2\pi i \theta} : \frac{d}{k-1} \leq \theta \leq \frac{d+1}{k-1}\},$$

where d is chosen in the unique way such that $\Gamma := \varphi(\gamma) \cup \{\infty\}$ separates V from ξ . Let Γ'_1 be the component of $f^{-1}(\Gamma)$ containing ∞ . It is easy to see that $\varphi^{-1}(\Gamma'_1 \cap \mathbb{C})$ consists of exactly two connected components, hence Γ'_1 contains one pole on $\partial U_\xi \cap V$ and $\deg(f : \Gamma'_1 \rightarrow \Gamma) = 2$ (compare Figure 4.3). Let $\Gamma'_2, \dots, \Gamma'_m$ be the other preimage components of Γ in V . Clearly, $\deg(f : \Gamma'_i \rightarrow \Gamma)$ equals the number of poles on Γ'_i , counting multiplicities, for each i . Conversely, each pole in V is contained in some Γ'_i . Now it follows from Lemma 4.3.3 that $\overline{V} \subset \widehat{\mathbb{C}}$ contains $p + 1$ fixed points. Since the simple fixed point $\infty \in \partial V$ was included in the count, the claim follows. $\square$

REMARK. If f is not postcritically fixed, we cannot use a global Böttcher coordinate, but since the construction of γ is local, the proof should extend to the general case with little change.

Corollary 4.3.5 (Fixed Points in Complement). *Let Δ be the channel diagram of f and V a component of $\widehat{\mathbb{C}} \setminus \Delta$. If V contains p poles of f , then $\partial V \cap \mathbb{C}$ contains $p + 1$ fixed points.*

Proof. If V is the only component of $\widehat{\mathbb{C}} \setminus \Delta$, the claim follows trivially. If there is exactly one fixed point ξ on ∂V whose immediate basin U_ξ separates the plane, then the claim follows directly from Theorem 4.3.4. Indeed, let in this case R_1, R_2 be the fixed internal rays of U_ξ that are on ∂V and let V_1 be the component of $\mathbb{C} \setminus (R_1 \cup R_2 \cup \{\xi\})$ such that $V \subset V_1$. Then, V_1

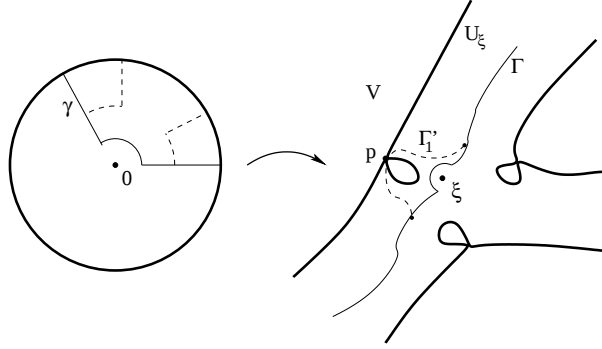


Figure 4.3: Illustrating the proof of Theorem 4.3.4: the construction of $\gamma \subset \mathbb{D}$ on the left (the dashed curves are components of $\varphi^{-1}(\Gamma'_1)$), the curve $\Gamma \subset U_\xi$ on the right. The dashed curve on the right indicates where Γ'_1 differs from Γ .

also contains p poles and by Theorem 4.3.4, V_1 contains p fixed points. Since $\xi \in \partial V$ as well, the claim follows.

Now suppose that ∂V contains fixed points $\xi_1, \dots, \xi_k$ whose immediate basins all separate the plane. Let R_1, R_2 be the fixed internal rays of U_{ξ_1} on ∂V and let V_1 be as above. Let m be the number of poles in V_1 and $m' = m - p > 0$. For $j = 2, \dots, k$, denote by $V_j^1, \dots, V_j^{i_j}$ all complementary components of the fixed internal rays of U_{ξ_j} that do not contain V . It is easy to see that all V_j^i combined contain m' poles and hence, $\partial V \cap \mathbb{C}$ contains $p + 1$ fixed points, including ξ_1 . $\square$

Corollary 4.3.6 (Existence of Shared Poles). *Let Δ be the channel diagram of f and V a component of $\widehat{\mathbb{C}} \setminus \Delta$. There is at least one pair of fixed points $\xi_1, \xi_2 \in \partial V \cap \mathbb{C}$ such that ∂U_{ξ_1} and ∂U_{ξ_2} intersect in a pole.*

Proof. Let U_ξ be an immediate basin. Clearly, the components of $\partial U_\xi \cap \mathbb{C}$ are separated by the accesses to ∞ . In Böttcher coordinates, there are prefixed internal rays in between any two fixed rays, and since all rational rays land, every component of $\partial U_\xi \cap \mathbb{C}$ contains at least one pole (Figure 4.2 shows that there can be more: the right boundary component of the central immediate basin contains two poles). By Corollary 4.3.5, there has to be at least one pole in V that is on the boundary of at least two immediate basins. If a pole was on the boundary of more than two immediate basins, then f

cannot preserve the cyclic order of the immediate basins near that pole, a contradiction. $\square$

4.3.3 Extending Maps on Finite Graphs

The channel diagram motivates the definition of a Newton graph. For this, we first need to introduce some notation regarding maps on imbedded graphs and their extensions to $\mathbb{S}^2$, compare [BFH, Chapter 6]. We assume in the following without explicit mention that all graphs are imbedded into $\mathbb{S}^2$.

Definition 4.3.7 (Graph Map). *Let $\Gamma_1, \Gamma_2 \subset \mathbb{S}^2$ be two finite graphs and $g : \Gamma_1 \rightarrow \Gamma_2$ continuous. We call g a graph map if it is injective on each edge of Γ_1 and forward and inverse images of vertices are vertices.*

An orientation-preserving branched covering map $\bar{g} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is called a regular extension of g if $\bar{g}|_{\Gamma_1} = g$ and $\bar{g}$ is injective on each component of $\mathbb{S}^2 \setminus \Gamma_1$.

Lemma 4.3.8 (Isotopic Graph Maps). *[BFH, Corollary 6.3] Let $g, h : \Gamma_1 \rightarrow \Gamma_2$ be two graph maps that coincide on the vertices of Γ_1 such that if $\gamma \subset \Gamma_1$ is an edge, then $g(\gamma) = h(\gamma)$ as a set. Suppose that g and h have regular extensions $\bar{g}, \bar{h} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Then there exists a homeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, isotopic to the identity relative the vertices of Γ_1 , such that $\bar{g} = \bar{h} \circ \psi$. $\square$*

Let $g : \Gamma_1 \rightarrow \Gamma_2$ be a graph map. For the next proposition, we will assume without loss of generality that each vertex v of Γ_1 has a neighborhood $U_v \subset \mathbb{S}^2$ such that all edges of Γ_1 that enter U_v terminate at v ; we may also assume that in local coordinates, U_v is a round disk of radius 1 centered at v , that all edges entering U_v are radial lines and that $g|_{U_v}$ is length-preserving. We make analogous assumptions for Γ_2 . Then, we can extend g to each U_v as in [BFH]: for a vertex $v \in \Gamma_1$, let γ_1 and γ_2 be two adjacent edges ending there. In local coordinates, these are radial lines at angles, say, θ_1, θ_2 such that $0 < \theta_2 - \theta_1 \leq 2\pi$ (if v is an endpoint of Γ_1 , then set $\theta_1 = 0, \theta_2 = 2\pi$). In the same way, choose arguments θ'_1, θ'_2 for the image edges in $U_{g(v)}$ and extend g to a map $\tilde{g}$ on $\Gamma_1 \cup \bigcup_v U_v$ by mapping

$$(\rho, \theta) \mapsto \left(\rho, \frac{\theta'_2 - \theta'_1}{\theta_2 - \theta_1} \cdot \theta \right),$$

where (ρ, θ) are polar coordinates in the sector bounded by the rays at θ_1 and θ_2 . In other words, sectors are mapped onto sectors in an orientation-preserving way. Then, the following holds.

Proposition 4.3.9 (Regular Extension). [BFH, Proposition 6.4] *The map $g : \Gamma_1 \rightarrow \Gamma_2$ has a regular extension if and only if for every vertex $y \in \Gamma_2$ and every component U of $\mathbb{S}^2 \setminus \Gamma_1$, the extension $\tilde{g}$ is injective on*

$$\bigcup_{v \in g^{-1}(\{y\})} U_v \cap U.$$

In this case, the regular extension $\bar{g}$ may have critical points only at the vertices of Γ_1 . $\square$

4.3.4 The Newton Graph

With these preparations, we are ready to introduce the concept of a Newton graph. It is a generalization of the *Newton tree* introduced by Janet Head [He] for postcritically finite cubic Newton maps and by Jiaqi Luo [Lu] for higher-degree postcritically finite Newton maps with only one free critical value.

Definition 4.3.10 (Abstract Newton Graph). *Let $\Gamma \subset \mathbb{S}^2$ be a connected graph and $g : \Gamma \rightarrow \Gamma$ a graph map. The pair (Γ, g) is called an abstract Newton graph if it satisfies the following conditions:*

1. *There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that g fixes each vertex and each edge of Δ .*
2. *If $v_0, \dots, v_{d_\Gamma}$ are the vertices of Δ , then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq 0$.*
3. *there are exactly $2d_\Gamma - 2$ vertices of Γ at which g is not injective, counting multiplicities.*
4. *There exists a minimal $N_\Gamma \in \mathbb{N}$ such that $g^{\circ N_\Gamma}(\Gamma) \subset \Delta$.*
5. *The graph $\overline{\Gamma \setminus \Delta}$ is connected.*
6. *For every vertex $y \in \Gamma$ and every component U of $\mathbb{S}^2 \setminus \Gamma$, the extension $\tilde{g}$ is injective on*

$$\bigcup_{v \in g^{-1}(\{y\})} U_v \cap U.$$

If (Γ, g) is an abstract Newton graph, g can be extended to a branched covering map $\bar{g} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ by Proposition 4.3.9. Condition (3) and the Riemann-Hurwitz formula ensure that $\bar{g}$ has degree d_Γ . An immediate consequence of Lemma 4.3.8 is that $\bar{g}$ is unique up to Thurston equivalence.

If Δ is the channel diagram of the postcritically fixed Newton map f , we denote by Δ_n the component of $f^{-n}(\Delta)$ that contains Δ and by Δ'_n the set of vertices of Δ_n . With these definitions, we can formulate the first part of our main result.

Theorem 4.3.11 (Realization). *Every abstract Newton graph is realized by a postcritically fixed Newton map which is unique up to automorphism of $\widehat{\mathbb{C}}$.*

More precisely, let (Γ, g) be an abstract Newton graph and X the set of vertices of Γ . Then, there exists a postcritically fixed Newton map f with channel diagram $\tilde{\Delta}$ and a subset $Y \subset \tilde{\Delta}'_{N_\Gamma}$ such that $(\bar{g}, X)$ and (f, Y) are Thurston equivalent as marked branched coverings.

Proof. Let $\Delta \subset \Gamma$ be the abstract channel diagram in Γ . Observe that by condition (2) of Definition 4.3.10, the vertices $v_1, \dots, v_{d_\Gamma}$ of Δ are branch points of $\bar{g}$. Since $d_\Gamma \geq 3$, $\bar{g}$ has hyperbolic orbifold and it suffices to show that $(\bar{g}, X)$ has no irreducible Thurston obstruction: it then follows from Theorem 4.2.6 that $\bar{g}$ is Thurston equivalent to a rational map f of degree d_Γ , which is unique up to Möbius transformation. Then, f has $d_\Gamma + 1$ fixed points, d_Γ of which are superattracting because $\bar{g}$ has the marked branch points $v_1, \dots, v_{d_\Gamma}$. The last fixed point is repelling [Mi, Corollary 12.7 & 14.5] and after possibly conjugating f with a Möbius transformation, we may assume that it is at ∞ . Now it follows from Proposition 4.1.2 that f is a Newton map.

So suppose by way of contradiction that Π is an irreducible Thurston obstruction for $(\bar{g}, X)$ and let $\gamma \in \Pi$. Then, γ is a non-peripheral simple closed curve in $\mathbb{S}^2 \setminus X$. It is easy to see that each edge λ of Δ forms an irreducible arc system, hence Theorem 4.2.10 implies that $\gamma \cdot (\bar{g}^{-n}(\lambda) \setminus \lambda) = 0$ for all $n \geq 1$. Since this is true for all edges of Δ , we get that $\gamma \cdot (\bar{\Gamma} \setminus \bar{\Delta}) = 0$. But since $\bar{\Gamma} \setminus \bar{\Delta}$ is connected and contains $X \setminus \{v_0\}$, this means that γ is peripheral, a contradiction. $\square$

4.4 The Channel Diagram of a Newton Map

In order to complete the classification of postcritically fixed Newton maps, it remains to prove the following theorem. Let Δ be the channel diagram of f and recall that Δ_n denotes the connected component of $f^{-n}(\Delta)$ containing Δ .

Theorem 4.4.1 (Newton Map Generates Newton Graph). *Every postcritically fixed Newton map f gives rise to an abstract Newton graph.*

More precisely, there exists $N \in \mathbb{N}$ such that (Δ_N, f) is an abstract Newton graph.

We will need the rest of this section to prove Theorem 4.4.1. It is clear that all (Δ_n, f) satisfy conditions (1), (2) and (4) of Definition 4.3.10. We will show that for sufficiently large N , Δ_N will contain all critical points of f , and hence satisfy (3) and (6). Then, condition (5) will follow from Lemma 4.4.5.

So it remains to show that all critical points of f will be contained in Δ_N for some $N \in \mathbb{N}$. First observe that Δ connects every fixed point of f to ∞ . Since f is postcritically fixed, each critical point of f is connected to some prepole by an iterated preimage of Δ . It thus suffices to show that for sufficiently large n , all poles of f are connected to ∞ through Δ_n ; the claim then follows by induction.

If Δ_1 contains all poles of f , then we are done. So assume in the following that there exists a component C_1 of $f^{-1}(\Delta)$ such that $C_1 \cap \Delta_1 = \emptyset$ (Figures 4.2 and 4.4 suggest that this may occur whenever $\deg(f) \geq 4$). Equivalently, we may assume that there exists a component V_0 of $\widehat{\mathbb{C}} \setminus \Delta$ and a component V_1 of $f^{-1}(V_0)$ such that V_1 is multiply connected. Then, C_1 intersects ∂V_1 .

Denote by C_n the component of $f^{-n}(\Delta)$ containing C_1 . Our standing assumption will be that $C_n \cap \Delta_n = \emptyset$ for all $n \in \mathbb{N}$ (otherwise we would be done). We will lead this assumption to a contradiction.

Lemma 4.4.2 (Properties of V_1). *V_1 is unbounded and satisfies $V_1 \subset V_0$.*

Proof. Since $\Delta \subset f^{-1}(\Delta)$, we either have $V_1 \subset V_0$ or $V_1 \cap V_0 = \emptyset$. In the latter case, let $\gamma \subset V_0$ be a simple closed curve that avoids all critical values and has the property that there exists a component γ' of $f^{-1}(\gamma) \cap V_1$ that is not contractible in V_1 (the image of a sufficiently large circle under any Riemann map of V_0 will do). Let D be the component of $\widehat{\mathbb{C}} \setminus \gamma'$ that does

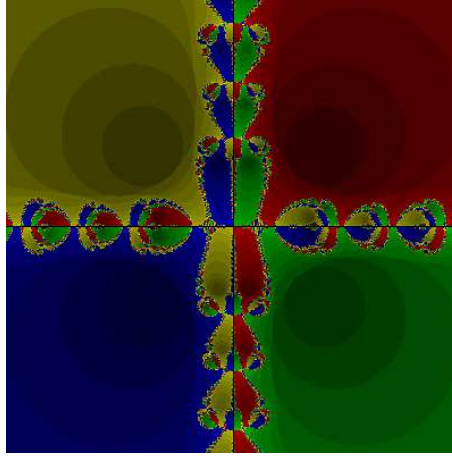


Figure 4.4: This Newton map of degree 4 has a pole which is not on any immediate basin boundary. The polynomial underlying the picture is a slight perturbation of $z \mapsto z^4 - 1$.

not intersect $\overline{V_0}$. By Lemma 4.3.3, $\overline{D}$ contains a fixed point of f . This is a contradiction, because $\overline{D}$ is separated from Δ .

Now suppose that V_1 is bounded. Then, either $\overline{V_1} \subset V_0$ or V_1 is contained in a bounded component W of $\widehat{\mathbb{C}} \setminus \Delta_1$. In the first case, we can construct a fixed point in $\overline{V_1}$ as before and arrive at a contradiction. In the second case, it follows from our standing assumption that $C_2 \subset W$ as well and since $f(C_2) = \Delta_1$, there exists a component $W' \subset \overline{W'} \subset W$ of $\widehat{\mathbb{C}} \setminus C_2$ such that $f(W') = W$. Again by Lemma 4.3.3 (or by observing that $f : W' \rightarrow W$ is polynomial-like), we arrive at the contradiction that $\overline{W'}$ must contain a fixed point outside Δ . $\square$

Corollary 4.4.3 (Free Pole in Unbounded Nest). *For each $n \in \mathbb{N}$, there exists an unbounded component V_n of $f^{-n}(V_0)$ such that $V_n \subset V_{n-1}$, $C_n \cap \partial V_n \neq \emptyset$ and $f : V_n \rightarrow V_{n-1}$ is a proper map of degree $d_n \geq 2$.*

Proof. Let $n \geq 2$. By induction on Lemma 4.4.2, we may suppose that V_{n-1} is unbounded. Recall that ∞ is fixed under f and has positive real multiplier, and that Δ_n is the only unbounded component of $f^{-n}(\Delta)$. Hence for each unbounded end of V_{n-1} , there is a component of $f^{-1}(V_{n-1})$ that contains this end. If the boundaries of all these components were disjoint from C_n , then

C_n would be contained in a bounded component of $\widehat{\mathbb{C}} \setminus \Delta_n$. As in Lemma 4.4.2, this leads to a contradiction.

Let V_n be the unbounded component of $f^{-1}(V_{n-1})$ whose boundary intersects C_n (by our standing assumption, this defines V_n uniquely). Clearly, $f : V_n \rightarrow V_{n-1}$ is proper and ∂V_n intersects at least two components of $f^{-1}(\Delta_{n-1})$: Δ_n and C_n . Hence $d_n \geq 2$. $\square$

The unbounded component of ∂V_n is clearly contained in Δ_n . We call this component the *outer boundary* of V_n and denote it with B_n . Now we are ready to prove the following.

Proposition 4.4.4 (All Poles in Δ_n). *There exists $n \in \mathbb{N}$ such that Δ_n contains all poles of f .*

Proof. The idea of the proof is that the disk bounded by B_n that contains C_n shrinks as $n \rightarrow \infty$, while C_n itself grows, yielding a contradiction. The proof will be in several steps.

1) First observe that while all V_n are unbounded, V_1 has strictly less unbounded ends than V_0 : Corollary 4.3.6 implies that there exist fixed points $\xi_1, \xi_2 \in \mathbb{C}$, a pole $p \in \mathbb{C}$ and internal rays $R_1^1 \in U_{\xi_1}, R_1^2 \in U_{\xi_2}$ such that $p \in X_1 := \overline{R_1^1} \cup \overline{R_1^2} \subset B_1$ is connected and separates some unbounded end of V_0 from V_1 .

2) Let X_n be the component of $f^{-n}(X_1) \cap V_n$ whose closure contains ξ_1 (since f is conjugate to some $z \mapsto z^k$ near ξ_1 , this defines X_n uniquely). If such a component does not exist, then ξ_1 must be separated from V_n by a similar “bridge” between two fixed points on B_n . In this case, replace X_n by this bridge and start the argument over. Since f has only finitely many fixed points, we may conclude that X_n exists as desired for all sufficiently large n , possibly after choosing new ξ_i finitely many times.

3) Let us show that $\xi_2 \in \overline{X_n}$ for at most finitely many n . Suppose this was not true. Recall that then, X_n consists of two internal rays $R_n^1 \in U_{\xi_1}, R_n^2 \in U_{\xi_2}$ such that $f^{o(n-1)}(R_n^i) = R_1^i$ and a continuum X'_n that satisfies $f^{o(n-1)}(X'_n) \subset X_1$ (X'_n is a prepole, unless p was a postcritical point. In that case, X'_n may consist of a bounded number of internal rays in some Fatou components of f . In either case, it follows from [Mi, Corollary 19.4] that $\text{diam}(X'_n) \rightarrow 0$ as $n \rightarrow \infty$). We know that the dynamics of f within the U_{ξ_i} is conjugate to some $z \mapsto z^k$; therefore, R_n^i converges to a fixed ray $R_0^i \subset \Delta \cap B_n$. Consider the unbounded component U_n of $\widehat{\mathbb{C}} \setminus \overline{R_0^1 \cup R_0^2 \cup X_n}$ that contains C_n . Now, $\widehat{\mathbb{C}} \setminus C_n$ has bounded components of some definite

area. On the other hand, $\text{area}(U_n) \rightarrow 0$, because its boundary converges to the arc $\overline{R_0^1 \cup R_0^2}$. This is a contradiction.

4) Hence, $\xi_2 \notin \overline{X_n}$ for all sufficiently large n . Then, the component Y_n of $f^{-n}(X_1) \cap V_n$ whose closure contains ξ_2 is disjoint from X_n . If these two sets are in the same component of $\Delta_n \setminus \Delta_{n-1}$, then let X_n be this component and show as in step (3) that this can only happen for finitely many n . Also, by steps (2) and (3), X_n and Y_n cannot connect to other fixed points except ξ_1, ξ_2 for infinitely many n .

5) Let $\{\xi_2^i\}$ be the points in $f^{-1}(\{\xi_2\}) \cap \overline{V_{n-1}}$, where n is chosen minimal such that $\overline{X_m} \cap \overline{B_m} = \{\xi_1\}$ for all $m \geq n$. Such an n exists by (4). Since $\overline{X_n}$ must contain some ξ_2^i , it follows that it connects a “free” pole to Δ_n . In other words, Δ_n contains at least one pole more than Δ_{n-1} . Since there is also another preimage of ξ_1 connected to this pole in $f^{-n}(\Delta)$ and preimages cannot just end, it follows that at time n , there exists a curve segment in B_n that connects ξ_1 to some finite fixed point in Δ_n through this “new” pole.

6) Repeating steps (2)–(5) with X_1 being this new connection, we can show that this will also have to break up and connect a new free pole to $\Delta_{n'}$ for some n' sufficiently large. Since f has only finitely many poles, we arrive at a contradiction to our standing assumption after finitely many steps. This finishes the proof. $\square$

In order to prove Theorem 4.4.1, it only remains to show that $\overline{\Delta_N \setminus \Delta}$ is connected if N is large enough such that Δ_{N-1} contains all critical points of f . Then, (Δ_N, f) will be a Newton graph of f . (We need to pull back this additional step to ensure that the critical points of f are actually branch points of f on Δ_N .)

Lemma 4.4.5 (Newton Graph Connected in $\mathbb{C}$). *Let $N \in \mathbb{N}$ minimal such that Δ_{N-1} contains all critical points of f . Then, $\overline{\Delta_N \setminus \Delta}$ is connected.*

Proof. Suppose that the bounded set $\overline{\Delta_{N-1} \setminus \Delta}$ is not connected (otherwise, we are done). Then, there exists an unbounded component V of $\widehat{\mathbb{C}} \setminus \Delta_{N-1}$ that separates the plane, i.e. V has at least two accesses to ∞ . Let W be a neighborhood of ∞ and let $V_1, \dots, V_k$ be the components of $V \cap W$. If W is sufficiently small, f acts injectively on each V_i and there exists a branch g_i of f^{-1} that maps V_i into itself (recall that ∞ is a repelling fixed point of f , so it is attracting for the g_i). By assumption, V is simply connected and contains no critical values of f , so the g_i extend to all of V by holomorphic continuation on lines. Since $\Delta_{N-1} \subset \Delta_N$, we get $g_i(V) \subset V$ for all i . If there

are $i \neq j$ such that $g_i(V) \cap g_j(V) \neq \emptyset$, then it follows that $g_i = g_j$ and we have found a holomorphic self-map of V with two attracting boundary fixed points, contradicting the Denjoy-Wolff theorem [Mi, Theorem 5.4].

Hence, the $g_i(V)$ are pairwise disjoint and if $w \in \partial g_i(V)$ for some i , then $f(w) \in \partial V$, for otherwise the map g_i would be defined in a neighborhood of $f(w)$. Hence $w \in f^{-1}(\Delta_{N-1})$ and since all $g_i(V)$ are open disks, we even get that $w \in \Delta_N$. It follows that no component of $\widehat{\mathbb{C}} \setminus \Delta_N$ has more than one access to ∞ . $\square$

Chapter 5

Outlook

In this chapter, we discuss some questions that arise naturally from the research outlined in Chapters 2–4. These questions show possible directions for new research, and might lead to a more complete understanding of the dynamics of Newton maps.

5.1 Root Finding

While we have not directly discussed effective root finding with Newton’s method, this problem has always been a motivation for our work. Building on the results of Hubbard, Schleicher and Sutherland [HSS], we ask if it is possible to construct universal sets of starting values for more general Newton maps. More precisely, let $f = pe^h$, where p and h are polynomials. Then, we have seen in Proposition 2.2.11 that the Newton map N_f is rational. Assume that p is normalized in such a way that all roots of p are contained in $\mathbb{D}$.

Question 5.1.1 (Global Starting Points). *Can one explicitly construct a finite set $\mathcal{S}'_d$ that intersects all immediate basins for all maps N_f , where f has the above form and $\deg p = d$?*

A more bold version of this question would allow h to be transcendental. In this case, N_f is transcendental with finitely many fixed points, all of which are contained in $\mathbb{D}$.

In the case of general transcendental Newton maps with infinitely many fixed points, this question would be ill-posed. But locally, the question still makes sense.

Question 5.1.2 (Local Starting Points). *Let f be an entire function and N_f its Newton map. Suppose that $\gamma \subset \mathbb{C}$ is a (smooth) Jordan curve that is disjoint from all roots of f , and let $n > 0$ be the number of roots of f in the bounded component U of $\mathbb{C} \setminus \gamma$.*

Is it possible to construct a finite set $\mathcal{S} \subset \gamma$ that depends only on γ and n , but not on N_f , such that $\mathcal{S}$ intersects all immediate basins of the fixed points in U ?

Even under additional hypotheses, progress in this direction might be useful in making the notion of a Newton map local. This may lead to a theory of *Newton-like* mappings in the spirit of polynomial-like mappings [DH2].

5.2 Virtual Immediate Basins

The assumption on the degree of N_f within V in Theorem 2.5.1 was made for purely technical reasons. Hence it is natural to ask the following.

Question 5.2.1 (Infinite Degree). *Using the notation of Theorem 2.5.1, is it true that V always contains an immediate basin or a virtual immediate basin, even if there exists $z \in \widehat{\mathbb{C}}$ such that $N_f^{-1}(\{z\}) \cap V$ is an infinite set?*

Alternatively, does there exist a transcendental Newton map N and a set V as above that contains neither an immediate nor a virtual immediate basin?

Our Lefschetz-type fixed point estimate 2.4.6 allowed us to show that a Newton map has no invariant Herman rings (Corollary 2.4.7). Ruling out non-trivial cycles of Herman rings seems less straightforward. Taixes (personal communication) has recently accomplished this using methods of holomorphic surgery.

Question 5.2.2 (Herman Rings). *Is it possible to prove the non-existence of cycles of Herman rings for Newton maps with the methods of Section 2.4?*

Assuming that Shishikura's theorem [Sh] extends to the transcendental case and that the Julia set of a Newton map is indeed always connected, we ask if there is a proof of this fact that does not use surgery.

Our last question in this section aims to strengthen Theorem 3.6.1.

Question 5.2.3 (Hyperbolic Virtual Basins). *What is the numerical value (or an upper bound thereof) of the constant H in Theorem 3.6.1?*

5.3 Classification of Newton Maps

For the construction of the Newton graph in Chapter 4, we have only shown that (Δ_n, f) is an abstract Newton graph if n was sufficiently large.

Question 5.3.1 (Number of Pull-Backs). *With the notation of Section 4.4, what is the minimum number of pull-backs needed to ensure that all critical points of f are contained in Δ_n ?*

A very natural extension of our results in Chapter 4 and a major breakthrough in the theory of rational dynamics would be a classification of *all* postcritically finite Newton maps. This might need a combination of the theory of Newton graphs with the theory of polynomial-like mappings [DH2] and their Hubbard trees.

Question 5.3.2 (Classification of Newton Maps). *Is it possible to obtain a combinatorial classification of all postcritically finite Newton maps (i.e. including preperiodic and periodic dynamics of the free critical points), e.g. by combining the Newton graph with several Hubbard trees?*

A positive answer to that question would yield a description of all hyperbolic components of the space of Newton maps (of any given degree). Then, one could extend the successful polynomial theory and investigate local connectivity of the Newton-bifurcation locus, density of hyperbolicity, etc.

Beyond that, one might try to generalize the idea of a channel diagram so that it makes sense for transcendental maps with infinitely many fixed points.

Question 5.3.3 (Transcendental Classification). *Is there a combinatorial object that can be used to classify transcendental Newton maps or some reasonable subclass thereof?*

Appendix A

On Questions of Fatou and Eremenko Concerning Escaping Points of Entire Functions

This chapter presents an excerpt of the results proved in [RRRS]. We show that for a large class of bounded-type entire functions, in particular those of finite order, every escaping point can be connected to ∞ by a curve of escaping points. This gives a partial positive answer to a question of Eremenko, and answers a question of Fatou from 1926.

A.1 Introduction

The dynamical study of transcendental entire functions was initiated by Fatou in 1926 [Fa]. As well as being a fascinating field of its own, the topic has recently received increasing interest partly because transcendental phenomena seem to be deeply linked with the behavior of polynomials in cases where the degree gets large.

In his seminal 1926 article, Fatou observed that the Julia sets of certain explicit entire functions, such as $z \mapsto r \sin(z)$, $r \in \mathbb{R}$, contain curves of points which escape to infinity under iteration. He then remarks

Il serait intéressant de rechercher si cette propriété n'appartiendrait pas à des substitutions beaucoup plus générales.¹

¹“It would be interesting to study whether this property holds for much more general

Sixty years later, Eremenko [Er2] was the first to undertake a thorough study of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : |f^n(z)| \rightarrow \infty\}$$

of an arbitrary entire transcendental function. In particular, he showed that every component of $\overline{I(f)}$ is unbounded, and asks whether in fact each component of $I(f)$ is unbounded (we will call this problem *Eremenko's conjecture*, or more precisely, the *weak form* of Eremenko's conjecture). He also states that

It is plausible that the set $I(f)$ always has the following property: every point $z \in I(f)$ can be joined with ∞ by a curve in $I(f)$.

This can be seen as making Fatou's original question more precise, and will be referred to in the following as the *strong form* of Eremenko's conjecture.

These problems are of particular importance since the existence of such curves can be used to study entire functions using combinatorial methods. This is analogous to the notion of "dynamic rays" of polynomials introduced by Douady and Hubbard [DH1], which has proved to be one of the fundamental tools for the successful study of polynomial dynamics.

We show that, in general, the answer to Fatou's question (and thus also to Eremenko's conjecture in its strong form) is negative, even when restricted to the class $\mathcal{B}$ of entire functions with a bounded set of singular values. For such functions, all escaping points lie in the Julia set.

Theorem A.1.1 (Entire Functions without Rays). *There exists a hyperbolic entire function $f \in \mathcal{B}$ such that every path component of $J(f)$ is bounded.*

For a proof of this theorem, we refer to [RRRS]. On the other hand, we show that the strong form of Eremenko's conjecture does hold for a large subclass of $\mathcal{B}$. Recall that f has *finite order* if $\log \log |f(z)| = O(\log |z|)$ as $|z| \rightarrow \infty$.

Theorem A.1.2 (Entire Functions With Rays). *Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to ∞ by a curve γ such that $f^n|_\gamma \rightarrow \infty$ uniformly.*

functions."

REMARK. Observe that while $\mathcal{B}$ is invariant under finite compositions, the property of having finite order is not.

We remark that our methods are purely local in that they only use the existence of a logarithmic singularity over ∞ . Therefore, they will apply to *any* function having a logarithmic singularity over ∞ .

A.1.1 Previous Results

In the early 1980s, Devaney gave a complete description of the Julia set of the map $z \mapsto \lambda \exp(z)$ with $\lambda \in (0, 1/e)$; i.e., real exponential maps with an attracting fixed point (see [DK]). This seems to have been the first entire function for which it was discovered that the escaping set (and in fact the Julia set) consists of curves to ∞ . Devaney, Goldberg and Hubbard [DGH] proved the existence of some curves to ∞ in $I(f)$ for arbitrary exponential maps $z \mapsto \lambda \exp(z)$ and championed the idea that these should be thought of as analogs of dynamic rays for polynomials. Devaney and Tangerman [DT] proved a similar result for a subclass of $\mathcal{B}$ consisting of functions whose *tracts* (see Section A.2) behave in essentially the same way as those of the exponential map. It seems that it was partly these developments that led Eremenko to pose the abovementioned questions in his 1989 paper.

In [SZ], it was shown that every escaping point of every exponential map can be connected to ∞ by a curve consisting of escaping points. This seems to have been the first time that a complete classification of all escaping points, and with it a positive answer to either of Eremenko's questions, was given for a complete family of transcendental functions. This result was carried over to the cosine family $z \mapsto a \exp(z) + b \exp(-z)$ in [RoS].

A very interesting and surprising case in which the weak form of Eremenko's question has a positive answer was discovered by Rippon and Stallard [RS2]. For the case of an entire function with a multiply-connected wandering domain, they show that $I(f)$ consists of a single and unbounded connected component. Such a function, however, is not in class $\mathcal{B}$. Moreover, they showed that for any transcendental entire function, the subset $A(f) \subset I(f)$ introduced by Bergweiler and Hinkkanen [BH] has only unbounded components.

Notation

Throughout this chapter, we denote the Riemann sphere by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the right half plane by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also, we write

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\} \quad \text{and} \quad \mathbb{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}.$$

If $A \subset \mathbb{C}$, the closures of A in $\mathbb{C}$ and $\widehat{\mathbb{C}}$ are denoted $\overline{A}$ and $\widehat{A}$, respectively.

Euclidean length and distance are denoted ℓ and dist , respectively. If a domain $V \subset \mathbb{C}$ omits at least two points of the plane, we similarly denote hyperbolic length and distance in V by ℓ_V and dist_V .

A.2 Bounded-Type Entire Functions

A.2.1 Tracts

Let $f \in \mathcal{B}$, and let R_0 be sufficiently large ($R_0 > |f(0)| + \sup\{|s| : s \in \operatorname{sing}(f^{-1})\}$ will suffice). Setting $W_{R_0} := \{z \in \mathbb{C} : |z| > R_0\}$, it is easy to see that every component V of

$$\mathcal{V} := f^{-1}(W_{R_0})$$

is an unbounded Jordan domain, and that $f : V \rightarrow W_{R_0}$ is a universal covering. (In other words, f has only logarithmic singularities over ∞ .) The components of $\mathcal{V}$ are called the *tracts* of f . Observe that each compact $K \subset \mathbb{C}$ will intersect at most finitely many tracts of f .

A.2.2 Logarithmic Coordinates

To study logarithmic singularities, it is natural to apply a logarithmic change of coordinates (compare [EL, Section 2]). More precisely, let $\rho_0 := \log R_0$, $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and

$$\mathbb{H}_{\rho_0} := \exp^{-1}(W_{R_0}) = \{z \in \mathbb{C} : \operatorname{Re} z > \rho_0\}.$$

Then there is a function $F : \mathcal{T} \rightarrow \mathbb{H}_{\rho_0}$ (the *logarithmic transform* of f) such that the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{T} & \xrightarrow{F} & \mathbb{H}_{\rho_0} \\
 \exp \downarrow & & \downarrow \exp \\
 \mathcal{V} & \xrightarrow{f} & W_{R_0}.
 \end{array}$$

The components of $\mathcal{T}$ are also called the *tracts* of F .

By construction, the function F and its domain $\mathcal{T}$ have the following properties:

1. every component T of $\mathcal{T}$ is an unbounded Jordan domain;
2. $\overline{\mathcal{T}}$ can be written as the disjoint union

$$\overline{\mathcal{T}} = \bigcup_{T \text{ component of } \mathcal{T}} \overline{T}.$$

3. for every component T of $\mathcal{T}$, $F : T \rightarrow \mathbb{H}_{\rho_0}$ is a conformal isomorphism, and F extends continuously to the closure $\overline{T}$ of T in $\mathbb{C}$;
4. for every component T of $\mathcal{T}$, $\exp|_T$ is injective;
5. $\mathcal{T}$ is invariant under translation by $2\pi i$; in particular, for every tract T of $\mathcal{T}$, $F|_T$ is unique up to translation by $2\pi i n_T$, where $n_T \in \mathbb{Z}$ can be chosen independently for each tract T .
6. $|F'(z)| \geq \frac{1}{4\pi}(\operatorname{Re} F(z) - \rho_0) \geq 2$, provided that R_0 was chosen large enough.

Property (6) is [EL, Lemma 2.1] and follows from a simple application of Koebe's distortion theorem; in the following, we will refer to this property as *expansivity* of F . Furthermore, applying the change of variable $w = z/R_0$, we may suppose without loss of generality that $R_0 = 1$; i.e., $\rho_0 = 0$.

We will denote by $\mathcal{B}_{\log}$ the class of all functions $F : \mathcal{T} \rightarrow \mathbb{H}$ such that $\mathcal{T}$ and F satisfy (1) to (6), regardless of whether they arise from an entire function $f \in \mathcal{B}$ or not. In particular, if $f : U \rightarrow \widehat{\mathbb{C}}$ is any holomorphic

function with a logarithmic singularity $U \subset \mathbb{C}$ over ∞ , then we can associate to f a function $F \in \mathcal{B}_{\log}$ which encodes the behavior of f in its logarithmic singularities.

If $F \in \mathcal{B}_{\log}$ and T is a tract of F , we denote the inverse of the conformal map $F : T \rightarrow \mathbb{H}$ by F_T^{-1} .

A.2.3 Combinatorics in $\mathcal{B}_{\log}$

Let $F \in \mathcal{B}_{\log}$; we denote the Julia set and the set of escaping points of F by

$$\begin{aligned} J(F) &:= \{z \in \overline{T} : F^{\circ n}(z) \text{ is defined and in } \overline{T} \text{ for all } n \geq 0\} \quad \text{and} \\ I(F) &:= \{z \in J(F) : \operatorname{Re} F^{\circ n}(z) \rightarrow \infty\}. \end{aligned}$$

If $f \in \mathcal{B}$ and F is its logarithmic transform, then it is not hard to show that $\exp(J(F)) \subset J(f)$ and $\exp(I(F)) \subset I(f)$. Furthermore, every escaping point of f eventually maps to some point in $\exp(I(F))$. For $K > 0$ we also define more generally

$$J^K(F) := \{z \in \overline{T} : F^{\circ n}(z) \text{ is defined and } \operatorname{Re} F^{\circ n}(z) \geq K \text{ for all } n \geq 0\}.$$

The partition of the domain of F into tracts suggests a natural way to assign symbolic dynamics to points in $J(F)$. More precisely, let $z \in J(F)$ and, for $j \geq 0$, let T_j be the tract of F with $f^j(z) \in \overline{T_j}$. Then the sequence

$$\underline{s} := T_0 T_1 T_2 \dots$$

is called the *external address* of z . More generally, we refer to any sequence of tracts of F as an external address (of F). If $\underline{s}$ is such an external address, we define the closed set

$$J_{\underline{s}} := \{z \in J(F) : z \text{ has address } \underline{s}\};$$

we define $I_{\underline{s}}$ and $J_{\underline{s}}^K$ in a similar fashion.

We denote the one-sided *shift-operator* on external addresses by σ ; i.e. $\sigma(T_0 T_1 T_2 \dots) = T_1 T_2 \dots$.

Definition A.2.1 (Dynamic Rays, Ray Tails). *Let $F \in \mathcal{B}_{\log}$. A ray tail with address $\underline{s}$ is an injective curve*

$$\gamma_{\underline{s}} : [0, \infty) \rightarrow I_{\underline{s}}$$

such that $\lim_{t \rightarrow \infty} \operatorname{Re} \gamma_{\underline{s}}(t) = +\infty$. A ray tail whose image in $I_{\underline{s}}$ is maximal with respect to inclusion is called a dynamic ray.

A.3 General Properties of Class $\mathcal{B}_{\text{log}}$

In this section, we prove some general results for functions in class $\mathcal{B}_{\text{log}}$. The first of these strengthens the aforementioned expansion estimate of [EL, Lemma 2.1] by showing that such a function expands distances exponentially.

Lemma A.3.1 (Exponential Separation of Orbits). *Let $F \in \mathcal{B}_{\text{log}}$ and T a tract of F . If $\omega, \zeta \in T$ such that $|\omega - \zeta| > 2$, then*

$$|F(\omega) - F(\zeta)| \geq \exp(|\omega - \zeta|/8\pi) \cdot \min\{\text{Re}F(\omega), \text{Re}F(\zeta)\}.$$

Proof. Suppose without loss of generality that $\text{Re}F(\omega) \geq \text{Re}F(\zeta)$. By assumption and the standard estimate of hyperbolic distance, it follows that

$$|\omega - \zeta|/2\pi \leq \text{dist}_T(\omega, \zeta) = \text{dist}_{\mathbb{H}}(F(\omega), F(\zeta)).$$

We will estimate the euclidean distance s between the point $F(\zeta)$ and a point ξ that satisfies $\text{dist}_{\mathbb{H}}(F(\zeta), \xi) = \text{dist}_{\mathbb{H}}(F(\omega), F(\zeta))$ and $\text{Re}F(\zeta) = \text{Re}\xi$. Then, $|F(\omega) - F(\zeta)| \geq s$. Let γ be the unique curve that consists of straight line segments parallel to the coordinate axes and connects $F(\zeta)$ with ξ through $F(\zeta) + s$ and $\xi + s$, see Figure A.1.

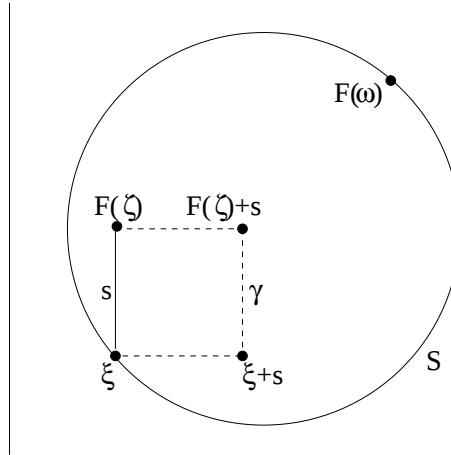


Figure A.1: The set $S = \{z \in \mathbb{H} : \text{dist}_{\mathbb{H}}(z, F(\zeta)) = \text{dist}_{\mathbb{H}}(F(\zeta), F(\omega))\}$ is a euclidean circle. Clearly, ξ is the euclidean closest point to $F(\zeta)$ on this circle that satisfies $\text{Re}\xi \geq \text{Re}F(\zeta)$. We use the dashed line γ to estimate s .

Each of the horizontal parts of $\gamma \subset \mathbb{H}$ has hyperbolic length

$$\log((\operatorname{Re} F(\zeta) + s)/\operatorname{Re} F(\zeta)) ,$$

and it is easy to see that the hyperbolic length of the vertical part is at most 1. Hence, we get

$$\frac{|\omega - \zeta|}{2\pi} \leq 2 \log \frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)} + 1 ,$$

and therefore

$$|F(\omega) - F(\zeta)| \geq s \geq \left(\exp \left(\frac{|\omega - \zeta|}{4\pi} - \frac{1}{2} \right) - 1 \right) \cdot \operatorname{Re} F(\zeta) .$$

Since $e^{x-1/2} - 1 > e^{x/2}$ for $x > 2$, the claim follows. $\square$

REMARK. It follows from expansivity of F that for any two distinct points w, z with the same external address, there exists $k \in \mathbb{N}$ such that $|F^{\circ k}(w) - F^{\circ k}(z)| > 2$. Hence Lemma A.3.1 will apply eventually.

Lemma A.3.2 (Escape of Some Real Parts). *Let $F \in \mathcal{B}_{\log}$. Then there is $K > 0$ with the following property: if $\zeta, \omega \in J^K(F)$ are distinct points with the same external address, then*

$$\lim_{k \rightarrow \infty} \max(\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega)) = \infty .$$

Proof. Choose K large enough so that no bounded component of $\mathbb{H} \cap \overline{\mathcal{T}}$ intersects the line $\{z \in \mathbb{C} : \operatorname{Re} z = K\}$ (this is possible because up to translation, only finitely many tracts intersect $\partial \mathbb{H}$). Let $\alpha : (0, 1) \rightarrow \mathbb{H}_K$ be a curve such that $\lim_{t \rightarrow 0} \operatorname{Re} \alpha(t) = K$, $\lim_{t \rightarrow 1} \operatorname{Re} \alpha(t) = \infty$ and that is disjoint from all tracts of F (since all tracts are isolated, we may follow the boundary of a tract to ∞ and then perturb this curve outward of $\overline{\mathcal{T}}$).

Now let $\zeta, \omega \in J^K(F)$ have the same external address and set $\zeta_k := F^{\circ k}(\zeta)$, $\omega_k := F^{\circ k}(\omega)$. The family of curves $\{\alpha + 2\pi i l\}_{l \in \mathbb{Z}}$ partitions $\mathbb{H}_K$ into countably many domains W_l that are unbounded to the right, intersect any left-half plane $\mathbb{H}_S^- := \{z \in \mathbb{C} : \operatorname{Re}(z) \leq S\}$ ($S > K$) in a bounded set and have the property that for all $k \geq 1$ there exists $l \in \mathbb{Z}$ such that $\zeta_k, \omega_k \in \overline{W_l}$. Note that $d_S := \operatorname{diam}(W_l \cap \mathbb{H}_S^-)$ only depends on S and not on l , because the W_l are vertical translates of each other.

It follows from expansivity of F that $|\omega_k - \zeta_k| \rightarrow \infty$. Thus for sufficiently large k , we have $|\omega_k - \zeta_k| > d_S$. So at least one of the points ω_k and ζ_k does not belong to $\mathbb{H}_{\underline{s}}$, or in other words

$$\max(\text{Re}\omega_k, \text{Re}\zeta_k) \geq S. \quad \square$$

As mentioned in the introduction, Rippon and Stallard [RS2] showed that the escaping set of every entire function f contains unbounded connected sets. The following theorem is a version of this result for functions in $\mathcal{B}_{\text{log}}$.

Theorem A.3.3 (Existence of Unbounded Continua in $J_{\underline{s}}$). *For every $F \in \mathcal{B}_{\text{log}}$ there exists $K \geq 0$ with the following property. If $z_0 \in J^K(F)$ and $\underline{s}$ is the external address of z_0 , then there exists an unbounded closed connected set $A \subset J_{\underline{s}}$ with $\text{dist}(z_0, A) \leq 2\pi$.*

Proof. Similarly as in the previous lemma, choose K large enough so that no bounded component of $\mathbb{H} \cap \overline{T}$ intersects the line $\{z \in \mathbb{C} : \text{Re}z = K\}$. Let $\underline{s} = T_0 T_1 \dots$ be the external address of z_0 . We set $z_k := F^{\circ k}(z)$ and consider the disks $B_{2\pi}(w_k)$. If $S \subset \mathbb{C}$ is an unbounded set such that $S \setminus B_{2\pi}(w_k)$ has exactly one unbounded component, let us denote this component by $X_k(S)$.

We claim that $X_k(\overline{T_k})$ is contained in $\mathbb{H}$ for all $k \geq 1$. (However, this set is *not* necessarily contained in $\mathbb{H}_K$.) Indeed, this is trivial if $\overline{T_k} \subset \mathbb{H}$. Otherwise, let α^- and α^+ denote the two unbounded components of $\mathbb{H} \cap \partial T_k$. Since T_k does not intersect its $2\pi i\mathbb{Z}$ -translates, both α^- and α^+ must intersect the vertical line segment $L := z_k + i[-2\pi, 2\pi]$. It follows easily that the unbounded component of $T_k \setminus L$ is contained in $\mathbb{H}$.

In particular, we can pull back the set $X_k(T_k)$ into T_{k-1} using $F_{T_{k-1}}^{-1}$. By expansivity of F , this pullback has distance at most π from z_{k-1} . Continuing inductively, we obtain the sets

$$A_k := X_0(F_{T_0}^{-1}(X_1(F_{T_1}^{-1}(\dots(X_{k-1}(F_{T_{k-1}}^{-1}(X_k(\overline{T_k}))))\dots)))$$

for $k \geq 1$; let $A_0 = X_0(\overline{T_0})$. Each $\widehat{A}_k \subset \widehat{\mathbb{C}}$ is a continuum, has distance at most 2π from z_0 and contains $\widehat{A}_{k+1}$. Hence, the set $A' = \bigcap_{k \geq 0} \widehat{A}_k$ has the same properties and there exists a component A of $A' \setminus \{\infty\}$ with $\text{dist}(A, z_0) \leq 2\pi$. By definition, A is closed and connected, and it is unbounded by the Boundary Bumping theorem [Na, Theorem 5.6]. $\square$

A.4 Functions Satisfying a Head-Start Condition

Throughout this section, we will fix some function $F \in \mathcal{B}_{\log}$. We will introduce the “head-start condition” which ensures that every escaping orbit will eventually land on a ray tail. In the subsequent section, we give sufficient conditions on F under which the head-start condition is satisfied.

Definition A.4.1 (Head-Start Condition). *Let T and T' be tracts of F and let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a (not necessarily strictly) monotonically increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}$. We say that the pair (T, T') satisfies the head-start condition for φ if, for all $z, w \in T$ with $F(z), F(w) \in T'$,*

$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \implies \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z)).$$

An external address $\underline{s}$ satisfies the head-start condition for φ if all consecutive pairs of tracts (T_k, T_{k+1}) satisfy the head-start condition for φ , and if for all $z, w \in J_{\underline{s}}$, there exists $M \in \mathbb{N}$ such that either $\operatorname{Re} F^{\circ M}(z) > \varphi(\operatorname{Re} F^{\circ M}(w))$ or $\operatorname{Re} F^{\circ M}(w) > \varphi(\operatorname{Re} F^{\circ M}(z))$.

We say that F satisfies a head-start condition if every external address of F satisfies the head-start condition for some φ . If the same function φ can be chosen for all external addresses, we say that F satisfies the uniform head-start condition for φ .

Theorem A.4.2 (Ray Tails). *Suppose that F satisfies a head-start condition. Then for every escaping point w , there exists $k \in \mathbb{N}$ such that $F^{\circ k}(w)$ is on a ray tail γ . Under the condition $\gamma(0) = F^{\circ k}(w)$, this ray tail is unique up to parametrization and satisfies $\operatorname{Re} F^{\circ n}(z) \rightarrow \infty$ uniformly on γ .*

We will devote the remainder of this section to the proof of Theorem A.4.2.

If $\underline{s}$ satisfies any head-start condition, the points in $J_{\underline{s}}$ are eventually ordered by their real parts: for any two points $z, w \in J_{\underline{s}}$, there is a $K \in \mathbb{N}$ such that after K iteration steps, the orbit of z remains to the right of w or vice versa.

Definition and Lemma A.4.3 (Speed Ordering). *Let $\underline{s}$ be an external address satisfying the head-start condition for φ . For $z, w \in J_{\underline{s}}$, we say that $z \succ w$ if there exists $K \in \mathbb{N}$ such that $\operatorname{Re} F^{\circ K}(z) > \varphi(\operatorname{Re} F^{\circ K}(w))$. We extend this order to $\hat{J}_{\underline{s}} = J_{\underline{s}} \cup \{\infty\}$ by the convention that $\infty \succ z$ for all $z \in J_{\underline{s}}$.*

With this definition, $(\widehat{J}_{\underline{s}}, \succ)$ is a totally ordered space.

REMARK. Note that if $z \succ w$, then $\text{Re}F^{\circ k}(z) > \varphi(\text{Re}F^{\circ k}(w))$ for all $k \geq K$.

Proof. By definition, $\text{Re}F^{\circ k}(z) < \varphi(\text{Re}F^{\circ k}(z))$ for all $k \in \mathbb{N}$ and $z \in J_{\underline{s}}$. Hence “ $\succ$ ” is non-reflexive.

Let $a, b, c \in J_{\underline{s}}$ such that $a \succ b$ and $b \succ c$. Then, there exist $k, l \in \mathbb{N}$ such that $\text{Re}F^{\circ k}(a) > \varphi(\text{Re}F^{\circ k}(b))$ and $\text{Re}F^{\circ l}(b) > \varphi(\text{Re}F^{\circ l}(c))$. Setting $m := \max\{k, l\}$, we get from the head-start condition that $\text{Re}F^{\circ m}(a) > \varphi(\text{Re}F^{\circ m}(b)) > \text{Re}F^{\circ m}(b) > \varphi(\text{Re}F^{\circ m}(c))$. Hence $a \succ c$ and “ $\succ$ ” is transitive.

By assumption, for any distinct $z, w \in J_{\underline{s}}$ there exists $k \in \mathbb{N}$ such that $\text{Re}F^{\circ k}(w) > \varphi(\text{Re}F^{\circ k}(z))$ or $\text{Re}F^{\circ k}(z) > \varphi(\text{Re}F^{\circ k}(w))$. It follows that any two points are comparable under “ $\succ$ ”. This completes the proof. $\square$

Corollary A.4.4 (Growth of Real Parts). *Let $\underline{s}$ be an external address that satisfies the head-start condition for φ and $z, w \in J_{\underline{s}}$. If $w \succ z$, then $w \in I(F)$. In particular, with at most one exception every point in $J_{\underline{s}}$ escapes.*

Proof. This is an immediate corollary of Lemma A.3.2 and the definition of “ $\succ$ ”. $\square$

Proposition A.4.5 (Arcs in $J_{\underline{s}}$). *Let $\underline{s}$ be an external address satisfying the head-start condition for φ . Then the topology of $\widehat{J}_{\underline{s}}$ as a subset of the Riemann sphere $\widehat{\mathbb{C}}$ agrees with the order topology induced by $\succ$. In particular,*

1. *every component of $\widehat{J}_{\underline{s}}$ is homeomorphic to a (possibly degenerate) compact interval, and*
2. *there exists $K > 0$ such that if $J_{\underline{s}}^K \neq \emptyset$, then $J_{\underline{s}}$ has a unique unbounded component, which is a closed arc to infinity.*

Proof. Let us first show that $\text{id} : \widehat{J}_{\underline{s}} \rightarrow (\widehat{J}_{\underline{s}}, \succ)$ is continuous. Since $\widehat{J}_{\underline{s}}$ is compact and the order topology on $\widehat{J}_{\underline{s}}$ is Hausdorff, this implies that id is a homeomorphism and that both topologies agree. It suffices to show that sub-basis elements for the order topology of the form $U_a^- := \{w \in J_{\underline{s}} : a \succ w\}$ and $U_a^+ := \{w \in \widehat{J}_{\underline{s}} : w \succ a\}$ are open in $\widehat{J}_{\underline{s}}$ for any $a \in \widehat{J}_{\underline{s}}$. We will only give a proof for the sets U_a^- ; the proof for U_a^+ is analogous.

Let $w \in U_a^-$ and $k \in \mathbb{N}$ minimal such that $\operatorname{Re} F^{\circ k}(a) > \varphi(\operatorname{Re} F^{\circ k}(w))$. Since φ , Re and $F^{\circ k}$ are continuous, this is true for a neighborhood V of w . It follows that U_a^- is a neighborhood of w in $\widehat{J}_{\underline{s}}$, because $V \cap \widehat{J}_{\underline{s}} \subset U_a^-$.

Thus the topology of $\widehat{J}_{\underline{s}}$ agrees with the order topology. Every connected component C of $\widehat{J}_{\underline{s}}$ is compact; it follows from [Na, Theorems 6.16 & 6.17] that C is either a point or an arc. This proves (a). To prove (b), observe that existence follows from Theorem A.3.3, while uniqueness follows because ∞ is the largest element of $(\widehat{J}_{\underline{s}}, \succ)$. $\square$

Proposition A.4.6 (Points in the Unbounded Component of $J_{\underline{s}}$). *Let $\underline{s}$ be an external address that satisfies the head-start condition for φ . Then there exists $K' \geq 0$ such that $J_{\underline{s}}^{K'}$ is contained in the unbounded component of $J_{\underline{s}}$, which is an arc. (The value K' depends on F and φ , but not on $\underline{s}$.)*

Proof. Let K be the constant from Theorem A.3.3, set $K' := \max\{\varphi(0) + 1, K\}$ and let $z_0 \in J_{\underline{s}}^{K'}$. For each $k \geq 0$, we let $z_k := F^{\circ k}(z_0)$ and consider the set

$$S_k := \{w \in J_{\sigma^k(\underline{s})} : w \succ z_k\} \cup \{z_k\}.$$

By Proposition A.4.5, this set has a unique unbounded component A_k which is a closed arc. By Theorem A.3.3, A_k satisfies $\operatorname{dist}(z_k, A_k) \leq 2\pi$.

Let us show $A_k \subset \mathbb{H}$, so that we may apply F^{-1} to it. Indeed, if $w \in J_{\sigma^k(\underline{s})}$ with $\operatorname{Re} w \leq 0$, then the choice of K' and monotonicity of φ yield that $\operatorname{Re} z_k > \varphi(0) \geq \varphi(\operatorname{Re} w)$, and therefore $z_k \succ w$. Thus, $w \notin S_k$. We conclude that $F_{T_{k-1}}^{-1}(A_k) \subset A_{k-1}$, because it is unbounded and contained in S_{k-1} . Since F is expanding, this means that

$$\operatorname{dist}(A_0, z_0) \leq 2^{-k} \operatorname{dist}(z_k, A_k) \leq 2^{-(k-1)} \pi$$

for all $k \geq 0$. Thus $z_0 \in A_0$, as required. That A_0 is an arc follows from Proposition A.4.5. $\square$

Proof of Theorem A.4.2. Let w be an escaping point for F and $\underline{s}$ its external address. By hypothesis, there exists $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\underline{s}$ satisfies the head-start condition for φ . If K' is the constant from Proposition A.4.6, then by the same proposition there exists $k \geq 0$ such that $F^{\circ k}(w) \in J_{\underline{s}}^{K'}$ and $\gamma_k := \{z \in I_{\sigma^k(\underline{s})} : z \succ F^{\circ k}(w)\} \cup \{F^{\circ k}(w)\}$ is an injective curve connecting $F^{\circ k}(w)$ to ∞ , i.e. a ray tail. Note that K' can be chosen independently from $\underline{s}$.

Let γ' be another curve connecting $F^{\circ k}(w)$ to ∞ . It follows from the construction of γ_k that there exists $\ell \in \mathbb{N}$ such that $F^{\circ \ell}(\gamma')$ is not contained in $T_{\ell+k}$. Hence γ' must intersect $\mathbb{H} \setminus F^{-\ell}(\mathcal{T})$ and is thus not a ray tail. This implies uniqueness. It remains to show that the real parts of all points on γ_k grow uniformly. Observe that for all $z \in \gamma$ and $\ell \in \mathbb{N}$, $\operatorname{Re} F^{\circ \ell}(z) \geq \inf\{\varphi^{-1}(\operatorname{Re} F^{\circ(k+\ell)}(w))\}$. Hence the $\operatorname{Re} F^{\circ \ell}(z)$ converge uniformly to ∞ as $\ell \rightarrow \infty$. $\square$

Theorem A.4.7 (Existence of Absorbing Brush). *Suppose that F satisfies a head-start condition. Then there exists a closed subset $X \subset J(F)$ with the following properties:*

1. $F(X) \subset X$;
2. each connected component C of X is an arc to infinity, all of whose points except possibly the finite endpoint escape;
3. every escaping point of F enters X after finitely many iterations. If F satisfies the uniform head-start condition for some function, then there exists $K' > 0$ such that $J^{K'}(F) \subset X$.

Proof. Let X denote the union over all external addresses of all unbounded components of $J(F)$. Since $\widehat{X}$ is the unbounded connected component of the compact set $J(F) \cup \{\infty\}$, X is a closed set. Clearly X is F -invariant, and satisfies (2) and (3) by Propositions A.4.5 and A.4.6. $\square$

A.5 Geometry, Growth & Head-Start

This section discusses geometric properties of tracts which imply a head-start condition. Moreover, we present a class of functions in $\mathcal{B}_{\log}$ that satisfies these properties.

Let $K > 1$ and $M > 0$. We say that $\underline{g}$ satisfies the *linear head-start condition* with constants K and M if it satisfies the head-start condition for

$$\varphi(t) := K \cdot t^+ + M,$$

where $t^+ = \max\{t, 0\}$.

We will restrict our attention to functions whose tracts do not grow too quickly in the imaginary direction.

Definition A.5.1 (Bounded Slope). Let $F \in \mathcal{B}_{\log}$. We say that the tracts of F have bounded slope (with constants $\alpha, \beta > 0$) if

$$|\operatorname{Im} z - \operatorname{Im} w| \leq \alpha \max\{\operatorname{Re} z, \operatorname{Re} w, 0\} + \beta$$

whenever z and w belong to a common tract of F . We denote the class of all functions with this property by $\mathcal{B}_{\log}(\alpha, \beta)$.

REMARK. This condition is equivalent to the existence of a curve $\gamma : [0, \infty) \rightarrow \mathcal{T}$ with $\operatorname{Re} F(\gamma(t)) \rightarrow \infty$ and $\limsup |\operatorname{Im} \gamma(t)| / \operatorname{Re} \gamma(t) < \infty$. Hence if one tract of F has bounded slope, then property (5) of $\mathcal{B}_{\log}$ implies that all tracts do.

The bounded slope condition means, roughly, that the distance between two points grows in proportion to the difference of their real parts. It turns out that this implies a linear head-start condition.

Lemma A.5.2 (Linear Separation of Orbits). Let $F \in \mathcal{B}_{\log}(\alpha, \beta)$ and $K, M > 0$. Then, there exists $\delta = \delta(\alpha, \beta, K, M) > 0$ with the following property. If $\underline{s}$ is an external address and $z, w \in J_{\underline{s}}$ with $|z - w| > \delta$, then

$$\operatorname{Re} F^{\circ k}(z) > K \operatorname{Re} F^{\circ k}(w) + M \quad \text{or} \quad \operatorname{Re} F^{\circ k}(w) > K \operatorname{Re} F^{\circ k}(z) + M$$

for all $k \in \mathbb{N}$.

Proof. Again, let $z_k = F^{\circ k}(z)$ and $w_k = F^{\circ k}(w)$. Set $\delta' = \sqrt{2 + (\alpha + \beta)^2}$, $\delta = \max\{\delta', 16\pi \log \delta', M\}$ and fix $k \in \mathbb{N}$. By possibly enlarging δ further, we may assume that $e^{t/16\pi} > K + t + 1/2$ for $t \geq \delta - 1/2$.

Suppose that $\operatorname{Re} w_{k+1} \geq \operatorname{Re} z_{k+1}$. Using the bounded slope condition, it is easy to estimate $|w_{k+1} - z_{k+1}|^2 \leq (\operatorname{Re} w_{k+1})^2(2 + \alpha^2) + 2\alpha\beta \operatorname{Re} w_{k+1} + \beta^2$. If $\operatorname{Re} w_{k+1} < 1$, then this implies that $|w_{k+1} - z_{k+1}| \leq \delta'$, contradicting expansivity of F . Hence $\operatorname{Re} w_{k+1} \geq 1$ and we conclude $|w_{k+1} - z_{k+1}| \leq \delta' \operatorname{Re} w_{k+1}$. Since $\delta' > 2$, Lemma A.3.1 now yields

$$\operatorname{Re} w_{k+1} \geq \frac{\exp(|w_k - z_k|/8\pi)}{\delta'} \cdot \operatorname{Re} z_{k+1} \geq e^{\frac{1}{16\pi}|w_k - z_k|} \cdot \operatorname{Re} z_{k+1}, \quad (\text{A.1})$$

because $\exp(x/8\pi)/\delta' > \exp(x/16\pi)$ for all $x > 16\pi \log \delta'$.

If $\operatorname{Re} z_{k+1} \geq 1$, then equation (A.1) implies

$$\operatorname{Re} w_{k+1} \geq e^{\frac{|w_k - z_k|}{16\pi}} \operatorname{Re} z_{k+1} > (K + |w_k - z_k| + \frac{1}{2}) \operatorname{Re} z_{k+1} \geq K \operatorname{Re} z_{k+1} + |w_k - z_k|.$$

Otherwise, let $z' = F_T^{-1}(1 + \text{Im}z_{k+1})$. Then $|z_k - z'| < 1/2$, so $|z' - w_k| \geq |w_k - z_k| - 1/2$. Applying the above argument to z' and w_k , we again have

$$\text{Re}w_{k+1} > K\text{Re}F(z') + |z' - w_k| + 1/2 \geq K\text{Re}z_{k+1} + |w_k - z_k|.$$

After possibly interchanging the roles of w and z , the claim follows, because $|w_k - z_k| > M$. $\square$

Corollary A.5.3 (Linear Head-Start is Preserved by Composition).

Let $F : \mathcal{T}_F \rightarrow \mathbb{H}$ and $G : \mathcal{T}_G \rightarrow \mathbb{H}$ be in $\mathcal{B}_{\log}$. If F and G have tracts of bounded slope and satisfy linear head-start conditions, then so does $F \circ G$.

Proof. Since $\mathcal{B}_{\log}(\alpha', \beta') \subset \mathcal{B}_{\log}(\alpha, \beta)$ whenever $\alpha \geq \alpha'$ or $\beta \geq \beta'$, we may suppose that $F, G \in \mathcal{B}_{\log}(\alpha, \beta)$ for some $\alpha, \beta > 0$. Since $\mathcal{T}_{F \circ G} = F^{-1}(\mathcal{T}_G \cap \mathbb{H}) \subset \mathcal{T}_F$, we get $F \circ G \in \mathcal{B}_{\log}(\alpha, \beta)$.

Let K_F, M_F and K_G, M_G be the constants for the linear head-start conditions of F and G , respectively, and set $K = \max\{K_F, K_G\}$ and $M = \delta$, where $\delta = \delta(\alpha, \beta, K, \max\{M_F, M_G\})$ is chosen as in Lemma A.5.2. Let T be a tract of F and $w, z \in T$, such that $\text{Re}w > K\text{Re}z + M$. Then, $|w - z| \geq \text{Re}w - \text{Re}z > M \geq \delta$, and the head-start condition of F gives $\text{Re}F(w) > K_F\text{Re}F(z) + M_F \geq \text{Re}F(z)$. Now, Lemma A.5.2 gives that

$$\text{Re}F(w) > K\text{Re}F(z) + M.$$

The same applies for G , so we may suppose that both F and G satisfy the head-start condition with constants K and M . Now it follows by definition that $F \circ G$ satisfies the head-start condition with constants K and M . $\square$

REMARK. Note that in general, the image of $F \circ G$ will not equal $\mathbb{H}$, so strictly speaking, $F \circ G$ is not an element of $\mathcal{B}_{\log}$. But since only finitely many tracts of G (up to translation) will leave $\mathbb{H}$, the set $\mathcal{T}_G \setminus \mathbb{H}$ is compact and hence, the image of $\mathcal{T}_{F \circ G}$ under $F \circ G$ contains a right-half plane. So after possibly restricting $\mathcal{T}_{F \circ G}$ and shifting it to the left, $F \circ G$ will be in $\mathcal{B}_{\log}$ up to conjugation with a translation.

Definition A.5.4 (Bounded Wiggling). Let $F \in \mathcal{B}_{\log}$, and let T be a tract of F . We say that T has bounded wiggling if there exist $K > 1$ and $\mu > 0$ such that for every $z_0 \in T$, every point z on the hyperbolic geodesic of T that connects z_0 to ∞ satisfies

$$\text{Re}z > \frac{1}{K}\text{Re}z_0 - \mu.$$

We say that $F \in \mathcal{B}_{\log}$ has uniformly bounded wiggling if the wiggling of all tracts of F is bounded by the same constants K, μ .

Proposition A.5.5 (Functions with Linear Head-Start Conditions).

Let $F \in \mathcal{B}_{\log}(\alpha, \beta)$, $\underline{s} = T_0 T_1 T_2 \dots$ be an external address, and let $K > 1$. Then the following are equivalent:

1. For some $\mu > 0$, all tracts T_k have bounded wiggling with constants K and μ ;
2. for some $M > 0$, $\underline{s}$ satisfies the linear head-start condition with constants K and M .

The relation between μ and M depends on K, α and β , but not on $\underline{s}$.

Proof. Suppose that (1) holds and let $M = \delta(\alpha, \beta, K, K(\mu + 2\pi(\alpha + \beta)))$ be the constant from Lemma A.5.2. Let $k \in \mathbb{N}$ and choose $z, w \in T_k$, such that $F(z), F(w) \in T_{k+1}$ and $\operatorname{Re} w > K \operatorname{Re} z + M$. Then, $|z - w| > M$ and by Lemma A.5.2, it suffices to show that $\operatorname{Re} F(w) \geq \operatorname{Re} F(z)$.

So suppose by way of contradiction that $\operatorname{Re} F(z) > \operatorname{Re} F(w)$. In this case, the same arguments as in Lemma A.5.2 give that $\operatorname{Re} F(z) \geq 1$. Set $\Gamma := \{F(w) + t : t \geq 0\}$ and $\gamma := F_{T_k}^{-1}(\Gamma)$; in other words, γ is the geodesic of T_k connecting w to ∞ . The bounded slope condition ensures that

$$\operatorname{dist}_{T_k}(z, \gamma) = \operatorname{dist}_{\mathbb{H}}(F(z), \Gamma) \leq \frac{|\operatorname{Im} F(z) - \operatorname{Im} F(w)|}{\operatorname{Re} F(z)} \leq \alpha + \beta.$$

Therefore, $\operatorname{dist}(z, \gamma) \leq 2\pi(\alpha + \beta)$ and consequently, $\operatorname{Re} z \geq \min_{\zeta \in \gamma} \{\operatorname{Re} \zeta\} - 2\pi(\alpha + \beta)$. By the bounded wiggling condition, we have $\operatorname{Re} \zeta \geq \frac{1}{K} \operatorname{Re} w - \mu$ for all $\zeta \in \gamma$. Thus

$$\operatorname{Re} w \leq K(\operatorname{Re} z + \mu + 2\pi(\alpha + \beta)) \leq K \operatorname{Re} z + M,$$

a contradiction.

For the converse direction, suppose that (2) holds. Let T be a tract of F and $z \in T$. Observe that there exists a constant $\kappa > 0$ such that $\operatorname{dist}(F(z), I(F)) < \kappa$ (this is because some right half-plane contains the $2\pi i\mathbb{Z}$ -translates of a curve connecting some escaping point to ∞). Pulling back, we find that there exists an escaping point $\zeta \in T$ such that z and ζ can be joined in T with a curve of euclidean length at most κ . Let $\underline{s}$ be the external

address of ζ and $\gamma \subset \{w \in I_{\underline{s}} : w \succ \zeta\}$ be a curve of escaping points that join ζ to ∞ . By the head-start condition, every $w \in \gamma$ satisfies

$$\operatorname{Re} w \geq \frac{\operatorname{Re} \zeta}{K} - \frac{M}{K}.$$

Hence there exists a curve $\gamma' \subset T$ that connects z to ∞ such that for every $w \in \gamma'$,

$$\operatorname{Re} w \geq \frac{\operatorname{Re} \zeta}{K} - \frac{M}{K} - \kappa \geq \frac{\operatorname{Re} z}{K} - \frac{\kappa}{K} - \frac{M}{K} - \kappa.$$

Now the claim follows from Lemma A.6.3. $\square$

Finally, let us consider functions of finite order.

Definition A.5.6 (Finite Order). *We say that $F \in \mathcal{B}_{\log}$ has finite order if*

$$\log \operatorname{Re} F(w) = O(\operatorname{Re} w)$$

as $\operatorname{Re} w \rightarrow \infty$ in $\mathcal{T}$.

Note that this definition ensures that $f \in \mathcal{B}$ has finite order if and only if its logarithmic transform $F \in \mathcal{B}_{\log}$ has finite order.

Theorem A.5.7 (Finite Order Functions Have Good Geometry). *Suppose that F has finite order. Then the tracts of F have bounded slope and (uniformly) bounded wiggling.*

Proof. By the Ahlfors non-spiralling theorem A.6.1, $F \in \mathcal{B}_{\log}(\alpha, \beta)$ for some constants α, β , and by the finite-order condition, there are ρ and M such that $\log \operatorname{Re} F(z) \leq \rho \operatorname{Re} z + M$ for all $z \in \mathcal{T}$. Let T be a tract of F and $z \in T$.

Suppose first that $\operatorname{Re} F(z) \geq 1$. Consider the geodesic $\gamma(t) := F_T^{-1}(F(z) + t)$ (for $t \geq 0$). Since the hyperbolic distance between z and $\gamma(t)$ is at most $\log t$, we have

$$\operatorname{Re} z - \operatorname{Re} \gamma(t) \leq 2\pi \log t \leq 2\pi \log \operatorname{Re} F(\gamma(t)) \leq 2\pi(\rho \operatorname{Re} \gamma(t) + M).$$

In other words, $\operatorname{Re} z \leq (1 + 2\pi\rho)\operatorname{Re} \gamma(t) + 2\pi M$, i.e.

$$\operatorname{Re} \gamma(t) \geq \frac{1}{1 + 2\pi\rho} \operatorname{Re} z - \frac{2\pi M}{1 + 2\pi\rho}.$$

Since z was chosen arbitrarily, F has uniformly bounded wiggling with constants $1/(1 + 2\pi\rho)$ and $2\pi M/(1 + 2\pi\rho)$.

If $\operatorname{Re} F(z) < 1$, we can connect z to a point $w \in T$ with $\operatorname{Re} F(w) \geq 1$ by a curve of bounded length. $\square$

Proof of Theorem A.1.2. Let $f = g_n \circ \cdots \circ g_1$, where $g_1, \dots, g_n \in \mathcal{B}$ have finite order, and let $G_1, \dots, G_n$ be the logarithmic transforms of $g_1, \dots, g_n$. Then, each G_i has bounded slope and satisfies a linear head-start condition by Theorem A.5.7 and Proposition A.5.5. By Corollary A.5.3, $F = G_n \circ \cdots \circ G_1$ also satisfies a linear head-start condition and it is easy to see that F is a logarithmic transform of f , after possibly restricting its domain and enlarging the critical disk W_{R_0} of f .

Now let X be the absorbing collection of ray tails from Theorem A.4.7. Then, $X' = \exp(X) \subset I(f)$ is absorbing and there exists $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in X'$. It follows from Theorem A.4.7 that there exists a unique arc $\gamma_k : [0, \infty) \rightarrow X'$ with $\gamma_k(0) = f^{\circ k}(z)$ and $\lim_{t \rightarrow \infty} \gamma_k(t) = \infty$. It satisfies $f^{\circ n}|_{\gamma_k} \rightarrow \infty$ uniformly.

Now let $T \in (0, \infty]$ be maximal such that there is a curve $\gamma_{k-1} : [0, T) \rightarrow \mathbb{C}$ with $\gamma_{k-1}(0) = f^{\circ(k-1)}(z)$ and $f(\gamma_{k-1}(t)) = \gamma_k(t)$. If $T = \infty$, then clearly $\gamma_{k-1}(t) \rightarrow \infty$ as $t \rightarrow \infty$. Otherwise, $w = \lim_{t \rightarrow T} \gamma_{k-1}(t)$ exists in $\hat{\mathbb{C}}$. If $w \neq \infty$, we could extend γ_{k-1} further (choosing any one of the possible branches of f^{-1} in the case where w is a critical point), contradicting the maximality of T . Thus $w = \infty$ (and, in particular, $\gamma_k(T)$ is an asymptotic value of f).

In either case, we have found a curve $\gamma_{k-1} \subset f^{-1}(\gamma_k) \subset I(f)$ which connects $f^{\circ(k-1)}(z)$ to infinity. Continuing this method inductively, we find a curve γ_0 with the required properties. $\square$

A.6 Some Hyperbolic Geometry

The Ahlfors spiral theorem [Hay, Theorem 8.21] states that any entire function of finite order has controlled spiralling. We give a simple proof of this result for functions in class $\mathcal{B}_{\log}$.

Theorem A.6.1 (Spiral Theorem). *Suppose that $F \in \mathcal{B}_{\log}$ has finite order. Then the tracts of F have bounded slope.*

Proof. Let T be a tract of F , set $\rho := \sup\{\frac{\log \operatorname{Re} F(z)}{\operatorname{Re} z} : z \in \mathbb{H} \cap \mathcal{T}\} < \infty$, and consider the central geodesic $\gamma : [1, \infty) \rightarrow T; t \mapsto F_T^{-1}(t)$. Then for every $t \geq 1$,

$$|\gamma(t)| - |\gamma(1)| \leq |\gamma(t) - \gamma(1)| \leq 2\pi \ell_T(\gamma([1, t])) = 2\pi \log t \leq 2\pi \rho \operatorname{Re} \gamma(t).$$

Thus we have proved the existence of an asymptotic curve γ satisfying $|\operatorname{Im}\gamma(t)| \leq |\gamma(t)| \leq K\operatorname{Re}\gamma(t) + M$ for $K = 2\pi\rho$ and $M = |\gamma(1)|$, which is equivalent to the bounded slope condition. $\square$

Proposition A.6.2 (Geodesics are short). *There is a constant $C > 0$ with the following property. Let $V \subset \mathbb{C}$ be an unbounded Jordan domain and $F : V \rightarrow \mathbb{H}$ a conformal map with $F(\infty) = \infty$, and let $\gamma(t) := F^{-1}(t)$ be the central geodesic.*

1. *For every $t \in \mathbb{R}$ there exists a geodesic α connecting t to the positive imaginary axis such that*

$$\ell(F^{-1}(\alpha)) \leq C \operatorname{dist}(\gamma(t), \partial V) .$$

The same is true for the negative imaginary axis.

2. *Let $1 \leq R_1 < R_2$ with $R_2/R_1 \geq 2$. Then any curve $\alpha \subset V$ connecting the two vertical geodesics $F^{-1}(\{|z| = R_1\})$ and $F^{-1}(\{|z| = R_2\})$ satisfies*

$$C \operatorname{diam}(\alpha) \geq \ell(\gamma([R_1, R_2])) .$$

3. *Let $z, w \in \overline{V}$, let α_1 be the geodesic of V connecting z and w and let $\alpha_2 \subset V$ be any curve connecting z and w . Then*

$$\operatorname{diam}(\alpha_1) \leq C \operatorname{diam}(\alpha_2) .$$

$\square$

A proof of these inequalities can be found in [Po, Section 4.5].

Lemma A.6.3 (Domains with bounded wiggling). *Let V be an unbounded Jordan domain such that $\exp|_V$ is injective, and let $F : V \rightarrow \mathbb{H}$ a conformal isomorphism with $F(\infty) = \infty$. Suppose that there are $K, M > 0$ such that every $z_0 \in V$ can be connected to ∞ by a curve $\gamma \subset V$ satisfying*

$$\operatorname{Re} w \geq \frac{\operatorname{Re} z_0}{K} - M$$

for all $w \in \gamma$. Then there is $M' > 0$ which depends only on M such that, for every $z_0 \in V$,

$$\operatorname{Re} z \geq \frac{\operatorname{Re} z_0}{K} - M'$$

for all z on the geodesic connecting z_0 to ∞ .

Proof. Let $z_0 \in V$, let γ be a curve as in the statement, and let

$$\gamma' := \{F^{-1}(F(z_0) + r) : r \geq 0\}$$

be the hyperbolic geodesic connecting z_0 to ∞ , and let $z \in \gamma'$. Then by the previous proposition, there is a crosscut of V passing through z which separates z_0 from ∞ and has diameter at most $C \operatorname{dist}(z, \partial V) \leq 2\pi C$.

The curve γ must intersect this crosscut in some point w . We thus have

$$\operatorname{Re} z \geq \operatorname{Re} w - 2C\pi \geq \operatorname{Re} z_0/K - M - 2\pi C. \quad \square$$

Appendix B

The Mechoui — A Family Recipe

The Mechoui—a whole lamb roasted over an open fire—is a traditional North-African dish. The idea of preparing a Mechoui at special occasions was introduced to the holomorphic dynamics community by Adrien Douady and passed on to John Hubbard and Dierk Schleicher.

During a visit of Adrien Douady at International University Bremen in September 2004, we used the opportunity to learn how to prepare a Mechoui from him. In this chapter, we would like to lay out how to successfully prepare this dish and what preparations are necessary in advance.

A Mechoui is the perfect meal for an outdoors-style campfire dinner or a BBQ evening on the porch. Served with some tasty side-dishes and dessert, one lamb easily feeds 30 adults. However, we will in the following concentrate on the lamb itself and not go into sides, drinks, dessert, or anything else necessary for a full meal.

B.1 Equipment

In order to cook a Mechoui, a decent fireplace and a rotating spit are the only equipment necessary.

The spit should be made of stainless steel and 1.6m in length, pointed at one end and with a crank at the other. Since rotation of the lamb will only be necessary at discrete time intervals, it should be possible to arrest the spit in any position. In order to fix the lamb on the spit, it must have

two movable awls parallel to the spit and at least 10cm in length.

There are two choices of fire one can make under the lamb: charcoal or wood. A charcoal fire is advisable to be made in a barbecue grill of dimensions at least 100×50 cm. It burns hotter than wood and the Mechoui will be finished faster, but the spit must be placed lower over the coal than with wood. For a wood fire, any fireplace (a shallow hole in the ground, a ring of rocks or metal, etc.) of diameter not larger than 120cm will do.

In either case, a contraption to fix the spit over the fire is necessary. Depending on individual skills, this can be built from wood or metal or bought from a blacksmith. The device should be resting so firmly on the ground that a load of 20kg can be rotated on the spit without tilting. It should allow for the spit to be placed at various heights between 40 and 120cm above ground. Figure B.1 describes the device that has been successfully used at International University Bremen. It was custom built by a local blacksmith.

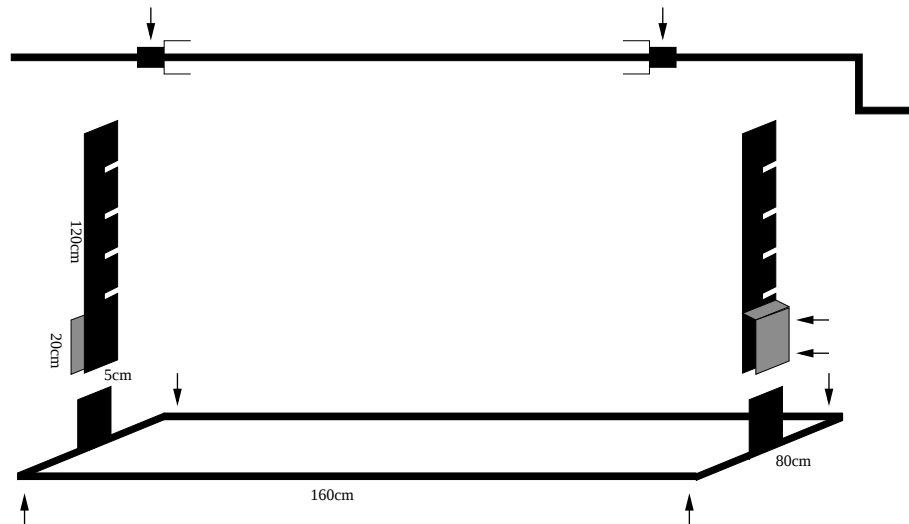


Figure B.1: The Mechoui grill of Dierk Schleicher. The bottom frame consists of 4 separate pieces of square steel pipe (edge length 20mm) and can be adjusted in length (all arrows indicate screws). The frame goes around the campfire or BBQ.

The only other permanent acquisition necessary is a meat thermometer to measure the progress of the meat over the fire. Experienced chefs may

replace this tool by instinct.

B.2 Shopping

This section contains a list of items necessary to make the Mechoui.

- The **lamb** should be ordered a week in advance from a trusted butcher. It should weigh about 20kg, skinned and with head and innards removed (except possibly the kidneys). The breastbone must not be split, and the meat should hang for several days before the event. It must not be marinated at delivery.
- 200kg **firewood** (preferably beech), cut into logs, or 20kg **charcoal**, depending on the desired fire and cooking time.
- A large clean **tablet** to place the finished lamb on for cutting.
- A sharp **knife** and a large **fork** to cut the meat.
- **Firestarters**, matches, old paper etc. to get the fire going.
- Several large **trashbags**.
- A long **stick**.
- Several meters of thin, non-insulated **wire** and a pair of **pliers**.
- A **shovel** to stir the fire.
- 2kg **onions**.
- About 60 cloves of **garlic** (400g).
- 200g pitted **olives**.
- A **knife** and **cutting board**.
- **Rosemary**, **thyme**, ground **pepper** and **salt**.
- 3 liters of vegetable **oil**.
- A clean piece of **cloth**.

- A large **bowl**.
- Several meters **aluminum foil**.
- **Detergent** and other cleaning equipment at will.

B.3 Over the Fire

For an open woodfire, a cooking time of 5–6 hours should be allocated; over a charcoal grill, the lamb reaches its target temperature after 3–4 hours. The Mechoui can succeed in slight rainfall; under such conditions, the fire needs to be hotter (allocate more fuel) and the lamb needs approximately an hour extra.

The fire should be started an hour before the lamb is put on. After it burns, peel the onions and the garlic (cut the onions into quarters; the cloves of garlic need not be cut). Insert the spit into the lamb through throat and rear end and attach the lamb firmly with the awls. To give it more stability when turning, the backbone should be attached to the spit using several short pieces of wire: the wire can be poked through the meat around spit and backbone and twisted tight. Then, fill the hollow belly with onions, garlic and olives and sew it shut with a long piece of wire, poking it through both edges of the meat in short spacing. Several leftover pieces of garlic may be wedged into small cuts in the outside of the lamb to add to the flavor.

Now, the lamb can go over the fire, which needs to be watched for the entire time and fed so that it remains so hot that one cannot hold a hand near the lamb.

Fill the oil into the bowl, add spices and stir. Attach the piece of cloth to the long stick with the wire. Dipping the cloth into the bowl, the oil and spice can be applied to the lamb brush-style without burning one's hand. This should be done in regular intervals (e.g. every 15min.), depending on the heat of the fire.

Now the lamb needs to cook for 3–6 hours as discussed above. Approximately every five minutes, the spit can be rotated $1/8$ turn. The target temperature for the meat is between 50° and 60°C *everywhere*, meaning especially deep down inside the lamb near the bones.

Should the lamb be done too early or run danger of burning, wrap it with a layer of aluminum foil (kept in place with more wire).

When dinner is ready, remove the lamb from the fire, cut into pieces and enjoy!



Figure B.2: Adrien Douady and Dierk Schleicher grilling a lamb on IUB campus (September 2004).

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