# Surjectivity of Gaussian Maps for Line Bundles of Large Degree on Curves

by

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## Introduction.

Let $C$ be a smooth complex projective curve of genus $g$, let $L$ and $N$ be line bundles on $C$, and denote by $R(L,N)$ the space of relations between $L$ and $N$:

$$R(L, N) = \ker\{ H^0(C, L) \otimes H^0(C, N) \longrightarrow H^0(C, N \otimes L) \}.$$

Then, writing $\Omega$ for the canonical bundle, one can define a homomorphism

$$\gamma_{L,N} : R(L, N) \longrightarrow H^0(C, \Omega \otimes L \otimes N)$$

by making sense of

$$s \otimes t \longmapsto s\,dt - t\,ds..$$

(When $L = N$, $\gamma_{L,L}$ vanishes on symmetric tensors, and so becomes simply a map $\gamma_L : \Lambda^2 H^0(L) \longrightarrow H^0(L^2 \otimes \Omega)$.) These so-called Gaussian or Wahl maps have attracted considerable attention ever since Wahl [W2] made the surprising observation that if $C$ lies on a K3 surface, then $\gamma_\Omega$ cannot be surjective. It seems likely that these maps will arise in other natural contexts as well (c.f. [Griff, Chapt 9]). It is therefore of some interest to obtain surjectivity statements for the $\gamma_{L,N}$ analogous to classical theorems of Castelnuovo et. al. (c.f. [M] or [G]) concerning the maps $H^0(L) \otimes H^0(N) \longrightarrow H^0(L \otimes N)$, for which the best possible uniform results are known.

A number of theorems in this direction have already appeared. First, Ciliberto, Harris and Miranda [CHM] gave in passing a very simple argument to show that if $\deg(L) \geq 4g + 6$, then $\gamma_L$ is surjective. Wahl [W3] proved that $\gamma_{L,N}$ is surjective provided that $\deg(L) \geq 5g + 1$ and $\deg(N) \geq 2g + 2$. He also showed that if $\deg(L) \geq 5g + 2$ then $\gamma_{\Omega,L}$ is surjective. The latter result is particularly interesting because it has a deformation-

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theoretic interpretation (c.f. [Wi]). In fact, if L is normally generated, then the surjectivity of $\gamma_{\Omega,L}$ implies that in the linearly normal embedding $C \subset \mathbb{P}^r = \mathbb{P}H^0(L)$ defined by L, C is not the hyperplane section of any variety $Y \subset \mathbb{P}^{r+1}$ other than a cone over C. (And so for instance C is not a very ample divisor with normal bundle L on any regular surface.) Further theorems along these lines appear in [T].

The purpose of this note is to record some strengthenings of these results. First we show that a small adaptation of the argument in [CHM] -- along lines suggested in a different context by Wahl -- leads to the optimal bound in the non-special case:

Theorem 1. Let L and N be bundles on C of degrees d and e respectively. Assume that $d, e \geq 2g + 2$.

(i). If $d + e \geq 6g + 3$, then $\gamma_{L,N}$ is surjective.

(ii). If C is non-hyperelliptic, then $\gamma_{L,N}$ is surjective provided that $d + e \geq 6g + 2$.

(iii). If C is hyperelliptic, then given L of degree $2g + 2 \leq d \leq 4g$ there exists a line bundle N on C of degree $6g + 2 - d$ for which $\gamma_{L,N}$ fails to be surjective.

So for instance $\gamma_L$ is surjective as soon as $\deg(L) \geq 3g + 2$, and this is best-possible if (and only if) C is hyperelliptic.

Our second theorem shows that for the map $\gamma_{\Omega,L}$ one can bring into play the intrinsic geometry of C as measured by its Clifford index. The theme is that as C becomes increasingly general, one obtains a progressively stronger surjectivity statement:

Theorem 2 Assume that C is neither hyperelliptic, trigonal, nor a plane quintic (i.e. assume that $\mathrm{Cliff}(C) \geq 2$). If $\deg(L) \geq 4g + 1 - 2\cdot\mathrm{Cliff}(C)$, then $\gamma_{\Omega,L}$ is surjective. If moreover $\mathrm{Cliff}(C) \geq 3$ (i.e. if in addition that C is neither quadrigonal nor a smooth plane sextic), then $\gamma_{\Omega,L}$ is surjective as soon as $\deg(L) \geq 4g + 1 - 3\cdot\mathrm{Cliff}(C)$.

We refer for instance to [GL] or [L, §2] for the definition and basic properties of the Clifford index Cliff(C) of C. Similar but somewhat weaker results were obtained by Tendian [T].

It would be interesting to know whether the elementary methods of this paper can be adapted to say anything about the Wahl map $\gamma_\Omega$ for the canonical bundle. It is proved by degenerational techniques in [CHM] that on a generic curve C, $\gamma_\Omega$ is surjective for $g = 10$ or $g \geq 12$. A conceptual approach to this theorem appears in Voisin's beautiful paper [V], where it is shown that the failure of $\gamma_\Omega$ to be surjective is "explained" by the presence of many non-projectively normal line bundles on C. Other results concerning $\gamma_\Omega$ appear in [CM1], [CM2], and [Mir].

The proof of Theorem 1 occupies §1. We also give a generalization to certain higher-order Gaussian maps. §2 is devoted to the proof of Theorem 2. We explain there how the geometric consequence of the surjectivity of $\gamma_{\Omega,L}$ mentioned above follows from an interesting theorem of L'voveskii [L'v] and F. L. Zak.

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## §1. The Gaussian Map for non-special line bundles.

In this section we adapt an argument from [CHM] to prove Theorem 1. As above, C is a smooth complex projective curve of genus g. We start by defining the Gaussian maps $\gamma_{L,N}$ more formally. To this end, denote by $p\,,\ q : C \times C \longrightarrow C$ the two projections, and let $\Delta \subset C \times C$ be the diagonal. Given a coherent sheaf F on C, we set:

$$F_1 = p^* F \quad \text{and} \quad F_2 = q^* F,$$

so that $F_1$ and $F_2$ are sheaves on $C \times C$. Suppose now that L and N are line bundles on C, and consider the exact sequence:

$$(1.1) \qquad 0 \longrightarrow L_1 \otimes N_2(-2\Delta) \longrightarrow L_1 \otimes N_2(-\Delta) \longrightarrow L_1 \otimes N_2 \otimes \mathcal{O}_\Delta(-\Delta) \longrightarrow 0.$$

Then as is well known, the Wahl map

$$\gamma_{L,N} : R(L\,,N) = H^0(L_1 \otimes N_2(-\Delta)) \longrightarrow H^0(L_1 \otimes N_2 \otimes \mathcal{O}_\Delta(-\Delta)) = H^0(\Omega \otimes L \otimes N)$$

is simply the homomorphism on global sections deduced from the restriction map in (1.1). In particular, to show that $\gamma_{L,N}$ is surjective, it is enough to prove that $H^1(L_1 \otimes N_2(-2\Delta)) = 0$. The idea of [CHM] is in effect to study this group geometrically.

To this end, suppose that A is a base-point free pencil on C defining a branched covering $\Phi_A : C \longrightarrow \mathbb{P}^1$. If $s, t \in H^0(A)$ is a basis, then

$$p^*s \otimes q^*t - p^*t \otimes q^*s \in H^0(C \times C\,, A_1 \otimes A_2)$$

vanishes on $\Delta$ and hence canonically defines a section $s_A \in H^0(C \times C\,, A_1 \otimes A_2(-\Delta))$; denote the divisor of this section by $\Gamma_A \subset C \times C$. More geometrically we may describe $\Gamma_A$ as the curve residual to the diagonal $\Delta$ in the fibre product $C \times_{\mathbb{P}^1} C$.

Example 1.2. If $C$ is hyperelliptic, and $A$ is the hyperelliptic pencil on $C$, then $\Gamma_A = \mathrm{graph}(i) \subset C \times C$, where $i : C \longrightarrow C$ is the hyperelliptic involution.

It is quite standard to analyze the geometry of $\Gamma_A$:

Lemma 1.3. Assume that $\Phi = \Phi_A : C \longrightarrow \mathbb{P}^1$ is a simple covering, i.e. assume that for any branch point $b \in \mathbb{P}^1$, its preimage $\Phi^{-1}\{b\}$ contains only one ramification point $p$, at which the local degree of $\Phi$ is 2: $e_{\Phi(p)} = 2$. Then $\Gamma_A$ is smooth and irreducible. If moreover $\deg(A) = n$, then the genus of $\Gamma_A$ is given by

$$g(\Gamma_A) = (n-1)(n + 2g - 3) + (1-g),$$

where as above $g$ is the genus of $C$.

Proof. We argue to begin with that $\Gamma_A$ is irreducible. To this end, let $G \subset S_n$ be the monodromy group of the covering $\Phi$. One may view $\Gamma_A$ as the closure of the set of all pairs $(x,y) \in C \times C$ with $x \neq y$ such that $\Phi(x) = \Phi(y)$, and hence it is certainly enough to show that $G = S_n$ is the whole symmetric group. Now $G$ is transitive since $C$ is irreducible, and it is generated by simple transpositions thanks to the simplicity of $\Phi$. But as F. Cukierman pointed out to us, the only transitive subgroup of $S_n$ generated by simple transpositions is $S_n$ itself, and therefore $\Gamma_A$ is irreducible. A computation in local coordinates shows that it is smooth. Finally, recalling that $\Gamma_A \in |A_1 \otimes A_2(-\Delta)|$, one computes $g(\Gamma_A)$ using the adjunction formula. ∎

In order to apply the Lemma, we will need a stock of pencils defining simple coverings:

Lemma 1.4. (i). Let $A \in W^1_{g+1}(C)$ be a general pencil of degree $g+1$. Then $\Phi_A$ defines a simple covering, and consequently $\Gamma_A$ is smooth and irreducible, of genus $3g^2 - 3g + 1$.

(ii). Assume that $C$ is non-hyperelliptic, of genus $g \geq 3$, and let $A \in W^1_g(C)$ be a general pencil of degree $g$. Then $A$ is base-point free, and again $\Phi_A$ defines a simple covering. The corresponding curve $\Gamma_A$ has genus $3g^2 - 7g + 4$.

Proof. The only point which isn't well-known is that if $C$ is non-hyperelliptic, then a general pencil $A$ of degree $g$ defines a simple covering. To check this, one needs to show that if $A \in W^1_g(C)$ is sufficiently general, then:

(a). $\forall\, x \in C$, $h^0(A(-3x)) = 0$ (no triple ramification); and

(b). $\forall\, x, y \in C$, $h^0(A(-2x-2y)) = 0$ (no two ramification points in one fibre).

But these follow from elementary dimension counts. For example, suppose that (a) fails. Then $W^1_g(C)$ is contained in the image of the map $t : \mathrm{Sym}^{g-3}(C) \times C \longrightarrow \mathrm{Pic}^g(C)$ given by $t(D, x) = D + 3x$. In this case, $\mathrm{Im}(t) = W^1_g(C)$ since both have the same dimension. Dually, this means that for all $x, x_1, \ldots, x_{g-3} \in C$, $h^0(\Omega(-x_1-\ldots-x_{g-3}-3x)) \geq 1$. But this is absurd, since for a general point $x \in X$, $h^0(\Omega(-3x)) = g-3$. The proof of (b) is similar. ∎

We now give the

Proof of Theorem 1. Let $A$ be any base-point free pencil on $C$. Recalling once again that $\Gamma_A \in |A_1 \otimes A_2(-\Delta)|$, observe that multiplication by $\Gamma = \Gamma_A$ gives rise on $C \times C$ to an exact sequence:

(1.5)

$$0 \longrightarrow (L \otimes A^*)_1 \otimes (N \otimes A^*)_2 \otimes \mathcal{O}(-\Delta) \longrightarrow L_1 \otimes N_2(-2\Delta) \longrightarrow \mathcal{O}_\Gamma(L_1 \otimes N_2 - 2\Delta)) \longrightarrow 0$$

The strategy is to use this sequence to study $H^1(L_1 \otimes N_2(-2\Delta))$. We consider separately the three statements of the Theorem.

(i). Take $A \in W^1_{g+1}(C)$ to be a general pencil. We claim that then the outer terms in (1.5) have vanishing $H^1$. This will show that $H^1(L_1 \otimes N_2(-2\Delta)) = 0$, and by the remarks at the beginning of the section, the surjectivity of $\gamma_{L,N}$ follows.

For the term on the left, it is enough to prove that

$$(*) \qquad H^0(C, L \otimes A^*) \otimes H^0(C, N \otimes A^*) \longrightarrow\!\!\!\!\!\rightarrow H^0(C, L \otimes N \otimes A^{-2}).$$

In fact, since $d, e \geq 2g+2$, the bundles $L \otimes A^*$, $N \otimes A^*$ both have degree $\geq g+1$. Hence by choosing $A$ generally, we may assume that they are base-point free and non-special. Moreover $\deg(L \otimes A^*) + \deg(N \otimes A^*) = d + e - 2g - 2 \geq 4g + 1$ by hypothesis. Then the required surjectivity (*) is well-known (e.g. by [EKS] or [G, (4.e.4)]).

It remains to verify that $H^1(\Gamma, \mathcal{O}_\Gamma(L_1 \otimes N_2(-2\Delta))) = 0$. But

$$\begin{aligned} \deg\{\mathcal{O}_\Gamma(L_1 \otimes N_2(-2\Delta))\} &= g(d + e - 8) + 4 \\ &\geq g(6g - 5) + 4 \\ &\geq 2 \cdot g(\Gamma_A) - 1, \end{aligned}$$

and the required vanishing follows.

(ii). The argument is similar to the one just given except that one works with a general $A \in W^1_g(C)$. We leave the details to the reader.

(iii). Let $A$ be the hyperelliptic pencil on $C$, and denote by $R$ the ramification divisor of the hyperelliptic covering $\Phi = \Phi_A : C \longrightarrow \mathbb{P}^1$, so that $\deg(R) = 2g + 2$. Fix $L$ of degree $2g + 2 \le d \le 4g$, and put $N = \Omega(2R) \otimes i^*L^*$, where $i : C \longrightarrow C$ is the hyperelliptic involution. We assert that $\gamma_{L,N}$ is not surjective. In fact, certainly $H^0(L\otimes A^*) \otimes H^0(N\otimes A^*) \longrightarrow\!\!\!> H^0(L\otimes N\otimes A^{-2})$, and therefore

$$H^1(C\times C\,,\ (L\otimes A^*)_1 \otimes (N\otimes A^*)_2 \otimes \mathcal{O}(-\Delta)) = 0\,.$$

So by (1.5) we are reduced to showing that $H^1(\Gamma\,, \mathcal{O}_\Gamma(L_1\otimes N_2(-2\Delta))\,) \neq 0$, where as usual $\Gamma = \Gamma_A$ is the divisor associated to $A$. But if we define $f : C \longrightarrow \Gamma$ to be the isomorphism $x \longmapsto (\,x\,, i(x)\,)$, then $f^*\,\mathcal{O}_\Gamma(L_1\otimes N_2(-2\Delta)) = L \otimes i^*N \otimes \mathcal{O}_C(-2R) = \Omega$, and we are done. ∎

<u>Remark</u>. Keeping the notation of part (iii) of the previous proof, note that if $L$ is a line bundle of degree $3g + 1$ on a hyperelliptic curve $C$, then $L \otimes i^*L = \Omega(2R)$. Hence it follows from the proof that $C$ is hyperelliptic $\Leftrightarrow$ $\gamma_L$ fails to be surjective for <u>some</u> line bundle of degree $3g + 1$ $\Leftrightarrow$ $\gamma_L$ fails to be surjective for <u>every</u> line bundle $L$ of degree $3g + 1$.

Finally, we indicate the analogue of Theorem 1 for higher order Gaussian maps. Specifically, set $\gamma^1_{L,N} = \gamma_{L,N}$, and fix an integer $k \ge 2$. Then an in [Griff, Chapt 9] there are naturally defined homomorphisms

$$\gamma^k_{L,N} :\ \ker \gamma^{k-1}_{L,N} \longrightarrow H^0(L\otimes N\otimes \Omega^{\otimes k})\,,$$

arising for example as the map induced on global sections by the restriction in the sequence

$$(1.6)\qquad 0 \longrightarrow L_1 \otimes N_2\,(-(k+1)\Delta) \longrightarrow L_1 \otimes N_2\,(-k\Delta) \longrightarrow L_1 \otimes N_2 \otimes \mathcal{O}_\Delta(-k\Delta) \longrightarrow 0.$$

Arguing as in the proof of Theorem 1, one obtains by induction the following

<u>Theorem 1.7. Let</u> $L$ <u>and</u> $N$ <u>be bundles on</u> $C$ <u>of degrees</u> $d$ <u>and</u> $e$ <u>respectively. Assume that</u> $d\,, e \ge (k+1)(g+1)$.

(i). <u>If</u> $d + e \ge (k+1)(2g+2) + 2g-1$, <u>then</u> $\gamma^k_{L,N}$ <u>is surjective.</u>

(ii). <u>If</u> $C$ <u>is non-hyperelliptic, then</u> $\gamma^k_{L,N}$ <u>is surjective provided that</u> $d + e \ge (k+1)(2g+2) + 2g-2$.

(iii). <u>If</u> $C$ <u>is hyperelliptic, suppose that</u> $L$ <u>and</u> $N$ <u>satisfy the relation</u> $L \otimes i^*N = \Omega((k+1)R)$, <u>where</u> $i : C \longrightarrow C$ <u>is the hyperelliptic involution, and</u> $R$ <u>the divisor of</u>

branch points of the hyperelliptic covering $C \longrightarrow \mathbb{P}^1$. Then $d + e = (k+1)(2g+2) + 2g-2$, and $\gamma^k_{L,N}$ fails to be surjective.

## §2. The Gaussian map $\gamma_{\Omega,L}$

It seems most natural to prove Theorem 2 using some elementary vector bundle techniques. We start by defining the bundles that come into play, which essentially arise as direct images of the sheaves occuring in §1. As above, $C$ is a smooth complex projective curve of genus $g$. To avoid problems in the definition of the Clifford index, we assume throughout this section that $g \geq 4$; we leave it to the interested reader to make the necessary adjustments to handle low genera.

Given a very ample line bundle $L$ on $C$, set

$$M_L = p_*\{q^*L \otimes \mathcal{O}_{C\times C}(-\Delta)\}$$

and

$$R_L = p_*\{q^*L \otimes \mathcal{O}_{C\times C}(-2\Delta)\},$$

where as in §1 $p, q : C\times C \longrightarrow C$ are the projections. Denoting by $P^1(L)$ the rank two locally free sheaf of first-order principal parts of $L$, these vector bundles are tied together by three exact sequences:

$$(2.1) \qquad 0 \longrightarrow M_L \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow L \longrightarrow 0$$

$$(2.2) \qquad 0 \longrightarrow R_L \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow P^1(L) \longrightarrow 0$$

and

$$(2.3) \qquad 0 \longrightarrow R_L \longrightarrow M_L \longrightarrow \Omega \otimes L \longrightarrow 0$$

The right-hand maps in (2.1) and (2.2) are the canonical evaluation homomorphisms, and (2.3), which is the direct image of (1.1) under $p$, may alternatively be deduced from (2.1), (2.2) and the standard sequence relating $P^1(L)$ to $L$ and $\Omega \otimes L$. Remark that (2.1) is a twist of the pull-back of the Euler sequence on $\mathbb{P}H^0(L)$.

Note that the Gaussian $\gamma_{L,N}$ is just the homomorphism $H^0(M_L \otimes N) \longrightarrow H^0(\Omega \otimes N \otimes L)$ on global sections obtained by twisting (2.3) by $N$. Therefore, much as in §1:

Lemma 2.4. If $H^1(C, R_L \otimes N) = 0$ then $\gamma_{L,N}$ is surjective ■

Observe also (from (2.1) and (2.2)) that

(2.5) $R_L = \mathcal{N}^* \otimes L$, where $\mathcal{N}^* = \mathcal{N}^*_{C/\mathbb{P}H^0(L)}$ is the conormal bundle to $C$ in $\mathbb{P}H^0(L)$ under the embedding defined by the complete linear series associated to $L$.

Hence:

Lemma 2.6. If $C$ is scheme-theoretically cut out by quadrics in the embedding $C \subset \mathbb{P}H^0(L)$ defined by $L$, then $R_L \otimes L$ is generated by its global sections. ∎

We will need one further property of the bundles $R_L$. Namely, suppose that $x_1, \ldots, x_m \in C$ are points such that $L(-\Sigma x_i)$ is still very ample (or at least immersive). Then, setting $D = \Sigma x_i$, there is an exact sequence:

(2.7) $$0 \longrightarrow R_{L(-D)} \longrightarrow R_L \longrightarrow \oplus\, \mathcal{O}_C(-2x_i) \longrightarrow 0\,.$$

This may be proved for instance as in [L, §1.4] and [E]. We remark that this sequence is the basis, from the vector bundle point of view, of Wahl's "immersive pencil trick" in [W3].

Proof of Theorem 2. Let $e = \mathrm{Cliff}(C)$. We will assume that $e \geq 3$, the case $e = 2$ being similar but simpler. Choose $(e-2)$ points $x_1, \ldots, x_{e-2} \in C$, and put $D = D_x = x_1 + \ldots + x_{e-2}$. We wish to apply to $\Omega(-D)$ a result of Green and the third author [L,(2.4.2)] to the effect that if $A$ is a very ample line bundle on $C$, with $\deg(A) \geq 2g + 2 - 2\cdot h^1(A) - \mathrm{Cliff}(C)$, then $C$ is scheme-theoretically cut out by quadrics in the embedding $C \subset \mathbb{P}H^0(A)$ defined by $A$ provided that $C$ has no tri-secant lines in $\mathbb{P}H^0(A)$. To this end, we claim:

> The line bundle $\Omega(-D)$ is very ample, and $C$ has no tri-secant lines in the embedding $C \subset H^0(\Omega(-D))$ defined by $\Omega(-D)$.

In fact, suppose that $y_1, y_2 \in C$ fail to impose independent conditions on $\Omega(-D)$. Then the degree $e$ divisor $D + y_1 + y_2$ spans a $\mathbb{P}^{e-2}$ in canonical space $\mathbb{P}^{g-1}$. Therefore $D + y_1 + y_2$ moves in a pencil, but this would force $\mathrm{Cliff}(C) \leq e-2$. A similar argument proves the second assertion.

We conclude from the result just quoted that $C \subset \mathbb{P}H^0(\Omega(-D))$ is cut out by quadrics. It then follows from Lemma 2.6 that $R_{\Omega(-D)}$ sits in an exact sequence

(2.8) $$\oplus\, \Omega^*(D) \longrightarrow R_{\Omega(-D)} \longrightarrow 0.$$

We claim next:

(2.9) If $\deg(L) = d \geq 4g + 1 - 3e$, then for a general choice of the $x_i$,

$$H^1(L\otimes\Omega^*(D)) = 0.$$

In fact, suppose to the contrary that $H^0(\Omega^2\otimes L^*(-D)) \neq 0$. Then in the first place

$$(*) \qquad \deg(\Omega^2\otimes L^*(-D)) \leq g-3.$$

[Proof: Otherwise $4g-4-d-e+2 \geq g-2$, which leads to the inequality $d \leq 3g-e$; but since in any event $2e \leq g-1$, this contradicts our hypothesis on $d$.] It follows from (*) that $h^1(\Omega^2\otimes L^*(-D)) \geq 2$. Since also $h^0(\Omega^2\otimes L^*(-D)) \geq 1$ for general choices of the $x_i$, we deduce that $h^1(\Omega^2\otimes L^*) \geq 2$ and $h^0(\Omega^2\otimes L^*) \geq e-1 \geq 2$. Therefore $\Omega^2\otimes L^*$ contributes to the Clifford index of $C$. But $\mathrm{Cliff}(\Omega^2\otimes L^*) \leq 4g-4-d-2(e-2) \leq e-1$, the last inequality coming from the hypothesis $d \geq 4g+1-3e$. This is a contradiction, and (2.9) is proved.

Twisting (2.8) by L, it follows from (2.9) that if $D$ is sufficiently general, then $H^1(C\,,R_{\Omega(-D)}\otimes L\,) = 0$. But $L(-2x_i)$ is non-special for reasons of degree, so $H^1(R_\Omega\otimes L) = 0$ thanks to the exact sequence (2.7). In view of (2.4), this competes the proof. ∎

<u>Remark.</u> Note that the theorem implies that if $\mathrm{Cliff}(C) \geq 3$, then $\gamma_{\Omega,\Omega^2}$ is surjective, a fact proved by other methods in [T]. It would be interesting to know how close Theorem 2 comes to being optimal.

Finally, we wish to explain the connection with an interesting theorem of L′vovskii [L′v] and Zak which we learned about at the Chicago conference. We start with:

<u>Definition 2.10</u>. A non-singular variety $X \subset \mathbb{P}^r$ of dimension $n$ is said to be **k - extendable** if there is a possibly singular non-degenerate projective variety $Y \subset \mathbb{P}^{k+r}$ of dimension $n+r$, which is not a cone, such that $X$ is the intersection of $Y$ with a codimension $k$ linear subspace of $\mathbb{P}^{r+k}$. We say that $X$ is **extendable** if it is at least 1-extendable.

Consider now a smooth non-degenerate variety $X \subset \mathbb{P}^r$, and let $\mathcal{N} = \mathcal{N}_{X/\mathbb{P}^r}$ denote the normal bundle to $X$ in $\mathbb{P}^r$. Clearly $h^0(\mathcal{N}(-1)) \geq r+1$. The theorem of Zak and L′vovskii concerns the situation when equality comes close to holding:

<u>Theorem</u>. ([L′v], and Zak, to appear). <u>Assume</u> $\mathrm{codim}(X,\mathbb{P}^r) \geq 2$. <u>If</u> $h^0(\mathcal{N}(-2)) = 0$ <u>and</u> $h^0(\mathcal{N}(-1)) \leq r+k$, <u>then</u> $X \subset \mathbb{P}^r$ <u>is not</u> $k$-<u>extendable. In particular, if</u> $h^0(\mathcal{N}(-1)) = r+1$, <u>then</u> $X$ <u>is not extendable, i.e.</u> $X \subset \mathbb{P}^r$ <u>is not the hyperplane section of any variety</u> $Y \subset \mathbb{P}^{r+1}$ <u>other than a cone over</u> $X$.

Concerning the second statement, one shows that if $h^0(\mathcal{N}(-1)) = r+1$, then necessarily $h^0(\mathcal{N}(-2)) = 0$.

Suppose now that $L$ is a very ample line bundle on a curve $C$, and denote by $\mathcal{N}$ the normal bundle to $C$ in $\mathbb{P}H^0(L)$. As we are assuming that $g(C) \geq 1$, the natural map $H^0(\Omega) \otimes H^0(L) \longrightarrow H^0(\Omega \otimes L)$ is automatically surjective (c.f. [G],(3.c.1)). It then follows by duality from the sequences (2.1) - (2.3) upon twisting by $\Omega$ that

(2.11) $$h^0(\mathcal{N} \otimes L^*) = h^0(L) + \operatorname{corank}(\gamma_{\Omega,L}).$$

Combining this with Theorem 2 and the L'vovskij-Zak theorem we obtain:

<u>Corollary 2.12</u>. <u>If</u> $\operatorname{Cliff}(C) \geq 3$ <u>and</u> $\deg(L) \geq 4g+1-3\cdot\operatorname{Cliff}(C)$, <u>then</u> $C$ <u>is not extendable in</u> $\mathbb{P}H^0(L)$. ∎

More generally:

<u>Corollary 2.13</u> <u>If</u> $\deg(L) > 2g+2$, <u>and if</u> $\operatorname{corank}(\gamma_{\Omega,L}) = k$, <u>then</u> $C \subset \mathbb{P}H^0(L)$ <u>is not</u> $(k+1)$- <u>extendable</u>.

<u>Proof</u>. It follows from Theorem 1 that $\gamma_{L,\Omega\otimes L}$ is surjective. Therefore $h^1(\mathcal{N}^* \otimes \Omega \otimes L) = h^0(\mathcal{N} \otimes L^{-2}) = 0$. So the corollary follows from (2.11) and L'vovskii-Zak's theorem. ∎

<u>Remark</u>. It also follows from Zak's theorem and (2.11) that if $S$ is a smooth regular surface (e.g. a K3 surface), and if $C \subset S$ is a very ample divisor with normal bundle $L$, then $\gamma_{\Omega,L}$ is not surjective. It was this observation (proved with deformation theory rather than L'vovskii-Zak's theorem) that was the starting point of Wahl's work [W1] [W2] in this area.

<u>References.</u>

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