Semiconjugacies in Complex Dynamics with Parabolics

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Abstract and Acknowledgments

In this thesis we investigate degeneration of rational maps and generation of parabolic cycles. There are two chapters as follows:

Chapter 1: Semiconjugacies between the Julia sets of geometrically finite rational maps.[1]

A rational map f is called *geometrically finite* if every critical point contained in its Julia set is eventually periodic. If a perturbation of f into another geometrically finite rational map is horocyclic and preserves the critical orbit relations with respect to the Julia set of f, then we can construct a semiconjugacy or a topological conjugacy between their dynamics on the Julia sets.

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Chapter 2: Regular leaf spaces of parabolic quadratic polynomials. The method of *tessellation* is developed. For a quadratic polynomial with a parabolic cycle, we construct pinching semiconjugacies from certain hyperbolic quadratic polynomials. These semiconjugacies describe degeneration and bifurcations of their associating regular leaf spaces.

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Chapter 1

Semiconjugacies between the Julia sets of geometrically finite rational maps

1.1 Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. We call such a map *geometrically finite* if all critical points contained in the Julia set $J(f)$ are eventually periodic. A geometrically finite rational map can have (super)attracting and parabolic basins, but no Siegel disks or Herman rings. In particular, if a rational map is (sub)hyperbolic or parabolic, then it is geometrically finite.

In this chapter, we discuss perturbations of a geometrically finite rational map f within Rat_d, the space of all rational maps of degree d. The topology of this space is defined by uniform convergence on the sphere with respect to the spherical distance $d_\sigma(\cdot,\cdot)$. Our aim is to study the dynamical stability of f on its Julia set; that is, structural stability of f restricted on the Julia set.

Perturbations of f. Let us consider a family of rational maps of degree $d \geq 2$, $\{f_\epsilon \in \mathrm{Rat}_d : \ \epsilon \in [0,1]\}$ with the following conditions:

- $f_0 = f$; and
- $\sup_{x \in \hat{\mathbb{C}}} d_\sigma(f_\epsilon(x), f(x)) \to 0$ as $\epsilon \searrow 0$.

We represent this family in the convergence form, $f_\epsilon \to f$, and call it a *perturbation* of f.

For this perturbation $f_\epsilon \to f$, let us consider whether the dynamics on $J(f)$ is perturbed continuously to that on $J(f_\epsilon)$. More precisely, we consider the existence of a map $h_\epsilon : J(f_\epsilon) \to J(f)$ for each $\epsilon \in [0,1]$ such that

- h_ϵ is a homeomorphism with $h_\epsilon \circ f_\epsilon = f \circ h_\epsilon$ on $J(f_\epsilon)$; and

- $h_\epsilon^{-1} : J(f) \to J(f_\epsilon)$ tends to $\mathrm{id} : J(f) \to J(f)$ as $\epsilon \to 0$.

Such an h_ϵ with the first condition is called a *(topological) conjugacy* between f_ϵ and f on their respective Julia sets. In addition, for the first condition, if h_ϵ is not a homeomorphism but merely continuous and surjective, then such an h_ϵ is called a *semiconjugacy* between f_ϵ and f on their respective Julia sets.

By the Mañé-Sad-Sullivan theory[15], if f has a connected neighborhood $U \subset \mathrm{Rat}_d$ where each $f_\epsilon \in U$ has the same number of attracting cycles as f, then for each $f_\epsilon \in U$ there exists a unique quasiconformal conjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ as above. This means any small perturbations of f have desired conjugacies. For example, hyperbolic rational maps have this property.

On the other hand, when f is geometrically finite f can have parabolic cycles: As we will describe, those parabolic cycles may change into attracting cycles under some perturbations. Thus the number of attracting cycles may change and we cannot apply the Mañé-Sad-Sullivan theory. Moreover, by a perturbation of parabolic cycles into attracting cycles, the topology of $J(f)$ may change and we cannot even hope that $J(f)$ and $J(f_\epsilon)$ are homeomorphic in general.

However, in our main theorem (Theorem 1.1.1), we will give a sufficient condition for perturbations $f_\epsilon \to f$ to be accompanied by such conjugacies as above or best possible semiconjugacies between the dynamics on their Julia sets.

Parabolic points. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, and let a be a periodic point of f with period l and multiplier $(f^l)'(a) =: \lambda$. We say a is a *parabolic (periodic) point* if λ is a root of unity.

Now let us suppose that a is a parabolic point and λ is a primitive q-th root of unity. Taking a local coordinate near a which maps a to 0, we obtain

$$f^{lq}(z) = z + A_{p+1} z^{p+1} + O(z^{p+2}) \tag{1.0}$$

with $A_{p+1} \neq 0$ and $p \geq 1$. (Moreover, we can normalize A_{p+1} to be 1 by using a linear transformation.) It is known that p is a multiple of q which does not depend on the choice of local coordinates. We call $p = p(a)$ the *petal number* of a. We also say that *a has p petals.*

Note that a is a fixed point of f^{lq} of multiplicity $p+1$. By a perturbation of f into f_ϵ, a splits into $p+1$ fixed points of f_ϵ^{lq} counting with multiplicity. This may cause drastic change of the dynamics, so we have to control the perturbation in order to change the original dynamics tamely.

Horocyclic perturbations. After C. McMullen, we say a perturbation $f_\epsilon \to f$ is *horocyclic* if each parabolic point a of f as above satisfies the following:

(a) There are fixed points a_ϵ of f_ϵ^l with multipliers $(f_\epsilon^l)'(a_\epsilon) = \lambda_\epsilon$ satisfying $a_\epsilon \to a$ and $\lambda_\epsilon \to \lambda$;

(b) There is a neighborhood D of a with local coordinates $\phi_\epsilon,\ \phi : D \to \mathbb{C}$ such that:

1. $a_\epsilon \in D$ and $\phi_\epsilon(a_\epsilon) = \phi(a) = 0$;
2. $\phi_\epsilon \to \phi$ uniformly on D; and
3. If we represent the actions of f_ϵ^{lq} and f^{lq} on D by ϕ_ϵ and ϕ respectively, we obtain the local representation of the perturbation as:

$$f_\epsilon^{lq}(z) = \lambda_\epsilon^q z + z^{p+1} + O(z^{p+2}) \to f^{lq}(z) = z + z^{p+1} + O(z^{p+2}). \quad (1.1)$$

(c) If we set $\exp(L_\epsilon + i\theta_\epsilon) := \lambda_\epsilon^q$, which tends to 1 as $\epsilon \to 0$, then $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon,\ \theta_\epsilon \to 0$.

Form (1.1) implies that the symmetry of the local dynamics near a is preserved by the perturbation. In particular, $\phi,\ \phi_\epsilon$ are not necessarily conformal, can be just homeomorphisms from D to their images. By condition (c), a avoids being perturbed into an irrationally indifferent periodic point. See §2 for more details.

Horocyclic perturbation was originally defined as *horocyclic convergence* of rational maps, to study the continuity of the Hausdorff dimensions of the Julia sets of geometrically finite rational maps[12, §7-9].

J-critical relations. A geometrically finite rational map may have critical points in its Julia set. Here we introduce a condition which controls the perturbations of the orbits of such critical points.

Let $c_1, \dots, c_N$ be all critical points of f contained in $J(f)$, where N is counted *without* multiplicity. A *J-critical relation of f* is a set of non-negative integers (i, j, m, n) such that $f^m(c_i) = f^n(c_j)$.

Let $\deg(f, x)$ denote the local degree of f at x. We say a perturbation $f_\epsilon \to f$ *preserves the J-critical relations of f* if:

- For all $i = 1, \dots, N$, the maps f_ϵ have critical points $c_i(\epsilon)$ (may be in the Fatou set) satisfying $c_i(\epsilon) \to c_i$ and $\deg(f_\epsilon, c_i(\epsilon)) = \deg(f, c_i)$ as $\epsilon \to 0$; and
- For each J-critical relation (i, j, m, n) of f, f_ϵ satisfies $f_\epsilon^m(c_i(\epsilon)) = f_\epsilon^n(c_j(\epsilon))$.

If f is geometrically finite, then the maps f_ϵ are also geometrically finite. If f is hyperbolic or parabolic, then $C(f) \cap J(f) = \emptyset$ and any small perturbation of f automatically preserves its J-critical relations.

Our main result is:

Theorem 1.1.1 *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a geometrically finite rational map of degree d, and $f_\epsilon \to f$ a horocyclic perturbation which preserves the J-critical relations of f.*

For each ϵ which is sufficiently small, there exists a unique semiconjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ with the following properties:

1. *If* $\mathrm{card}(h_\epsilon^{-1}(y)) \geq 2$ *for some* $y \in J(f)$*, then there exists an* n *such that* $f^n(y)$ *is a parabolic point of* f *and* $\mathrm{card}(h_\epsilon^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$.

2. h_ϵ *can be arbitrarily close to the identity on* $J(f_\epsilon)$*. That is, if we fix an arbitrarily small* $r > 0$*, then for all sufficiently small* ϵ*,* h_ϵ *satisfies*

$$\sup\{d_\sigma(h_\epsilon(x), x) : x \in J(f_\epsilon)\} < r.$$

Property 1 implies that the injectivity of h_ϵ may break on the backward orbits of parabolic points of f. Since such points are countable, we say that h_ϵ is *almost bijective*. However, even though f has parabolic points, h_ϵ can give a topological conjugacy. The precise condition for this is described in Corollary 1.7.3. In addition, Property 2 implies:

Corollary 1.1.2 *For* $f_\epsilon \to f$ *as above,* $J(f_\epsilon)$ *converges to* $J(f)$ *in the Hausdorff topology.*

For a given geometrically finite rational map, the existence of such perturbations is guaranteed by [10].

Example 1. Let us consider perturbations of a geometrically finite map $f(z) = z(1+z)^m$ with $m \geq 2$. Now -1 is a preparabolic critical point and 0 is a parabolic fixed point with one petal. Here are two typical perturbations:

- $f_\epsilon(z) = \lambda_\epsilon z(1+z)^m$ with real $\lambda_\epsilon \searrow 1$
- $f_\epsilon(z) = \lambda_\epsilon z(1+z)^m$ with real $\lambda_\epsilon \nearrow 1$

For both cases, 0 is split into a pair of attracting and repelling fixed points, 0 and $-1 + 1/\sqrt[m]{\lambda_\epsilon}$. For the first case, 0 is the repelling one, and for the second case, the attracting one. In Figure 1.1, curves roughly show the shape of the Julia sets for $m = 3$. These split fixed points and their first preimages are shown by heavy dots. Figure 1.2 shows the equipotential curves in the Fatou sets.

Both two perturbations are horocyclic and preserving the J-critical relations of f. For the first case, we obtain h_ϵ as a topological conjugacy. For the second case, h_ϵ is a semiconjugacy which pinches the backward images of $-1 + 1/\sqrt[m]{\lambda_\epsilon}$ onto those of 0. The injectivity is broken only at these points.

Remark on the Goldberg-Milnor conjecture. Theorem 1.1.1 gives a partial and affirmative answer to the following Goldberg-Milnor conjecture[6]: *For a polynomial* f *which has a parabolic cycle, there exists a small perturbation of* f *such that*

- *the immediate basin of the parabolic cycle is converted to basins of some attracting cycles; and*

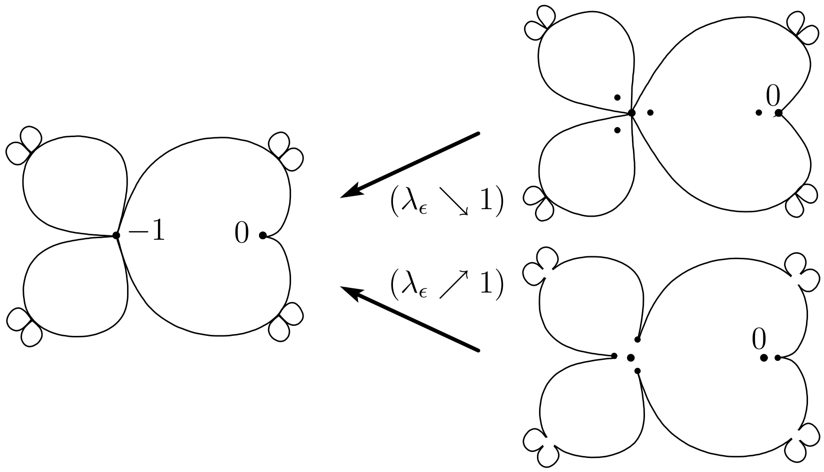


Figure 1.1: The perturbations $f_\epsilon(z) = \lambda_\epsilon z(1+z)^3$ with real $\lambda_\epsilon \to 1$

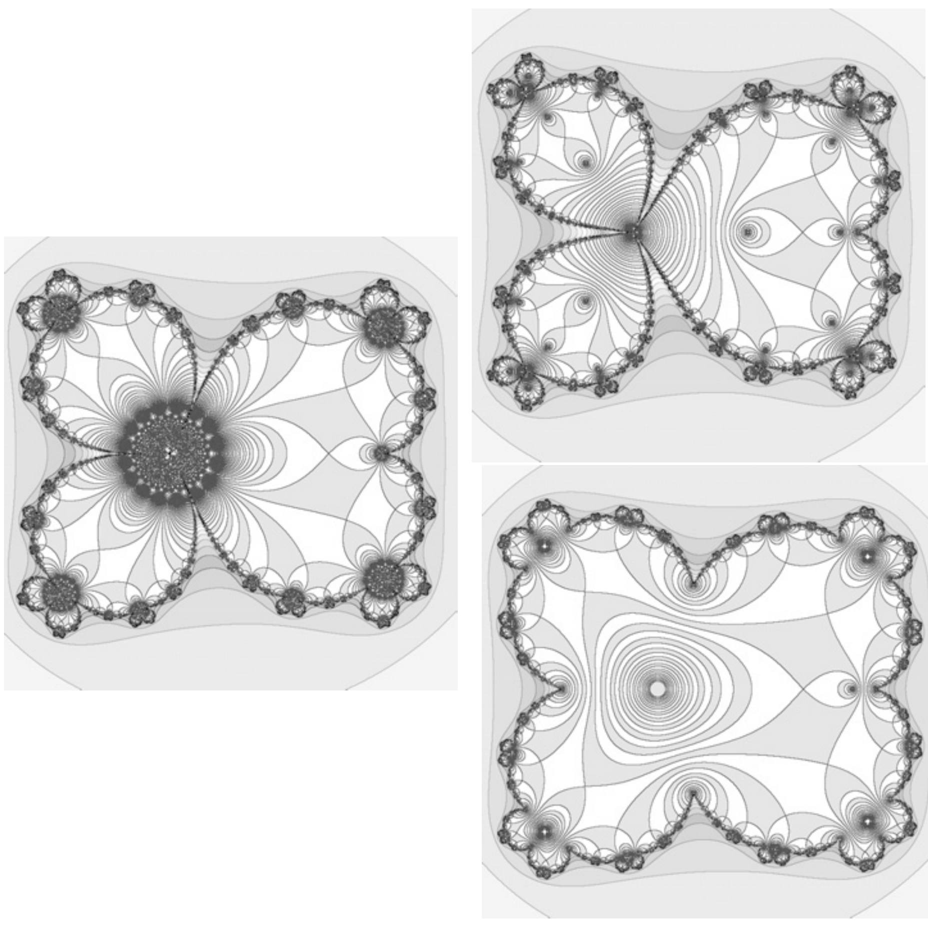


Figure 1.2: Equipotential curves for the Fatou sets of f_ϵ and f.

- *the perturbed polynomial on its Julia set is topologically conjugate to the original polynomial f on $J(f)$.*

Some horocyclic perturbations of a geometrically finite polynomial explicitly give such perturbations. For example, the first perturbation in Example 1 gives an affirmative answer to this conjecture for $f(z) = z(1+z)^m$.

In general, any geometrically finite rational map has such a perturbation. See [10]. For other partial solutions of this conjecture, see [3] and [7].

Example 2. Let us consider a Blaschke product $f(z) = (3z^2+1)/(3+z^2)$ with a parabolic fixed point at $z = 1$, which has 2 petals. The critical points of f are 0 and ∞. The Julia set is the unit circle and the Fatou set is the parabolic basin of $z = 1$.

Let us consider perturbations of f of the form

$$f_\epsilon(z) = \frac{(2+\lambda_\epsilon)z^2 + 2 - \lambda_\epsilon}{2 + \lambda_\epsilon + (2-\lambda_\epsilon)z^2} \quad \text{with real} \quad \lambda_\epsilon \to 1.$$

For $\epsilon \ll 1$, f_ϵ are also Blaschke products and the Julia sets are contained in the unit circle. By this perturbation, the parabolic point $z = 1$ of f splits into the following three fixed points (counting with multiplicity): $z_0 = 1$ with multiplier λ_ϵ, $z_1 = (-\lambda_\epsilon + 2\sqrt{-1+\lambda_\epsilon})/(-2+\lambda_\epsilon)$ and $z_2 = (-\lambda_\epsilon - 2\sqrt{-1+\lambda_\epsilon})/(-2+\lambda_\epsilon)$ with the same multipliers $-1 + 2/\lambda_\epsilon$.

Now consider the case of real λ_ϵ with (a)$\lambda_\epsilon \searrow 1$ or (b)$\lambda_\epsilon \nearrow 1$ (See Figure 1.3). For each cases, one can check that $f_\epsilon \to f$ is a horocyclic perturbation.

When (a), $z_0 = 1$ is repelling and z_1, z_2 are attracting. The Julia set of f_ϵ is also the unit circle. By Theorem 1.1.1, there is a conjugacy between f_ϵ and f on the unit circle.

When (b), $z_0 = 1$ is attracting and z_1, z_2 are repelling. The Julia set of f_ϵ is a Cantor set contained in the unit circle. By Theorem 1.1.1, there is a semiconjugacy between f_ϵ and f on their respective Julia sets. Note that the semiconjugacy maps a Cantor set *onto* the unit circle.

Sketch of the proof of the main theorem. Let us roughly sketch the proof of Theorem 1.1.1; the construction of the semiconjugacy between f_ϵ and f on their respective Julia sets.

Let f be a geometrically finite rational map and let $f_\epsilon \to f$ be a horocyclic perturbation which preserves the J-critical relations of f. We investigate the properties of such a perturbation in §2.

In §3, we prepare the ingredients for the semiconjugacy. For f, we construct a compact set Ω such that $J(f) \subset \Omega \subset f(\Omega)$. Correspondingly, for each fixed f_ϵ, we construct a compact set Ω_ϵ such that $J(f_\epsilon) \subset \Omega_\epsilon \subset f_\epsilon(\Omega_\epsilon)$. We also construct a certain surjective map $h_0 (= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$ as the "0-th" step to the semiconjugacy.

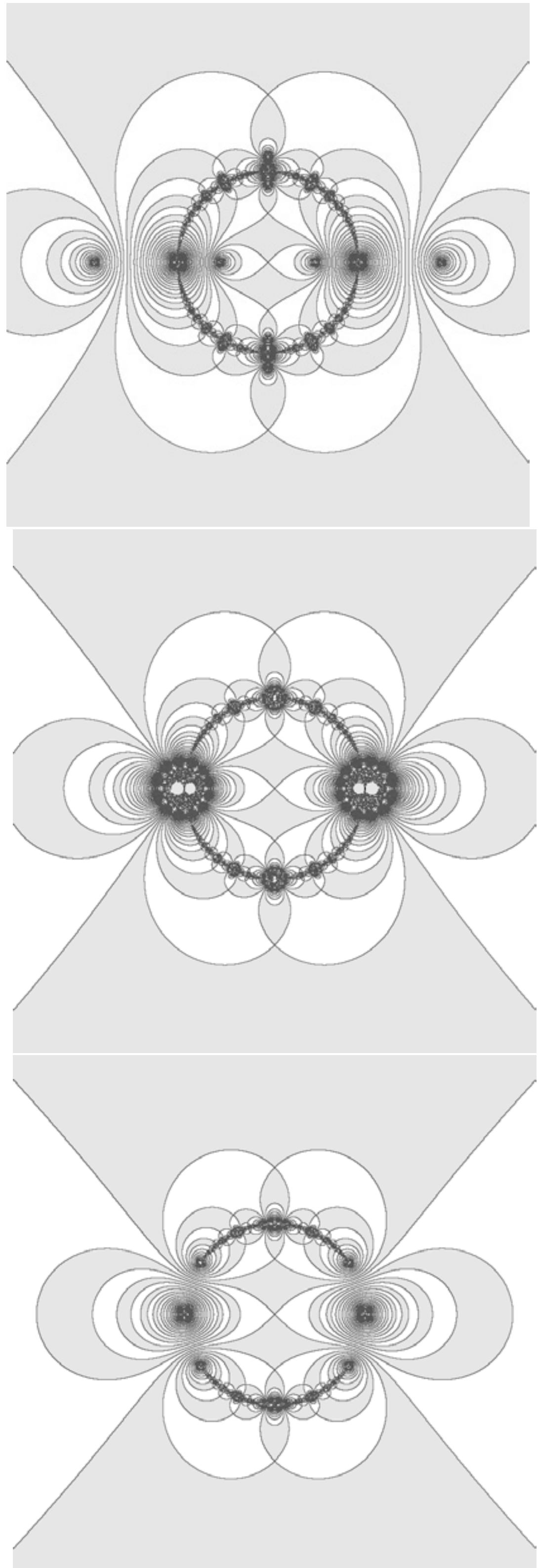


Figure 1.3: The equipotential curves for the Fatou sets of f_ϵ of type (a), f, and f_ϵ of type (b).

Then in §4, we inductively construct a sequence of "lifts"

$$\left\{h_n(=h_{n,\epsilon}) : f_\epsilon^{-n}(\Omega_\epsilon) \to f^{-n}(\Omega)\right\}_{n=1}^{\infty}$$

satisfying $f \circ h_{n+1} = h_n \circ f_\epsilon$. In §5, we investigate the expanding property of f; in other words, the contracting property of f^{-1}. By using this property, in §6, we show that $\{h_n\}$ converges uniformly to a surjective map h_ϵ on $J(f_\epsilon)$ if $\epsilon \ll 1$.

$$\begin{array}{ccc} \vdots & & \vdots \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ f_\epsilon^{-2}(\Omega_\epsilon) & \xrightarrow{h_2} & f^{-2}(\Omega) \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ f_\epsilon^{-1}(\Omega_\epsilon) & \xrightarrow{h_1} & f^{-1}(\Omega) \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega_\epsilon & \xrightarrow{h_0} & \Omega \end{array} \qquad \begin{array}{ccc} J(f_\epsilon) & \xrightarrow{h_\epsilon} & J(f) \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ J(f_\epsilon) & \xrightarrow{h_\epsilon} & J(f) \end{array}$$

In §7, we check that h_ϵ satisfies the properties in Theorem 1.1.1. To simplify the argument, from §3 to §7, we suppose that $J(f) \neq \hat{\mathbb{C}}$. The case of $J(f) = \hat{\mathbb{C}}$ is treated in §8.

Notes.

1. For the basic properties of the Julia sets and parabolic points, refer to [1], [2] and [5], etc.

2. If f is hyperbolic, we obtain h_ϵ as a topological conjugacy. In particular, by uniqueness, h_ϵ coincides with the quasiconformal conjugacy obtained by using λ-Lemma in [15]. In general, for a perturbation $f_\epsilon \to f$ as Theorem 1.1.1, if each f_ϵ for $\epsilon \in (0,1]$ is hyperbolic, then each h_ϵ is characterized as a uniform limit of quasiconformal conjugacies.

3. If a rational map f has no Siegel disks or Herman rings and $f_\epsilon \to f$ horocyclically, it is known that $J(f_\epsilon) \to J(f)$ in the Hausdorff topology[8],[12, Theorem 9.1]. Corollary 1.1.2 gives another proof of this fact in a special case by using the existence of the semiconjugacy.

4. Theorem 1.1.1 is an improvement of an author's result on horocyclic perturbation of parabolic rational maps in [9] or [8].

Notation. Here we list some notation used throughout this chapter.

- $\sigma := 2|dz|/(1+|z|^2)$ is the spherical metric on the Riemann sphere $\hat{\mathbb{C}}$.
- $d_\sigma(\cdot,\cdot)$: the spherical distance measured in σ.
- $B_\sigma(x,r) := \{y \in \hat{\mathbb{C}} : d_\sigma(x,y) < r\}$
- $F(f)$: the Fatou set of f
- $C(f)$: the set of all critical points of f.
- $P(f) := \overline{\{f^n(c) : c \in C(f), n = 1, 2, \ldots\}}$; the postcritical set of f.
- For any map f, f^0 denotes the identity map on the domain of f.
- $n \gg 0$ means that $n > 0$ is sufficiently large.
- $\epsilon \ll 1$ means that $\epsilon > 0$ is sufficiently small.

1.2 Horocyclic perturbations

Bifurcations of parabolic periodic points have a strong effect on the local dynamics as well as the global dynamics. In this section, we describe a horocyclic perturbation $f_\epsilon \to f$ of a geometrically finite rational map f in further detail. In particular, we introduce the notion of planet and satellite for periodic points generated by perturbation of parabolic points. Roughly speaking, a planet is the central periodic point which determines the properties of the perturbed local dynamics. Satellites accompany a planet. Moreover, we will show a key lemma on horocyclic perturbation (Lemma 1.2.2), and see the local dynamics near parabolic points change tamely under such perturbations.

1.2.1 Planets and satellites.

First we consider condition (b)-3 of horocyclic perturbation. Let a be a parabolic point of f as in the preceding section, which has a local representation as (1.0).

As we will see afterward, condition (b)-3 is important to keep the original symmetry of the local dynamics for the petals of a. However, if we suppose only conditions (a), (b)-1 and (b)-2 for $f_\epsilon \to f$, we just obtain a local representation of the convergence near a as the following:

$$\begin{aligned} f_\epsilon^{lq}(z) &= \lambda_\epsilon^q z + A_{\epsilon,r} z^r + \cdots + A_{\epsilon,p+1} z^{p+1} + O(z^{p+2}) \\ \to f^{lq}(z) &= z + A_{p+1} z^{p+1} + O(z^{p+2}) \quad (\epsilon \to 0), \end{aligned} \tag{2.1}$$

where $2 \leq r \leq p$. In [12, §7], C. McMullen gave some conditions which insure form (2.1) becomes form (1.1) by taking suitable local coordinates. One of such conditions is:

Proposition 1.2.1 *If $q = p$, then through a continuous change of coordinates near a, we obtain the normalized form of the convergence as (1.1).*

Proof. For the local representation as (2.1), consider a coordinate change by

$$\zeta = \phi_{\epsilon,r}(z) = z - B_{\epsilon,r} z^r \ , \quad B_{\epsilon,r} = \frac{A_{\epsilon,r}}{\lambda_\epsilon(\lambda_\epsilon^{r-1} - 1)}.$$

Since λ is a primitive p-th root of unity and $\lambda_\epsilon \to \lambda$, we obtain $\lambda_\epsilon^{r-1} \neq 1$ for all $\epsilon \ll 1$. Thus $B_{\epsilon,r} \to 0$ as $A_{\epsilon,r} \to 0$ and $\phi_{\epsilon,r} \to \mathrm{id}$ uniformly near the origin. For each ϵ, changing the coordinate by $\phi_{\epsilon,r}$, we obtain

$$\phi_{\epsilon,r} \circ f_\epsilon^{lp} \circ \phi_{\epsilon,r}^{-1}(\zeta) = \lambda_\epsilon^p \zeta + O(\zeta^{r+1}).$$

So we can continue the discussion by replacing r with $r+1$ until $r+1$ becomes $p+1$. Finally, take a linear coordinate change so that $A_{p+1} = A_{\epsilon,p+1} = 1$. ■

The key point of the proof above is that $B_{\epsilon,r}$ does not diverge as $\epsilon \to 0$. Here we used the condition that λ is a primitive p-th root of unity, however, we can replace this by the condition that $B_{\epsilon,r}$ *converges as* $\epsilon \to 0$ *for each step of* $r = 2, \ldots, p$. In the original definition of horocyclicity by C. McMullen, he formulated and studied this condition as *dominant convergence* of analytic germs[12, §7-9]. For example, by using [12, Proposition 7.1], we can improve Proposition 1.2.1 as follows: *For the form (2.1) above, if $A_{\epsilon,i}/(\lambda_\epsilon^q - 1)$ converges as $\epsilon \to 0$ for each $r \leq i \leq p$, then through a continuous change of coordinates near a, we obtain the normalized form of the convergence as (1.1).*

Planets and satellites. Next, we consider the effect of condition (c) of horocyclic perturbation. Let $f_\epsilon \to f$ be a horocyclic perturbation. Now $\lambda_\epsilon^q = \exp(L_\epsilon + i\theta_\epsilon)$, with the assumption that $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon, \ \theta_\epsilon \to 0$. By this relation, $L_\epsilon = 0$ implies $\theta_\epsilon = 0$. In other words, if $|\lambda_\epsilon^q| = 1$ then a_ϵ is persistently a parabolic point of f_ϵ with the same multiplier λ as a. This means, perturbations of a into another kind of indifferent periodic point are prohibited.

Let us look the relation $\theta_\epsilon^2 = o(|L_\epsilon|)$ in the complex plane. If we fix a pair of arbitrarily small closed disks on the both sides of the imaginary axis, so that they are tangent to the axis at the origin, then they contain $L_\epsilon + i\theta_\epsilon$ for all $\epsilon \ll 1$. Thus $L_\epsilon + i\theta_\epsilon$ cannot converge to 0 along the imaginary axis, but can converge along a curve tangent to the imaginary axis with order < 2.

From (1.1), the solutions of the equation $f_\epsilon^{lq}(z) = z$ near the origin are $z = 0$ and $z \approx (1 - \lambda_\epsilon^q)^{1/p}$ and they correspond to the symmetrically arrayed fixed points of f_ϵ^{lq} generated by the perturbation of a (See Figure 2). We classify them into two types: *planet* and *satellite*.

First, we consider the case of multiple petals: That is, $p \geq 2$. Then we have the following three cases corresponding to $L_\epsilon = 0, < 0$, or > 0:

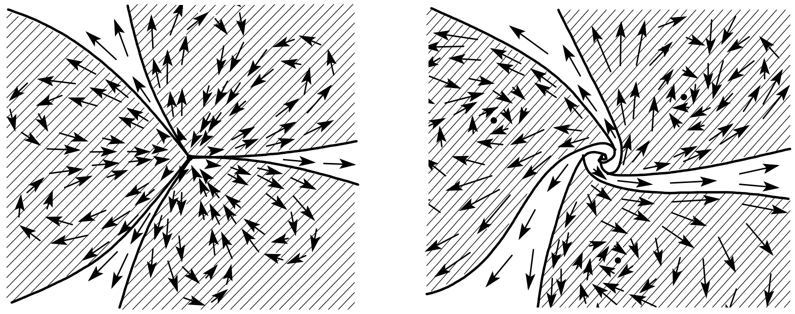


Figure 1.4: A horocyclic perturbation of a parabolic fixed point of f^{lq} of 3 petals (left) into a repelling fixed point of f_ϵ^{lq} (right).

(1) a_ϵ is persistently a parabolic point with p petals and the multiplier $\lambda_\epsilon = \lambda$;

(2) a_ϵ is an attracting periodic point, and there are p symmetrically arrayed repelling periodic points near a_ϵ; or

(3) a_ϵ is a repelling periodic point, and there are p symmetrically arrayed attracting periodic points near a_ϵ.

For cases (2) and (3), these symmetrically arrayed periodic points have the same period lq and the multipliers $\approx \lambda_\epsilon^{-pq}$. Moreover, they are contained in an open ball centered at a_ϵ with radius $O(|1-\lambda_\epsilon^q|^{1/p})$. We call them the *satellites* of a_ϵ and a_ϵ itself the *planet*. In particular, for case (2), we say that *the parabolic point a is perturbed into an attracting planet a_ϵ*. As we will see in the following sections, attracting planets are the cause of non-injectivity of the semiconjugacies. For case (1), we also call a_ϵ the planet, although it has no satellite.

Next, we consider the case of one petal. Now $p = 1$, then automatically $q = 1$ and $\lambda = 1$. If $\lambda_\epsilon = \lambda(= 1)$, a_ϵ is persistently a parabolic point with one petal. In this case, we also call a_ϵ the planet. If $\lambda_\epsilon \neq \lambda$, a splits into a pair of repelling and attracting periodic points. Which one is suitable for the planet? To define the planet in this case, we need to consider the J-critical relations.

Preparabolic critical orbits in $J(f)$. Let b be a preimage of a such that $a = f^i(b) = f^{i+l}(b)$. If $\deg(f^i, b) = m$, we can take a local coordinate near b such that $\zeta(b) = 0$ and

$$f^{-i} \circ f^{lq} \circ f^i(\zeta) = \zeta + \zeta^{mp+1} + O(\zeta^{mp+2}),$$

with a suitable branch of f^{-i}. This implies that there are mp petals attached to b as preimages of the petals of a.

Let us suppose that a horocyclic perturbation $f_\epsilon \to f$ preserves the J-critical relations of f. Then there exists b_ϵ such that $a_\epsilon = f_\epsilon^i(b_\epsilon) = f_\epsilon^{i+l}(b_\epsilon)$ and $\deg(f_\epsilon^i, b_\epsilon) =$

m. Taking a suitable local coordinate near b_ϵ such that $\zeta(b_\epsilon) = 0$, we obtain the corresponding normalized form of f_ϵ;

$$f_\epsilon^{-i} \circ f_\epsilon^{lq} \circ f_\epsilon^i(\zeta) = \lambda_\epsilon^q \zeta + \zeta^{mp+1} + O(\zeta^{mp+2}).$$

If $\lambda_\epsilon^q \neq 1$ (that is, $L_\epsilon \neq 0$) and $p \geq 2$, there are symmetrically arrayed mp "satellites" near b_ϵ as the preimages of the satellites of a_ϵ. Recall that a_ϵ may be attracting: this implies, b_ϵ may be in the Fatou set.

Now let us return to the definition of the planet when a has one petal. In the case of $\lambda_\epsilon = \lambda(= 1)$, it has been defined by a_ϵ. In the case of $\lambda_\epsilon \neq \lambda$, a splits into a pair of repelling and attracting fixed points of f_ϵ^l, say a_ϵ^+ and a_ϵ^- respectively. If a has a critical point in its preimages, then either a_ϵ^+ or a_ϵ^- has a critical point in its preimages because the J-critical relations are preserved. In this case, we define the planet as one containing a critical point in its preimages, and the satellite bas the other one. In particular, if a_ϵ^- is the planet, we also say that *a is perturbed into an attracting planet a_ϵ^-*. If a has no critical point in its preimages, then we formally define the planet as a_ϵ^+ and the satellite as a_ϵ^-.

Example. Let us consider perturbations of $f(z) = z(1+z)^m$ with $m > 1$ again. Recall that 0 is a parabolic fixed point with one petal.

For both perturbations in Example 1, 0 is the planet and $-1 + 1/\sqrt[m]{\lambda_\epsilon}$ is the satellite (See Figure 1). For the second perturbation, 0 is perturbed into an attracting planet.

On the other hand, for a trivial perturbation $f_\epsilon(z) = z(1+\lambda_\epsilon z)^m$ with $\lambda_\epsilon \to 1$, where f_ϵ are conjugate to f by linear transformations, 0 is the planet with no satellite.

Prerepelling critical orbits in $J(f)$. By geometric finiteness of f, some critical orbits in $J(f)$ land on repelling cycles. Since the J-critical relations are preserved, such repelling cycles are perturbed into repelling cycles of f_ϵ for $\epsilon \ll 1$. Let us consider local representations of the perturbations near such cycles.

Let b be a repelling periodic point of f in $P(f) \cap J(f)$, with multiplier λ and period l. Then there exists a repelling periodic point b_ϵ of f_ϵ in $P(f_\epsilon) \cap J(f_\epsilon)$, with multiplier λ_ϵ and period l, such that $b_\epsilon \to b$ and $\lambda_\epsilon \to \lambda$. By using a fundamental fact about linearization near repelling fixed points, we can take suitable local coordinates ψ_ϵ, ψ on a neighborhood of b such that $\psi_\epsilon(b_\epsilon) = \psi(b) = 0$ and

$$\psi_\epsilon \circ f_\epsilon^l \circ \psi_\epsilon^{-1}(z) = \lambda_\epsilon z \to \psi \circ f^l \circ \psi^{-1}(z) = \lambda z, \tag{2.2}$$

where ψ_ϵ converges to ψ uniformly near b. See [5, 8.3 Remark].

1.2.2 Key lemma on horocyclic perturbation.

Here we show a key lemma on horocyclic perturbation, which describes the perturbation of an orbit which accumulates on parabolic periodic points. We will see how horocyclic perturbations control the parabolic bifurcations.

Let a_0 be a periodic point of f with period l. The *cycle* α of a_0 is defined by

$$\alpha := \{a_0, f(a_0), \ldots, f^{l-1}(a_0)\}.$$

When a_0 is parabolic (resp. attracting, etc.), we call α a *parabolic (resp. attracting, etc.) cycle*.

Let us fix an $x \in \hat{\mathbb{C}}$ whose orbit accumulates on a parabolic cycle α. For an *arbitrarily* small $\delta > 0$, set $\Delta = \Delta(\delta) := \bigcup_{a \in \alpha} B_\sigma(a, \delta)$, and take $N_0 = N_0(x, \delta) \gg 0$ such that $f^n(x)$ are contained in Δ for all $n \geq N_0$. Now the key lemma is described as:

Lemma 1.2.2 *If the perturbation $f_\epsilon \to f$ is horocyclic, then there exists an $N \geq N_0$ such that $f_\epsilon^n(x)$ are contained in Δ for all $n \geq N$ and all $\epsilon \ll 1$.*

To simplify the proof of this lemma, we use "linearization" of parabolic bifurcations due to C. McMullen[12].

Proof. We begin the proof with constructing a simpler representation of the perturbation.

Linearizing parabolics. Let us take an integer k so that $f^k(a) = a$ and $(f^k)'(a) = 1$ for any $a \in \alpha$, and replace f by f^k. Then we may assume that $\alpha = \{a\}$ is a fixed point with multiplier 1 and that $\Delta = B_\sigma(a, \delta)$. It is sufficient to prove the statement in this case.

From the conditions of horocyclic perturbation, there exists a fixed point a_ϵ of f_ϵ converging to a. We may assume $\epsilon \ll 1$ such that a_ϵ is contained in Δ and sufficiently close to a. Now we set

$$\lambda_\epsilon = \exp(L_\epsilon + i\theta_\epsilon) := 1/f'_\epsilon(a_\epsilon),$$

which tends to 1 with $\theta_\epsilon^2 = o(|L_\epsilon|)$.

By replacing $\Delta = \Delta(\delta)$ with smaller δ and the definition of horocyclic perturbation, we can take a normalized convergent form on Δ as (1.1);

$$f_\epsilon(z) = \lambda_\epsilon^{-1} z + z^{p+1} + O(z^{p+2}) \to f(z) = z + z^{p+1} + O(z^{p+2})$$

where $z(a_\epsilon) = z(a) = 0$ and p is the petal number of a. Moreover, we take a simpler form of the convergence as follows.

First, by using local coordinates such that $z(a_\epsilon) = z(a) = \infty$, we obtain

$$f_\epsilon(z) = \lambda_\epsilon z + z^{1-p} + O(z^{-p}) \to f(z) = z + z^{1-p} + O(z^{-p}) \qquad (2.3)$$

as a normal form of the convergence. Next, by using [12, Theorem 8.3] and additional linear conjugacies, we can show that there exist quasiconformal maps

$\phi_{\epsilon,0}$, ϕ_0 with $\phi_{\epsilon,0} \to \phi_0$ near infinity and $\phi_{\epsilon,0}(\infty) = \phi_0(\infty) = \infty$ such that

$$\begin{aligned} T_\epsilon(z) &:= \phi_{\epsilon,0} \circ f_\epsilon \circ \phi_{\epsilon,0}^{-1}(z) = (\lambda_\epsilon^p z^p + 1)^{1/p} \\ \to T(z) &:= \phi_0 \circ f \circ \phi_0^{-1}(z) = (z^p + 1)^{1/p}. \end{aligned} \tag{2.4}$$

Where p-th roots are taken so that $(\lambda_\epsilon^p z^p + 1)^{1/p} = \lambda_\epsilon z + O(1)$ and $(z^p + 1)^{1/p} = z + O(1)$. Note that T_ϵ and T are p-fold branched coverings of linear transformations $\tilde{T}_\epsilon(w) = \lambda_\epsilon^p w + 1$ and $\tilde{T}(w) = w + 1$ respectively (where $w = z^p$). We call this form (2.4) a *linearized model* of the perturbation $f_\epsilon \to f$ near a.

Let ϕ_ϵ (resp. ϕ) be the composition of local coordinates of a_ϵ (resp. a) as (2.3) with $\phi_{\epsilon,0}$ (resp. ϕ_0) as (2.4). Then we obtain $\phi_\epsilon \to \phi$, a uniformly convergent family of local coordinates near a, which satisfies $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$ and conjugates $f_\epsilon \to f$ to $T_\epsilon \to T$. Finally, by replacing $\Delta = \Delta(\delta)$ with much smaller δ, we may assume that Δ is the domains of ϕ_ϵ and ϕ.

Now let us show the lemma by using the linearized model as (2.4). Take a constant $R \gg 0$ and a closed disk $D := \{|z| \geq R\}$, such that D is contained in both $\phi_\epsilon(\Delta)$ and $\phi(\Delta)$. Then there exists an $N_1 \geq N_0$ such that $\phi(f^n(x)) \in D$ for all $n \geq N_1$. Moreover, by uniform convergence of $f_\epsilon \to f$ and $\phi_\epsilon \to \phi$, we may assume that $\phi_\epsilon(f_\epsilon^{N_1}(x)) \in D$. To prove the lemma, it is enough to show that there exists an $N \geq N_1$ such that $\phi_\epsilon(f_\epsilon^n(x)) \in D$ for all $n \geq N$.

The proof breaks into the cases of $p = 1$ and $p \geq 2$.

Case 1: $p = 1$**.** Now $\phi_\epsilon \to \phi$ conjugates $f_\epsilon \to f$ to

$$T_\epsilon(z) = \lambda_\epsilon z + 1 \to T(z) = z + 1 \tag{2.5}$$

on D, with $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$. (See Figure 3. The four regions are centered at infinity.)

When $\lambda_\epsilon = 1$, T_ϵ is still parabolic and

$$T_\epsilon^k(\phi_\epsilon(f_\epsilon^{N_1}(x))) = \phi_\epsilon(f_\epsilon^{N_1}(x)) + k \in D$$

for all $k \geq 0$. This implies that $f_\epsilon^{N_1+k}(x)$ never escapes from $\phi_\epsilon^{-1}(D) \subset \Delta$ for all $k \geq 0$. Hence we take N_1 as N in this case.

We henceforth assume that $|\lambda_\epsilon| \neq 1$. By the perturbation, a splits into a pair of attracting and repelling fixed points. We may suppose that a_ϵ is the repelling one, and let b_ϵ denote the attracting one. (Here we do not consider which the planet is.) Then $|1/\lambda_\epsilon| = |f_\epsilon'(a_\epsilon)| > 1$, that is, $L_\epsilon \nearrow 0$. Moreover, in the linearized model (2.5), $\phi_\epsilon(b_\epsilon)$ must be the attracting fixed point of T_ϵ; thus $\phi_\epsilon(b_\epsilon) = (1 - \lambda_\epsilon)^{-1} =: b_\epsilon'$, and the multiplier of b_ϵ' is λ_ϵ.

Since the real part of $T^n(z)$ tends to infinity, there exists an integer $N \geq N_1$ such that $\phi(f^N(x))$ is in $D \cap \{|\arg z| < \pi/4\}$. By uniform convergence of $f_\epsilon \to f$

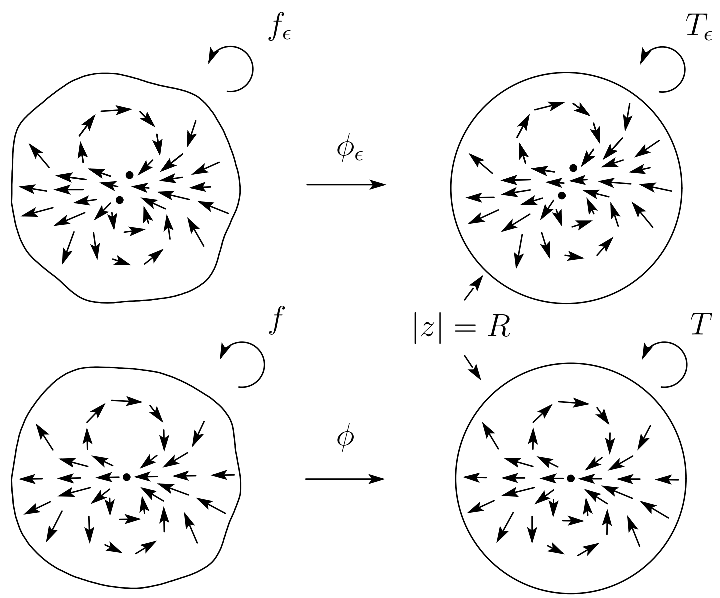


Figure 1.5: The dynamics on a neighborhood of infinity.

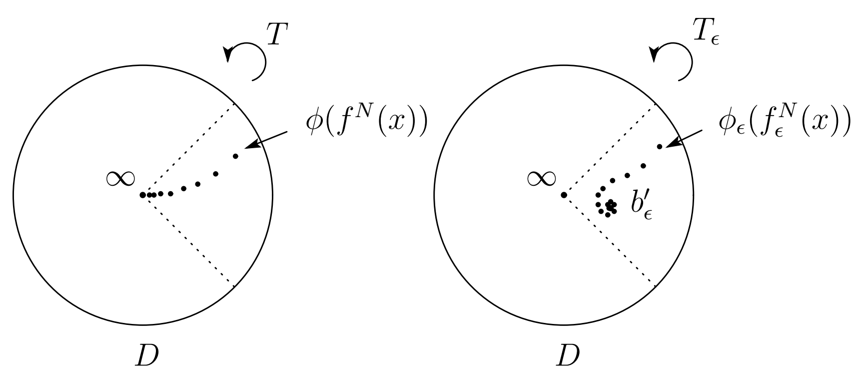


Figure 1.6: The orbits of $f^N(x)$ and $f_\epsilon^N(x)$ in the model.

and $\phi_\epsilon \to \phi$, we may also assume that $y := \phi_\epsilon(f_\epsilon^N(x))$ is in $D \cap \{|\arg z| < \pi/4\}$ for all $\epsilon \ll 1$(Figure 4).

To see the dynamics of T_ϵ in detail, we take a Möbius conjugacy of T_ϵ by

$$w = \psi_\epsilon(z) = \frac{z - b'_\epsilon}{y - b'_\epsilon},$$

which maps $\infty \mapsto \infty$, $b'_\epsilon \mapsto 0$ and $y \mapsto 1$. This conjugates the action of T_ϵ to $w \mapsto \lambda_\epsilon w$ with $|\lambda_\epsilon| < 1$. Hence $1 = \psi_\epsilon(y)$ is attracted to $0 = \psi_\epsilon(b'_\epsilon)$ by the iteration of $w \mapsto \lambda_\epsilon w$.

Now we claim: *For any fixed $\epsilon \ll 1$, $f_\epsilon^n(x)$ is contained in Δ for all $n \geq N$, and converges to b_ϵ as $n \to \infty$.* In other words, the whole orbit of $1 = \psi_\epsilon(y)$ is contained in $\psi_\epsilon(D)$ where the conjugation between T_ϵ and $w \mapsto \lambda_\epsilon w$ holds.

Set $B := \hat{\mathbb{C}} - D$ and $B' := \psi_\epsilon(B)$. Then B' is defined by this inequality:

$$\left| w - \frac{b'_\epsilon}{b'_\epsilon - y} \right| < \frac{R}{|b'_\epsilon - y|}. \tag{2.6}$$

We will show that the orbit of 1, that is, $\{1 = \psi_\epsilon(y),\ \lambda_\epsilon,\ \lambda_\epsilon^2, \ldots\}$, never enters B'.

For all $\epsilon \ll 1$, the center $b'_\epsilon/(b'_\epsilon - y)$ of B' is approximately $1 - y(L_\epsilon + i\theta_\epsilon)$. On the other hand, for any k such that $|k(L_\epsilon + i\theta_\epsilon)| \ll 1$, λ_ϵ^k is approximately $1 + k(L_\epsilon + i\theta_\epsilon)$. Since $|\arg y| < \pi/4$, the direction of first several points of the orbit $\{1,\ \lambda_\epsilon,\ \lambda_\epsilon^2, \ldots\}$ is opposite to the center of B' with respect to 1. This means, at least, the orbit does not go to B' immediately (Figure 5).

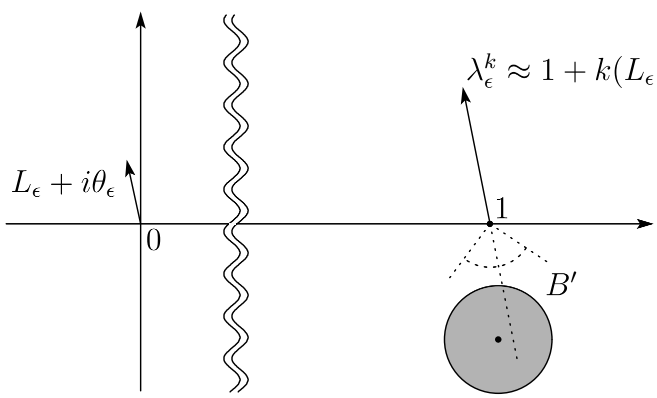


Figure 1.7: The orbit of $1 = \psi_\epsilon(y)$ near 1

Suppose that $\theta_\epsilon = 0$. Then the orbit of 1 accumulates on 0 along the real axis, and it is disjoint from B'.

Suppose that $\theta_\epsilon \neq 0$. We may assume that $\theta_\epsilon > 0$ because the signature of θ_ϵ determines only the direction of the rotation by the action of $w \mapsto \lambda_\epsilon w$. Then the orbit of 1 returns near the positive real axis by nearly $2\pi/\theta_\epsilon$ times iterations of $w \mapsto \lambda_\epsilon w$. Now we have to handle the case where the order of $\theta_\epsilon \searrow 0$ is lower

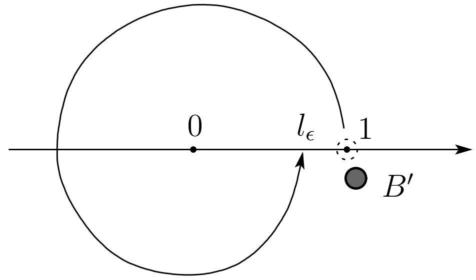


Figure 1.8: The orbit of 1

than that of $L_\epsilon \nearrow 0$: Then the orbit might touch B'. However, we will show that it cannot occur if $\epsilon \ll 1$.

Now note that the following two facts: when the orbit of 1 returns near the positive real axis, the distance between 0 and the orbit is nearly $l_\epsilon := \exp(2\pi L_\epsilon/\theta_\epsilon)$; on the other hand, by (2.6), B' is contained in a ball centered at 1 with radius $O(|L_\epsilon + i\theta_\epsilon|)$, that is, every point in B' tends to 1 as $\epsilon \to 0$.

By these facts, if $\liminf |L_\epsilon/\theta_\epsilon| \neq 0$, l_ϵ does not tend to 1 and the orbit of 1 never touches B' (Figure 6).

Otherwise we can take a decreasing sequence $\epsilon_n \searrow 0$ such that $L_{\epsilon_n}/\theta_{\epsilon_n} \to 0$. Now $l_{\epsilon_n} \to 1$ as $n \to \infty$. In this case, $|1 - l_{\epsilon_n}| \approx 2\pi|L_{\epsilon_n}|/\theta_{\epsilon_n}$ for $n \gg 0$ thus

$$\frac{O(|L_{\epsilon_n} + i\theta_{\epsilon_n}|)}{|1 - l_{\epsilon_n}|} = O\big(\big|\theta_{\epsilon_n} + i\theta_{\epsilon_n}^2/|L_{\epsilon_n}|\big|\big) \to 0 \quad (\epsilon_n \to 0). \tag{2.7}$$

This means, for any choice of $\{\epsilon_n\}$, every point in B' tends to 1 faster than l_{ϵ_n} does. Note that the order of convergence in (2.7) depends only on the order of L_ϵ, $\theta_\epsilon \to 0$ (not on the choice of $\{\epsilon_n\}$). Hence for $\epsilon \ll 1$, the orbit of 1 is attracted to 0 without entering B'.

Case 2 : $p \geq 2$. Now $\phi_\epsilon \to \phi$ with $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$ conjugates $f_\epsilon \to f$ to

$$T_\epsilon(z) = (\lambda_\epsilon^p z^p + 1)^{1/p} \to T(z) = (z^p + 1)^{1/p} \tag{2.8}$$

on D. As in the case of $p = 1$, we may assume that

$$\phi(f^N(x)) \in \bigcup_{j=0}^{p-1} \left\{ \left| \arg z - \frac{2\pi j}{p} \right| < \frac{\pi}{4p} \right\}$$

for an $N \geq N_1$, and

$$y = \phi_\epsilon(f_\epsilon^N(x)) \in \bigcup_{j=0}^{p-1} \left\{ \left| \arg z - \frac{2\pi j}{p} \right| < \frac{\pi}{4p} \right\}$$

for all $\epsilon \ll 1$.

Let us consider a semiconjugation of T_ϵ by a branched covering $w = \pi(z) = z^p$. Then the dynamics of T_ϵ on D is reduced to the dynamics of $\tilde{T}_\epsilon(w) = \lambda_\epsilon^p w + 1$ on $\pi(D) = \{|w| \geq R^p\}$(Figure 7). Similarly, $\pi(z)$ gives a semiconjugacy from $T(z)$ on D to $\tilde{T}(w) = w + 1$ on $\pi(D)$.

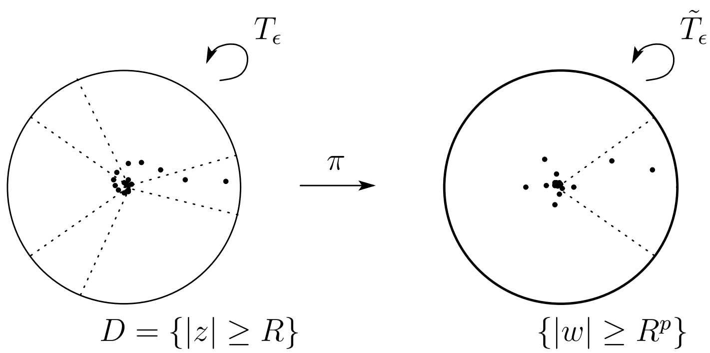


Figure 1.9: $w = \pi(z) = z^p$

By the same argument as the case of $p = 1$, when $\lambda_\epsilon = 1$, the orbit of $\pi(y)$ tends to $w = \infty$ and never escapes from $\pi(D)$. Similarly, if $|\lambda_\epsilon| \neq 1$, the orbit of $\pi(y)$ tends to an attracting fixed point, which is either $w = \infty$ or $w = 1/(1 - \lambda_\epsilon^p)$, and never escapes from $\pi(D)$. Thus the original orbit of $\phi_\epsilon(f_\epsilon^N(x))$ by T_ϵ never escapes from D. ■

Remark. One can easily check that the same result holds if we replace x with a compact set in the parabolic basin of a. We will use this in the proof of Proposition 1.3.2.

1.3 Construction of Ω and Ω_ϵ

In this section, we prepare the ingredients for the construction of the semiconjugacy; Ω, Ω_ϵ and $h_0 : \Omega_\epsilon \to \Omega$.

To simplify the arguments, *from this section to* §*7, we assume that* $J(f) \neq \hat{\mathbb{C}}$. The case of $J(f) = \hat{\mathbb{C}}$ is treated in §8.

Let us introduce some notation. Let A denote the finite set of all parabolic points of f. We define the sets of all preperiodic critical orbits in the Julia sets by

$$Z := \bigcup_{n=1}^{\infty} f^n(C(f) \cap J(f)), \quad Z_\epsilon := \bigcup_{n=1}^{\infty} f_\epsilon^n(C(f_\epsilon) \cap J(f_\epsilon)).$$

In addition, we set $Z^1 := f^{-1}(Z)$ and $Z_\epsilon^1 := f_\epsilon^{-1}(Z_\epsilon)$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, $\mathrm{card}(Z_\epsilon) \leq \mathrm{card}(Z) < \infty$ in general. The equality holds precisely if none of the parabolic points of f is perturbed into an attracting planet.

1.3.1 Construction of Ω.

Here we construct a compact set Ω for f.

Proposition 1.3.1 *There exists a finitely connected compact set $\Omega \subset \hat{\mathbb{C}}$ with the following properties:*

1. $\Omega \cap (P(f) \cup C(f)) = J(f) \cap (P(f) \cup C(f))$. *This set is the union of A and all critical orbits in $J(f)$.*
2. $J(f) \subset \Omega$ *and* $f^{-1}(\Omega) \subset \mathrm{Int}(\Omega) \cup A$.

Proof. To define the compact set Ω, we will construct two open sets F and V which consist of finitely many simply connected components.

Let a be an attracting or parabolic periodic point of f and α the cycle of a. First, we construct F: If α is attracting, we take a small disk neighborhood F_a for each $a \in \alpha$ such that $f(\overline{F_a}) \subset F_{f(a)}$. Here we can take $\{F_a\}$ to be pairwise disjoint. If α is parabolic, we take F_a for each point $a \in \alpha$ to be a small "flower" (that is, a union of attracting petals for each attracting directions of a) such that $f(\overline{F_a} - \{a\}) \subset F_{f(a)}$. Here we can also take $\{F_a\}$ to be pairwise disjoint, and each ∂F_a to be tangent to the repelling directions.

Now we set
$$F := \bigcup_{\alpha} \bigcup_{a \in \alpha} F_a$$
where α ranges over all attracting and parabolic cycles. Note that $f(\overline{F} - A) \subset F$.

Next, we construct V: Let $C(f, \alpha)$ denote the set of all critical points of f whose orbits accumulate on α but never land on it. Now let us set $F_\alpha := \bigcup_{a \in \alpha} F_a$. For each $c \in C(f, \alpha)$, there exists a natural number $N = N(c)$ such that $f^n(c) \in F_\alpha$ for all $n \geq N$. Then we can take a family of open disks $\{V_c^i\}_{i=0}^{N}$ satisfying the following conditions (See Figure 1.10):

- V_c^i is a small disk-neighborhood of $f^i(c)$;
- $V_c^i \cap V_c^j = \emptyset$ for $i \neq j$;
- $V_c^N \subset F_\alpha$; and
- $f(\overline{V_c^i}) \subset V_c^{i+1}$ for all $i < N$.

Now we set
$$V := \bigcup_{\alpha} \bigcup_{c \in C(f,\alpha)} \bigcup_{i=0}^{N(c)} V_c^i$$
where α ranges over all attracting and parabolic cycles. Note that $f(\overline{V}) \subset V \cup F$.

Using F and V, we define Ω as $\hat{\mathbb{C}} - (F \cup V)$. Then we can easily check that Ω satisfies the conditions in the statement. ■

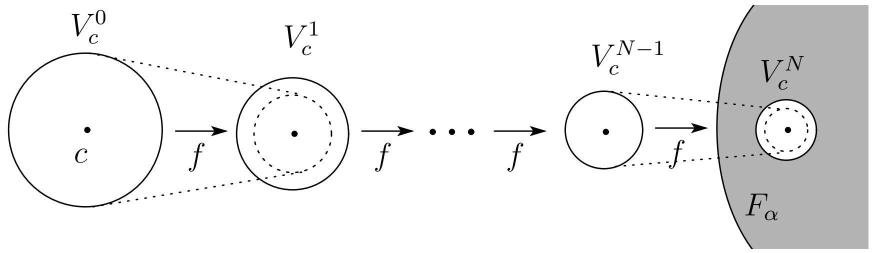


Figure 1.10: The orbit of c and $\{V_c^i\}$

1.3.2 Construction of Ω_ϵ and the "0-th" map h_0.

Next we consider a horocyclic perturbation $f_\epsilon \to f$ preserving the J-critical relations of f. For each f_ϵ, we construct a compact set Ω_ϵ corresponding to $\Omega = \hat{\mathbb{C}} - (F \cup V)$, and the correspondence is represented by the map $h_0(= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$.

Proposition 1.3.2 *For each $\epsilon \ll 1$, there exists a compact set $\Omega_\epsilon \subset \hat{\mathbb{C}}$ and a continuous map $h_0(= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$ with the following properties:*

1. *$\Omega_\epsilon \cap (P(f_\epsilon) \cup C(f_\epsilon)) = J(f_\epsilon) \cap (P(f_\epsilon) \cup C(f_\epsilon))$, and this set is the union of all parabolic points of f_ϵ and all critical orbits in $J(f_\epsilon)$.*
2. *$J(f_\epsilon) \subset \Omega_\epsilon$ and $f_\epsilon^{-1}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$.*
3. *$h_0 : \Omega_\epsilon \to \Omega$ is surjective.*
4. *If there exists $y \in \Omega$ such that $\mathrm{card}(h_0^{-1}(y)) \geq 2$ then y is a parabolic point and $\mathrm{card}(h_0^{-1}(y)) = p(y)$. Moreover, y is perturbed into an attracting planet and $h_0^{-1}(y)$ is the set of $p(y)$ repelling satellites of the attracting planet.*
5. *For each $b_\epsilon \in Z_\epsilon^1$, there exists a unique $b \in Z^1$ such that $b_\epsilon \to b$, and*
$$h_0(b_\epsilon) = b.$$

Moreover, for any fixed $r > 0$, we can make h_0 satisfy

$$\sup \{d_\sigma(h_0(x), x) : x \in \Omega_\epsilon\} \leq r$$

for all $\epsilon \ll 1$.

For example, suppose that f is hyperbolic; that is, both A and $J(f) \cap C(f)$ are empty. For $\epsilon \ll 1$, f_ϵ is a very small perturbation of f, thus every attracting cycle of f is perturbed into an attracting cycle of f_ϵ. By uniform convergence of $f_\epsilon \to f$, we obtain $f_\epsilon(\overline{F}) \subset F$ for all $\epsilon \ll 1$. Similarly, if $\epsilon \ll 1$, V satisfies $f_\epsilon(\overline{V}) \subset V \cup F$. Hence we can set $\Omega_\epsilon := \Omega = \hat{\mathbb{C}} - (F \cup V)$ and $h_0 := \mathrm{id}$.

For general geometrically finite rational maps, to construct Ω_ϵ for $f_\epsilon \to f$, we need to modify F; in particular, certain parts of the flowers $\{F_a\}_{a\in A}$. We also need additional modification near the critical orbits in the Julia set.

Let us fix an $r > 0$ and set $B_x := B_\sigma(x, r/2)$ for each $x \in A \cup Z^1$. We suppose that r is sufficiently small so that $B_x \cap B_{x'} = \emptyset$ for different $x,\ x' \in A \cup Z^1$ and that $B_x \subset \mathrm{Int}(\Omega)$ for $x \in Z^1 - A$.

Modification of Ω near the parabolics. Fix a parabolic point of f, say $a \in A$. Set $E_a := \Omega \cap \overline{B_a}$. We may assume that E_a is a union of $p(a)$ narrow cusps near the repelling directions.

Lemma 1.3.3 *For each $\epsilon \ll 1$, there exists a compact set E'_a and a map $h_a : E'_a \to E_a$ with the following conditions:*

- $\partial E_a \cap \partial B_a = \partial E'_a \cap \partial B_a$*, and h_a is the identity on this set.*
- $f_\epsilon^{-1}(E'_{f(a)}) \cap B_a \subset E'_a$.
- $B_a - E'_a \subset F(f_\epsilon)$.
- $h_a : E'_a \to E_a$ *is continuous and surjective.*
- *If $y \in E_a$ and $\mathrm{card}(h_a^{-1}(y)) \geq 2$, then $y = a$. In this case, a is perturbed into an attracting planet a_ϵ and $h_a^{-1}(y)$ is the set of all repelling satellites of a_ϵ.*
- $d_\sigma(h_a(x), x) \leq r$ *for any $x \in E'_a$.*

Proof. For simplicity, here we only treat the case where a is a fixed point with multiplier 1. The case of a with multiplier $\neq 1$ or period $\neq 1$ is similar.

As $f_\epsilon \to f$ horocyclically, suppose that a is perturbed into the planet a_ϵ, a fixed point of f_ϵ.

Let us consider the local dynamics by f^{-1} and f_ϵ^{-1} restricted near B_a. We denote by g (resp. g_ϵ) the branch of f^{-1} (resp. f_ϵ^{-1}) near B_a which fixes a (resp. a_ϵ). Then a is still a parabolic fixed point of g and a_ϵ is a fixed point of g_ϵ with multiplier $1/f'_\epsilon(a_\epsilon)$. Note that $g_\epsilon \to g$ is a locally defined horocyclic perturbation, thus we can apply Lemma 1.2.2.

Set $p := p(a)$, the petal number of a. The construction of E'_a and h_a breaks into the cases of $p = 1$ and $p \geq 2$.

Case 1 : $p = 1$. In this case, we may assume that a_ϵ is an attracting or parabolic fixed point of g_ϵ. (Here we need not distinguish planet from satellite.)

Now $\partial E_a \cap \partial B_a$ is an arc. Let e_1 and e_2 be its end points. Since r is sufficiently small, we may assume that e_1 and e_2 are enough close to the attracting direction for g, and that their orbits by g accumulate on a within E_a. Then we may apply the argument in Lemma 1.2.2 to the orbits of e_1 and e_2 by g_ϵ. For $\epsilon \ll 1$, joining the orbits of e_i $(i = 1, 2)$ by g_ϵ contained in B_a, we obtain a piecewise smooth Jordan arcs η_i with the following properties:

- Joining from e_i to a_ϵ.
- $g_\epsilon(\eta_i) \subset \eta_i \subset B_a \cup \{e_i\}$ and $f_\epsilon(\eta_i) - B_a \subset F_a$
- $\eta_1 \cap \eta_2 = \{a_\epsilon\}$.

In fact, joining e_i and $g_\epsilon(e_i)$ by nearly straight curve and taking the union of their forward images by g_ϵ, we obtain such a curve η_i. We define E'_a as the closure of the region in B_a enclosed by η_1, η_2 and $\partial E_a \cap \partial B_a$. Then we see that $f_\epsilon^{-1}(E'_a) \cap B_a \subset E'_a$.

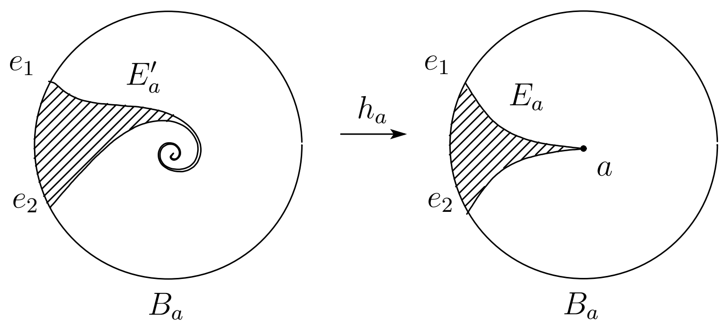


Figure 1.11: Construction of E'_a

We claim that $B_a - E'_a \subset F(f_\epsilon)$ for $\epsilon \ll 1$. Let us take an arbitrary $x \in B_a - E'_a$.

If the orbit of x never escapes from B_a and is attracted to the parabolic or attracting point of f_ϵ in B_a, then $x \in F(f_\epsilon)$. So we consider the case where the orbit of x escapes from B_a. Then for some $i > 0$, $f_\epsilon^i(x)$ is contained in the compact set $\overline{F_a - B_a} \subset F(f)$.

By the local dynamics in F_a, there exists $N \gg 0$ such that $f^N(\overline{F_a - B_a})$ is contained in B_a and is sufficiently near the attracting direction of a. By uniform convergence of $f_\epsilon \to f$, we may suppose the same holds for $f_\epsilon^N(\overline{F_a - B_a})$. Furthermore, since $f^n(\overline{F_a - B_a})$ converges uniformly to a within B_a as n tends to infinity, we may apply the argument in Lemma 1.2.2 to the forward images of $f_\epsilon^N(\overline{F_a - B_a})$ by f_ϵ; thus $f_\epsilon^n(\overline{F_a - B_a})$ converges uniformly to the parabolic or attracting point of f_ϵ within B_a. This implies $x \in F(f_\epsilon)$.

Finally we define the map $h_a : E'_a \to E_a$: Let us take a Riemann map $R_\epsilon : \mathrm{Int}(E'_a) \to \mathbb{D}$, here $\mathbb{D}$ is the unit disk. Since the boundary of E'_a is a Jordan curve, R_ϵ is extended to a homeomorphism $R_\epsilon : E'_a \to \overline{\mathbb{D}}$. Similarly, we take an extended Riemann map $R : E_a \to \overline{\mathbb{D}}$. By choosing a suitable topological map $H_\epsilon : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, we obtain $h_a := R^{-1} \circ H_\epsilon \circ R_\epsilon$ such that:

- $h_a : E'_a \to E_a$ is a homeomorphism;

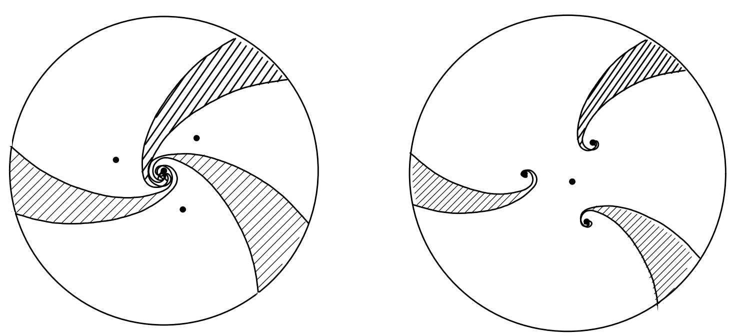
- $h_a|(\partial E'_a \cap \partial B_a) = \mathrm{id}$; and
- $h_a(a_\epsilon) = a$.

Furthermore, since the radius of B_a is $r/2$, we obtain $d_\sigma(h_a(x), x) \leq r$ for any $x \in E'_a$.

Case 2 : $p \geq 2$**.** Now E_a is the union of p narrow cusps which intersect only at a. We distinguish these p cusps as $\{E_1, \ldots, E_p\}$; that is, each E_j is a union of $\{a\}$ and one of the p connected components of $E_a - \{a\}$. Let e_{1j} and e_{2j} be the end points of $\partial E_j \cap \partial B_a$ for $j = 1, \ldots, p$.

As in the case of $p = 1$, let us apply the argument in Lemma 1.2.2. Then we can take g_ϵ-invariant path η_{ij} which joins e_{ij} and a parabolic or attracting point of g_ϵ generated in B_a by the perturbation of a. We define E'_j as the compact set in $\overline{B_a}$ enclosed by η_{1j}, η_{2j}, and $\partial E_j \cap \partial B_a$. Note that we obtain the following three cases:

1. The planet a_ϵ is a parabolic fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $f'_\epsilon(a_\epsilon) = 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to a_ϵ and $\bigcap_{j=1}^{p} E'_j = \{a_\epsilon\}$.
2. The planet a_ϵ is a repelling fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $|f'_\epsilon(a_\epsilon)| > 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to a_ϵ and $\bigcap_{j=1}^{p} E'_j = \{a_\epsilon\}$ (Figure 1.12).
3. The planet a_ϵ is an attracting fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $|f'_\epsilon(a_\epsilon)| < 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to one of the symmetrically arrayed repelling satellites of a_ϵ and $\bigcap_{j=1}^{p} E'_j = \emptyset$ (Figure 1.12).



$|f'_\epsilon(a_\epsilon)| > 1$ $|f'_\epsilon(a_\epsilon)| < 1$

Figure 1.12: Cases 2 and 3 of E'_a

Now we set $E'_a := \bigcup_{j=0}^{p-1} E'_j$. We can show $B_a - E'_a \subset F(f_\epsilon)$ for $\epsilon \ll 1$ by the same argument as the case of $p = 1$.

For each E'_j, let us take a homeomorphism $h_{a,j} : E'_j \to E_j$ in the same way as h_a for $p = 1$, and define a continuous map $h_a : E'_a \to E_a$ by $h_a|E'_j = h_{a,j}$. Then h_a has the following properties:

- $h_a|(\partial E'_a \cap \partial B_a) = \mathrm{id}$;
- $h_a : E'_a \to E_a$ is surjective; and
- if $y \in E_a$ and $\mathrm{card}(h_a^{-1}(y)) \geq 2$, then $y = a$. Moreover, a is perturbed into the attracting planet a_ϵ, and $h_a^{-1}(y)$ consist of p repelling satellites of a_ϵ.

In particular, we also obtain $d_\sigma(h_a(x), x) \leq r$ for any $x \in E'_a$. ■

Finally let us show the existence of Ω_ϵ.

Proof(Proposition 1.3.2). For each fixed $\epsilon \ll 1$, set

$$\Omega_\epsilon := \left(\Omega - \bigcup_{a \in A} B_a\right) \cup \bigcup_{a \in A} E'_a.$$

By the construction of E'_a, one can easily check that $J(f_\epsilon) \subset \Omega_\epsilon$ and $f_\epsilon^{-1}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$.

To check that $\Omega_\epsilon \cap (P(f_\epsilon) \cup C(f_\epsilon)) = J(f_\epsilon) \cap (P(f_\epsilon) \cup C(f_\epsilon))$, it is sufficient to show that the critical orbits in the Fatou set never land on Ω_ϵ.

Let us take $c_\epsilon \in C(f_\epsilon) \cap F(f_\epsilon)$. Then there exists $c \in C(f)$ such that $c_\epsilon \to c$ $(\epsilon \to 0)$.

If $c \in J(f)$, by geometric finiteness of f, the orbit of c lands on a parabolic or repelling cycle, say α. Since the J-critical relations of f are preserved, c_ϵ also lands on a cycle. By our assumption that $c_\epsilon \in F(f_\epsilon)$, α must be parabolic and the orbit of c_ϵ must land on an attracting cycle which is generated by the perturbation of α. Thus the orbit of c_ϵ never lands on Ω_ϵ by the definition of $\bigcup_{a \in A} E'_a$.

If $c \in F(f)$, the orbit of c accumulates on a parabolic or attracting cycle. By the construction of Ω, c is not contained in Ω. Similarly, by the definition of Ω_ϵ, we may assume that $c_\epsilon \notin \Omega_\epsilon$. Let us suppose that $f_\epsilon^n(c_\epsilon) \in \Omega_\epsilon$ for some n. Then $c_\epsilon \in f_\epsilon^{-n}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$ and it is a contradiction. Thus $f_\epsilon^n(c_\epsilon) \notin \Omega_\epsilon$ for all n.

Finally we define $h_0 : \Omega_\epsilon \to \Omega$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, we may assume that for any $b \in Z^1 - A$, B_b contains only one point of Z^1_ϵ, say b_ϵ, such that $b_\epsilon \to b$. Recall that $B_b \subset \mathrm{Int}(\Omega)$, by the assumption for r. Let $h_b : B_b \to B_b$ be an arbitrary topological map which satisfies $h_b(b_\epsilon) = b$ and $h_b|\partial B_b = \mathrm{id}$. Then we obtain $d_\sigma(h_b(x), x) \leq r$ for $x \in B_b$.

Let us define $h_0 : \Omega_\epsilon \to \Omega$ by

$$\begin{aligned} h_0 &= h_a \quad \text{on } E'_a \text{ for } a \in A, \\ h_0 &= h_b \quad \text{on } B_b \text{ for } b \in Z^1 - A, \text{and} \\ h_0 &= \mathrm{id} \quad \text{otherwise.} \end{aligned}$$

■

1.4 Construction of h_n

For Ω_ϵ and Ω constructed in §3, we set

$$\Omega_\epsilon^n := f_\epsilon^{-n}(\Omega_\epsilon) \ \text{ and } \ \Omega^n := f^{-n}(\Omega) \quad (n = 0, 1, 2, \ldots).$$

In addition, we set $U_\epsilon^n := \mathrm{Int}(\Omega_\epsilon^n)$ and $U^n := \mathrm{Int}(\Omega^n)$. By the construction of these sets, $f_\epsilon : \Omega_\epsilon^{n+1} \to \Omega_\epsilon^n$ and $f : \Omega^{n+1} \to \Omega^n$ are branched covering maps, where the critical values are contained in Z_ϵ and Z respectively. Note that $\{\Omega_\epsilon^n\}$ and $\{\Omega^n\}$ form the decreasing sequences as below:

$$\Omega_\epsilon = \Omega_\epsilon^0 \supsetneq \Omega_\epsilon^1 \supsetneq \cdots \supsetneq \Omega_\epsilon^n \supsetneq \Omega_\epsilon^{n+1} \supsetneq \cdots \supsetneq J(f_\epsilon),$$
$$\Omega = \Omega^0 \supsetneq \Omega^1 \supsetneq \cdots \supsetneq \Omega^n \supsetneq \Omega^{n+1} \supsetneq \cdots \supsetneq J(f).$$

In this section, we inductively construct a sequence of lifts of $h_0 : \Omega_\epsilon^0 \to \Omega^0$,

$$\{h_n(= h_{n,\epsilon}) : \Omega_\epsilon^n \to \Omega^n\}_{n=1}^\infty$$

satisfying $f \circ h_{n+1} = h_n \circ f_\epsilon$.

Proposition 1.4.1 *For an $n \geq 0$, assume that there exists $h_n(= h_{n,\epsilon}) : \Omega_\epsilon^n \to \Omega^n$ satisfying the following properties:*

$(1, n)$ *h_n is continuous and surjective.*

$(2, n)$ *h_n maps U_ϵ^n onto U^n homeomorphically. Moreover, if there exists $y \in \Omega^n$ such that $\mathrm{card}(h_n^{-1}(y)) \geq 2$ then $f^n(y)$ is a parabolic point of f perturbed into an attracting planet and $\mathrm{card}(h_n^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$.*

$(3, n)$ *For any $b_\epsilon \in Z_\epsilon^1$, there exists a unique $b \in Z^1$ such that*

$$h_n(b_\epsilon) = b.$$

Under these assumptions, there exists $h_{n+1}(= h_{n+1,\epsilon}) : \Omega_\epsilon^{n+1} \to \Omega^{n+1}$ satisfying

$$f \circ h_{n+1} = h_n \circ f_\epsilon$$

and properties $(1, n+1)$, $(2, n+1)$ and $(3, n+1)$.

Recall that the map $h_0 : \Omega_\epsilon^0 \to \Omega^0$ has properties (1, 0), (2, 0), and (3, 0). Thus this proposition gives us desired $\{h_n : \Omega_\epsilon^n \to \Omega^n\}_{n=1}^\infty$.

Proof. The proof breaks into 3 steps.

Step 1: Interior correspondence. The first step is to try to construct a homeomorphism between U_ϵ^{n+1} and U^{n+1}. To begin with, we construct h_{n+1} such that the following diagram commutes:

$$\begin{array}{ccc} U_\epsilon^{n+1} - Z_\epsilon^1 & \xrightarrow{h_{n+1}} & U^{n+1} - Z^1 \\ f_\epsilon \big\downarrow & & \big\downarrow f \\ U_\epsilon^n - Z_\epsilon & \xrightarrow{h_n} & U^n - Z \end{array}$$

Here $f|(U^{n+1} - Z^1)$ and $f_\epsilon|(U_\epsilon^{n+1} - Z_\epsilon^1)$ are d-sheeted covering maps. Moreover, by properties $(2, n)$ and $(3, n)$, $h_n|(U_\epsilon^n - Z_\epsilon)$ is a homeomorphism. We will construct prospective h_{n+1} in the diagram by lifting this $h_n|(U_\epsilon^n - Z_\epsilon)$. Note that U_ϵ^n and U^n for $n \geq 1$ are either connected or finitely many connected components. (For example, suppose that $J(f)$ is a Cantor set.) Hence we construct h_{n+1} on each connected component of $U_\epsilon^{n+1} - Z_\epsilon^1$.

Let Q_ϵ^1 be a connected component of $U_\epsilon^{n+1} - Z_\epsilon^1$, and take a base point $x_0^1 \in Q_\epsilon^1$. Set $Q_\epsilon := f_\epsilon(Q_\epsilon^1)$, a connected component of U_ϵ^n, and set $x_0 := f_\epsilon(x_0^1) \in Q_\epsilon$. Moreover, set $Q := h_n(Q_\epsilon)$ and $y_0 := h_n(x_0) \in Q$.

Let $y_0^1 \in U^{n+1}$ be the closest point to x_0^1 in $f^{-1}(y_0)$. Such y_0^1 is uniquely determined, since critical values in the Fatou sets stay a bounded distance away from Q_ϵ and Q. Let Q^1 denote a connected component of $f^{-1}(Q)$ containing y_0^1. We will lift h_n to h_{n+1} such that the following diagram commutes:

$$\begin{array}{ccc} (Q_\epsilon^1, x_0^1) & \xrightarrow{h_{n+1}} & (Q^1, y_0^1) \\ f_\epsilon \big\downarrow & & \big\downarrow f \\ (Q_\epsilon, x_0) & \xrightarrow{h_n} & (Q, y_0) \end{array}$$

Take a point $x^1 \in Q_\epsilon^1$ and a curve $\eta_\epsilon : [0, 1] \to Q_\epsilon^1$ such that $\eta_\epsilon(0) = x_0^1$ and $\eta_\epsilon(1) = x^1$. Then the curve $h_n(f_\epsilon(\eta_\epsilon))$ has the initial point y_0. We lift this curve to $\eta : [0, 1] \to Q^1$ with the initial point y_0^1, and define $h_{n+1}(x^1)$ as its end point $\eta(1)$.

Since $h_n|Q_\epsilon$ is a homeomorphism and the J-critical relations of f are preserved, for the fundamental groups $\pi_1(Q_\epsilon^1, x_0^1)$ and $\pi_1(Q^1, y_0^1)$,

$$(h_n)_* : (f_\epsilon)_* \pi_1(Q_\epsilon^1, x_0^1) \to f_* \pi_1(Q^1, y_0^1)$$

is a group isomorphism. Hence the above definition of $h_{n+1}(x^1)$ gives the homeomorphism $h_{n+1} : (Q_\epsilon^1, x_0^1) \to (Q^1, y_0^1)$ as a lift of $h_n : (Q_\epsilon, x_0) \to (Q, y_0)$ (See [11, Ch.III]).

Now we have a homeomorphism $h_{n+1} : U_\epsilon^{n+1} - Z_\epsilon^1 \to U^{n+1} - Z^1$. For $x \in U_\epsilon \cap Z_\epsilon^1$, let us set $h_{n+1}(x) := h_n(x)$. Then we obtain a homeomorphism $h_{n+1} : U_\epsilon^{n+1} \to U^{n+1}$ as a natural lift of $h_n : U_\epsilon^n \to U^n$.

Step 2: Boundary correspondence. The second step is to extend h_{n+1} defined on U_ϵ^{n+1} to the boundary $\partial U_\epsilon^{n+1} = \partial\Omega_\epsilon^{n+1}$, in a natural way. Here we should be careful about the boundary correspondence near the preimages of a parabolic point which is perturbed into an attracting planet. Note that the injectivity of h_n has already been broken at some of these points.

To construct $h_{n+1}|\partial\Omega_\epsilon^{n+1}$, it suffices to construct $h_{n+1}|\partial Q_\epsilon^1$ for each Q_ϵ^1 in Step 1. For $x_0^1 \in Q_\epsilon^1$ and $x^1 \in \partial Q_\epsilon^1$, take a curve $\eta_\epsilon : [0,1] \to Q_\epsilon^1 \cup \{x^1\}$ with $\eta_\epsilon(0) = x_0^1$ and $\eta_\epsilon(1) = x^1$. Now the value of h_{n+1} at x^1 is defined by

$$h_{n+1}(x^1) := \lim_{t\to 1} h_{n+1}(\eta_\epsilon(t)) \in \partial Q^1.$$

One can easily check that this value does not depend on the choice of η_ϵ.

By this definition, if $a \in \partial Q^1$ is a parabolic point with $p \geq 2$ petals and is perturbed into an attracting planet, then $h_{n+1}^{-1}(a)$ is p distinct points in ∂Q_ϵ^1 corresponding to p distinct accesses to a in E_a. The case of k-th preimages of a with $k \leq n+1$ is similar. Moreover, note that $h_{n+1}(x^1) = h_n(x^1)$ if $x^1 \in \partial Q_\epsilon^1 \cap Z_\epsilon^1$.

Step 3: Checking the properties. Now we have already defined a continuous map $h_{n+1} : \Omega_\epsilon \to \Omega$. For the last step, we check that h_{n+1} has properties $(1, n+1)$, $(2, n+1)$ and $(3, n+1)$.

Note that $h_{n+1}|Q_\epsilon^1$ is a homeomorphism and $h_{n+1}|\overline{Q_\epsilon^1}$ is continuous. Thus bijectivity of h_{n+1} may break only at the boundary points. For a boundary point y^1 of Q^1, take a curve $\eta : [0,1] \to Q^1 \cup \{y^1\}$ such that $\eta(0) = y_0^1$ and $\eta(1) = y^1$. Then the limit of $h_{n+1}^{-1}(\eta(t))$ as $t \to 1$ determines an element of $h_{n+1}^{-1}(y^1)$ which is contained in the boundary of Q_ϵ^1. Hence $h_{n+1}|\partial Q_\epsilon^1$ is surjective and we obtain property $(1, n+1)$.

Next, suppose that $q := \mathrm{card}(h_{n+1}^{-1}(y^1)) \geq 2$. Note that η determines an access to y^1 within Q^1 and an element of $h_{n+1}^{-1}(y^1)$. Thus $q \geq 2$ means that there are two or more distinct accesses to y^1 (more precisely, there are two or more distinct prime ends of Q^1 at y^1). By the definition of Ω^{n+1}, $f^{n+1}(y^1)$ must be a parabolic point with $p \geq 1$ petals such that $q = p \cdot \deg(f^{n+1}, y^1) \geq 2$. By the definition of Ω_ϵ^{n+1}, such a must be perturbed into an attracting planet, since otherwise all possible η determines the same element of $h_{n+1}^{-1}(y^1)$. Thus we obtain property $(2, n+1)$.

Finally, we obtain property $(3, n+1)$ by the fact that $h_{n+1}(x^1) = h_n(x^1)$ if $x^1 \in Z_\epsilon^1$. ■

1.5 Contracting property of f^{-1}

By the construction above, h_n is one of the branches of $f^{-n} \circ h_0 \circ f_\epsilon^n$. This implies, to obtain the convergence of $\{h_n\}$ on $J(f)$, it is necessary to use some kind of contracting property of the branches of f^{-1} (in other words, some kind of expanding property of f) near the Julia set. In this section, to obtain such a property of f, we follow [16, *Step* 2-5] with brief sketches of the proofs. The idea is originally due to A. Douady and J. H. Hubbard[1, Exposé No.X].

1.5.1 Branched covering of Ω

There exists a function $v : \Omega \to \mathbb{N}$ such that $v(x)$ is the multiple of $v(y) \cdot \deg(f, y)$ for each $y \in f^{-1}(x)$. For example,

$$v(x) = \prod_{f^n(y)=x} \deg(f, y)$$

satisfies this condition. Here we take v as the function which takes minimal possible values. Note that $Z = \{x \in \Omega : v(x) \geq 2\}$.

Let O be an open δ-neighborhood of Ω with $\delta \ll 1$. Then O contains a neighborhood of each $a \in A$. For $x \in O - \Omega$, set $v(x) = 1$. Let us take an N-sheeted branched covering $q : O^* \to O$ such that:

- O^* is connected;
- there are $N/v(x)$ points over $x \in O$; and
- for any $y \in q^{-1}(x)$, $\deg(q, y) = v(x)$.

Now set $U := \mathrm{Int}(\Omega)$, $U^* := q^{-1}(U)$ and $\Omega^* := q^{-1}(\Omega)$. For U^* let us take the universal covering $\pi : \mathbb{D} \to U^*$, where $\mathbb{D}$ is the unit disk. Then we obtain a branched covering $p := q \circ \pi : \mathbb{D} \to U$.

Let Γ be the fundamental group of U^* and $\Lambda(\Gamma)$ the limit set of Γ. By lifting paths in Ω^* terminating at boundary points, we can continuously extend π to the ideal boundary, $\pi|(\partial\mathbb{D} - \Lambda(\Gamma)) \to \partial\Omega^*$. Thus we obtain a branched covering $p : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \Omega$.

Remark. For a parabolic point a of f with multiple petals, every component of $E_a - \{a\}$ defines a different access to a. For such accesses, corresponding ideal boundary points of $\partial\mathbb{D} - \Lambda(\Gamma)$ over a are distinct.

1.5.2 Lifting f^{-1}

Next, we lift f^{-1} to the branched covering $\overline{\mathbb{D}} - \Lambda(\Gamma)$ of Ω.

Proposition 1.5.1 *There is a holomorphic map $g : \mathbb{D} \to \mathbb{D}$ such that $f \circ p \circ g = p$. Moreover, g can be extended to $g : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \overline{\mathbb{D}} - \Lambda(\Gamma)$ continuously.*

Sketch of the proof. For $x \in \Omega$, we take a small disk neighborhood B_x. Let G be one of the components of $q^{-1}(B_x)$, and H that of $(f \circ q)^{-1}(B_x)$. Then there exists a unique y such that $\{y\} = f^{-1}(x) \cap q(H)$. By taking suitable local coordinates, $q|G \to B_x$ and $(f \circ q)|H \to B_x$ are represented as $z \mapsto z^{v(x)}$ and $z \mapsto z^{v(y)\deg(f,y)}$ respectively. Thus we can define the unique map $g_{GH} : G \to H$ which has the form

$$z \mapsto z^{v(x)/(v(y)\deg(f,y))}$$

as a branch of $(f \circ q)^{-1} \circ q$.

Let us fix $x_0 \in \Omega - Z$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Let η be a curve $\eta : [0,1] \to \Omega^*$ with $\eta(0) = \pi(\tilde{x}_0)$ and $\eta((0,1)) \subset U^*$, and η' be the unique lifting of η by π with $\tilde{\eta}(0) = \tilde{x}_0$. Now we consider analytic continuation of the function elements $\{g_{GH}\}$ along $\tilde{\eta}$. Let $g_{G_0H_0}$ be a function element at $\pi(\tilde{x}_0)$. Since $\overline{\mathbb{D}} - \Lambda(\Gamma)$ is simply connected, the analytic continuation of $g_{G_0H_0}$ along $\tilde{\eta}$ determines a unique function element at $\tilde{\eta}(1)$. Next, by ranging over all possible η, we obtain $g : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \overline{\mathbb{D}} - \Lambda(\Gamma)$. It is clear that $g|\mathbb{D}$ is holomorphic.■

1.5.3 The metric ρ

Proposition 1.5.2 *There exists a piecewise continuous metric ρ with the following properties:*

- *ρ is defined on $U - Z$ and small disk neighborhoods for each parabolic point of f.*
- *For every C^1 curve $\eta \subset f^{-1}(\Omega) = \Omega^1$,*

$$\mathrm{length}_\rho(f \circ \eta) > \mathrm{length}_\rho(\eta).$$

So f is expanding for ρ in the sense of this inequality.

Sketch of the proof. Let $\rho_0 = u_0(z)|dz|$ be a metric of $U - Z$ induced from the Poincaré metric of $\mathbb{D}$ by the branched covering $p : \mathbb{D} \to U$. Note that $u_0(z) \asymp |z - b|^{-1+1/v(b)}$ near $b \in Z$. Thus any rectifiable curve $\eta : [0,1] \to U$ passing through Z has finite length with respect to ρ_0.

However, any curve in $f^{-1}(\Omega)$ terminating at A has infinite length with respect to ρ_0. So we try to modify ρ_0 so that such a curve has finite length.

For a sufficiently small $\delta > 0$ and for each $a \in A$, set $\mathcal{D}_a := B_\sigma(a,\delta)$ and $\mathcal{D} := \bigcup_{a \in A} \mathcal{D}_a$. Note that $\Omega \cap \mathcal{D}$ is a finite union of narrow cusps near the repelling directions. Thus on each $\mathcal{D}_a$, we can take a suitable local coordinate ζ_a such that f is strictly expanding from the metric $|d\zeta_a|$ to the metric $|d\zeta_{f(a)}|$ on any compact subset of $f^{-1}(\Omega \cap \mathcal{D}_{f(a)}) \cap \mathcal{D}_a - \{a\}$. Furthermore, we take a sufficiently large $M > 0$ so that for any $a \in A$, f is expanding from ρ_0 to $M|d\zeta_a|$ on a relatively compact set $f^{-1}(\Omega \cap \mathcal{D}_a - Z) - \mathcal{D}$. Set $u_a(z)|dz| := |d\zeta_a|$. Then we define the metric $\rho = u(z)|dz|$ on $U \cup \mathcal{D} - Z$ by $u(z) := \min\{u_0(z), Mu_a(z)\}$ for $z \in \mathcal{D}_a$, and by $u(z) := u_0(z)$ otherwise.

By construction, it is not difficult to show

$$u(f(z))|f'(z)| > u(z)$$

for $z \in f^{-1}(\Omega - Z) - A$. This implies

$$\text{length}_\rho(f \circ \eta) > \text{length}_\rho(\eta).$$

for every C^1 curve $\eta \subset f^{-1}(\Omega)$. ■

1.5.4 Continuous modulus

Let $\tilde{\rho}$ be the lifting of ρ on $p^{-1}(U - Z)$. Since $f^{-1}(\Omega) = \Omega^1$ has one or more connected components, $p^{-1}(\Omega^1)$ is either connected or has countably many connected components. Take one of the components of $p^{-1}(\Omega^1)$, say Q, and take $x,\ y \in Q$. We define the distance by

$$d_{\tilde{\rho}}(x, y) := \inf_{\tilde{\eta}} \text{length}_{\tilde{\rho}}(\tilde{\eta}),$$

where $\tilde{\eta}$ ranges over all rectifiable curves such that

$$\tilde{\eta} : [0, 1] \to p^{-1}(\Omega^1),\ \tilde{\eta}(0) = x,\ \text{and } \tilde{\eta}(1) = y.$$

Note that such $\tilde{\eta}$ has finite length with respect to $\tilde{\rho}$. Now $(Q, d_{\tilde{\rho}})$ is a complete metric space. For different components Q and Q' of $p^{-1}(\Omega^1)$, we formally define $d_{\tilde{\rho}}(x, y) := \infty$ if $x \in Q$ and $y \in Q'$.

For g, a lifting of f^{-1}, we define a function $\tau_g : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau_g(s) := \sup\left\{d_{\tilde{\rho}}(g(x), g(y)) : x,\ y \in p^{-1}(\Omega^1),\ d_{\tilde{\rho}}(x, y) \le s\right\}.$$

Furthermore, we define $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau(s) := \sup\left\{\tau_g(s) : g \text{ a lifting of } f^{-1}\right\}.$$

Then we obtain:

Proposition 1.5.3 *τ has the following properties:*

(*i*) *τ is an increasing and right-continuous function;*

(*ii*) *$s > \tau(s)$ for any s;*

(*iii*) *the function $s \mapsto s - \tau(s)$ is also increasing; and*

(*iv*) *For any $x,\ y \in p^{-1}(\Omega^1)$ and any lifting g of f^{-1},*

$$d_{\tilde{\rho}}(g(x), g(y)) \le \tau(d_{\tilde{\rho}}(x, y)).$$

Sketch of the proof. If we replace τ by τ_g, then (i), (ii) and (iv) are almost clear by definition. (iii) follows from the fact that $\tau_g(s_1+s_2) \leq \tau_g(s_1)+\tau_g(s_2)$. A calculation shows that there exist d distinct liftings of f^{-1}, say $g_1, \ldots, g_d$, such that any τ_g coincide with one of $\tau_{g_1}, \ldots, \tau_{g_d}$. Thus

$$\tau(s) = \sup\{\tau_{g_i}(s) : 1 \leq i \leq d\},$$

and satisfies properties (i)-(iv). ■

1.6 Convergence of h_n

In this section, we give the proof of the convergence of the sequence $\{h_n : \Omega^n_\epsilon \to \Omega^n\}_{n=0}^{\infty}$. Here the expanding property of f with respect to ρ plays an important role. For instance, we can easily show the convergence when f is hyperbolic:

Proposition 1.6.1 *Suppose that f is hyperbolic. For $\epsilon \ll 1$, the sequence h_n converges uniformly to the limit h_ϵ on $J(f_\epsilon)$ which satisfies $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$.*

Proof. Since f has no parabolic point nor critical point in $J(f)$, the metric ρ in Proposition 1.5.2 is the Poincaré metric on U. Now $\Omega^1 \subset U$ thus there is a constant C such that $f^*\rho/\rho \geq C > 1$ on Ω^1.

Note that the constant

$$M := \sup\left\{d_\rho(h_0(x), h_1(x)) : x \in \Omega^1_\epsilon\right\}$$

is finite since $h_0(\Omega^1_\epsilon) \subset U$. For any $x \in \Omega^2_\epsilon$, we obtain

$$\begin{aligned} &C d_\rho(h_1(x), h_2(x)) \\ \leq& d_\rho(f(h_1(x)), f(h_2(x))) = d_\rho(h_0(f_\epsilon(x)), h_1(f_\epsilon(x))) \\ \leq& M, \end{aligned}$$

thus $d_\rho(h_1(x), h_2(x)) \leq M/C$. Similarly, for any $x \in J(f_\epsilon)$, we obtain

$$d_\rho(h_n(x), h_{n+1}(x)) \leq M/C^n \to 0 \quad (n \to \infty).$$

(Recall that $J(f_\epsilon) \subset \Omega^n_\epsilon$ and thus $h_n|J(f_\epsilon)$ are defined for any $n \geq 0$.) Hence h_n converges uniformly and rapidly to the limit h_ϵ on $J(f_\epsilon)$. The relation $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$ follows from $f \circ h_{n+1} = h_n \circ f_\epsilon$. ■

Let us consider the general case. When f has parabolic points, it is not uniformly expanding on Ω^1. However, since it is uniformly expanding on each compact subset of Ω^1 with respect to the metric ρ, h_n converges slowly to the limit:

Proposition 1.6.2 *For $\epsilon \ll 1$, the sequence h_n converges uniformly to the limit h_ϵ on $J(f_\epsilon)$ which satisfies $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$. Moreover, h_ϵ can be arbitrarily close to the identity map: That is, for arbitrarily small $r > 0$, if $\epsilon \ll 1$, h_ϵ satisfies*

$$\sup\{d_\sigma(h_\epsilon(x), x) : x \in J(f_\epsilon)\} < r.$$

Proof. Let us fix an arbitrary $L > 0$. Then we may assume that

$$d_\rho(h_0(x), h_1(x)) < L - \tau(L)$$

for any $x \in J(f_\epsilon)$. In fact, by the construction of h_0 and h_1, if $\epsilon \ll 1$, $d_\rho(h_0(x), h_1(x))$ can be arbitrarily small for any $x \in J(f_\epsilon)$.

We claim that $d_\rho(h_0(x), h_n(x)) < L$ for any $n \geq 1$ and any $x \in J(f_\epsilon)$. If $n = 1$, $d_\rho(h_0(x), h_1(x)) < L - \tau(L) < L$. For $n = k$, let us assume that $d_\rho(h_0(x), h_k(x)) < L$ for any $x \in J(f_\epsilon)$. We first show that

$$d_\rho(h_1(x), h_{k+1}(x)) < \tau(L).$$

By assumption, we can take a rectifiable curve $\eta : [0, 1] \to \Omega^1$ such that

- $\eta(0) = h_0(f_\epsilon(x))$ and $\eta(1) = h_k(f_\epsilon(x))$;
- $\eta \cap Z = \emptyset$; and
- $L > \mathrm{length}_\rho(\eta)$.

Fix $z_0 \in p^{-1}(h_0(f_\epsilon(x)))$, and let $\tilde{\eta}$ be the lifting of η by p whose initial point is z_0. Then the end point over $h_k(f_\epsilon(x))$ is uniquely determined, say z_1, and

$$\begin{aligned} L &> \mathrm{length}_\rho(\eta) = \mathrm{length}_{\tilde{\rho}}(\tilde{\eta}) \\ &> d_{\tilde{\rho}}(z_0, z_1). \end{aligned}$$

By using the function τ,

$$\tau(L) > \tau(d_{\tilde{\rho}}(z_0, z_1)) \geq d_{\tilde{\rho}}(g(z_0), g(z_1)),$$

where g is a lifting of f^{-1} such that $p \circ g(z_0) = h_1(x)$. Then we can take a curve $\tilde{\eta}' : [0, 1] \to \overline{\mathbb{D}} - \Lambda(\mathit{\Gamma})$ such that

- $\tilde{\eta}'(0) = g(z_0)$ and $\tilde{\eta}'(1) = g(z_1)$;
- $\tilde{\eta}' \cap p^{-1}(Z) = \emptyset$; and
- $\tau(L) > \mathrm{length}_{\tilde{\rho}}(\tilde{\eta}')$.

Hence

$$\begin{aligned} \tau(L) &> \mathrm{length}_{\tilde{\rho}}(\tilde{\eta}') = \mathrm{length}_\rho(p \circ \tilde{\eta}') \\ &> d_\rho(p(g(z_0)), p(g(z_1))) = d_\rho(h_1(x), h_{k+1}(x)). \end{aligned}$$

Then for $n = k + 1$ and for any $x \in J(f_\epsilon)$,

$$\begin{aligned} d_\rho(h_0(x), h_{k+1}(x)) &\leq d_\rho(h_0(x), h_1(x)) + d_\rho(h_1(x), h_{k+1}(x)) \\ &< L - \tau(L) + \tau(L) = L. \end{aligned}$$

Thus we have shown the claim by induction on n.

Let us show the convergence. By the same argument as above, for sufficiently large integer $l,\ m$,

$$\begin{aligned} d_\rho(h_l(x), h_{m+l}(x)) &< \tau^l\big(d_\rho(h_0(f_\epsilon^l(x)), h_m(f_\epsilon^l(x)))\big) \\ &< \tau^l(L) \to 0 \qquad (l \to \infty). \end{aligned}$$

Because we can take arbitrary $x \in J(f_\epsilon)$, h_n converges uniformly on $J(f_\epsilon)$ with respect to the distance d_ρ. Since the topology of Ω^n defined by d_ρ is equivalent to the topology defined by the spherical distance d_σ, h_n also converges uniformly on $J(f_\epsilon)$ with respect to d_σ. By continuity of each h_n, the limit h_ϵ is also continuous. The relation $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$ follows from $f \circ h_{n+1} = h_n \circ f_\epsilon$.

Finally we show the last part of the statement. Let us fix any $r > 0$ and suppose that $\epsilon \ll 1$. Then we can take h_0 such that $d_\sigma(x, h_0(x)) < r/2$ for any $x \in J(f_\epsilon)$. On the other hand, by the claim above, we obtain $d_\rho(h_0(x), h_\epsilon(x)) \leq L$ for arbitrarily small L. Since we may also suppose that L is sufficiently small such that $d_\sigma(h_0(x), h_\epsilon(x)) < r/2$ for any $x \in J(f_\epsilon)$, we obtain

$$d_\sigma(x, h_\epsilon(x)) \leq d_\sigma(x, h_0(x)) + d_\sigma(h_0(x), h_\epsilon(x)) < r.$$

■

1.7 Almost bijectivity and uniqueness of h_ϵ

In this section, we prove that the continuous map h_ϵ in Proposition 1.6.2 maps $J(f_\epsilon)$ onto $J(f)$ "almost bijectively"; that is, there are at most countably many points in $J(f)$ where h_ϵ is not one-to-one. Furthermore we prove the uniqueness of such an h_ϵ.

First we show:

Proposition 1.7.1 *h_ϵ maps $J(f_\epsilon)$ to $J(f)$.*

Proof. Let X denote the set of all repelling periodic points of f_ϵ. Since $h_\epsilon \circ f_\epsilon^n = f^n \circ h_\epsilon$ for any n, h_ϵ maps X to a set of periodic points of f in Ω, which must be a subset of $J(f)$. Since h_ϵ is continuous and $J(f_\epsilon) = \overline{X}$, h_ϵ maps $J(f_\epsilon)$ into $J(f)$. ■

Next, we complete the proof of Theorem 1.1.1 under the assumption that $J(f) \neq \hat{\mathbb{C}}$. For fixed ϵ, let $A_- = A_{-,\epsilon} \subset A$ be the set of all parabolic points of f which are perturbed into attracting planets of f_ϵ.

Proposition 1.7.2 *If $\epsilon \ll 1$, $h_\epsilon : J(f_\epsilon) \to J(f)$ has the following properties:*

- *(Surjectivity) h_ϵ is surjective.*

- *(Almost injectivity) If $h_\epsilon(x) = h_\epsilon(x')$ for distinct $x,\ x' \in J(f_\epsilon)$, then there exists an integer N such that $f_\epsilon^N(x)$ and $f_\epsilon^N(x')$ are repelling satellites of an attracting planet a_ϵ generated by the perturbation of a point in A_-.*

- *(Uniqueness) h_ϵ is the unique semiconjugacy between f_ϵ and f on their respective Julia sets which satisfies properties 1 and 2 in Theorem 1.1.1.*

By the almost injectivity above, we obtain the precise condition for h_ϵ to be a topological conjugacy.

Corollary 1.7.3 *h_ϵ is a topological conjugacy if and only if $A_- = \emptyset$; that is, none of the parabolic points of f is perturbed into an attracting planet.*

Proof of Proposition 1.7.2: Surjectivity. Fix any $y \in J(f)$. By surjectivity of h_n, there is a sequence $x_n \in \Omega_\epsilon^n \subset \Omega_\epsilon$ such that $h_n(x_n) = y$. Since Ω_ϵ is compact, $\{x_n\}$ has an accumulation point $x \in \Omega_\epsilon$ and we can choose a subsequence x_{n_k} so that $x_{n_k} \to x$ $(k \to \infty)$. Now we claim that $x \in J(f_\epsilon)$. If $x \in F(f_\epsilon)$, $f_\epsilon^n(x)$ is attracted to an attracting or parabolic cycle as $n \to \infty$. Thus there exists an N and a small disk neighborhood D such that $f_\epsilon^n(D)$ is outside of Ω_ϵ for all $n \geq N$. On the other hand, for all $k \gg 0$, we have $n_k \geq N$, $x_{n_k} \in D$, and $f_\epsilon^{n_k}(x_{n_k}) \in \Omega_\epsilon$. This is a contradiction.

Since $h_n \to h_\epsilon$ uniformly and the family $\{h_n\}$ is clearly equicontinuous, the inequality

$$d_\rho(y, h_\epsilon(x)) \leq d_\rho(h_{n_k}(x_{n_k}), h_{n_k}(x)) + d_\rho(h_{n_k}(x), h_\epsilon(x))$$

implies $y = h_\epsilon(x)$. Thus h_ϵ is surjective.

Preliminary to the almost injectivity and uniqueness. Since f is geometrically finite and the assumption that $J(f) \neq \hat{\mathbb{C}}$, f has at least one critical point in the Fatou set, and so does f_ϵ. Now we take suitable conjugations of $f_\epsilon \to f_0 = f$ by rotations of $\hat{\mathbb{C}}$ so that $\infty \in C(f_\epsilon) \cap F(f_\epsilon)$. By the construction of Ω_ϵ, there exist $R \gg 0$ such that $D(R) := \hat{\mathbb{C}} - \{|z| \leq R\}$ is a disk neighborhood of ∞ which is not contained in Ω_ϵ for all $0 \leq \epsilon \ll 1$. Then Ω_ϵ and $J(f_\epsilon)$ are bounded sets in the complex plane.

For $\delta > 0$ and $x \in \mathbb{C}$, we set

$$B(x, \delta) := \{z \in \mathbb{C} : |z - x| < \delta\},$$

which is an open Euclidean ball. Now we fix δ to be sufficiently small so that the set

$$\mathcal{B} := \bigcup_{x \in A \cup Z^1} B(x, \delta)$$

is a disjoint union of balls satisfying the following conditions:

- if an $x \in A \cup Z^1$ is periodic, then there exists a local chart on $B(x, \delta)$ as (2.2) or (2.4); and

- for $x \in Z^1 - A$, $P(f) \cap B(x, \delta) = \{x\}$.

Set $\tilde{s} := d(P(f), J(f) - \mathcal{B})$, where $d(\cdot, \cdot)$ is the distance between sets measured by Euclidean distance. Since f is geometrically finite, every critical orbit either accumulates on an attracting or parabolic cycle, or is already contained in Z^1. Hence we obtain $0 < \tilde{s} \le \delta$.

Now we claim that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$ for all $\epsilon \ll 1$. It suffices to restrict our attention to the perturbation of the critical orbits accumulating on A or Z^1. First, take a parabolic cycle $\alpha \subset A$ and a critical orbit accumulating to α. By horocyclicity of $f_\epsilon \to f$, we may apply Lemma 1.2.2. That is, for $\epsilon \ll 1$, the corresponding perturbed critical orbit of f_ϵ is contained in $\cup_{a \in \alpha} B(a, \delta) \subset \mathcal{B}$ except finitely many points in the orbit. Since $f_\epsilon \to f$ uniformly, such finitely many points are very close to the original ones. On the other hand, $J(f_\epsilon)$ is very close to $J(f)$ with respect to the Hausdorff topology, since h_ϵ maps $J(f_\epsilon)$ onto $J(f)$ and r-neighborhood of $J(f)$ with respect to the spherical distance contains $h_\epsilon^{-1}(J(f)) = J(f_\epsilon)$. (Recall that r is fixed and arbitrarily small for $\epsilon \ll 1$.) Thus such finitely many points stay away from $J(f_\epsilon) - \mathcal{B}$ for $\epsilon \ll 1$, and the distance can be at least $\tilde{s}/2$. Next, take $b \in Z^1$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, for all $\epsilon \ll 1$, we may suppose that there exists a unique $b_\epsilon \in f_\epsilon^{-1}(P(f_\epsilon))$ such that $|b - b_\epsilon| < \delta/2$. For such b_ϵ, $d(b_\epsilon, J(f_\epsilon) - \mathcal{B}) \ge \delta/2 \ge \tilde{s}/2$. Thus we conclude the claim.

Replacing f_ϵ (resp. f) by its suitable iteration, we may consider the extreme case where every point in $h_0^{-1}(A) \cup Z_\epsilon$ (resp. $A \cup Z$) is a fixed point of f_ϵ (resp. f), and the multipliers of all parabolic points are 1. Then Z_ϵ and Z are the sets of all critical values of f_ϵ and f on their respective Julia sets.

Set $\Gamma_- = \Gamma_{-,\epsilon} := h_0^{-1}(A_-)$, the set of all repelling satellites generated by the perturbation of parabolic points in A_-. Note that now every element in A_- or Γ_- is a fixed point of f or f_ϵ respectively. Also, note that Γ_- and Z_ϵ are disjoint.

Almost injectivity. Now let us start the discussion on the almost injectivity of h_ϵ. We suppose that $h_\epsilon(x) = h_\epsilon(x')$ for distinct $x,\ x' \in J(f_\epsilon)$. Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. Then $h_\epsilon(x_n) = h_\epsilon(x'_n)$ because $f^n \circ h_\epsilon = h_\epsilon \circ f_\epsilon^n$. Recall that $d_\sigma(x, h_\epsilon(x)) < r$ for any $x \in J(f_\epsilon)$. Thus we obtain

$$d_\sigma(x_n, x'_n) \le d_\sigma(x_n, h_\epsilon(x_n)) + d_\sigma(h_\epsilon(x'_n), x'_n) < 2r$$

and it implies $|x_n - x'_n| = O(r)$. Indeed, since the Julia set is contained in $\hat{\mathbb{C}} - D(R)$, there exists a constant $M \approx 1 + R^2$ such that $|x_n - x'_n| \le Mr$ for sufficiently small r. Now we set

$$\tilde{r} := \sup_n |x_n - x'_n| \ \ (\le Mr).$$

Then we may suppose that r is sufficiently small such that $\tilde{r} \leq Mr < \tilde{s}/2$ for $\epsilon \ll 1$. Note that $\tilde{r} \leq Mr < \delta/2$ also holds.

For the orbit of the x and x', we consider the following three cases:

1. Both x_n and x'_n land on Γ_-.

2. x_n lands on Γ_- but x'_n never lands on Γ_-.

3. Both x_n and x'_n never land on Γ_-.

Case 1: Suppose that x_n lands on $h_\epsilon^{-1}(a)$ for some $a \in A_-$ when $n = N$. Here $h_\epsilon^{-1}(a) \subset \Gamma_-$ is a set of repelling fixed points contained in $B_\sigma(a, r)$. By the facts that

$$B_\sigma(a, r) \subset B(a, Mr) \subset B(a, \delta/2)$$

and $\tilde{r} < \delta/2$, x'_n must be contained in $B(a, \delta)$ for all $n \geq N$. If $x'_N \notin h_\epsilon^{-1}(a)$, by the local dynamics of f_ϵ on $B(a, \delta)$ in the form (2.4), x'_n goes out of $B(a, \delta)$. Thus $x'_N \in h_\epsilon^{-1}(a)$; that is, x_n and x'_n simultaneously land on repelling satellites in $h_\epsilon^{-1}(a)$, when $n = N$.

Hence we need to show that the other cases cannot occur.

Case 2: We suppose again that x_n lands on $h_\epsilon^{-1}(a)$ for some $a \in A_-$ when $n = N$. By the same argument as Case 1, x'_n must be contained in $B(a, \delta)$ for all $n \geq N$. However, $x'_n \notin h_\epsilon^{-1}(a) \subset \Gamma_-$, and thus by the local dynamics of f_ϵ on $B(a, \delta)$ in the form (2.4), x'_n goes out of $B(a, \delta)$. This is a contradiction.

Case 3: Furthermore we need to consider the following three cases:

I. x_n lands on Z_ϵ but x'_n never lands on Z_ϵ.

II. Both x_n and x'_n land on Z_ϵ.

III. Both x_n and x'_n never land on Z_ϵ.

Case 3-I: Suppose that x_n lands on $h_\epsilon^{-1}(b)$ for some $b \in Z$ when $n = N$. Here $h_\epsilon^{-1}(b) \subset Z_\epsilon$ is a repelling or parabolic fixed point of f_ϵ contained in $B_\sigma(b, r)$. By the same argument as above, x'_n must be contained in $B(b, \delta)$ for $n \geq N$. Now x'_n never lands on Z_ϵ. This implies, by the local dynamics of f_ϵ on $B(b, \delta)$ in the form (2.2) or (2.4), x'_n goes out of $B(b, \delta)$. This is also a contradiction.

Case 3-II: Since $\tilde{r} < \delta/2$ and all elements of Z_ϵ^1 remain at leaset δ apart, the orbits of x and x' have merged before landing on Z_ϵ: That is, there exist two integers N_1 and N_2 with $N_1 < N_2$ such that

- $x_{N_1} \neq x'_{N_1}$ and $x_{N_1+1} = x'_{N_1+1}$, and

- $x_{N_2} = x'_{N_2} \in Z_\epsilon^1$ and $x_{N_2+1} = x'_{N_2+1} \in Z_\epsilon$.

Set $w := x_{N_1+1} = x'_{N_1+1}$. Since w is not contained in Z_ϵ, which is the set of critical values, the inverse image $f_\epsilon^{-1}(w)$ consists of d distinct points. (Recall that d is the degree of f.) Similarly, by the construction of h_ϵ, $h_\epsilon(w) =: z$ is not contained in Z and $f^{-1}(z)$ also consists of d distinct points. Moreover, since h_ϵ is surjective, $h_\epsilon^{-1}(f^{-1}(z))$ must consist of at least d points.

Note that $f_\epsilon^{-1}(w) \subset h_\epsilon^{-1}(f^{-1}(z))$. Since $h_\epsilon(x_{N_1}) = h_\epsilon(x'_{N_1})$ for distinct x_{N_1}, $x'_{N_1} \in f_\epsilon^{-1}(w)$, there exists an $x'' \in h_\epsilon^{-1}(f^{-1}(z)) - f_\epsilon^{-1}(w)$ which satisfies $f_\epsilon(x'') \neq w$ and $h_\epsilon(f_\epsilon(x'')) = z$. Setting $w' := f_\epsilon(x'')$, we obtain $h_\epsilon(w) = h_\epsilon(w')$ for $w \neq w'$. Let us replace x and x' by w and w' respectively. This reduces Case 3-II with $x_{N_2} \in Z_\epsilon^1$ to Case 3-I or 3-II with $x_{N_2-N_1-1} \in Z_\epsilon^1$.

However, as we have seen, Case 3-I implies a contradiction. In Case 3-II, we can repeat the argument above. Hence we eventually consider the case where $h_\epsilon(x) = h_\epsilon(x')$ for $x \neq x'$ with $x \in Z_\epsilon^1$.

Suppose that $f_\epsilon(x) = h_\epsilon^{-1}(b)$ for some $b \in Z$. Then $f_\epsilon(x)$ is contained in $B_\sigma(b, r)$ and is a repelling or parabolic fixed point. On the other hand, since the elements of Z_ϵ^1 remain separated, $x \neq x'$ implies $x' \notin Z_\epsilon^1$, and thus $f_\epsilon(x') \notin Z_\epsilon$. By the local dynamics of f_ϵ on $B(b, \delta)$ in the form (2.2) or (2.4), $f_\epsilon(x')$ is not a fixed point and goes out of $B(b, \delta)$. This is a contradiction.

Case 3-III: If either x_n or x'_n lands in $\mathcal{B}$, it goes out of $\mathcal{B}$ by finitely many iterations of f_ϵ. Now we take a subsequence $\{n_k\}$ of $\{n\}$ so that each x_{n_k} is never contained in $\mathcal{B}$; that is, $x_{n_k} \in J(f_\epsilon) - \mathcal{B}$. Recall that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$. For any s satisfying $\tilde{r} < s < \tilde{s}/2$ and for any $k \gg 0$, there exists a branch g_{n_k} of $f_\epsilon^{-n_k}$ on $B(x_{n_k}, s)$ which is univalent and $g_{n_k}(x_{n_k}) = x$. Set $V_{n_k} := g_{n_k}(B(x_{n_k}, \tilde{r}))$. Then V_{n_k} contains x and x'. By applying the Koebe distortion theorem to g_{n_k} on $B(x_{n_k}, s)$, we obtain

$$\operatorname{diam} V_{n_k} = O(|g'_{n_k}(x_{n_k})|) = O(1/|(f_\epsilon^{n_k})'(x)|).$$

If $|P(f_\epsilon)| < 3$, f_ϵ is conjugate to $z \mapsto z^{\pm d}$, and thus it is hyperbolic. On the Julia set, $|(f_\epsilon^{n_k})'(x)| \to \infty$ as $k \to \infty$ hence $\lim(\operatorname{diam} V_{n_k}) = 0$. It contradicts $x \neq x'$.

If $|P(f_\epsilon)| \geq 3$, let ρ_ϵ be the Poincaré metric of $\hat{\mathbb{C}} - P(f_\epsilon)$. By [13, Theorem 3.6], since $x_n \notin P(f_\epsilon)$ for any n, we obtain

$$\|(f_\epsilon^n)'(x)\|_{\rho_\epsilon} = \frac{\rho_\epsilon(f_\epsilon^n(x))|(f_\epsilon^n)'(x)|}{\rho_\epsilon(x)} \to \infty \quad (n \to \infty).$$

Now recall again that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$. Since x_{n_k} and x stay away from $P(f_\epsilon)$, $\rho_\epsilon(f_\epsilon^{n_k}(x))$ and $\rho_\epsilon(x)$ are bounded. Hence $|(f_\epsilon^{n_k})'(x)| \to \infty$ as $k \to \infty$, then $\lim(\operatorname{diam} V_{n_k}) = 0$. It contradicts $x \neq x'$ again.

Uniqueness. From Proposition 1.6.2, Proposition 1.7.1 and the proof of the almost bijectivity above, it is easy to check that h_ϵ satisfies properties in Theorem 1.1.1. In particular, we obtain property 1 in Theorem 1.1.1 from the almost injectivity discussed above and property $(2, n)$ of h_n in Proposition 1.4.1.

Let h'_ϵ be another semiconjugacy between f_ϵ and f on their respective Julia sets with properties 1 and 2 in Theorem 1.1.1. Take a repelling periodic point x of f_ϵ which has period more than one. By our assumption that $h_\epsilon^{-1}(A) \cup Z_\epsilon$ is a set of fixed points, x does not belong to $\Gamma_- \cup Z_\epsilon$. By surjectivity of h'_ϵ, there exists an $x' \in J(f_\epsilon)$ such that

$$h_\epsilon(x) = h'_\epsilon(x').$$

It is easy to see that $h_\epsilon(x)$ and x' are also repelling periodic points with the same period as x.

Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. Then $h_\epsilon(x_n) = h_\epsilon(x'_n)$ because h_ϵ and h'_ϵ are semiconjugacies. Moreover, we obtain $d_\sigma(x_n, x'_n) < 2r$ from property 2 in Theorem 1.1.1. Thus

$$|x_n - x'_n| \leq Mr < \delta/2$$

for all n and we may suppose that x'_n belongs to $\Gamma_- \cup Z_\epsilon$ as well as x_n.

Now we can apply the same argument as Case 3-III of the proof of the almost injectivity, and we conclude that $x = x'$. This means that $h_\epsilon = h'_\epsilon$ on the dense subset of $J(f_\epsilon)$, because repelling periodic points are dense in the Julia set. Since h_ϵ and h'_ϵ are continuous, h'_ϵ must coincide with h_ϵ on $J(f_\epsilon)$. ■

1.8 Geometrically finite maps with the empty Fatou set

In this section, we prove Theorem 1.1.1 for a geometrically finite rational map f with $J(f) = \hat{\mathbb{C}}$ by using the same idea as in the case of $J(f) \neq \hat{\mathbb{C}}$.

Now f has no parabolic or (super)attracting periodic point. Moreover, by the geometric finiteness, every critical point of f is preperiodic; that is, f is postcritically finite. Then we can consider the orbifold $\mathcal{O}_f$ with base space $\hat{\mathbb{C}}$ which is parabolic or hyperbolic type[13, §A]. This $\mathcal{O}_f$ has an orbifold metric $\rho = \rho(z)|dz|$ which is induced from the Euclidean or hyperbolic metric of the universal covering. In both cases, there exists a constant $C > 1$ such that

$$\|f'\|_\rho := \frac{f^*\rho}{\rho} \geq C.$$

(See the argument in [13, Theorem A.6]). Note that ρ has singularity at $b \in P(f)$ as $|d(z-b)^{1/v(b)}|$.

Let us consider a horocyclic perturbation $f_\epsilon \to f$ preserving the J-critical relations of f. Since f has no parabolic point, horocyclicity is trivial. By the

J-critical relations of f, f_ϵ is also postcritically finite. Since f has no attracting or superattracting periodic point, f_ϵ has no superattracting periodic point: This implies $J(f_\epsilon)$ is also the whole sphere (See [13, Theorem A.6] again).

Now let us begin the construction of h_ϵ.

Proof of Theorem 1.1.1 in the case of $J(f) = \hat{\mathbb{C}}$**.** First, set $\Omega := \hat{\mathbb{C}}$ and $\Omega_\epsilon := \hat{\mathbb{C}}$. We take $h_0 : \Omega_\epsilon \to \Omega$ as a homeomorphism which satisfies condition 5 of Proposition 1.3.2. For any fixed $r > 0$, if $\epsilon \ll 1$, such h_0 satisfies $d_\sigma(h_0(x), x) < r$ for all $x \in \hat{\mathbb{C}}$.

Next, we lift h_0 to the family of homeomorphism $\{h_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}_{n=1}^{\infty}$ as in Proposition 1.4.1. We can show that h_n converges to the limit h_ϵ in the same way as Proposition 1.6.1. In fact, we may replace the Poincaré metric in the proof of Proposition 1.6.1 with the orbifold metric ρ of $\mathcal{O}_f$. Furthermore, we can also lift h_0^{-1} to the uniformly convergent sequence of homeomorphisms $\{h_n^{-1}\}$. The limit must be surjective and thus $h_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism.

Finally, we show the uniqueness in the same way as Proposition 1.7.2: Let h'_ϵ be another conjugacy with property 2 in Theorem 1.1.1, and x be a repelling periodic point of f_ϵ which does not belong to $P(f)$. Since h'_ϵ is a homeomorphism, there exists a unique x' such that $h_\epsilon(x) = h'_\epsilon(x')$. Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. By using the uniformly expanding property of f_ϵ with respect to the orbifold metric ρ_ϵ of $\mathcal{O}_{f_\epsilon}$, $d_{\rho_\epsilon}(x, x')$ is bounded by $d_{\rho_\epsilon}(x_n, x'_n)/C_\epsilon^n$ with $C_\epsilon > 1$. This implies $x = x'$. Thus $h_\epsilon = h'_\epsilon$ on a dense subset of the sphere, which is a set of repelling periodic points. By continuity of h_ϵ and h'_ϵ, we obtain $h_\epsilon = h'_\epsilon$ on the whole sphere. ■

Remark. If the orbifold $\mathcal{O}_f$ does *not* have signature $(2, 2, 2, 2)$, by Thurston's theorem([5], [13, Theorem B.2]), h_ϵ is a Möbius transformation which conjugates f_ϵ to f. Here we gave a general construction of the conjugacy h_ϵ including such a particular case of signature (2,2,2,2).

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Chapter 2

Regular leaf spaces of parabolic quadratic polynomials

2.1 Introduction

As an analogy to hyperbolic 3-orbifolds associated with Kleinian groups, Lyubich and Minsky[3] introduced hyperbolic orbifold 3-laminations associated with rational maps. For a given rational map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of degree ≥ 2, considering its natural extension $\mathcal{N}_f$ and regular leaf space $\mathcal{R}_f$ is the first step to the construction of such a hyperbolic orbifold 3-lamination. The natural extension $\mathcal{N}_f$ is the set of all backward orbits ("history") of the dynamics. The regular leaf space $\mathcal{R}_f$ is an analytically well behaved part of $\mathcal{N}_f$. The leaves of $\mathcal{R}_f$ are Riemann surfaces and the natural lift $\hat{f}$ of f acts leafwise isomorphically.

However, the global structures of the regular leaf spaces of rational maps are not precisely known except only a few examples. Here are some of such examples. For $f_c(z) = z^2 + c$ with c in the main cardioid of the Mandelbrot set, all regular leaf spaces of f_c are topologically the same as that of $f_0(z) = z^2$, which is 2-dimensional extension of 2-adic solenoid[4, Example 2][3, §11].

In [2], the author introduced the method of *tessellation* for f_c with $c \in (0, 1/4]$ and describe the structure of the regular leaf space of $f_{1/4}$ as a degeneration of that of f_c with $c \in (0, 1/4)$. Such an f_c and $f_{1/4}$ have topologically the same dynamics on and outside the Julia sets, and thus their natural extentions have topologically the same parts. Such a part of $\mathcal{N}_{f_{1/4}}$ contains the backward orbit staying at the parabolic fixed point on the Julia set. The intriguing fact is, the backward orbit is *not* in $\mathcal{R}_{f_{1/4}}$, while corresponding backward orbit in $\mathcal{N}_{f_c}$ staying at the repelling fixed point on the Julia set is in $\mathcal{R}_{f_c}$. To describe this phenomenon, we need to investigate the degeneration of the dynamics inside the Julia sets. The tessellation is defined for the interiors of the filled Julia sets and works like external rays of the dynamics outside the Julia sets. Then we obtain a precise description of the degeneration and we can lift it to their natural extentions. Now we have a clear picture of the phenomenon.

In this chapter, we develop the method of tessellation to treat the case where f_c has a parabolic fixed point of multiple petals. In §2, we survey some of basic notion on the dynamics of quadratic polynomials. In §3, we show a fundamental lemma which is necessary for the definition of tessellation. The tessellation for a quadratic polynomial with an attracting or parabolic fixed point is defined in §4.

In §5, we construct a semiconjugacy $H : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from a hyperbolic $f = f_c$ to a parabolic $g = f_\sigma$, by gluing tile-to-tile homeomorphisms and the topological conjugacy outside the Julia sets induced from Böttcher coordinates. Then we have the precise description of the degeneration of the dynamics.

In §6, we first survey the basics of natural extensions and regular leaf spaces. By lifting the semiconjugacy H above to $\hat{H} : \mathcal{N}_f \to \mathcal{N}_g$, we describe how the regular leaf space degenerates, in detail. The significant degeneration happens only on the periodic leaves corresponding to the repelling directions of the parabolic fixed point of g. We construct an analytic model of these degenerating periodic leaves.

In §7, we apply the method of tessellation to some quadratic polynomials with attracting cycles.

2.2 Dynamics of quadratic polynomials

In this section we first recall some basic facts on the dynamics of quadratic polynomials on the Riemann sphere.

2.2.1 Douady-Hubbard theory of quadratic polynomials

In [1], Douady and Hubbard developed the theory of complex polynomial dynamics. Here we survey some basic results and notions used throughout this chapter.

The Julia set. Let us set $f(z) = f_c(z) = z^2 + c$ $(c \in \mathbb{C})$ and consider it as a rational map on the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $f(\infty) = \infty$. The *filled Julia set* K_f of f is defined by

$$K_f := \left\{ z \in \bar{\mathbb{C}} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded} \right\}.$$

The *Julia set* J_f of f is the boundary of K_f. One can easily check that those sets are forward and backward invariant under the action of f.

Böttcher coordinate and external rays. Now suppose that K_f is connected. (Thus so is J_f.) We denote the unit disk by $\mathbb{D}$. For the outside of K_f, there exists a unique conformal map $\phi_f : \bar{\mathbb{C}} - K_f \to \bar{\mathbb{C}} - \bar{\mathbb{D}}$ such that

- $\phi_f(f_c(z)) = \phi_f(z)^2$; and

- $\phi_f(z)/z \to 1$ as $z \to \infty$.

For $\theta \in \mathbb{R}/\mathbb{Z}$, the *external ray of angle* θ is defined by the following set:

$$R_f(\theta) = \left\{\phi_f^{-1}(re^{2\pi i\theta}) : 1 < r < \infty\right\}.$$

If the limit of $\phi_f^{-1}(re^{2\pi i\theta})$ as $r \to 1$ exists, it is called the *landing point* of $R_f(\theta)$, and we denote it by $\gamma_f(\theta)$.

If J_f is locally connected, ϕ_f continuously extends to $\bar{\phi}_f : \bar{\mathbb{C}} - K_f^\circ \to \bar{\mathbb{C}} - \mathbb{D}$. In this case, $\gamma_f(\cdot)$ defines a semiconjugacy $\gamma_f : \mathbb{R}/\mathbb{Z} \to J_f$ from $\theta \mapsto 2\theta$ to $f|_{J_f}$. γ_f is a conjugacy if and only if J_f is a Jordan curve.

Linearizing coordinates. Suppose that $f = f_c$ has an attracting fixed point α with multiplier $\lambda \neq 0$. (That is, we take c from the main cardioid of the Mandelbrot set other than the origin.) Then K_f° is its attracting basin and contains the critical point $z = 0$. Moreover, J_f is known to be a quasicircle, and thus is a Jordan curve.

On a small neighborhood of α, there exists a linearizing coordinate Φ_f which analytically conjugates the action of f near α to $w \mapsto \lambda w$ near the origin. Moreover, we can extend this map to $\Phi_f : K_f^\circ \to \mathbb{C}$, and it is unique up to multiplication by a constant[5, §8]. Now let us normalize it as follows:

- $\Phi_f(f(z)) = \lambda\Phi_f(z)$;
- $\Phi_f(\alpha) = 0,\ \Phi_f(0) = 1$; and
- Φ_f is an infinitely branched covering whose branch points are $\bigcup_{k\geq 0} f^{-k}(\{0\})$, and their ramified points (critical value of Φ_f) are $\{1, \lambda^{-1}, \lambda^{-2}, \ldots\}$.

In this chapter, by *the linearizing coordinate* of α we mean this extended and normalized Φ_f.

2.3 Internal landing lemma

In this section we deal with the case of $f_c(z) = z^2 + c$ with an attracting fixed point. We will show "Internal landing lemma" for such an f, which gives a nice invariant arc system in the filled Julia set. In the case of $f_0(z) = z^2$, the external rays naturally penetrate the Julia set (the unit circle) and land at the origin. The lemma gives a similar fact in the case of $c \neq 0$.

Combinatorial rotation number. We assume from now on that p and q are relatively prime positive integers. (That is, $(p, q) = 1$ where we allow $p = q = 1$.) Then the following is well-known:

Lemma 2.3.1 *For p and q above, there is a set of q distinct angles $\Theta := \{\theta_1, \dots, \theta_q\}$ in $\mathbb{Q}/\mathbb{Z}$ with $0 \le \theta_1 < \cdots < \theta_q < 1$ such that:*

(1) For each $\theta_j \in \Theta$, there exists $\theta_k \in \Theta$ such that $\theta_k = 2\theta_j$ in $\mathbb{Q}/\mathbb{Z}$; and

(2) for such j and k as above, $k \equiv j + p \mod q$.

Then Θ is a periodic cycle of period q under doubling. In particular, such a Θ is determined uniquely by the value $p/q \in \mathbb{Q}/\mathbb{Z}$.

We consider that the subscripts $\{1, \dots, q\}$ of angles of Θ are the elements of $\mathbb{Z}/q\mathbb{Z}$. For $\Theta = \Theta(p/q)$ above, $p/q \in \mathbb{Q}/\mathbb{Z}$ is called the *(combinatorial) rotation number*. Note that each $\theta_j \in \Theta$ has the form $n/(2^q - 1) \in \mathbb{Q}/\mathbb{Z}$.

Let $g(z) := f_\sigma(z) = z^2 + \sigma$ be a quadratic polynomial which has a parabolic fixed point of multiplier $\omega := \exp(2\pi i p/q)$. Note that $\sigma = \omega/2 - \omega^2/4$. Now let us fix an $r \in (0,1)$ and take a value $c := r\omega/2 - (r\omega)^2/4$ from the main cardioid of the Mandelbrot set. Then $f(z) := f_c(z) = z^2 + c$ has an attracting fixed point of multiplier $\lambda := r\omega$ and J_f is a Jordan curve. The dynamics on J_f is topologically the same as that of $f_0(z) = z^2$ on the unit circle.

For the rotation number p/q, let $\mathcal{F}(p/q)$ denote the family of such an f_c, that is,

$$\mathcal{F}(p/q) := \big\{ f_c : c = r\omega/2 - (r\omega)^2/4,\ r \in (0,1) \big\}.$$

For example, $\mathcal{F}(0) = \mathcal{F}(1) = \{f_c : c \in (0, 1/4)\}$ and $\mathcal{F}(1/2) = \{f_c : c \in (-3/4, 0)\}$.

By Douady-Hubbard theory[1], above lemma implies:

Lemma 2.3.2 *For $f = f_c \in \mathcal{F}(p/q)$ and $\Theta = \Theta(p/q) = \{\theta_1, \dots, \theta_q\}$ above, f maps $R_f(\theta_j)$ onto $R_f(\theta_k)$ univalently iff $k \equiv j + p \mod q$. Thus each $R_f(\theta_j)$ has period exactly q, that is, $f^q(R_f(\theta_j)) = R_f(\theta_j)$.*

Note that $\gamma_f(\theta_j)$ is a repelling periodic point of period q. In the case of $g = f_\sigma$, the external rays $R_g(\theta_1), \dots, R_g(\theta_q)$ also have the same properties as (1) and (2) though they have the same landing point at the parabolic fixed point, say β. The set of angles of external rays landing at β is exactly $\Theta = \{\theta_1, \dots, \theta_q\}$, and is called the *portrait* of β.

Internal landing lemma. For $f \in \mathcal{F}(p/q)$, those rays $R_f(\theta_1), \dots, R_f(\theta_q)$ above continuously extend to the inside of the Julia set, and meet at the attracting fixed point:

Lemma 2.3.3 (Internal landing) *Let α be the attracting fixed point of f. For $\theta_1, \dots, \theta_q$ as above, there exist open arcs $I(\theta_1), \dots, I(\theta_q)$ such that:*

- *For each j modulo q, $I(\theta_j)$ joins α and $\gamma_f(\theta_j)$.*
- *f maps $I(\theta_j)$ onto $I(\theta_k)$ univalently iff $k \equiv j + p \mod q$.*

Proof. For $w \in \mathbb{C}$, set $T(w) := \lambda w = (r\omega)w$. Let $\Phi_f : K_f^\circ \to \mathbb{C}$ be the linearizing coordinate of α, that is, $\Phi_f(\alpha) = \Phi_f(0) - 1 = 0$ and $\Phi_f(f(z)) = T(\Phi_f(z))$. Note that the critical points of Φ_f are $\bigcup_{k>0} f^{-k}(0)$, and thus the critical values are the form $T^{-k}(1) = \lambda^{-k}$ $(k = 1, 2, \ldots)$.

Set

$$U_0 := \mathbb{C} - \bigcup_{k=0}^{q-1} \{t\omega^k : t \in (1, \infty)\}; \quad \text{and}$$
$$U_1 := \mathbb{C} - \bigcup_{k=0}^{q-1} \{t\omega^k : t \in (r, \infty)\}.$$

Note that $T(U_0) = U_1 \subsetneq U_0$. Let ρ_0 and ρ_1 denote the Poincaré metric on U_0 and U_1 respectively. Since $T : U_0 \to U_1$ is a conformal isomorphism,

$$\frac{T^*\rho_1}{\rho_1} \leq \frac{T^*\rho_1}{\rho_0} = 1$$

by Schwartz-Pick.

Note that U_0 does not contain critical value of Φ_f. Thus we can take a univalent branch Ψ of $(\Phi_f|_{U_0})^{-1}$ such that $\Psi(0) = \alpha$. Set

$$U_i' := \Psi(U_i) \text{ and } \rho_i' := \Psi^* \rho_i \quad (i = 0, 1).$$

Then U_i' are f-invariant regions in K_f° and ρ_i' are their respective Poincaré metric with $f^*\rho_1'/\rho_1' \leq 1$ on U_1'.

For each integer k modulo q, set

$$I_k = \{t \exp((2k-1)\pi i/q) : t \in (0, \infty)\} \subset U_1,$$

and set $I_k' := \Psi(I_k) \subset U_1'$. Now it is clear that f maps I_j' onto I_k' univalently iff $k \equiv j + p \mod q$. We claim that I_k' is one of $I(\theta_1), \ldots, I(\theta_q)$ in the statement.

First we show that each I_k' lands at a periodic point in the Julia set J_f. By f^q, I_k' is mapped univalently onto itself. Take $\{z_n\}_{n\geq 1}$ in I_k' such that $f^q(z_{n+1}) = z_n$. Set $w_n := \Phi_f(z_n)$.

Now let η_n denote the line segment of I_k which joins w_n and w_{n+1}. Then $\text{length}_{\rho_1}(\eta_n)$ are bounded for all n since $(T^q)^*\rho_1/\rho_1 \leq 1$. By pushing forward by Ψ, $\Psi(\eta_n)$ is getting uniformly closer to J_f since f is hyperbolic. Thus if we set $\rho_1'(z) = u(z)|dz|$, for any $z \in \Psi(\eta_n)$, $u(z)$ uniformly tends to $+\infty$ as $n \to \infty$. Thus $|z_n - z_{n+1}| \to 0$ as $n \to 0$.

Let $\zeta \in J_f$ be an accumulation point of z_n. By taking a subsequence $\{n_j\} \subset \{n\}$, we may assume that $z_{n_j} \to \zeta$. By continuity, we also have $z_{n_j - 1} = f^q(z_{n_j}) \to f^q(\zeta)$. Thus

$$\begin{aligned} &|f^q(\zeta) - \zeta| \\ \leq &|f^q(\zeta) - f^q(z_{n_j})| + |z_{n_j-1} - z_{n_j}| + |z_{n_j} - \zeta| \\ \to &0 \quad (j \to \infty). \end{aligned}$$

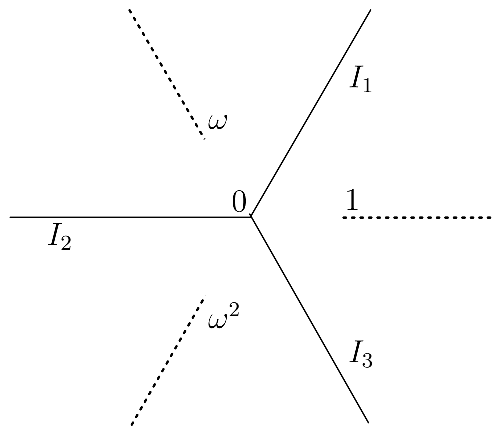

Figure 2.1: U_0 and I_k in the case of $p/q = 1/3$ ($\omega = e^{2\pi i/3}$). The dotted lines are removed from $\mathbb{C}$.

This implies $f^q(\zeta) = \zeta$. It is not difficult to show that any accumulation point of I'_k is that of z_n. Since the set of accumulation points of I'_k is connected [5, Problem 5-b] and fixed points of f^q are finite, I'_k accumulates only on ζ above. In other words, I'_k lands on $\zeta \in J_f$, a fixed point of f^q. Since $\zeta \in J_f$ and J_f is a Jordan curve, there exists an angle θ'_k such that $\zeta = \gamma_f(\theta'_k)$.

If f maps $\gamma_f(\theta'_j)$ to $\gamma_f(\theta'_k)$, then $\theta'_k = 2\theta_j$ by the dynamics on the Julia set and $k \equiv j + p \mod q$ by the dynamics of $I'_1, \ldots, I'_q$. Thus $\left\{\theta'_1, \ldots, \theta'_q\right\}$ has the combinatorial rotation number p/q and thus $\left\{\theta'_1, \ldots, \theta'_q\right\} = \{\theta_1, \ldots, \theta_q\}$. By shifting subscripts such that $0 \le \theta'_1 < \cdots < \theta'_q < 1$, we have $\theta'_j = \theta_j$ for all j and then I'_j satisfies the conditions of $I(\theta_j)$ in the statement. ■

Degenerating arc system. For $\Theta = \{\theta_1, \ldots . \theta_q\}$, set

$$I(\Theta) := \bigcup_{j=1}^{q} \overline{I(\theta_j)} = \{\alpha\} \cup \bigcup_{j=1}^{q} (I(\theta_j) \cup \{\gamma_f(\theta_j)\}).$$

Since this set contains no critical orbit, its preimages are univalently spread around in K_f°. Let I_f denote $\bigcup_{n \ge 0} f^{-n}(I(\Theta))$. We call I_f the *degenerating arc system* of f with rotation number p/q (See Remark below). Note that I_f is a forward and backward invariant set of f.

For each connected component I of I_f, there is a unique set of q distinct angles $\Theta' = \left\{\theta'_1, \ldots, \theta'_q\right\}$ such that:

(1) there exists an $n \ge 0$ such that $\theta_j = 2^n \theta'_j$ for all $j = 1, \ldots q$; and

(2) $I \cap J_f = \left\{\gamma_f(\theta'_1), \ldots, \gamma_f(\theta'_q)\right\}$.

We denote such an I by $I(\Theta')$. By $I(\theta'_j)$ we denote the open arc in $I(\Theta')$ which is an n-th preimage of $I(\theta_j)$ joining α' and $\gamma_f(\theta'_j)$. In addition, I contains a unique α' such that $f^n(\alpha') = \alpha$. Thus we abuse the term "portrait" and call Θ' *the portrait of α' with rotation number p/q*, or simply, the *portrait* of α' in our situation.

Now we may consider that I_f degenerates to $\bigcup_{n\geq 0} g^{-n}(\{\beta\})$ as $r \to 1$, and denote it by I_g. For Θ' as above, there is a unique $\beta' \in I_g$ which is the landing point of external rays $R_g(\theta'_1), \ldots, R_g(\theta'_q)$ and satisfies $g^n(\beta') = \beta$. Thus we also call Θ' the *portrait* of β'.

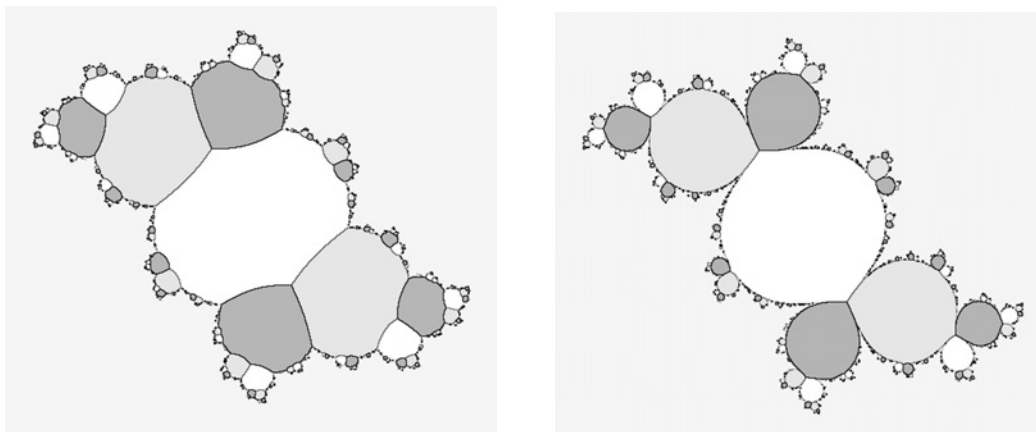

Figure 2.2: Left, the Julia set for an $f \in \mathcal{F}(1/3)$ with its degenerating arc system with rotation number 1/3 drawn in. Right, the Julia set for g with rotation number 1/3. Colors distinguish the regions mapped to distinct copies of $\mathbb{C}$ in the linearized models (§4).

Remark. It is known that for any two c, c' in the main cardioid of the Mandelbrot set other than the origin, f_c and $f_{c'}$ are topologically conjugate. Thus for any $f_c(c \neq 0)$ with an attracting fixed point, the degenerating arc system with any rotation number exists.

2.4 Tessellation: Making tiles

In this section, we develop the method in [2] and we tessellate the interior of the filled Julia sets for such f and g in the proceeding section. Tiles are parameterized by an address, which consists of an angle $\in \mathbb{Q}/\mathbb{Z}$, a level $\in \mathbb{Z}$, and a signature $\in \{+,-\}$. Let $\tilde{\Theta} = \tilde{\Theta}(p/q)$ be the set of angles which eventually land on one of the angles in Θ by iteration of angle doubling. For each $\theta \in \tilde{\Theta}$ and $m \in \mathbb{Z}$, we will define the tile $T_f(\theta, m, \pm)$ with the property

$$f(T_f(\theta, m, \pm)) = T_f(2\theta, m+1, \pm).$$

We will also define the tiles for g having the same property.

2.4.1 Tiles of K_f°

Linearized model. Let $\Phi_f : K_f^\circ \to \mathbb{C}$ be the linearizing coordinate of α with multiplier $\lambda = r\omega$ and with portrait $\Theta = \{\theta_1, \ldots, \theta_q\}$. Recall that $\Phi_f(I(\theta_j)) = I_j$ for each j modulo q, which is renumbered in the proof of Lemma 2.3.3. Now $\{0\} \cup \bigcup_j I_j$ divides the plane into q open sectors. For each j modulo q, let Σ_j^* denote the union of I_j and one of the q sectors bounded by I_j and I_{j+1}. We also set $\Sigma_j := \Sigma_j^* \cup \{0\}$.

Let $\mathbb{C}_j$ be a copy of $\mathbb{C}$. For $w \in \Sigma_j$, we define $\chi : \Sigma_j \to \mathbb{C}_j$ by

$$W = \chi(w) := \frac{1}{1-R}(1 - w^q) \ \in \ \mathbb{C}_j,$$

where $R := r^q = \lambda^q \in (0,1)$. Note that $\chi(\Sigma_j^*) = \mathbb{C}_j - \{1/(1-R)\}$ and $1/(1-R)$ is fixed by the map $W \mapsto RW + 1$. Set $a := 1/(1-R)$. Now χ naturally glues the copies $\mathbb{C}_1, \ldots, \mathbb{C}_q$ of $\mathbb{C}$ along $\chi(I_1), \ldots, \chi(I_q)$ and at $\chi(0)$. Thus we consider that χ is not branched at $w = 0$. Let $\bigcup \mathbb{C}_j$ denote this glued set homeomorphic to $\mathbb{C} = \bigcup \Sigma_j$. Let us define $F : \bigcup \mathbb{C}_j \to \bigcup \mathbb{C}_j$ by

$$\mathbb{C}_j \ \ni \ W \ \overset{F}{\longmapsto} \ RW + 1 \ \in \ \mathbb{C}_{j+p}.$$

Then χ conjugates $w \mapsto \lambda w$ on $\mathbb{C} = \bigcup \Sigma_j$ and F on $\bigcup \mathbb{C}_j$:

$$\begin{array}{ccc} K_f^\circ & \xrightarrow{f} & K_f^\circ \\ \Phi_f \big\downarrow & & \big\downarrow \Phi_f \\ \mathbb{C} = \bigcup \Sigma_j & \xrightarrow{\cdot\lambda} & \mathbb{C} = \bigcup \Sigma_j \\ \chi \big\downarrow & & \big\downarrow \chi \\ \bigcup \mathbb{C}_j & \xrightarrow{F} & \bigcup \mathbb{C}_j \end{array}$$

Fundamental semi-annuli. For $m \in \mathbb{Z}$ and j modulo q, set

$$A(m,+)_j := \left\{W \in \mathbb{C}_j - \chi(I_j) : R^{m+1}a \le |W - a| \le R^m a, \ \operatorname{Im} W \ge 0\right\}$$
$$A(m,-)_j := \left\{W \in \mathbb{C}_j - \chi(I_j) : R^{m+1}a \le |W - a| \le R^m a, \ \operatorname{Im} W \le 0\right\}$$

and we call them the *fundamental semi-annuli*.

Note the following three facts:

- F maps $A(m,\pm)_j$ onto $A(m+1,\pm)_{j+p}$ univalently.
- $\chi \circ \Phi_f$ maps the grand orbit of 0 (critical point) to vertices of fundamental semi-annuli on the q copies of the interval $(-\infty, a)$. In particular, all of the ramified points (critical values) of $\chi \circ \Phi_f$ are on the q copies of the interval $(-\infty, 0]$.

- For any $\theta \in \tilde{\Theta}$, $I(\theta)$ is mapped univalently onto one of the copies of the interval (a, ∞) by $\chi \circ \Phi_f$.

For the boundary of $A(m, \pm)_j$, we call the edge on the interval $(-\infty, a)$ (resp. $[a, \infty)$) the *critical-edge (resp. degenerating-edge)*. We call the edges shared by $A(m-1, \pm)_j$ or $A(m+1, \pm)_j$ the *circular edges*. Note that the degenerating edge is not contained in $A(m, \pm)_j$.

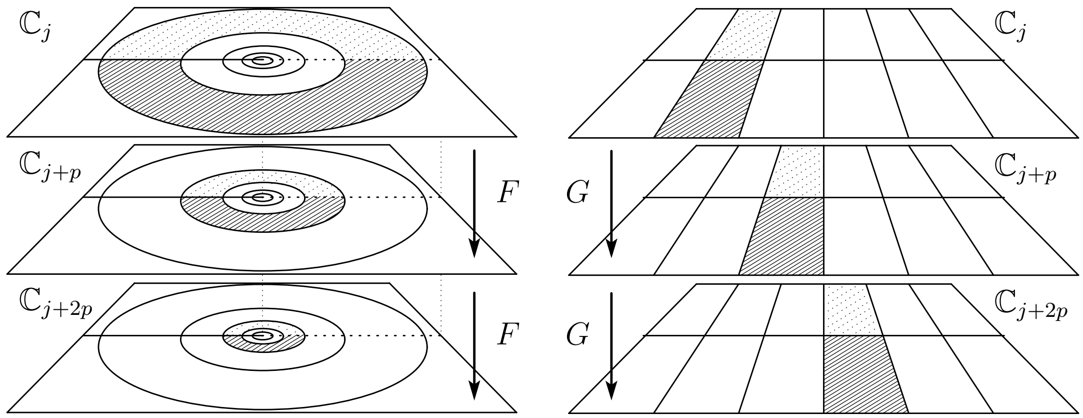

Figure 2.3: Linearized models for f and g.

Definition of tiles. Let α' be a preimages of α such that $f^n(\alpha') = \alpha$ for some $n \geq 0$. Then $\Phi_f(\alpha') = 0$ by the definition. Since $U_0 \subset \mathbb{C}$ in the proof of Lemma 2.3.3 does not contain ramified points (critical values) of Φ_f, $\Phi_f^{-1} : U_0 \to K_f^\circ$ is a multivalued function with univalent branches. Now we take such a branch $\Psi : U_0 \to K_f^\circ$ such that $\Psi(0) = \alpha'$. Let $\Theta' = \{\theta'_j\}$ be the portrait of α'. Then we may assume that $\Psi(I_{j-np}) = I(\theta'_j)$.

For $m \in \mathbb{Z}$ and j modulo q, $\Psi \circ \chi^{-1}$ maps the interior of $A(m, +)_j$ into K_f° univalently. Since $\Psi \circ \chi^{-1}$ extends to the whole $A(m, +)_j$ homeomorphically, the set

$$T_f(\theta'_j, m, +) := \Psi \circ \chi^{-1}(A(m, +)_j) \subset K_f^\circ$$

is well defined. Similarly, we set

$$T_f(\theta'_{j+1}, m, -) := \Psi \circ \chi^{-1}(A(m, -)_j) \subset K_f^\circ.$$

For any $\theta \in \tilde{\Theta}$ and $m \in \mathbb{Z}$, we can define $T_f(\theta, m, \pm)$ in this way and we call it the *tile of address* $(\theta, m, \pm)$. Now one can easily check the desired property:

$$f(T_f(\theta, m, \pm)) = T_f(2\theta, m + 1, \pm).$$

For the boundary of $T = T_f(\theta, m, +)$ or $T_f(\theta, m, -)$, the critical, degenerating and circular edges are defined by the edges corresponding to the critical, degenerating, circular edges of $A(m,\pm)_j$. Note that ∂T has degenerating edge on $I(\theta')$ while T does not contain the edge itself.

We call the family of tiles

$$\mathcal{T}_f := \left\{ T_f(\theta, m, *) : \theta \in \tilde{\Theta}, m \in \mathbb{Z}, * \in \{+,-\} \right\}$$

defined as above the *tessellation* of K_f° with rotation number p/q. Indeed, $K_f^\circ - I_f$ is tessellated by $\mathcal{T}_f$ and K_f is the closure of the union $\bigcup_{T \in \mathcal{T}_f} T$.

2.4.2 Tiles of K_g°

Let β be the parabolic fixed point of g with multiplier $\omega = e^{2\pi i p/q}$ and with portrait $\Theta = \{\theta_j\}$. Now $\{\beta\} \cup \bigcup R_g(\theta_j)$ divide $\mathbb{C}$ into q sectors. For each j modulo q, let S_j denote the sector bounded by $R_g(\theta_j)$ and $R_g(\theta_{j+1})$. (That is, the union of external rays with angles satisfying $\theta_j \leq \theta \leq \theta_{j+1} (< \theta_j + 1)$.) S_j contains an attracting petal $\Pi_j \subset K_g^\circ$ such that $g^q(\Pi_j) \subset \Pi_j$. Set $\tilde{\Pi}_j := \bigcup_{n=0}^{\infty} g^{-nq}(\Pi_j)$. Note that $K_g^\circ = \bigsqcup \tilde{\Pi}_j$. We take q copies $\mathbb{C}_1, \ldots, \mathbb{C}_q$ of $\mathbb{C}$ again.

Let us fix k modulo q such that S_k contains the critical point 0 of g. On $\tilde{\Pi}_k$, there is a unique Fatou coordinate $\Phi_k : \tilde{\Pi}_k \to \mathbb{C}_k$ such that

- $\Phi_k(g^q(z)) = \Phi_k(z) + q$;
- $\Phi_k(0) = 0$; and
- Φ_k is an infinitely branched covering whose branch points are $\bigcup_{m \geq 0} g^{-mq}(\{0\})$, and their ramified points (critical value of Φ_k) are $\{0, -q, -2q, \ldots\}$.

([5, §10]. We used the fact that $w \mapsto w+1$ is conjugate to $w \mapsto w+q$.) We extend Φ_k to $\Phi_g : K_g^\circ \to \bigsqcup \mathbb{C}_j$ as following: For any j modulo q, there is an n such that $k \equiv j + pn \mod q$, that is, $g^n(\tilde{\Pi}_j) = \tilde{\Pi}_k$. We define Φ_g on $\tilde{\Pi}_j$ by

$$\tilde{\Pi}_j \ \ni \ z \ \stackrel{\Phi_g}{\longmapsto} \ \Phi_k(g^n(z)) - n \ \in \ \mathbb{C}_j.$$

Then for $z \in \mathbb{C}_j$, we have $\Phi_g(g(z)) = \Phi_g(z) + 1 \in \mathbb{C}_{j+p}$. We define $G : \bigsqcup \mathbb{C}_j \to \bigsqcup \mathbb{C}_j$ by

$$\mathbb{C}_j \ \ni \ W \ \stackrel{G}{\longmapsto} \ W + 1 \ \in \ \mathbb{C}_{j+p},$$

and then Φ_g semiconjugates g on K_g° and G on $\bigsqcup \mathbb{C}_j$:

$$\begin{array}{ccc} K_g^\circ & \xrightarrow{g} & K_g^\circ \\ \downarrow{\scriptstyle \Phi_g} & & \downarrow{\scriptstyle \Phi_g} \\ \bigsqcup \mathbb{C}_j & \xrightarrow{G} & \bigsqcup \mathbb{C}_j \end{array}$$

Fundamental semi-cylinders. For $m \in \mathbb{Z}$ and $j = 1, \ldots, q$, set

$$C(m,+)_j := \{W \in \mathbb{C}_j : m \le \operatorname{Re} W \le m+1,\ \operatorname{Im} W \ge 0\}$$
$$C(m,-)_j := \{W \in \mathbb{C}_j : m \le \operatorname{Re} W \le m+1,\ \operatorname{Im} W \le 0\}$$

and we call them the *fundamental semi-cylinders*.

Note the following two facts, and compare with the case of f:

- G maps $C(m,+)_j$ onto $C(m+1,+)_{j+p}$ univalently.
- Φ_g maps the grand orbit of 0 to the vertices of fundamental semi-cylinders on the q copies of the real axis $(-\infty,\infty)$. In particular, all of the ramified points of Φ_g are on the q copies of the interval $(-\infty, 0]$.

For the boundary of $C(m,\pm)_j$, we call the edge on the real axis the *critical-edge*. We also call the edges shared by $C(m-1,\pm)_j$ or $C(m+1,\pm)_j$ the *circular edges*. Note that $C(m,\pm)_j$ has no edges corresponding to degenerating edges of fundamental semi-annuli.

Definition of tiles. Let β' be a preimage of β such that $g^n(\beta') = \beta$ for some $n \ge 0$, and $\Theta' = \left\{\theta'_j\right\}$ be the portrait of β' with $\theta_j = 2^n \theta'_j$ for each j modulo q. Note that $\{\beta'\} \cup \bigcup R_g(\theta'_j)$ divide the plane into q sectors. For each j modulo q, one of the q sectors bounded by $R_g(\theta'_j)$ and $R_g(\theta'_{j+1})$ contains a component Π' of $\tilde{\Pi}_j$ attached to β'. Let $(-\infty,0]_j$ denote the copy of $(-\infty,0]$ in $\mathbb{C}_j$. Since $\mathbb{C}_j - (-\infty,0]_j$ does not contain ramified points (critical values) of Φ_g, $\Phi_g^{-1} : \mathbb{C}_j - (-\infty,0]_j \to \tilde{\Pi}_j$ is a multivalued function with univalent branches. Now we take a branch $\Psi : \mathbb{C}_j - (-\infty,0]_j \to K_g^\circ$ of Φ_g^{-1} above such that $\Psi(\mathbb{C}_j - (-\infty,0]_j) \subset \Pi'$.

For $m \in \mathbb{Z}$ and j modulo q, Ψ maps the interior of $C(m,+)_j$ into K_g° univalently. Since Ψ extends to the whole $C(m,+)_j$ homeomorphically, the set

$$T_g(\theta'_j, m, +) := \Psi^{-1}(C(m,+)_j) \subset K_g^\circ$$

is well defined. Similarly, we set

$$T_g(\theta'_{j+1}, m, -) := \Psi^{-1}(C(m,-)_j) \subset K_g^\circ.$$

For any $\theta \in \tilde{\Theta}$ and $m \in \mathbb{Z}$, we can define $T_g(\theta, m, \pm)$ in this way and we call it the *tile of address* $(\theta, m, \pm)$. Now one can easily check the desired property:

$$g(T_g(\theta, m, \pm)) = T_g(2\theta, m+1, \pm).$$

For the boundary of $T = T_g(\theta, m, +)$ or $T_g(\theta, m, -)$, the critical and circular edges are defined by the edges which are mapped to the critical and circular edges of fundamental semi-cylinders by Φ_g. Note that T has no edge corresponding to the degenerating edges of $\{T_f(\theta, m, \pm)\}$.

We call the family of tiles

$$\mathcal{T}_g := \Big\{ T_g(\theta, m, *) : \theta \in \tilde{\Theta}, m \in \mathbb{Z}, * \in \{+, -\} \Big\}$$

defined as above the *tessellation* of K_g° with rotation number p/q. Indeed, K_g° is tessellated by $\mathcal{T}_g$ and K_g is the closure of the union $\bigcup_{T \in \mathcal{T}_g} T$.

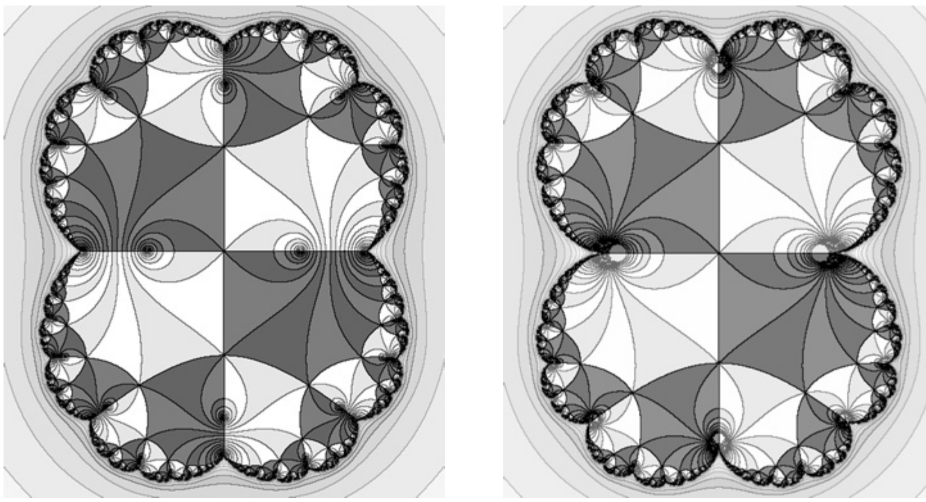

Figure 2.4: The tessellation for an $f \in \mathcal{F}(1/1)$ and $z^2+1/4$, which has a parabolic fixed point with one petal.

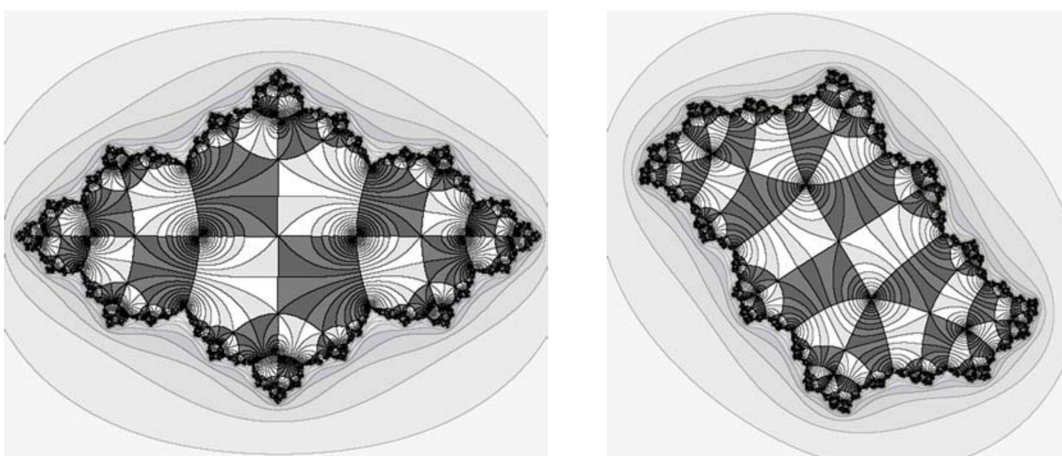

Figure 2.5: The tessellation for an $f \in \mathcal{F}(1/2)$ and anther $f \in \mathcal{F}(1/3)$.

2.4.3 Edge sharing

Here we describe how tiles share their edges with one another.

Circular edges. For f and g, by the definition of $\mathcal{T}_f$ and $\mathcal{T}_g$, one can easily check the following:

> *For $\theta \in \tilde{\Theta}$, $m \in \mathbb{Z}$ and $* \in \{+,-\}$, the tile of address $(\theta, m, *)$ shares its circular edges with the tiles of addresses $(\theta, m-1, *)$ and $(\theta, m+1, *)$.*

Degenerating edges. Only tiles in $\mathcal{T}_f$ have degenerating edges. By the definition, one can also check the following:

> *For $\theta \in \tilde{\Theta}$ and $m \in \mathbb{Z}$, $T_f(\theta, m, +)$ shares its degenerating edge with $T_f(\theta, m, -)$.*

Critical edges in K_f°. The combinatorics of tiles are essentially determined by the connection of critical edges. Here we consider the critical edges of tiles in $\mathcal{T}_f$.

We begin with some notation. Let δ denote the angle doubling map on $\mathbb{R}/\mathbb{Z}$ to itself. For $\Theta = \Theta(p/q)$ and $n = 0, 1, \ldots$, set $\Theta_{-n} := \delta^{-n}(\Theta)$. Then Θ_{-n} consists of $2^n q$ angles. We denote them by $\theta_1^{(-n)}, \ldots, \theta_{2^n q}^{(-n)}$ with cyclic order $\theta_1^{(-n)} < \cdots < \theta_{2^n q}^{(-n)} < \theta_1^{(-n)} + 1$ and with subscripts modulo $2^n q$. One can easily check that $\Theta_{-n} \subset \Theta_{-n-1}$ and $\tilde{\Theta} = \bigcup_n \Theta_{-n}$.

First we consider the insular part of the tessellation. Let us take an α' such that $f^n(\alpha') = \alpha$ with minimal $n \geq 0$. Then the portrait Θ' of α' is a subset of Θ_{-n}. In the w-plane which is the target space of $\Phi_f : K_f^\circ \to \mathbb{C}$, we take a closed disk $B_{-n} := \{|w| \leq r^{-n+1}\}$. By the definition of Φ_f, there exists a univalent branch $\Psi : B_{-n} \to K_f^\circ$ of $\Phi^{-1}|_{B_{-n}}$ with $\Phi(0) = \alpha'$. The image $\Psi(B_{-n})$ consist of the tiles of addresses of the form $(\theta, m, \pm)$ with $\theta \in \Theta'$ and $m > -n$ which are univalent pull-backs of fundamental semi-annuli in B_{-n}. Thus we have:

> *Such a tile $T_f(\theta, m, +)$ with $\theta \in \Theta'$ and $m > -n$ shares the critical edge with $T_f(\theta', m, -)$ where θ' is the angle next to θ in the cyclic order of Θ'. More precisely, if we set $\Theta' = \left\{\theta'_1, \ldots, \theta'_q\right\}$ with cyclic order $\theta'_1 < \cdots < \theta'_q < \theta'_1 + 1$ and with subscripts modulo q, then $\theta = \theta'_j$ and $\theta' = \theta'_{j+1}$ for some j.*

Next we consider the other part of tessellation. Take the univalent branch Ψ of Φ_f^{-1} on the unit disk of the w-plane such that $\Psi(0) = \alpha$. Let C_0 denote the pull-back of the circle $\{|w| = \sqrt{r}\}$ by Ψ, which is a simple closed curve in K_f° passing through each tile of address $(\theta_j, 0, \pm)$, where $\theta_j \in \Theta$. Let D_0 denote the topological disk bounded by C_0, which contains α. For $n = 0, 1, \ldots$, we set $C_{-n} := f^{-n}(C_0)$ and $D_{-n} := f^{-n}(D_0)$. Then we have:

- Each C_{-n} is also a simple closed curve, passing though the tiles of addresses $(\theta_j^{(-n)}, -n, \pm)$, where $\theta_j^{(-n)} \in \Theta_{-n}$.
- $D_{-n} \Subset D_{-n-1}$.

- $f : D_{-n-1} \to D_{-n}$ is a proper 2-fold branched covering.

Since C_0 intersects $I(\theta_j)$ once for each j modulo q in the cyclic order of Θ, C_{-n} intersects $I(\theta_j^{(-n)})$ once for each j modulo $2^n q$ in the cyclic order of Θ_{-n}. Thus we have:

Such a tile $T_f(\theta, -n, +)$ with $\theta \in \Theta_{-n}$ shares the critical edge with $T_f(\theta', -n, -)$ where θ' is the angle next to θ in the cyclic order of Θ_{-n}. That is, $\theta = \theta_j^{(-n)}$ and $\theta' = \theta_{j+1}^{(-n)}$ for some j modulo $2^n q$.

More precisely, the angle θ' is given as following: Now $2^n\theta = \theta_j \in \Theta$ for some j modulo q. Then $2^n\theta'$ must be $\theta_{j+1} \in \Theta$. Let ℓ denote the length of the interval of angle $[\theta_j, \theta_{j+1}]$. Then θ' is given by

$$\theta' = \theta + \frac{\ell}{2^n}.$$

Critical edges in K_g°. The same argument works for the tiles in $\mathcal{T}_g$ with a little modification. Instead of the insular part of $\mathcal{T}_f$, we use the "flower part" of the $\mathcal{T}_g$. More precisely, instead of α' and $\Psi(B_{-n})$ in the argument above, which is the union

$$\{\alpha'\} \cup \bigcup \{T_f(\theta, m, \pm) : \theta \in \Theta', m > -n\},$$

we take $\beta' \in I_g$ with portrait Θ' and use the union

$$\{\beta'\} \cup \bigcup \{T_g(\theta, m, \pm) : \theta \in \Theta', m > -n\}.$$

Instead of the simple closed curve C_0 and the topological disk D_0, we may use the curve C_0' and the topological disk D_0' constructed as following: First take attracting petals $\Pi_1, \ldots, \Pi_q$ as in the construction of $\mathcal{T}_g$ such that Φ_g univalently maps each petal Π_j onto the half plane $\{W \in \mathbb{C}_j : \mathrm{Re}\, W > 1/2\}$. Then the boundary of each Π_j passes through the tiles $T_f(\theta_j, 0, +)$ and $T_f(\theta_{j+1}, 0, -)$. Next we take a small open disk centered at β, say Δ. Then the boundary circle of Δ intersects each boundary of Π_j twice, and each $R_g(\theta_j)$ once. Now $D_0' := \Delta \cup \bigsqcup \Pi_j$ is a topological disk containing β as desired. Let C_0' be the boundary curve of D_0'. One can easily check that $C_{-n}' := g^{-n}(C_0')$ and $D_{-n}' := g^{-n}(D_0')$ have similar properties to C_{-n} and D_{-n}, and we can apply the same argument.

2.5 Pinching semiconjugacy

In this section we construct a semiconjugacy $H : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ by gluing tile-to-tile homeomorphisms inside the Julia sets and the topological conjugacy induced from Böttcher coordinates outside the Julia sets.

Theorem 2.5.1 *For f, g as above, there exists a semiconjugacy $H : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g such that*

- *H maps $\bar{\mathbb{C}} - I_f$ to $\bar{\mathbb{C}} - I_g$ homeomorphically and is a topological conjugacy between $f|_{\bar{\mathbb{C}}-I_f}$ and $g|_{\bar{\mathbb{C}}-I_g}$;*
- *For each $\alpha' \in \bigcup_n f^{-n}(\alpha)$ with portrait Θ', H maps $I(\Theta')$ onto a point $\beta' \in I_g$ with portrait Θ'.*

Proof. The rest of this section is devoted to the proof of this theorem. The proof breaks into four steps.

Conjugacy on the fundamental semi-annuli and semi-cylinders. First we make a topological map $h : \bigsqcup(\mathbb{C}_j - \chi(I_j \cup \{0\}) \to \bigsqcup \mathbb{C}_j$ which maps $A(m, \pm)_j$ to $C(m, \pm)_j$ homeomorphically. Note that each $\mathbb{C}_j - \chi(I_j \cup \{0\})$ is a copy of $\mathbb{C} - [a, \infty)$. For j modulo q and $W \in \mathbb{C}_j - \chi(I_j \cup \{0\})$, set $W := a + \rho e^{it}$ where $\rho > 0$ and $0 < t < 2\pi$. We define the map h by

$$h(W) := \frac{\log \rho - \log a}{\log R} + i \tan \frac{\pi - t}{2} \in \mathbb{C}_j.$$

Then one can check that h conjugates the action of F on $\bigsqcup(\mathbb{C}_j - \chi(I_j \cup \{0\}))$ to that of G on $\bigsqcup \mathbb{C}_j$ and h maps $A(m, \pm)_j$ to $C(m, \pm)_j$ homeomorphically.

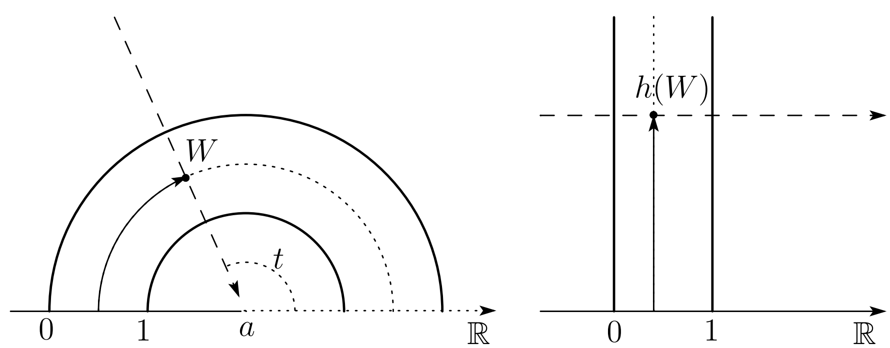

Figure 2.6: h maps $A(0, +)_j$ to $C(0, +)_j$.

Tile-to-tile conjugation. Fix a $\beta' \in I_g$ with portrait $\Theta' = \{\theta'_j\}$. For j modulo q, the boundary of $T = T_g(\theta'_j, m, +)$ contains $\gamma_g(\theta'_j)$, and T itself is contained in the sector bounded by $R_g(\theta'_j)$ and $R_g(\theta'_{j+1})$. In particular, $T \subset \tilde{\Pi}_j$. Since Φ_g does not branch over $\mathbb{C}_j - (-\infty, 0]_j$, there exist a univalent branch $\Psi_g = \Psi_g[\theta'_j] : \mathbb{C}_j - (-\infty, 0]_j \to \tilde{\Pi}_j$ which maps the interior of $C(m, +)_j$ to that of T. By extending Ψ_g to the edges of $C(m, +)_j$, we have a tile-to-tile homeomorphism Ψ_g :

$C(m,+)_j \to T_g(\theta'_j, m, +)$. In the same way, Ψ_g also extends to $\Psi_g : C(m,-)_j \to T_g(\theta'_{j+1}, m, -)$. Now we define tile-to-tile homeomorphisms

$$H|\ T_f(\theta'_j, m, +) \to T_g(\theta'_j, m, +) \quad \text{and}$$
$$H|\ T_f(\theta'_{j+1}, m, -) \to T_g(\theta'_{j+1}, m, -)$$

by $H := \Psi_g \circ h \circ \Phi_f$. By gluing such tile-to-tile homeomorphisms along the edges of tiles, we obtain the topological conjugacy $H : K_f^\circ - I_f \to K_g^\circ$. (Here we used the fact that the combinatorics of $\mathcal{T}_f$ and $\mathcal{T}_g$ are the same.)

Continuous extension to the Julia set. For $\beta' \in I_g$ with portrait Θ' above, we define $H(I(\Theta')) := \beta'$. Then H maps I_f onto I_g and $H : K_f^\circ \cup I_f \to K_g^\circ \cup I_g$ semiconjugates $f|_{K_f^\circ \cup I_f}$ to $g|_{K_g^\circ \cup I_g}$. Now we claim that H continuously extends to $H : K_f \to K_g$.

Take $z_n \in K_f^\circ \cup I_f$ converging to a point $\zeta \in J_f$. Since J_f is a Jordan curve, there exists $\theta \in \mathbb{R}/\mathbb{Z}$ such that $\zeta = \gamma_f(\theta)$. We show that $w_n := H(z_n) \in K_g^\circ \cup I_g$ converges to $\gamma_g(\theta) \in J_g$. (Recall that J_g is locally connected and $\gamma_g(\theta) \in J_g$ exists.)

Take a small interval of angle $[t, t']$ containing θ, where $t,\ t' \in \Theta_{-m}$ with $m \gg 0$. Then $\gamma_f(t)$ and $\gamma_f(t')$ bound a small piece of J_f, and the piece, say J'_f, is a Jordan arc containing ζ. Take an open arc $C \subset K_f^\circ$ joining $\gamma_f(t)$ and $\gamma_f(t')$ via $I(t)$, C_{-m}, and $I(t')$. Let V denote the small open set with $\partial V = C \cup J'_f$. By the definition of H, $\overline{H(V)} \cap J_g =: J'_g$ is a small piece of J_g which is the set of all landing points of external rays of angles in $[t, t']$.

Since $z_n \in V \cup J'_f$ for all $n \gg 0$, $w_n \in H(V) \cup J'_g$ for all $n \gg 0$. If there exists a subsequence $\{n_i\} \subset \{n\}$ such that w_{n_i} converges to a point in K_g°, then $z_{n_i} \to \zeta \in K_f^\circ - I_f$ by the definition of H. This contradicts $\zeta \in J_f$. Thus w_n accumulates on J'_g. Since t and t' are arbitrarily close to θ, w_n must converges to $\gamma_g(\theta)$.

Global extension. Finally we define H outside the Julia set by

$$\begin{aligned} H : \bar{\mathbb{C}} - K_f &\to \bar{\mathbb{C}} - K_g \\ z &\mapsto \phi_g^{-1} \circ \phi_f(z), \end{aligned}$$

which gives a topological conjugacy on the domain, and continuously extends to the semiconjugacy $H : \bar{\mathbb{C}} - K_f^\circ \to \bar{\mathbb{C}} - K_g^\circ$. Then H inside and outside J_f are continuously glued along J_f. Now $H : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a desired semiconjugacy. ■

2.6 Degeneration of the regular leaf spaces

2.6.1 The regular leaf space

We first survey the basic notion on the regular leaf spaces of quadratic polynomials. We follow [3, §3].

The natural extension. For general $f = f_c$ ($c \in \mathbb{C}$), let us consider the set of all possible backward orbits

$$\mathcal{N}_f := \left\{ \hat{z} = (z_0, z_{-1}, \dots) : \ z_0 \in \bar{\mathbb{C}}, \ f(z_{-n-1}) = z_{-n} \right\}.$$

This set is called the *natural extension* of f, and is equipped with a topology from $\bar{\mathbb{C}} \times \bar{\mathbb{C}} \times \cdots$. On this natural extension, the lift of f and a natural projection are defined by

$$\hat{f}(\hat{z}) := (f(z_0), z_0, z_{-1}, \dots) \ \text{ and}$$
$$\pi_f(\hat{z}) := z_0.$$

It is clear that $\hat{f}$ is a homeomorphism, and satisfies $\pi_f \circ \hat{f} = f \circ \pi_f$. For a fixed point $\zeta \in \bar{\mathbb{C}}$ of f, set $\hat{\zeta} := (\zeta, \zeta, \dots) \in \mathcal{N}_f$.

The regular leaf space. An element $\hat{z} = (z_0, z_{-1}, \dots) \in \mathcal{N}_f$ is *regular* if there exists a neighborhood U_0 of z_0 such that its pull-back U_{-n} along the backward orbit $\hat{z}$ are eventually univalent. For example, $\hat{\infty} = (\infty, \infty, \dots)$ is *not* regular for any $f = f_c$ ($c \in \mathbb{C}$).

Let $\mathcal{R}_f$ denote the set of regular points in $\mathcal{N}_f$. $\mathcal{R}_f$ is called the *regular leaf space* of f. A *leaf* of $\mathcal{R}_f$ is a path connected component of $\mathcal{R}_f$. By [3, Lemma 3.1], leaves of $\mathcal{R}_f$ are Riemann surfaces:

Lemma 2.6.1 *Leaves of $\mathcal{R}_f$ have following properties:*

- *For each leaf L, we can introduce a complex structure such that $\pi_f : L \to \bar{\mathbb{C}}$ is an analytic map.*
- *$\pi_f : L \to \bar{\mathbb{C}}$ branches at $\hat{z} = (z_0, z_{-1}, \dots) \in L$ if and only if $\hat{z}$ contains a critical point in $\{z_{-n}\}$.*
- *$\hat{f}$ maps a leaf to a leaf isomorphically.*

This lemma holds for any $c \in \mathbb{C}$. In our case, we have:

Proposition 2.6.2 *Suppose f_c has an attracting or parabolic fixed point ζ. Then $\mathcal{R}_{f_c}$ has the following properties:*

- $\mathcal{R}_{f_c} = \mathcal{N}_{f_c} - \{\hat{\infty}, \hat{\zeta}\}$

- *Each leaf of $\mathcal{R}_{f_c}$ is isomorphic to $\mathbb{C}$.*

Thus the regular leaf spaces of f and g in the preceding sections have these properties. This proposition is immediate from lemmas in [3, §3].

2.6.2 Semiconjugacy on the natural extensions

Here we investigate the structure of $\mathcal{R}_g$, the regular leaf space of g. We begin with some notation and remarks.

For the portrait $\Theta = \{\theta_j\}$ of the attracting fixed point α of f, set $\gamma_j := \gamma_f(\theta_j)$, and

$$\hat{\gamma}_j := (\gamma_j, \gamma_{j-p}, \gamma_{j-2p}, \ldots).$$

Then $\hat{\gamma}_1, \ldots, \hat{\gamma}_q$ are periodic cycle of period q under the action of $\hat{f}$ and contained in $\mathcal{R}_f$. On the other hand, for g, the lift of the parabolic fixed point $\hat{\beta} = (\beta, \beta, \ldots)$ is *not* regular and thus $\hat{\beta} \notin \mathcal{R}_g$.

For each j modulo q, we set

$$\begin{aligned} L_j &:= \{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_f : z_{-nq} \to \gamma_j\} \\ L'_j &:= \left\{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_g : z_{-nq} \to \Pi_j^+ \text{ for all } n \gg 0\right\}, \end{aligned}$$

where Π_j^+ is a repelling petal of β containing the end of $R_g(\theta_j)$ near J_g. Then each L_j (resp. L'_j) is invariant under the action of $\hat{f}^q$ (resp. $\hat{g}^q$), and actually is a leaf isomorphic to $\mathbb{C}$. (We will construct the isomorphisms later.) In particular, $\hat{f}$ (resp. $\hat{g}$) maps L_j (resp. L'_j) to L_{j+p} (resp. L'_{j+p}) isomorphically, and thus L_j (resp. L'_j) is periodic leaf of period q.

For each j modulo q, we define a component $\hat{I}_j$ of $\pi_f^{-1}(I(\theta_j))$ in L_j by

$$\hat{I}_j := \{(z_0, z_{-1}, \ldots) \in \mathcal{R}_f : z_{-n} \in I(\theta_{j-np})\} \subset L_j.$$

Then each $\hat{I}_j$ is an open arc in $\mathcal{N}_f$ which joins $\hat{\alpha}$ and $\hat{\gamma}_j$.

Let us set $\mathcal{I}_f := \pi_f^{-1}(I_f)$ and $\mathcal{I}_g := \pi_g^{-1}(I_g)$. Take a backward orbit $\hat{\beta}' = (\beta'_0, \beta'_{-1}, \ldots) \in \mathcal{I}_g$. Then it uniquely determines a sequence $\hat{\Theta}' := (\Theta'_0, \Theta'_{-1}, \ldots)$ of portraits of each β'_{-n}. We call $\hat{\Theta}'$ the portrait of $\hat{\beta}'$. On the other hand, $\hat{\Theta}'$ bijectively corresponds to a component of $\mathcal{I}_f$ which consists of backward orbits $(z_0, z_{-1}, \ldots)$ with $z_{-n} \in I(\Theta'_{-n})$. We denote this component by $\hat{I}(\hat{\Theta}')$. Set $\hat{\Theta} = (\Theta, \Theta, \ldots)$. Then $\hat{\beta}$ has the portrait $\hat{\Theta}$ and $\hat{I}(\hat{\Theta})$ contains $\hat{\alpha}$. Note that $\hat{\beta}$ and $\hat{\alpha}$ are irregular points. However, $\hat{I}(\hat{\Theta}) - \{\hat{\alpha}\} = \bigsqcup(\hat{I}_j \cup \{\hat{\gamma}_j\})$ is contained in the regular leaf space $\mathcal{R}_f$. Now the main result is:

Theorem 2.6.3 *For f and g as above, there exists a semiconjugacy $\hat{H} : \mathcal{N}_f \to \mathcal{N}_g$ from $\hat{f}$ to $\hat{g}$ with the following properties:*

(1) $\hat{H} : \mathcal{N}_f - \mathcal{I}_f \to \mathcal{N}_g - \mathcal{I}_g$ *is a topological conjugacy between* $\hat{f}|_{\mathcal{N}_f - \mathcal{I}_f}$ *and* $\hat{g}|_{\mathcal{N}_g - \mathcal{I}_g}$.

(2) For any $\hat{\beta}'$ *with portrait* $\hat{\Theta}'$ *as above,* $\hat{H}^{-1}(\hat{\beta}') = \hat{I}(\hat{\Theta}')$*. In particular,* $\hat{H}^{-1}(\hat{\beta}) = \hat{I}(\hat{\Theta})$.

(3) For each j *modulo* q, $\hat{H}^{-1}(L'_j) = L_j - \hat{I}_j \cup \{\hat{\gamma}_j\}$.

(4) $\hat{H}$ *maps a leaf of* $\mathcal{R}_f - \bigsqcup L_j$ *onto a leaf of* $\mathcal{R}_g - \bigsqcup L'_j$.

(5) For a leaf L *of* $\mathcal{R}_g - \bigsqcup L'_j$, $\hat{H}^{-1}(L)$ *is a leaf of* $\mathcal{R}_f - \bigsqcup L_j$.

Proof. For $\hat{z} = (z_0, z_{-1}, \dots) \in \mathcal{N}_f$, set

$$\hat{H}(\hat{z}) := (H(z_0), H(z_{-1}), \dots) \in \mathcal{N}_g.$$

Since H is a semiconjugacy from f to g, one can easily check that $\hat{H}$ is surjective, continuous, and satisfies $\hat{H} \circ \hat{f} = \hat{g} \circ \hat{H}$. Thus $\hat{H}$ is a semiconjugacy from $\hat{f}$ to $\hat{g}$ on their respective natural extensions. In particular, since $H : \bar{\mathbb{C}} - I_f \to \bar{\mathbb{C}} - I_g$ is a topological conjugacy, corresponding lift to the natural extensions $\hat{H} : \mathcal{N}_f - \mathcal{I}_f \to \mathcal{N}_g - \mathcal{I}_g$ is also a topological conjugacy. Thus we obtain property (1).

Property (2) comes from the definition of $\hat{H}$ above and the one-to-one correspondence between $\hat{\beta}'$ with portrait $\hat{\Theta}'$ and $\hat{I}(\hat{\Theta}')$.

Now let us show properties (3) to (5), by using the idea of [3, Lemma 3.2]. Take a leaf L' in $\mathcal{R}_g$, and fix two distinct points $\hat{z}' = (z'_0, z'_{-1}, \dots)$ and $\hat{w}' = (w'_0, w'_{-1}, \dots)$ in L'. Let $\hat{\eta}'$ be a path in L' joining $\hat{z}'$ and $\hat{w}'$. Then $\eta'_{-n} := \pi_g \circ \hat{f}^{-n}(\hat{\eta}')$ is a path joining z'_{-n} and w'_{-n}, and η'_{-n} has a neighborhood U_{-n} whose pull-back along $\hat{z}'$ and $\hat{w}'$ is eventually univalent. (That is, η'_{-n} ($n \gg 0$) does not pass through $\hat{\beta}$ and $\hat{\infty}$.)

Choose any $\hat{z} = (z_0, z_{-1}, \dots) \in \hat{H}^{-1}(\hat{z}')$ and $\hat{w} = (w_0, w_{-1}, \dots) \in \hat{H}^{-1}(\hat{w}')$. For $N \gg 0$, even if η'_{-N} passes through I_g, $H^{-1}(\eta'_{-N})$ is a path connected set by the definition of H. Since z_{-N} and w_{-N} are contained in $H^{-1}(\eta'_{-N})$, we can choose a path η_{-N} joining z_{-N} and w_{-N}. Since we may assume that η'_{-N} contains neither β nor ∞, we may assume that η_{-N} contains neither $I(\Theta)$ nor ∞. Then we can take a neighborhood of η_{-N} whose pull-back along $\hat{z}$ and $\hat{w}$ is eventually univalent. Since we can lift paths $\{\eta_{-N-n}\}$ to a path in $\mathcal{N}_f$ joining $\hat{z}$ and $\hat{w}$, $\hat{z}$ and $\hat{w}$ are in the same leaf in $\mathcal{R}_f$, say L. Now we have $\hat{H}^{-1}(L') \subset L$, and thus $L' \subset \hat{H}(L)$.

Case 1: Suppose that $\hat{H}(L)$ contains either $\hat{\beta}$ or $\hat{\infty}$. Since $\hat{H}^{-1}(\hat{\beta}) = \hat{I}(\hat{\Theta})$ and $\hat{H}^{-1}(\hat{\infty}) = \hat{\infty}$, it is equivalent to $L \cap \hat{I}(\Theta) \neq \emptyset$, that is, $L = L_j$ for some j modulo q. Since $\hat{\beta}$ and L' are disjoint, we have

$$\hat{H}^{-1}(L') \subset L_j - \hat{H}^{-1}(\hat{\beta}) = L_j - \hat{I}(\hat{\Theta}) = L_j - \hat{I}_j \cup \{\hat{\gamma}_j\}.$$

Let us set $L_j^- := L_j - \hat{I}_j \cup \{\hat{\gamma}_j\}$ for simplicity. Then we have $L' \subset \hat{H}(L_j^-)$. Since L_j^- is path connected, so is $\hat{H}(L_j^-)$ and thus contained in a leaf of $\mathcal{R}_g$, which must be L'. Thus we have $\hat{H}(L_j^-) = L'$ and it implies

$$L_j^- \subset \hat{H}^{-1}(\hat{H}(L_j^-)) = \hat{H}^{-1}(L') \subset L_j^-.$$

Let us show (3) by checking $L' = L_j'$. Set

$$\hat{R}_j := \{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_f : z_{-n} \in R_f(\theta_{j-np})\} \text{ and}$$
$$\hat{R}_j' := \{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_g : z_{-n} \in R_g(\theta_{j-np})\}.$$

Then $\hat{R}_j \subset L_j^-$ and $\hat{R}_j' \subset L_j'$. Moreover, $\hat{H}$ maps $\hat{R}_j$ onto $\hat{R}_j'$ univalently. Thus $L' = \hat{H}(L_j^-)$ must be L_j'.

Case 2: Suppose that $\hat{H}(L)$ contains neither $\hat{\beta}$ nor $\hat{\infty}$. It is equivalent to $L \neq L_j$ for any j modulo q. Since $\hat{H}(L) \subset \mathcal{R}_g$ is path connected, there is a leaf of $\mathcal{R}_g$ containing $\hat{H}(L)$, which must be L'. In particular, by property (3), $L' \neq L_j'$ for any j modulo q. Now we have $\hat{H}(L) = L'$ and thus

$$L \subset \hat{H}^{-1}(\hat{H}(L)) = \hat{H}^{-1}(L') \subset L.$$

Hence we conclude property (5).

Property (4) comes from (3) and (5). Take a leaf $L \in \mathcal{R}_f - \bigsqcup L_j$. Then $\hat{H}(L)$ is path connected and thus contained in a leaf $L' \in \mathcal{R}_g - \bigsqcup L_j'$. Then we have $L \subset \hat{H}^{-1}(L') = L$ by (5), and it implies $\hat{H}(L) = L'$, a leaf in $\mathcal{R}_g - \bigsqcup L_j'$. ■

2.6.3 Degeneration of periodic leaves.

Let us describe property (3) in further detail. For any j modulo q, L_j compactly contains all but one component of $\mathcal{I}_f \cap L_j$. The exception is $\hat{H}^{-1}(\hat{\beta}) \cap L_j = \hat{I}_j \cup \{\gamma_j\} \subset \hat{I}(\Theta)$. Since $\hat{I}_j \cup \{\hat{\gamma}_j\}$ and $\hat{\beta}$ are invariant under the action of $\hat{f}^q$ and $\hat{g}^q$ respectively, the map

$$\hat{H}|\ L_j - \hat{I}_j \cup \{\hat{\gamma}_j\} = L_j^- \to L_j'$$

is a semiconjugacy from $\hat{f}^q|_{L_j^-}$ to $\hat{g}^q|_{L_j'}$. Let us describe this semiconjugacy more precisely.

An analytic model. We start with an analytic model of the dynamics on $\bigsqcup L_j$ and $\bigsqcup L_j'$. Let $\mathbb{C}_1, \ldots, \mathbb{C}_q$ be q copies of $\mathbb{C}$ again, taking subscripts modulo q. Set

$$\tilde{\lambda} := \sqrt[q]{f'(\gamma_1) \cdots f'(\gamma_q)}$$

where the q-th root is taken to be the closest to 1. Set $\tilde{a} := 1/(1-\tilde{\lambda})$. Then $\tilde{a}$ is fixed by the linear map $S(W) = \tilde{\lambda}(W-\tilde{a})+\tilde{a} = \tilde{\lambda}W+1$. Note that as $r \to 1$ ($f \to g$), $|\tilde{a}| \to \infty$ and S converges to $W \mapsto W+1$ on any compact subset of $\mathbb{C}_j$. Now we define a "linear map" $\tilde{F} : \bigsqcup \mathbb{C}_j \to \bigsqcup \mathbb{C}_j$ by

$$\mathbb{C}_j \ \ni \ W \ \overset{\tilde{F}}{\longmapsto} \ S(W) \ \in \ \mathbb{C}_{j+p}.$$

Then for each j modulo q, $\tilde{F}^q | \ \mathbb{C}_j \to \mathbb{C}_j$ is the same as $S^q(W) = \tilde{\lambda}^q(W-\tilde{a})+\tilde{a}$.

On the other hand, we define a map $\tilde{G}$ as a copy of $G : \bigsqcup \mathbb{C}_j \to \bigsqcup \mathbb{C}_j$ in the construction of $\mathcal{T}_g$. Then for each j modulo q, $\tilde{G}^q | \ \mathbb{C}_j \to \mathbb{C}_j$ is the same as $W \mapsto W+q$.

Simultaneous uniformization. For f (resp. g), take a linearizing (resp. Fatou) coordinate Φ_1 on a neighborhood V_1 (resp. repelling petal Π_1^+) of γ_1 (resp. β) such that the action of f^q (resp. g^q) is conjugate to $S^q(w) = \tilde{\lambda}^q(w-\tilde{a})+\tilde{a}$ (resp. $w \mapsto w+q$). In particular, for $\rho > 1$ sufficiently close to 1, V_1 (resp. Π_1^+) contains $\zeta_0 = \phi_f^{-1}(\rho e^{2\pi i \theta_1})$ (resp. $\phi_g^{-1}(\rho e^{2\pi i \theta_1})$) and $\Phi_1(\zeta_0) = 0$. Then for any $\hat{z} = (z_0, z_{-1}, \ldots) \in L_1$, there exists an N such that $z_{-nq} \in V_1$ (resp. Π_1^+) for any $n \geq N$. By [3, §4], an isomorphism between L_1 and $\mathbb{C}_1$ is given by:

$$\hat{\Phi}_f|_{L_1}(\hat{z}) := (S)^{Nq}(\Phi_1(z_{-Nq})).$$

Similarly, an isomorphism between L_1' and $\mathbb{C}_1$ is given by:

$$\hat{\Phi}_g|_{L_1'}(\hat{z}) := \Phi_1(z_{-Nq}) + Nq.$$

One can easily check that they do not depend on the choice of N. For $k = 1, \ldots, q-1$, we define $\hat{\Phi}_f : L_{1+kp} \to \mathbb{C}_{1+kp}$ and $\hat{\Phi}_g : L_{1+kp}' \to \mathbb{C}_{1+kp}$ by

$$\hat{\Phi}_f := \tilde{F}^k \circ \hat{\Phi}_f|_{L_1} \circ \hat{f}^{-k} \ \text{ and } \ \hat{\Phi}_g := \tilde{G}^k \circ \hat{\Phi}_f|_{L_1'} \circ \hat{g}^{-k}.$$

Then for each j modulo q, $\hat{\Phi}_f| \ L_j \to \mathbb{C}_j$ and $\hat{\Phi}_g| \ L_j' \to \mathbb{C}_j$ give isomorphisms respectively. Moreover, $\hat{\Phi}_f : \bigsqcup L_j \to \bigsqcup \mathbb{C}_j$ has a property that for any $\hat{z} \in L_j$, $\hat{\Phi}_f(\hat{f}(\hat{z})) = \tilde{\lambda}\hat{\Phi}_f(\hat{z}) + 1 \in \mathbb{C}_{j+p}$. On the other hand, $\hat{\Phi}_g : \bigsqcup L_j' \to \bigsqcup \mathbb{C}_j$ also has a property that $\hat{\Phi}_g(\hat{g}(\hat{z})) = \hat{\Phi}_g(\hat{z}) + q \in \mathbb{C}_{j+p}$ for any $\hat{z} \in L_j'$. Informally, $\hat{\Phi}_f(\hat{\gamma}_j) = \tilde{a} \in \mathbb{C}_j$ tends to "∞" as $f \to g$ and $\tilde{I}_j := \hat{\Phi}_f(\hat{I}_j) \subset \mathbb{C}_j$ is an open path joining $\tilde{a} \in \mathbb{C}_j$ and "∞" which is invariant under the action of $\tilde{F}^q$.

Now let us consider the map

$$\hat{\Phi}_g^+ \circ \hat{H} \circ (\hat{\Phi}_f)^{-1} : \bigsqcup (\mathbb{C}_j - \hat{I}_j \cup \{\tilde{a}\}) \to \bigsqcup \mathbb{C}_j,$$

which is a semiconjugacy from $\tilde{F}|_{\bigsqcup(\mathbb{C}_j - \hat{I}_j \cup \{\tilde{a}\})}$ to $\tilde{G}$. The "slits" $\tilde{I}_j \cup \{\tilde{a}\}$ of each $\mathbb{C}_j$ are just like pinched and pushed away to "infinity". Topologically the same thing happens on the periodic leaves. By $\hat{H}$, the slits $\hat{I}_j \cup \{\hat{\gamma}_j\}$ are pinched, and pushed away to their common "point at infinity" $\hat{\beta}$. As a result, each $\pi_g^{-1}(J_g) \cap L_j'$ is split into two components. (See Figure 2.7)

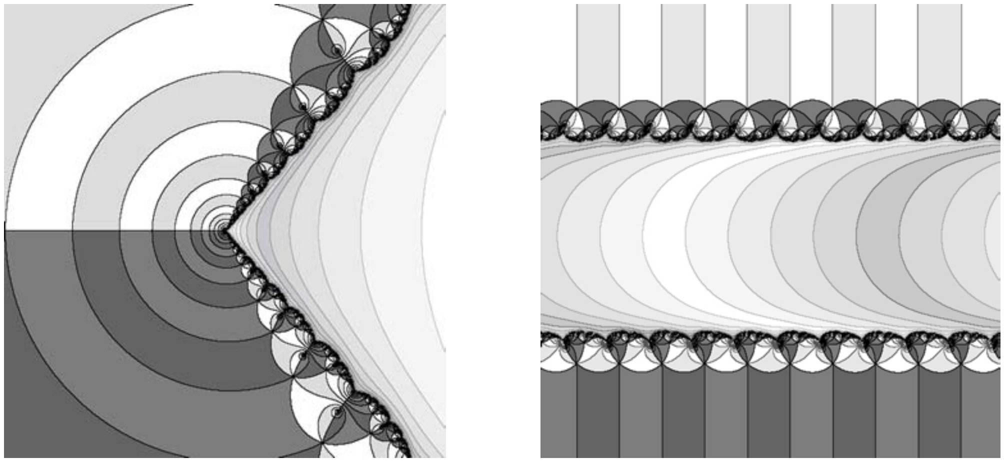

Figure 2.7: Invariant leaves of an $f \in \mathcal{F}(1/1)$ and $g = f_{1/4}$, parabolic with one petal.

Notes.

1. Both $\mathcal{R}_f$ and $\mathcal{R}_g$ have the structures of Riemann surface lamination. More precisely, each point of $\mathcal{R}_f$ (resp. $\mathcal{R}_g$) has a neighborhood homeomorphic to $D \times T$, where D is a topological disk and T is a Cantor set, and each $t \in T$, $D \times \{t\}$ corresponds to a topological disk on a leaf of $\mathcal{R}_f$ (resp. $\mathcal{R}_g$). (See [3, §2].) $\hat{H}$ preserves the Cantor set direction of such neighborhoods, and the holonomies of fibers of π_f and π_g.

2. The hyperbolic 3-lamination of f is constructed by adding "height" to the leaves of $\mathcal{R}_f$ to obtain leaves isomorphic to $\mathbb{H}^3$. Though the actual construction in [3] is very complicated, we may hope that the pinching $\hat{H}$ will naturally extend to this hyperbolic 3-lamination and describe the degeneration as f tends to g.

2.7 Bifurcation of the regular leaf spaces

Next we investigate the regular leaf space of another f_c which has an attracting cycle of period q generated by bifurcation of the parabolic fixed point β of $g = f_\sigma$ in preceding sections.

By Douady and Hubbard theory, σ in the parameter space is the root point of p/q-wake. Let $\mathcal{H} = \mathcal{H}(p/q)$ be the hyperbolic component attaching to the main cardioid at σ. Then it is known that for any $c \in \mathcal{H}$, f_c has an attracting cycle of period q, and there is a canonical homeomorphism from the unit disk $\mathbb{D}$ to $\mathcal{H}$ which parameterize the multiplier of the attracting cycles. For fixed $0 < R < 1$ (which is distinct from R in §4), we take the unique $c \in \mathcal{H}$ such that $f = f_c$ has an attracting cycle with multiplier R^q. For any $c' \in \mathcal{H}$ other than the center (that is, the image of the origin by the canonical homeomorphism above), $f_{c'}$ are quasiconformally conjugate to f. Thus the structure of the regular leaf spaces are topologically the same, and it is enough to consider the structure of $\mathcal{R}_f$.

We start with some notation. Let $\alpha_1, \ldots, \alpha_q$, taking subscript modulo q, be the attracting cycle of f with $f(\alpha_j) = \alpha_{j+p}$. Let γ be the repelling fixed point with portrait $\Theta = \Theta(p/q)$. Here the term *portrait* means the set of angles of external rays landing at the point, just as in the case of β. For any preimage γ' of γ, we also use this term. For each j modulo q, set

$$\hat{\alpha}_j := (\alpha_j, \alpha_{j-p}, \alpha_{j-2p}, \ldots) \in \mathcal{N}_f.$$

Then Proposition 2.6.2 easily extends to the following:

Proposition 2.7.1 *$\mathcal{R}_f$ is a Riemann surface lamination with the following properties:*

- $\mathcal{R}_f = \mathcal{N}_f - \{\hat{\infty}, \hat{\alpha}_1, \ldots, \hat{\alpha}_q\}$
- *Each leaf of $\mathcal{R}_f$ is isomorphic to $\mathbb{C}$.*

According to the method described in the preceding sections, let us describe the structure of $\mathcal{R}_f$ for this new f by reconstructing the semiconjugacy $\hat{H} : \mathcal{N}_f \to \mathcal{N}_g$.

2.7.1 Linearizing coordinate and tessellation

Linearizing coordinate. For each j modulo q, let V_j be the attracting basin of α_j by the action of f^q. Take q copies $\mathbb{C}_1, \ldots, \mathbb{C}_q$ of $\mathbb{C}$ again, and define the "isomorphism" $F : \bigsqcup \mathbb{C}_j \to \bigsqcup \mathbb{C}_j$ by the same map as in §4. Suppose that V_k contains the critical point 0 of f. There is a unique linearizing coordinate $\Phi_k : V_k \to \mathbb{C}_k$ such that $\Phi_k(f^q(z)) = R^q(\Phi_k(z) - a) + a$ and $\Phi_k(0) = 0$, where $a = 1/(1-R)$. For any $n = 0, \ldots, q-1$, we redefine $\Phi_f : K_f^\circ = \bigsqcup V_j \to \bigsqcup \mathbb{C}_j$ by

$$V_{k-np} \ \ni \ z \ \overset{\Phi_f}{\longmapsto} \ F^{-n} \circ \Phi_k \circ f^n \in \ \mathbb{C}_{k-np}.$$

Tessellation of K_f°. For each j modulo q, take a univalent branch $\Psi_j : \mathbb{C}_j - (-\infty, 0]_j \to V_j$ of Φ_f such that $\Psi_j(a) = \alpha_j$. Let I_j be the copy of the interval (a, ∞) in $\mathbb{C}_j$. Then one can check that $I'_j := \Psi_j(I_j)$ is invariant under the action of f^q and is an open arc joining α_j and γ. The rays $R_f(\theta_1), \ldots, R_f(\theta_q)$ divide the plane into q sectors, and now we may suppose that I'_j is contained in one of the q sectors bounded by $R_f(\theta_j)$ and $R_f(\theta_{j+1})$. We also denote I'_j by $I(\theta_j)$. For the portrait Θ of γ, we redefine $I(\Theta)$ by

$$I(\Theta) := \bigcup_{j=1}^{q} \overline{I(\theta_j)} = \{\gamma\} \cup \bigcup_{j=1}^{q} (I(\theta_j) \cup \{\alpha_j\}),$$

and the degenerating arc system I_f by $\bigcup_{n\geq 0} f^{-n}(I(\Theta))$.

For each j modulo q and $m \in \mathbb{Z}$, we redefine $A(m,\pm)_j$ by replacing $\chi(I_j)$ in the previous definition in §4 by this I_j. Let $\gamma' \in f^{-n}(\gamma)$ $(n = 1, 2, \ldots)$ with portrait $\Theta' = \{\theta'_1, \ldots, \theta'_q\}$ satisfying $2^n\theta'_j = \theta_j$. Then there is a component V of V_j attached to γ' contained in the sector bounded by $R_f(\theta_j)$ and $R_f(\theta_{j+1})$. On $\mathbb{C}_j - (-\infty, 0]_j$, there is a univalent branch Ψ of Φ_f^{-1} which maps I_j into V. By extending Ψ on the interiors of $A(m,\pm)_j$ to their edges, we define the tiles in K_f° by

$$T_f(\theta'_j, m, +) := \Psi^{-1}(A(m,+)_j) \subset V_j$$
$$T_f(\theta'_{j+1}, m, -) := \Psi^{-1}(A(m,-)_j) \subset V_j.$$

For $\theta \in \tilde{\Theta}$ and $m \in \mathbb{Z}$, the family $\{T_f(\theta, m, \pm)\}$ gives the tessellation $\mathcal{T}_f$ of $K_f^\circ - I_f$.

Edge sharing. Tiles of $\mathcal{T}_f$ has the same property of edge sharing as those of $\mathcal{T}_g$. To check the fact, one can start with the closed path C_0 defined as following. For each j modulo q, take a path $\eta_j \subset \mathbb{C}_j$ which comes from $+\infty$ along the real axis in the negative direction, turn around the circle $|W - a| = \sqrt{R}\,a$ anticlockwise, and then return to $+\infty$ along the real axis in the positive direction. Then the pull-back η'_j of η_j by Φ_j is a path in V_j and the union $C_0 := \{\gamma\} \cup \bigcup \eta'_j$ is a closed path. Now we may consider C_0 as a map $C_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ with $C_0(\theta_j) = \gamma$ for any j modulo q. Let C_{-n} be the pull-back of this path by f^{-n}, and then the same argument in the case of $\mathcal{T}_g$ works.

Note that for a preimage γ' of γ with portrait $\Theta' = \{\theta'_1, \ldots, \theta'_q\}$ as above, $T_f(\theta'_j, m, +)$ shares its degenerating edge with $T_f(\theta'_{j+1}, m, \pm)$.

Remark. We can simplify the tessellation above and $\mathcal{T}_g$ without changing the combinatorics of tiles. For each angle and signature, glue q tiles along their circular edges such that the two vertices of the critical edge of this new tile are contained in the grand orbit of the critical point 0. (Compare Figure 2.5 and Figure 2.8.)

2.7.2 Semiconjugacies

By gluing tile-to-tile homeomorphisms and the conjugacy outside the Julia sets induced from Böttcher coordinates, we have a semiconjugacy $H : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g which corresponds to the semiconjugacy in Theorem 2.5.1. In particular, H pinches a component of I_f containing a preimage γ' of γ into $\beta' \in I_g$ with the same portrait as γ'.

Let us consider the pinching in the natural extentions. Now we can redefine $\hat{I}(\hat{\Theta}')$, $\mathcal{I}_f$ and $\mathcal{I}_g$ in the same way as §6. Note that $\hat{I}(\hat{\Theta})$ contains $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_q\}$, however, $\hat{I}(\hat{\Theta}) - \{\hat{\alpha}_1, \ldots, \hat{\alpha}_q\}$ is in $\mathcal{R}_f$. In fact, if we set

$$L_f := \{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_f : z_{-n} \to \gamma\},$$

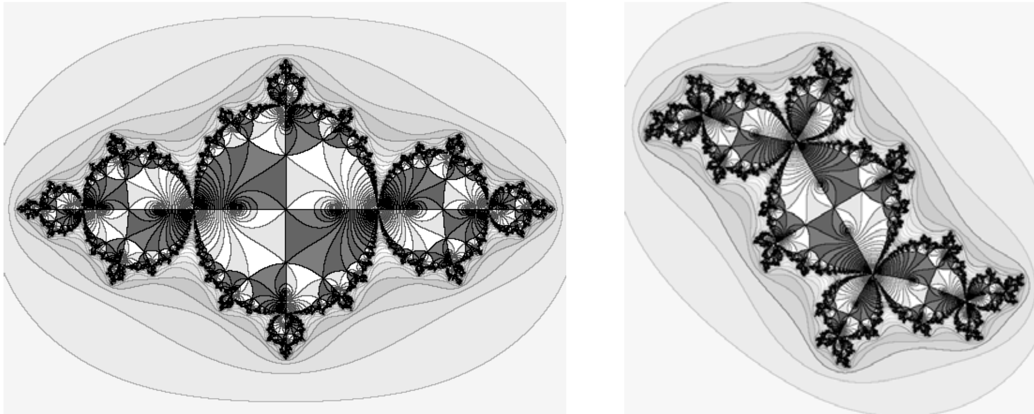

Figure 2.8: Simplified tessellation for an $f \in \mathcal{H}(1/2)$ and another $f \in \mathcal{H}(1/3)$. (Here we identify c in the parameter space with f_c.)

which is an invariant leaf isomorphic to $\mathbb{C}$, then L_f contains $\hat{I}(\hat{\Theta}) - \{\hat{\alpha}_1, \ldots, \hat{\alpha}_q\}$ *non-compactly*. In addition, the action of $\hat{f}$ on L_f is conjugate to that of $W \mapsto f'(\gamma)W$ on $\mathbb{C}$. The result corresponding to Theorem 2.6.3 is:

Theorem 2.7.2 *For f and g as above, there exists a semiconjugacy $\hat{H} : \mathcal{N}_f \to \mathcal{N}_g$ from $\hat{f}$ to $\hat{g}$ with the following properties:*

(1) $\hat{H} : \mathcal{N}_f - \mathcal{I}_f \to \mathcal{N}_g - \mathcal{I}_g$ is a topological conjugacy between $\hat{f}|_{\mathcal{N}_f - \mathcal{I}_f}$ and $\hat{g}|_{\mathcal{N}_g - \mathcal{I}_g}$.

(2) For any $\hat{\beta}'$ with portrait $\hat{\Theta}'$, $\hat{H}^{-1}(\hat{\beta}') = \hat{I}(\hat{\Theta}')$. In particular, $\hat{H}^{-1}(\hat{\beta}) = \hat{I}(\hat{\Theta})$.

(3) For each j modulo q, $\hat{H}^{-1}(\bigsqcup L'_j) = L_f - \hat{I}(\hat{\Theta})$.

(4) $\hat{H}$ maps a leaf of $\mathcal{R}_f - L_f$ onto a leaf of $\mathcal{R}_g - \bigsqcup L'_j$.

(5) For a leaf L of $\mathcal{R}_g - \bigsqcup L'_j$, $\hat{H}^{-1}(L)$ is a leaf of $\mathcal{R}_f - L_f$.

Sketch of the theorem. Follow the argument in Theorem 2.6.3. To show (3), (4) and (5), take a leaf L' in $\mathcal{R}_g$. Then $\hat{H}^{-1}(L')$ is contained in a leaf L of $\mathcal{R}_f$. If $H(L)$ intersects the irregular points (Case 1), then $L = L_f$ and $H^{-1}(L') \subset L_f - \hat{I}(\hat{\Theta}) := L_f^-$. Note that L_f^- is the union of q sectors divided by $\hat{I}(\hat{\Theta})$. By the correspondence of $\hat{R}_j \subset L_f^-$ to $\hat{R}'_j \subset L'_j$, we obtain $\hat{H}(L_f^-) = \bigsqcup L'_j$ and this implies property (3). If $H(L)$ and the irregular points are disjoint (Case 2), then $L \neq L_f$. Now (4) and (5) follows as in Theorem 2.6.3. ■

Structure of $\mathcal{R}_f$. As c of f_c changes from 0 to the center of the $\mathcal{H}$, the transversal Cantor set direction of the Riemann surface lamination $\mathcal{R}_{f_c}$ is preserved. However, the periodic leaves $L_1, \ldots, L_q$ of $f \in \mathcal{F}(p/q)$ with an affine loxodromic dynamics are pinched to be the periodic leaves $L'_1, \dots, L'_q$ of g with an affine parabolic dynamics, and then $L'_1, \dots, L'_q$ merge into the invariant leaf L_f of $f \in \mathcal{H}(p/q)$ with an affine loxodromic dynamics.

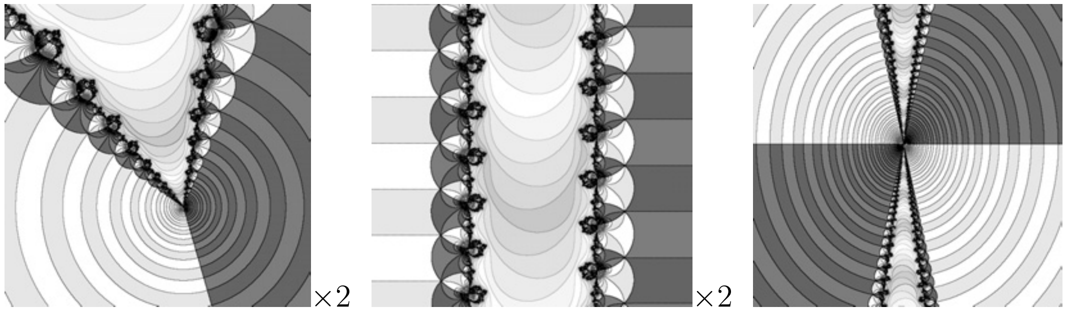

Figure 2.9: Periodic leaves of $f \in \mathcal{F}(1/2)$ become those of parabolic $g \in \overline{\mathcal{F}(1/2)} \cap \mathcal{H}(1/2)$, and merge into an invariant leaf of another $f \in \mathcal{H}(1/2)$.

Note. For any quadratic polynomial with an attracting cycle, we can consider its degeneration to a parabolic cycle with multiple petals. To investigate the associated degeneration of the regular leaf spaces, the method developed in this chapter is useful. For any quadratic polynomial with an attracting or parabolic cycle, we can define the tessellation of the interior of its filled Julia set by using the notion of *orbit portrait*. The degeneration of tiles induces a semiconjugation from a hyperbolic map to a parabolic map, and we can naturally lift it to their natural extensions. Then the lifted semiconjugation gives us essential information about the degeneration (or bifurcation) of the regular leaf spaces.

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