Dynamics of Quadratic Polynomials: Geometry and Combinatorics of the Principal Nest

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Abstract of the Dissertation,

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In order to study the combinatorics of non-renormalizable recurrent quadratic polynomials, we supplement the definition of principal nest with a *frame system*. This provides enough information to describe the *combinatorial type* for every level of the nest. As a consequence, we give necessary and sufficient conditions for the admissibility of a type and prove that given a sequence of non-renormalizable finite admissible types, there is a polynomial whose nest realizes it. In particular, we give a classification of *maximal* hyperbolic components of the Mandelbrot set according to combinatorial type.

In Chapter 4, we present some families of maps that are easily described in terms of their frame descriptions. For any irrational ρ we give infinite families of maps whose postcritical sets behave like adding machines of variable stepsize and are semi-conjugate to the

circle rotation of angle ρ. Then, for superattracting polynomials Q in a prime hyperbolic component we introduce the class of Q-recurrent maps.

In Chapter 5, we develop the geometric properties of the last class. The nest pieces of a Q-recurrent polynomial f are shown to converge in shape to the filled Julia set of Q. From this result we can deduce the exact rate of convergence of the principal moduli.

As a particular example, we give a complete characterization of the family of complex quadratic Fibonacci polynomials: They are precisely the set of (z^2-1)-recurrent polynomials. This is a Cantor set of Hausdorff dimension 0 possessing a natural dyadic structure.

In Chapter 6 we transfer the previous results to the parameter plane. There, the paranest pieces around a Q-recurrent parameter c_Q converge to K_Q.

As a consequence of the parametric results, we obtain the following auto-similarity result: Let $c_1, c_2 \in \partial M$ be two parameters such that f_{c_2} has no parabolic points or Siegel disks. Then there exists a sequence of parapieces $\{\Upsilon_1, \Upsilon_2, \ldots\}$ (most likely not nested) converging to c_1 as compact sets, but such that $\Upsilon_n \longrightarrow K_{c_2}$ in shape.

To Olga

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Chapter 1

Introduction

We will consider quadratic polynomials $f_c(z) : z \mapsto z^2 + c$ with recurrent critical orbit $\mathcal{O} = \{0, c, f(c), \ldots\}$ and study the behavior of $\mathcal{O}$ by means of the *principal nest*. This is a collection of strictly nested domains (a subcollection of the set of *puzzle pieces*) around the critical point 0, determined by the combinatorics of $\mathcal{O}$. At every level there is a central piece surrounding 0 and (possibly) some lateral pieces nearby enclosing portions of $\mathcal{O}$. The number of pieces in each level plus the way they map into pieces of the previous level, provide most of the combinatorial information that distinguishes between different behaviors.

Each level of the nest can be given the structure of a *generalized quadratic map*. This idea opens the door for various analytic techniques, but does not make full use of the structure of the underlying Julia set. For real polynomials, this is not a problem since the invariant set is an interval. Because of this, the structure of a nest level is given by just deciding whether each piece is on the left or the right side of the critical point plus giving the orientation of its first return map. Compare the discussion in Section 3.4.

For the complex case the situation is more involved as the nest pieces can be scattered arbitrarily around 0. In this work, we provide the nest with the extra structure of a *frame system*. This will give us a language to describe the relative positions of pieces within a level and their interaction with previous levels. We will exploit this construction to classify combinatorial types.

1.1 Historical background

One of the most sought for open problems in Complex Dynamics is the MLC Conjecture stating that the Mandelbrot set is locally connected.

MLC was proved for quasi-hyperbolic parameters by Douady and Hubbard (see [DH1]), and for boundaries of hyperbolic components by Yoccoz. Later, Yoccoz introduced his puzzle partition to prove MLC at finitely renormalizable parameters.

The concept of *puzzle* originates in [BH] where the authors consider cubic polynomials with one critical orbit escaping to ∞. In this setting, appropriate pull-backs of a chosen equipotential curve around the Julia set K will enclose a nested collection of topological disks. Then, certain combinatorial properties ensure that the intersection of these *pieces* shrinks to a single point $x \in K$. This proves that K is a Cantor set. The more refined version of puzzles given by Yoccoz, is adapted to quadratic polynomials with both fixed points repelling; in this case, the definition of the puzzle is more involved, as the pieces do not enclose whole components of the Julia set.

The puzzle of Yoccoz decomposes a neighborhood of a quadratic Julia set K_c in pieces that map onto each other by f_c. The pieces centered around 0 define a nested sequence of annuli (some degenerate) and Yoccoz showed that the sum of their moduli goes to ∞ when the map is not infinitely renormalizable. This result implies local connectivity of K_c and can be translated to M, to obtain MLC at the parameter c. In particular, this established combinatorial rigidity for these maps.

In [L3], Lyubich introduced the principal nest as a tool to provide the first examples of infinitely renormalizable parameters satisfying MLC. The nest is well defined for recurrent critical orbits. A sequence of central pieces is determined by consecutive first returns to the preceding level and such pieces form non-degenerate annuli over whose moduli it is possible to exert better control. The main result of [L3] states the following.

Theorem L1. (Lyubich) *Let $\kappa(n)$ count the levels of the principal nest up to level n, which are non-central. Then the moduli of the principal annuli grow linearly:*

$$\mu_{(n+2)} \geq B \cdot \kappa(n) + C.$$

where the constant B depends only on the initial modulus.

This fundamental property of the nest lies at the heart of many analytic results of Lyubich, including a proof of the Feigenbaum-Collet-Tresser Conjecture, the Theorem on the measure-theoretic attractor, density of real hyperbolic maps and local connectivity of infinitely renormalizable parameters of bounded type.

A fundamental step in the proof of Theorem L1 is obtaining the result for the particular case of Fibonacci combinatorics, which can be thought of as the worst case of moduli growth. The defining behavior requires exactly one lateral piece at every level of the nest, mapping directly onto the central piece of previous level. [1]

The Fibonacci pattern of recursion comes closest to be renormalizable at every level without actually being renormalizable. Because of this, it was believed to possess a very wild recursion pattern and it was extensively studied for some time. In [LM], the authors show that Fibonacci combinatorics are rather tame. They prove that there is a unique real quadratic polynomial $x \mapsto x^2 + c_{\text{fib}}$ with such behavior and compute the exact rate $r = \frac{\ln 2}{3}$ of linear growth of its scaling factors.

The last result has a different interpretation when we regard $z \mapsto z^2 + c_{\text{fib}}$ as a complex map. From the same paper it follows that the principal moduli for this parameter grow exactly at a linear rate. Then, in [W], L. Wenstrom shows that this growth has the same coefficient r. This result is a consequence of a precise control on the shapes of central pieces. In fact, in [L1] Lyubich showed that these pieces converge in shape to the filled Julia set of f_{-1}. Wenstrom translated this result to the Mandelbrot set M to obtain similar scaling results around the parameter c_{fib}. These include linear growth of the paramoduli with rate $2r$ and hairiness of M at c_{fib}. Our work has been greatly influenced by those ideas.

1.13 Overview

The purpose of this work is to present a discussion of combinatorics in the principal nest. The frame system will allow us to describe the itinerary of the critical orbit within nest levels and is hence, an excellent tool to refine some geometric results obtained from the study of the principal nest.

In Chapter 2, we introduce briefly the basic concepts of Complex Dynamics and fix notation. In particular, we describe the puzzle construction of Yoccoz and the principal nest according to Lyubich.

In Chapter 3 we define the frame system $\mathcal{F}$ associated to the nest. The construction requires care in order to ensure that nest levels and frame levels go hand by hand. Then we describe a labeling of frame cells and produce a language to describe combinatorial types. The main geometric result of this Chapter states that for an arbitrary sequence of admissible types there exists a parameter whose frame types at every level are as given. A corollary of this result is a classification of *maximal* hyperbolic components of M according to their combinatorial type.

[1] The name Fibonacci is due to the fact that the corresponding combinatorics forces the critical orbit to have closest returns to 0 exactly when the iterates are the Fibonacci numbers.

In Chapter 4 we give some examples of maps with interesting combinatorics. Both *complex rotation-like maps* and *Q-recurrent maps* are generalizations of the Fibonacci pattern.

Real rotation-like maps are constructed in [BKP]. For an irrational rotation number ρ, there is a real polynomial whose postcritical set is conjugate to certain adding machine of variable stepsize and admits a semi-conjugacy to a rotation of the circle by angle ρ. The Fibonacci parameter corresponds to the case when ρ is the golden ratio. For each rotation number, the Theorem on admissible combinatorics allows us to construct an uncountable family of complex maps whose postcritical set has an analogous behavior.

To define Q-recurrent maps, we fix first the superattracting map Q of period m associated to a prime hyperbolic component of M. A map f is called Q-recurrent if the sequence of its *frames* at every level has exactly the same structure as the sequence of *puzzles* of Q. In this manner, the combinatorial behavior of the first return maps is essentially identical to the action of Q on its own Julia set. The key to further results is that every first return map g_n turns out to be the composition of the m previous returns: $g_n = g_{n-m} \circ \ldots \circ g_{n-1}$.

The last example of Chapter 4 is a real parameter whose frames mimic the puzzle structure of the Chebyshev polynomial $z^2 - 2$. The main difference from Q-recurrent maps is that for every n, the first return g_n is the composition of all previous returns: $g_n = g_1 \circ \ldots \circ g_{n-1}$. Due to this difference, the methods of Chapter 5 cannot be used to establish convergence results. However, by describing the combinatorics carefully, we can provide a kneading sequence and show its admissibility; hence establishing the existence of a parameter with such behavior.

Chapter 5 is devoted to the geometric properties of Q-recurrent maps. Our first result is a complete characterization of complex quadratic Fibonacci polynomials: They are precisely the set of $(z^2 - 1)$-recurrent maps. This is a Cantor set, symmetric with respect to the real axis and intersecting it at one point. It can be given a natural dyadic structure in terms of the bifurcation of returns at each level.

For any Q-recurrent map f, the central nest pieces of f can be rescaled to constant diameter, so that the first return maps g_n induce functions G_n. The main result of this Chapter is the fact that these rescaled maps converge to the quadratic polynomial Q. As a consequence, the shape of the nest pieces approaches exponentially fast to the shape of the Julia set K_Q.

This control over the shape of pieces allows us to obtain precise analytic results. In particular, we can compute the exact rate of growth of the principal moduli of f. In the case of Fibonacci parameters, the growth is linear. For any other W-recurrent polynomial, the moduli grow exponentially at a rate that depends only on the period of Q..

The dyadic structure of the set of Fibonacci parameters is a combinatorial result belonging naturally to Chapters 3 and 5. Yoccoz's Theorem implies that they form a Cantor set, but we need to translate our results to the Mandelbrot set M in order to estimate its Hausdorff dimension. This is done in Chapter 6. There we show that the paranest pieces surrounding a Q-recurrent parameter also converge in shape to the filled Julia set of Q. From this result we can compute the rates of growth of paramoduli. This shows that the set of Fibonacci parameters (or any other Q-recurrent family) is a Cantor set of Hausdorff dimension 0.

As a final application of the results in parameter space, we obtain the following auto-similarity result: Let $c_1, c_2 \in \partial M$ be two parameters such that f_{c_2} has no parabolic points or Siegel disks. Then there exists a sequence of parapieces $\{\Upsilon_1, \Upsilon_2, \ldots\}$ (most likely not nested) converging to c_1 as compact sets, but such that $\Upsilon_n \longrightarrow K_{c_2}$ in shape.

An Appendix contains brief summaries of all the Complex Analysis tools that are used. Finally, let us mention that some of the pictures were generated with the PC program `mandel.exe` of Wolf Jung [J].

Chapter 2

Basics

2.1 Basic notions

In order to fix notation, let us start by defining the basic notions of complex dynamics that will be used; we refer the reader to [DH1] and [Mi] for details on this introductory material.

We focus attention on the *quadratic family* $\mathcal{Q} = \{f_c : z \mapsto z^2 + c\}$ depending on one complex parameter $c \in \mathbb{C}$ equal to the critical value. For every c, the compact sets $K_c = \{z \mid \text{the sequence } \{f_c^{\circ n}(z)\} \text{ is bounded}\}$ and $J_c = \partial K_c$ are called the **filled Julia set** and **Julia set** respectively. They are perfect sets with 2-fold symmetry around 0 and satisfy an important dichotomy: Depending on whether the orbit of the critical point 0 is bounded or not, J_c and K_c are connected or totally disconnected. Thus, in the latter case, $J_c = K_c$ is a Cantor set. Moreover, when the critical orbit is bounded, K_c is in fact cellular. We define the **Mandelbrot set** as $M = \{c \mid c \in K_c\}$; that is, the set of parameters with bounded critical orbit; see Figure 1. M is also a perfect set and cellular.

A component of $\operatorname{int} M$ that contains a superattracting parameter will be called a **hyperbolic component**[1]. The boundary of a hyperbolic component can either be real analytic, or fail to be so at one cusp point. The later kind are called **primitive**. Actually, these boundaries closely resemble round circles and cardioids. In particular, the hyperbolic component associated to $z \mapsto z^2$ is bounded by a true cardioid which we call the **main cardioid**.

The structure of M is extremely complicated. It contains small homeomorphic copies of itself, densely distributed around ∂M. In fact, every hyperbolic component H other than the main one is the base of one such small copy M'. H is called **prime** if it is not contained in any other small copy. Prime components attached to the main cardioid are called **immediate** components.

[1] It is conjectured that all interior components are hyperbolic.

Any other prime component is also primitive. To simplify later statements, a component that is both primitive and prime will be called **maximal**.

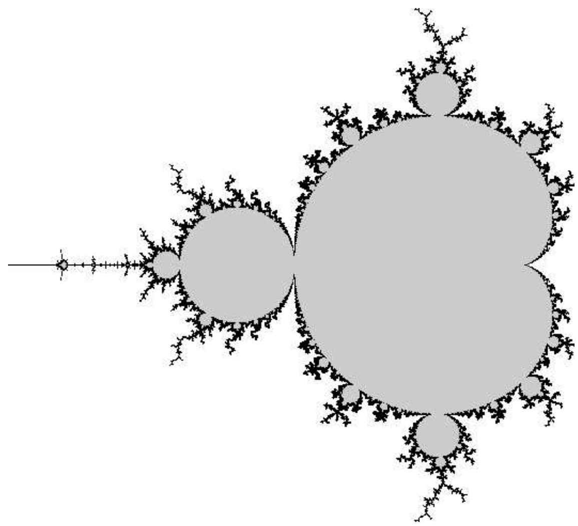

Figure 1: *The Mandelbrot set.*

2.2 External rays

The behavior of a polynomial near ∞ is very simple; we can treat f_c as a rational function $f_c : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ for which $f_c^{-1}(\infty) = \{\infty\}$. Then, ∞ is just a fixed critical point and a result of Böttcher yields a change of coordinates that conjugates f_c to $z \mapsto z^2$ in a neighborhood of ∞. We can extend this map to $\varphi_c : N_c \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}_R$ where $\mathbb{D}_R$ is the disk of maximal radius $R \geq 1$ on which the extension is possible. The map φ_c can be normalized by the requirement that the derivative of φ_c at ∞ is 1.

It can be shown that $N_c = \mathbb{C} \setminus K_c$ and $R = 1$ whenever $c \in M$. Otherwise, N_c is the exterior of a figure 8 curve that is real analytic, symmetric with respect to 0. When this is the case, $R > 1$ and K_c is contained in the two bounded regions determined by the curve.

Consider the system of radial lines and concentric circles in $\mathbb{C} \setminus \mathbb{D}_R$ characteristic of polar coordinates. Pulling these back by φ_c, we obtain a collection of **external rays** r_θ ($\theta \in [0, 2\pi)$) and **equipotential curves** e_s (here $s \in (1, \infty)$ is called the **radius** of e_s) on N_c. These form two orthogonal foliations that behave nicely under dynamics: $f_c(r_\theta) = r_{2\theta}$, $f_c(e_s) = e_{s^2}$. When $c \in M$, we say that a ray r_θ **lands** at $z \in J_c$ if z is the only point of accumulation of r_θ on J_c.

A similar coordinate system exists around the Mandelbrot set. For $c \notin M$, we define the map

$$\Phi_M(c) = \varphi_c(c). \tag{2.1}$$

In [DH1] it is shown that $\Phi_M : \overline{\mathbb{C}} \setminus M \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is a conformal homeomorphism tangent to the identity at ∞. This yields connectivity of M and allows us to define **parametric external rays** and **parametric equipotentials** as in the dynamical case. Since there is little risk of confusion, we will use the same notation (r_θ, e_s) to denote these curves and say that a parametric ray lands at a point $c \in \partial M$ if c is the only point of accumulation of the ray on M.

For the rest of this work, all rays considered, whether in dynamical or parameter plane, will have rational angles. These are enough to work out our combinatorial constructions and satisfy rather neat properties. In order to introduce our constructions, it will be enough to mention the following general result about landing rays.

Proposition 2.1. *If $\theta = \frac{p}{q} \in \mathbb{Q}$, the external ray r_θ lands, both in the parametric and the dynamical situations. In the dynamical case, the landing point is preperiodic with the period and preperiod determined by the binary expansion of θ. A point in J_c (respectively M) can be landing point of, at most, a finite number of rays (respectively parametric rays). If this number is larger than 1, each component of the plane split by the rays intersects J_c (respectively M).*

2.3 Wakes and limbs

When $c \neq \frac{1}{4}$, f_c has two distinct fixed points. If $c \in M$, these can be distinguished since one of them is always the landing point of the ray r_0. We call this fixed point β; note that it is always repelling, as it lies on J_c. The second fixed point, called α, can be attracting, indifferent or repelling, depending on whether the parameter c belongs to the region $\heartsuit$ enclosed by the main cardioid, to the cardioid itself or is outside.

The map $\psi_0 : \heartsuit \longrightarrow \mathbb{D}$ given by $c \mapsto f_c'(\alpha_c)$ is the Riemann map of $\heartsuit$ normalized by $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. Since the cardioid is a real analytic curve except at $\frac{1}{4}$, ψ_0 extends to $\overline{\heartsuit}$. The fixed point α is parabolic exactly at parameters $c_\eta \in \partial\heartsuit$ of the form $c_\eta = \psi_0^{-1}\left(e^{2\pi i\eta}\right)$ where $\eta \in \mathbb{Q} \cap [0, 1)$.

If $\eta \neq 0$, the Mandelbrot set M is separated in two pieces at these points. More precisely, each c_η is the landing point of two rays $r_{t^-(\eta)}$ and $r_{t^+(\eta)}$ in such a way that M intersects both components of $\mathbb{C} \setminus \left(r_{t^-(\eta)} \cup c_\eta \cup r_{t^+(\eta)}\right)$.

Definition 2.2. *The closure of the component of* $\mathbb{C} \setminus \left(r_{t^-(\eta)} \cup c_\eta \cup r_{t^+(\eta)}\right)$ *that does not contain* $\heartsuit$ *is called the* η***-wake*** *and is denoted* W_η. *The* η***-limb*** *is defined as* $L_\eta = M \cap W_\eta$.

Definition 2.3. $\mathcal{P}(\frac{p}{q})$ *denotes the unique set of rational ray angles whose behavior under doubling is a cyclic permutation with combinatorial rotation number* $\frac{p}{q}$.

If $\mathcal{P}(\frac{p}{q}) = \{t_1, \ldots, t_q\}$, then for any parameter $c \in L_{p/q}$, the corresponding point α splits K_c in q parts, separated by q rays $\{r_{t_1}, \ldots, r_{t_q}\}$ that land at α. The two rays whose angles span the shortest arc separate the critical point 0 from the critical value c; these two angles turn out to be $t^-(\eta)$ and $t^+(\eta)$.

2.4 Yoccoz puzzles

The Yoccoz **puzzle** is well defined for maps with both fixed points repelling; that is, $c \in L_{p/q}$ for some p and q. If 0 is not a preimage of α, the puzzle is defined at infinitely many depths and these are the parameters we will consider. Since we describe properties of a general parameter, we will omit the subscript and write f instead of f_c and so on.

Let us fix the neighborhood U of K bounded by the equipotential of radius 2. The rays that land at α determine a partition of $U \setminus \{r_{t_1}, \ldots, r_{t_q}\}$ in q connected components. We will call the closures $Y_0^{(0)}, Y_1^{(0)}, \ldots Y_{q-1}^{(0)}$ of these components, **puzzle pieces** of depth 0. At this stage the labeling is chosen so that $0 \in Y_0^{(0)}$ and $f\left(K \cap Y_j^{(0)}\right) = K \cap Y_{j+1}^{(0)}$; where the labels are understood as numbers modulo q. In particular $Y_1^{(0)}$ contains the critical value c and the angles of its bounding rays are precisely $t^-(\frac{p}{q}), t^+(\frac{p}{q})$.

Now we can define the puzzle pieces $Y_i^{(n)}$ of higher depths as the closures of connected components in $f^{\circ(-n)}\left(\bigcup \operatorname{int} Y_j^{(0)}\right)$; see Figure 2. At each depth n, there is a unique piece which contains the critical point and we will always choose the indices so that $0 \in Y_0^{(n)}$. The resulting family $\mathcal{Y}_c$ of puzzle pieces of all depths, has the following two properties:

- Any two puzzle pieces either are nested (with the piece of higher depth contained in the piece of lower depth), or have disjoint interiors.
- The image of any piece $Y_j^{(n)}$ $(n \geq 1)$ is a piece $Y_i^{(n-1)}$ of the previous depth $n-1$. The restricted map $f : \operatorname{int} Y_j^{(n)} \longrightarrow \operatorname{int} Y_i^{(n-1)}$ is a 2 to 1 branched covering or a conformal homeomorphism, depending on whether $j = 0$ or not.

These properties characterize $\mathcal{Y}_c$ as a Markov family and endow the puzzle partition with dynamical meaning.

Note that the collection of ray angles at depth n consists of all $f_c^{\circ n}$-preimages of $\{r_{t_1}, \ldots, r_{t_q}\}$ under angle doubling. The union of all pieces of depth n is the region enclosed by the equipotential $e_{\left(2^{2^{-n}}\right)}$. Note also that any piece Y eventually maps onto some piece of level 0. By further iterating it, Y will map onto a region determined by the same rays as $Y_0^{(0)}$ and a possibly larger equipotential. This means that there exists a 1 to 1 correspondence between puzzle pieces and preimages of 0. Such distinguished point inside every piece is called the **center**.

We will use the symbol P_n to denote the collection of all puzzle pieces at depth n. In order to describe its structure, we need a combinatorial model that reflects the adjacency between pieces.

Definition 2.4. *Given a set of puzzle pieces P, we consider the* ***dual graph*** *$\Gamma(P)$. It is an abstract graph whose set of vertices is in one to one correspondence with the puzzle pieces of P and whose edges join vertices corresponding to pieces that share an arc of external ray. We will regard $\Gamma(P)$ with its natural embedding in the plane.*

In the case $P = P_n$, we call $\Gamma(P_n)$ the ***puzzle graph*** *of depth n. It has two distinguished vertices; namely those containing the critical point and the critical value. The vertex that corresponds to the central piece $Y_0^{(n)}$, will be denoted ξ_n and the vertex for the piece around the critical value $f_c(0)$ is η_n.*

Definition 2.5. *The vertices ξ_n and η_n determine two partial orders on Γ_n as follows: We write $a \succ_{\eta_n} b$ when there is a path from a to η_n that passes through b. We write $a \succ_{\xi_n} b$ when there is a path from a to ξ_n that passes through b or its symmetric image with respect to ξ_n.*

The following are natural consequences of the definitions; see Figure 2 for reference.

Proposition 2.6. *The puzzle graphs of f satisfy:*

1. *$\Gamma(P_0)$ is a q-gon whenever $c \in L_{p/q}$. For $n \geq 1$, $\Gamma(P_n)$ consists of 2^n q-gons joined by their vertices in a tree-like structure; i.e. the only cycles on this graph are the q-gons themselves.*

2. *The graphs $\Gamma(P_n)$ have 2-fold central symmetry around the critical vertex. To be precise, removing ξ_n and its edges, splits $\Gamma(P_n)$ in two isomorphic graphs A_n and B_n, with A_n containing η_n. We define the graphs Puzz_n^- and Puzz_n^+ by including ξ_n again into each of A_n and B_n, then adding the corresponding edges to each.*

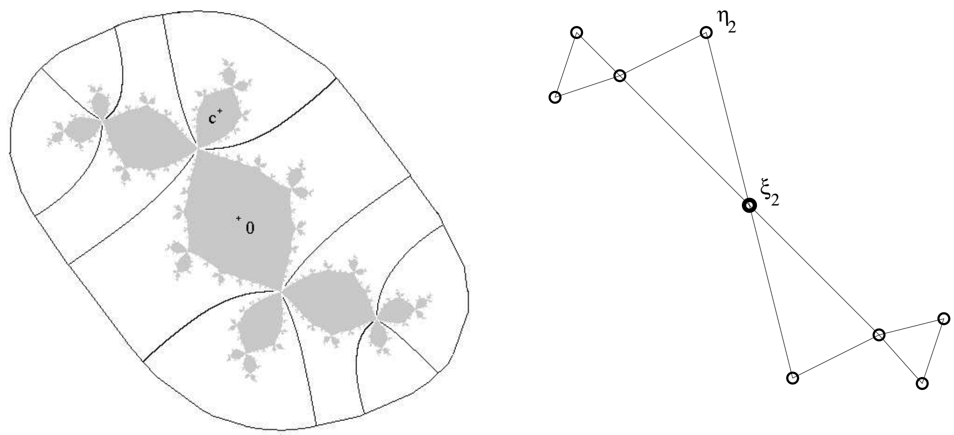

Figure 2: *Puzzle of depth 2 with its corresponding graph. Splitting the graph at* ξ_2 *we obtain the graphs* Puzz_2^- *and* Puzz_2^+*; both shaped like a bow tie and isomorphic to* $\Gamma_1(P)$*.*

3. $\Gamma(P_{n+1})$ *is completely determined by* $\{\Gamma(P_n), \eta_n\}$*. In fact, both* Puzz_{n+1}^- *and* Puzz_{n+1}^+ *are isomorphic to* $\Gamma(P_n)$ *with* $\xi_{n+1} \in \mathrm{Puzz}_{n+1}^{\pm}$ *corresponding to* η_n*.*

4. *There are two natural maps,* $D : \Gamma(P_{n+1}) \longrightarrow \Gamma(P_n)$ *induced by* f*, and* $R : \Gamma(P_{n+1}) \longrightarrow \Gamma(P_n)$ *induced by inclusion between pieces of consecutive depths. Moreover,* D *is 2 to 1 except at* ξ_{n+1} *and sends* $\mathrm{Puzz}_{n+1}^{\pm}$ *onto* $\Gamma(P_n)$*.*

5. *The map* $D : (\Gamma(P_{n+1}), \succ_{\xi_{n+1}}) \longrightarrow (\Gamma(P_n), \succ_{\eta_n})$ *respects order. That is,* $a \succ_{\xi_{n+1}} b \Rightarrow D(a) \succ_{\eta_n} D(b)$*.*

Definition 2.7. *Let* Γ *be a graph isomorphic to a subgraph of* Γ_{n+1} *and* Γ' *a graph isomporphic to a subgraph of* Γ_n*. Consider a map* $E : \Gamma \longrightarrow \Gamma'$ *that satisfies properties (1) and (2).* E *will be called* ***admissible*** *if it also respects order in the sense of property (5).*

Proof of Proposition 2.6: Properties (1), (4) and the first part of (2) are obvious. Property (3) and the second part of (2) follow when we consider the action of f on P_{n+1}. By the symmetry, every piece of P_{n+1} except the central one has a symmetric partner and they both map in a 1 to 1 fashion to the same piece of P_n. This justifies the existence of A_n and B_n. Since there is a cyclic permutation of pieces around α, f maps the central piece to a non-central one containing the critical value, so we are justified in selecting A_n as the graph containing η_n. The isomorphisms mentioned in (3) should be clear now.

To prove (5), let us construct a new graph Γ'_n with 2 to 1 central symmetry by collapsing every q-gon into a single vertex. The order in $\Gamma(P_n)$ is induced by the order in the tree Γ'_n. The corresponding map $D' : \Gamma'_{n+1} \longrightarrow \Gamma'_n$ is a 2 to 1 map on trees taking each half of Γ'_{n+1} injectively into a sub-tree of Γ'_n. Thus D' respects order and so does D. □

2.5 Parapuzzle

The puzzle is a dynamical construction in the sense that it reflects some characteristics of a given critical orbit. However, fixing a parameter c and depth n, there is a neighborhood of c where every parameter has the same behavior up to depth n. The **parapuzzle** encodes these similarities at every scale: For every wake of M, we define a partition in pieces of increasing depths, with the property that all parameters inside a given *parapiece* share the same initial critical orbit pattern.

Definition 2.8. *Consider a wake $W_{p/q}$ and let $n \geq 0$ be given. Call W^n the wake $W_{p/q}$ truncated by the equipotential $e_{\left(2^{2^{-n}}\right)}$ and consider the set of angles $\mathcal{P}_n(\frac{p}{q}) = \{t \mid 2^n t \in \mathcal{P}(\frac{p}{q})\}$ (refer to Definition 2.3). The* ***parapieces*** *of $W_{p/q}$ at depth n are the closures of the components of $W^n \setminus \{r_t \mid t \in \mathcal{P}_n(\frac{p}{q})\}$.*

Note: Even though the critical value $f_c(0)$ is just c, it will be convenient to write $c \in \Delta$ when Δ is a parapiece and $f_c(0) \in V$ when V is a piece in the dynamical plane of f_c.

We will also use the notation OBJ$[c]$ to refer to any dynamical object OBJ that is associated to a specific parameter c.

Definition 2.9. *If the boundary of a dynamical piece A is described by the same equipotential and ray angles as those of a parapiece B, we denote this relation by $\partial A \stackrel{\circ}{=} \partial B$.*

Definition 2.10. *Consider $c \in M$ with puzzles defined up to depth n. We denote by $\mathrm{CV}_n(c) \in P_n(f_c)$ be the piece of depth n such that $f_c(0) \in \mathrm{CV}_n(c)$.*

A consequence of formula 2.1 is the well known fact that follows, a proof of which can be found in [DH2] or [R]; also, see the Appendix for the definition of holomorphic motions.

Proposition 2.11. *Let Δ be a parapiece of depth n in some wake W. Then the family $\{c \mapsto \mathrm{CV}_n[c] \mid c \in \Delta\}$ is well defined; it determines a holomorphic motion of the critical value pieces and $\{c \mapsto f_c(0)\}$ is a section with winding number 1.*

The ray angles that constitute $\partial\,\mathrm{CV}_n[c]$ for a given parameter c, are completely determined by the orbit of $f_c(0)$ in the next $n-1$ iterations, together with the information about which wake contains c. Thus, we have the following.

Corollary 2.12. *Let $c_0 \in \Delta$, at depth n. Then, the combinatorial structure of $P_n(f_c)$ will remain unchanged for any other $c \in \Delta$ and only for those parameters.*

We can mention the following examples of combinatorial properties that depend on the behavior of the first n iterates of 0. The fact that these objects remain unchanged for $c \in \Delta$ follows from the above and will be useful to us in the next chapters.

- The graph $\Gamma(P_n(f_c))$.
- The combinatorial boundary of every piece of depth $\leq n$.
- The location in $P_n(f_c)$ of the first n iterates of the critical orbit.

From the general results of [L4], we can say more about the geometric objects associated to the above examples.

Proposition 2.13. *Each of the sets listed below moves holomorphically as c varies in Δ:*

- *The boundary of every piece of depth $\leq n$.*
- *The first n iterates of the critical orbit.*
- *The collection of j-fold preimages of α and β ($j \leq n$).*

In the case of the holomorphic motion $\{c \mapsto \mathrm{CV}_n(c)\}$, the proper section $\{c \mapsto f_c(0)\}$ has winding number 1; refer to [D2] and Proposition 3.3 of [L4]. We can interpret this result as loosely saying that, as c goes once around $\partial\Delta$, the critical value $f_c(0)$ goes once around $\partial\,\mathrm{CV}_n$. A consequence of this, which we will exploit in Chapter 6, is the following.

Proposition 2.14. *Let $r \geq 1$. Suppose that for all $0 < r' \leq r$, none of the pieces $f_{c_0}^{\circ r'}(\mathrm{CV}_n(c_0))$ contains the critical value $f_{c_0}(0)$. Then, for any fixed iterate $0 \leq s \leq r$, the family $\{c \mapsto f_c^{\circ s}(\mathrm{CV}_n(c)) \mid c \in \Delta_n(c_0)\}$ determines a holomorphic motion in which the section $\{c \mapsto f_c^{\circ s}(0)\}$ has winding number 1.*

Proof: It follows from Proposition 2.11 since $f_c^{\circ s}$ is univalent at $\mathrm{CV}_n(c)$. $\square$

2.6 Principal nest

The principal nest is well defined for parameters c that belong neither to $\overline{\heartsuit}$ nor to a prime component. The first condition means that both fixed points are repelling, while the second condition characterizes those polynomials that do not admit an *immediate renormalization* (to be defined shortly). When c is also recurrent, the nest is infinite. This conditions will justify themselves as we describe the nest.

In order to explain the construction of the nest, we need a more detailed description of the puzzle partition at depth 1 (use figure 3 for reference). As a note of warning, the pieces of depth 1 will be renamed to reflect certain properties of $P_1(f)$. That is, we will not use the symbols $Y_j^{(1)}$.

P_1 consists of $2q-1$ pieces of which $q-1$ are the restrictions of the pieces $Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_{q-1}^{(0)}$ to a lower equipotential. Such pieces cluster around α and will simply be denoted $Y_1, Y_2, \ldots, Y_{q-1}$. The restriction of $Y_0^{(0)}$ however, is further divided into the union of the critical piece $Y_0^{(1)}$ with $q-1$ pieces $Z_1, Z_2, \ldots, Z_{q-1}$ around $-\alpha$ which are symmetric to the corresponding Y_j. The indices are again determined by the rotation number of α, so that $f(Z_j)$ is opposite to Y_j; that is, $f(Z_j) = Y_{j+1}^{(0)}$.

Note that $f^{\circ q}(0) \in Y_0^{(0)}$, so we face two possibilities. It may happen that $f^{\circ jq}(0) \in Y_0^{(1)}$ for all j, in which case we can find a thickening of $Y_0^{(1)}$ leading to an **immediate renormalization** as defined by Douady and Hubbard; or else, we can find the least k for which the orbit of 0 under $f^{\circ q}$ escapes from $Y_0^{(1)}$. In this case, $f^{\circ kq}(0) \in Z_\nu$ for some ν and we call kq the **first escape time.**

We will define V_0^0 as the (kq)-fold pull-back of Z_ν along the critical orbit, so that $0 \in V_0^0$ and $f^{\circ kq}(V_0^0) = Z_\nu$. Since $0 \in V_0^0$ and kq is the first escape time, we see that $f_{|V_0^0}$ is a 2 to 1 branched cover but $f^{\circ kq-1}$ is univalent on the image $f(V_0^0)$. V_0^0 is the initial piece of the nest.

Note that $Z_\nu \Subset Y_0^{(0)}$ so $V_0^0 \Subset Y_0^{(1)}$; i.e. $\mathrm{int}\, Y_0^{(1)} \setminus V_0^0$ is a non-degenerate annulus. In fact, V_0^0 *is the first central piece that is compactly contained in* $Y_0^{(1)}$.

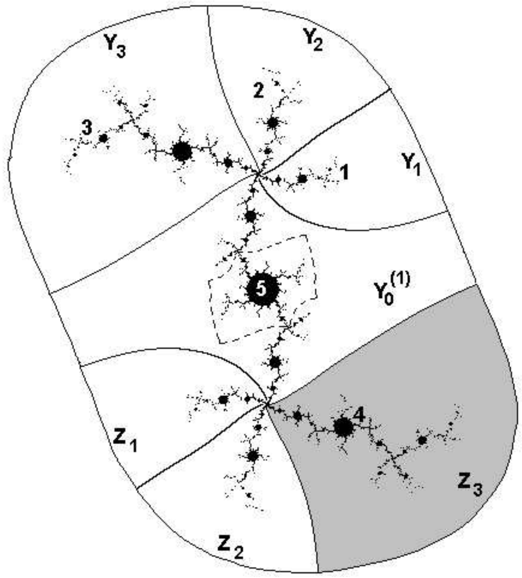

Figure 3: *Puzzle $P_1(f_c)$ of depth 1, where $c = (0.35926...) + i(0.64251...)$ is the center of the component of period 5 in $L_{1/4}$. The first escape is $f_c^{\circ 4}(0) \in Z_3$ and the pull-back V_0^0 is shown in dotted lines. Note that $f^{\circ 5}(0) \in V_0^0$. This already creates the piece $V_0^1 \Subset V_0^0$ around the central Fatou component (V_0^1 is not shown).*

Now we can define the principal nest inductively. Suppose that the pieces $V_0^0 \Supset V_0^1 \Supset \ldots V_0^n$ have been already constructed. If the critical orbit does not return to V_0^n then the nest is finite. Otherwise, there is a first return time ℓ_n such that $f^{\circ \ell_n}(0) \in V_0^n$; then we define V_0^{n+1} as the critical piece that maps to V_0^n under $f^{\circ \ell_n}$.

It may happen that $\ell_{n+1} = \ell_n$; this means that not only does 0 return to V_0^n under $f^{\circ \ell_n}$, but even deeper to V_0^{n+1} without further iteration. In this case we say that the return is **central** and we call a chain of consecutive central returns $\ell_n = \ell_{n+1} = \ldots = \ell_{n+s}$ a **cascade of central returns**. An infinite cascade means that $\{\ell_n\}$ is eventually constant, so $f^{\circ \ell_n}(0) \in \bigcap_{j=n}^{\infty} V_0^j$. By definition, $f^{\circ \ell_n} : V_0^{n+1} \longrightarrow V_0^n$ is a **simple renormalization** of f; that is, a 2 to 1 branched cover of V_0^n such that the orbit of the critical point is defined for all iterates.

The return to V_0^n, however, can be non-central; in fact, it is possible to have several returns to V_0^n before the critical orbit lands at V_0^{n+1} for the first time. When the return is non-central, we complete the description of the nest by introducing the **lateral** pieces $V_k^n \in V_0^{n-1} \setminus V_0^n$. These encode the relation between different levels of the nest. Let $\mathcal{O} \subset K$ stand for the post-critical set $\mathcal{O} = \overline{\{f^{\circ j}(0)|j \geq 0\}}$; then take a point $z \in \mathcal{O} \cap V_0^{n-1}$ whose forward orbit returns to V_0^{n-1}. If we call $r_{n-1}(z)$ the first return time of z back to V_0^{n-1}, we can define $V^n(z)$ as the unique puzzle piece that satisfies $z \in V^n(z)$ and $f^{\circ r_{n-1}(z)}(V^n(z)) = V_0^{n-1}$. In particular, it is clear that $V^n(0)$ is just the same as V_0^n and that all the pieces created by this process are disjoint.

Definition 2.15. *The collection of all pieces $V^n(z)$ for $z \in \mathcal{O} \cap V_0^{n-1}$ which actually contain a point of the orbit of* 0, *is denoted $\mathcal{V}^n$ and referred to as the* ***level*** n *of the nest. The annuli $V_0^{n-1} \setminus V_0^n$ are called* ***principal annuli*** *and will be denoted A_n.*

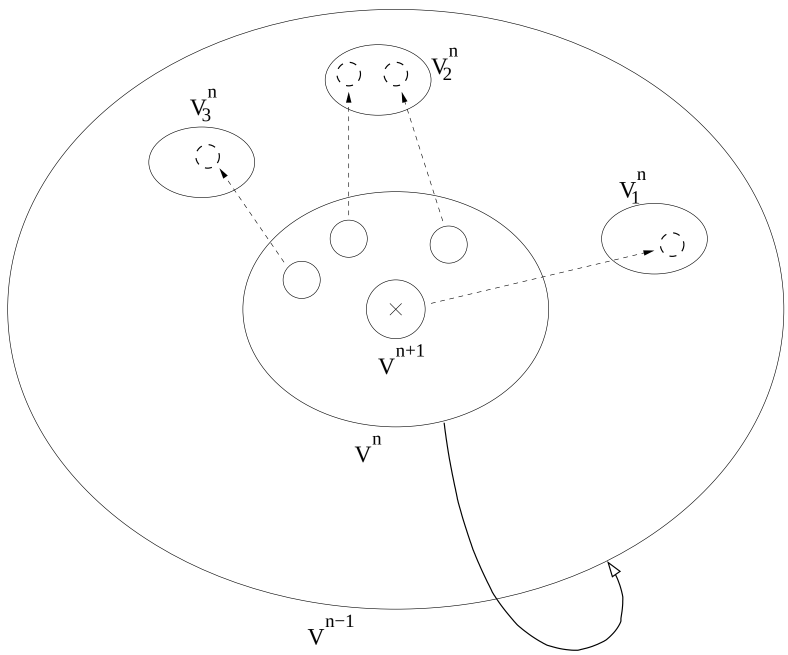

Figure 4: *Relation between consecutive nest levels. The curved arrow represents the first return map $g_n : V_0^n \longrightarrow V_0^{n-1}$ which is 2 to 1. The dotted arrows show a possible effect of this map on each nest piece of level $n+1$. Each V_j^{n+1} may require a different number of additional iterates in order to return to this level and map on V_0^n.*

Under the condition that c is recurrent, the principal nest has infinitely many levels. Let us assume that c is not periodic. Then it is called **reluctantly recurrent** if for some central piece V_0^n there are arbitrarily long sequences of univalent f_c-pull-backs of V_0^n along preorbits in the postcritical set $\mathcal{O}$. Otherwise, the parameter c is called **persistently recurrent**.

Lemma 2.16. (see [L2],[Ma]) *If f_c is persistently recurrent, $\mathcal{O}$ is a Cantor set and the action of $f_{c|_{\mathcal{O}}}$ is minimal.*

When f_c is not renormalizable, c is reluctantly recurrent if and only if some central piece V_0^n has infinitely many quadratic-like pull-backs along a segment of the critical orbit.

In particular, if c is non-renormalizable but every level of the principal nest has a finite number of pieces, then f_c acts minimally on the postcritical set. In this situation, we can name the pieces $\mathcal{V}^n = \{V_0^n, V_1^n, \ldots, V_{m_n}^n\}$ in such a way that the first visit of the critical orbit to V_i^n occurs before the first visit to V_j^n if and only if $i < j$.

Obviously, the value of $r_{n-1}(z)$ is independent of $z \in V_k^n$; we will denote it $r_{n,k}$.

Definition 2.17. *When $\mathcal{V}^n$ is finite, we define the map:*

$$g_n : \bigcup_{\mathcal{V}^n} V_k^n \longrightarrow V_0^{n-1},$$

given on each V_k^n by $g_{n|_{V_k^n}} = f^{\circ r_{n,k}}$.

The map g_n satisfies the properties of a *generalized quadratic-like* (**gql**) *map,* i.e.:

- $|\mathcal{V}^n| < \infty$.
- $\bigcup_{\mathcal{V}^n} V_k^n \Subset V_0^{n-1}$ and all the pieces of $\mathcal{V}^n$ are pairwise disjoint.
- $g_{n|_{V_k^n}} : V_k^n \longrightarrow V_0^{n-1}$ is a 2 to 1 branched cover or a conformal homeomorphism depending on whether $k = 0$ or not.

Note that g_n may be the result of a different number of iterates of f when restricted to different V_k^n. However, since we use often the map g_n restricted to individual pieces, it is typographically convenient to introduce the notation

Definition 2.18. *The map $g_{n|_{V_k^n}} = f^{\circ r_{n,k}}$ will be denoted $g_{n,k}$.*

Thus, $g_{n,k}(V_k^n) = V_0^{n-1}$ is a 2 to 1 branched cover or a homeomorphism depending on whether $k = 0$ or not.

From this moment, we will assume that the principal nest is infinite, excluding the possibility of an infinite cascade of central returns so that f is non-renormalizable. This property is called **combinatorial recurrence.**

2.7 Paranest

The *paranest* is also well defined around parameters c outside the main cardioid that are not immediately renormalizable. It consists of a nested sequence of parapieces around c, encoding the combinatorial information of the principal nest. When c is critically recurrent, the paranest is infinite.

Definition 2.19. *The* ***paranest*** *piece* $\Delta^n(c)$ *(for* $n \geq 0$*) is defined by the condition* $\partial\Delta^n(c) \mathrel{\mathring{=}} \partial f_c(V_0^n)$*, where* V_0^n *is the central piece of level* n *in the principal nest of* f_c*.*

The definition of principal nest, together with Proposition 2.14 imply that when $c' \in \Delta^n(c)$, the principal nests of f_c and $f_{c'}$ are identical until the first return $g_n(0)$ to V_0^{n-1} (creating V_0^n), and the relevant pieces move holomorphically as c' moves in Δ^n. Moreover, the set of initial ℓ_n iterates of 0 (recall that $g_n \equiv f^{\circ \ell_n}$) moves holomorphically without crossing piece boundaries. This means that the entire combinatorial structure of the nest, up to level n, remains unchanged.

In the notation of [L4], the family $\{g_n[c'] : V_0^n \longrightarrow V_0^{n-1} \mid c' \in \Delta^n(c)\}$ is a proper DH quadratic-like family with winding number 1. The last property follows from Proposition 2.14 since g_n is the first return to a critical piece at this level.

From Definition 2.19 it follows, since the central nest pieces are compactly nested, that the pieces of the paranest are compactly nested as well. It follows that $\operatorname{int} \Delta^n \setminus \Delta^{n-1}$ is a non-degenerate annulus. One of our concerns will be to estimate its modulus or, as is usually called, the **paramodulus.**

Chapter 3

Frame system

Let f_c have an infinite principal nest. Although the nest categorizes the combinatorics of the critical orbit, it falls short of providing a complete picture since it does not account for the relative positions between lateral pieces. In contrast to the real case, where bounded orbits form an interval, the Julia set of a complex polynomial displays a complicated structure that varies with the parameter. For this reason, a record of the relative positions of nest pieces must be preceded by a description of the combinatorial structure around them.

In this Chapter we enhance the principal nest with the addition of a *frame system.* This provides the necessary language to locate the lateral nest pieces and describe as a consequence, the behavior of the critical orbit. *The idea is to split a nest piece in smaller regions whose future behavior can be differentiated.*

For convenience, let us summarize certain aspects of the construction before giving it in detail. Recall that V_0^0 was defined in an intricate manner to guarantee that $Y_0^{(1)} \setminus V_0^0$ is a non-degenerate annulus. Because of this, and since our purpose is that frame levels correspond to nest levels, we need to pay individual attention to the construction of the first three frame levels. Figure 5 illustrates these initial steps. In particular, it is a good idea to keep in mind our convention of distinguishing between puzzle depths and nest levels. In accordance, frames will be also stratified in levels since their definition depends on the same pull-backs as the nest. To distinguish between nest pieces and frame pieces, the latter will be referred to as *cells.* As a final note of warning, we will abuse our notation and use F_n to refer to the frame as well as to the system of curves that it describes. In particular, we will use ∂F_n to describe the union of curves that form the boundary of the union of all cells in F_n. The context will always make clear which meaning is intended.

3.1 Frames

As we mentioned, some attention will be devoted to the construction of the frames F_0, F_1 and F_2 so that the properties in Proposition 3.4 hold. After this, the frames of higher level are defined inductively.

Consider the puzzle partition at depth 1 and recall that kq denotes the first escape of the critical orbit to Z_ν. Figure 3 provides a useful reference. The **initial frame** F_0 is the collection of nest pieces $F_0 = \{Y_0^{(1)}\} \cup \{\bigcup_{j=1}^{q}\{Z_j\}\}$, each of which is called a frame **cell**. Thus, $\Gamma(F_0)$ is a q-gon. The frame F_1 is the collection of pull-backs of cells in F_0 along the $f^{\circ kq}$-orbit of 0.

From the definition, F_1 contains the central piece V_0^0 that maps 2 to 1 onto $Z_\nu \in F_0$. The pull-back of any other cell $A \in F_0$ consists of two symmetrically opposite cells, each mapping univalently onto A. We say that F_1 is a *well defined unimodal* pull-back of F_0.

Lemma 3.1. *All the cells of F_1 are contained in $Y_0^{(1)}$.*

Proof: $f^{\circ kq}(Y_0^{(1)})$ is an extension of $Y_0^{(0)}$ to a larger equipotential. Thus, it contains all cells of F_0 and its pull-back by $f^{\circ kq}$ results in a proper map. □

Let λ be the first return time of 0 to a cell of F_1. By Lemma 3.1, the collection F_2' of pull-backs of cells in F_1 along the $f^{\circ\lambda}$-orbit of 0 is well defined and 2 to 1. Unfortunately, it does not cover every point of J_f inside V_0^0. We will give first some results about F_2' and define afterward a complete frame of level 2.

Lemma 3.2. *The temporary frame F_2' satisfies:*

1. *All cells of F_2' are contained in V_0^0.*

2. *V_0^1 is contained in the central cell of F_2'.*

Proof: First note that $\lambda = kq + (q - \nu)$ is the first return of 0 to $Y_0^{(1)}$ after the first escape to Z_ν. We have $kq < \lambda \leq \ell_0$, where the second inequality is true since $V_0^0 \in F_1$. Then the first return to F_1 occurs no later than the first return to V_0^0. By definition, $f^{\circ\lambda}(V_0^0)$ is just $Y_0^{(0)}$ extended to a larger equipotential. Since all cells of F_1 are inside $Y_0^{(1)} \subset f^{\circ\lambda}(V_0^0)$, the first assertion follows.

Now, V_0^1 is central. By the Markov properties of $\mathcal{Y}_c$, either V_0^1 is contained in the central cell C of F_2' or vice versa. However, both $f^{\circ\ell_0}(V_0^1)$ and $f^{\circ\lambda}(C)$ belong to F_1. Since $\ell_0 \geq \lambda$, the first possibility is the one that holds. This proves property (2). □

Our intention is to extend F_2' to a frame that covers the intersection $J_f \cap V_0^0$. To do this, we just need to add the $f^{\circ\lambda}$-pull-backs of the pieces Z_ν. The union of those pull-backs with the cells of F_2' is the frame F_2.

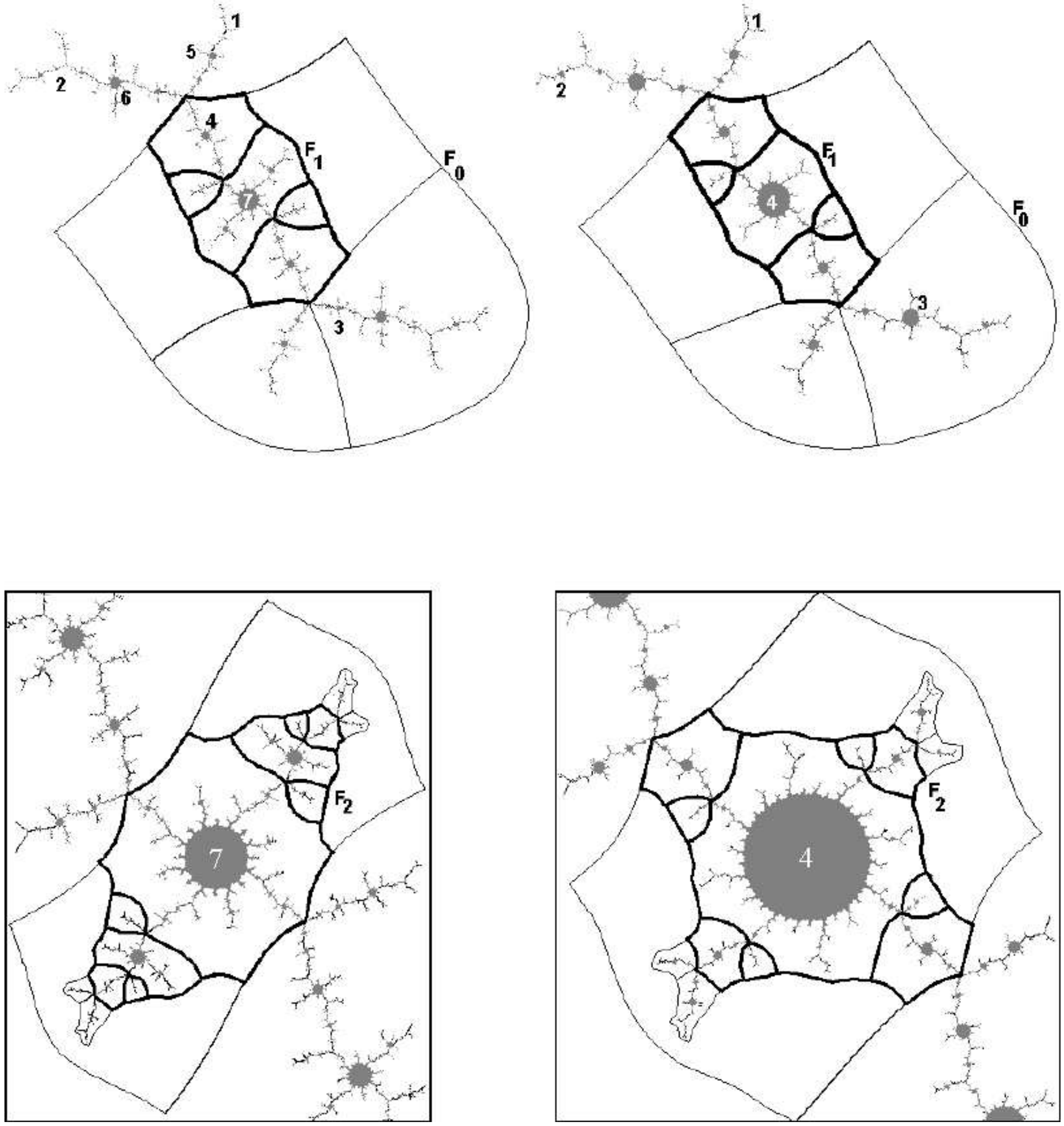

Figure 5: *Both of these parameters belong to the left antenna of* $L_{1/3}$*; they are centers of components of periods 7 and 4. Above we can see that the structures of the frames of levels 0 and 1 coincide between the two examples. Still, the first return to* F_1 *falls in each case on a different cell, producing dissimilar frames of level 2. The pull-back of cells in* F_1 *produces a preliminary frame* F_2'*, shown in heavy line on the second row. The complete frame* F_2*, inside* V_0^0 *has* $2(q-1)$ *additional cells (here* $q=3$*) in order to cover all of* $J_f \cap V_0^0$.

After introducing the first frames and relating them to the initial levels of the nest, we can give the complete definition of the *frame system.* The driving idea of this discussion is that the internal structure of a frame F_{n+2}, represented by the graph $\Gamma(F_{n+2})$, provides a decomposition of $J_f \cap V_0^n$ that helps to describe the combinatorial type of the nest at level $n+1$.

Definition 3.3. *For $n \geq 0$ consider the first return $g_n(0) \in V_0^n$ and define F_{n+3} as the collection of g_n-pull-backs of cells in F_{n+2} along the critical orbit. The family $\mathcal{F}_c = \{F_0, F_1, \ldots\}$ is called a **frame system** for the principal nest of f_c and each piece of a frame is called a **cell**.*

*The dual graph $\Gamma(F_n)$ (see Definition 2.4) is called the **frame graph.** As in the case of the puzzle graph, we consider $\Gamma(F_n)$ with its natural embedding in the plane.*

Let us mention now some properties of frame systems.

Proposition 3.4. *The frame system satisfies:*

1. *Frames exist at all levels.*
2. *The union of cells $\bigcup_{C_i \in \mathcal{F}_n} C_i$ forms a cover of $K_{f_c} \cap V_0^{n-2}$.*
3. *The central cell of F_n contains the nest piece V_0^{n-1}.*
4. *Each F_n has 2-fold central symmetry around 0.*

Figure 5 should help to clarify the definition of frames. In particular, it is important to make the following observation. As follows from the comment after Lemma 3.1, the union of cells in $\mathcal{F}_2$ covers exactly the intersection of K_f with the nest piece V_0^0. This is because V_0^0 can be described as the pull-back of $Y_0^{(0)}$ under the first return map to F_1. Then, we can think of this union of cells as a single piece, determined by the same rays as V_0^0, but cut off by a lower equipotential.

Proof: F_0 and F_1 are easily seen to exist from their construction. Since F_1 covers the central part of K_f between α and $-\alpha$, there will definitely be a return to it, creating F_2'. As we saw already, this frame is contained inside V_0^0, so its pull-backs are well defined as long as there are new levels of the nest. In particular, this already proves claim 2. Since the principal nest is infinite, the critical point is recurrent or the map is renormalizable. Either case creates critical returns to central nest pieces of arbitrarily high level, so F_{n+1} is defined.

The piece V_0^0 is actually the central cell of F_1. Now, the first return to F_1 cannot occur later than the first return to V_0^0, so the central cell C of F_2 is of lower depth than V_0^1; thus, $V_0^1 \subset C$. Afterwards, the depth from V_0^{n-1} to V_0^n increases by ℓ_{n-1}, while the depth from F_n to F_{n+1} increases ℓ_{n-2}. Inductively, since $V_0^{n-1} \subset F_n$ and $\ell_{n-2} \leq \ell_{n-1}$, we obtain $V_0^n \subset F_{n+1}$.

Finally, each F_n is a well defined 2 to 1 pull-back of F_{n-1}, a cell C belongs to F_n if and only if its symmetric $-C \in F_n$. □

Our next objective is to introduce a labeling system for pieces of the frame. This will allow us to describe the relative position of pieces of the nest within a central piece of the previous level. Unlike the case of unimodal maps, where nest pieces are always located left or right of the critical point, the possible labels for vertices of $\Gamma(F_n)$ will depend on the combinatorics of the critical orbit. Only after determining the labeling, it becomes possible to describe the location of nest pieces in a systematic manner.

Observe that the structure of F_{n+1} is trivially determined once we know F_n and the location of $g_n(0)$. A graphic way of seeing this is as follows. Say that the first return $g_{n-1}(0)$ to V_0^{n-2} falls in a cell $X \in F_n$. Let L_n and R_n be two copies of $\Gamma(F_n)$ with disjoint embeddings in the plane. Now connect L_n and R_n with a curve γ that does not intersect either graph. Suppose that one extreme of γ lands at the vertex of L_n that corresponds to X and the other extreme lands at the corresponding vertex of R_n *approaching it from the same access.*

Lemma 3.5. *If we collapse γ by a homotopy of the whole ensemble, the resulting graph is isomorphic to* $\Gamma(F_{n+1})$.

Note: The above construction provides $\Gamma(F_{n+1})$ with a natural plane embedding; see lemma 3.6 below.

A label at level n will be a chain of $n+1$ symbols taken from the alphabet { 0,1, ..., (q-1), l, r, e, b, t }. First, put the labels { '0', '1', ..., '(q-1)' } on the cells of F_0, starting at the central piece $Y_0^{(0)}$ and moving counterclockwise.

Let σ_0 be the label of the cell that holds the first return of 0 to F_0 and, in general, let σ_n denote the label of the cell in $\Gamma(F_n)$ that holds the first return of 0. In order to label $\Gamma(F_{n+1})$, assume that we know the number q of pieces in F_0, and the *label sequence* $(q; \sigma_0, \dots, \sigma_{n-1})$ that identify the location of first returns of 0 to levels $0, \dots, n-1$ of the nest. In particular, all frames up to $\Gamma(F_n)$ have been successfully labeled.

Duplicate in L_n the labels of $\Gamma(F_n)$, but concatenate an extra 'l' at the beginning. Do a similar labeling on R_n by concatenating an extra 'r' to the duplicated labels. Note that the labels of the two vertices corresponding to

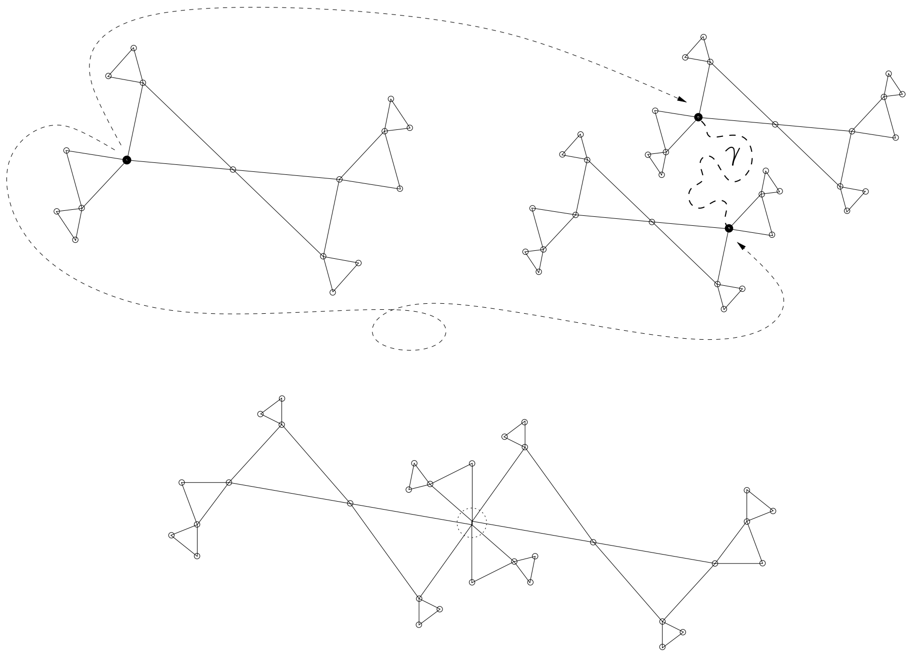

Figure 6: *The curve γ joins two copies of the same frame graph approaching the selected vertex from the same direction. The new frame graph is obtained after γ is contracted to a point.*

X are $'\mathsf{l}'\sigma_n$ and $'\mathsf{r}'\sigma_n$. The labels on $\Gamma(F_{n+1})$ will be the same as those in the union of L_n and R_n except that we change the label of the identified vertex, to become $'\mathsf{0}'\sigma_n$.

Note: The above procedure does not give labels to the additional cells of F_2 that do not come from a pull-back. These are the cells that are not drawn in heavy line in Figure 5 Being cells of level 2, their labels should have 3 symbols for consistency with the rest. The easiest way to do this is simply to impose the labels 'et1', 'et2', ..., 'et(q-1)' and 'eb1' ,'eb2', ..., 'eb(q-1)' in their natural order in the plane ('et' stands for **e**xtra piece on **t**op and 'eb' for **e**xtra piece on **b**ottom), then extend the labeling to higher levels as described.

Clearly, f induces a map $f_* : \Gamma(F_{n+1}) \longrightarrow \Gamma_n$ for $n \geq 2$, that acts by forgetting the leftmost symbol of each label. This is the case also for the induced map on the temporary frame F_2'.

3.2 Properties of frame labelings

Under certain conditions, label sequences give a complete characterization of the entire combinatorial structure. This is the content of Theorem 3.11. Before stating it, we need to review some properties of the frame and its labels.

Lemma 3.6. *The plane embedding of Γ does not depend on the homotopy class of the curve γ in lemma 3.5.*

Proof: Since we regard $\Gamma = \Gamma(F_n)$ as embedded in the sphere, the exterior of Γ is simply connected, so there is a natural cyclic order of accesses to vertices (some vertices can be accessed from more than one direction). In this order, all accesses to L_n are grouped together, followed by the vertices of R_n. □

It is important to mention that the resulting labeling of $\Gamma(F_n)$ **does** depend on the access to ξ_n approached by γ. However, the final unlabeled graphs are equivalent as embedded in the plane.

As we just mentioned, some vertices are accessible from ∞ in two or more directions. These are precisely the vertices whose label contains the symbol '0' (for $n \geq 1$). Since such a vertex represents a frame cell that maps (eventually) to a central frame cell, the tail of a label with '0' at position j must be σ_j. On the other hand, for every j there must be labels with a '0' in position j. It follows that the set of labels of $\Gamma(F_n)$ and the sequence $(q; \sigma_0, \dots, \sigma_n)$ can be recovered from each other.

3.3 Frames and nest together

The definition of frame system was conceived to satisfy the properties of Proposition 3.4. An extension of the argument used to prove those properties shows that every piece V_j^n of the nest is contained in a frame cell of level $n+1$. Moreover, we would like to extend the definition of frames so that each V_j^n can be partitioned by a pull-back of an adequate central frame. For this, we must recall first that $g_{n,j}(V_j^n) = V_0^{n-1} \supset F_{n+1}$.

Definition 3.7. *The frame $F_{n,k}$ is the collection of pieces inside V_k^{n-2} obtained by the $g_{n-2,k}$-pull-back of F_{n-1}. Elements of the frame $F_{n,k}$ are called **cells** and we will write $F_{n,0}$ instead of F_n, when there is a need to stress that a property holds in $F_{n,k}$ for every k.*

If a puzzle piece A is contained in a cell $B \in F_{n,k}$, we denote B by $\Phi_{n,k}(A)$.

We have described already how to label F_n. The other frames $F_{n,k}$ $(k \geq 1)$, mapping univalently onto F_{n-1}, have a natural labeling induced from that of F_{n-1} by the corresponding $g_{n-2,k}$-pull-back.

Let us describe now the itinerary of a piece V_j^n. Since $V_j^n \subset V_0^{n-1}$, the map g_{n-1} takes V_j^n inside some piece $V_{k_1(j)}^{n-1} \subset V_0^{n-2}$. Then, $g_{n-1,k_1(j)}$ takes $g_{n-1}(V_j^n)$ inside a new piece $V_{k_2(j)}^{n-1}$ and so on, until the composition of returns of level $n-1$

$$(g_{n-1,k_r(j)} \circ \ldots \circ g_{n-1,k_1(j)} \circ g_{n-1})_{|V_j^n}$$

is exactly $g_{n,j} : V_j^n \mapsto V_0^{n-1}$. Of course, k_r is just 0, and we will write it accordingly.

We have extra information that deems this description more accurate. For the sake of typographical clarity, we will write k_i instead of $k_i(j)$. For $i \leq r$, let Φ_{n+1,k_i} be the cell in $F_{n+1,k_i} \subset V_{k_i}^{n-1}$ that contains

$$g_{n-1,k_i} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n)$$

and denote by λ_{n+1,k_i} the label of Φ_{n+1,k_i}.

Definition 3.8. *The **itinerary** of V_j^n is the list of piece-label pairs:*

$$\chi(V_j^n) = \Big([V_{k_1}^{n-1}; \lambda_{n+1,k_1}], [V_{k_2}^{n-1}; \lambda_{n+1,k_2}], \ldots, [V_{k_{r-1}}^{n-1}; \lambda_{n+1,k_{r-1}}], [V_0^{n-1}; \lambda_{n+1,0}]\Big) \tag{3.1}$$

up to the moment when V_j^n maps onto V_0^{n-1}.

Note first of all that the last label, $\lambda_{n+1,0}$, will start with '0' due to the fact that V_0^{n-1} is in the central cell of F_n. More importantly, the conditions

$$\begin{array}{lcl} V_{k_1}^{n-1} & \subset & g_{n-1}(\Phi_{n+1,0}) \\ V_{k_{i+1}}^{n-1} & \subset & g_{n-1,k_i}(\Phi_{n+1,k_i}) \quad 2 \leq i < r \end{array} \tag{3.2}$$

must hold since we know that $g_{n-1,k_{i-1}} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n) \subset \Phi_{n+1,k_i}$ and $g_{n-1,k_i} \circ \ldots \circ g_{n-1,k_1} \circ g_{n-1}(V_j^n) \subset V_{k_{i+1}}^{n-1}$.

Definition 3.9. *When we specify the sequence of frame labelings up to a given level n, the locations of the nest pieces and their (admissible) itineraries, we say that we have described the **combinatorial type** of the map at level n. If $|\mathcal{V}^n| < \infty$ we say that the type is **finite**; refer to Lemma 2.16 and Definition 2.17.*

*Condition 3.2 will be called the **frame admissibility condition**.*

3.4 Real frames

Let us digress momentarily in order to compare the above definitions with their counterparts in the real case.

When the parameter c is real, all the pieces of the nest intersect the real axis. Call I_j^n the intersection of V_j^n with $\mathbb{R}$. The combinatorial type of the nest is determined by how many intervals are there left and right of I_0^n, the sign (orientation) of each map $g_{n,j} : I_j^n \longrightarrow I_0^{n-1}$ and the itineraries of all I_j^n through intervals of the previous level. If we specify an arbitrary type, the *unimodal admissibility conditions* are necessary so that the type can be realized; these conditions require

- Since $g_{n-1,k}$ is supposed to take I_k^{n-1} onto I_0^{n-2}, the order of the intervals inside I_k^{n-1} is preserved or reversed according to the orientation of $g_{n-1,k}$.
- Since $g_{n,j} : I_j^n \longrightarrow I_0^{n-1}$ is supposed to be the composition of all g_{n-1,k_i} specified by the itinerary of I_j^n, the sign of $g_{n,j}$ must be the product of signs of the g_{n-1,k_i} when I_j^n is right of I_0^n and the negative of that sign when I_j^n is to the left of I_0^n (or the other way, if $g_{n,0}$ reverses orientation).

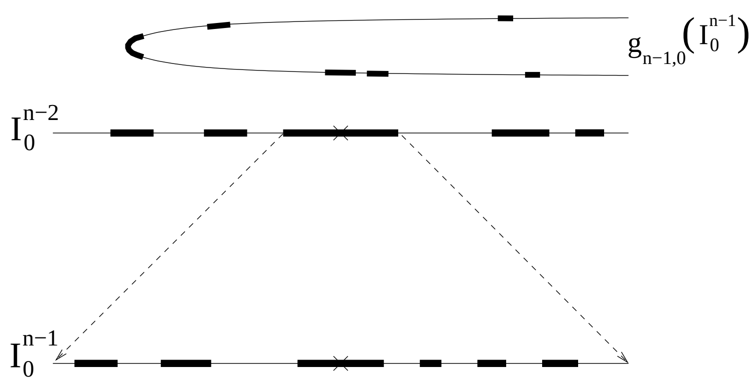

Figure 7: *Illustration of the unimodal admissibility conditions. The map $g_{n-1,0}$ spreads the intervals of level n inside some intervals of level $n-1$. However, the order of the right intervals is respected and that of the left intervals is reversed. Note that the orientation of each left interval is also reversed and that I_0^n maps to the leftmost position.*

We note first that both conditions emphasize the fact that $g_{n,0}$ is unimodal. The first map $g_{n-1,0}$ can mix left intervals with right intervals as in Figure 7, but the order of the right intervals is preserved and the order of the left ones is reversed (or vice-versa). The second condition specifies that the orientation of each $g_{n,j}$ is the product of the orientations of all intermediate steps *including the fact that $g_{n-1,0}$ has different orientations on each side of 0.* The important observation to make is that the simplicity of the unimodal admissibility conditions is due to the existence of a natural order on $\mathbb{R}$. In the more general case of complex polynomials, the order of intervals is replaced by relative locations of nest pieces within a frame. The requirement that relative orders are preserved is replaced by Conditions 3.2 and the rule of signs is replaced by a compatible choice of labels.

3.5 Combinatorial classification

We are ready to state the main theorem of this Chapter. In loose language, it states the existence within the quadratic family, of arbitrary admissible finite combinatorial types.

Definition 3.10. *We will say that two non-renormalizable polynomials are* **weakly combinatorially equivalent** *if they have the same combinatorial types at every level, so that they differ only by the orientation of their frames.*

Note: The point $g_n(0)$ is contained in V_0^{n-1}. In particular, it is possible to apply the map g_{n-1} to it and, in fact, we could keep composing first return maps of lower levels until the first return of the critical orbit to V_0^n. This argument shows that for weak combinatorially equivalent maps, g_{n+1} is formed by the same composition of previous levels first return maps and consequently, *the first returns to corresponding pieces happen at the same times.* In the next chapters we will make use of this property.

Theorem 3.11. *Consider a finite combinatorial type of level n, together with a parapiece Δ of parameters that satisfy it up to level $n-1$. Let ℓ be the level of the last lateral return prior to level n and let*

$$r = \begin{cases} 1 & \text{if } g_n \text{ is a central return} \\ 2^{n-\ell} & \text{if } g_n \text{ is lateral.} \end{cases}$$

Then there exist r parapieces inside Δ each consisting of parameters satisfying the same weak combinatorial type to level n.

Moreover, for any such parapiece Δ', the first returns $\{g_n[c] \mid c \in \Delta'\}$ form a full DH quadratic-like family.

Note: This property of accumulating powers of 2 during central cascades is related to the phenomenon that makes Theorem L1 possible. Namely, the fact that moduli grow linearly from lateral return to lateral return, even though they decrease by half on each central return.

Proof: We are already acquainted with the central symmetry of frames. It is obvious that the dual graph of a frame can be symmetric only at its critical vertex ξ. Because of this, the frame $F_{\ell+1}$ cannot be symmetric around the lateral cell C where $g_\ell(0) \in \left(V_0^{\ell-1} \setminus V_0^\ell\right)$ falls, so the pull-back $F_{\ell+2}$ cannot have more than 2-fold symmetry around the origin.

By definition, the (possibly empty) sequence $\{g_{\ell+1}, \ldots, g_{n-1}\}$ is the beginning of a cascade of central returns of length $n-\ell$. Therefore, the dual graph of the frame F_{n+2} has exactly $(2^{n-\ell})$-fold symmetry around ξ_n.

Let $c \in \Delta$. Every map $g_{n-1,k}$ takes its corresponding piece V_k^{n-1} onto V_0^{n-2}. Then the pull-back by $g_{n-1,k}$ of any region inside V_0^{n-2} is well defined and located inside V_k^{n-1}. In particular, for every piece V_j^n listed in the type of level n, the itinerary prescribes the sequence of returns $g_{n-1,0}, g_{n-1,k_1}, \ldots, g_{n-1,k_r}$, so the univalent pull-back of V_0^{n-1} under the composition $(g_{n-1,k_1} \circ \ldots \circ g_{n-1,k_r})$ is a well defined piece inside $V_{k_1}^{n-1}$. Let us name this piece U'_j.

Clearly $U'_0 \subset V_1^{n-1}$ because the itinerary of the critical piece V_0^n begins with the first return of 0 to level $n-1$. As c moves within Δ, this return can be made to fall in U'_0. All c with this property form a parapiece $\Delta^* \Subset \Delta$ that can be described as the set of parameters for which the itinerary of U_0 is as originally prescribed; i.e. $U_0 = V_0^n$. For the rest of the argument we will restrict c to Δ^*.

For $j \geq 1$, the $g_{n-1,0}$-pull-back of U'_j will be called U_j; however, $g_{n-1,0}$ is 2 to 1, so we have to decide on a frame orientation before locating these pieces inside F_{n+2}.

The combinatorial type of level n involves the label σ_{n+2} that specifies the cell in F_{n+2} containing the first return $g_n(0)$. If this return is central there is no choice: The return falls on the piece V_0^{n+1} inside the central cell. Otherwise, we need to recall the discussion above. After a (possibly vacuous) cascade of central returns, there are $\frac{r}{2} = 2^{n-\ell-1}$ cells of F_{n+1,k_1} that can be labeled with σ_{n+2} and contain U'_1. This comes from the $n-\ell-1$ choices of orientation taken from level $\ell+1$ to $n-1$. Assuming that the return $g_n(0)$ is lateral, there is one more choice of orientation to make, so F_{n+2} has $(2^{n-\ell})$ cells that can host U_j. Once this decision is made, the label orientation is determined and the rest of the pieces U_j are forcibly placed around the frame F_{n+2}.

We have constructed pieces $U_j \subset V_0^{n-1}$ that follow the given itineraries. It rests now to show that for some parameters $c \in \Delta^*$, the U_j can be made to coincide with the respective V_j^n. This can be shown as follows. The itinerary of V_0^n (and of 0) ends with the first return g_n of 0 to V_0^{n-1}. This return generates a full family for $c \in \Delta^*$, so we can choose a parapiece Δ^{**} of c such that

$g_n(0) \in U_1$.

The second return to V_0^{n-1} is specified by the itinerary of U_1. From this observation we conclude that $U_1 = V_1^n$ from the definition of nest. Also, this second return generates a full family for $c \in \Delta^{**}$, so we can choose an even smaller parapiece Δ^{***} of parameters c such that $g_{n,1}(0) \in U_2$. This argument can be pursued till the end to obtain the parapiece Δ' of values c for which every $U_j = V_j^n$. □

Repeated application of Theorem 3.11 yields the following.

Corollary 3.12. *Arbitrary infinite sequences of finite weak combinatorial types can be realized in the quadratic family, as long as they satisfy the admissibility condition at every level. The set of parameters satisfying the complete type is the intersection of a family of nested sequences of parapieces, with 2^n of them at every non-central level n.*

Proof: This is clear, since each Δ contains at least one parapiece Δ' that satisfies the combinatorial type at level n. The collection of first return maps of level n for parameters in Δ' forms a full family, so we can apply Theorem 3.11 again. An arbitrary choice of orientation at every level gives an infinite nested sequence of parapieces. Evidently, a parameter in the intersection satisfies the prescribed combinatorics at every level.

Every level accounts for one dyadic choice of orientation. Although they are not apparent during central cascades, the previous proof shows that they accumulate to display $2^{n-\ell}$ pieces of level n inside each of the 2^ℓ pieces of (lateral) level ℓ. □

The set of parameters that are combinatorially equivalent to a given one cannot be completely characterized without some amount of analytical information. Corollary 3.12 describes such set as a collection of nested sequences of parametric pieces, but it does not say whether they intersect in single points or in more complicated regions. The fact that the parapieces shrink to a unique parameter amounts to *combinatorial rigidity*; this was the strategy of Yoccoz to establish local connectivity in the case of non-renormalizable polynomials. For such parameters, he showed that the sum of paramoduli is infinite, so the set of parameters in the nested intersections of parapieces becomes a Cantor set. In particular, if the type includes no central returns, every parapiece contains exactly two pieces of the next level and the Cantor set has a natural dyadic structure. Thus, for some precise sequences of combinatorial types, the choice of frame orientations at every level may single out a unique parameter. In the next Chapter we will discuss examples of parameters of this kind, with particularly interesting behaviors.

Note: It should be remarked that alternative classifications of combinatorial properties are possible and indeed quite useful. Of particular notice is D. Schleicher's concept of *internal addresses* (see [LS]), describing a combinatorial type in terms of an irreducible sequence of hyperbolic components encoding the critical orbit information at increasing levels.

3.6 Maximal hyperbolic components

Consider an arbitrary combinatorial type up to some level n, with the property that the last return is not central. Upon specifying a frame orientation, there is a unique parapiece Δ consisting of parameters that satisfy the given combinatorics. Clearly, parapieces corresponding to different descriptions must be disjoint.

If the return to level $n+1$ is central, there is no need to orient the frame; that is, there is a unique piece $\Delta' \subset \Delta$ of parameters featuring this central return. When a parameter in Δ has an infinite cascade of central returns starting at level $n+1$, its combinatorial type will be completely determined by the initial n levels. The unique sequence of nested parapieces $\Delta \supset \Delta' \supset \ldots$ intersects in the set M' of renormalizable parameters whose first n nest levels are as prescribed. It is known that M' is quasi-conformally homeomorphic to M (see [DH1] and [L4]). In fact, this homeomorphism is given by **straightening**: For every $c \in M'$ the renormalized map $g_n[c]$ is **hybrid equivalent** to some quadratic polynomial f; that is, there is a quasi-conformal map h that realizes the conjugation $h \circ g_n = f \circ h$ and such that $\overline{\partial} h = 0$ on the small filled Julia set of $g_n[c]$.

Since the parameters of M' have a well defined nest the renormalization is not of immediate type. The base of such "small copy" of M is a primitive hyperbolic component H. Since H is a quasi-conformal deformation of $\heartsuit$, its boundary has a cusp point. Also, the parameters in H are exactly once renormalizable. Therefore H is maximal (see definitions at the beginning of Chapter 2).

The above discussion shows that any finite frame type is associated to a maximal hyperbolic component of M. Conversely, each maximal copy of M is encoded by the type of its frame, that is, by the associated graph $\Gamma(F_{n+1})$ or its label sequence. Note that the frame graph of level $n' > n$ consists of a bouquet of $2^{n'-n}$ copies of $\Gamma(F_{n+1})$ with their central vertices identified. This is illustrated in the right hand example in Figure 5.

From our point of view, the combinatorial information in these examples is finite, so the frame system cannot say much about the map. Therefore, the next step in dealing with these maps should be to start a new frame system associated to the renormalized small Julia set and try to keep control of the moduli from one step to the next.

Chapter 4

Examples

Let us start by introducing the Fibonacci parameter $c_{\text{fib}} = -1.8705286321\ldots$ In [LM], the authors prove the uniqueness of the real quadratic Fibonacci map $f_{c_{\text{fib}}}$ and describe in detail its asymptotic geometry. The real parameter $c_{\text{fib}} \in \mathbb{R}$ is determined by either of the two following equivalent conditions:

1. The closest returns to 0 of the critical orbit occur exactly when the iterates are the Fibonacci numbers.

2. For $n \geq 2$, each level of the principal nest consists of the central piece V_0^n and a unique lateral piece V_1^n. The first return map of previous level $g_{n-1} : V_0^{n-1} \longrightarrow V_0^{n-2}$ interchanges the central and lateral roles:

$$g_{n-1}(V_0^n) \Subset V_1^{n-1}, g_{n-1}(V_1^n) = V_0^{n-1}.$$

   Additionally, the first returns to $Y_0^{(1)}$ and V_0^0 happen on the third and fifth iterates respectively.

These conditions turn out to be equivalent as we will see later.

We can omit a precise description of the frame for the Fibonacci parameter, since it is a particular case in both the family of *rotation-like* maps and the family of *Q-recurrent* maps. To understand the critical orbit behavior, it is enough to note that for $f_{c_{\text{fib}}}$, every level of the nest has a unique lateral piece, and so, in a way, every first return comes as close as possible to being central without actually being central. This means that the map $f_{c_{\text{fib}}}$ is not renormalizable in the classical sense, although its combinatorics can be described as an infinite cascade of *Fibonacci renormalizations* in the space of **gql** maps with one lateral piece.

The simplicity of this description opened the doors to many analytic results; it features for instance, as a decisive case in the proof of Lyubich's

Theorem L1; see [L3].

In this Chapter we will present both rotation-like and Q-recurrent maps as two families of examples with interesting combinatorics described in terms of the frame system. After this we show an example of a persistently recurrent polynomial whose central nest pieces seem to converge in shape to an interval, but where there is no bound on the number of compositions of first return maps g_j ($j \leq n$) that are required to create the first return map g_{n+1}. Starting in the next Chapter, we will attempt a detailed understanding of the asymptotic properties of Q-recurrent maps.

4.1 Rotation-like maps

Let $S = (S_0, S_1, \ldots)$ be a strictly increasing sequence of numbers such that $\frac{S_{j+1}}{S_j} \leq 2$. The *$S$-odometer* is a symbolic dynamical system (Ω, T) defined as follows. For any nonnegative n there is a k such that $S_k \leq n < S_{k+1}$. Then $n = S_k + n_1$ with $n_1 < S_k$. By splitting further $n_1 = S_{k'} + n_2$ (with $k' < k$ and $n_2 < S_{k'}$) and so on, we obtain the decomposition

$$n = d_k \cdot S_k + \ldots + d_0 \cdot S_0$$

where each d_j is either 0 or 1. Letting $d_j = 0$ for $j > k$, we get the sequence

$$\langle n \rangle = (d_0, d_1, \ldots) \in \{0,1\}^{\mathbb{N}}.$$

We use $\langle \mathbb{N} \rangle$ to denote $\{\langle n \rangle \mid n \in \mathbb{N}\}$ and let Ω be the closure

$$\Omega = \overline{\langle \mathbb{N} \rangle} = \{\omega \in \{0,1\}^{\mathbb{N}} \mid \sum_{i=0}^{j} \omega_j S_i < S_{j+1} \text{ for all } j \geq 0\}.$$

The map $T : \langle \mathbb{N} \rangle \longrightarrow \langle \mathbb{N} \rangle$ is given by $T\langle n \rangle = \langle n+1 \rangle$. This map does not always extend uniquely to Ω. When there is an extension, the dynamical system (Ω, T) obtained from the sequence S is called a **S-odometer**. It can be described as an adding machine with variable stepsize.

Let us relate the above concept to interval dynamics. First, some definitions related to Hofbauer's towers; see [Ho].

Consider a unimodal map $f : I \longrightarrow I$ where $I = [c_1, c_2]$ and $\{0, c_1, c_2, \ldots\}$ is the critical orbit. Let $D_1 = [c_1, 0]$ and, for $n \geq 2$, define

$$D_{n+1} = \begin{cases} [c_{n+1}, c_1] & 0 \in D_n \\ f(D_n) & 0 \notin D_n \end{cases}$$

The sequence $S = (S_0, S_1, \ldots)$ of *cutting times* consists of those n such that $0 \in D_n$. Note that $S_0 = 1$. It is easy to show that $S_{k+1} - S_k$ is also a cutting time so we can define the **kneading map** $Q : \mathbb{N} \longrightarrow \mathbb{N}$ by the relation

$$S_{Q(k)} = S_{k+1} - S_k.$$

Lemma 4.1. *If S is the sequence of cutting times of a unimodal map f, the following characterization of Ω holds:*

$$\Omega = \{\omega \in \{0,1\}^{\mathbb{N}} \mid \omega_j = 1 \Rightarrow \omega_i = 0 \text{ for } Q(j+1) \leq i \leq j-1\}.$$

Also, if $Q(k) \longrightarrow \infty$, then T extends uniquely to Ω and is conjugate to the action of f on its postcritical set.

See [BKP] for proofs.

In the case of the Fibonacci polynomial, the above definitions correspond to the description of the critical orbit in Section 3 of [LM]. There it is shown that $(\Omega, T)_{c_{\text{fib}}}$ is semiconjugate to the circle rotation by $\rho = \frac{\sqrt{5}-1}{2}$. Real **rotation-like maps**, as defined in [BKP], are unimodal maps that generalize this behavior.

Let $\rho \in [0,1) \setminus \mathbb{Q}$ with continued fraction expansion $\rho = [a_1, a_2, \ldots]$ and denote its convergents with $\frac{p_i}{q_i}$ so that $\frac{p_0}{q_0} = \frac{0}{1}$ and $\frac{p_1}{q_1} = \frac{1}{a_1}$.

Theorem 4.2. [BKP] *Consider the sequence r_k starting with $r_1 = q_1 - 1$ and whose $(k+1)^{st}$ element is given recursively by $r_{k+1} = r_k + a_{k+1}$. Then the S-sequence given by*

$$\begin{aligned} S_{r_k} &= q_k \\ S_{r_k+j} &= (j+1)q_k \quad \text{for } 1 \leq j < a_{k+1} \end{aligned}$$

is realized as the sequence of cutting times of some quadratic polynomial. Moreover, the map

$$\Pi_\rho(\omega) = \sum \omega_j S_j \rho \,(\text{mod } 1)$$

from Ω to the unit circle is well defined and continuous. This map satisfies $\Pi_\rho \circ T = R_\rho \circ \Pi_\rho$, where R_ρ is the rotation by angle ρ, and is 1 to 1 everywhere except at the preimages of 0.

In terms of the principal nest, the behavior that characterizes rotation-like maps is a succession of central cascades followed by one lateral escape. That is, the critical orbit falls in $V_0^{S_k - 1}$ starting a central cascade. After iterating the first return map g_k for $a_k - 1$ turns, we get a lateral return on $V_1^{S_k}$. Next,

$g_{S_k,1}$ creates a new cascade and so on. In particular, the Fibonacci map is the special case of a rotation-like map where every central cascade has length 0.

Consider an arbitrary sequence $a_1, a_2, \ldots$ of positive integers. We will construct now a Cantor set of *complex* rotation-like parameters with central cascades of length $a_i - 1$. By theorem 3.11, it is only necessary to give an admissible description of labeling sequences and to show that it models the combinatorics mentioned above.

The initial labeling data for our map is $q = 2$ and $\sigma_0 = '1'$, so rotation-like maps will all be located in the 1/2-limb. Note also that on central return levels, $\sigma_{k+1} = '0'\sigma_k$. Therefore, we only need to specify the labels σ_{r_η} for $r_\eta = \sum a_j$.

Let $(\tau_1, \tau_2, \ldots)$ be a sequence of random chains of 'l''s and 'r''s so that τ_i has length $a_i + 1$. Set $\sigma_{r_1} = \tau_1'0'$ and $\sigma_{r_2} = \tau_2\sigma_{r_1-1} = \tau_2'00\ldots01'$. Now we can define inductively $\sigma_{r_j} = \tau_j'0'\sigma_{r_{j-1}-1}$.

Proposition 4.3. *The label sequence* $(q; \sigma_0, \sigma_1, \ldots)$ *defined above is admissible, it completely describes a combinatorial type and the corresponding map is rotation-like.*

Proof: The fact that the sequence of labels determines the type can be seen to be true since there are no consecutive lateral returns. This implies that the nest has exactly one lateral piece at those levels (and none elsewhere) so its position within the frame is completely determined by σ_{r_j}.

As mentioned above, $'0'\sigma_k$ (when $k \neq r_j$) is an admissible label since it corresponds to the central cell of F_{k+1}. Now consider what happens to the central cell labeled $'0'\sigma_{r_{j-1}-1}$. Since level r_{j-1} corresponds to a non-central return, $F_{r_{j-1}+1}$ has two preimages of that cell, labeled $'l0'\sigma_{r_{j-1}-1}$ and $'r0'\sigma_{r_{j-1}-1}$ respectively. On consecutive central returns, we double the number of pullbacks of such cells and thus, use all possible combinations of 'l' and 'r' to label them. A glance to the frame graph shows that these are the cells neighboring the central one. An eventual lateral return must fall precisely in one of these cells, and this is what happens when $\sigma_{r_j} = \tau_j'0'\sigma_{r_{j-1}-1}$. □

The real rotation-like maps studied in [BKP] correspond to a careful choice of the τ_j to ensure that the resulting parameter is real. This sequence of choices can be determined for instance, constructing the kneading sequence. The complex maps corresponding to other choices of τ_j's have the same weak combinatorial behavior, so the critical orbits of two maps with the same sequence $a_1, a_2, \ldots$ are conjugate. In particular we obtain the following result.

Corollary 4.4. *Given the sequence* $a_1, a_2, \ldots$ *there exists an infinite family of complex quadratic polynomials for which the postcritical set is conjugate to an* S*-odometer and semi-conjugate to the circle rotation of angle* $\rho = [a_1, a_2, \ldots]$.

4.2 Q-recurrent maps

The papers [L1] and [W] analyze an unexpected feature of the Fibonacci map. If we rescale the central pieces V_0^n to regions $\widetilde{V}^n$ of fixed size, the g_n induce maps $G_n : \widetilde{V}^n \longrightarrow \widetilde{V}^{n-1}$. On increasing levels, the criss-cross behavior that determines c_{fib} in condition (2) at the beginning of this Chapter, approximates with exponential accuracy the pattern of the critical orbit of $P_{-1}(z) = z^2 - 1$ (i.e. $0 \mapsto -1 \mapsto 0 \mapsto \ldots$). In fact, $G_n \longrightarrow P_{-1}$ locally uniformly in the C^1 norm. Also, since $\operatorname{diam} \widetilde{V}^n \asymp 1$, it is shown that the rescaled pieces converge in the Hausdorff metric to the filled Julia set of P_{-1}, that is, $\widetilde{V}^n \longrightarrow K_{-1}$.

In [W], Wenstrom translates this behavior to the Mandelbrot set and obtains pieces of the paranest around c_{fib} that asymptotically resemble K_{-1}; see Figure 1 of [W]. As consequences of this control on shape, he can compute the exact rate of linear growth of the principal moduli and prove hairiness around the parameter c_{fib}.

Q-recurrency greatly generalizes the above behavior. The general construction will be carried out here, while the next Chapter will focus on developing all the geometrical properties of this combinatorial type.

First, let us fix the center c of a prime hyperbolic component and its associated polynomial $Q(z)$.. The critical orbit is periodic (of least period m) and $Q^{\circ m}$ is the only renormalization of Q. Let us denote the orbit of 0 by $\mathcal{O} = \{0 \mapsto c \mapsto z_2 \mapsto \ldots \mapsto z_{m-1}\}$ and let the fixed point α of Q have combinatorial rotation number $\frac{p}{q}$. All Q-recurrent parameters will be located in the limb $L_{p/q}$.

We will use f to refer to an arbitrary Q-recurrent map. For these parameters, the critical orbit $\mathcal{O}$ never lands on a cell that is a preimage of an 'etj' or 'btj' cell of F_2; therefore, it will be convenient to forget completely about these cells and use the notation F'_n for the reduced frames. In what follows we will save notation by restricting the use of "P_n" to refer to the puzzle of Q and "V_j^n", "F'_n" for the nest and frames of the Q-recurrent map f.

We begin with the observation that $\Gamma(F_0)$ is isomorphic to $\Gamma(P_0)$ since both Q and f belong to the same limb. Let us label $\Gamma(P_0)$ with symbols '0' to 'q-1' starting at the critical point piece and moving counterclockwise; in this way, the labels match those of $\Gamma(F_0)$. Since P_{n+1} is a 2 to 1 pull-back of P_n, the graph $\Gamma(P_{n+1})$ is just two copies of $\Gamma(P_n)$ identified at η_n and we can start a labeling procedure identical to that for frames. For $Q \in L_{p/q}$ the label sequence starts with $(q; \text{'p'}, \ldots)$. Note that $\Gamma(P_n)$ is symmetric, but we can specify a canonical orientation by imposing that the label on η_n (corresponding to the critical value) begins with the symbol 'l'.

What we have produced is a labeling of the puzzle of Q. To define Q-recurrent maps, we just need to guarantee that the label sequence determining the frame system of f matches the label sequence of Q. If we can achieve this, it is clear by an induction argument that $\Gamma(F'_n)$ will be isomorphic to $\Gamma(P_n)$ as embedded in the plane.

Definition 4.5. *A critically recurrent polynomial $z \mapsto z^2 + c$ whose frame system has the same label sequence $(q; p, \sigma_1, \sigma_2, \ldots)$ as Q is called **Q-recurrent** if it satisfies the following condition. For any $n \geq 0$ and $2 \leq k \leq m-1$, the k^{th} return to V_0^n is the composition $(g_n \circ \ldots \circ g_{n+k-2} \circ g_{n+k-1})$.*

Observe that F'_n is symmetric so there are two choices for the homeomorphism identifying F'_n with the puzzle of Q. Once we select a frame orientation, we have an admissible label system.

Note: There is an annoying offset between nest levels and frame levels. Because of it, V_0^n is contained in the central cell of F_{n+1} and contains in turn the cells of $F_{n+2.}$ The notation suffers slightly when discussing return maps to several consecutive levels; hopefully this complication is balanced by the advantage of matching every frame level with the corresponding depth of the puzzle of Q.

Proposition 4.6. *For a Q-recurrent map every sufficiently high level n of the nest has exactly m pieces $V_0^n, V_1^n, \ldots, V_{m-1}^n$. For any $0 \leq j \leq m-1$, V_j^n is contained in the cell of F'_{n+1} corresponding to the piece in P_n that contains z_j.*

Proof: Choose N big enough so that the puzzle P_N isolates every point of $\mathcal{O}(Q)$ and let $n \geq N$. We will call L_j^n the piece of P_n containing z_j.

Consider the orbit of 0 under the composition $g_{n-2} \circ \ldots \circ g_{n+m-3}$. According to the label sequences, $g_{n+m-3}(0)$ falls in the cell of F'_{n+m-2} that corresponds to L_1^{n+m-2}. Next, $g_{n+m-4}(g_{n+m-3}(0))$ falls in the cell of F'_{n+m-3} corresponding to L_2^{n+m-3}. Continue in this manner, with $g_{n+m-3-j} \circ \ldots \circ g_{n+m-3}(0)$ (where $0 \leq j \leq m-2$) falling in the cell of level $n+m-2+j$ that corresponds to $L_{j+1}^{n+m-2+j}$. At every step, we jump out one nest level and create in the process (by adequate pull-backs) the nest pieces $V_1^{n+m-4}, V_2^{n+m-5}, \ldots, V_{m-1}^{n-2}$. Note that all these are lateral pieces since they are contained in a frame cell that is not central. In fact, $V_{j+1}^{n+m-4+j}$ is in the cell of $F'_{n+m-2+j}$ that corresponds to $L_j^{n+m-2+j}$; see figure 8.

The last map in this chain of compositions is g_{n-2}. It brings the critical orbit very nearly to the center, inside V_0^{n+m-3}. To see this, remember that the definition of Q-recurrency requires that the composition of maps $g_{n-2} \circ \ldots \circ g_{n+m-3} : V_0^{n+m-2} \longrightarrow V_0^{n+m-3}$ is the first return to V_0^{n+m-3}, i.e.

$g_{n-2} \circ \ldots \circ g_{n+m-3}$ is exactly the map g_{n+m-2}.

In summary, if a point of $\mathcal{O}$ falls in a piece V_j^n (for $j \leq m-1$), the next return falls inside V_{j+1}^{n-1}. If it falls on a piece V_{m-1}^n, the next return falls m levels deeper, inside V_0^{n+m-1} and is in fact, the first return to this piece. Repeating this procedure at $m-1$ consecutive levels creates various pieces of different levels. Among these, the $m-1$ lateral nest pieces of level n, each corresponding to a point z_j $(1 \leq j \leq m-1)$ of the critical orbit of Q. □

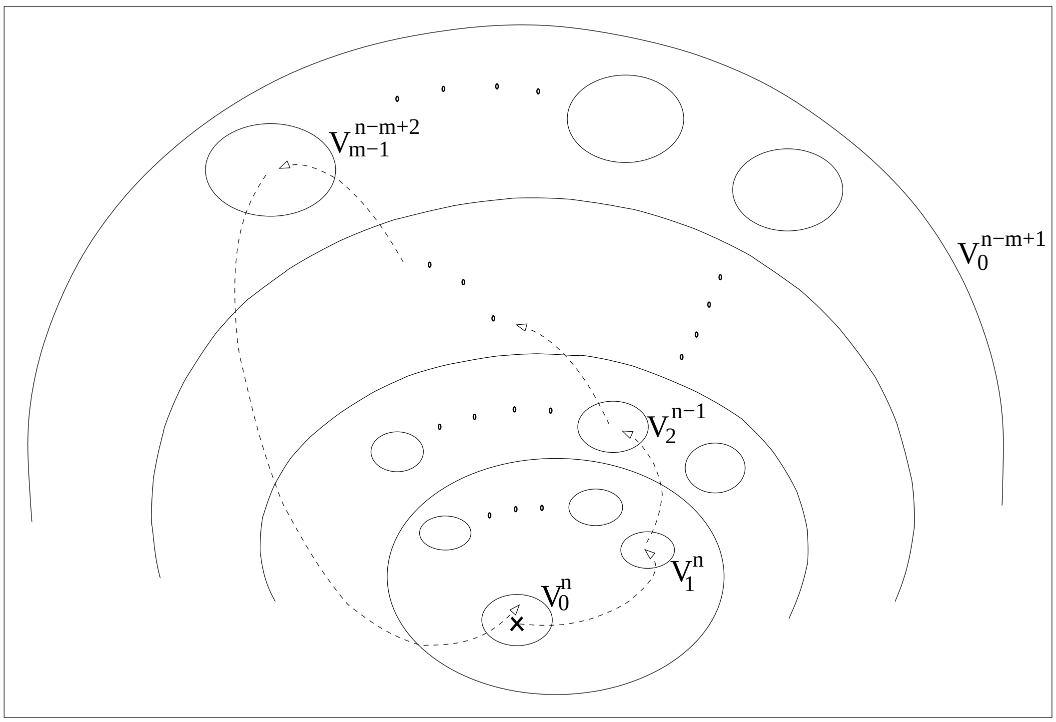

Figure 8: *In a Q-recurrent map there are $m-1$ lateral pieces at each level, where m is the period of 0 under Q. Under successive first return maps, the critical orbit jumps out to lower levels (as V_i^{n-i+1} goes inside V_{i+1}^{n-i}) until at step $m-1$ it returns to the center. See also Figure 10.*

Theorem 3.11 guarantees the existence of Q-recurrent maps. Observe that a first consequence of proposition 4.6 is the fact that the itinerary of V_j^n in the nest of level $n-1$ is (see Definition 3.8)

$$\begin{aligned} \chi(V_j^n) &= ([V_{j+1}^{n-1}; \lambda_{n+1,j+1}], [V_0^{n-1}; \lambda_{n+1,0}]) \text{ when } 0 \leq j < m-1 \\ \chi(V_{m-1}^n) &= ([V_0^{n-1}; \lambda_{n+1,0}]). \end{aligned} \tag{4.1}$$

This hints to a similarity between the actions of Q and g_n that will be made precise in the next Chapter, where we develop the asymptotic properties of Q-recurrent maps and their principal nests.

4.3 Meta-Chebyshev

Consider the polynomial $P_{\text{MCheb}}(z) = z^2 - 1.87450961730020085\ldots$ This parameter was constructed with the requirement that the graphs $\Gamma(F'_n)$ are isomorphic to $\Gamma(P_n(f_{-2}))$, where $f_{-2} : z \mapsto z^2 - 2$ is the Chebyshev polynomial. The motivation for this example is to determine what properties of Q-recurrent polynomials will hold for parameters that imitate the behavior of more general postcritically finite maps. In fact, the construction of P_{MCheb} is very similar to that of Q-recurrent polynomials, except for the fact that the critical orbit of f_{-2} does not go back to the center. This means that the first return to level $n+1$ must be delayed until after the composition $g_1 \circ \ldots \circ g_n$. At this point, the critical orbit falls in $Y_0^{(0)}$ and the previous combinatorics impose no restriction.

The parameter is chosen so that the first return to level $n+1$ occurs exactly at this moment; that is $g_{n+1} = g_1 \circ g_2 \circ \ldots \circ g_n$ for all n. Also, the choice of frame orientations that result in a real parameter imposes the label sequence $(2; '1', 'l0', 'lr0', 'lrr0', 'lrrr0', \ldots)$. By analogy with the critical orbit of Q, every nest has two lateral pieces and the itinerary of V_i^n includes an infinite number of visits to V_2^n after the first return to V_0^{n+1}.

If this combinatorial description is admissible, the results of Chapter 3 guarantee an uncountable set of complex parameters with the same combinatorics. By a result of Yoccoz, there cannot be other real polynomials in this class except P_{MCheb}.

To justify the existence of such map, consider the kneading sequence θ:

$$-\check{+}\check{+}-\check{-}-+--\check{-}-++-+-+--\check{-}-++---+--+-++-+-+--\check{-}\cdots$$

The checks are just an aid to construct the sequence by the following iterative procedure: Start with the chain $-\check{+}\check{+}-\check{-}$ Create a second, unchecked copy inverting the symbol at the second to last mark. Concatenate this copy to the right of the previous chain and put a check on the last symbol. Repeat.

The checks mark the first return moments to every level of the nest and the first symbols are fixed by the requirement of having a real map (that is, in $L_{1/2}$). Recall that the critical orbit of f_{-2} is $0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto \ldots$. If P_{MCheb} is to replicate this behavior, every visit to a level n must fall on V_2^n

except for the first return to V_1^n that should be located in the opposite side. This justifies inverting the middle symbol at each step. Note that the first returns occur at iterates $2, 3, 5, 10, 20, 40, \ldots$, i.e. doubling at each step after the third.

In order to verify that this sequence is admissible, we just have to describe the sequence $\epsilon_1 \epsilon_2 \ldots$ of accumulated products of symbols in θ. Then we have to prove that for any m, the least i such that $\epsilon_{m+i} \neq \epsilon_m \cdot \epsilon_i$ satisfies $\epsilon_i = -$ The sequence of ϵ_j begins:

$$-\check{-}\check{-}+\check{-}++-+\check{-}+++--++-+\check{-}+++-+--+--+++--++-+\check{-}\cdots$$

and the rule to construct it is as follows: Start with the chain $-\check{-}\check{-}+\check{-}$ Make a second unchecked copy and invert every symbol that appears to the left of the second to last check. Concatenate this copy to the right of the previous chain and put a check on the last symbol. Repeat.

This sequence starts with $-\check{-}\check{-}+\check{-}$ and every mark will be on a $-$ symbol. It follows that there cannot be more that three $+$ in a row so the admissibility condition is satisfied. Thus, the kneading sequence θ can be realized by a real polynomial and P_{MCheb} exists.

The methods we use to work with Q-recurrent polynomials are not enough to prove corresponding results for P_{MCheb}. In particular, we will rely on the fact that K_Q has a non-empty interior, which is not the case for K_{-2}. Nevertheless, the analogy is good enough that it is natural to ask the following.

Conjecture 1. *There are suitable rescalings of the nest pieces of P_{MCheb} such that the functions induced from the first return maps converge to f_{-2} and such that the rescaled pieces converge to the interval $[-2, 2]$ in the Hausdorff topology.*

Chapter 5

Asymptotics of Q-recurrency

This Chapter presents the geometric properties of Q-recurrent maps. Their explicit relation to the combinatorics of Q gives control over the shapes of nest pieces and this will yield very precise estimates of the analytic invariants of the nest.

For reference, let us state again the fundamental relation between levels of a Q-recurrent nest:

$$g_{n+m} = g_n \circ \ldots \circ g_{n+m-1} \tag{5.1}$$
$$g_{n+m}\left(V^{n+m}, V^{n+m-1}\right) = \left(V^{n+m-1}, V^{n-1}\right) \tag{5.2}$$

5.1 Complex Fibonacci maps

Definition 5.1. *A complex polynomial map, or equivalently its corresponding parameter, is said to be* ***Fibonacci*** *if the first return to each level of the nest happens exactly when the iterates are the Fibonacci numbers.*

Note: The first return to a level can be viewed as a close return in a combinatorial sense; that is, a return to a small central piece. Since Lyubich's Theorem guarantees that central pieces decrease in size, the property in Definition 5.1 is equivalent to its metric analogue, property (1) at the beginning of Chapter 4.

Our first result is a classification of the Fibonacci behavior in the complex case.

Definition 5.2. *The set of all* (z^2-1)*-recurrent parameters is denoted* $\mathbb{F}\text{ibo}$.

Theorem 5.3. *A parameter c is Fibonacci if and only if* $c \in \mathbb{F}\text{ibo}$.

Proof: All (z^2-1)-recurrent maps have the same weak combinatorial type. As was pointed out in the note after Definition 3.10, the first returns of high levels are just predetermined compositions of lower level ones. Thus, the number of iterates until the first return to a piece V_0^n is independent of the parameter $c \in \mathbb{F}\text{ibo}$. As it is known that the real parameter $c_{\text{fib}} \in \mathbb{F}\text{ibo}$ is Fibonacci, the first direction of the assertion follows.

To show the converse, it is only necessary to observe that the first return times in a Fibonacci nest are strictly increasing, so there are no central returns. Therefore the first return map g_{n+1} must be the composition of at least two first return maps of the two preceding levels. If the nest does not have (z^2-1)-recurrent type, there must be more than one lateral piece at some level n. Then the composition of maps generating g_{n+1} will actually contain more than two maps and the sequence $\{\ell_n\}$ of first return times grows faster than the sequence generated by the recursion $\ell_{n+1} = \ell_n + \ell_{n-1}$. This contradicts the assumption that the map is Fibonacci. □

Although Yoccoz's Theorem allows us to characterize $\mathbb{F}\text{ibo}$ as a Cantor set, we must wait until next Chapter to show that the relevant parapieces shrink exponentially fast, allowing us a complete description of the set $\mathbb{F}\text{ibo}$ as a Cantor set of Hausdorff dimension 0 on which we can impose a natural dyadic decomposition.

5.2 Shape

We want to study the shape of nest pieces in the following sense.

Definition 5.4. *A sequence of compact sets $\{C_j \subset \mathbb{C}\}$ is said to* ***converge in shape*** *to a compact K if there exist rescalings $\widetilde{C_j} = a_j \cdot C_j$ (with $a_j \in \mathbb{C}$) such that $\{\widetilde{C_j}\} \longrightarrow K$ in the Hausdorff metric.*

The main Theorem of this Chapter is a vast extension of the result on the shape of central pieces of $f_{c_{\text{fib}}}$ found in [L1]. In order to give the statement, we need to fix some notation.

Let $Q = Q(z)$ be the center of a prime hyperbolic component and c_0 a Q-recurrent parameter. Recall that f_{c_0} is described by a dyadic choice of labels ('l' and 'r') on every level. These frame orientations determine the sequence of paranest pieces $\{\Delta^n\}$ around c_0. If $c \in \Delta^n$ is any nearby parameter, the combinatorics of f_c are identical to those of f_{c_0} including the orientations of the homeomorphic frames, at all levels $j \leq n$. In particular, for any $c \in \Delta^n$ we can find a (unique) point s_j in F_j' corresponding to the fixed point α of Q. In what follows, we do not include in the notation the fact that the objects described depend on c. Let $\alpha_j = \frac{\alpha}{s_j}$ and define the complex rescalings $\widetilde{V}^j = \alpha_j \cdot V_0^j$ of the central nest pieces of f_c, up to level n. Then, the first return maps g_j induce maps $G_j : \widetilde{V}^j \longrightarrow \widetilde{V}^{j-1}$ on rescaled pieces whose action on the rescaled frame $\widetilde{F'}_{j-2} \subset \widetilde{V}^j$ is isotopic to the action of Q on its own puzzle.

Theorem 5.5. *Given $\varepsilon > 0$, there is an N such that for every parameter $c \in \Delta^n$ and level $n \geq N$, the maps $G_N, G_{N+1}, \ldots, G_n$ are all ε-close to Q in the C^1 topology inside the ball of radius $\frac{1}{\varepsilon}$.*

Corollary 5.6. *The sequence of central nest pieces $\{V_0^n\}$ of f_{c_0} converges in shape to the filled Julia set K_Q.*

Proof of Corollary 5.6: The point $\alpha \in K_Q$ is fixed under G_n and is surrounded by $\widetilde{V}^n$. This rescaled nest piece also surrounds the critical point 0 which attracts every point in $K_Q \setminus J_Q$. Now, $\widetilde{V}^n$ is the pull-back of $\widetilde{V}^{n-1}$ under G_n. By Theorem 5.5, G_n is a small perturbation of Q; since the rescaled pieces in the sequence $\{\widetilde{V}^{n+1}, \widetilde{V}^{n+2}, \ldots\}$ have bounded diameter, they become exponentially close to the regions in the sequence $\{Q^{\circ -1}(\widetilde{V}^n), Q^{\circ -2}(\widetilde{V}^n), \ldots\}$ converging to K_Q. This yields the result. $\square$

In particular, the central pieces of any complex quadratic Fibonacci map look like K_{-1}, although each one may be tilted at a bizarre angle (recall that the $\widetilde{V}^n$ are rescaled by a complex number). Other examples can be seen in Figure 9, showing puzzle pieces that approximate the behavior of different periodic orbits of period 3.

Notice that, since the frames are defined by the same sequence of pull-backs as the central nest pieces, the result of Corollary 5.6 holds also for frames; i.e. the union of cells in F'_n converges in shape to K_Q.

The proof of Theorem 5.5 depends on the convergence of Thurston's map on an appropriate Teichmüller space (see the Appendix for definitions). Let $\mathcal{O} \equiv \mathcal{O}(Q)$ and consider the surface S obtained by puncturing the plane at the critical orbit of Q; that is, $S = \mathbb{C} \setminus \mathcal{O}$. Since deformations are considered only up to an isotopy that leaves $\mathcal{O}$ invariant, the structure of a puzzle-like construction does not change. Thus, when h is a deformation in the class of id, the deformation $h(P(Q))$ of the puzzle of Q can be isotoped back to the puzzle $P(Q)$ itself without changing its configuration and without moving $\mathcal{O}$.

Thurston's map is best described via the alternate description of $\mathcal{T}_S$ in terms of Beltrami differentials. First we normalize every deformation h by an affine change of coordinates φ so that $\varphi \circ h$ leaves $0, c \in \mathcal{O}$ fixed. The Beltrami coefficient $\mu = \frac{\overline{\partial} h}{\partial h} \frac{d\overline{z}}{dz}$ determines a conformal structure associated to h.

Definition 5.7. *The map $\tau_Q : \mathcal{T}_S \longrightarrow \mathcal{T}_S$ induced on equivalence classes of conformal structures by the pull-back $\mu \mapsto Q^*\mu$ is called the* ***Thurston map*** *associated to Q.*

The action of τ_Q on a deformation class h is easy to describe. The class $\tau_Q([h])$ is represented by a deformation $\widetilde{h}$ such that the map $Q_h = h \circ Q \circ \widetilde{h}^{-1}$ is analytic. Because of conjugacy, Q_h replicates the critical orbit behavior of Q in a neighborhood of $\widetilde{h}(\mathcal{O})$. In particular, one can specify a puzzle-like structure around $\widetilde{h}(\mathcal{O})$ which pulls back according to the same combinatorics as Q. Since $\mathcal{O}$ is finite, and $Q^{\circ m}$ is not renormalizable, such a puzzle structure of high enough depth will isolate all the elements of the critical orbit in individual cells. We conclude that the isotopy class of $\widetilde{h}$ relative to punctures consists of those Q_h-pull-backs of $h(\mathcal{O})$ that keep the puzzle structure intact (however deformed).

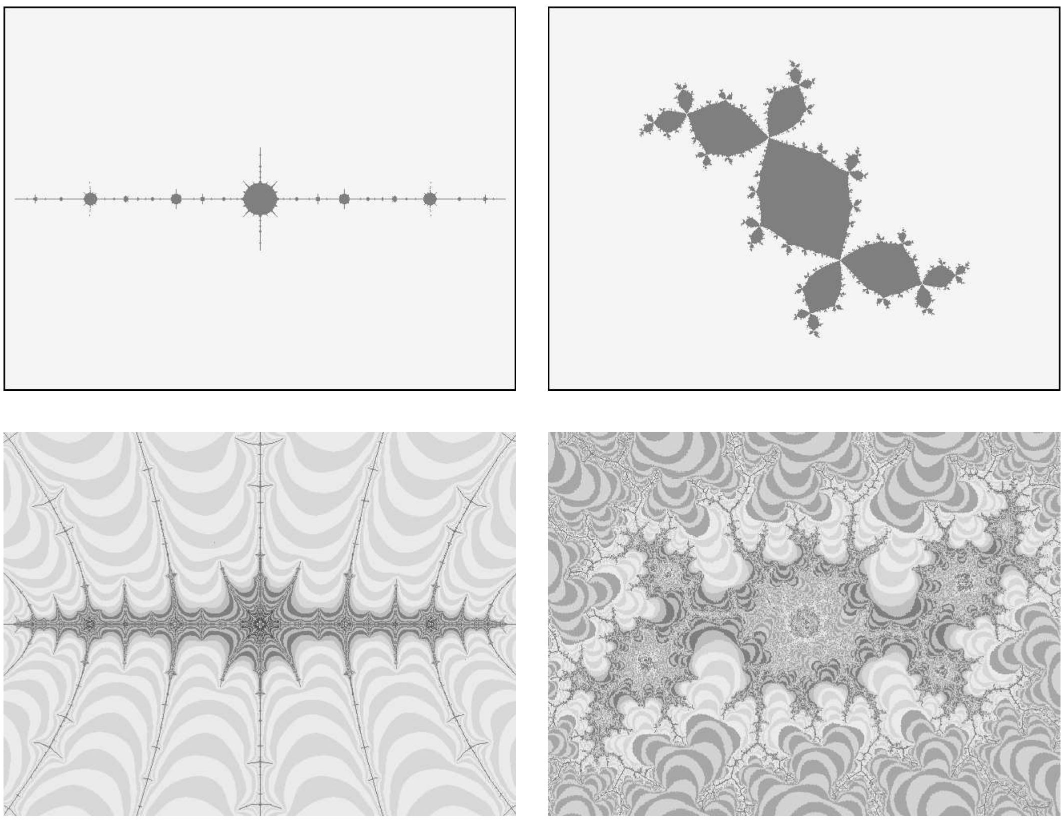

Figure 9: *Consider the maps $Q_1 : z \mapsto z^2 - 1.75487...$ (the "airplane") and $Q_2 : z \mapsto z^2 - (0.123...) + i(0.745...)$ (the "rabbit"), as displayed in the first row. Both maps have critical orbits of period 3. The pictures in the second row show close-ups near 0 of two other Julia sets. On the left, a Q_1-recurrent map ($c_1 = -1.87449300898719...$). On the right, a Q_2-recurrent map ($c_2 = -0.023918090959967... + i0.984732550113053...$). In each case, there is a central nest piece approximating the Julia set of the corresponding Q_i.*

Proof of Theorem 5.5: Let X be any finite collection of simply connected compact subsets of $\mathbb{C}$. By a **multicurve** Γ around X we mean a system of disjoint isotopy classes of simple closed curves in $\overline{\mathbb{C}} \setminus X$ such that each curve $\gamma_i \in \Gamma$ splits $\overline{\mathbb{C}}$ in two regions, each enclosing at least two elements of X (i.e. γ_i is non-peripheral). If $f : \overline{\mathbb{C}} \setminus X \longrightarrow \overline{\mathbb{C}} \setminus X$ fixes every element of X, we denote by Γ_f^{-1} the multicurve consisting of the classes of f-preimages of elements $\gamma_i \in \Gamma$ that are not peripheral. The multicurve Γ is said to be f**-stable** if $\Gamma_f^{-1} \subset \Gamma$.

Given a map $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ fixing the critical orbit $\mathcal{O}$ of Q and an arbitrary f-stable multicurve Γ around $\mathcal{O}$, we can construct the linear space $\mathbb{R}^\Gamma$ generated by the curves of Γ, and an induced linear map $\hat{f}_\Gamma : \mathbb{R}^\Gamma \longrightarrow \mathbb{R}^\Gamma$ given as follows. If $\gamma_i \in \Gamma$, let $\gamma_{i,j,k}$ denote the components of $f^{-1}(\gamma_i)$ that are in the class of $\gamma_j \in \Gamma_f^{-1}$. Then

$$\hat{f}_\Gamma(\gamma_i) = \sum_{j,k} \frac{1}{d_{i,j,k}} \gamma_j.$$

where $d_{i,j,k}$. denotes the degree of $f_{|\gamma_{i,j,k}} : \gamma_{i,j,k} \longrightarrow \gamma_i$.

An obstruction to the convergence of Thurston's map τ_f is represented by a f-stable multicurve around $\mathcal{O}$, for which $\hat{f}_\Gamma$ has an eigenvalue $\lambda \geq 1$. In our case, Q is a polynomial so it represents the fixed point of its own Thurston map. In particular, there are no obstructions to the convergence of τ_Q; see [DH3].

Now, since Q belongs to a prime hyperbolic component of period m, the map $Q^{\circ m}$ is a renormalization conjugate to $z \mapsto z^2$. By hyperbolicity, the central puzzle pieces of K_Q get arbitrarily close to the immediate basin of 0. In particular, there is a finite depth so that 0 is the only point of $\mathcal{O}$ inside the central piece. By further iteration, the same will be true of any point in $\mathcal{O}$.

Let us choose a level k high enough so that the puzzle P_{k-1} isolates all the points in the critical orbit of Q. Again, this is possible since $Q^{\circ m}$ is not renormalizable. Then, any Q-stable multicurve Γ can be represented with curves that are constructed from segments of the arcs defining P_{k-1}. In this way, Γ is described in terms of the structure of $P_{k'}$, for any level $k' \geq k-1$. Moreover, Γ_Q^{-1} is a multicurve around $\mathcal{O}$ that can be described in terms of the combinatorial structure of $P_{k'+1}$.

Now consider f_c with $c \in \Delta^k$. Any G_k-stable multicurve Γ' around the pieces $\widetilde{V}_j^{k+1}$ can be described with segments of curves in the boundary of the frame $\widetilde{F}'_{k-1}$. Since $\widetilde{F}'_{k-1}$ is isomorphic to P_{k-1}, there is a correspondence between G_k-stable multicurves around $\bigcup_j \left\{\widetilde{V}_j^{k+1}\right\}$ and Q-stable multicurves around $\mathcal{O}$. This means that the only possible obstructions for τ_{G_k} must form inside one of those pieces; that is, a multicurve realizing such obstruction would intersect at least one of the pieces $\widetilde{V}_j^{k+1}$. Note that such multicurve cannot be represented by curves that are close to the boundary of $\widetilde{F}'_{k-1}$.

By [L3], we know that the size of $\widetilde{V}_0^{k+2}$ with respect to $\widetilde{V}_0^{k+1}$ decreases exponentially as $k \to \infty$. Then, Koebe's Theorem shows that G_k is exponentially close to being quadratic; that is, it can be decomposed as $G_k = D_k \circ Q_{h_k}$, where the maps D_k become linear and the deformations h_k are given by iteration of the Thurston map τ_Q. Moreover, both Q_{h_k} and G_k fix α and send 0 close to itself, so we can conclude that $D_k \longrightarrow \mathrm{id}$. It follows that G_k rapidly approaches Q_{h_k}.

Select any Q-stable multicurve Γ' around $\mathcal{O}$. If there is a level k such that Γ' does not intersect any of the pieces $\widetilde{V}_j^{k+1}$, then Γ' can be pushed to the boundary of $\widetilde{F}'_{k-1}$ to represent a G_k-stable multicurve around the pieces $\widetilde{V}_j^{k+1}$. Since Γ' is not an obstruction for Q, we deduce that, outside the $\widetilde{V}_j^{k+1}$, the map G_k is isotopic to Q. However, the only possible Thurston obstructions are restricted to extremely small regions, then the distortion of h_k goes to 0 and the maps G_k converge to Q exponentially fast in a neighborhood of $K_Q \setminus \mathcal{O}$. We know that the Koebe space between V_0^{k+1} and V_0^k increases without bound, so we can claim convergence of the maps G_k in arbitrarily big neighborhoods of K_Q. □

Theorem 5.5 has broad implications since it provides excellent control of the shapes of nest pieces. In the next Section, we use our knowledge on the shape of the central pieces to compute the rate of growth of the principal moduli.

5.3 Growth of annuli

Here we study the moduli of the principal annuli in Q-recurrent maps. For this family, we can state a more precise version of Lyubich's Theorem L1 on the linear growth of moduli. The key ingredient in the proof is Theorem 5.5 giving control over the shape of pieces, in conjunction with the extended Grötzsch inequality as stated in the Appendix.

As a preparation for Theorem 5.9, we compute first the capacities of K_Q with respect to 0 and ∞.

Lemma 5.8. *Let $Q = Q(z)$ be the center of a hyperbolic component with* *critical period m. Then* $\operatorname{cap}_\infty(K_Q) = 0$ *and*

$$\operatorname{cap}_0(K_Q) = -(m-1)\ln 2 - \sum_{j=1}^{m-1} \ln \left| Q^{\circ j}(0) \right|.$$

Proof: K_Q is connected, so the Böttcher coordinate $\varphi : \overline{\mathbb{C}} \setminus K_Q \longrightarrow \overline{\mathbb{C}} \setminus \mathbb{D}$ sending 0 to ∞ is the Riemann mapping with derivative 1, so the first equality is obvious.

The capacity of K_Q at 0 is simply $\operatorname{cap}_0(U)$, where U is the immediate basin of attraction of 0. Consider the iterated polynomial $Q^{\circ m} : U \longrightarrow U$. It is a 2 to 1 map of a simply connected domain with fixed critical point. Therefore, there is a map $\psi : \mathbb{D} \longrightarrow U$ such that

$$\psi(z^2) = Q^{\circ m} \circ \psi(z) \tag{5.3}$$

and it is clear that $\operatorname{cap}_0(K_Q) = \operatorname{cap}_0(U) = \ln|\psi'(0)|$.

Equation 5.3 shows that $\psi'(0)$ is the inverse of the quadratic coefficient in the series expansion of $Q^{\circ m}(z)$. Since the constant term of $Q^{\circ j}(z)$ is just $Q^{\circ j}(0)$, it is easy to find recursively that $\frac{1}{\psi'(0)} = \prod_{j=1}^{m-1} 2Q^{\circ j}(0)$ and thus,

$$\operatorname{cap}_0(K_Q) = \ln \psi'(0) = -(m-1)\ln 2 - \sum_{j=1}^{m-1} \ln \left| Q^{\circ j}(0) \right|.$$

□

Recall that capacity and modulus are invariants that vary continuously with respect to the Carathéodory topology. Thus, given a sequence of topological disks around 0 converging in the Hausdorff topology to a set with pinched points, the sequence of capacities will converge to the capacity of the component of the limit set that contains 0. Similarly, for a sequence of annuli with adequate convergence, the limit of moduli detects only the modulus of the limit component that contains the limit closed geodesic.

Theorem 5.9. *For any parameter* $c \in \mathbb{F}\mathrm{ibo}$ *the principal moduli grow linearly at the rate*

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{\ln 2}{3}.$$

If $Q = Q(z)$ *is the center of a prime hyperbolic component with critical period* $m \geq 3$, *the rate of growth is exponential*

$$\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = \kappa_m,$$

where $\kappa_m \nearrow \frac{3}{2}$ *as the period* m *of* Q *increases.*

Proof: Fix a level N large enough so that the shape of rescaled nest pieces is already close to the shape of K_Q. In particular, $\text{cap}_0\left(\widetilde{V}^n\right) \sim \text{cap}_0\left((K_Q)_0\right)$ for all $n \geq N$, where $(K_Q)_0$ is the Fatou component of K_Q containing 0.

We also require the lateral pieces are small enough to sit in the center of their (almost pinched) regions, far away from the boundary. This is possible since Theorem L1 forces shrinking and Theorem 5.5 locates the nest pieces in positions that resemble the critical orbit of Q.

Theorem 5.5 also gives the recursion formula

$$g_{n+m} = g_n \circ \ldots \circ g_{n+m-2} \circ g_{n+m-1}.$$

On consecutive levels the first returns of a central piece V_0^{n+m+1} fall inside the pieces $V_1^{n+m}, V_2^{n+m-1}, \ldots, V_{m-1}^{n+2}$ and V_0^{n+m}. From this, we obtain the annuli relation

$$g_{n+m}^{-1}\left(V_0^n \setminus V_0^{n+m}\right) = \left(V_0^{n+m} \setminus V_0^{n+m+1}\right).$$

In order to estimate the modulus of $\left(V_0^{n+m} \setminus V_0^{n+m+1}\right)$, let us split the return map $g_{n+m}|_{\left(V_0^{n+m} \setminus V_0^{n+m+1}\right)}$ in the above mentioned composition of first returns. First, g_{n+m-1} is 2 to 1 on the annulus $\left(V_0^{n+m} \setminus V_0^{n+m+1}\right)$. Note that the image of V_0^{n+m+1} is deep inside V_1^{n+m}; in fact, these two pieces are separated by a nested sequence of preimages of the central pieces $V_0^{n+1}, \ldots, V_0^{n+m-1}$. Due to the pinching of pieces near repelling points, most of the modulus of $g_{n+m-1}\left(V_0^{n+m} \setminus V_0^{n+m+1}\right) \subset V_0^{n+m-1}$ is concentrated in a region of V_0^{n+m-1} that resembles the immediate basin of the critical value of Q. On this region, g_{n+m-2} is injective.

The remaining returns $g_{n+m-3}, \ldots, g_n$ behave in a similar manner, essentially preserving the modulus on regions around non-central pieces. We can conclude that

$$\text{mod}\left(V_0^{n+m} \setminus V_0^{n+m+1}\right) \asymp \frac{1}{2} \text{mod}\left(V_0^n \setminus V_0^{n+m}\right).$$

The right hand side can be extimated by applying the Extended Grötzsch Inequality to the annulus $\left(V_0^n \setminus V_0^{n+m}\right) = \left(V_0^n \setminus V_0^{n+1}\right) \cup \ldots \cup \left(V_0^{n+m-1} \setminus V_0^{n+m}\right)$, we obtain

$$\text{mod}\left(V_0^{n+m} \setminus V_0^{n+m+1}\right) \asymp \frac{1}{2}\left(\sum_{j=0}^{m-1} \text{mod}(V_0^j \setminus V_0^{j+1}) + \varepsilon_Q\right)$$

where $\varepsilon_Q \sim (m-1)\left|\text{cap}_0(K_Q)\right| = (m-1)\left|(m-1)\ln 2 + \sum_{j=1}^{m-1} \ln\left|Q^{\circ j}(0)\right|\right|$

The recursive formula $x_{n+m} = \frac{1}{2}(x_{n+m-1} + x_{n+m-2} + \dots x_n) + \varepsilon_Q$ has an asymptotic behavior that is ruled by the largest root of its characteristic polynomial

$$z^m - \frac{1}{2}(z^{m-1} + z^{m-2} + \dots + z + 1). \tag{5.4}$$

When $m = 2$, $Q(z) = z^2 - 1$, the largest root of $z^2 - \frac{1}{2}(x+1) = 0$ is 1 and $\varepsilon_Q = \ln 2$. Consequently, the growth of the moduli is dominated by a linear term $\mu_n \sim An + B$. The recursive relation $\mu_n \asymp \frac{\mu_{n-1}}{2} + \frac{\mu_{n-2}}{2} + \ln 2$ gives

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{\ln 2}{3}.$$

If $m \geq 3$, the largest root κ_m of 5.4 is strictly larger than 1. In fact, it strictly increases to $\frac{3}{2}$ as $m \to \infty$. The exponential growth of μ_n follows and it is clear that eventually it does not depend on the value of ε_Q. □

Note: One should contrast the above result with [AM]. There, the authors show that for almost every non-hyperbolic real parameter, the principal moduli grow at least as fast as a tower of exponentials. The "slower" growth displayed by Q-recurrent polynomials has immediate geometric consequences.

Definition 5.10. *We will say that a compact set K is* ***hairy*** *at a point $c \in K$ if there is a sequence $\{\varepsilon_1, \varepsilon_2, \dots\}$ converging to 0, such that $\frac{1}{\varepsilon_j} \cdot (K - c) \cap \overline{\mathbb{D}}$ becomes dense in $\overline{\mathbb{D}}$.*

If K is hairy at c for any sequence of scaling factors $\{\varepsilon_j\}$, we say that it satisfies ***hairiness at arbitrary scales****.*

By an observation of Rivera-Letelier ([R-L]), the construction of [W] can be extended to prove hairiness of M at any critically recurrent non-renormalizable parameter. The idea is as follows:

Since K_c is connected, it contains a path from 0 to β. This crosses every principal annuli from one boundary component to the other. Choosing a high enough level, we can rescale the annulus A_n to constant diameter containing a *hair* that connects the outer boundary with a small neighborhood of 0. The pull-backs by consecutive first return maps duplicate the number of hairs inside deeper annuli and this collection of hairs is equidistributed around 0 (control of geometry). The hairiness of K_c is then translated to the parameter plane to obtain the result.

Rivera-Letelier has announced a proof that the real quadratic Fibonacci polynomial displays hairiness at arbitrary scales. The proof relies in an essential way on the linear growth of moduli, so it holds true for any parameter in $\mathbb{F}$ibo . Other Q-recurrent polynomials miss this sharper property in account of the exponential growth of their principal moduli. It should be observed that this same property creates a somewhat embarrassing difficulty; since the moduli grow so fast, computer generated pictures fail to exhibit a convincingly hairy picture. In order to do so, it would be necessary to reach deep levels of the nest that may be out of the range of resolution of the software used.

Chapter 6

Parameter space

One of the most amazing attributes of complex quadratic dynamics is the replication of dynamical features in the parameter plane. For instance, the structure of a limb $L_{p/q}$ reflects the initial steps of the critical orbit for any parameter contained in it. In [T], Tan Lei showed that for a strictly preperiodic parameter c, the Julia set of f_c and the Mandelbrot set exhibit local asymptotic similarity around c.

As we have mentioned, a result of similar nature appears in [W] where Wenstrom shows that the paranest pieces around the real Fibonacci parameter c_{fib} are asymptotically similar to the central pieces in the principal nest of $f_{c_{\text{fib}}}$. Thus, $\Delta^n(c_{\text{fib}}) \longrightarrow K_{-1}$ in shape and the author exploits this geometric result to obtain hairiness of M around c_{fib}.

This Chapter discusses a generalization of the above results to the family of all Q-recurrent parameters. Note that the maps Q are dense in ∂M and that for each one we have an uncountable set of Q-recurrent parameters.

Theorem 6.1. *Let $Q = Q(z)$ be the center of a prime hyperbolic component with critical period m, and let c_Q be a Q-recurrent parameter. Then the paranest around c_Q is infinite and the parapieces $\{\Delta^j(c_Q)\}$ converge in shape to the filled Julia set K_Q.*

This will require translating the corresponding result that we obtained in the dynamical plane to the space of parameters. To do this, we need to introduce certain auxiliary parapieces; describe in detail the boundary of $\Delta^j(c_Q)$ and define a map $M_n : \Delta^n(c_Q) \longrightarrow \mathbb{C}$ that "rescales" Δ^n to a compact set close to K_Q.

From the above result follow the possibility of computing the rate of growth of the paramoduli. Since the paramoduli increase at least linearly, the set of Q-recurrent parameters is a Cantor set of Hausdorff dimension 0.

Lastly, in Section 6.3 we exploit the result on shapes of parapieces to present a new form of auto-similarity in the Mandelbrot set.

For the rest of this Chapter, unless explicitly mentioned, we will fix a map $Q = Q(z)$ in the center of a prime hyperbolic component such that the critical orbit has period m; also, c_Q will stand for a fixed Q-recurrent parameter.

6.1 Auxiliary parapieces

Consider the first return map $g_{n-1} : V_0^{n-1} \longrightarrow V_0^{n-2}$. Given that $V_1^n \Subset V_0^{n-1}$, we can study the effect of g_{n-1} on V_1^n; there, the condition of Q-recurrency gives:

$$g_{n-1}(V_1^n) \Subset V_2^{n-1}.$$

Then, we can apply g_{n-2} to obtain

$$g_{n-2} \circ g_{n-1}(V_1^n) \Subset g_{n-2}(V_2^{n-1}) \Subset V_3^{n-2}.$$

This procedure can continue further for a total of $m-2$ steps:

$$g_{n-m+2} \circ \dots g_{n-1}(V_1^n) \Subset V_{m-1}^{n-m+2},$$

where in fact, this image is contained in a nest of intermediate pieces inside V_{m-1}^{n-m+2}; see Figure 10. When we apply g_{n-m+1}, the piece V_{m-1}^{n-m+2} maps onto V_0^{n-m+1} so the intermediate pieces mentioned will map onto the central pieces $V_0^{n-m+2}, V_0^{n-m+3}, \dots, V_0^{n-2}$ and the combined effect on V_1^n will be (recall Definition 2.18):

$$g_{n-m+1} \circ \dots g_{n-1}(V_1^n) = g_{n,1}(V_1^n) = V_0^{n-1}. \tag{6.1}$$

Since $V_0^n \Subset V_0^{n-1}$, the following is well defined.

Definition 6.2. *We denote by $U^n \Subset V_1^n$ and $F_{n+2}^* \subset U^n$ the $g_{n,1}$-pull-backs of V_0^n and $F'_{n+2} \subset V_0^n$, respectively. Compare Figure 10.*

Note that F_{n+2}^* is known once the nest structure up to level n is given. However, if we assume (as is the case here) that the nest of our parameter displays the Q-recurrency type up to level $n+1$, we can say more. Since $g_{n+1}(0) = g_{n-m+1} \circ \dots g_{n-1} \circ g_n(0)$, we must have

$$g_n(0) \in F_{n+2}^*. \tag{6.2}$$

Let us pass to the parameter plane. Our initial goal is to obtain a precise control of the combinatorics inside relevant consecutive parapieces. In the first place, Δ^n is the set of parameters that have the same nest combinatorics as c_Q, up to the first return $g_n(0)$ to V_0^{n-1}.

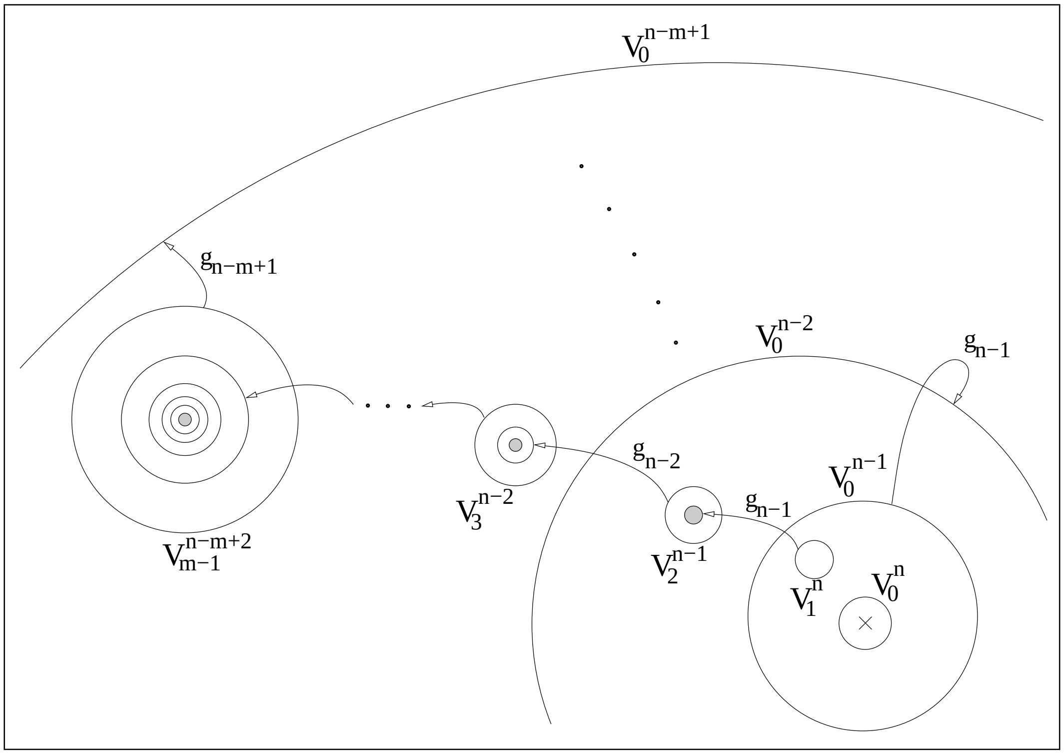

Figure 10: *Construction of U^n and F^*_{n+2}. The domain of g_{n-1} is V^{n-1}; in particular, it maps V_1^n inside V_2^{n-1}. Further maps $g_{n-2}, g_{n-3}, \dots$ take the current ensemble inside the next piece of previous level, until V_{m-1}^{n-m+2}. Note that g_{n-m+1} takes V_{m-1}^{n-m+2} onto the central piece V_0^{n-m+1}, instead of a lateral one. This maps the gray piece (the image of V_1^n) onto V_0^{n-1}. Then, we can pull V_0^n back all the way to the piece U^n inside V_1^n. Also, U^n has a frame F^*_{n+2} which is the corresponding pull-back of $F'_{n+2} \subset V_0^n$. Neither U^n nor F^*_{n+2} are drawn. See also Figure 8.*

Definition 6.3. *We introduce two new auxiliary parapieces.*

- Δ_*^{n+2} *is the set of parameters such that $g_n(0)$ falls inside the frame $F'_{n+1} \subset V_0^{n-1}$.*
- Ξ^{n+1} *is the set of parameters such that $g_n(0)$ falls in $V_1^n \Subset F'_{n+1}$.*

Each region Ξ^n is well defined as a **parapiece** since it represents the return to a single piece of the puzzle. On the other hand, Δ_*^{n+2} is actually the union of several parapieces; nevertheless, it is convenient to regard it as a parapiece to avoid longer descriptions. With this in mind, we are interested in the fact that parapieces of consecutive levels can be described in terms of a single first return map. Because of formulas 5.2, 6.1 and 6.2, we have:

$$\begin{array}{lllll} c \in \Delta^n & \Longleftrightarrow & g_n(0) \in V_0^{n-1} & \Longleftrightarrow & g_{n+1}(0) \in V_0^{n-m} \\ c \in \Delta_*^{n+2} & \Longleftrightarrow & g_n(0) \in F'_{n+1} & \Longleftrightarrow & g_{n+1}(0) \in F'_{n-m+2} \\ c \in \Xi^{n+1} & \Longleftrightarrow & g_n(0) \in V_1^n & \Longleftrightarrow & g_{n+1}(0) \in V_0^{n-1} \\ c \in \Delta^{n+1} & \Longleftrightarrow & g_n(0) \in U^n & \Longleftrightarrow & g_{n+1}(0) \in V_0^n \\ c \in \Delta_*^{n+3} & \Longleftrightarrow & g_n(0) \in F^*_{n+2} & \Longleftrightarrow & g_{n+1}(0) \in F'_{n+2} \end{array} \tag{6.3}$$

Since $F^*_{n+2} \subset U^n \Subset V_1^n \Subset F'_{n+1} \subset V_0^{n-1}$, we have the following parapiece inclusions:

$$\Delta_*^{n+3} \subset \Delta^{n+1} \Subset \Xi^{n+1} \Subset \Delta_*^{n+2} \subset \Delta^n. \tag{6.4}$$

6.2 Shape and paramoduli

In order to prove Theorem 6.1, let us introduce the map $M_n : \Xi^{n-1} \longrightarrow \mathbb{C}$, where Ξ^{n-1} belongs to the paraframe of the fixed parameter c_Q. Remember that $\alpha_{n-2} = \frac{\alpha_Q}{s_{n-2}}$ is the rescaling factor that defines $\widetilde{F}'_n[c_Q] = \alpha_{n-2} \cdot F'_n[c_Q]$.

For $c \in \Xi^{n-1}$, let

$$M_n(c) = \alpha_{n-2} \cdot g_{n-1}(0)[c].$$

Proof of Theorem 6.1: From Table 6.3, when $c \in \Delta_*^{n+1}$, the first return $g_{n-1}(0)[c]$ is in F'_n. We know that for n large, $\widetilde{F}'_n[c]$ is exponentially close to K_Q.

Fix an $\varepsilon > 0$ and find n big enough so that both rescalings $\alpha_{n-2}[c] \cdot F'_n = \widetilde{F}'_n$ and $\alpha_{n-2} \cdot F'_n$ are at most $\frac{\varepsilon}{2}$-close to each other and to K_Q. This means that $M_n(\Delta_*^{n+1})$ is a compact set ε-close to K_Q.

By definition, the parapiece Ξ^{n-1} is the set of parameters c for which $g_{n-2}(0)[c]$ falls on the lateral piece V_1^{n-2}. Since this map is a first return, Proposition 2.14 implies that the correspondence $c \mapsto g_{n-2}(0)[c]$ is univalent in Ξ^{n-1}. Moreover, $V_1^{n-2}[c]$ is at a definite distance away from the central piece $V_0^{n-2}[c]$ for all c, so the image of Ξ^{n-1} under $c \mapsto g_{n-2}(0)[c]$ is uniformly far from 0 and similarly for all further iterations up to the first return $g_{n-1}(0)[c]$. Again, we can use Proposition 2.14 to obtain that M_n is univalent in its entire domain.

Since n is big, the modulus of the annulus $(\operatorname{int} V_0^{n-3} \setminus F'_n)[c]$ is large for every $c \in \Xi^{n-1}$. In particular, since $\widetilde{F}'_n[c]$ has bounded diameter, this implies that the distance between the point $\alpha_{n-2}[c] \cdot g_{n-1}(0)[c] \in \partial\widetilde{V}_0^{n-3}[c]$ and the curve $\partial\widetilde{F}'_n[c]$ is exponentially big for $c \in \partial\Xi^{n-1}$. Let d_n be the minimum of these distances over all c. Then, $M_n(\partial\Xi^{n-1})$ and $M_n(\partial\Delta_*^{n+1})$ are at least a distance $(d_n - \varepsilon) \sim d_n \nearrow \infty$ apart. We can conclude that the modulus of $M_n(\operatorname{int} \Xi^{n-1} \setminus \Delta_*^{n+1})$ is arbitrarily large and so will be the modulus of

$\operatorname{int} \Xi^{n-1} \setminus \Delta_*^{n+1}$.

We have shown that the map M_n is univalent in its domain and the modulus of $\operatorname{int} \Xi^{n-1} \setminus \Delta_*^{n+1}$ is big. Then, by the Koebe distortion Theorem, M_n is asymptotically linear in a neighborhood of Δ_*^{n+1}. Since we got that $M_n(\Delta_*^{n+1})$ is ε-close to K_Q, we can conclude the proof. $\square$

As an immediate consequence of this control over the shape of parapieces, we can compute the rate of growth of principal paramoduli μ_n.

Corollary 6.4. *The annuli of consecutive parapieces in the nest of a (z^2-1)-recurrent map grow linearly at the rate*

$$\lim_{n\to\infty} \frac{\mu_n}{n} = 2\frac{\ln 2}{3}.$$

For any other Q-recurrent map (where Q has critical orbit of period $m \geq 3$) the moduli grow exponentially at the rate

$$\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = \kappa_m,$$

where κ_m is the same constant as in Theorem 5.9, growing to $\frac{3}{2}$ as the period m of Q increases.

Proof: First note that, although U^n is defined as a pull-back of V_0^n, relation 6.2 shows that this piece is just $g_n(V_0^{n+1})$.

Now, when $c \in \Delta^n$, the first return $g_n(0)$ falls in V_0^{n-1}. For $c \in \Delta^{n+1}$, $g_n(0)$ is in U^n. From the previous result, $c \mapsto g_n(0)[c]$ is an almost linear map taking the annulus $\left(\Delta^n \setminus \Delta^{n+1}\right)$ close to $\left(V_0^{n-1} \setminus U^n\right)[c_Q]$. Therefore

$$\operatorname{mod}\left(\Delta^n \setminus \Delta^{n+1}\right) \sim \operatorname{mod}\left(V_0^{n-1} \setminus U^n\right) \sim 2 \operatorname{mod}\left(V_0^n \setminus V_0^{n+1}\right).$$

The result follows from Theorem 5.9. $\square$

6.3 Auto-similarity in the Mandelbrot set

The discovery that parapieces around c_{fib} are similar to the Julia set of -1 revealed one more level of complexity in the structure of M since it relates two seemingly different parameters by the dynamics they generate. In this Section we use our results to take one further step. Having at our disposal an infinite collection of superattracting parameters, we reveal an interesting relation between two arbitrary parameters on ∂M whose combinatorics can be completely dissimilar.

Q-recurrency is not restricted to the Cantor sets described so far. As part of the proof of the next Theorem, we will show that parapieces whose shape approximates K_Q are dense on ∂M. This requires relaxing the definition of Q-recurrency which assumes that the correct combinatorics start from level 0. Instead, we allow critical orbits that behave arbitrarily for several levels before settling in the desired Q-recurrent pattern. This critical behavior is referred to as *generalized Q-recurrency* and its density in ∂M will follow from the description. After this we may conclude the assertion of Theorem 6.5, which can be interpreted as saying that the geometry of most Julia sets is replicated near arbitrary locations of the boundary of M.

Theorem 6.5. *Let $c_1, c_2 \in \partial M$ be two parameters such that f_{c_2} has no indifferent periodic orbits that are rational or linearizable. Then there exists a sequence of parapieces $\{\Upsilon_1, \Upsilon_2, \ldots\}$ (most likely not nested) converging to c_1 as compact sets, but such that $\Upsilon_n \longrightarrow K_{c_2}$ in shape.*

Proof: It is not difficult to obtain the result of Theorem 5.5 in more generality. In fact, inside any ball $B_\varepsilon(c)$ with $c \in \partial M$, we can find a system of parameters for which the first return maps converge (after scaling) to a given superattracting map Q.

To see this, simply consider a tuned copy of M contained in B_ε. All parameters in this copy M', are renormalizable by the same combinatorics. In particular, there will be parameters whose renormalization is hybrid equivalent to a Q-recurrent map. For these parameters a high level of the frame will contain a substructure whose graph is isomorphic to $\Gamma_0(Q)$ and we can start the same construction as in the proof of Theorem 5.5 to produce frame-like structures whose graphs are isomorphic to $\Gamma_n(Q)$. Since the combinatorics is prescribed by a polynomial, there can be no obstructions just as in the original case. Then, the rescaled first return maps will converge to Q as before. Moreover, we can translate the shape property to the parameter plane.

Note that this argument is equivalent to prescribing the itineraries of nest pieces arbitrarily on the initial levels and then proving that they can be admissibly extended on subsequent levels to match the pattern given in Formula (4.1).

Now consider the filled Julia set of f_{c_2}. We know from [D1] that there is a sequence $\{Q_1, Q_2, \ldots\}$ of superattracting polynomials in a prime hyperbolic component of M such that K_{c_2} can be arbitrarily approximated by filled Julia sets: $K_{Q_n} \longrightarrow K_{c_2}$. To fix ideas, let us choose subindices so that the Hausdorff distance is $\text{dist}_H(K_{Q_n}, K_{c_2}) < \frac{1}{2n}$.

For any n, consider the ball $B_{\frac{1}{n}}(c_1)$ and locate a generalized Q_n-recurrent parameter s_n. By going to a deep enough level, we can find some parapiece $\Upsilon_n \subset B_{\frac{1}{n}}$ around s_n whose shape is $(\frac{1}{2n})$-close to K_{Q_n}; that is, so that there is a rescaling $\widetilde{\Upsilon}_n$ of Υ_n for which $\text{dist}_H(\widetilde{\Upsilon}_n, K_{Q_n}) < \frac{1}{2n}$.

Since $\Upsilon_n \subset B_{\frac{1}{n}}$, the sequence $\{\Upsilon_n\}$ consists of parapieces that get arbitrarily small and converge to c_1, while at the same time $\text{dist}_H(\widetilde{\Upsilon}_n, K_{c_2}) < \frac{1}{n}$, so $\Upsilon_n \longrightarrow K_{c_2}$ in shape. □

Appendix

The theorems of previous chapters rely on several notions and results of complex analysis. In this Appendix we will describe the necessary ideas on which the text relies. For references and proofs of these results, the reader can consult [A], [DH3] and [LV].

A Carathéodory topology

A sequence of pointed disks $\{(U_n, x_n)\}$ is said to converge to the pointed disk (U, x) in the Carathéodory topology if:

a. $x_n \longrightarrow x$.

b. For every compact $K \subset U$ there is an N such that for $n > N$ $K \subset U_n$.

c. If $V \ni x$ is an open connected region and $V \subset U_n$ for infinitely many n, then $V \subset U$.

The interpretation of convergence in this topology is as follows. Consider the complements $\hat{\mathbb{C}} \setminus U_n$ which are compact sets converging to X in the Hausdorff topology. To satisfy the above conditions, $x \notin X$ and U is the component of $\hat{\mathbb{C}} \setminus X$ that contains x.

We can describe a similar convergence for a sequence of annuli $\{A_n\}$. In this case, we require that $\hat{\mathbb{C}} \setminus A_n \longrightarrow Y$ in the Hausdorff sense and that the core geodesics γ_n of the annuli A_n converge to a non-degenerate loop $\gamma \subset Y$. The Carathéodory limit will be the doubly connected component of $\hat{\mathbb{C}} \setminus Y$ containing γ.

B Modulus and capacity as conformal invariants

Let $R \subset \mathbb{C}$ be a doubly connected region in the plane. Consider the family Γ of curves $\gamma \subset R$ whose endpoints are on the boundary of R. Given a conformal metric ρ on R, we can define the length of Γ as $L_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma \rho|dz|$ and the area of R as $A_\rho(R) = \int\int_A \rho^2 dxdy$.

Definition B.1. *The modulus of R is*

$$\operatorname{mod}(R) = \sup_\rho \frac{\left(L_\rho(\Gamma)\right)^2}{A_\rho(R)}$$

where the supremum is taken over all conformal metrics ρ with non-degenerate area: $A_\rho(R) \neq 0, \infty$.

It can be shown that the modulus is a conformal invariant. As a consequence, we can give an alternative definition as follows. Consider the Riemann map $\varphi : R \longrightarrow S$ where S is the round annulus $S = \{1 < z < r\}$. Then $\operatorname{mod(R)} = 2\pi \ln r$.

Another conformal invariant, this time of topological disks, is the *capacity* or *conformal radius.*

Definition B.2. *Let $U \in \mathbb{C}$ be a simply connected domain, $z \in U$ and $\varphi : (\mathbb{D}, 0) \longrightarrow (U, z)$ the Riemann map from the unit disk to U that satisfies $\varphi'(0) > 0$. The* ***capacity*** *of U with respect to z is*

$$\operatorname{cap}_z(U) = \ln \varphi'(0).$$

Of interest to us, will be the following property of capacities and moduli.

Theorem B.3. *Both capacity and modulus are quantities that vary continuously with respect to the Caratéodory topology.*

C Grötzsch inequality

The following result (and its quantitative version) is essential to estimate the modulus of an annulus that is split in subannuli.

Theorem C.1. Extended Grötzsch Inequality: *Let $K \subset \mathbb{C}$ be a simply connected compact set and denote $\operatorname{int}_0 K$ the component of its interior that contains 0.*

a. Consider two topological disks $0 \in U \subset \operatorname{int}_0 K \subset V$*. Then*

$$\operatorname{mod}(V \setminus U) \geq \operatorname{mod}(K \setminus U) + \operatorname{mod}(V \setminus K).$$

b. Let $\{U_n\}$ *and* $\{V_n\}$ *be two sequences of nested topological disks satisfying*

- $0 \in U_n \subset \operatorname{int}_0 K$ *and* $\operatorname{diam} U_n \searrow 0$
- $K \subset V_n$ *and* $\operatorname{dist}(K, \partial V_n) \nearrow \infty$.

Then the deficit in the Grötzsch inequality tends to

$$\lim_{n\to\infty}\Big(\operatorname{mod}(V_n \setminus U_n) - \operatorname{mod}(K \setminus U_n) - \operatorname{mod}(V_n \setminus K)\Big) =$$

$$|\operatorname{cap}_0(\operatorname{int}_0 K)| + |\operatorname{cap}_\infty(K)|.$$

An important observation is the fact that equality in (a) is achieved if and only if $\partial K \subset V \setminus U$ maps to a centered circle under the Riemann map.

D Koebe distortion Theorem

Definition D.1. *Given an analytic univalent map* φ *between regions* U *and* V*, the* ***distortion*** *of* φ_0 *is defined as:*

$$\operatorname{Dist}(\varphi_0) = \sup_{x,y\in U} \frac{\varphi_0'(x)}{\varphi_0'(y)}.$$

Koebe's Theorem provides great control of the distortion when there is enough space between U and V.

Theorem D.2. *Let* U *and* V *be two topological disks with* $U \setminus V$*. Then there is a constant* C *such that for any univalent map* $\varphi(U) = V$*,*

$$\operatorname{Dist}(\varphi) < C.$$

Moreover, $C = 1 + O(e^{-\operatorname{mod}(V\setminus U)})$ *as the modulus goes to* ∞*.*

E Teichmüller space

The Teichmüller space of a Riemann surface carries a great deal of structural information. Here we focus in the case that the surface S is the complex plane punctured at a finite set $\mathcal{O}$. Then, the Teichmüller space $\mathcal{T}_S$ can be described as a quotient of the space of quasiconformal deformations of S (i.e. the family of maps $\{h : S \longrightarrow \mathbb{C} \,|\, h \text{ is a qc homeomorphism}\}$), where two deformations h_1 and h_2 are identified if and only if there is a conformal change of coordinates $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\varphi \circ h_1$ is isotopic to h_2 relative to the puncture set $h_2(\mathcal{O})$.

Note: The coordinate changes φ are affine maps, so the deformation of $\mathcal{O}$ within a class is determined up to translation and complex scaling. Therefore, we can normalize a deformation h by requiring that h fixes two distinguished points in $\mathcal{O}$. These could be, for instance, the critical point and critical value of Q in the case that $\mathcal{O}$ is the postcritical set of a hyperbolic map Q.

It is fundamental to consider an alternate description of $\mathcal{T}_S$ in terms of Beltrami differentials. Fix two almost complex structures on S determined by their Beltrami coefficients $\mu\frac{d\bar{z}}{dz}$ and $\nu\frac{d\bar{z}}{dz}$. Assume that they are related by $\nu = \overline{h}^* \mu$, where $\overline{h} : S \longrightarrow S$ is a quasiconformal self homeomorphism of S which is homotopic to id relative to $\mathcal{O}$. Then, the straightening maps h_μ and h_ν are two quasiconformal deformations of S in the same equivalence class in $\mathcal{T}_S$.

Conversely, we can associate to a deformation h the almost complex structure $h^*\sigma = \frac{\overline{\partial} h}{\partial h}\frac{d\bar{z}}{dz}$ where σ is the standard structure. It is easy to verify that this correspondence lifts to the equivalence classes where it induces a bijection.

F Holomorphic motions of puzzle pieces

Definition F.1. *Let $X_* \subset \overline{\mathbb{C}}$ be an arbitrary set and $\Delta \subset \mathbb{C}$ a simply connected domain with $*$ as a base point. A **holomorphic motion** of X_* over Δ is a family of injections $h_\lambda : X_* \longrightarrow \overline{\mathbb{C}}$ $(\lambda \in \Delta)$ such that for each fixed $x \in X_*$, $h_\lambda(x)$ is a holomorphic function of λ and $h_* = \mathrm{id}$.*

For every $\lambda \in \Delta$ we write X_λ to denote the set $h_\lambda(X_*)$.

Holomorphic motions are extremely versatile because of their regularity properties. The motion can always be extended beyond X_* and is transversally quasi-conformal. This is the content of the *λ-lemma.*

Theorem F.2. [Sl], [MSS] (**the λ-lemma**) *For every holomorphic motion $h_\lambda : X_* \longrightarrow \overline{\mathbb{C}}$, there is an extension to a holomorphic motion $H_\lambda : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$.*

The extension to the closure $\overline{h}_\lambda : \overline{X}_* \longrightarrow \overline{\mathbb{C}}$ is unique. Moreover, there is a function $K(r)$ approaching 1 as $r \to 0$ such that the maps h_λ are $K(r)$-quasi-conformal, where $r = \mathrm{d}_\Delta(*, \lambda)$ is the hyperbolic distance between $*$ and λ in Δ.

We are interested in the case when the holomorphic motion is defined over a parapiece Δ of M. In agreement with the notation used in the main body of this work, we use c instead of λ to denote parameters in Δ. When an object is defined for any $c \in \Delta$, we express its dependence on the parameter by writing OBJ$[c]$.

As mentioned in Corollary 2.12, Δ can be interpreted as the set of parameters for which *a certain combinatorial behavior holds, up to a return $g(0)$ of the critical orbit to some central piece V*. In particular, this description provides a natural base point for Δ. Namely, the superattracting parameter c_0 for which $g_{c_0}(0) = 0$. The little M-copy associated to Δ can be defined as the set of parameters for which the iterates $\{g_c(0), g_c^{\circ 2}(0), \ldots\}$ remain in $V[c]$ (refer to Section 3.6).

The dynamics in the region N_c (defined at the beginning of Section 2.2) is always conjugate to $z \mapsto z^2$, so varying the parameter $c \in \mathbb{C}$ provides a holomorphic motion of any specified ray or equipotential. When c is restricted to Δ, the combinatorics require that some rays land together, enclosing the boundary of $V[c]$. Since the intersection $\partial V \cap K$ is a collection of preimages of the fixed point α and these vary holomorphically with c, there is a natural holomorphic motion of $\partial V[c_0]$ over Δ. This can be extended to a holomorphic motion $h_c : V[c_0] \longrightarrow V[c]$.

The holomorphic motion of a puzzle piece can be viewed as a complex 1-dimensional foliation of the bi-disk

$$\mathbb{V} = \bigcup_{c \in \Delta} V[c] \in \mathbb{C}^2$$

whose leaves are the graphs of the functions $c \mapsto h_c(p)$ for every $p \in V[c_0]$. Under this interpretation we will write $\{c \mapsto V[c] \mid c \in \Delta\}$ to refer to the motion.

Definition F.3. *A correspondence $c \mapsto \phi(c)$ such that $\phi(c) \in V[c]$ determines a section $\phi : \Delta \longrightarrow \mathbb{V}$ of the holomorphic motion h. It is said to be a* ***proper holomorphic section*** *if it maps $\partial\Delta$ into the torus $\delta\mathbb{V} = \bigcup_{c \in \partial\Delta} \partial V[c]$.*

We say that a proper section $\{c \mapsto \phi(c)\}$ has ***winding number*** *n if the curve $\phi(\partial\Delta)$ has winding number n with respect to the vertical generator of the 1-dimensional homology of $\delta\mathbb{V}$.*

In the case $\phi(c) = g_c(0)$, this return map determines a proper section since $g_c(0) \in V[c]$ for all c and $c \in \partial\Delta \Rightarrow g_c(0) \in \partial V[c]$. Each return map $g_c : g_c^{-1}(V) \longrightarrow V$ is a quadratic-like map and the associated map

$$g_c : \mathbb{U} \longrightarrow \mathbb{V},$$

where $\mathbb{U} = \bigcup g_c^{-1}(V[c])$, is called a **DH quadratic-like family**. We can interpret intuitively the fact that a family has winding number n as saying that, as c goes once along $\partial\Delta$, the point $g_c(0)$ goes n times around the (moving) boundary of the piece $V[c]$.

An immediate consequence of extending the holomorphic motion of $\partial V[c_0]$, is the fact that $\{g_c \mid c \in \Delta\}$ is a full family; that is, there is a homeomorphism $\mathrm{Hyb} : \widetilde{M} \longrightarrow \Delta$ from a neighborhood $\widetilde{M}$ of M to Δ with the following property: For every parameter $c' \in \widetilde{M}$, $g_{\mathrm{Hyb}(c')}$ is hybrid equivalent[1] to $z \mapsto z^2 + c'$. This of course, justifies the existence of the small M-copy associated to Δ.

[1]see Section 3.6

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