Self-similarity of the Mandelbrot set and parabolic bifurcation

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Abstract of the Dissertation

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In this thesis, we extend Lyubich's result about the smoothness of the holonomy in the space of quadratic-like germs in the case of tripling essentially bounded combinatorics. This combinatorics are given by little copies of the Mandelbrot set which converge to the root point of a primitive real copy of the Mandelbrot set. This result implies self-similarity of the Mandelbrot set for this combinatorics.

To Sara

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Chapter 1

Introduction

In this thesis, we study the holonomy map of the lamination $\mathcal{F}$ in the space of quadratic-like germs given by the *hybrid classes*. (See §3 for definitions and background). Our purpose is to give conditions for smoothness of the holonomy for some special type of combinatorics, associated with parabolic bifurcation of a primitive parabolic point.

In this chapter, we first give some historical account of the study of the regularity of the holonomy map and its applications. Afterward, we state the main result and discuss some directions of further study.

1.1 The context

When we look at computer pictures of the Mandelbrot set, we can see that it is not everywhere self-similar. However, some self-similar features are still observable. For instance, we can observe many little Mandelbrot copies inside the Mandelbrot set. In order to explain this computer observable little Mandelbrot copies, the notion of *complex renormalization* was introduced by Douady and Hubbard [DH1]. Since the space of quadratic polynomials is not invariant under complex renormalization, a bigger space of complex analytic functions was needed. This was one of the reasons of the introduction of quadratic-like maps in holomorphic dynamics by [DH2].

It was then conjectured by Milnor that the little Mandelbrot sets around the Feigenbaum point of stationary type have asymptotically the same shape ([M2, Conjs. 3.1 and 3.3]).

In [L3], Lyubich supplied the space of quadratic-like germs with topology and complex analytic structure, modeled on families of Banach spaces and proved that the Douady-Hubbard hybrid classes form a foliation of the connectedness locus $\mathcal{C}$ with complex codimension-one analytic leaves. Then, a generalized version of the λ-lemma was used to show that this foliation is transversally quasi-conformal, that is, the holonomy map h between two transversals to a hybrid class, is locally a restriction of a quasi-conformal map.

However, Douady and Hubbard [DH2], as well as Lyubich [L3], showed that the foliation was not transversally smooth. For instance, consider the Ulam-Neumann quadratic map $P = P_{-2} : z \mapsto z^2 - 2$ and a quadratic-like map f in the hybrid class of P, i.e. $f \in \mathcal{H}_{-2}$. Approximate both maps with superattracting parameter values $c_n \to -2$ and $f_n \to f$, $f_n \in \mathcal{H}_{c_n}$, with different rates of convergence, then the holonomy $c_n \mapsto f_n$ is not smooth at -2. For the same reason the foliation is not smooth at other Misiurewicz points and quasiconformality seems to be the best transverse regularity of the foliation $\mathcal{F}$ which is satisfied everywhere.

Another important result in [L3] is Lyubich's Hyperbolicity Theorem for the renormalization operator of real bounded type. Using this powerful result, the foliation was shown to be transversally $C^{1+\alpha}$-conformal along the hybrid class $\mathcal{H}_c$ of a Feigenbaum point c (this is an expected regularity of a codimension-one stable foliation). As a consequence, some of Milnor's conjectures were proven, in particular, the following result stated here in the case of stationary combinatorics.

Theorem 1.1.1 (Self-Similarity Theorem [L3]). *Let M_0 be a real Mandelbrot copy and $\sigma : M_0 \to M$ be the homeomorphism of M_0 onto the whole Mandelbrot set M. Then σ has a unique real fixed point c. Moreover, σ is $C^{1+\alpha}$-conformal at c, with derivative at c equal to the Feigenbaum universal scaling constant $\lambda = \lambda_M > 1$.*

In the same work, this result was generalized for bounded combinatorics by proving that the foliation is $C^{1+\alpha}$-conformal at infinitely renormalizable points of bounded type. Moreover, in [L4] the Hyperbolicity Theorem was proved for any real combinatorics. In this case, there are three types of combinatorics to deal with: bounded, "essentially bounded" and "high". The notion of "essentially bounded" was introduced in [L2].

For any bounded combinatorics, Sullivan [S] and McMullen [McM2] constructed the corresponding renormalization horseshoe and its strong stable foliation. It was proven in [L3] that the renormalization horseshoe is hyperbolic.

The unbounded combinatorics can be split into two types: essentially bounded and high. In the former case, the unboundedness is produced by the saddle-node behavior of the critical point. This phenomenon can be analyzed by means of parabolic bifurcation theory. For instance, there is a special sequence of real little copies of the Mandelbrot set which converge to the cusp of a primitive real hyperbolic component [D2]. The parameters in this sequence of little copies have "essentially bounded" combinatorics. Loosely speaking this means that a big period of renormalized maps is created only by saddle-node behavior of the return maps.

A basic geometric quality of infinitely renormalizable maps is *a priori* bounds. By definition, an infinitely renormalizable quadratic map f has such bounds if there exists a lower bound $\mu > 0$ such that for every $n \in \mathbb{N}$, the renormalization $R^n f$ has a quadratic-like extension $U \to V$, whose fundamental annulus $V \setminus U$ has modulus at least μ.

For real, infinitely renormalizable quadratic-like maps with bounded combinatorics, this property was proven by Sullivan [S]. Complex bounds were proven for real quadratics of "essentially big type" by Lyubich [L2]. In [LY], complex bounds were proven for infinitely renormalizable maps with essentially bounded combinatorics, thus completing the story for real maps. Independently, it was done by Levin and van Strien [LS].

The study of geometric limits of renormalization of quadratic-like maps with essentially bounded type was carried out by Hinkle [Hi], based on the *a priori* bounds of [LY]. Using this result, Lyubich proved hyperbolicity of the renormalization operator with essentially bounded combinatorics [L4].

Let f be a quadratic-like germ which is infinitely renormalizable. Its *tuning invariant* is defined as $\tau(f) = \{M_0, M_1, M_2, ...\}$ where M_n is the maximal M-copy containing $\chi(R^n f)$ and χ is the straightening map. We say that f has real combinatorics if all M-copies in $\tau(f)$ are real. Let Σ be the space of all possible real combinatorial types $\tau = \{M_k\}_{k=-\infty}^{\infty}$, where the copies M_k are selected arbitrarily from the family of real maximal Mandelbrot copies.

Let us describe briefly the Renormalization Theorem, since it will be the main tool for the proof of our result. Let us say that an infinitely renormalizable quadratic-like map f is *completely nonescaping* under the renormalization if some full renormalization orbit $\{R^n f\}_{n=-\infty}^{\infty}$ is well defined, $R^n f \in \mathcal{C}$, and $\operatorname{mod}(f_n) \geq \mu = \mu(f) > 0$, $n \in \mathbb{Z}$. Let $\mathcal{A} \subset \mathcal{Q}$ stand for the set of completely nonescaping maps with real combinatorics. We call this set the *(full) renormalization horseshoe.* Then, for any real combinatorics, the renormalization operator is *hyperbolic* in the space of real quadratic-like germs over $\mathcal{A}$. In particular, the stable leaves $\mathcal{H}(f)$, $f \in \mathcal{A}$, which are codimension-one real analytic submanifolds in the space of real quadratic-like germs, form an R-invariant lamination. Moreover, if $g \in \mathcal{H}(f)$ and $\operatorname{mod}(g) \geq \nu$, then $||R^n f - R^n g||_V \leq C\rho^n$ for $n \geq N(\nu)$, where $\rho \in (0,1)$ is a constant and V is a neighborhood of the origin in $\mathbb{C}$[L4].

The natural question is for which other points in the Mandelbrot set or for which other combinatorics, we can have a better regularity of the holonomy map than just quasi-conformality. McMullen has given a geometric condition for which a quasi-conformal map from the Riemann sphere $\hat{\mathbb{C}}$ to itself, conformal on a measurable set, is $C^{1+\alpha}$-conformal at a point in the boundary of this measurable set; this geometric notion is known as "measurable deep points" [McM2]. For example, root points of primitive hyperbolic components are measurable deep points in the boundary of the Mandelbrot set. Since the homeomorphism σ between a primitive little Mandelbrot copy and the Mandelbrot set is holomorphic in the interior of the Mandelbrot set, and it can be extended in a quasi-conformal way in a neighborhood of the Mandelbrot set [L2], the holonomy map is $C^{1+\alpha}$-conformal at these points.

Let us also mention that the results of [ALM] show that, in appropriate Banach Spaces of analytic unimodal maps, the set of non-regular topological classes in the space of unimodal maps, form a lamination $\mathcal{L}$ with analytic leaves of codimension one and quasi-symmetric holonomy, at least almost everywhere. On the other hand, A. Avila and G. Moreira [AvM] have shown some other interesting properties of the lamination by topological classes: the stratification of the set of typical non-regular analytic unimodal maps by topological classes is singular, in the sense that it fails drastically to be absolutely continuous. The lamination $\mathcal{L}$ has automatically quasi-symmetric holonomy but quasi-symmetric maps are not always absolutely continuous (even though quasi-conformal maps are). It turns out that $\mathcal{L}$ is very far from being absolutely continuous, at least at the set of non-regular leaves. For example it is completely singular on the set of Collet-Eckmann maps satisfying the conclusions of Theorem A in [AvM].

1.2 Results

In the case of infinitely renormalizable quadratic maps with unbounded combinatorics, we will focus on the combinatorics for the *tripling* essentially bounded case. This case of essentially bounded combinatorics is related to the existence of a sequence of little Mandelbrot copies converging to the cusp of the real copy of the Mandelbrot set of period 3 [D2, Hi]. For this particular sequence, it is possible to estimate the rate of convergence to the cusp and the diameter of its elements. In particular, we give an estimation of the distance between two consecutive elements of this sequence of little copies of the Mandelbrot set. It turns out that points in these copies have bounded essential period, for which we can construct infinitely renormalizable parameters with essentially bounded combinatorics. In order to prove smoothness, we need to look for parameters with combinatorics given by subsequences that do not converge "too" fast to the cusp.

Next we state the theorem, which gives a condition on the combinatorics of a quadratic-like germ of tripling essentially bounded combinatorics that implies smoothness of the holonomy.

Theorem 1.2.1. *Let $\{M_k\}_{k=1}^{\infty}$ be the sequence of copies with tripling essentially bounded period, with corresponding periods $n_k = 3k+2$. Consider a real quadratic-like germ $f_* \in \mathcal{A}$ with tuning invariant $\tau(f_*) = \{M_{k_n}\}_{n=0}^{\infty}$, where this sequence is given by an arbitrary choice of elements in $\{M_k\}_{k=1}^{\infty}$. Let $\rho \in (0,1)$ be an upper bound of the contraction factor in the stable leaves. Suppose that*

$$\sum_{n\geq 1} \rho^n \log k_n < \infty.$$

Then the foliation $\mathcal{F}$ is transversally C^1-conformal (smooth) at the point $\chi(f_*)$.

Let us outline the proof of this theorem. Let $\mathcal{S}$ be a transversal to the leaf $\mathcal{H}_{\mathbb{R}}(f_*)$ through $f \in \mathcal{H}_{\mathbb{R}}(f_*)$. In order to check C^1-conformality of the holonomy, it is enough to check this property for the holonomy from $\mathcal{S}$ to the unstable manifold $\mathcal{W}^u \equiv \mathcal{W}^u_{loc}(f_*)$. Let $\mathcal{S}^m = R^m(\mathcal{S})$ be the m iterate of $\mathcal{S}$ under R, then it is sufficient to study the holonomy h_m from $\mathcal{S}^m$ to $\mathcal{W}^u(m)$: if h denote the holonomy from $\mathcal{S}$ to $\mathcal{W}^u$ then by the R-invariance of the foliation $\mathcal{F}$, $h = R^{-m} \circ h_m \circ R^m$ where R^m is a local conformal diffeomorphism.

The principal tool of the proof is the Renormalization Theorem for any real combinatorics. There exists an R^{-1}-invariant family of real analytic curves $\mathcal{W}^u_{loc}(f)$, $f \in \mathcal{A}$ ("local unstable leaves"). Also, there exists a sequence of bidisks $Q(n)$ centered at $R^n f_*$, where an iterate $T = R^N$ acts hyperbolically on this family (uniformly contracting in the horizontal direction and uniformly expanding in the vertical direction).

By the hyperbolicity of T, the transversals $\mathcal{S}^n$ can be eventually represented as graphs of analytic functions $\phi_n : E^u_n(\gamma) \to E^s_n(\gamma)$ with bounded vertical slope, where $E^{s/u}_n(\gamma)$ is the γ-ball in the tangent space $E^{u/s}(n)$ at $R^n f_*$. Moreover, these graphs are exponentially close to the corresponding unstable manifolds $\mathcal{W}^u(n)$. The local unstable manifolds $\mathcal{W}^u(R^n f_*) \cap Q(n)$ can be also parametrized in the same way by some functions ψ_n. By the hyperbolicity of T on the family of bidisks $Q(n)$, the manifolds $\mathcal{S}^n$ approach, exponentially fast, to the unstable manifolds $\mathcal{W}^u(n)$:

$$||\phi_n - \psi_n||_{C^1} \leq \kappa\rho^n,$$

where $\rho \in (0,1)$ is a strict upper bound of the contraction factor in the stable leaves.

In a first lemma, we prove that for big n, the holonomy from $\mathcal{S}^n$ to $\mathcal{W}^u(R^n f_*)$ has an exponentially small ratio distortion for points with comparable distances to $R^n f$, which follows by the Renormalization Theorem and the λ-lemma. We also need to have control in the distortion of the renormalization operator R^n. By the Koebe Distortion Theorem, the ratio distortion of $R^n f$ is of order $O(\epsilon)$ for points in the domain of R^n that are mapped to points of relative distances of order $O(\epsilon)$ from $R^n f$.

To prove smoothness of the holonomy, we need to give an estimation for the distortion of the holonomy for two arbitrary points. These two points may not have comparable distances to f, so we need to define a string of points between them, in such a way that every two consecutive points in this string have comparable distances to f; a natural way to do it, is by going half the distance to f at each step. The number of points from that string, in the domain L_n of R^n, but not in the domain L_{n+1} of R^{n+1} (see §5.2), will be comparable with $\log k_n$ by the estimation on the size of the diameter of M_{k_n} and by the Koebe Distortion Theorem. This factor will be added to our distortion estimate, and under the hypothesis of the combinatorics, we can see that the ratio distortion of the holonomy is controlled by the tail of a convergent series in ρ and $\log k_n$. Thus, the holonomy will be of order $o(1)$, which implies smoothness.

Corollary 1.2.2 (Self-similarity for essentially bounded combinatorics). *Let c be a real infinitely renormalizable parameter value with essentially bounded combinatorics satisfying the hypothesis in Theorem 1.2.1, and let $M_1 \in \mathcal{M}$ be the Mandelbrot copy containing c. Then the homeomorphism $\sigma : M_1 \to M$ is C^1-conformal at c.*

In the same way, we consider any real maximal little Mandelbrot copy M_0 of period $n \geq 3$. Now let $\{M_n\}_{n=1}^{\infty}$ be the sequence of little Mandelbrot copies with essentially bounded combinatorics which converges to the cusp of M_0 (see [§4]). Then, the holonomy map is smooth at parameters that satisfy the same conditions on the combinatorics as in Theorem 1.2.1.

In the last chapter, we look at conjugations between satellite parabolic maps and the map $z \mapsto z^2 + 1/4$. It is known that at those maps, the renormalization operator is not defined. However, some study has been done in order to conjugate them to the root of the Mandelbrot set ([H1], [H2], [BD]). Let us consider a satellite parabolic map, we can conjugate the dynamics on a component of its Fatou set to the dynamics on the Fatou set of $P(z) = z^2+1/4$, by a μ-conformal map: we give a construction of a continuous map, between a neighborhood of the Fatou set of $P(z) = z^2 + 1/4$ and a neighborhood of the Fatou component of $f(z) = z^2 - 3/4$ containing the critical value. This continuous map conjugates the dynamics in the Fatou set of P and it is a μ-conformal map. This result makes use of some techniques developed by [H1].

Theorem 1.2.3. *There is a neighborhood U of $K(P)$ and a continuous function $\psi : U \longrightarrow \mathbb{C}$ such that for all $z \in K(P)$, $\psi \circ P = f^{\circ 2} \circ \psi$. Moreover, $\psi \in W^{1,p}_{loc}$, for all $p < 2$ and $\overline{\partial}\psi = 0$ a.e. on $K(P)$.*

1.3 Further study

The result about the holonomy for tripling essentially bounded combinatorics stated before corresponds to the case of maps with only one central saddle-node cascade, which is related to the existence of a sequence of little copies approaching the tripling cusp. In general, we can consider essentially bounded parameters with more than one central cascade of this type, that is, we approach the tripling cusp with the sequence of little copies, then for every element in this sequence we can consider a corresponding sequence of little copies converging to its tripling cusp, and we do this procedure finitely many times. This correspond to the most general version of essentially bounded combinatorics. Therefore, the natural question is, for which of these combinatorics we have some regularity of the holonomy? Is it true, for instance, that the Lipschitz condition is always satisfied by the holonomy map in some cases?

Another direction is to consider the case of "high" unbounded combinatorics. One question is to give a characterization of infinitely renormalizable parameters with "high" combinatorics, for which the holonomy is at least smooth. Or, we can also study how singular the holonomy could be.

Yamposlky in [Y] has approached differently the study of polynomials satisfying the essentially bounded condition. He considers them as small perturbation of parabolic maps. Such a geometric consideration draws an instructive parallel with the critical circle maps case. Then, it would be interesting to study some properties of the holonomy in the space of parameters for circle maps.

Finally, let us mention that some of the pictures were generated with a series of programs called *Julia* created by C. McMullen. Also, the program *xfig* for linux was used.

1.4 Notation

- As usual $\hat{\mathbb{C}}$ denotes the Riemann sphere, $\mathbb{C}$ the complex numbers, $\mathbb{R}$ the real line and $\mathbb{N}$ the non-negative integers.

- $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$.

- $\bar{X}$ denotes the closure of a set X.
- if $U \Subset V$, we say that U is compactly contained in V ($\bar{U}$ is compact and $\bar{U} \subset V$).
- a *topological annulus* means a doubly connected domain in $\mathbb{C}$.
- if V is a simply connected domain and $U \subset V$ then $\mathrm{mod}(U, V) = \sup_A \mathrm{mod}(A)$ where A is an annulus separating U from ∂V.
- $\mathcal{B}$ will denote a Banach space, $\mathcal{B}_r(x)$ will stand for the ball of radius r with center at x and $\mathcal{B}_r \equiv \mathcal{B}_r(0)$.
- $P_c(z) = z^2 + c$.
- in a dynamical context, f^n denotes the n-th iterate of f.
- M will denote the Mandelbrot set.
- $a \asymp b$ means that the ratio a/b stays away from 0 and ∞.
- $a \sim b$ means that $a/b \to 1$.

Chapter 2

Background

2.1 Preliminaries

We discuss here some of the definitions and results that will be used through this work. All the results are well known (see [L1],[M1]).

2.1.1 The Carathéodory topology

A *disk* is an open simply connected region in $\mathbb{C}$. Consider the set $\mathcal{D}$ of pointed disks (U, u). The Carathéodory topology on $\mathcal{D}$ is defined as follows: $(U_n, u_n) \to (U, u)$ if and only if
(i) $u_n \to u$;
(ii) for any compact $K \subset U$, $K \subset U_n$ for all n sufficiently large;
(iii) for any open connected N containing u, if $N \subset U_n$ for infinitely many n, then $N \subset U$.

Equivalently, convergence means $u_n \to u$, and for any subsequence such that $(\hat{\mathbb{C}} - U_n) \to K$ in the Hausdorff topology on compact sets of the sphere, U is equal to the component of $(\hat{\mathbb{C}} - U_n) \to K$ containing u.

Carathéodory topology on functions. Let $\mathcal{H}$ be the set of all holomorphic functions $f : (U, u) \to \mathbb{C}$ defined on pointed disks $(U, u) \in \mathcal{D}$. We define the Carathéodory topology on $\mathcal{H}$ as follows. Let $f_n : (U_n, u_n) \to \mathbb{C}$ be a sequence in $\mathcal{H}$. Then f_n converges to $f : (U, u) \to \mathbb{C}$ if:
(i) $(U_n, u_n) \to (U, u)$ in $\mathcal{D}$.
(ii) for all n sufficiently large, f_n converges to f uniformly on compact subsets of U.

Any compact set $K \subset U$ is eventually contained in U_n, so f_n is defined on K for all n sufficiently large. We note the following facts

1. If the domains are K-quasidisks then the Carathéodory convergence of pointed domains is equivalent to the Hausdorff convergence of their closures.

2. The set of K-quasidisks in $\mathbb{C}$ containing a definite neighborhood of the origin and with bounded diameter is compact in the Hausdorff topology.

2.1.2 The Koebe Distortion Theorem

One important tool in the estimations of distortion of analytic maps is given by the set of Koebe distortion theorems. The Koebe distortion theorems make precise the fact that a univalent map has bounded geometry. The principle can be stated as follows:

Theorem 2.1.1 (Koebe Distortion). *The space of univalent maps $f : \mathbb{D} \to \bar{\mathbb{C}}$ is compact up to post-composition with automorphism of $\bar{\mathbb{C}}$.*

The Koebe principle also controls univalent maps defined on disks which are not round. For this case, one obtains bounded geometry after removing an annulus of definite modulus.

Theorem 2.1.2. *Let $D \subset U \subset \mathbb{C}$ be disks with $\mathrm{mod}(D, U) > m > 0$. Let $f : U \longrightarrow \mathbb{C}$ be a univalent map. Then there is a constant $C(m)$ such that for any $x, y, z \in D$*

$$\frac{1}{C(m)}|f'(x)| \leq \frac{|f(y) - f(z)|}{|y - z|} \leq C(m)|f'(x)|.$$

2.1.3 Quasi-conformal Mappings

Definition 2.1.3. *A homeomorphism $f : X \longrightarrow Y$ between Riemann surfaces X and Y is K-quasi-conformal, $K \geq 1$ if for all annuli $A \subset X$,*

$$\frac{1}{K}\,\mathrm{mod}(A) \leq \mathrm{mod}(f(A)) \;\leq K\,\mathrm{mod}(A).$$

Let us use the notation $dz = dx + idy$, $d\bar{z} = dx - idy$, and

$$f_z \equiv \frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} \equiv \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y).$$

The Jacobian $J(f)$ of f is given by $J(f) = |f_z|^2 - |f_{\bar{z}}|^2$, then f preserves orientation if and only if $|f_{\bar{z}}| < |f_z|$. We are concerned only with mappings that satisfy this condition.

There is also an equivalent analytic definition: f is K-quasi-conformal if locally has distributional derivatives in L^2, and if the complex dilatation μ, given locally by

$$\mu(z)\frac{d\bar{z}}{dz} = \frac{f_{\bar{z}}}{f_z} = \frac{\partial f/\partial \bar{z}}{\partial f/\partial z}\frac{d\bar{z}}{dz},$$

satisfies $|\mu| \leq (K-1)/(K+1)$ almost everywhere. The dilatation μ is also called the *Beltrami coefficient* of f, and the equation $f_{\bar{z}} = \mu f_z$ is the *Beltrami equation.* Note that $|\mu| < 1$ if f is orientation preserving and that $\mu = 0$ if and only if f is conformal. Also, we can associate to f an infinitesimal ellipse field by assigning to each z in the domain, the ellipse that is mapped to a circle by f. The argument of the major axis of this infinitesimal ellipse corresponding to f at z is $\pi/2 + \arg(\mu)/2$, and the eccentricity is $(|f_z| - |f_{\bar{z}}|)/(|f_z| + |f_{\bar{z}}|) = (1-|\mu|)/(1+|\mu|)$. Thus, we associate to any μ satisfying $|\mu| < 1$ an infinitesimal ellipse field, that is, a choice of direction and eccentricity at each point. Solving the Beltrami equation $f_{\bar{z}} = \mu f_z$ is then equivalent to finding f whose associated ellipse field coincides with that of μ. A mapping f is 1-quasi-conformal if and only if f conformal in the usual sense.

Measurable Riemann Mapping Theorem The great flexibility of quasi-conformal maps comes from the fact that any μ with $\|\mu\|_\infty < 1$ is realized by a quasi-conformal map.

Theorem 2.1.4 (Ahlfors-Bers). *For any L^∞ Beltrami differential μ on the plane with $\|\mu\|_\infty < 1$, there is a unique quasi-conformal map $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that ϕ fixes 0 and 1 and the complex dilatation of ϕ is μ.*

2.1.4 The λ-lemma

Let $X \subset \mathbb{C}$ be a subset of the complex plane. A *holomorphic motion* of X over a Banach domain $(\Lambda, 0)$ is a family of injections $h_\lambda : X \to \mathbb{C}$, $\lambda \in \Lambda$, with $h_0 =$id, holomorphically depending on $\lambda \in \mathcal{B}_1$ (for any given $z \in X$). The dual viewpoint on holomorphic motions is to consider the graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in X$, which form a lamination $\mathcal{F}$ in $\Lambda \times \mathbb{C}$ with complex codimension-one analytic leaves.

A basic fact about holomorphic motions usually known as the "λ-lemma", consists of two parts: extension and quasi-conformality.

Theorem 2.1.5. *The λ-lemma (Extension). A holomorphic motion $h_\lambda : X_* \to X_\lambda$, of a set $X_* \subset \mathbb{C}$ over a topological disk D admits an extension to a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole complex plane over D.*

Given two complex one-dimensional transversals $\mathcal{S}$ and $\mathcal{T}$ to the lamination $\mathcal{F}$ in $\mathcal{B}_1 \times \mathbb{C}$, we have a holonomy map $\mathcal{S} \to \mathcal{T}$. We say that this map is locally quasi-conformal if it admits local quasi-conformal extensions near any point.

Theorem 2.1.6. *The λ-lemma (Quasi-conformality). Holomorphic motion h_λ of a set X over a Banach ball $\mathcal{B}_1$ is transversally quasi-conformal. The local dilatation K of the holonomy from $p = (\lambda, u) \in \mathcal{S}$ to $q = (\mu, v) \in \mathcal{T}$ depends only on the hyperbolic distance ρ between the points λ and μ in $\mathcal{B}_1$. Moreover, $K = 1 + O(\rho)$ as $\rho \to 0$.*

2.2 Dynamics of Complex Polynomials

2.2.1 The Filled Julia set

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$, $f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d$.

Definition 2.2.1. *A point z such that $f^p(z) = z$ for some $p \geq 1$ is a periodic point for f. The least such p is the period of z. The multiplier of a point z of period p is the derivative $(f^p)'(z)$ of the first return map.*

If $f^p(z) = z$, we say the z is *repelling* if $|(f^p)'(z)| > 1$, *indifferent* if $|(f^p)'(z)| = 1$, *attracting* if $|(f^p)'(z)| < 1$ and *super-attracting* if $|(f^p)'(z)| = 0$. An indifferent point is parabolic if $(f^p)'(z)$ is a root of unity.

The *Fatou set* $F(f) \subset \hat{\mathbb{C}}$ is the largest open set such that the iterates $\{f^n : n \geq 1\}$ form a normal family. We denote by K_f the set of points with bounded orbit under f

$$K_f = \{z \in \mathbb{C} \mid f^{\circ n}(z) \nrightarrow \infty\}$$

and by $A_f(\infty)$ the set of points with unbounded orbit

$$A_f(\infty) = \mathbb{C} \backslash K_f = \{z \in \mathbb{C} \mid f^{\circ n}(z) \to \infty\}.$$

The set K_f is called the *filled Julia set*, the set $A_f(\infty)$ is called the attracting basin of ∞ and the common boundary $\partial K_f = \partial A_f(\infty) = J_f$ is called the *Julia set*. Note that the critical orbit is bounded if the critical point belongs to K_f and unbounded if the critical point belongs to $A_f(\infty)$. The following lemma states a basic property about K_f.

Lemma 2.2.2. *For any polynomial f of degree at least two, the set $K_f \subset \mathbb{C}$ is compact, with connected complement. It can be described as the union of the Julia set J_f together with all bounded components of the complement $\mathbb{C} \backslash J_f$.*

In particular, the Fatou set is the complement of the Julia set $F(f) = \mathbb{C}\backslash J_f$. A connected component of $\mathbb{C}\backslash J_f$ is called a *Fatou component.*

We define the function $G_f : \mathbb{C} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n}\log_+(|f^{\circ n}(z)|)$$

where $\log_+(|z|)$ =max$\{0,\log(|z|)\}$. The map G_f satisfies that is continuous on $\mathbb{C}$, harmonic on $\mathbb{C}\backslash K_f$ and equal to 0 on K_f. Moreover, $G_f(z) = \log(|z|)+O(1)$ when $|z| \to \infty$, and $G_f(f(z)) = dG_f(z)$. These properties show that the restriction of G_f to $\mathbb{C}\backslash K_f$ is the Green function of the filled Julia set K_f.

Equipotentials. Choose any $\eta > 0$. The set $G_f^{-1}(\eta)$ is called the *equipotential* of value η. Note that f maps each equipotential $G_f^{-1}(\eta)$ to the equipotential $G_f^{-1}(d\eta)$ by a d-to-one fold covering map.

2.2.2 The Böttcher Theorem

In a more general setting, we consider the dynamics of a holomorphic map in some small neighborhood of a fixed point. We assume that the fixed point is at $z = 0$, then we can describe the map by a power series of the form

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots,$$

which converges for $|z|$ sufficiently small. When $\lambda = 0$ the fixed point $z = 0$ is super-attracting, then the map takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $n \geq 2$ and $a_n \neq 0$.

Theorem 2.2.3 (Böttcher). *With f as above, there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, which conjugate f to the $n-$th power map $w \mapsto w^n$ throughout some neighborhood of zero. Furthermore, ϕ is unique up to multiplication by an $(n-1)-st$ root of unity.*

Thus, near any critical fixed point, f is conjugate to a map of the form $\phi \circ f \circ \phi^{-1} : w \mapsto w^n$, with $n \geq 2$. This theorem is applied in the case of a fixed point at infinity. For example, the point at ∞ is a super-attracting fixed point for any polynomial f of degree $d \geq 2$ extended to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Application to dynamics of polynomials. For a monic polynomial f of degree d, set $U_f = \{z \in \mathbb{C} : G_f(z) > G(f)\}$. Then, there exists a unique analytic isomorphism

$$\varphi_f : U_f \longrightarrow \mathbb{C} \backslash \bar{D}_{\exp G(f)}$$

satisfying $\varphi_f(z)/z \longrightarrow 1$ as $|z| \to \infty$ and conjugating f to the polynomial $f_0(z) = z^d$, i.e. $\varphi_f \circ f = f_0 \circ \varphi_f$. If all the critical points of f are contained in K_f then $U_f = \mathbb{C} \backslash K_f$ and K_f are connected.

Theorem 2.2.4. *Let f be a polynomial of degree $d \geq 2$. If the filled Julia set K_f contains all of the finite critical points of f, then both K_f and J_f are connected, and the complement of K_f is conformally isomorphic to the exterior of the closed unit disk $\overline{\mathbb{D}}$ under an isomorphism*

$$\varphi_f : \mathbb{C} \backslash K_f \longrightarrow \mathbb{C} \backslash \overline{\mathbb{D}}$$

which conjugate f on $\mathbb{C} \backslash K_f$ to the d-th power map $w \mapsto w^d$. On the other hand, if at least one critical point of f belongs to $\mathbb{C} \backslash K_f$, then both K_f and J_f have uncountably many connected components.

2.2.3 External Rays

Suppose that the set K_f is connected. Let $\varphi_f : \widehat{\mathbb{C}} \backslash K_f \longrightarrow \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ be as above. The orthogonal trajectories $\{z : \arg(\varphi_f(z)) = \text{constant}\}$ to the family of equipotentials curves are called *external rays* for K_f. The ray of *external argument* $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $\theta \in \mathbb{R}/\mathbb{Z}$, is defined by $R_\theta = \varphi_f^{-1}(\{re^{2\pi i\theta} | r > 1\})$. The external rays and equipotentials form two orthogonal invariant foliations of $\hat{\mathbb{C}} \backslash K_f$.

Definition 2.2.5. *An external ray R_θ is called rational if its angle $\theta \in \mathbb{R}/\mathbb{Z}$ is rational; and periodic if θ is periodic under multiplication by the degree d so that $d^p\theta \equiv \theta (mod 1)$ for some $p \geq 1$.*

Some particular rational rays are essential in the construction of the puzzle in the quadratic polynomial case. If K_f is not connected, one can still consider the orthogonal trajectories to the equipotentials in $\mathbb{C} \backslash K_f$ minus the critical points of G_f. Some rays branch at critical points of G_f, the others continue unbroken toward K_f.

2.3 The Quadratic Family

Now, let $P(z) = az^2 + bz + c$ be a quadratic polynomial. It can be conjugated by $w = \lambda z$ to a monic polynomial $z^2 + \alpha z + \beta$. This can be further conjugated by a translation that moves any given point to 0. If we move one of the fixed points to 0, we have conjugated P to the form $\lambda z+z^2$, where λ is the multiplier of the fixed point. This does not determine the conjugacy class uniquely, as we can place the second fixed point at 0. If we move the critical point to 0, we have conjugated P to the form

$$P_c(z) = z^2 + c, \quad c \in \mathbb{C}$$

and different choice of c corresponds to different conjugacy class. Thus, we can regard the c-plane as representing conjugacy classes of quadratic polynomials. The dynamical behavior of c plays a crucial role in determining the dynamics of P_c and the topology of K_c. By Theorem 2.2.4, if $P_c^n(0) \to \infty$ as $n \to \infty$, the Julia set J_c is totally disconnected. Otherwise, $P_c^n(0)$ is bounded and the Julia set is connected. This dichotomy is reflected in the definition of the Mandelbrot set M.

2.3.1 The Mandelbrot set

The *Mandelbrot set* M is defined as the set of $c \in \mathbb{C}$ for which $J(P_c)$ is connected, that is,

$$M = \{c \in \mathbb{C} : P_c^n(0) \not\to \infty \ , \ n \to \infty\}$$

Thus, $c \in M$ if and only if 0 does not belong to the basin of attraction of the super-attracting fixed point at ∞. If $c \in \mathbb{C}\backslash M$, then $J(P_c)$ is a Cantor set.

The Mandelbrot set itself is connected (see [DH1],[CG]). This is proven by constructing explicitly the Riemann mapping $\Phi_M : \mathbb{C} \setminus M \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ defined as

$$\Phi_M(c) = \varphi_c(c),$$

where φ_c is the Böttcher function of P_c.

A quadratic polynomial P_c with $c \in M$ is called *hyperbolic* if it has an attracting cycle. The set of hyperbolic parameter values is the union of some components of the interior of M called *hyperbolic components.* The *main cardioid* of M is defined as the set of points c for which P_c has a neutral fixed point. It encloses the *main hyperbolic component* where P_c has an attracting fixed point.

We also consider the family of quadratic maps mentioned above, $f_\lambda(z) = z^2 + \lambda z, \ \ \lambda \in \mathbb{C}$ in which the multipliers at the two fixed point are λ at $z = 0$, and $2 - \lambda$ at $z = 1 - \lambda$. The set of parameters in the λ-plane for which the corresponding orbit of 0 stays bounded under the dynamics of f_λ is a doubled branched covering of M, with the only branched point at 1.

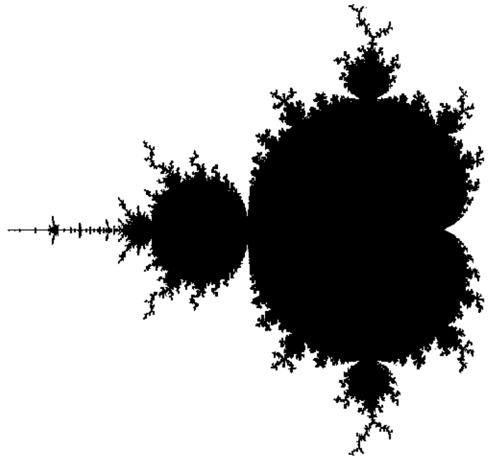

Figure 2.1: The Mandelbrot set

2.3.2 Limbs and Wakes of the Mandelbrot set

The boundary of each hyperbolic component Ω of the Mandelbrot set M can be parametrized by a map $\gamma_\Omega : [0,1) \longrightarrow \partial\Omega$ so that, at $c = \gamma_\Omega(t)$, the indifferent periodic orbit has multiplier $e^{2\pi it}$. The point $c = \gamma_\Omega(0)$ is called the root of the hyperbolic component Ω. The largest hyperbolic component Ω_0 consists of all parameter values c for which P_c has an attracting fixed point. Its boundary is the main cardioid mentioned before. At each boundary point $\gamma_{\Omega_0}(p/q)$, for any $p/q \in (0,1) \cap \mathbb{Q}$, there is attached a hyperbolic component $\Omega_{p/q}$ of period q. We define the *p/q-limb* of M, $M_{p/q}$, as the union of $c = \gamma_{\Omega_0}(p/q)$ and the connected component of $M \setminus \overline{\Omega_0}$ attached to the main cardioid at the point $c = \gamma_{\Omega_0}(p/q)$.

Using the Riemann map Φ_M, we can define the *parameter external rays* and *equipotentials* as the preimages of the straight rays going to ∞ and round circles centered at 0. Define the parameter external ray of external argument θ as $R_M(\theta) = \Phi_M^{-1}(\{re^{2\pi i\theta} : 1 < r < \infty\})$. If $R_M(\theta)$ has a limit $c \in \partial M$ when $r \to 1$, we say that $R_M(\theta)$ lands at c. It is known that all external rays with rational arguments land at either a root of a hyperbolic component or at a Misiurewicz point, i.e. a parameter value $c \in \partial M$ for which $w = 0$ is strictly preperiodic under P_c.

There are exactly two external rays landing at each root point in M (except at $c = 1/4$). Given $p/q \in (0,1)\cap\mathbb{Q}$, we denote by $\theta^-_{p/q}$ and $\theta^+_{p/q}$ the arguments of the two external rays landing at the root point of $\Omega_{p/q}$, i.e., at $\gamma_{\Omega_0}(p/q) \in \partial\Omega_0$. Then, we define the *$p/q$-wake* of M, $W_{p/q}$, as the open subset of $\mathbb{C}$ that contains the p/q-limb of M and is bounded by these two rays.

2.3.3 The Dynamics of $P_c(z) = z^2 + c$ for $c \in W_{1/2}$

The polynomial $P_c(z) = z^2 + c$ has one critical point $w = 0$. For $c \in M$ we denote by β_c the fixed point satisfying $\beta_c = \gamma_c(0)$ and by β'_c the other preimage of β_c with $\beta'_c = \gamma_c(1/2) = -\beta_c$. Let α_c be the other fixed point of $P_c(z)$. For $c \in W_{1/2}$, $\alpha_c = \gamma_c(1/3) = \gamma_c(2/3)$. We denote by α'_c the other preimage of α_c, then $\alpha'_c = \gamma_c(1/6) = \gamma_c(5/6) = -\alpha_c$. Choose $\eta > 0$ and set

$$W_i = W_i^c = \{z \in \mathbb{C} : G_c(z) \le \frac{\eta}{2^i}\}, \quad W = W_0.$$

The lines $\ell_c = R_c(1/3) \cup \{\alpha_c\} \cup R_c(2/3)$ and $\ell'_c = R_c(1/6) \cup \{\alpha'_c\} \cup R_c(5/6)$ decompose W into 3 compact subsets V, V', V'' with $w \in V$, $\beta_c \in V'$, $\beta'_c \in V''$. Set $V_i = V \cap W_i$, etc. (The V_i are called the central pieces of the puzzle.)

The polynomial P_c induces a homeomorphism from V'_{i+1} onto $V'_i \cup V_i$, a homeomorphism from V''_{i+1} onto $V'_i \cup V_i$ and a mapping of degree 2 from V_{i+1} onto V''_i.

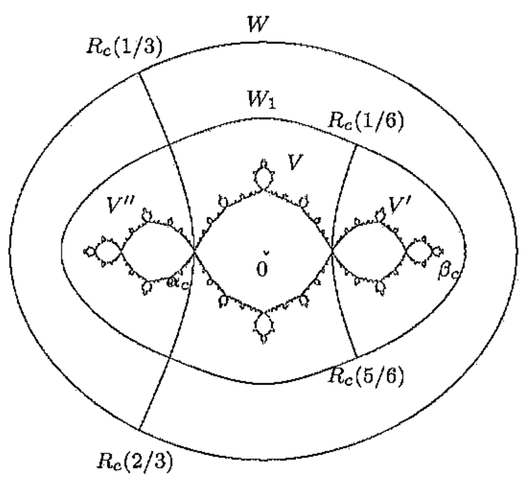

Figure 2.2: The first two levels in the puzzle for $c \in W_{1/2}$

2.4 Parabolic Periodic points

This section is mostly based in the works of [**S1, S2, O**]. Here we explain briefly the theory of parabolic points, parabolic bifurcation, and the existence of Fatou coordinates.

2.4.1 Change of coordinates

We consider functions $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$ which are defined and holomorphic in some neighborhood of the origin, with the multiplier λ at the fixed point equal to a root of unity, $\lambda^q = 1$. Such a fixed point is said to be *parabolic* provided that $f^{\circ q}$ is not the identity map. Consider the case $\lambda = 1$. Then, we write the map f as

$$f(z) = z(1 + az^n + (\text{higher terms})) = z + az^{n+1} + (\text{higher terms})$$

with $n \geq 1$ and $a \neq 0$. The integer $n+1$ is called the *multiplicity* of the fixed point.

There are n equally spaced repelling directions at the origin, separated by n equally spaced attracting directions. Note that the repelling directions for f are just the attracting directions for the inverse map f^{-1}, which is also well defined and holomorphic in a neighborhood of the origin.

We can use the change of coordinates $w = \phi(z) = c/z^n$, where $c = -1/na$. If we write $f(z) = z(1 + az^n + o(z^n))$, where the notation $o(z^n)$ stands for a remainder term depending on z, which tends to zero faster than z^n so that $o(z^n)/z^n \to 0$ as $z \to 0$, the corresponding transformation in the w-plane is $w \mapsto F(w) = \phi \circ f \circ \phi^{-1}(w)$ where $\phi^{-1}(w) = \sqrt[n]{c/w}$, taking the branch of the n-th root in an appropriate domain. Note that

$$f \circ \phi^{-1}(w) = \sqrt[n]{\frac{c}{w}}(1 + a\frac{c}{w} + o(\frac{1}{w})),$$

then

$$F(w) = w(1 + a\frac{c}{w} + o(\frac{1}{w}))^{-n} = w(1 + \frac{-nac}{w} + o(\frac{1}{w})).$$

Since $nac = -1$, this can be written as $F(w) = w + 1 + o(1)$ where $o(1)$ stands for the remainder term which tends to zero as $|w| \to \infty$.

There is a more precise statement for the asymptotes of the map F, written the map f as $f(z) = z(1 + az^n + bz^{n+1} + O(z^{n+1}))$ then

$$\begin{aligned} F(w) &= c[(\frac{c}{w})^{\frac{1}{n}}(1 + a\frac{c}{w} + b(\frac{c}{w})^{\frac{n+1}{n}} + \cdots)]^{-n} \\ &= w(1 + a\frac{c}{w} + b(\frac{c}{w})^{\frac{n+1}{n}} + \cdots)^{-n} \\ &= w(1 - na\frac{c}{w} - nb(\frac{c}{w})^{\frac{n+1}{n}} + \cdots) \\ &= w + 1 + Cw^{-\frac{1}{n}} + \cdots = w + 1 + O(1/\sqrt[n]{w}) \end{aligned}$$

as $|w| \to \infty$.

That is, there exist constants r and C so that $|F(w) - w - 1| \leq C/\sqrt[n]{|w|}$ whenever $|w| \geq r$. In particular, it follows that $\mathrm{Re}(F(w)) > \mathrm{Re}(w) + 1 - \eta$ when $|w| \geq r_\eta$.

For simplicity, we only treat the case where f has a parabolic fixed point with multiplier 1, i.e. $f_0(z) = z + z^2 + ...$, and $F(w) = w + 1 + \frac{C}{w} + ...$ as $w \to \infty$.

The topology that we will use is the Compact-Open topology together with domain of definition. For any holomorphic map f defined on a subset of $\hat{\mathbb{C}}$, let $D(f)$ denote the domain of definition of f. Now set

$$\mathcal{H} = \left\{ f : D(f) \to \hat{\mathbb{C}} \mid f \text{ is holomorphic and } \partial \overset{\circ}{D}(f) = \partial D(f) \right\}$$

where two functions are considered to be distinct if they have different domain of definition. We can construct a non-Hausdorff topology on $\mathcal{H}$ so that $f_m \to f$ if and only if for every compact set $K \subset \overset{\circ}{D}(f)$ there is an n_0 so that $K \subset \overset{\circ}{D}(f_m)$ for every $m \geq n_0$, and $f_m|_K \to f|_K$ uniformly as $m \to +\infty$.

We take a very small $r_0 > 0$ so that the closed disk $K_0 = \overline{D_{2r_0}} \subset \overset{\circ}{D}(f_0)$. This disk K_0 will remain fixed. Then, we take a small neighborhood N_0 of f_0 in the compact open topology. Assuming this is small enough, then $K_0 \subset D(f)$ for every $f \in N_0$.

Given $r_0 > 0$, we define $z_- = -r_0$ and $z_+ = r_0$ so that z_- is in the attracting direction of f_0 and z_+ is in the repelling direction.

Lemma 2.4.1. *(Fundamental regions for f_0). Let $\gamma_{s,0}$ be the maximal trajectory passing through z_s for the vector field $\dot{z} = i(f_0(z) - z)$, where $s \in \{+, -\}$. Then $\gamma_{s,0}$ is well defined on $\mathbb{R}$ and $\gamma_{s,0}(t) \in \overset{\circ}{K_0}$ for all $t \in \mathbb{R}$. Also $\gamma_{s,f}(t) \to 0$ as $t \to \pm\infty$. None of these paths intersect each other.*

Proof. The existence is given by the existence and uniqueness of solution curves. Also, for any autonomous differential equation the trajectories will never intersect, unless they coincide everywhere.

On the set $T_s = \{z \in K_0 \setminus \{0\} \ : \ |\arg(z/z_s)| < 3\pi/8\}$ we make the change of coordinates

$$w = I(z) = -\frac{1}{z},$$

and in this coordinate we get a function $F_s = I \circ f_0 \circ I|_{T_s}^{-1}$,

$$\begin{aligned} F_s(w) = I(f_0(z)) &= -\frac{1}{z}(1 + z + O(z^2))^{-1} \\ &= w(1 - \frac{1}{w} + O(w^{-1-1}))^{-1} \\ &= w(1 + \frac{1}{w} + O(w^{-1-1})) \\ &= w + 1 + O(w^{-1}) \ \text{ as } w \ \to \infty. \end{aligned}$$

If we define $\Gamma_{s,0} = I \circ \gamma_{s,0}$ then we have

$$\Gamma'_{s,0}(t) = I'(\gamma_{s,0}(t))\gamma'_{s,0}(t) = \frac{i[f_0(z) - z]}{z^2} \approx i,$$

where $z = \gamma_{s,0}(t)$.

Then $\Gamma_{\pm,0}(t)$ will be an almost vertical line in the w-coordinate, passing through $w_\pm = I(z_\pm) = \mp 1/r_0 \in \mathbb{R}$. Thus $\Gamma_{\pm,0}(t) \to \infty$ as $t \to \pm\infty$, implying that $\gamma_{\pm,0}(t) \to 0$ as $t \to \pm\infty$ (since $w = \infty$ corresponds to $z = 0$).

Since $\Gamma = \Gamma_{\pm,0}$ is an almost vertical line through $\pm 1/r_0$, $|\Gamma(t)|$ is always large and we must have $F_s \approx w + 1$. This implies that $\Gamma(\mathbb{R})$ cannot intersect $F_s(\Gamma(\mathbb{R}))$, which in turn implies that $\gamma(\mathbb{R})$ does not intersect $f_0(\gamma(\mathbb{R}))$. Since $z \to 0$ when $w \to \infty$, the $\gamma_{\pm,0}$ must be loops with their ends at 0. ☐

In addition, if we define $\ell_{s,0} = \gamma_{s,0}(\mathbb{R})$, then $f_0(\ell_{-,0})$ lies inside the loop $\ell_{-,0} \cup \{0\}$ and $f_0(\ell_{+,0})$ lies outside the loop $\ell_{+,0} \cup \{0\}$. We denote by $S_{s,0}$ the closed set bounded by $\ell_{s,0} \cup f_0(\ell_{s,0}) \cup \{0\}$.

We need the following standard theorem [**BR**].

Theorem 2.4.2 (Continuous dependence of solutions). *Let $D \subset \mathbb{C}$ be a subset of the complex plane and $f,\ g : D \longrightarrow \mathbb{C}$ be continuous. Also let $z(t)$, $w(t)$ be differentiable solutions of $\dot{z} = f(z)$ and $\dot{w} = g(w)$ on an open interval I containing t_0.*

If f is k-Lipschitz in D and $|f(z) - g(z)| \le \mu$ for all $z \in D$, then

$$|z(t) - w(t)| \le |z(t_0) - w(t_0)|e^{k|t-t_0|} + \frac{\mu}{k}(e^{k|t-t_0|} - 1)$$

for $t \in I$.

Then, the next lemma follows as an application.

Lemma 2.4.3. *Suppose that $h \in N_0$ where N_0 is a very small neighborhood of $f_0(z) = z + z^2 + \ldots$, and that $f \mapsto z_0(f) \in D_{5r_0/4} \setminus D_{3r_0/2}$ is continuous on N_0. For $f \in N_0$ let p_f be the maximal trajectory for $\dot{z} = i[f(z) - z]$ with $p_f(0) = z_0(f)$. If $p_h(t) \in D_{5r_0/4} \setminus D_{3r_0/2}$ for all $t \geq 0$ then $p_{f_0}(t) \to 0$ as $t \to +\infty$.*

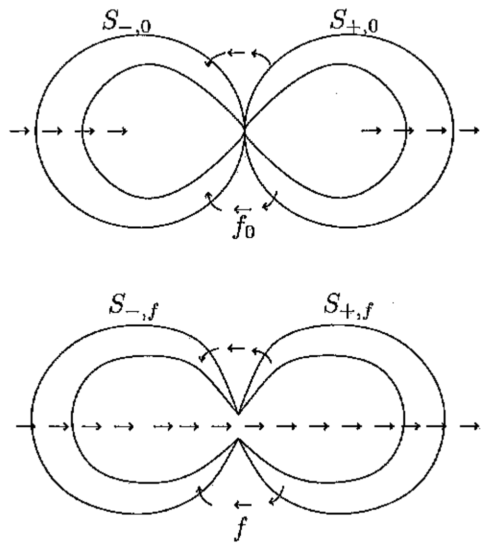

Figure 2.3: Dynamics of a parabolic map and a perturbation.

Given $f \in N_0$, we let $\gamma_{s,f} : I \longrightarrow \mathbb{C}$ be the maximal solution of the vector field $\dot{z} = i(f(z) - z)$ (defined on K_0) satisfying $\gamma_{s,f}(0) = z_s$. Also let $\ell_{s,f} = \gamma_{s,f}(I)$. Then, the "Continuous dependence of solutions" tells us that if $f \in N_0$, and N_0 is sufficiently small, then all forward and backward trajectories for the vector field $\dot{z} = i(f(z) - z)$ will enter the open disk $D_{r_0/2}$ (since the same is true for f_0).

Consider now all $f \in N_0$ such that if $f'(0) = \exp(2\pi\alpha(f))$, where $\alpha(f) \in \mathbb{C}$ and $-\frac{1}{2} < \mathrm{Re}\alpha(f) \leq \frac{1}{2}$, then $|\arg\alpha(f)| < \pi/4$. By [O], if $f \in N_0$ and $|\arg\alpha(f)| < \pi/4$ then every forward and backward trajectory for the vector field $\dot{z} = i(f(z) - z)$ passing through z_s stays in $D_{r_0/2}$ once it has entered that disk: for $s \in \{+, -\}$ there are some $t_-, t_+ \in \mathbb{R}$ such that $t_- < 0 < t_+$ and

$$\gamma_{s,f}((-\infty, t_-)) \subset D_{r_0/2}, \quad \gamma_{s,f}([t_-, t_+]) \subset K_0 \setminus D_{r_0/2}, \quad \gamma_{s,f}((t_+, \infty)) \subset D_{r_0/2}.$$

Proposition 2.4.4. *With f as before, the following hold.*

1. *Every* $\gamma_{s,f}(t)$ *converges to a fixed point (close to* 0*) as* $t \to \pm\infty$

2. *For any fixed point* σ *of* f *in* K_0 *and* $s \in \{+,-\}$*, then either* $\gamma_{s,f}(+\infty) = \sigma$ *or* $\gamma_{s,f}(-\infty) = \sigma$*.*

3. *We have* $\gamma_{-,f}(+\infty) = \gamma_{+,f}(+\infty)$ *and* $\gamma_{-,f}(-\infty) = \gamma_{+,f}(-\infty)$*.*

4. *For* $s \in \{+,-\}$*, the closure of* $\{\ell_{s,f}\} \cup \{\ell_{\bar{s},f}\}$ *is homeomorphic to a circle where* $s \neq \bar{s} \in \{+,-\}$*.*

5. *For* $s \in \{+,-\}$ *we have* $\ell_{s,f} \cap f(\ell_{s,f}) = \emptyset$ *and the closure of* $\ell_{s,f} \cup f(\ell_{s,f})$ *is a Jordan contour which bounds a closed Jordan domain* $S_{s,f}$*. These* $S_{s,f}$ *can only intersect each other at the fixed points.*

Proof. This follows from the previous lemma. □

2.4.2 Fatou Coordinates

We set $S'_{s,f} = S_{s,f} \setminus \{\gamma_{-,f}(+\infty), \gamma_{-,f}(-\infty)\}$, we call these sets the fundamental regions for f.

Notice that if $\sigma = \gamma_{-,f}(+\infty)$ then $Imf'(\sigma) > 0$ and "the dynamics of f rotates anti-clockwise around σ". Similarly if $\sigma = \gamma_{-,f}(-\infty)$ then $Imf'(\sigma) < 0$ and the dynamics of f rotates clockwise around σ".

Lemma 2.4.5. *Suppose that* $Q_F \subset \mathbb{C}$ *is a region bounded by either one or two (non-intersecting) differentiable paths* $\gamma_i : \mathbb{R} \longrightarrow \mathbb{C}$ *where arg* $\gamma_i'(t) \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ *for* $t \in \mathbb{R}$ *and each* i*. If* $F : Q_F \longrightarrow \mathbb{C}$ *is analytic, univalent and satisfies*

$$|F(w) - (w+1)| \leq \frac{1}{4},$$

$$|F'(w) - 1| \leq \frac{1}{4}$$

on Q_F*, and* Q_F *contains both* $\ell = i\mathbb{R}$ *and* $F(\ell)$*, then there is an analytic, univalent map* $\Phi_F : Q_F \longrightarrow \mathbb{C}$ *satisfying*

$$\Phi_F(F(w)) = \Phi_F(w) + 1, \quad if\ w, F(w) \in Q_F.$$

The map Φ_F *will be unique up to addition by a constant. If* $F \mapsto w_0(F)$ *is continuous in a neighborhood of* F_0 *in* $\mathcal{H}$*, and* $w_0(F) \in Q_{F_0}$*, then for* F *close to* F_0 *we can always normalize* Φ_F *by requiring that* $\Phi_F(w_0(F)) = 0$*. Then* $F \mapsto \Phi_F$ *will be continuous with respect to the compact-open topology in a neighborhood of* F_0*.*

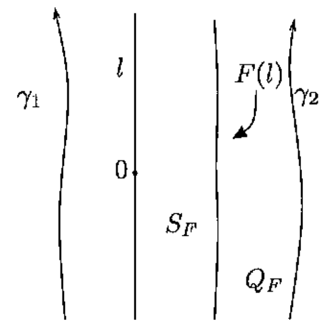

Figure 2.4: The domain of F.

Proof. See [**L1**],[**S2**]. It depends on the Ahlfors-Bers Theorem. $\square$

Lemma 2.4.6 (Existence of Fatou Coordinates). *Let K be a closed Jordan domain and let $N \subset \mathcal{H}$ such that every $f \in N$ is defined in a neighborhood of K. Suppose that $f \mapsto z_0(f) \in \mathring{K}$ is a continuous map on N.*

Now let M' be the set of $f \in N$ such that the following are satisfied.

1. *$|f(z) - z| < \frac{1}{10}$ and $|f'(z) - 1| < \frac{1}{10}$ for every $z \in K$;*
2. *$\gamma_f : \mathbb{R} \longrightarrow \mathring{K}$ solves $\dot{z} = i[f(z) - z]$ with $\gamma_f(0) = z_0(f)$ and $\gamma_f(t) \to \sigma_\pm(f) \in \mathring{K}$ as $t \to \pm\infty$ (for some $\sigma_-(f)$, $\sigma_+(f)$;*
3. *$f(\ell_f) \subset \mathring{K}$ and $\ell_f \cap f(\ell_f) = \emptyset$, where $\ell_f = \gamma_f(\mathbb{R})$.*

Then, for all $f \in M'$, we can let S_f be the closed set bounded by the loop $\ell_f \cup f(\ell_f) \cup \{\sigma_-(f),\ \sigma_+(f)\}$ and let $S'_f = S_f \setminus \{\sigma_-,\ \sigma_+\}$ (a fundamental region).

There is an analytic, injective map $\Phi_f : S'_f \longrightarrow \mathbb{C}$ such that

$$\Phi_f(f(z)) = \Phi_f(z) + 1, \quad \textit{for every } z \in \ell_f,$$

and Φ_f is unique up to addition by a constant. We call Φ_f a Fatou Coordinate. Also, the Écalle Cylinder S'_f/f is isomorphic to $\mathbb{C}/\mathbb{R}$. We can normalize Φ_f such that $\Phi_f(z_0(f)) = 0$.

The map $f \mapsto S_f$ is Hausdorff lower semi-continuous on M'. Also, the map $f \mapsto \Phi_f$ is continuous on M'.

Proof. First let $\Psi_f : S'_f \longrightarrow \mathbb{C}$ be defined as

$$\Psi_f(z) = \int_{z_0(f)}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$$

where $z \in S'_f$. We let $F_f = \Psi_f \circ f \circ \Psi_f^{-1} : \Psi_f(S'_f) \longrightarrow \mathbb{C}$ and we will prove that

1. $\Psi_f(\ell_f)$ is the vertical line $\{it : t \in \mathbb{R}\}$;
2. $|F_f(w) - (w+1)| < \frac{1}{4}$ if $w,\ F(w) \in Q_F$;
3. $|F'_f(w) - 1| < \frac{1}{4}$ if $w \in Q_F$.

If the last three conditions hold, we can apply lemma 2.4.5 to get $\Phi_{F_f} : \Psi_f(S'_f) \longrightarrow \mathbb{C}$ and then we let $\Phi_f = \Phi_{F_f} \circ \Psi_f : S'_f \longrightarrow \mathbb{C}$.

1. The first condition holds on $\Psi_f(S'_f)$ because

$$\begin{aligned}
\Psi_f(\gamma_f(t)) &= \int_{\gamma_{f(0)}}^{\gamma_f(t)} \frac{d\zeta}{f(\zeta) - \zeta} \\
&= \int_0^t \frac{\gamma'_f(s)}{f(\gamma_f(s)) - \gamma_f(s)} ds = \int_0^t i ds = it.
\end{aligned}$$

2. If $z \in \ell$ is fixed and $p(t) = (1-t)z + tf(z)$ where $t \in [0,1]$ then we can show that $p([0,1]) \subset S_f$ (using the condition 1 of the lemma). And also

$$\begin{aligned}
|[f(p(t)) - p(t)] - [f(z) - z]| &= |[f - id](p(t)) - [f - id](p(0))| \\
&= |\int_0^t ([f - id] \circ p)'(t) ds| \\
&= |\int_0^t ([f - id]')p(t) \cdot p'(t) ds| \\
&\leq \int_0^t |f'(p(t)) - 1| \cdot |f(z) - z| ds \\
&\leq \frac{1}{10} |f(z) - z|,
\end{aligned}$$

since $|f'(z) - 1| \leq \frac{1}{10}$. This implies that

$$|\frac{f(p(t)) - p(t)}{f(z) - z} - 1| \leq \frac{1}{10} \tag{2.1}$$

If $w,\ F_f(w) \in Q_F$ and $w = \Psi_f(z)$, we get

$$F_f(w) - w = \Psi_f(f(z)) - \Psi_f(z) = \int_z^{f(z)} \frac{d\zeta}{f(\zeta) - \zeta} = \int_0^1 \frac{f(z) - z}{f(p(t)) - p(t)} dt,$$

then by equation (1), we see that $|F_f(w) - (w+1)| \leq \frac{1}{4}$.

3. We can also see that

$$F_f'(w) = \Psi_f'(f(z)) \cdot f'(z) \cdot (\Psi_f^{-1})'(w) = \frac{\Psi_f'(f(z))}{\Psi_f'(z)} f'(z) = \frac{f(z) - z}{f^2(z) - f(z)} f'(z).$$

Applying (1) again with $t = 1$, we must have $|F_f'(w) - 1| \leq \frac{1}{4}$.

Therefore, $\Phi_f : S_f' \longrightarrow \mathbb{C}$ exists. By the continuous dependence of solutions we have that $f \mapsto \gamma_f(t)$ is continuous for every t. Since $\ell_f = \{\gamma_f(t) : t \in \mathbb{R}\}$, $f \mapsto \overline{\ell_f}$ must be lower semi-continuous on M', then it follows that $f \mapsto S_f$ is lower semi-continuous on M'. Also, if $f_1 \in M'$, then given any compact $G \subset \overset{\circ}{S}_{f_1}$, we will have $G \subset \overset{\circ}{S}_f$ if f is sufficiently close to f_1.

We can show that $f \mapsto (\Psi_f : S_f' \longrightarrow \Psi_f(S_f'))$ and $f \mapsto (F_f : \Psi_f(S_f') \longrightarrow \mathbb{C})$ are continuous. So if we set $w_0(f) = \Psi_f(z_0(f))$ then Lemma 2.4.5 tells us that $f \mapsto (\Phi_{F_f} : \Psi_f(S_f') \longrightarrow \mathbb{C})$ is continuous. Thus $f \mapsto \Phi_F \circ \Psi_f = \Phi_f$ is continuous also.

□

Chapter 3

Hybrid lamination and Renormalization

3.1 Quadratic-like germs

In this section we give a summary of [L3, §§3, 4] and [L4, §2].

3.1.1 Quadratic-like maps

A holomorphic map $f : U \to U'$ is called *quadratic-like* if it is a double branched covering between topological disks U, U' in $\mathbb{C}$ such that $U \Subset U'$. It has a single critical point which is assumed to be located at the origin 0. We will also make the following assumptions:

- The boundaries ∂U and $\partial U'$ are quasicircles. Hence f is continuous on $\bar{U}$ and maps ∂U onto $\partial U'$ as a double covering.
- U is symmetric with respect to the origin and f is even, i.e. $f(-z) = f(z)$.

The filled Julia set of f is defined as the set of non-escaping points under iteration by f: $K(f) = \{z : f^n z \in U,\ n = 0, 1, ...\}$. Its boundary is called the Julia set, $J(f) = \partial K(f)$. The set $K(f)$ and $J(f)$ are connected if and only if $0 \in K(f)$. Otherwise, these sets are Cantor. The fundamental annulus of a quadratic-like map $f : U \to U'$ is the annulus between the domain and the range of f, $A = U' \backslash \bar{U}$. We let mod$(f)$ =mod(A). Any quadratic-like map f has two fixed points counted with multiplicity, that in the case of connected Julia set can be dynamically distinguished. One of them, α, is either non-repelling or if you remove it, makes the Julia set disconnected. The other one, β, does not disconnected the Julia set if it is removed.

A quadratic-like map $f : U \to U'$ is called *real* if the domains U and U' are $\mathbb{R}$-symmetric and f commutes with the conjugacy $z \mapsto \bar{z}$.

3.1.2 Space of quadratic-like germs

A quadratic-like map $g : V \to V'$ is an *adjustment* of another quadratic-like map $f : U \to U'$ if $V \subset U$, $g = f|V$ and $\partial V' \subset \bar{U}'\setminus U$. (In particular, we can restrict f to $V = f^{-1}U$, provided $f(0) \in U$). Let us say that two quadratic-like maps f and $\tilde{f}$ represent the same quadratic-like germ if there is a sequence of quadratic-like maps $f = f_0, f_1, ..., f_n = \tilde{f}$, such that f_{i+1} is obtained by an adjustment of f_i or the other way around. By [McM1 §5.5], a quadratic-like germ has a well-defined Julia set.

We will consider quadratic-like maps/germs up to affine conjugacy (rescaling), so that near the origin they can be normalized as follows:

$$f(z) = c + z^2 + \sum_{k=2}^{\infty} a_k z^{2k},$$

which will still be called "quadratic-like maps/germs".

Let $\mathcal{QM}$ be the union of the space of normalized quadratic-like maps and the quadratic family $\mathcal{QP} = \{P_c(z) = z^2 + c\}_{c\in\mathbb{C}}$. We have the following convergence structure on $\mathcal{QM}$ [McM1,§5.1]: a sequence of maps $f_n : V_n \to V_n'$ converges to a map $f : V \to V'$ if the pointed domains $(V_n', V_n, 0)$ Carathéodory converge to $(V', V, 0)$, and $f_n \to f$ uniformly on compact subsets of V.

A quadratic-like germ is called real if it has a real representative. Any quadratic polynomial P_c determines a quadratic-like germ by restricting it to the preimage $P_c^{-1}(\mathbb{D}_r)$ of a sufficiently big round disk $\mathbb{D}_r$. These germs will be called "quadratic polynomials".

3.1.3 Complex structure on the space of quadratic-like germs

Let $\mathcal{Q}$ stand for the space of quadratic-like germs, and let $\mathcal{C}$ be its connectedness locus, the subset of germs with connected Julia set. Let $\mathcal{Q}_\mathbb{R}$ stand for the space of real quadratic-like germs. We will endow $\mathcal{Q}$ with topology and complex analytic structure modeled on a family of Banach spaces $\mathcal{B}_V$.

Let $\mathbb{V}$ be the set of topological disks $V \ni 0$ with piecewise smooth boundary symmetric with respect to the origin. Let $\mathcal{B}_V$ denote the affine space of normalized even analytic functions $f(z) = c + z^2 + \sum_{k>1} a_k z^{2k}$ on $V \in \mathbb{V}$ continuous up to the boundary supplied with the sup-norm $||\cdot||_V$. Let $\mathcal{B}_V(f, \epsilon)$ stand for the ϵ-ball in $\mathcal{B}_V$ centered at f.

If $f : V \to V'$ is a quadratic-like map, then all nearby maps $g \in \mathcal{B}_V$ are also quadratic-like on a slightly smaller domain. Thus, we have a natural inclusion j_V of some Banach ball $\mathcal{B}_V(f, \epsilon)$ into $\mathcal{Q}$. We will call it a Banach ball or a

Banach slice of $\mathcal{Q}$ and we will denote it by $\mathcal{Q}_V$. The inclusions $j_V : \mathcal{Q}_V \to \mathcal{Q}$ play a role of Banach charts on $\mathcal{Q}$ (though $\mathcal{Q}$ is not going to be a Banach manifold). For $U \subset V$, let $j_{U,V} : \mathcal{B}_V \to \mathcal{B}_U$ stand for the restriction operator, where $j_{U,V} = j_U^{-1} \circ j_V$.

Lemma 3.1.1. *The family of local charts j_V satisfies the following properties.*

- *P1: Countable base and compactness. There exists a countable family of Banach slices $\mathcal{Q}_{V_n}$ with the following property: for any $f \in \mathcal{Q}_V$, there is a $\delta > 0$ and a Banach slice $\mathcal{Q}_{V_n}$ such that $V_n \Subset V$, and the Banach ball $\mathcal{B}_V(f, \delta) \subset \mathcal{Q}$ is compactly embedded into $\mathcal{Q}_{V_n}$.*
- *P2: Analyticity. For $W \subset V$, the inclusion $j_{W,V} : \mathcal{Q}_V \to \mathcal{B}_W$ is complex analytic.*
- *P3: Density. If $W \subset V$, then the space $\mathcal{B}_V$ is dense in $\mathcal{B}_W$. (The differential $Dj_{W,V}(f)$ has a dense image in $\mathcal{B}_W$).*

Given a set $\mathcal{X} \subset \mathcal{Q}$, the intersections $\mathcal{X}_V = \mathcal{X} \cap \mathcal{Q}_V = j_V^{-1}\mathcal{X}$ will be called a Banach slice of $\mathcal{X}$ (f and ϵ are implicit in this notation). By the intrinsic (or Banach) topology/metric on the slice $\mathcal{Q}_V$ we will mean the topology/metric induced from the Banach space $\mathcal{B}_V$. We endow $\mathcal{Q}$ with the finest topology which makes all the local charts j_V continuous, i.e. a set $\mathcal{V} \subset \mathcal{Q}$ is open if and only if all its Banach slices $\mathcal{V}_V$ are intrinsically open.

Let us say that two metrics ρ and d on the same space $\mathcal{K}$ are *Hölder equivalent* if there exist constants $C > 0$ and $\delta > 0$ such that

$$C^{-1}\rho(x,y)^{1/\delta} \leq d(x,y) \leq C\rho(x,y)^{\delta}.$$

Lemma 3.1.2. *The topological space $\mathcal{Q}$ satisfies the following properties:*

- *(i) A sequence $\bar{f} = \{f_n\}$ in $\mathcal{Q}$ converges to $f \in \mathcal{Q}$ if and only if there exists a finite family of Banach slices $\mathcal{Q}_i = \mathcal{Q}_{V_i}$ such that $f \in \cap \mathcal{Q}_i$, $\bar{f} \subset \cup \mathcal{Q}_i$, and the corresponding subsequences $\bar{f}^i = \bar{f} \cap \mathcal{Q}_i$ converge to f in the intrinsic topology of $\mathcal{Q}_i$.*
- *(ii) A set $\mathcal{K} \subset \mathcal{Q}$ is compact (or sequentially compact) if and only if there exists a finite family of Banach slices $\mathcal{Q}_i$ and intrinsically compact subsets $\mathcal{K}_i \subset \mathcal{Q}_i$ such that $\mathcal{K} \subset \cup \mathcal{K}_i$. Thus, compactness and sequential compactness in $\mathcal{Q}$ are equivalent.*
- *(iii) A compact set $\mathcal{K} \subset \mathcal{Q}$ is metrizable with a "Montel metric"* $\mathrm{dist}_{\mathrm{Mon}}$ *induced from some $\mathcal{B}_V$ containing $\mathcal{K}$. The Montel metrics induced from different domains V are Hölder equivalent.*

The family of local charts j_V endows $\mathcal{Q}$ with the complex analytic structure modeled on the family of Banach spaces $\mathcal{B}_V$. For a germ $f \in \mathcal{Q}$, $\mathbb{V}_f$ is the set of topological disks $V \in \mathbb{V}$ such that f has a quadratic-like representative $f_V : V \to f(V)$ in the space $\mathcal{B}_V$.

Given a quadratic-like germ f, let $\mathrm{mod}(f) = \sup \mathrm{mod}(A)$, where A runs over the fundamental annuli of quadratic-like representatives of f. For $\mu > 0$, let $\mathcal{Q}(\mu, \rho)$ stand for the set of normalized quadratic-like germs which have representatives $f : V \to V'$ such that the curves ∂V and $\partial V'$ are ρ-quasicircles, $\mathrm{mod}(V', V) \geq \mu$, $|f(0)| \leq \rho$, and $\mathrm{dist}_{\mathrm{hyp}}(0, f(0)) \leq \rho$, where the hyperbolic distance is measured in V'. Let $\mathcal{Q}(\mu) = \{f \in \mathcal{Q} : \mathrm{mod}(f) \geq \mu\}$. Note that $\mathcal{C}(\mu) \equiv \mathcal{Q}(\mu) \cap \mathcal{C} \subset \mathcal{Q}(\mu, R(\mu))$. Finally we state the following compactness lemma.

Lemma 3.1.3. *A subset $\mathcal{K}$ of $\mathcal{Q}$ (resp. of $\mathcal{C}$) is pre-compact if and only if it is contained in some $\mathcal{Q}(\mu, R)$ (resp. in $\mathcal{C}(\mu)$). Any compact set $\mathcal{K}$ sits in a union of finitely many Banach slices.*

3.1.4 Conjugacies and Hybrid classes

Two quadratic-like maps $f : U \to U'$ and $\tilde{f} : V \to V'$ are called *topologically conjugate* if there exists a homeomorphism $h : (U', U) \to (V', V)$ such that $h(fz) = \tilde{f}(hz)$, $z \in U$. Two quadratic-like germs f and $\tilde{f}$ are called topologically conjugate if there is a choice of topologically conjugate quadratic-like representatives.

Two maps/germs are called *quasi-conformally/smoothly* etc. conjugate if they admit a conjugacy h with the corresponding regularity. If two maps/germs are qc conjugate with $\bar{\partial} h = 0$ almost everywhere on the filled Julia set, then f and $\tilde{f}$ are called *hybrid equivalent*. Let $\mathcal{H}(f)$ stand for the hybrid class of $f \in \mathcal{Q}$. The relation between points in $\mathcal{C}$ and the quadratic family is given by the following theorem.

Theorem 3.1.4 (Straightening [DH2]). *If f is a quadratic-like germ with connected Julia set then its hybrid class $\mathcal{H}(f)$ contains a unique quadratic polynomial $P : z \mapsto z^2 + \chi(f)$, where $c = \chi(f)$ is a point of the Mandelbrot set M.*

Moreover, the polynomial P and the quasi-conformal map h, which conjugates P and f, are uniquely determined by the choice of an equivariant quasi-conformal map

$$H : \mathbb{C}\backslash U \to \mathbb{C}\backslash \mathbb{D}_r, \quad H(fz) = P_0(Hz) \quad \text{for } z \in \partial U$$

where $P_0(z) = z^2$. Such a map H is called a *tubing* of the fundamental annulus $U'\backslash U$, and the quadratic polynomial P is called the *straightening of f*.

Then, given a quadratic-like map with connected Julia set, we can define *external rays* and *equipotentials* near the filled Julia set by conjugating it to a polynomial and transferring the corresponding curves.

3.1.5 Hybrid lamination

By the Straightening Theorem, every hybrid class $\mathcal{H}(f)$ in $\mathcal{C}$ intersects the quadratic family $\mathcal{QP}$ at a single point $c = \chi(f)$ of the Mandelbrot set M. We denote such hybrid classes as $\mathcal{H}_c$, $c \in M$.

It is proven in [L3] that the hybrid classes $\mathcal{H}_c$, $c \in M$, are connected codimension-one holomorphic submanifolds of $\mathcal{Q}$.

Theorem 3.1.5 (Leaves). *The hybrid classes $\mathcal{H}_c$, $c \in M$, are connected codimension-one complex analytic submanifolds of $\mathcal{Q}$. The quadratic family $\mathcal{QP}$ is a complex one dimensional submanifold of $\mathcal{Q}$, transversal to these submanifolds.*

Horizontal foliation of $\mathcal{C}$. These leaves form a foliation (or rather a lamination) $\mathcal{F}$ called horizontal. This foliation will be transversally quasi-conformal everywhere, and holomorphic on int $\mathcal{C}$.

Lemma 3.1.6 ([L3]). *The partition of int $\mathcal{C}$ into the hybrid classes is a complex analytic foliation.*

The hybrid classes in the connectedness locus will also be called the leaves of $\mathcal{F}$. A Banach slice $\mathcal{F}_V$ of the foliation $\mathcal{F}$ is the restriction of $\mathcal{F}$ to the Banach space $\mathcal{B}_V$, in such a way that the leaves of $\mathcal{F}_V$ are $\mathcal{H}_V(f) = \mathcal{H}(f) \cap \mathcal{B}_V$, $f \in \mathcal{C}$. Then, sufficiently deep Banach slices of $\mathcal{F}$ are still foliations with complex codimension-one analytic leaves in the corresponding Banach space.

Lemma 3.1.7 ([L3]). *For any $f_0 \in \mathcal{C}$ there exists a domain $V_0 \in \mathbb{V}_{f_0}$ such that for any $V \subset V_0$, $V \in \mathbb{V}_{f_0}$, the slice $\mathcal{F}_V$ near f_0 is a foliation in $\mathcal{B}_V$ with complex codimension-one analytic leaves.*

Also, the foliation $\mathcal{F}_V$ admits a local smooth extension beyond $\mathcal{C}$, the leaves of this foliation are given by the position of the critical point in the appropriate local chart.

Theorem 3.1.8 ([L3]). *For any $f_0 \in \mathcal{C}$ and any Banach slice $\mathcal{B}_U \ni f_0$, $U \in \mathbb{V}_f$, as in Lemma 3.1.7, there exists a Banach neighborhood $\mathcal{U} \subset \mathcal{B}_U$ of f_0 such that the foliation $\mathcal{F}_U$ admits an extension to $\mathcal{U}$ with codimension-one complex analytic leaves which is smooth on $\mathcal{U}\backslash\mathcal{C}$.*

3.1.6 $\mathcal{F}$ is transversally quasi-conformal

Definition 3.1.9. *The foliation $\mathcal{F}$ is transversally quasi-conformal if the holonomy between two transversals $\mathcal{S}_1$ and $\mathcal{S}_2$ is locally a restriction of a qc map.*

Thus, take two hybrid equivalent germs $f_i \in \mathfrak{C}$, and two holomorphic transversals $\mathcal{S}_i$ to the leaf $\mathcal{H} \equiv \mathcal{H}(f_i)$ through f_i. The holonomy $\gamma : \mathcal{C} \cap \mathcal{S}_1 \to \mathcal{C} \cap \mathcal{S}_2$ along $\mathcal{F}$ is called *locally quasi-conformal* at f_i if it admits a local qc extension $\tilde{\gamma} : \Omega_1 \to \Omega_2$, where $\Omega_i \subset \mathcal{S}_i$ are neighborhoods of the f_i in the transversals S_i. The local dilatation of γ at f_i is defined as inf $\mathrm{Dil}(\tilde{\gamma})$ where the infimum is taken over all local qc extensions $\tilde{\gamma}$ of γ. For a transversal $\mathcal{S}$, let $\mathrm{mod}(\mathcal{S}) = \inf_{f \in \mathcal{S}} \mathrm{mod}\,(f)$. Then the following holds.

Theorem 3.1.10 ([L3]). *The foliation $\mathcal{F}$ is transversally quasi-conformal. The dilatation of the holonomy between two transversals $\mathcal{S}_1$ and $\mathcal{S}_2$ depends only on $\mu = \min(\mathrm{mod}(\mathcal{S}_1), \mathrm{mod}(\mathcal{S}_2))$. Moreover, if the transversals $\mathcal{S}_i$ are represented by holomorphic one-parameter families $\{f_{i,\lambda}\}$ of quadratic-like maps such that $\mathrm{mod}(f_{i,\lambda}) \geq \mu > 0$, then the local dilatation of γ at f_i is bounded by $K(\mu)$.*

Proof. Let us take two transversals $\mathcal{S}_1$ and $\mathcal{S}_2$ to a leaf $\mathcal{H}$ of the foliation and a Beltrami path γ in $\mathcal{H}$ joining two intersection points. Since this path is compact, it is contained in finitely many Banach slices, whose number N depends only on μ. By Theorem 3.1.8, this path can be covered by finitely many Banach balls $\mathcal{B}_{V_i}(f_i, \epsilon_i)$ such that $\mathcal{F}$ extends to the bigger balls $\mathcal{B}_{V_i}(f_i, 2\epsilon_i)$. Hence, the holonomy between $\mathcal{S}_1$ and $\mathcal{S}_2$ can be decomposed into N Banach holomorphic motions, which extend to the twice bigger domains. By the λ-lemma, each of the Banach motions is locally transversally quasi-conformal with uniform dilatation. ☐

In particular, if we take the quadratic family $\mathcal{QP}$ as one of the transversals, we obtain that the straightening $\chi : \mathcal{S} \to \mathcal{QP}$ is locally K-quasi-conformal, with K depending only on $\mathrm{mod}(\mathcal{S})$, and $K \to 1$ as $\mathrm{mod}(\mathcal{S}) \to \infty$.

On the other hand, the foliation $\mathcal{F}$ is not generally transversally smooth. Take the Ulam-Neumann quadratic polynomial $P = P_{-2} : z \mapsto z^2 - 2$ with a postcritical fixed point $\beta = 2$. We can take a convergent sequence of superattracting parameter values $c_n \to -2$ in such a way that $c_n - 2 \asymp 4^{-n}$ where 4 is the multiplier of β. But we can take another map $f \in \mathcal{H}_{-2}$ and a sequence of maps $f_n \to f$, $f_n \in \mathcal{H}_{c_n}$, with rate of convergence λ^{-n}, where λ is the multiplier of the β-fixed point of f. Since λ can be made different from 4, the holonomy $c_n \mapsto f_n$ is not smooth at -2. For the same reason the foliation is not smooth at other Misiurewicz points. Thus, quasi-conformality is the best transverse regularity of $\mathcal{F}$ which is satisfied everywhere. However, Lyubich proved that $\mathcal{F}$ is transversally smooth at Feigenbaum points. Also, after McMullen, a relation between conformality and deep points in the Mandelbrot set implies that the foliation is transversally smooth at parabolic points, roots of hyperbolic components of the Mandelbrot set.

3.1.7 Quadratic-like families

For a background in the theory of quadratic-like families see [DH2]. Let us consider a domain $\Lambda \Subset \mathbb{C}$. A domain $\mathbb{V} \subset \Lambda \times \mathbb{C}$ is called a *topological bidisk* over Λ if it is homeomorphic over Λ to a straight bidisk $\Lambda \times \mathbb{D}$. Let $V_\lambda = \pi^{-1}\{\lambda\}$ stand for the vertical fibers of a bidisk $\mathbb{V}$, where $\pi : \mathbb{V} \to \Lambda$ is the natural projection. We will assume that they are quasi-disks containing 0. Denote by $\partial^h \mathbb{V} = \cup_{\lambda \in \Lambda} \partial V_\lambda$ the *horizontal boundary* of $\mathbb{V}$.

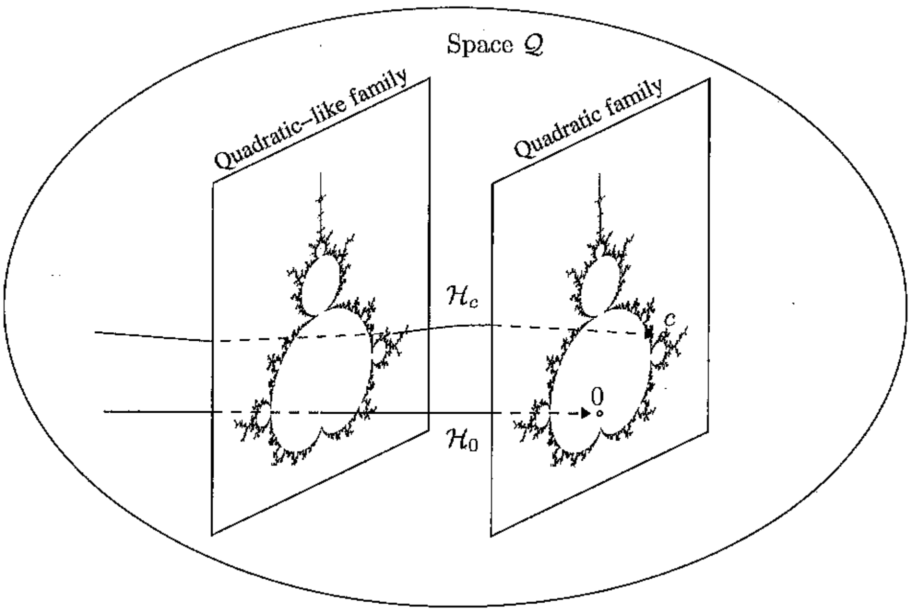

Figure 3.1: The space of quadratic-like germs.

A map $\mathbf{f} : \mathbb{V} \to \mathbb{V}'$ between two bidisks $\mathbb{V} \subset \mathbb{V}'$ over Λ is called a *quadratic-like family* over $\Lambda = \Lambda_\mathbf{f}$ if $\mathbf{f}$ is a holomorphic endomorphism preserving the fibers such that every fiber restriction $f_\lambda : V_\lambda \to V'_\lambda$, $z \mapsto z^2 + c(\lambda) + ...$, is a normalized quadratic-like map with the critical point at 0. Any quadratic-like family $\mathbf{f}$ represents a holomorphic curve in $\mathcal{Q}$.

A quadratic-like family $\mathbf{f} : \mathbb{V} \to \mathbb{V}'$ over $(\Lambda, *)$ is *equipped* if the base map f_* is equipped with a tubing H_* (see 3.1.4) and there is an equivariant holomorphic motion $\mathbf{h}$,

$$h_\lambda : (\mathbb{C}, \mathrm{cl}(V'_*\backslash V_*)) \to (\mathbb{C}, \mathrm{cl}(V'_\lambda \backslash V_\lambda)), \quad \lambda \in \Lambda,$$

where equivariance means that $h_\lambda(f_* z) = f_\lambda(h_\lambda z)$ for $z \in \partial V_*$.

Let $M^0_{\mathbf{f}} = \{\lambda \in \Lambda : 0 \in K(f_\lambda)\} = \{\lambda \in \Lambda : f_\lambda \in \mathcal{C}\}$ stand for the Mandelbrot set of $\mathbf{f}$. The family $\mathbf{f}$ is *full* if $M^0_{\mathbf{f}}$ is compact.

Let $\phi(\lambda) = f_\lambda(0)$ denote the critical value of f_λ, and let $\Phi(\lambda) = (\lambda, \phi(\lambda))$. Let $\Lambda^1 \equiv \Lambda^1_{\mathbf{f}} = \{\lambda \in \Lambda : \phi(\lambda) \in V_\lambda\}$. Consider the natural map

$$\eta \equiv \eta_{\mathbf{f}} : \Lambda \backslash \Lambda^1 \to V'_* \backslash V_*, \qquad \eta(\lambda) = h_\lambda^{-1}(\phi(\lambda))$$

from the parameter region $\Lambda\backslash\Lambda^1$ to the dynamical annulus $V'_*\backslash V_*$. A family $\mathbf{f}$ is called *proper* if the map η is proper, i.e., $\eta(\lambda) \to \partial V'_*$ as $\lambda \to \partial\Lambda$. Any proper family is full.

For a full family, one defines the *winding number* $w(\mathbf{f})$ as the winding number of the curve $\lambda \mapsto \phi(\lambda)$ about the origin, as λ goes once anti-clockwise around a Jordan curve $\gamma \subset \Lambda\backslash M^0_{\mathbf{f}}$ surrounding $M^0_{\mathbf{f}}$. A full family is called *unfolded* if $w(\mathbf{f}) = 1$.

The straightening provides the continuous map

$$\chi = \chi_{\mathbf{f}} : (\Lambda, M^0_{\mathbf{f}}) \to (\mathbb{D}_r, M).$$

If $\mathbf{f}$ is full and unfolded, then $\chi : M^0_{\mathbf{f}} \to M$ is a homeomorphism. If $\mathbf{f}$ is proper then χ is a homeomorphism in the whole domain Λ.

3.2 Renormalization of quadratic-like germs

The notion of complex renormalization was introduced by Douady and Hubbard [DH2] in order to explain computer observable little Mandelbrot copies inside the Mandelbrot set.

Let f be a quadratic-like map. Assume that we can find topological disks $U \Subset U'$ around 0 and an integer p such that $g = f^p : U \to U'$ is a quadratic-like map with connected Julia set. Assume that the little Julia sets $f^k J(g), k = 0, ..., p-1$, are pairwise disjoint except, perhaps, touching at their β-fixed points. Then, the map f is called renormalizable of period p and the map g is called its pre-renormalization. The quadratic-like germ of g, considered up to rescaling, is called a renormalization Rf of f

Take a quadratic-like representative $f : V \to V'$. If the pre-renormalization $g = f^p : U \to U'$ above is selected in such a way that $f^k U \Subset V$, $k = 0, 1, ..., p-1$, then we say that g is *subordinate* to V. In the same way, we can define the germ of subordinate pre-renormalizations Rf_V.

3.2.1 The Renormalization in $\mathcal{QP}$

A parameter value $c \in \mathbb{C}$ in the Mandelbrot set M is called super-attracting if 0 is periodic under P_c. To each super-attracting parameter $c \neq 0$, there is associated a copy of M containing c called the Mandelbrot set tuned by c, little Mandelbrot copy or an M-copy, canonically homeomorphic to the whole set M, denoted by $c * M$. The root of $c * M$, r_c, is the point corresponding to the cusp $1/4$ and the center is the point c. A little Mandelbrot copy is called primitive if it is not attached at its root point to any other hyperbolic component. Otherwise, it is called satellite. For every copy $c * M$, there is a $p > 1$ such that for any $c' \in c * M$, except possibly the root, and any $f \in \mathcal{H}(c')$ there is a domain $U \ni 0$ such that $f^p|_U$ is a quadratic-like map. Then, the map $f^p|_U$ is a (complex) pre-renormalization of f and f is said to be renormalizable of period p. This pre-renormalization is always simple, i.e. the iterates of $J(f^p|_U)$ under f are either disjoint or intersect only along the orbit of $\beta(f^p|_U)$, the β-fixed points.

The period of the copy, $p(c * M)$, is the maximal such p and we say that $c * M$ is maximal if there is only one such p or, equivalently, if it does not belong to any other copy, except M itself. These copies are disjoint, and any other copy, except M itself, belongs to a unique maximal one. All maximal copies are primitive except for the ones attached to the main cardioid.

We say that $c * M$ is real if c is real. The only real maximal M-copy for which the root point is not renormalizable is the period two copy $M^{(2)}$. Also, all real maximal Mandelbrot copies are primitive except for the period two copy $M^{(2)}$. We will denote the real period three copy by $M^{(3)}$. Define $\mathcal{H}(c * M)$ as the set of renormalizable maps $f \in \chi^{-1}(c * M)$.

All the copies $M' \neq M$ are obtained from M by iterated tunings

$$M' = c_l * \ldots * c_1 * M$$

where c_k is the center of the maximal Mandelbrot copy $M_k = M_{c_k}$. Thus, any two M-copies are either disjoint or nested.

Let $\hat{M}' = M'$ in the primitive case and $\hat{M}' = M' \backslash \{r_{M'}\}$ otherwise. For a copy M', let $\mathcal{T}_{M'} = \chi^{-1} M' \subset \mathcal{C}$ (resp. $\mathcal{T}_{\hat{M}'} = \chi^{-1}\hat{M}' \subset \mathcal{T}_{M'}$) stand for the union of the hybrid classes via M' (resp. $\hat{M}'$). These sets will be called the *(horizontal) renormalization strips.* The strips $\mathcal{T}_{M'}$ are closed. The renormalization strip is called maximal if it corresponds to a maximal Mandelbrot copy. Note that the maximal renormalization strips are pairwise disjoint.

There is a canonical renormalization operator $R_{M'} : \mathcal{T}_{\hat{M}'} \to \mathcal{C}$ defined as the $p = p_M$-fold iterate of f restricted to an appropiate neighborhood U of the critical point, up to rescaling. This neighborhood is selected in such a way that $g = f^p|_U$ is a quadratic-like map with connected Julia set, and the "little

Julia sets" $f^k J(g)$, $k = 0, 1, ..., p-1$, are pairwise disjoint except, perhaps, touching at their β-fixed points. The maps $f \in \mathcal{T}_{\hat{M}'}$ are called renormalizable with combinatorics M'.

Let $c * M$ be a maximal M-copy with period p and suppose $f \in \mathcal{H}(c * M)$. If $f^p|_U$ and $f^p|_{U'}$ are two pre-renormalizations then $[f^p|_U] = [f^p|_{U'}]$. Hence we can define the renormalization $R(f)$ to be the normalized quadratic-like germ of any pre-renormalization of period p. Thus, the renormalization of a germ $R([f])$ is the renormalization of a quadratic-like representative. A quadratic-like map f is infinitely renormalizable if $R^n(f)$ is defined for all $n \geq 0$, i.e. $\chi(f)$ is contained in infinitely many M-copies. The tuning invariant of an infinitely renormalizable map f is

$$\tau(f) = \{M_0, M_1, M_2, ...\}$$

where M_n is the maximal M-copy containing $\chi(R^n(f))$. We say f has real combinatorics if all M-copies in $\tau(f)$ are real. One says that an infinitely renormalizable map f has a bounded type if all the periods $p(M_n)$ are bounded.

3.2.2 Real renormalizations strips

A Mandelbrot copy M' is called *real* if it is centered on the real line. The real slice $J = M' \cap \mathbb{R} \subset (-2, 1/4)$ of a real Mandelbrot copy is an interval called the *renormalization window.* Denote by $\mathcal{M}$ the family of maximal real Mandelbrot copies. The set of maximal renormalization windows (formally coinciding with $\mathcal{M}$) will be denoted by $\mathcal{J}$.

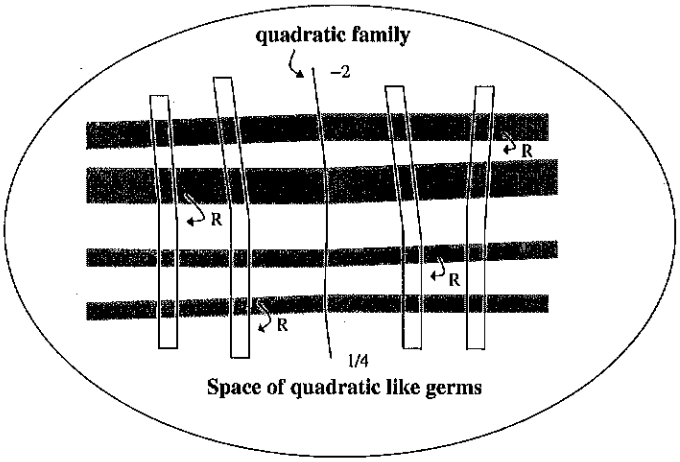

Figure 3.2: The real horseshoe.

We say that the maps $f \in \mathcal{T}_{M'}$ are renormalizable with real combinatorics encoded by the little Mandelbrot set $M' \in \mathcal{M}$. Thus, the renormalization operator R is canonically defined on the union of all renormalization strips. We let $R_{M'} \equiv R|\mathcal{T}_{M'}$.

Remember from 3.1.3, $\mathcal{Q}_{\mathbb{R}}$ is the space of real quadratic-like germs. The real slice of the renormalization strip $\mathcal{T}_{M'}$ will be denoted as $\mathcal{T}_J$, where $J = M' \cap \mathbb{R}$ is the corresponding renormalization window. Thus the union of $\mathcal{T}_J$, $J \in \mathcal{J}$ forms the domain of definition of the renormalization R in the space $\mathcal{Q}_{\mathbb{R}}$ of real quadratic-like maps.

Injectivity of R.

Lemma 3.2.1 ([MvS]). *The renormalization operator*

$$R : \cup_{J \in \mathcal{J}} \mathcal{T}_J \to \mathcal{Q}_{\mathbb{R}}$$

is injective.

3.2.3 Complex bounds

Definition 3.2.2. *An infinitely renormalizable map $f : V \to V'$ is said to have a priori bounds if* $\mathrm{mod}(R^n f_V) \geq \nu > 0$, $n = 0, 1, \ldots$, *where the $R^n f_V$ are the subordinate renormalizations of f_V. We say that a map $f \in \mathcal{C}$ is close to the cusp if $|\chi(f) - 1/4| < \epsilon$.*

The following two theorems establish combinatorial rigidity of infinitely renormalizable maps with real combinatorics and complex bounds. Moreover, complex bounds are proven to exist for real quadratic-like maps.

Theorem 3.2.3 (A *priori* bounds [L3],[LY]). *Let $f : V \to V'$ be an n-times renormalizable real quadratic-like map with* $\mathrm{mod}(V' \backslash V) \geq \mu > 0$. *Then*

$$\mathrm{mod}(R^n f_V) \geq \nu_n(\mu) \geq \nu(\mu) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. Moreover, $\liminf \nu_n(\mu) \geq \nu > 0$, *where ν is an absolute constant. Thus, all real infinitely renormalizable maps have a priori bounds.*

Theorem 3.2.4 ([L3]). *If P_c and $P_{c'}$ are two infinitely renormalizable quadratic polynomials with complex a priori bounds and the same real combinatorics then $c = c'$.*

3.2.4 Analytic extension

Any $R_{M'}$ admits a complex analytic extension to Banach neighborhoods of maps $f \in \mathcal{T}_{M'}$. If $R_{M'} f_V = f^p : U \to U'$ is a subordinated quadratic-like prerenormalization of $f \in \mathcal{C}_V$, then any nearby map $g \in \mathcal{C}_V$ admits a quadratic-like return map $g^p : U_g \to U'$ with the same range. Since g^p analytically depends on g, this provides us with the desired extension. ([L3,§5.3]).

3.3 Yoccoz Puzzle

This section is based in [L2, §2, 3]

Let $f : U' \longrightarrow U$ be a quadratic-like map with both fixed points α and β repelling, where α is the dividing fixed point with rotation number q/p, $p > 1$. Let E be an equipotential sufficiently close to $K(f)$ (so that both E and fE are closed curves). Let R_α denote the union of external rays landing at α and let $R'_\alpha = -R_\alpha$ the symmetric configuration. Let Ω be the component of $\mathbb{C}\backslash(E \cup R_\alpha \cup R'_\alpha)$ containing the critical point 0. Let us consider the domain $\Omega' \subset \Omega$, the component of $f^{-p}\Omega$ attached to α. If $0 \in \Omega'$ then $f^p : \Omega' \to \Omega$ is a double covering map. Moreover, if 0 does not escape Ω' under iterates of f^p, we say that f is *DH immediately renormalizable.*

The rays R_α cut the domain bounded by E into p closed topological disks $Y_i^{(0)}$, $i = 0, ..., p-1$ called puzzle pieces of zero depth, such that $f\partial Y_j^{(0)}$ is outside of $\bigcap \mathrm{int} Y_i^{(0)}$.

Now define *puzzle pieces* $Y_i^{(n)}$ *of depth* n as the closures of the connected components of $f^{-n}\ \mathrm{int} Y_k^{(0)}$. They form a partition of the neighborhood of $K(f)$ bounded by $f^{-n}E$. If the critical orbit does not land at α, then for every depth there is a single puzzle piece containing the critical point. It is called *critical* and is labeled as $Y^{(n)} \equiv Y_0^{(n)}$.

Let $\mathcal{Y}_f$ denote the family of all puzzle pieces of f of all levels. They satisfy the following conditions (Markov):

i) Any two puzzle pieces are either nested or have disjoint interiors (the puzzle piece of bigger depth is contained in the one of smaller depth).

ii) The image of any puzzle piece $Y_i^{(n)}$ (of depth $n > 0$) is a puzzle piece $Y_k^{(n-1)}$ of the previous depth. Moreover, $f : Y_i^{(n)} \longrightarrow Y_k^{(n-1)}$ is a two-to-one branched covering or a conformal isomorphism depending on whether $Y_i^{(n)}$ is critical or not.

On depth 1 we have $2p-1$ puzzle pieces: $Y^{(1)} \equiv Y_0^{(1)}$, $p-1$ non central $Y_i^{(1)}$ attached to α (cuts of $Y_i^{(0)}$ by the equipotential $f^{-1}E$), and $p-1$ symmetric ones $Z_i^{(1)}$ attached to α'. Moreover, $f|Y^{(1)}$ two-to-one covers $Y_1^{(1)}$, $f|Y_i^{(1)}$

univalently covers $Y^{(1)}_{i+1}$, $i = 1, ..., p-2$ and $f|Y^{(1)}_{p-1}$ univalently covers $Y^{(1)} \cup \bigcup_i Z^{(1)}_i$. Thus $f^p|Y^{(1)}$ truncated by $f^{-1}(E)$ is the union of $Y^{(1)}$ and $Z^{(1)}_i$.

3.3.1 Lyubich's Principal Nest

Given a set $W = \mathrm{cl}(\mathrm{int}\, W)$ and z such that $f^k z \in \mathrm{int} W$, we define the pull-back of W along the orbit $orb_k(z)$ as the chain of sets $W_0 = W$, $W_{-1} \ni f^{k-1}z, ..., W_{-k} \ni z$ such that W_{-m} is the closure of the component of $f^{-m}(\mathrm{int} W)$ containing $f^{k-m}z$.

If $z \in \mathrm{int} W$ and $k > 0$ is the moment of first return of the orbit of z back to $\mathrm{int} W$, then we will refer to the pull-backs corresponding to the first return of $orb(z)$ to $\mathrm{int} W$.

Let us consider the puzzle pieces of depth 1: $Y^{(1)}$, $Y^{(1)}_i$ and $Z^{(1)}_i$, $i = 1, ..., p-1$. If $z \in Y^{(1)}$ then $f^p z$ is either in $Y^{(1)}$ or in one of $Z^{(1)}_i$. Hence either $f^{kp}0 \in Y^{(1)}$ for all $k = 0, 1, ..$ or there is a smallest $t > 0$ and ν such that $f^{tp}0 \in Z^{(1)}_\nu$. Thus, either f is immediately DH-renormalizable, or the critical point escapes through one of the non-critical pieces attached to α'.

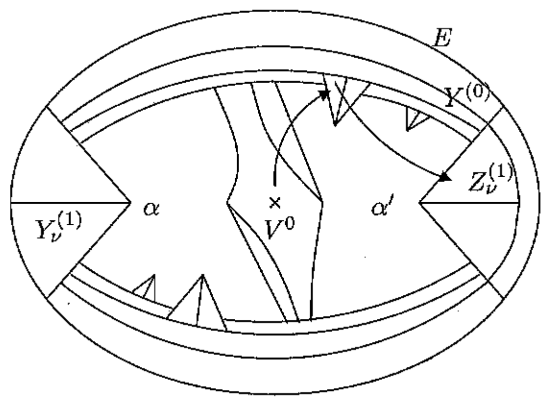

Figure 3.3: First levels of the Principal Nest.

In the former case, the principal nest of puzzle pieces consists of just $Y^{(0)}$. In the escaping case we will construct the *principal nest*

$$Y^{(0)} \supset V^{(0)} \supset V^{(1)} \supset ...$$

as follows: let $V^{(0)} \ni 0$ be the pull-back of $Z^{(1)}_\nu$ along $orb_{tp}0$. Let us define V^{n+1} as the pullback of V^n corresponding to the first return map of the critical point 0 back to $\mathrm{int} V^n$. If the critical point never returns back to $\mathrm{int} V^n$ we stop, and the principal nest is finite. This case is called *combinatorially non-recurrent.*

Let $s = s(n)$ be the first return time of the critical point back to $\text{int}V^{n-1}$, then $g_n = f^{s(n)} : V^n \to V^{n-1}$ is a two-to-one branched covering ($f^k V^n \cap \text{int}V^{n-1} = \emptyset$, for $k = 1, ..., s-1$ so $f : f^k V^n \to f^{k+1} V^n$ are univalent for those $k's$).

3.3.2 Central Cascades

Let us call a return to level $n-1$ central if $g_n 0 \in V^n$, i.e. $s(n) = s(n+1)$. Thus $g_n = g_{n+1}$ in V^{n+1}.

Definition 3.3.1. *The sequence $n, n+1, ..., n+N-1$ (with $N \geq 1$) of levels of the principal nest form a central cascade if the returns to all levels $n, n+1, ..., n+N-2$ are central, while the return to level $n+N-1$ is non-central. In this case $g_{n+k}|V^{n+k} = g_{n+1}|V^{n+k}$, $k = 1, ..., N$ and $g_{n+1}0 \in V^{n+N-1} \setminus V^{n+N}$.*

Thus, all the maps $g_{n+1}, ..., g_{n+N}$ are the same quadratic-like maps with shrinking domains of definition. The number N of levels in the cascade is its length. A cascade of length 1 consists of a single non-central level. The cascade is maximal if the return to level $n-1$ is non-central.

The whole principal nest, except $Y^{(0)}$, is the union of disjoint maximal cascades, this number is called the *height* $\chi(f)$ of f. In other words, $\chi(f)$ is the number of different quadratic-like maps among the g_n's.

3.3.3 Renormalization and central cascades

We state a proposition which relates the notion of central cascade with renormalization of a quadratic-like map. Let n_k count the non-central levels for a quadratic-like map f; if this sequence is infinite, f is non-renormalizable.

Proposition 3.3.2 ([L2]). *A quadratic-like map is renormalizable if and only if it is either immediately renormalizable, or the principal nest $V^0 \supset V^1 \supset ...$ ends with an infinite cascade of central returns. Thus the height $\chi(f)$ is finite iff f is either renormalizable or combinatorially non-recurrent.*

The above proposition shows that there is a well-defined first renormalization Rf with the biggest Julia set, and it can be constructed in the following way. If f is immediately renormalizable, then Rf is obtained by thickening $Y^{(1)} \to Y^{(0)}$. Otherwise, the principal nest ends up with the infinite central cascade $V^{m-1} \supset V^m \supset ...$, and $Rf = g_m : V^m \to V^{m-1}$.

In this case, $\chi(Rf)$, the internal class of the first renormalization, belongs to a maximal copy M_0 of the Mandelbrot set.

3.3.4 Generalized Renormalization

Now, let $\{U_i\}$ be a finite or countable family of topological disks with disjoint interiors strictly contained in a topological disk U. We call a map $g : \bigcup U_i \to U$ a *generalized quadratic-like map* if $g : U_i \to U$ is a branched covering of degree 2 which is univalent on all but one U_i, i.e., it has a single (and non-degenerate) critical point; we can normalize in such a way that 0 is the critical point with $0 \in U_0$. The generalized quadratic–like map g is of "finite type" if its domain consists of finitely many disks U_i. We define in this case the filled Julia set $K(g)$ as the set of all non-escaping points, and $J(g) = \partial K(g)$ (quadratic-like maps correspond to the case of a single disk U_0). For instance, the principal sequence g_n of the first return maps are in this class.

3.4 Mandelbrot-like families

In this section, we will describe the equivalent to the Mandelbrot set in a family of quadratic-like maps.

We consider a family of quadratic-like maps $\mathbf{h} = \{h_\lambda\}_{\lambda \in W}$ with w_λ the critical point, equipped with a tubing Θ [DH2],[L3]. More precisely we assume:

• W is a Jordan domain in $\mathbb{C}$, with C^1 boundary.

• For each $\lambda \in W$, the map Θ_λ is a quasi-conformal embedding of $\bar{A}_{R^2,R}$ in $\mathbb{C}$; for each $z \in \bar{A}_{R^2,R}$ the point $\Theta_\lambda(z)$ depends holomorphically on λ. Let $C_\lambda = \Theta_\lambda(S^1_{R^2})$ and $C'_\lambda = \Theta_\lambda(S^1_R)$; we denote by U_λ (resp. U'_λ) the Jordan domain bounded by C_λ (resp. C'_λ).

• h_λ is a quadratic-like map $U'_\lambda \to U_\lambda$; the map $h : (\lambda, z) \mapsto (\lambda, h_\lambda(z))$ is $\mathbb{C}$-analytic and maps in a proper way $\mathcal{U}' \to \mathcal{U}$, where $\mathcal{U} = \{(\lambda, z) : \lambda \in W$ and $z \in U_\lambda\}$ is an open set and $\mathcal{U}'$ is defined similarly.

• $\Theta_\lambda(z^2) = h_\lambda(\Theta_\lambda(z))$ for $z \in S^1_R$.

• The map h extends continuously to a map $\overline{\mathcal{U}'} \to \overline{\mathcal{U}}$, and $\Theta : (\lambda, z) \mapsto (\lambda, \Theta_\lambda(z))$ extends continuously to a map $\overline{W} \times \bar{A}_{R^2,R} \to \overline{\mathcal{U}}$ such that Θ_λ is injective on $A_{R^2,R}$ for $\lambda \in \partial W$. The map $\lambda \mapsto w_\lambda$ extends continuously to $\overline{W}$.

• For $\lambda \in \partial W$, we have $h_\lambda(w_\lambda) \in C_\lambda$.

• When λ ranges over ∂W making 1 turn, the vector $h_\lambda(w_\lambda) - w_\lambda$ makes 1 turn around 0.

We denote by $M_\mathbf{h}$ the connectedness locus of the family $\mathbf{h}$. It is proved in [DH2] that $M_\mathbf{h}$ is homeomorphic to the Mandelbrot set M, by a map χ using the Straightening Theorem. This map is an homeomorphism of W onto an open set in $\mathbb{C}$. Lyubich [L3] improved this result by showing that χ is quasi-conformal on W' for any W' relatively compact in W.

Chapter 4

Essentially bounded combinatorics

4.1 Little copies of the Mandelbrot set

In this section we give a size estimation and speed of convergence for a particular sequence of little copies of the Mandelbrot set, which converges to the root point of a primitive copy of the Mandelbrot set. It is based in [D2].

The Model

Let $\mathbf{f} = (f_\lambda : U'_\lambda \longrightarrow U_\lambda)_{\lambda\in\Lambda}$ be an analytic family of quadratic-like maps, where Λ is an open set in $\mathbb{C}$ and $\lambda_0 \in \Lambda$. We suppose that f_{λ_0} has a fixed point α_0 with derivative 1; this implies that f_{λ_0} is hybrid equivalent to $z^2 + \frac{1}{4}$. We are only interested in values of λ which are close to λ_0.

Let $g_\lambda : V_\lambda \longrightarrow U_\lambda$ be an analytic isomorphism, depending analytically on λ, with V_λ relatively compact in $U_\lambda \backslash \overline{U}'_\lambda$. We suppose that the open sets are Jordan domains with a C^1 boundary undergoing a holomorphic motion.

Let w_λ be the critical point of f_λ, and let $a_n(\lambda) = f_\lambda^n(w_\lambda)$ and $b_\lambda = g_\lambda^{-1}(w_\lambda)$. We denote by F_λ the map $U'_\lambda \cup V_\lambda \rightarrow U_\lambda$ which induces f_λ on U'_λ and g_λ on V_λ; thus, $\{F_\lambda\}_{\lambda\in\Lambda}$ is a family of generalized quadratic-like maps. We define $K(F_\lambda)$ as the set of points z such that $F_\lambda^n(z)$ is defined and belongs to $U'_\lambda \cup V_\lambda$ for all n. $K(F_\lambda)$ is a full compact set in $\mathbb{C}$.

Denote by M_F the set of values of λ for which $w_\lambda \in K(F_\lambda)$. M_F is a closed set in Λ. Note that $\lambda \in M_F$ does not imply that $K(F_\lambda)$ is connected, in fact it is easy to see that $K(F_\lambda)$ is never connected.

We can normalize the functions, by a change of variables, in such a way that $\lambda_0 = 0$, $\alpha_0 = 0$, and $f_\lambda(z) = z + z^2 + \lambda + O(z^3)$.

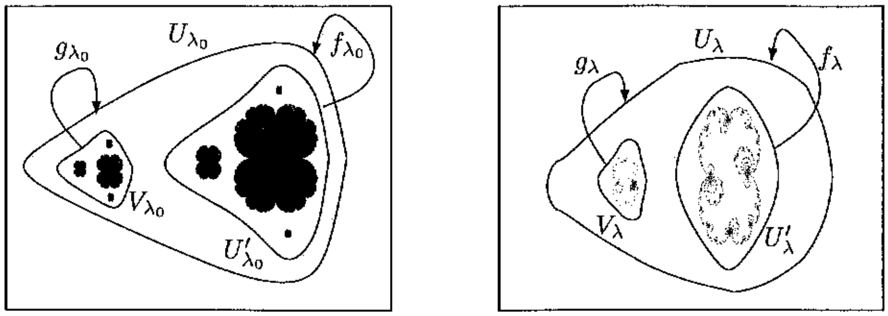

Figure 4.1: The maps F_0 and F_λ.

Fatou coordinates

For the map $P_0 : z \mapsto z + z^2$, one can define two Fatou coordinate $\Phi^+_{P_0} : \Omega^+_{P_0} \to \mathbb{C}$ and $\Phi^-_{P_0} : \Omega^-_{P_0} \to \mathbb{C}$ such that $\Omega^+_{P_0}$ (resp. $\Omega^-_{P_0}$) contains the disk $D^+_r = \{z : |z - r| < r\}$ (resp. $D^-_r = \{z : |z + r| < r\}$), for $r > 0$ small enough. As a holomorphic function, $\Phi^-_{P_0}$ extends to all of $\mathring{K}(P_0)$, but is no longer injective there. A maximal domain for $\Phi^+_{P_0}$ is $\mathbb{C} \backslash \mathbb{R}_-$.

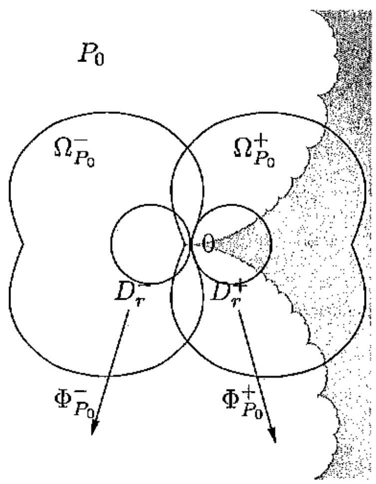

Figure 4.2: The Fatou coordinates for P_0.

In our setting, the map f_0 is quasi-conformally conjugate to P_0. We can choose a Fatou coordinate ϕ^-_0 whose domain Ω^-_0 contains D^-_r (for $r > 0$ small enough) and the points $a_n = f^n_0(w_0)$ for $n \geq 1$.

Following the construction of the quasi-conformal conjugacy between f_0 and P_0, we can find a Fatou coordinate ϕ^+_0 for f_0 whose domain Ω^+_0 contains

$\overline{V}_0$ and $\mathrm{mod}(\Omega_0^+ \backslash \overline{V}_0) > \mu$ for some $\mu > 0$.

Now choose $r_0 > 0$ small and $\theta_0 \in (0, \pi)$, and define the sector S as the set of values of λ such that $|\lambda| \le r_0$ and $|\arg(\lambda)| \le \theta_0$. We set $S^* = S\backslash\{0\}$. Moreover, if r_0 is small enough, we have $S \subset \Lambda$, and for $\lambda \in S$ there exist Fatou coordinates $\phi_\lambda^- : \Omega_\lambda^- \to \mathbb{C}$ and $\phi_\lambda^+ : \Omega_\lambda^+ \to \mathbb{C}$ such that:

i) the fixed points of f_λ belong to the boundaries $\partial\Omega_\lambda^+$, $\partial\Omega_\lambda^-$;

ii) the set $\Omega^\pm = \{(\lambda, z) | \lambda \in S, \ \ z \in \Omega_\lambda^\pm\}$ is open in $S \times \mathbb{C}$;

iii) for $\lambda \in S^*$ the intersection $\Omega_\lambda^+ \cap \Omega_\lambda^-$ is the domain of a Fatou coordinate;

iv) (normalization) $\phi_\lambda^-(a_n(\lambda)) = n$, $\phi_\lambda^+(b_\lambda) = 0$.

Restricting S if necessary, we have $\overline{V}_\lambda \subset \Omega_\lambda^+$ for all $\lambda \in S$, with definite modulus between them since they converge to V_0 and Ω_0^+ respectively. Moreover, with the above conditions, the function $(\lambda, z) \mapsto \phi_\lambda^\pm(z)$ is continuous on $\Omega^\pm$ and $\mathbb{C}$-analytic on $\Omega^\pm \cap (S^* \times \mathbb{C})$.

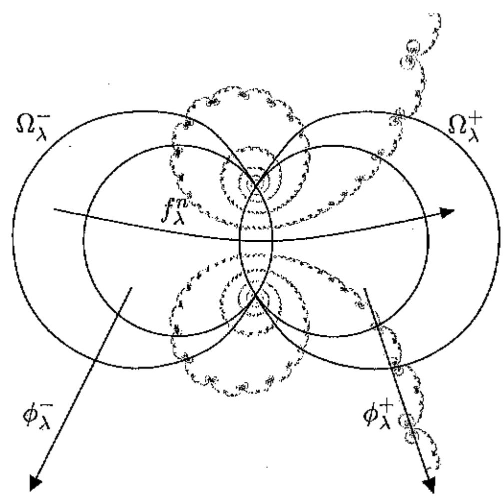

Figure 4.3: Perturbed Fatou coordinates for P_0.

The phase map

For $\lambda \in S^*$, the functions $\phi_\lambda^\pm$ both induce a Fatou coordinate on $\Omega_\lambda^+ \cap \Omega_\lambda^-$, then $\phi_\lambda^+ - \phi_\lambda^-$ is a constant function with value $\tau(\lambda) \in \mathbb{C}$. This defines a function $\tau : S^* \to \mathbb{C}$ called the *lifted phase*, which is holomorphic and tends to ∞ as λ tends to 0.

Lemma 4.1.1. *The lifted phase map satisfies*

$$\tau(\lambda) = \frac{-\pi}{\sqrt{\lambda}} + O(1).$$

Proof. We will give the proof when $f : z \mapsto z + z^2 + \lambda$, $\lambda > 0$. In this case, the fixed points of f are $\alpha^\pm = \pm i\sqrt{\lambda}$ with multipliers $\rho^\pm = 1 \pm 2i\sqrt{\lambda}$. Let us make the change of variable $\zeta = \frac{1}{2i\sqrt{\lambda}} \log \frac{z-\alpha^+}{z-\alpha^-}$, where the branches of logarithm differ by $\pi/\sqrt{\lambda}$ on $\Omega_\lambda^+ \cap \Omega_\lambda^-$. Under this change of variable, f is conjugated to

$$F_\lambda(z) = z + \frac{\log \rho^+}{2i\sqrt{\lambda}} + \frac{1}{2i\sqrt{\lambda}} \log \frac{1 - e^{2i\sqrt{\lambda}z}/\rho^+}{1 - \rho^- e^{2i\sqrt{\lambda}z}}.$$

The function F_λ satisfy the following properties:

i) $F_\lambda(z) = z + 1 + V_\lambda(z)$, $|V_\lambda(z)| \le \frac{1}{4}$, $|V_\lambda'(z)| \le \frac{1}{4}$ outside a fixed disk.

ii) $F_\lambda(z) - z \to \frac{\log \rho^\pm}{2i\sqrt{\lambda}}$ as $\mathrm{Im} z \to \pm\infty$.

Then, there exists $\phi_\lambda^\pm(z)$ conjugating F_λ to $z \mapsto z+1$ on $V_\lambda^\pm$, where $\phi_\lambda^\pm(\Omega_\lambda^\pm) = V_\lambda^\pm$ is a right\left region in the complex plane. Let $z^- \in V_\lambda^-$ and let U be the region bounded by the (left)boundary of V_λ^- and the (right)boundary of the region $T_{\frac{\pi}{\lambda}}(V_\lambda^+)$, where $T_{\frac{\pi}{\lambda}}(z) = z + \frac{\pi}{\lambda}$. Finally, let ϕ_λ be the Fatou coordinate on U. Then $\phi_\lambda^+ = \phi_\lambda \circ T_{\frac{\pi}{\lambda}} + \sigma_\lambda^+$ on V_λ^+ and $\phi_\lambda^- = \phi_\lambda + \sigma_\lambda^-$ on V_λ^-, with σ_λ^+ and σ_λ^- chosen in such a way that $\phi_\lambda^\pm$ converges to $\phi^\pm$, the Fatou coordinates of $z \mapsto z + z^2$. Then, the phase map is $\sigma_\lambda^+ - \sigma_\lambda^- = \sigma_\lambda$. For $\mathrm{Im}\, z > 0$ large, ϕ_λ^+ can be extended to the disk of center z and radius R, $D(z, R)$, where it is univalent.

Statement: $\phi_\lambda'(z) \to \frac{2i\sqrt{\lambda}}{\log \rho^\pm}$ *as* $\mathrm{Im}\, z \to \pm\infty$. Since

$$1 = \phi_\lambda(F_\lambda(z)) - \phi_\lambda(z) = w(z)\phi_\lambda'(z) + \frac{w(z)^2}{2}\phi_\lambda''(z) + ...$$

where $w(z) = F_\lambda(z) - z$, it follows that $|\phi_\lambda'(z) - \frac{1}{w(z)}| \le \frac{C}{R}$ by the Cauchy formula (or Koebe Distortion Theorem). The claim follows, since $\frac{1}{w(z)} \to \frac{2i\sqrt{\lambda}}{\log \rho^+}$.

Let $h_\lambda = \phi_\lambda^- \circ (\phi_\lambda^+)^{-1}$, then

$$h_\lambda = \phi_\lambda^- \circ (\phi_\lambda^+)^{-1} = T_{\sigma_\lambda^-} \circ \phi_\lambda \circ (T_{-\frac{\pi}{\lambda}} \circ \phi_\lambda^{-1} \circ T_{-\sigma_\lambda^+}) = T_{\sigma_\lambda^-} \circ \tilde{h}_\lambda \circ T_{-\sigma_\lambda^+}$$

where $\tilde{h}_\lambda = \phi_\lambda \circ T_{-\frac{\pi}{\lambda}} \circ \phi_\lambda^{-1}$. Let $w^+ = \phi_\lambda(z^+)$, then we get

$$\begin{aligned} \tilde{h}_\lambda(w^+) - w^+ &= \phi_\lambda \circ T_{-\frac{\pi}{\lambda}} \circ \phi_\lambda^{-1}(w^+) - w^+ = \phi_\lambda(z^+ - \frac{\pi}{\lambda}) \\ &= \phi_\lambda(z^-) - \phi_\lambda(z^+) = -\int_{z^-}^{z^+} \phi_\lambda' dz. \end{aligned}$$

Since

$$-\int_{z^-}^{z^+} \phi_\lambda' dz = -\frac{\pi}{\lambda}\frac{2i\sqrt{\lambda}}{\log \rho^+} + ...,$$

we conclude that $h_\lambda(w^+) = w^+ - \sigma_\lambda - \frac{2i\pi}{\log \rho^+} + ...$, and $-\sigma_\lambda - \frac{2i\pi}{\log \rho^+} \to b_0$ as $\lambda \to 0$ for some b_0; therefore, $\sigma_\lambda \sim \frac{2i\pi}{\log(1+2i\sqrt{\lambda})} \sim -\frac{\pi}{\sqrt{\lambda}}$ $\square$

Reducing θ_0 and r_0 if necessary, the map τ induces a $\mathbb{C}$-analytic isomorphism $\lambda \mapsto \tilde{\lambda}$ of S^* onto a closed set $\tilde{S}$. From $\tau(\lambda) = \frac{-\pi}{\sqrt{\lambda}} + O(1)$, we obtain that $\tilde{\lambda} \sim \frac{-\pi}{\sqrt{\lambda}}$. Then $\lambda \sim \frac{\pi^2}{\tilde{\lambda}^2}$ and $\frac{d\lambda}{d\tilde{\lambda}} \sim \frac{-2\pi^2}{\tilde{\lambda}^3}$ when λ tends to 0, or equivalently, when $\tilde{\lambda} = \tau(\lambda)$ tends to ∞ in $\tilde{S}$.

A sequence of Mandelbrot-like families

Let $T_w(z) = z + w$ be the complex translation by $w \in \mathbb{C}$. We set $\tilde{V_\lambda} = \phi_\lambda^+(V_\lambda)$ for $\lambda \in S$, and $\tilde{W}_n = \{\tilde{\lambda} \in \tilde{S} : \tilde{\lambda} + n \in \tilde{V_\lambda}\}$ where $\lambda = \tau^{-1}(\tilde{\lambda})$. We set $W_n = \tau^{-1}(\tilde{W}_n)$. Then the sets W_n are disjoint. More precisely, W_n is the set of values of λ such that $a_i(\lambda) \in \Omega_\lambda^- \cup \Omega_\lambda^+$ for $1 \leq i \leq n$ and $a_n(\lambda) \in V_\lambda$. In order to see this, note that for $\lambda \in W_n$, $\tilde{\lambda} + n \in \tilde{V_\lambda}$ implies that there is $z \in V_\lambda$ such that $\phi_\lambda^+(z) = \tilde{\lambda} + n$; since $\phi_\lambda^+(z) - \phi_\lambda^-(z) = \tilde{\lambda}$, then $\phi_\lambda^-(z) = n$. By the normalization of ϕ_λ^- we conclude that $z = a_n(\lambda)$.

We choose base points in all these sets: w_λ for U_λ, b_λ for V_λ, $0 = \phi_\lambda^+(b_\lambda)$ for $\tilde{V_\lambda}$, $\tilde{\lambda}_n = -n$ for $\tilde{W}_n$, and $\lambda_n = \tau^{-1}(\tilde{\lambda}_n)$ for W_n, so that $f_{\lambda_n}^n(w_{\lambda_n}) = b_{\lambda_n}$.

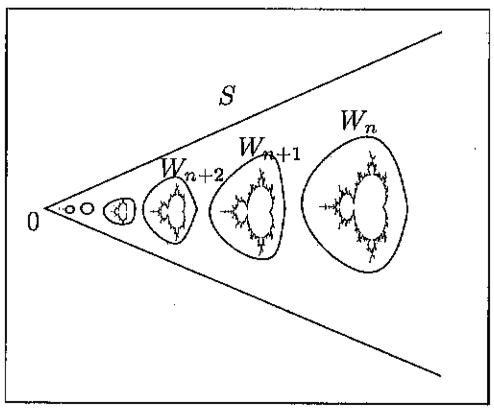

Figure 4.4: The sequence of domains W_n in S.

When n tends to infinity, the open set $T_n(\tilde{W}_n)$ tends to $\tilde{V}_0$, with convergence of the boundaries for the Hausdorff distance. In particular

$$\operatorname{diam}(\tilde{W}_n) = \operatorname{diam}(T_n(\tilde{W}_n)) \to \delta_0 = \operatorname{diam}(\tilde{V}_0).$$

It follows that

$$\lambda_n \sim \frac{\pi^2}{n^2} \quad \text{and} \quad \operatorname{diam}(W_n) \sim \frac{2\pi^2 \delta_0}{n^3},$$

for which $n^3\text{diam}(W_n) = 2\pi^2\delta_0 + O(1)$.

The set W_n consists of parameters $\lambda \in \Lambda$ with disconnected Julia sets for which the critical point escapes U'_λ exactly after n iterations of f_λ, and escapes exactly through the corresponding set V_λ. But we can consider parameters close to W_n which still escape in the same time but not through the set V_λ. We define the set $A_n = \{\lambda \in S : f_\lambda^n(w_\lambda) \in \Omega_\lambda^+ \backslash U'_\lambda\}$, then $W_n \subset A_n$. We set $\tilde{A}_n = \{\tilde{\lambda} \in \tilde{S} : \tilde{\lambda} + n \in \phi_\lambda^+(\Omega_\lambda^+ \backslash U'_\lambda)\}$, where $\tau^{-1}(\tilde{\lambda}) = \lambda$; since $\phi_\lambda^+(f_\lambda^n(w_\lambda)) - \phi_\lambda^-(f_\lambda^n(w_\lambda)) = \tilde{\lambda}$, then $\phi_\lambda^+(f_\lambda^n(w_\lambda)) = \tilde{\lambda} + n$, i.e. $\lambda \in A_n$. We conclude that $\tau^{-1}(\tilde{A}_n) = A_n$.

Now, we have that $T_n(\tilde{A}_n)$ tends to $\phi_0^+(\Omega_0^+ \backslash U'_0)$, with convergence of the boundaries. In particular

$$\text{diam}(\tilde{A}_n) = \text{diam}(T_n(\tilde{A}_n)) \to \gamma_0 = \text{diam}(\phi_0^+(\Omega_0^+ \backslash U'_0)).$$

By the same estimations as above, we have that $\text{diam}(A_n) \sim \frac{2\pi^2\gamma_0}{n^3}$, i.e.

$$n^3\,\text{diam}(A_n) = 2\pi^2\gamma_0 + O(1).$$

By the choice of the sets V_λ and the choice for the domains U_λ, we have $\gamma_0 - \delta_0 > 0$, since $\bar{V}_0 \subset U_0 \backslash \bar{U}'_0$.

Even though parameters in W_n have disconnected Julia sets, we still can construct a connected locus inside these sets by changing the map f_λ. For $\lambda \in W_n$, we denote by V'_λ the connected component of $f_\lambda^{-n}(V_\lambda)$ which contains w_λ, and we set $G_\lambda = g_\lambda \circ f_\lambda^n : V'_\lambda \to U_\lambda$.

Proposition 4.1.2 ([D2]). *Take n big enough. Then for $\lambda \in W_n$, the map $G_\lambda : V'_\lambda \to U_\lambda$ is quadratic-like, and the family $\mathbf{G_n} = (G_\lambda)_{\lambda \in W_n}$ is Mandelbrot-like.*

We fix $R > 1$. For each n, we can choose a tubing $\Theta_n = (\Theta_\lambda)_{\lambda \in W_n}$ for $\mathbf{G_n}$, with $\Theta_\lambda : \overline{A}_{R^2,R} \to \bar{U}_\lambda \backslash V'_\lambda$ quasi-conformal. Then the straightening $\chi_n = \chi_{\Theta_n} : W_n \to W_M$ is a quasi-conformal map where W_M is a neighborhood of the Mandelbrot set.

Primitive copies

Let c_0 be the root point of a primitive hyperbolic component of the interior of the Mandelbrot set M, and denote by Q_λ the map $z \mapsto z^2 + c_0 + \lambda$. Let Δ be the the component of the interior of K_{c_0} which contains 0, and k be the period of Δ, so that Q_0^k induces a proper map $\Delta \to \Delta$ of degree 2. We suppose $c_0 \neq \frac{1}{4}$. There is a parabolic point $\alpha_0 \in \partial\Delta$ of period k with $(Q_0^k)'(\alpha_0) = 1$.

Using the conjugation of the map Q_0 to $z \mapsto z+1$ by the Fatou coordinates, it is possible to construct a sequence $(b_n)_{n \in \mathbb{Z}}$ such that

i) $b_{n+1} = Q_0(b_n)$ for all n
ii) $b_N = 0$ for some N
iii) $b_n \notin \Delta$ for $n < N$
iv) $b_{mk} \to \alpha_0$ when $m \to \infty$.

Lemma 4.1.3 ([D2]). *One can find Jordan domains U_0, U_0', V_0 with C^1 boundary such that:*
i) $\bar{\Delta} \subset U_0' \subset \bar{U}_0' \subset U_0$;
ii) Q_0^k induces a proper map $f_0 : U_0' \to U_0$ of degree 2;
iii) $\bar{V}_0 \subset U_0 \backslash \bar{U}_0'$;
iv) there is $l = -mk < N$ for some $m \in \mathbb{N}$ such that Q_0^{N-l} induces an isomorphism $g_0 : V_0 \to U_0$.

In fact one can choose U_0 contained in an arbitrary small neighborhood of Δ, then we take for instance $U_\lambda = U_0$ for every $\lambda \in \mathbb{C}$. For λ close to 0, let U_λ' (*resp.* V_λ) be the Jordan domain bounded by the component of $Q_\lambda^{-k}(\partial U_\lambda)$ (*resp.* $Q_\lambda^{-l}(\partial U_\lambda)$ close to $\partial U_0'$ (*resp.* ∂V_0).

There is a simply connected neighborhood Λ of 0 such that $\partial U_\lambda'$ and ∂V_λ undergo a holomorphic motion when λ ranges in Λ, and that the inclusions $\overline{U}_\lambda' \subset U_\lambda$, $\bar{V}_\lambda \subset U_\lambda \backslash \overline{U}_\lambda'$ hold for $\lambda \in \Lambda$. Then Q_λ^k and Q_λ^l induce maps $f_\lambda : U_\lambda' \to U_\lambda$ and $g_\lambda : V_\lambda \to U_\lambda$ and we are in the situation described above.

Finally, we have a map $\Xi : \Lambda \to \mathbb{C}$ given by $\lambda \mapsto c_0 + \lambda$ which sends 0 to c_0, M_F to M and ∂M_F to ∂M. In particular the little copy of the Mandelbrot set generated in W_n is mapped to M_n, a little copy of the Mandelbrot set. The center λ_n of W_n satisfies that $\lambda_n \sim \frac{\pi^2}{n^2}$. Therefore, M_n approaches the root point c_0 as $\frac{1}{n^2}$ approaches to 0. Similarly we have that $\operatorname{diam}(M_n) \sim \frac{1}{n^3}$.

The size estimations

Consider the lifted phase map $\tau(\lambda)$, we will calculate the estimations given in section 4.1.4 for the family of maps in S.

First, from $\lambda_n = \tau^{-1}(-n)$ we get $-n = \frac{-\pi}{\sqrt{\lambda_n}} + O(1)$, then $\frac{-\pi}{\sqrt{\lambda_n}} \sim -n$ and $\lambda_n \sim \frac{\pi^2}{n^2}$.

Lemma 4.1.4. *The distance between two consecutive little Mandelbrot families, in the collection $\{\mathbf{G_n}\}$, is of order $O(1/(n+1)^3)$ as $n \to \infty$.*

Proof. First, since $\lambda_n = \tau^{-1}(-n)$, we have that $-n = \frac{-\pi}{\sqrt{\lambda_n}} + O(1)$, then $\frac{-\pi}{\sqrt{\lambda_n}} \sim -n$; thus $\lambda_n \sim \frac{\pi^2}{n^2}$. For the derivative, we have that

$$\frac{d\lambda}{d\tilde{\lambda}}\Big|_{\tilde{\lambda_n}} \sim \frac{2\pi^2}{n^3}.$$

Using the estimation of the difference between the diameters of W_n and A_n,

$$n^3(\mathrm{diam}(A_n) - \mathrm{diam}(W_n)) = 2\pi^2(\gamma_0 - \delta_0) + O(1)$$

we can conclude that $\mathrm{diam}(A_n) - \mathrm{diam}(W_n) \sim \frac{2\pi^2}{n^3}(\gamma_0 - \delta_0)$ and then

$$\mathrm{dist}(W_n, W_{n+1}) \geq \frac{2\pi^2}{(n+1)^3}(\gamma_0 - \delta_0)$$

with the corresponding estimate between the little Mandelbrot copies,

$$\mathrm{dist}(M_n, M_{n+1}) = O(1/(n+1)^3).$$

$\square$

4.2 Essentially bounded combinatorics

We will consider a central cascade $\mathcal{C} \equiv \mathcal{C}^{m+N}$:

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N}, \quad g_{m+1}0 \in V^{m+N-1} \backslash V^{m+N}.$$

The double covering $g_{m+1} : V^{m+1} \to V^m$ can be viewed as a small perturbation of a quadratic-like map g_* with a definite modulus and with non-escaping critical point. Here, we use the Carathéodory topology and the fact that $\mathcal{Q}(\mu)$ is a compact set. By Theorem II in [L2], all the return maps g_{m+1} of the principal nest belong to $\mathcal{Q}(\bar{\mu})$. Let $\mathcal{Q}_N$ (or $\mathcal{Q}_N(\mu)$) denote the space of quadratic-like maps $g : U' \to U$ from $\mathcal{Q}$ (or $\mathcal{Q}(\mu)$) such that $g^n 0 \in U$, $n = 0, 1, .., N$. Since $\cap_N \mathcal{Q}_N(\mu) = \mathcal{Q}_\infty(\mu)$, for any neighborhood $\mathcal{U} \supset \mathcal{Q}_\infty(\mu)$, there is an N such that $\mathcal{Q}_N(\mu) \subset \mathcal{U}$. In this sense any double map $g \in \mathcal{Q}_N(\mu)$ is close to some quadratic-like map g_* with connected Julia set, in particular, the return map g_{m+1} generating a cascade of length N. Moreover, since g_{m+1} has an escaping fixed point, the neighborhood of g_* containing g_{m+1} also contains a quadratic-like map with hybrid class $c(g_*) \in \partial M$.

4.2.1 Preliminaries

Let us restrict our discussion to the real line. Let $I' \subset I$ be two intervals, a map $f : (I', \partial I') \to (I, \partial I)$ is called quasi-quadratic if it is S-unimodal (negative Schwartzian derivative) and has quadratic-like critical point $0 \in \mathrm{int} I'$.

Let $I^0 = [\alpha, \alpha']$ be the interval between the dividing fixed point α and the symmetric one. Let $\mathcal{Y}_f$ denote the Markov family of pull-backs of the $\mathrm{int} I^0$. Given a critical interval $J \ni 0$, we can define a (generalized) renormalization

$T_J f$ on J as the first return map to J restricted to the component of its domain meeting the post-critical set $w(0)$. If f admits a unimodal renormalization $Rf \equiv T_J f$ for some J, then there are only finitely many such components.

Let $I^0 \supset I^1 \supset ... \supset I^{t+1}$ be the real principal nest of intervals until the next quadratic-like level. Let us look at real cascades of central returns. The return to level $n-1$ is called high or low if $g_n I^n \supset I^n$ or $g_n I^n \cap I^n = \emptyset$ correspondingly. Let us classify a central cascade $\mathcal{C} \equiv \mathcal{C}^{m+N}$, $I^m \supset I^{m+1} \supset ... \supset I^{m+N}$, $g_{m+1}0 \in I^{m+N-1} \backslash I^{m+N}$ as Ulam-Neumann or saddle-node according as the return to level $m+N-1$ is high or low, which is equivalent to the condition $0 \in g_{m+1} I^{m+1}$ or otherwise. In the former case, the map $g_{m+1} : I^{m+1} \to I^m$ is combinatorially close to the Ulam-Neumann map $z \mapsto z^2 - 2$, while in the latter, it is close to the saddle node map $z \mapsto z^2 + \frac{1}{4}$.

Unlike the complex situation, on the real line we observe only two types of cascades since there are only two boundary points in the "real Mandelbrot set" $[-2, \frac{1}{4}]$. The next lemma shows that for a long saddle-node cascade, the map $g_{m+1} : I^{m+1} \to I^m$ is a small perturbation of a map with a parabolic fixed point.

Lemma 4.2.1 ([L2]). *Let $g_k : U_k \to V_k$ be a sequence of real-symmetric quadratic-like maps with* $\operatorname{mod}(g_k) \geq \epsilon > 0$ *having saddle-node cascade of length $l_k \to \infty$. Then any limit point of this sequence in the Carathéodory topology $f : U \to V$ is hybrid equivalent to $z \mapsto z^2 + 1/4$, and thus has a parabolic fixed point.*

Proof. It takes l_k iterates for the critical point to escape U_k under iterates of g_k. Hence the critical point does not escape U under iterates of f. By the kneading theory [MT] f has on the real line topological type of $z^2 + c$ with $-2 \leq c \leq 1/4$. Since small perturbations of f have escaping critical point, the choice for c is only two boundary parameter values, $1/4$ and -2. Since cascades of g_k are of saddle-node type, $c = 1/4$. □

Let us consider the orbit $J_k \equiv f^k I^n$, $k = 0, ...s(n)$, of I^n until its first return to I^{n-1}, i.e. $f^{s(n)} I^n \subset I^{n-1}$. Let us see how this orbit passes through a saddle-node cascade. The level $m+s$ of the cascade is "branched" if for some interval $J_k \subset I^m \backslash I^{m+1}$ we have $g_{m+1} J_k \subset I^{m+s-1} \backslash I^{m+s}$.

4.2.2 Essential Period

There is a especial type of combinatorics related to the parabolic bifurcation which usually requires a special treatment. In this section we will briefly recall the definition of the essential period of a renormalizable unimodal map. We will follow the work of Lyubich, Hinkle and Yampolsky.

Let f be a renormalizable unimodal map. Consider its principal nest of intervals

$$[\alpha(f), -\alpha(f)] \equiv I^0 \supset I^1 \supset I^2 \supset ...$$

where $\alpha(f)$ is the dividing fixed point of f, and $I^m \ni 0$ is the central component of the first return map of I^{m-1},

$$g_m : \cup I_i^m \to I^{m-1}.$$

Set $m(0) = 0$, and let $m(0) < m(1) < ... < m(\kappa)$ be the sequence of non-central levels. Then the map $g_{m(\kappa)+1}|_{I^{m(\kappa)+1}} \equiv pRf$ is a pre-renormalization of f.

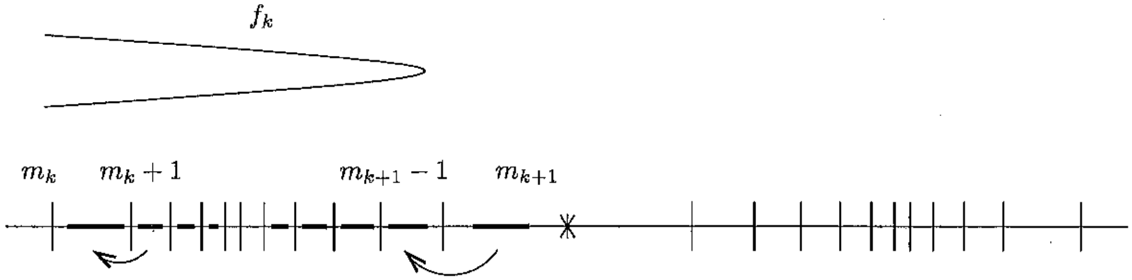

Figure 4.5: A saddle-node cascade.

For $0 \le k < \kappa$, let $x \in P(f) \cap (I^{m(k)} \backslash I^{m(k)+1})$ and set $d_k(x)$ =min$\{j - m(k), m(k+1) - j\}$, where $g_{m(k)+1} \in I^j \backslash I^{j+1}$. This number shows how deep the image of x lands inside the cascade. Now we define d_k as the maximum of $d_k(x)$ over all points $x \in P(f) \cap (I^{m(k)} \backslash I^{m(k)+1})$. For a saddle node-cascade the levels l such that $m(k) + d_k < l < m(k+1) - d_k$ are *neglectable.* Define the essential period as follows. Set $J = I^{m(\kappa)+1}$, and let p be its period. Consider the orbit $J_0 \equiv J, J_i = f^i(J_0)$, $i \le p-1$. Suppose that J_k lands at a neglectable level of a central cascade generated by the branch of $g_m|_{I^m} \equiv f^{l_m}$. We will call the iterates $J_k, J_{k+1}, ..., J_{k+l_m-1}$, which constitute one iterate by the cascade, neglectable. The number of non-neglectable intervals in the orbit $\{J_i\}_{i=0}^{p-1}$ is the *essential period*, $p_e(f)$.

4.2.3 Tripling essentially bounded combinatorics

There is a simple example of an infinitely renormalizable map of unbounded but essentially bounded combinatorial type. This map is constructed in such a way that its every renormalization is a small perturbation of a unimodal map with period 3 parabolic orbit. Closeness to a parabolic map will give high renormalization periods, but all the essential periods will be bounded.

First, let us consider the dynamics of the quadratic map $f_0 : z \mapsto z^2 - 1.75$. This polynomial has a parabolic orbit of period 3 on the real line. Let z_0 be the element of this orbit which is closer to 0. Let $I^0 = [\alpha(f), \alpha' f]$, and I^1 be the central component of the domain of the first return map $g : I^0 \to I^0$. For this map we have $g|_{I^1} \equiv f^3$, $z_0 \in I^1$, and $f^{3n}(0) \to z_0$. The map g has two non-central components: I_1^1 the one whose boundary contains $\alpha(f)$, then $g = f^2 : I_1^1 \to I^0$. For small perturbation of f, $f_\epsilon(z) = z^2 - 1.75 + \epsilon$, $\epsilon > 0$, the orbit of 0 under f_ϵ eventually escapes I^1. Let us define ϵ_n as the parameter value for which $f_{\epsilon_n}^{3i}(0) \in I^1$, $i \leq n-1$, $f_{\epsilon_n}^{3n}(0) \in I_1^1$, and $f_{\epsilon_n}^{3n+2}(0) = 0$. These maps correspond to the centers of a sequence of small copies $M_n^{(3)}$ of the Mandelbrot set converging to the cusp $c = -1.75$ of the real period 3 copy $M^{(3)}$. The existence of these maps follows from [Hi].

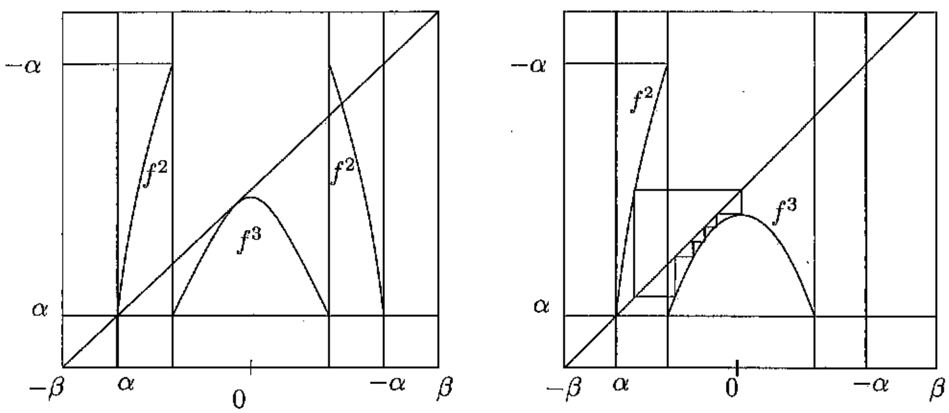

Figure 4.6: The first return map for $z^2 - 1.75$ and $z^2 + \epsilon_5$.

We will show now that the essential period for all these maps is bounded. For each n, f_{ϵ_n} will have only one central cascade, with length n, therefore we need to check for neglectable levels only in this cascade. Consider the map f_{ϵ_n}, its principal nest is given by

$$I^0 \supset I^1 \supset ... \supset I^{n-1} \supset I^n \supset I^{n+1}$$

where all the returns of level $n = 1, 2, ..., n-1$ are central and the level n is non-central. Moreover, the level $n+1$ is the renormalization level. Then the sequence of non-central levels is $m(0) = 0 < m(1) = n$ for which $I^1 \supset I^2 \supset ... \supset I^n$ is the only central cascade of length $m(1) - m(0) = n$; it is clear that the cascade is saddle-node since 0 eventually escapes I^1.

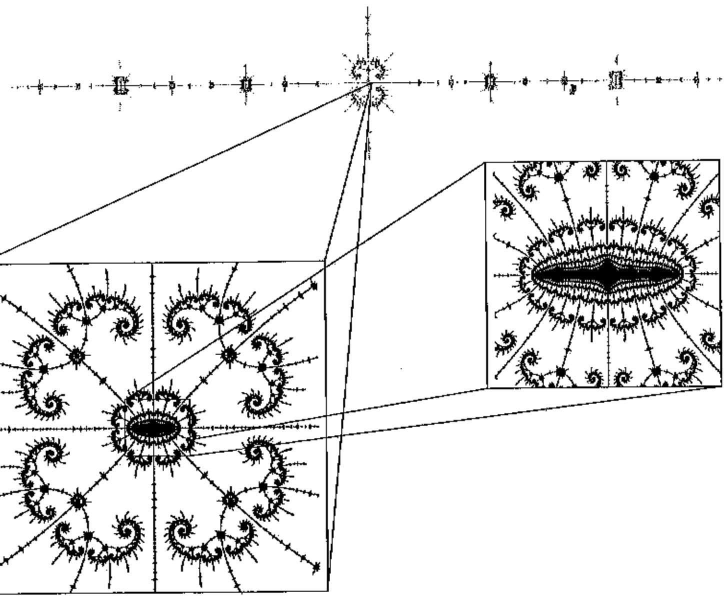

Figure 4.7: A blow-up of a Julia set: an airplane inside of an airplane.

Now consider $x \in P(f_{\epsilon_n}) \cap (I^0 \backslash I^1)$, then $x = f_{\epsilon_n}^{3n}(0)$ and $d_0(x)$ =min$\{j - 0, n - j\}$=0 since x never comes back to the cascade. Thus, all the levels $m(0) = 0 < l < m(1) = n$ are neglectable, i.e. all of the levels of the central cascade of f_{ϵ_n} are neglectable. Denote by $J = I^{m(2)}$, the renormalization interval of f_{ϵ_n} with period $p = p(f_{\epsilon_n})$, which is some large number if ϵ_n is small. To calculate the essential period of the orbit $\{J_i \equiv f_{\epsilon_n}^i(J)\}_{i=0}^{p-1}$, we have to ignore all the iterates, but five: J_0, J_1, J_2 and J_{p-2}, J_{p-1}. These correspond to the first iterate of the orbit of J by the cascade generated by the central branch of $g|_{I^1} \equiv f^3$, and the one iterate by the non-central branch $g|_{I_1^1} \equiv f^2$, which, after the interval has run through the whole cascade, maps it back over the critical point. Thus, the essential period $p_e(f_{\epsilon_n}) = 5$, on the other hand $p(f_{\epsilon_n}) \to \infty$.

This is the desired example, we can select h in the real quadratic family, by choosing an infinitely renormalizable parameter value $c \in M$ such that $\chi(R^k(f_c)) \in M_{n_k}^{(3)}$, with $n_k \to \infty$. Thus, we can blow-up a small copy $M_{n_1}^{(3)}$, find its period 3 cusp and the corresponding sequence of small copies converging to this cusp, then blow-up one of them and find its period 3 cusp; continue

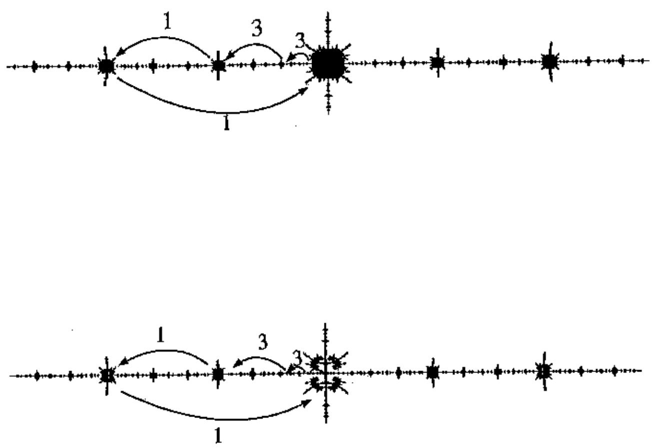

Figure 4.8: The Julia sets of $z^2 - 1.75$ and $z^2 + \epsilon_n$ for some large n.

this procedure to infinity.

Relation between $p_e(f) = 5$ and little copies of M We give now a relation between the little copies constructed before and bounded essential period.

Lemma 4.2.2. *The family of little Mandelbrot copies $\{M_n \subset W_n\}_{n \geq N}$ coincide with the family of copies $\{M_n^{(3)}\}_{n \geq N}$ for some big N.*

Proof. Given a long saddle-node cascade of length N, the map G obtained from $g_{m+1} : I^{m+1} \to I^m$ by rescaling I^m to the unit size, must be close to a saddle node quasi-quadratic map. In [L2], G can be reduced to the form $z \mapsto z + \epsilon + \psi(z)$, where $\psi(z) > 0$ is uniformly comparable with z^2. Then, the quasi-symmetric class of the cascade is determined by ϵ, which in turn is related to the length of the cascade by $N \asymp 1/\sqrt{\epsilon}$. In our case we have $n \asymp 1/\sqrt{\epsilon_n}$.

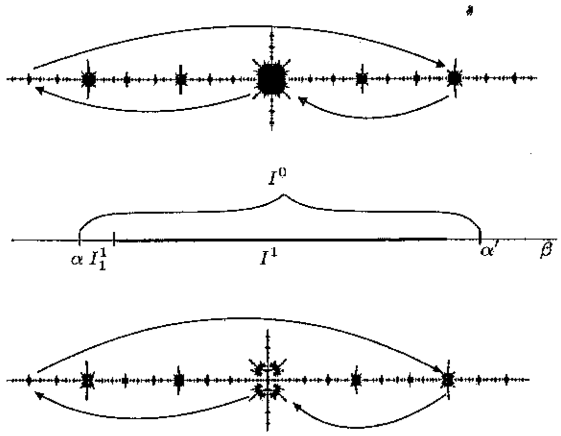

Figure 4.9: The map $z \mapsto z^2 - 1.75$, and its small perturbation.

By our estimation of the center λ_n of the set W_n, we get that $\lambda_n \sim \frac{\pi^2}{n^2}$. Since $n^2 \asymp 1/\lambda_n$ and the parameters in $M_n^{(3)}$ have cascades of length n, we conclude that $\epsilon_n \sim \lambda_n$. $\square$

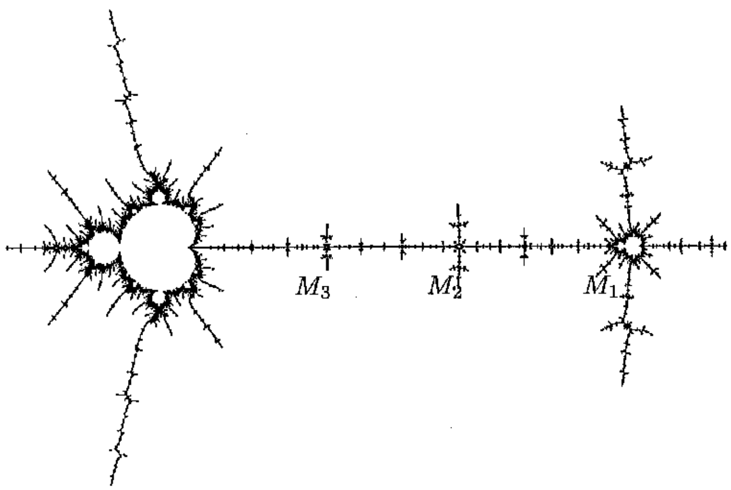

Figure 4.10: The sequence $\{M_n\}$.

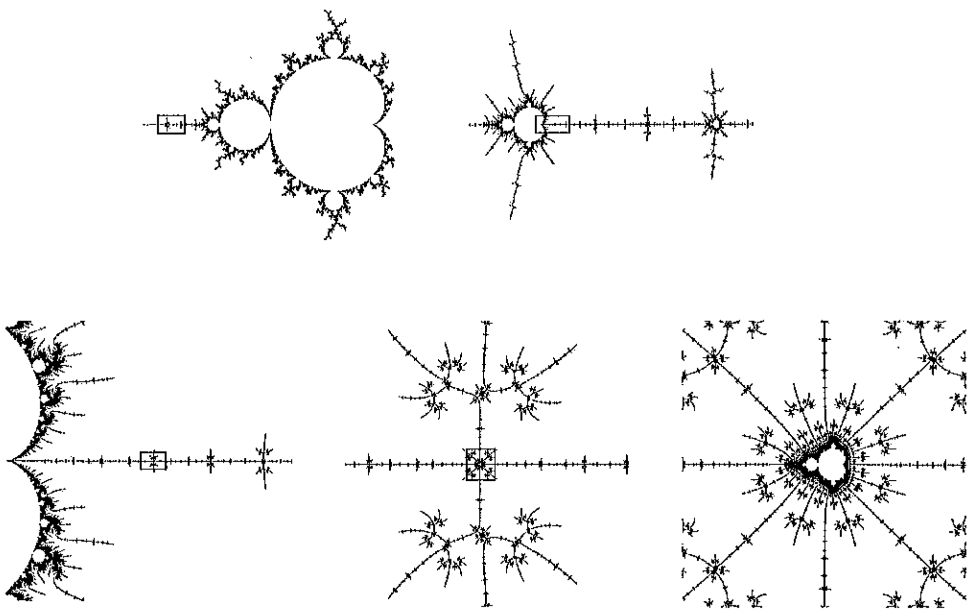

Figure 4.11: Blow-up of M_n for big n.

Chapter 5

Smoothness of the holonomy

5.1 The Full Renormalization Horseshoe

Let f be an infinitely renormalizable quadratic-like germ. Its tuning invariant is defined as $\tau(f) = \{M_0, M_1, M_2, ...\}$ where M_n is the maximal M-copy containing $\chi(R^n f)$. We say that f has real combinatorics if all M-copies in $\tau(f)$ are real.

Let Σ be the space of all possible real combinatorial types $\tau = \{M_k\}_{k=-\infty}^{\infty}$, where the $M_k \in \mathcal{M}$ are selected arbitrarily from the family of real maximal Mandelbrot copies. Supply Σ with the weak topology.

Let us say that an infinitely renormalizable quadratic-like map f is *completely nonescaping* under the renormalization if some full renormalization orbit $\{R^n f\}_{n=-\infty}^{\infty}$ is well defined, $R^n f \in \mathcal{C}$, and $\mathrm{mod}(f_n) \geq \mu = \mu(f) > 0$, $n \in \mathbb{Z}$.

Let $\mathcal{A} \subset \mathcal{Q}$ stand for the set of completely nonescaping maps with real combinatorics. We call this set the *(full) renormalization horseshoe.*

5.1.1 The Renormalization Theorem

Remember that, for $J \in \mathcal{J}$, $\mathcal{T}_J$ is a real renormalization strip, whose union over all J, forms the domain of definition of the renormalization R in the space $\mathcal{Q}_{\mathbb{R}}$ of real quadratic-like maps ($\mathcal{J}$ is the collection of of real maximal renormalization windows). We now state Lyubich's Renormalization Theorem.

Theorem 5.1.1 ([L4]). *There is a constant $\rho \in (0, 1)$ and a neighborhood V of the origin in $\mathbb{C}$ such that:*

(i) $\mathcal{A}$ is precompact in $\mathcal{Q}_{\mathbb{R}}$, R-invariant, and $R|\mathcal{A}$ is topologically conjugate to the two-sided shift $\omega : \Sigma \to \Sigma$ in countably many symbols.

(ii) *The topological classes $\mathcal{H}_{\mathbb{R}}(f)$, $f \in \mathcal{A}$ are codimension-one real analytic submanifolds in $\mathcal{Q}_{\mathbb{R}}$ (stable leaves) which form an R-invariant lamination in $\mathcal{Q}_{\mathbb{R}}$. Moreover, if $g \in \mathcal{H}_{\mathbb{R}}(f)$ and $\operatorname{mod}(g) \geq \nu$, then $R^n g \in \mathcal{B}_V$ and $||R^n f - R^n g||_V \leq C\rho^n$ for $n \geq N(\nu)$.*

(iii) *There exists an R^{-1}-invariant family of real analytic curves $W^u_{\mathbb{R}}(f)$, $f \in \mathcal{A}$ (unstable leaves) which transversally pass through all real hybrid classes $c \in [-2, 1/4 - \epsilon]$ and such that*

$$||R^{-n} f - R^{-n} g||_V \leq C\rho^n, \qquad n \geq 0,$$

provided $g \in W^u_{\mathbb{R}}(f)$.

(iv) *The renormalization operator has uniformly bounded distortion with respect to the Montel metric on the unstables leaves.*

(v) *The stable lamination is transversally quasi-symmetric.*

In the proof of the Renormalization Theorem there are finer properties which will be used in our result.

5.1.2 Absolute a priori bounds

Let $\mathcal{C}_{\mathbb{R}} \subset \mathcal{C}$ be the union of complex quadratic-like germs f with real straightening, $\chi(f) \in [-2, 1/4]$. Let $\mathcal{C}_{\mathbb{R}}(\mu) = \mathcal{C}_{\mathbb{R}} \cap \mathcal{C}(\mu)$ and let $\mathcal{C}_{\mathbb{R}}(\mu, n)$ be the set of n times renormalizable germs of $\mathcal{C}_{\mathbb{R}}(\mu)$. Then, the following lemma gives an absolute *a priori bounds* in $\mathcal{C}_{\mathbb{R}}$:

Lemma 5.1.2 ([L4]). *There is an absolute $\mu > 0$ such that if the germ of $f : V \to V'$ belongs to $\mathcal{C}_{\mathbb{R}}(\mu, n+1)$, then $\operatorname{mod}(R^m(f_V)) \geq \mu$ for $m = N(\nu), ..., n$, where $R^m(f_V)$ is the subordinate renormalization.*

Using the above result, it is possible to select a family of Banach slices invariant with respect to some iterate of the renormalization:

Lemma 5.1.3 ([L4]). *Let μ be an absolute bound from Lemma 5.1.2, and $0 < \nu \leq \mu$. There exist $N = N(\nu)$, $\delta > 0$, and a family of quadratic-like representatives $f : V(f) \to V'(f)$ of germs $f \in \mathcal{C}(\nu)$, with the following properties:*

- $\operatorname{mod}(V'(f) \backslash V(f)) > \gamma(\nu) > 0$;
- *If $f \in \mathcal{C}_{\mathbb{R}}(\nu, n+1)$ and $g \in \mathcal{B}_f(\delta) \cap \mathcal{H}(f)$, then $R^N g \in \mathcal{B}'_{R^N f}(\rho)$, where $\rho = \rho(\nu, \delta) \to 0$ as $\delta \to 0$ (ν being fixed), $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$ and $\mathcal{B}_f(\delta) \equiv \mathcal{B}_{V(f)}(f, \delta)$.*

In particular the set $\mathcal{A}$ belongs to $\mathcal{C}(\mu)$ and $R : \mathcal{A} \to \mathcal{A}$ is a homeomorphism.

Let $\mathcal{O}^s = \{E^s_f \subset \mathcal{B}_f\}$ stand for the field of tangent subspaces to $\mathcal{F}$ over $\mathcal{A}$ (" the horizontal field"), and let $\mathcal{O}^u = \{E^u_f \subset \mathcal{B}_f\}$, $f \in \mathcal{A}$ be the continuous invariant tangent line field transverse to $\mathcal{F}$. Then, the renormalization operator $R : \mathcal{A} \to \mathcal{A}$ is uniformly hyperbolic with $\mathcal{O}^s$ and $\mathcal{O}^u$ serving as the stable and unstable fields. Consider a number $N = N(\nu)$, a bound $\gamma = \gamma(\nu)$ and the family of Banach slices $\mathcal{B}_f$, $f \in \mathcal{C}(\nu)$ from Lemma 5.1.3; by [L4, §3.7.2], there exists a family of *special bidisks* $Q_f \subset \mathcal{B}_f(\delta)$ centered at f for some small δ and $f \in \mathcal{A}$ such that $R^N Q_f \subset \mathcal{B}_{R^N f}$ (the family is invariant), and R^N has uniform horizontal contraction and uniform vertical expansion.

Finally, let us denote by $E^{s/u}_f(\delta) \equiv E^{s/u}_f(f, \delta)$ the δ-balls in the spaces $E^{s/u}_f$ centered at f. In fact Q_f is generated by a small topological disk $S_f \subset E^u_f$ containing f, and a holomorphic motion of some local holonomy $\mathcal{F}_f$ over a set homeomorphic to $E^s_f(\delta)$.

5.1.3 Transverse control of the renormalization

The following statement shows that the renormalization has transversally bounded distortion with respect to the Montel metric in quadratic-like families.

Lemma 5.1.4 ([L4]). *Consider a quadratic-like family* $\mathbf{f}$ *of class* $\mathcal{G}_{C,\mu}$. *Take a little Mandelbrot set* $M \in \mathcal{M}$ *and let* $M_{\mathbf{f}}$ *be the corresponding set in the family* $\mathbf{f}$. *If* $p(M) > 2$, *then there is* $\lambda = \lambda(C, \mu) > 0$ *and a domain* $\Omega_{\mathbf{f}} \subset \mathbf{f}$ *of the renormalization* R_M *with* $\mathrm{mod}(\Omega_{\mathbf{f}} \backslash M_{\mathbf{f}}) \geq \lambda$ *such that the curve* $R_M(\Omega_{\mathbf{f}})$ *is uniformly transverse to the foliation* $\mathcal{F}$, *and* R_M *on* $\Omega_{\mathbf{f}}$ *has* $K(C, \mu)$*-bounded distortion (independent of* M*) with respect to the Montel metric.*

It follows from the fact that the Montel metric has bounded distortion with respect the hyperbolic metric and from the Koebe Distortion Theorem.

5.2 Holonomy for tripling essentially bounded combinatorics

Let $\{M_k\}_{k=1}^{\infty}$ be the sequence of little copies of the Mandelbrot set of periods $n_k = 3k + 2$ constructed above, with essentially bounded combinatorics. Let consider the sequence of corresponding real slices $S_0 = \{J_k = M_k \cap \mathbb{R}\}_{k=1}^{\infty}$. Let f_* be a real infinitely renormalizable quadratic-like germ with tuning invariant $\tau(f_*) = \{M_{k_n}\}_{n=0}^{\infty}$ and let $S = \{J_{k_n} = M_{k_n} \cap \mathbb{R}\}_{n=0}^{\infty}$ be the corresponding sequence of real slices, where this sequence is given by an arbitrary choice of elements of S_0. Let c_n stand for the center of the corresponding copy M_{k_n}.

First, let us mention here that when the combinatorics of f_* is given by S with $k_n \to \infty$, we have the following result by Hinkle.

Theorem 5.2.1 ([Hi]). *There is a unique quadratic-like germ F such that $R^n f \to F$ for any $f \in \mathcal{H}_{\mathbb{R}}(f_*)$. Any quadratic-like representative of F is hybrid equivalent to $z^2 - 1.75$ and hence has a period three parabolic orbit.*

Let us look closer at these real slices and the combinatorics of the map f_*. In J_{k_0} there is a sequence of real slices of little copies of the Mandelbrot set which correspond to $\{J_k\}_{k=1}^{\infty}$ under tuning: if $\{J_k^{(0)}\}_{k=1}^{\infty}$ are these slices, which we call of level 0, then $c_0 * J_k = J_k^{(0)}$ for all k, where $d_0 \equiv c_0$ is the center of M_{k_0}. Between those real slices, there is one $J_{k_1}^{(0)}$, such that $c_0 * J_{k_1} = J_{k_1}^{(0)}$ with center d_1.

Similarly, in $J_{k_1}^{(0)}$ there is a sequence of real slices $\{J_k^{(1)}\}_{k=1}^{\infty}$, of level 1, which under tuning correspond to $\{J_k\}_{k=1}^{\infty}$: $d_1 * J_k = J_k^{(1)}$; between the real slices of level 1, there is one, $J_{k_2}^{(1)}$, such that $d_1 * J_{k_2} = J_{k_2}^{(1)}$ with center d_2; in the same way we can define the corresponding slices of level n inside $J_{k_n}^{(n-1)}$. Thus, we have a sequence of real slices of little Mandelbrot copies

$$J_{k_0} \supset J_{k_1}^{(0)} \supset J_{k_2}^{(1)} \supset \ldots$$

such that $\chi(f_*) \in \bigcap_{s=1}^{\infty} J_{k_s}^{(s-1)}$ and $\chi(R^n(J_{k_n}^{(n-1)})) = J_{k_n}$ for $n \geq 1$.

5.3 Smoothness Condition

Definition 5.3.1. *A map $h : (M_1, 0) \to (M_2, 0)$ between two subsets in $\mathbb{C}$ is called C^1-conformal (or smooth) at the origin if there exists $\tau \neq 0$ such that $h(u) = \tau u(1 + o(1))$ for $u \in M_1$ near 0.*

Recall that $\mathcal{C}_{\mathbb{R}} \subset \mathcal{C}$ stand for the union of (complex) quadratic-like germs f with connected Julia set and real straightening, i.e., such that $\chi(f) \in [-2, 1/4]$.

Definition 5.3.2. *The foliation $\mathcal{F}$ is transversally C^1-conformal (smooth) at a point $c \in M$ (or along a leaf $\mathcal{H}_c$) if for any two transversals $\mathcal{S}$ and $\mathcal{T}$ to the leaf $\mathcal{H}_c$, the holonomy $h : \mathcal{M}_{\mathcal{S}} \to \mathcal{M}_{\mathcal{T}}$ is C^1-conformal at the points of intersection with $\mathcal{H}_c$, where $\mathcal{M}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{C}_{\mathbb{R}}$ and $\mathcal{M}_{\mathcal{T}} = \mathcal{T} \cap \mathcal{C}_{\mathbb{R}}$.*

By Lemma 5.3, there exist a sequence of quadratic-like representatives $R^n f_* : V_*(n) \to V'_*(n)$ and a natural number $N = N(\nu)$ such that the family of Banach slices $(\mathcal{B}_{V_*(n)}, R^n f_*)$ is invariant under $T = R^N$ and the orbit of

$f \in \mathcal{H}(f_*)$ under T is uniformly exponentially asymptotic to the orbit of f_* in these Banach slices.

By the Renormalization Theorem, there exists an R^{-1}-invariant family of real analytic curves $\mathcal{W}^u_{loc}(f)$, $f \in \mathcal{A}$ ("local unstable leaves") with $\mathcal{W}^u_{loc}(f) \subset \mathcal{B}_f$. In particular for $f_* \in \mathcal{A}$, let us define $\mathcal{W}^u_{loc}(f_*) \equiv \mathcal{W}^u$ and $\mathcal{W}^u_{loc}(R^n f_*) \equiv \mathcal{W}^u(n)$ for $n > 1$.

Also, there exists a sequence of bidisks $Q(n) \subset \mathcal{B}_{V_*(n)}$ centered at $R^n f_*$, where the iterate T acts hyperbolically on this family (uniformly contracting in the horizontal direction and uniformly expanding in the vertical direction).

Let $\mathcal{S}$ be a transversal to the leaf $\mathcal{H}_{\mathbb{R}}(f_*)$ through $f \in \mathcal{H}_{\mathbb{R}}(f_*)$. In order to check C^1-conformality of the holonomy, it is enough to check this property for the holonomy from $\mathcal{S}$ to the unstable manifold $\mathcal{W}^u \equiv \mathcal{W}^u_{loc}(f_*)$. Let us take a Banach slice $\mathcal{B}_U \ni f$ locally containing the transversal $\mathcal{S}$. Then, there is a neighborhood $\mathcal{U} \subset \mathcal{B}_U$ of f which is mapped by R^m into some Banach slice $\mathcal{B}_{V_*(m)}$ as above. Thus, the curve $\mathcal{S}^m = R^m(\mathcal{S})$ locally sits in $\mathcal{B}_{V_*(m)}$.

It is sufficient to study the holonomy h_m from $\mathcal{S}^m$ to $\mathcal{W}^u(m)$: if h denotes the holonomy from $\mathcal{S}$ to $\mathcal{W}^u$ then by the R-invariance of the foliation,

$$h = R^{-m} \circ h_m \circ R^m \tag{5.1}$$

where $R^m : (\mathcal{S}, f) \to (\mathcal{S}^m, R^m f)$ is a local conformal diffeomorphism [L3, §5.4].

Then the situation reduces to the Banach set up, and without loss of generality we can assume that $\mathcal{S}$ itself belongs to the Banach neighborhood (bidisk) $Q(1) \subset \mathcal{B}_{V_*(1)}$ of f_*.

Let $\mathcal{S}^n$ denote the connected component of $Q(n) \cap T^n \mathcal{S}$ containing $f_n = T^n f_*$. By the hyperbolicity of T, the transversals $\mathcal{S}^n$ can be eventually represented as graphs of analytic functions $\phi_n : E^u_n(\gamma) \to E^s_n(\gamma)$ with bounded vertical slope, where $E^{s/u}_n(\gamma)$ is the γ-ball in the tangent space $E^{u/s}_n$ at f_n. Moreover, these graphs are exponentially close to the corresponding unstable manifolds $\mathcal{W}^u(n)$. The local unstable manifolds $\mathcal{W}^u(n) \cap Q(n)$ can be also parametrized in the same way by some functions ψ_n.

By the hyperbolicity of T on the family of bidisks $Q(n)$, the manifolds $\mathcal{S}^n$ approach, exponentially fast, to the unstable manifolds $\mathcal{W}^u(n)$:

$$||\phi_n - \psi_n||_{C^1} \leq \kappa \rho^n \tag{5.2}$$

where $\rho \in (0, 1)$ is an upper bound of the contraction factor in the stable leaves. Moreover, $\kappa > 0$ can be a priori selected arbitrary small (replace $\mathcal{S}$ by some $\mathcal{S}^m$ with $m = m(\kappa)$).

Let us use the projections $p : \mathcal{S}^n \to E^u_n$ as analytic charts on $\mathcal{S}^n$. By the Koebe Theorem, they have distortion $O(\epsilon)$ in scale ϵ with a uniform constant (independent of n). We will use the following notation: for $u, v \in \mathcal{S}^n$,

$u - v$ means the difference between the local coordinates: $p(u) - p(v)$. Let $B_r(g)$ stand for the neighborhood of radius r around g in $\mathcal{M}_S$. Note that we can assume also $T = R$ since the combinatorics of T are included in the combinatorics of R.

Lemma 5.3.3. *Let $\mathcal{M}_S = \mathcal{S} \cap \mathcal{C}_{\mathbb{R}}$ and $\mathcal{M}_S^m = \mathcal{S}^m \cap \mathcal{C}_{\mathbb{R}}$ be the truncated iterates of $\mathcal{M}$, such that $\mathcal{S}^m$ is at a distance $O(\rho^m)$ from $\mathcal{W}^u(m)$. Then the holonomy from $\mathcal{S}^m$ to $\mathcal{W}^u(m)$ has an exponentially small ratio distortion on points with comparable distance to $R^m(f) = g$.*

Proof. Let z_1, z_2 be two points in $\mathcal{M}_S$ such that $R^m(z_1) = \zeta_1$, $R^m(z_2) = \zeta_2$ are defined and have comparable distances to $R^m(f) = g$. We can extend the foliation $\mathcal{F}_V$ to a neighborhood of $R^m f_*$ in $\mathcal{B}_{V_*(m)}$ by Theorem 3.1.8. If γ is sufficiently small then the bi-disk $E_f^s(2\gamma) \times E_f^u(2\gamma)$ is contained in the domain of the extended foliation. Consider the holonomy $h_m : \mathcal{S}^m \to \mathcal{W}^u(m)$ along the extended foliation. By the λ-lemma, h_m is K_m-qc with $K_m = 1 + O(\rho^m)$. Then by the distortion estimates for qc maps in [LV] and the fact that the holonomy preserves the angles at $R^m f$ up to order $O(\rho_1^m)$, with any $\rho_1 > \rho$, we obtain

$$\frac{h_m(\zeta_1) - h_m(g)}{h_m(\zeta_2) - h_m(g)} = \frac{\zeta_1 - g}{\zeta_2 - g}(1 + O(\rho_1^m)) \tag{5.3}$$

for any $\rho_1 > \rho$.

$\square$

There is an alternative way to compute the distortion in the last Lemma. Consider the cross-ratio of four points in $\hat{\mathbb{C}}$,

$$[u_1, u_2, u_3, u_4] = \frac{u_1 - u_3}{u_1 - u_2} \frac{u_2 - u_4}{u_3 - u_4}.$$

Taking $u_1 = g$, $u_2 = \zeta_1$, $u_3 = \zeta_2$ and $u_4 = \infty$ we obtain $[u_1, u_2, u_3, u_4] = \frac{\zeta_1 - g}{\zeta_2 - g}$. Consider the K_m- qc map h_m and let μ be the its Beltrami differential, $||\mu||_\infty < 1$. Let $S_\mu \equiv B_{1/||\mu||_\infty}(0)$ be the open ball of radius $1/||\mu||_\infty$ in $\mathbb{C}$. For $\lambda \in B_{1/||\mu||_\infty}(0)$ we have that $\lambda\mu$ is a Beltrami differential with $||\lambda\mu||_\infty < 1$. By Theorem 2.4, there is a quasi-conformal map h_λ with Beltrami differential $\lambda\mu$. Moreover, $h_\lambda(z)$ is holomorphic for each $z \in \mathbb{C}$.

We consider the map $T : S_\mu \to \mathbb{C}\backslash\{0, 1\} \equiv S'$ given by

$$\lambda \mapsto [h_\lambda(g), h_\lambda(\zeta_1), h_\lambda(\zeta_2), \infty],$$

which is a holomorphic map between two hyperbolic Riemann surfaces. If d_{S_μ} and $d_{S'}$ are the hyperbolic metrics in S_μ and S' respectively, for a compact set $K \subset B_{1/||\mu||_\infty}(0)$ containing $0, 1$ we have that $d_{S'}(T(\lambda_1), T(\lambda_2)) \leq$

$c_K d_{S_\mu}(\lambda_1, \lambda_2)$. In particular,

$$d_{S'}(T(0), T(1)) = d_{S'}\left(\frac{\zeta_1 - g}{\zeta_2 - g}, \frac{h_m(\zeta_1) - h_m(g)}{h_m(\zeta_2) - h_m(g)}\right) \leq c_K d_{S_\mu}(0,1)$$

where $d_{S_\mu}(0,1)$ is independent of g, ζ_1 and ζ_2. Since $d_{S_\mu}(0,1)$ is of order $O(\rho^m)$, we obtain the estimation.

From formula (2), we need to control the distortion of the renormalization operator R^m. The following lemma gives the first partial estimation of the distortion. It follows directly from the Koebe Distortion Theorem.

Lemma 5.3.4. *Let $\epsilon > 0$. Let $z_1, z_2 \in \mathcal{M}_S$ be two points in the domain of R^m, such that they go to points of relative distances of order $O(\epsilon)$ from $R^m f$. Then the ratio distortion of $R^m f$ is given by*

$$\frac{R^m(z_1) - R^m(f)}{R^m(z_2) - R^m(f)} = \frac{z_1 - f}{z_2 - f}(1 + O(\epsilon)). \tag{5.4}$$

That is, the Koebe Theorem gives distortion of order ϵ for any two points whose images under R^m have relative distances of order ϵ from $R^m f$. But in order to apply Lemma 5.3.3, we also need to have comparable distances between the images to $R^m f$. The next lemma gives an estimation for the distortion of the holonomy for two arbitrary points: these two points may not have comparable distances to f, so we need to define a string of points between them in such a way that every two consecutive points in this string have comparable distances to f; a natural way to do it, is by going half the distance to f at each step.

Lemma 5.3.5. *Let $\epsilon > 0$ and $m \in \mathbb{N}$ as in Lemma 5.3.3. Let $\delta > 0$ such that if $u \in \mathcal{M}_S \cap B_\delta(f)$, its image under R^m has relative distance of order ϵ from $R^m f$. Then for any two points $u, v \in \mathcal{M}_S \cap B_\delta(f)$, with $|v - f| \leq |u - f| < \delta$, the ratio distortion of the holonomy is given by*

$$\frac{h(u) - f_*}{u - f} = \frac{h(v) - f_*}{v - f}(1 + O(k(u,v)(\epsilon + \rho^m))), \tag{5.5}$$

where $k(u,v) \asymp \log \frac{|u-f|}{|v-f|}$.

Proof. Let $u_0 \equiv u, u_1, ..., u_k \equiv v$ be a string of points in $\mathcal{M}_S \cap B_\delta(f)$ such that $|u_i - f| = |u_{i-1} - f|/2$ for $i < k$ and $|u_{k-1} - f|/2 \leq |u_k - f| \leq |u_{k-1} - f|$. Such a string exists since the Mandelbrot set M is connected and the holonomy from M to $\mathcal{M}_S$ is continuous. Hence the set $\mathcal{M}_S$ intersects every circle around

f with sufficiently small radius. The number of points in this sequence satisfy $k \asymp \log \frac{|u-f|}{|v-f|} / \log 2$. We would like to estimate the following quotients

$$Q(u_i, u_{i+1}) = \frac{h(u_i) - f_*}{u_i - f} : \frac{h(u_{i+1}) - f_*}{u_{i+1} - f},$$

for $0 \leq i < k$, where h is the holonomy between the two transversals. For all i, we have that the ratio $Q(u_i, u_{i+1})$ is the product of the following three quotients

$$\frac{R^m(u_i) - R^m(f)}{R^m(u_{i+1}) - R^m(f)} : \frac{u_i - f}{u_{i+1} - f} \tag{5.6}$$

$$\frac{h_m(R^m(u_i)) - h_m(R^m(f))}{h_m(R^m(u_{i+1})) - h_m(R^m(f))} : \frac{R^m(u_i) - R^m(f)}{R^m(u_{i+1}) - R^m(f)} \tag{5.7}$$

$$\frac{h(u_i) - f_*}{h(u_{i+1}) - f_*} : \frac{h_m(R^m(u_i)) - h_m(R^m f)}{h_m(R^m(u_{i+1})) - h_m(R^m f)} \tag{5.8}$$

By Lemma 5.3.4 the first quotient is equal to $1 + O(\epsilon)$. Moreover, $R^m(u_i)$ and $R^m(u_{i+1})$ have comparable distances to $R^m f$ by the Koebe Theorem, since we start with points of comparable distances $|u_i - f| = |u_{i-1} - f|/2$. Then by Lemma 5.3.3, the second quotient is equal to $1 + O(\rho^m)$.

To estimate the last one, we need to apply the inverse map $R^{-m} : \mathcal{W}^u(m) \to \mathcal{W}^u$ to the points $h_m(R^m(u_i))$, $h_m(R^m(u_{i+1}))$ and $h_m(R^m f)$. Since the foliation $\mathcal{F}_V$ is R-invariant, we obtain the points $h(u_i)$, $h(u_{i+1})$ and f_*. By the Koebe Theorem, the ratio distortion of this transition is $O(\epsilon')$ with any $\epsilon' = \sqrt{\epsilon} > \epsilon$ (the renormalization operator has uniformly bounded distortion on the unstable leaves). Combining all this we obtain that

$$Q(u_i, u_{i+1}) = \frac{h(u_i) - f_*}{u_i - f} : \frac{h(u_{i+1}) - f_*}{u_{i+1} - f} = 1 + O(\epsilon + \rho^m).$$

Finally, since

$$Q(u, v) = \prod_{i=0}^{k-1} Q(u_i, u_{i+1})$$

we obtain that the ratio distortion estimate for the holonomy $h : \mathcal{M}_S \to \mathcal{W}^u$ is given by

$$\frac{h(u) - f_*}{u - f} = \frac{h(v) - f_*}{v - f}(1 + O(k(u, v)(\epsilon + \rho^m))) \tag{5.9}$$

when $u, v \in B_\delta(f)$. $\square$

Remark 5.3.6. *Note that we can improve the factor in the ratio distortion estimate of h by computing the distortion of the holonomy map h_m in the image, under the renormalization operator R^m, of the string of points $u_0, u_1, ..., u_k$, then adding the distortion factor of R^m only at the points u and v; in such a way, we get distortion of order $O(\epsilon + \rho^m k(u,v))$.*

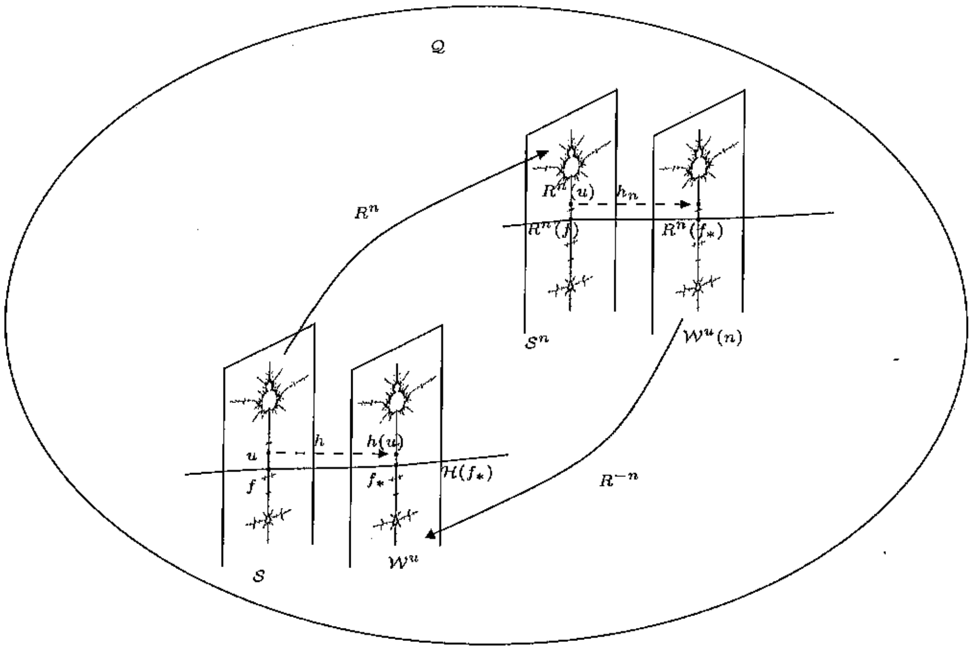

Figure 5.1: The factorization of the holonomy map.

Now, we state the main theorem, which gives a condition on the combinatorics of a quadratic-like germ of essentially bounded combinatorics that implies smoothness of the holonomy.

Theorem 5.3.7. *Let $\{M_k\}_{k=1}^{\infty}$ be the sequence of copies with tripling essentially bounded period, with corresponding periods $n_k = 3k+2$. Consider a real quadratic-like germ $f_* \in \mathcal{A}$ with tuning invariant $\tau(f_*) = \{M_{k_n}\}_{n=0}^{\infty}$, where this sequence is given by an arbitrary choice of elements in $\{M_k\}_{k=1}^{\infty}$. Let $\rho \in (0,1)$ be an upper bound of the contraction factor in the stable leaves. Suppose that*

$$\sum_{n\geq 1} \rho^n \log k_n < \infty.$$

Then the foliation $\mathcal{F}$ is transversally C^1-conformal (smooth) at the point $\chi(f_*)$.

Proof. As we mention before, it is enough to check C^1-conformality of the holonomy from a transversal $\mathcal{S}$ via $f \in \mathcal{H}_{\mathbb{R}}(f_*)$ to the unstable manifold $\mathcal{W}^u \equiv \mathcal{W}^u_{loc}(f_*)$. Since $f \in \mathcal{H}_{\mathbb{R}}(f_*)$, $\chi(f) \in \bigcap_{s=1}^{\infty} J^{(s-1)}_{k_s}$.

Our assumption of the convergence of the series $\sum_{n\geq 1} \rho^n \log k_n$ implies that

$$\limsup_{n\to\infty} (\rho^n \log k_n)^{1/n} \leq 1,$$

then if $r = \limsup_{n\to\infty} (\log k_n)^{1/n}$, it follows that $\rho\, r \leq 1$ or $r \leq 1/\rho$; then the maximum growth of k_n is of order e^{r^n}.

Choose $\epsilon > 0$, then for any N big enough (for instance in order to have that $\mathcal{S}^N$ is at a distance $O(\rho^N)$ from $\mathcal{W}^u(N)$), there exist $\delta \equiv \delta(\epsilon, N) > 0$ so that if we take two points $u, v \in \mathcal{M}_{\mathcal{S}} \cap B_\delta(f)$ on distances of the same order from f, after N iterations of R, they go to points $R^N(u), R^N(v), f_N = R^N(f) \in \mathcal{M}^N_{\mathcal{S}}$ with relative distances of order ϵ. By lemma 5.3.4 we have that

$$\frac{R^N(u) - f_N}{R^N(v) - f_N} = \frac{u - f}{v - f}(1 + O(\epsilon)), \tag{5.10}$$

in fact N is at least $\log(\epsilon/\delta)/\log\lambda$ where $\lambda > 1$ is an upper bound for the unstable eigenvalue λ_*. Later we will choose N in a particular way.

Let us define $L_n = J^{(n-1)}_{k_n}$ for all n. Note that L_n belongs to the sequence $\{J^{(n-1)}_k\}_{k=1}^{\infty}$ of real slices of little copies of the Mandelbrot set of level $n-1$, that correspond to the original sequence $\{J_k\}_{k=1}^{\infty}$ under tuning, with all the elements of the sequence inside $J^{(n-2)}_{k_{n-1}}$. In particular, we have that $R^n(L_n) = J_{k_n}$ for all n.

For a point $u \in \mathcal{M}_{\mathcal{S}} \cap B_\delta(f)$ we have the following:

i) there is $n \in \mathbb{N}$ such that $\chi(u) \in L_n \backslash L_{n+1}$, with $L_{n+1} \subset \chi(B_\delta(f)) \subset L_n$

ii) if δ small enough, we have $n \geq N$

iii) $R^m u$ is defined for all $1 \leq m \leq n+1$ with $\chi(R^m u) \in J_{k_m}$ for all $m \in \{1, .., n\}$.

iv) $\chi(R^{n+1}u) \notin J_{k_{n+1}}$

Thus, u is at least $n+1$ times renormalizable and its tuning invariant up to level $n+1$ coincides with the first $n+1$ terms of the tuning invariant of f.

Now take two points $u, v \in \mathcal{M}_{\mathcal{S}}$ with $|v - f| \leq |u - f| < \delta$, with δ sufficiently small that satisfy the conditions i)-iv) above. Then $u \in L_n \backslash L_{n+1}$ for some $n \geq N$ and $v \in L_{n+t}$ with t maximum. Let $v_0 \equiv u, v_1, ..., v_k \equiv v$ be a string of points in $\mathcal{M}_{\mathcal{S}}$ such that $|v_i - f| = |v_{i-1} - f|/2$ for $i < k$ and $|v_{k-1} - f|/2 \leq |v_k - f| \leq |v_{k-1} - f|$. The number of points in this sequence satisfy $k \asymp \log \frac{|u-f|}{|v-f|}$. From this string of points, we will get a substring $u_0, u_1, ..., u_t$ by an inductive construction, to which we will apply Lemma 5.3.5.

First, we want to have distortion of order $O(\epsilon/m^n)$, $m \geq 2$, for R^n. We need to estimate how many points v_i between u_j and u_{j+1} belong to each level L_i for $n \leq i \leq n+t$. Let $S_n = \{v_i : v_i \in L_n \backslash L_{n+1}\}$ and let $|S_n| = s(n)$.

For all $v_i, v_{i+1} \in L_n \backslash L_{n+1}$ we have that

$$\frac{h(v_i) - f_*}{h(v_{i+1}) - f_*} = \frac{v_i - f}{v_{i+1} - f}(1 + O(\epsilon + \rho^N))$$

since $|v_i - f|$ and $|v_{i+1} - f|$ are comparable $\left(\frac{|v_i - f|}{|v_{i+1} - f|} = 2\right)$.

We start with $v_0 \equiv u_0$. Eventually, our string of points reaches the level L_{n+1}. Since we want to have distortion of order $O(\epsilon/m^n)$ for the map R^n, we need to go deeper in L_{n+1} applying R^N to our string of points until we reach a point $v_j \equiv u_1 \in L_{n+1}$, so that its image under R^{n+1} goes to $R^{n+1}(u_1)$ with relative distance of order ϵ/m^{n+1} from $R^{n+1}f$. At this point of the string we start applying R^{n+1} instead of R^N, so we need $\log(m^{n+1}/\epsilon)$ further steps inside L_{n+1} to reach those points. Suppose we have constructed u_j inside L_{n+j}; to construct u_{j+1}, we apply R^{n+j} to the string of points v_k inside $L_{n+j} \backslash L_{n+j+1}$, which are at most $s(n+j)$, then we continue inside L_{n+j+1} until we reach a point $v_l \equiv u_{j+1}$ for which its image under R^{n+j+1} goes to $R^{n+j+1}(u_{j+1})$ with relative distance of order ϵ/m^{n+j+1} from $R^{n+j+1}f$. Before this change on the iterate of R, we have applied R^{n+j} to at most $s(n+j) + \log(m^{n+j+1}/\epsilon)$ points that will give distortion of order $O(\epsilon/m^{n+j})$. We continue with this procedure until we reach the last level $n+t$. By Lemma 5.3.5, we conclude that the ratio distortion of h is

$$\frac{h(u) - f_*}{h(v) - f_*} = \frac{u - f}{v - f}\left(1 + O\left(\sum_{j=n}^{n+t} \frac{\epsilon}{m^j} + \sum_{j=n}^{n+t}(s(j) + \log\frac{m^{j+1}}{\epsilon})\rho^j\right)\right)$$

Note that $s(m) \sim \log \frac{|d_m - f|}{|d_{m+1} - f|} / \log 2$ and

$$\frac{R^{m+1}(d_m) - R^{m+1}(f)}{R^{m+1}(d_{m+1}) - R^{m+1}(f)} = \frac{d_m - f}{d_{m+1} - f}(1 + O(\kappa))$$

by the Koebe Theorem, here we take $\kappa \in (0, 1/8)$ and restricted R^{m+1} to $A = (R^{m+1})^{-1}(\chi^{-1}((-1.75 - \kappa, \kappa)))$, then $\chi(R^{m+1}(A)) = (-1.75 - \kappa, \kappa)$ for all $m \in \mathbb{N}$, that is, we have a definite Koebe space in the image of R^{m+1}, then we conclude that $\log \frac{|d_m - f|}{|d_{m+1} - f|} \sim 3 \log k_{m+1}$, since $\chi(R^{m+1}(d_m)) = 0$, $\chi(R^{m+1}(d_{m+1})) = c_{m+1} \in M_{k_{m+1}}$ and $\operatorname{diam}(M_{k_{m+1}}) \sim \frac{1}{k^3_{m+1}}$.
We get the following series, which we need to show that is convergent,

$$\sum_{j \geq 0} (\log k_{j+1} + \log \frac{m^{j+1}}{\epsilon})\rho^j. \tag{5.11}$$

The first part of (5.11) is the series $\sum_{j\geq 0} \rho^j \log k_{j+1}$, which is convergent since

$$\sum_{j\geq 0} \rho^j \log k_{j+1} = \frac{1}{\rho} \sum_{j\geq 0} \rho^{j+1} \log k_{j+1}$$

and the last series is convergent.

The second series in (5.11) is $\sum_{j\geq 0} \rho^j \log \frac{m^{j+1}}{\epsilon}$, which is convergent since

$$\begin{aligned}
\limsup_{j\to\infty} \frac{\rho^{j+1} \log \frac{m^{j+2}}{\epsilon}}{\rho^j \log \frac{m^{j+1}}{\epsilon}} &= \limsup_{j\to\infty} \rho \frac{(j+2)\log m - \log\epsilon}{(j+1)\log m - \log\epsilon} \\
&= \limsup_{j\to\infty} \rho \left(\frac{\log m}{(j+1)\log m - \log\epsilon} + 1 \right) \\
&= \rho < 1
\end{aligned}$$

Finally, we can choose N sufficiently big such that the two partial sums

- $\sum_{j=n}^{n+t} \rho^j \log k_{j+1}$,
- $\sum_{j=n}^{n+t} \rho^j \log \frac{m^{j+1}}{\epsilon}$.

are sufficiently small since $n \geq N$. Thus,

$$\frac{h(u) - f_*}{h(v) - f_*} = \frac{u - f}{v - f}(1 + O(s(\epsilon)))$$

with $s(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, we have that the limit for $\frac{h(v)-f_*}{v-f}$ as $v \to f$ cannot be 0 or ∞. Fix δ and $u \in \mathcal{M}_S \cap B_\delta(f)$, let v_n be a sequence of points in $\mathcal{M}_S \cap B_\delta(f)$ satisfying:

i) $v_n \to f$ as $n \to \infty$,

ii) for all $n \in \mathbb{N}$, $|v_n - f| \leq |u - f|$

iii) $\lim_{n\to\infty} \frac{h(v_n) - f_*}{v_n - f} = \tau$

Then passing to the limit as $n \to \infty$, we conclude that

$$\frac{h(u) - f_*}{u - f} = \tau(1 + O(s(\epsilon)).$$

Therefore $h(u) - f_* = \tau(u - f) + o(1)$. $\square$

Corollary 5.3.8 (Self-similarity for essentially bounded combinatorics). *Let c be a real infinitely renormalizable real parameter value with essentially bounded combinatorics satisfying the hypothesis in the theorem, and let $M_1 \in \mathcal{M}$ be the Mandelbrot copy containing c. Then the homeomorphism $\sigma : M_1 \to M$ is C^1-conformal at c.*

Proof. The holonomy $h : M_1 \to \mathcal{W}^u$ locally conjugates σ to the renormalization: $h^{-1} \circ R \circ h = \sigma$. Since this holonomy is C^1-conformal, σ is C^1-conformal at c. $\square$

Chapter 6

A μ-conformal conjugation

6.1 Preliminaries

Let $U \subset \mathbb{C}$ be an open set. We denote by $C(U)$ the vector space of continuous functions from U to $\mathbb{C}$. Let $C_c(U)$ be the functions on U with compact support. The space $L^2(U)$ is defined as the completion of $C_c(U)$ with the norm L^2 defined by

$$||f||_{L^2}^2 = \int\int_{z \in U} |f(z)|^2 dxdy$$

where $z = x + iy$. Finally, let $L^2_{loc}(U)$ be the space of functions on U that locally belong to $L^2(U)$. We will write $L^2 \equiv L^2(\mathbb{C})$.

We need to introduce the class of G. David's μ-conformal maps. Let $W^{1,p}_{loc}(\mathbb{C})$ be the space of maps $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\varphi \in L^2_{loc}$ and $D\varphi \in L^p_{loc}$ in the sense of distributions. We note that a $W^{1,1}_{loc}(\mathbb{C})$ map is ACL (absolutely continuous on lines) ([A], p.28). For these maps, the dilatation ratio is defined as $K_\varphi = \frac{|\varphi_z| + |\varphi_{\bar{z}}|}{\text{Jac}\,\varphi}$.

Definition 6.1.1. *A μ-conformal map $\varphi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is an orientation preserving homeomorphism such that:*

(i) $\varphi \in W^{1,p}_{loc}$ for all $p < 2$, and

(ii) there exist C, $a > 0$ and $K_0 \geq 1$ such that, for all $K \geq K_0$,

$$\text{Area}\{z \in \mathbb{C} \ : \ K_{\varphi(z)} \geq K\} \leq C \exp(-aK).$$

(Remark: Inverse map of a μ-conformal map need not be μ-conformal)

We have the following counterpart for the Measurable Riemann Mapping Theorem in the case of μ-conformal maps.

Theorem 6.1.2 (G. David). *Let μ be a Beltrami coefficient on $\mathbb{C}$. If there exist constants C, $a > 0$ and $K_0 \geq 1$ such that, for $K \geq K_0$,*

$$\text{Area}\{z \in \mathbb{C} \ : \ K_\mu(z) > K\} < C \exp(-aK).$$

then there exists an ACL homeomorphism $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$, unique up to composition by a conformal map, such that, for all $p < 2$, $\varphi \in W^{1,p}_{loc}(\mathbb{C})$ and $\mu_\varphi = \mu$ a.e.

6.2 Maps between germs

We are interested in defining local homeomorphism of the origin that are extensions of partial conjugacies between two maps on invariant sectors. These maps will be defined in neighborhoods of repelling and parabolic points.

6.2.1 Repelling point and invariant sectors

Let f be a holomorphic function defined on a neighborhood D of zero, such that $f(0) = 0$ and $f'(0) = \lambda$ with $|\lambda| > 1$. In a simply connected neighborhood of 0 , there exists a linearizing coordinate, i.e. an isomorphism $h : D \longrightarrow h(D) \subset \mathbb{C}$ such that $h(f(z)) = \lambda h(z)$, for $z \in f^{-1}(D) \cap D$. Let $\mathbb{T}_f = (D \backslash \{0\})/(f)$ be the quotient torus which is isomorphic to the torus $\mathbb{C}/(\mathbb{Z} \log \lambda \oplus 2i\pi\mathbb{Z})$. Denote by $\eta : D\backslash\{0\} \longrightarrow \mathbb{T}_f$ the canonical projection. Any annulus $A \subset \mathbb{T}_f$ such that $S = \eta^{-1}(A)$ is connected, defines an f-invariant sector. Its edges will be the lift of the boundary of A.

Two repelling germs are always quasiconformally conjugate: take à quasiconformal map between one quotient torus onto the other and then lift the map to a neighborhood of the fixed points.

6.2.2 Parabolic point and invariant sectors

Let $g_\nu(z) = z + z^{\nu+1} + O(z^{\nu+2})$, $\nu \geq 1$, be a holomorphic germ defined in a neighborhood of the origin. There exist the Fatou coordinates which conjugate the germ on 2ν overlapping sectors of the origin to the translation $z \mapsto z + 1$ in a half plane. Consider a repelling sector, the quotient space is a cylinder (isomorphic to $\mathbb{C}/\mathbb{Z}$) in which one can consider subannuli for which their lifts are connected. These lifts define invariant sectors.

Now, we are going to define local homeomorphisms of the origin that are extensions of partial conjugacies between f and g_ν on invariant sectors: the conjugacy being defined on sectors, the extension will map circle arcs centered at the origin to circle arcs centered at the origin. The extension is always piecewise C^1.

Let us consider an open sector $S = \{z \in \mathbb{C} \,:\, |\arg z| < \theta \text{ and } 0 < |z| < 1\}$, where $0 < \theta < \pi$ and $f : z \mapsto \lambda z$, $\lambda > 1$ (repelling model). The map

$\psi(z) = \log\lambda/\log z$ conjugates f to $g(z) = \frac{z}{1+z}$ (parabolic model) defined on the cusp $C = w(S)$ with vertex at the origin.

Let Q_n^f be the quadrilateral bounded by the segments in the unit disk $[(1/\lambda^{n+1})e^{\pm i\theta}, (1/\lambda^n)e^{\pm i\theta}]$ and the circle arcs of radii $1/\lambda^{n+1}$ and $1/\lambda^n$ contained in $\mathbb{D}\backslash S$, then $Q_n^f = f^{-n}(Q_0^f)$.

In the case of a parabolic germ g_ν, we define Q_n^g as the quadrilaterals outside a repelling cusp bounded by circle arcs of radii of order $1/n^{1/\nu}$ and $1/(n+1)^{1/\nu}$. Between two parabolic germs, one can define holomorphic conjugacies between two repelling petals and we want to understand the regularity of one of its extensions.

Lemma 6.2.1 ([H1]). *There exists a piecewise C^1 homeomorphism Ψ, an extension of ψ in a neighborhood of the origin, such that $K_\Psi \asymp n$ on Q_n^f. In particular, Ψ is μ-conformal and $\Psi \in W^{1,2}$.*

Proof. Define $\Psi : \mathbb{D}\backslash\overline{S} \longrightarrow \mathbb{D}\backslash\overline{C}$ by

$$\rho e^{it} \mapsto \frac{\log\lambda}{|\log\rho + i\theta|} \exp ia_\rho(t),$$

where $a_\rho(t)$ is an affine map in t, for fixed ρ.

This map Ψ is a homeomorphism from $\mathbb{D}\backslash S$ onto $\mathbb{D}\backslash C$, which maps circles centered at the origin to circles centered at the origin. Let $Q_n^g = \Psi(Q_n^f)$. The dilatation ratio of Ψ can be calculated by estimating ratios of moduli of quadrilaterals as follows: $\operatorname{mod} Q_n^f \asymp 1$ and $\operatorname{mod} Q_n^g \asymp \log(1 + 1/n) \asymp 1/n$; on Q_n^f we have $K_\Psi \asymp \frac{\operatorname{mod} Q_n^f}{\operatorname{mod} Q_n^g} = K_n \asymp n$. Hence for n_0 big enough,

$$\operatorname{Area}\{z \in \mathbb{D} \ : \ K_\Psi(z) \geq n_0\} \asymp \operatorname{Area} \cup_{n\geq n_0} Q_n^f \leq c/\lambda^{2n_0},$$

where $c > 0$ is independent of n_0.

Note that the $\operatorname{Area} Q_n^g \asymp 2\pi\frac{1}{n}(\frac{1}{n} - \frac{1}{n+1}) = \frac{2\pi}{n}(\frac{1}{n}\frac{1}{n+1}) \asymp \frac{1}{n^3}$. Moreover, since Ψ is orientation preserving, it suffices to show that $|\partial_z\Psi|^2 \asymp K_\Psi \operatorname{Jac}\Psi$ belong to L^1:

$$\int |\partial_z\Psi|^2 \asymp \operatorname{Area} C + \sum_{n\geq 1} \int_{Q_n^f} K_\Psi \cdot \operatorname{Jac}\Psi$$

and since

$$\sum_{n\geq 1} \int_{Q_n^f} K_\Psi \cdot \operatorname{Jac}\Psi \asymp \sum_{n\geq 1} n \cdot \operatorname{Area} Q_n^g \asymp \sum_{n\geq 1} 1/n^2 < \infty,$$

we obtain the result. $\square$

We observe that between two parabolic germs, we can always define holomorphic conjugacies ψ between two repeling petals. In this case we have the following.

Lemma 6.2.2. *For ψ a holomorphic map between two repelling petals of two parabolic germs, g_1 and g_ν, the extension Ψ is quasiconformal.*

Proof. In the case of a parabolic germ g_1, let Q_n^1 be the quadrilaterals outside a repelling cusp bounded by circle arcs of radii of order $1/n$ and $1/(n+1)$. For the parabolic germ g_ν, let Q_n^ν be the quadrilaterals outside a repelling cusp bounded by circle arcs of radii of order $1/n^{1/\nu}$ and $1/(n+1)^{1/\nu}$.

The dilatation ratio of Ψ can be calculated by estimating ratios of moduli of quadrilaterals as follows: $\operatorname{mod} Q_n^1 \asymp \log(1+1/n) \asymp 1/n$ and $\operatorname{mod} Q_n^\nu \asymp \frac{1}{\nu}\log(1+1/n) \asymp \frac{1}{\nu}\frac{1}{n}$; on Q_n^1 we have $K_\Psi \asymp \frac{\operatorname{mod} Q_n^1}{\operatorname{mod} Q_n^\nu} \asymp 1$. Therefore Ψ is a quasiconformal map. □

6.2.3 Coverings of the unit circle

Let us consider a holomorphic map (h, X) defined in a neighborhood of $\mathbb{S}^1$ such that $h(\mathbb{S}^1) = \mathbb{S}^1$ and where $X \subset \mathbb{S}^1$ is an invariant set. Let us assume that X contains, at least, all the critical orbits of h and all the parabolic points on $\mathbb{S}^1$. Also, we will assume that:

i) h is topological expanding, i.e. for every subinterval $I \subset \mathbb{S}^1$, there exists $n \geq 1$ such that $h^n(I) \supset \mathbb{S}^1$;

ii) h is combinatorially finite, i.e. X_h is finite.

Thus, h has a finite number of parabolic points and of strictly preperiodic critical orbits.

Dynamical Partition Given a topologically expanding real analytic map (h, X) of $\mathbb{S}^1$, we define a partition of a neighborhood of $\mathbb{S}^1$ which isolates the non expanding dynamics of its extension $H : W' \longrightarrow W$, which is a holomorphic map, where the domains are annuli and symmetric with respect the unit circle. Assume that H is an extension on an annulus $(1/R < |z| < R)$, $R > 1$.

We first define sectors with vertices coming from X. Then we will fill up the partition with quadrilaterals. Let $x_1, \ldots, x_k$ be a periodic orbit on X.

1) If the orbit is repelling, then looking in the quotient torus, we can define two symmetric annuli which separate both components of the projection of $\mathbb{S}^1$. If their lifts are S_i (sectors with vertex x_i) then they are H^k-invariant.

2) If the orbit is parabolic, $\mathbb{S}^1$ belongs to two repelling petals (topological expanding property). Define two symmetric sectors with respect to $\mathbb{S}^1$ and

disjoint from it such that their edges are invariant curves living in the repelling petals which contain $\mathbb{S}^1$. These sectors are also H^k-invariant.

Then one can pull back the sectors already defined. We need to be careful here if a point is critical, but in our case there won't be critical points in $\mathbb{S}^1$. If W is small enough, then all sectors and their backward orbits along $\mathbb{S}^1$ are pairwise disjoint. In particular, it follows that the map H is expansive [DH1].

Let (S_i) denote the sectors just defined with vertex in X. Write $V = W\backslash(\cup S_i)$ and $V' = W'\backslash(\cup S_i)$. Since H is expansive, we get $V' \subset V$ and the common boundary comes from the invariant edges. Let $\gamma_0 = \partial W$. Define $\gamma_1 = H^{-1}(\gamma_0) \cap V$. The curves γ_0 and γ_1 together with the edges of the sectors bound in V a finite union of quadrilaterals Q_0. By induction define $Q_n = H^{-1}(Q_{n-1})$. Then $V\backslash(\cup Q_n) = \cup H^{-n}(S_i)$ are the backward orbits of S_i.

Let $f(z) = P_{-3/4}(z) = z^2 - \frac{3}{4}$. We have that the connected components of the interior of $K(f)$ are quasidisks ([CJY]). Let U be the component of $K(f)$ containing the critial value $c_1 = -\frac{3}{4}$, then U is invariant under $f^{\circ 2}$. Let $R : U \longrightarrow \mathbb{D}$ be the Riemann map from U to $\mathbb{D}$ such that $R(c_1) = 0$ and $R(-\frac{1}{2}) = 1$ This map conjugates $f^{\circ 2}$ to the Blaschke product $B(z) = \frac{z^2+\frac{1}{3}}{1+\frac{1}{3}z^2}$, where $B(1) = 1$ is a parabolic fixed point of multiplicity 3 with two attracting petals. In fact, B is conjugate to the map $h(z) = \frac{z}{z^2+1}$ with a parabolic fixed point at 0 and $h(z) = z - z^3 + z^5 - \cdots = z(1 - z^2 + O(z^4))$.

Since U is a quasidisk we can extend R to a K-quasiconformal map of $\overline{\mathbb{C}}$, such that $R|U$ conjugates conformally $f^{\circ 2}$ to B. In particular $\overline{\partial} R = 0$ on U. Let $g = B|S^1$, then g is an expanding covering of degree 2 with only one parabolic fixed point. We want to isolate the non-expanding dynamics of its extension $G : W' \longrightarrow W$. In particular, (B, X) with $X = \{1\}$ is an expanding holomorphic map on $\mathbb{S}^1$. Consider the corresponding partition of a neighborhood of $\mathbb{S}^1 = J(B)$, starting with only one invariant sector (the one with vertex at 1). Let us consider the map $P(z) = P_{1/4}(z) = z^2 + \frac{1}{4}$.

Theorem 6.2.3. *There is a neighborhood U of $K(P)$ and a continuos function $\psi : U \longrightarrow \mathbb{C}$ such that for all $z \in K(P)$, $\psi \circ P = f^{\circ 2} \circ \psi$. Moreover, $\psi \in W^{1,p}_{loc}$, for all $p < 2$ and $\overline{\partial}\psi = 0$ a.e. on $K(P)$.*

Proof. We want to construct a partition of a neighborhood of the Julia set of P, "the cauliflower". Let us take U' and U, two domains bounded by two equipotentials such that $P^{-1}(U) = U'$. Let C_0 be an invariant sector with vertex at $1/2$ in the exterior of $K(P)$ that can be decomposed in fundamental domains D_n. Let $T = U\backslash C_0$, $T' = U'\backslash C_0$, $\Gamma_0 = \partial U$ and $\Gamma_1 = P^{-1}(\Gamma_0) \cap T$.

The curves Γ_0 and Γ_1, together with the edges of the sector C_0, bound in T a quadrilateral Q'_0. By induction define $Q'_n = P^{-1}(Q'_{n-1})$, then $T\backslash(\cup Q'_n) = \cup P^{-n}(C_0)$ are the backward orbits of C_0, and let C_n stand for the inverse image of C_0 with vertex at $P^{-n}(1/2)$. We obtain a partition of quadrilaterals Q'_n and preimages of C_0 which is combinatorially equivalent to the partition for (B, X).

Define two diffeomorphisms $\psi_0 : \gamma_0 \longrightarrow \Gamma_0$ and $\psi_1 : \gamma_1 \longrightarrow \Gamma_1$, such that $\psi_0 \circ B = P \circ \psi_1$. Extend both maps quasiconformally to Q_0 and denote it by ψ. Using inverse branches of B and P and the combinatorics, extend this map to be quasiconformal homeomorphism $\psi : V \backslash (B^{-n}(S_0)) \longrightarrow T \backslash (P^{-n}(C_0))$ which conjugate B to P.

On the sector S_0, extend ψ using Lemma 6.2.2 in the following way. For the parabolic germ defined on the sector S_0, one can find a germ g_ν and a conformal map which maps S_0 outside a "repelling cusp" C. Since the sector C_0 is also parabolic, we have a μ-conformal extension between the two sectors.

Then, one can define ψ on their preimages by a pull back argument. This extension is a priori not quasiconformal nor compatible with the dynamics. But we end up with a homeomorphism from V to T which conjugates B to P.

By the construction of ψ we need to compute the norm L^p of $\partial_z \phi$ in the exterior of the cauliflower where $\phi = \psi^{-1}$. Note that the restriction of ϕ to the domain D_n has dilatation $K_\phi \asymp 1$.

$$||\partial_z \phi||^p_{L^p} \asymp \int (K_\phi \operatorname{Jac}_\phi)^{p/2} = \sum_n \int_{C_n} (K_\phi \operatorname{Jac}_\phi)^{p/2} + C$$

and

$$\int_{C_n} (K_\phi \operatorname{Jac}_\phi)^{p/2} = \int_{C_0} (K_\phi(P^{-n}) \operatorname{Jac}_\phi(P^{-n}))^{p/2} |(P^{-n})'|^2$$

which is equal to

$$\int_{C_0} (K_\phi \operatorname{Jac}_\phi)^{p/2} |(B^{-n})'(1)|^p |(P^{-n})'|^{2-p}$$

since $\operatorname{Jac}_\phi(P^{-n}) |(P^{-n})'|^2 = \operatorname{Jac}_\phi |(B^{-n})'|^2$.

Let $\alpha \in (1,2)$ be the Hausdorff dimension of $J(P)$, and let $q \in (1, \frac{\alpha}{\alpha+p-2})$. By the Koebe theorem

$$||\partial_z \phi||^p_{L^p} \leq C \int_{C_0} (K_\phi \operatorname{Jac}_\phi)^{p/2} \sum |(B^{-n})'(1)|^p \cdot |(P^{-n})'(1/2)|^{2-p},$$

and by the Hölder inequality the sum is bounded by

$$(\sum |(B^{-n})'(1)|^{pq})^{1/q} (\sum |(P^{-n})'(1/2)|^{\frac{2-p}{1-1/q}})^{1-1/q}.$$

By [McM3], since $pq > 1$ and $\frac{2-p}{1-1/q} > \alpha$, the sums are finite. Moreover,

$$\int_{C_0} (K_\phi \operatorname{Jac}_\phi)^{p/2} \leq C \sum_n n^{p/2} (\operatorname{Area} D_n)^{1-p/2} (\int_{D_n} \operatorname{Jac} \phi)^{p/2}.$$

We conclude that

$$\int_{C_0} (K_\phi \operatorname{Jac}_\phi)^{p/2} \leq C \sum_n (1/n)^{4-(1/2)p} < \infty.$$

Hence $\phi \in W^{1,p}$. Note that $\partial_z \psi \in W^{1,2}$ since $||\partial_z \phi||_{L^2}^2 \asymp \sum n \int_{\phi(D_n)} \operatorname{Jac} \phi^{-1} \asymp \sum 1/n^3 < \infty.$ □

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