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R. Lazarsfeld and A. Van de Ven

# Topics in the Geometry of Projective Space

## Recent Work of F. L. Zak

With an Addendum by
F. L. Zak

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# Preface

The main topics discussed at the D.M.V. Seminar were the connectedness theorems of Fulton and Hansen, linear normality and subvarieties of small codimension in projective spaces. They are closely related; thus the connectedness theorem can be used to prove the inequality-part of Hartshorne's conjecture on linear normality, whereas Deligne's generalisation of the connectedness theorem leads to a refinement of Barth's results on the topology of varieties with small codimension in a projective space.

The material concerning the connectedness theorem itself (including the highly surprising application to tamely ramified coverings of the projective plane) can be found in the paper by Fulton and the first author: W. Fulton, R. Lazarsfeld, Connectivity and its applications in algebraic geometry, Lecture Notes in Math. 862, p. 26—92 (Springer 1981). It was never intended to be written out in these notes.

As to linear normality, the situation is different. The main point was an exposition of Zak's work, for most of which there is no reference but his letters. Thus it is appropriate to take an extended version of the content of the lectures as the central part of these notes.

In the lectures on varieties of small codimension, a detailed proof was presented for a theorem of Barth and the second author, saying that a smooth variety in a projective space is a complete intersection as soon as its dimension is (sufficiently) much larger than its degree. Originally we intended to include this proof in the notes, since the only reference is Barth's short address at the Conference in Vancouver. But very recently, Z. Ran has improved this result considerably, using new methods, so we have decided to skip this part and reproduce instead Zak's letters. For his permission to do so we are most grateful.

We also very much indebted to the D.M.V. and Prof. G. Fischer who has organised the seminar.

*R. Lazarsfeld*
*A. Van de Ven*

## Introduction

The purpose of these notes is to outline Zak's classification of smooth subvarieties of projective space with maximally degenerate secant varieties.

Consider a non-singular complex projective variety

$$X \subseteq \mathbb{P}^m$$

of dimension $n$, not contained in any hyperplane. Given a point $P \in \mathbb{P}^m - X$, projection from $P$ defines a finite map

$$\pi_P \colon X \longrightarrow \mathbb{P}^{m-1},$$

and we may ask whether for generic $P$, $\pi_P$ gives an *embedding* of $X$ into $\mathbb{P}^{m-1}$. This is always the case if $m > 2n+1$, whereas when $m \leqq 2n+1$ the map $\pi_P$ will usually have double points. For example, when $n=2$ and $m=5$, a classical theorem of Severi asserts that (up to projective equivalence) there is exactly one surface $X \subseteq \mathbb{P}^5$—namely, the Veronese surface—which projects smoothly to $\mathbb{P}^4$. When $n=2$ and $m=4$, $\pi_P$ is never an embedding.

In the course of his work on subvarieties of projective space of low codimension [H], Hartshorne was led on the basis of a few examples to suggest how this state of affairs for surfaces should generalize to higher dimensions. Hartshorne's conjecture was proved by Zak [Z1] in 1979, and we may state the result as

**Zak's Theorem on Linear Normality.** *Keeping notation as above, if* $3n > 2(m-2)$ *then* $\pi_P$ *is never an embedding.*

(The title of the theorem will be explained below; for an exposition, and an alternative proof, see [FL].)

The question then arises to classify all examples on the boundary of Zak's theorem, which Zak calls *Severi varieties*. Thus a Severi variety is a smooth non-degenerate subvariety

$$X \subseteq \mathbb{P}^m$$

of dimension $n = \frac{2}{3}(m-2)$, such that for generic $P \in \mathbb{P}^m - X$, $\pi_P$ is an embedding. Zak has answered the classification question with the following remarkable result:

**Zak's Classification Theorem.** ([Z2, Z3]). *Up to projective equivalence there are exactly four Severi varieties, to wit:*

(1) $n=2$: $X=V\subseteq\mathbb{P}^5$, *the Veronese surface.*

(2) $n=4$: $X=\mathbb{P}^2\times\mathbb{P}^2\subseteq\mathbb{P}^8$, *the Segre four-fold.*

(3) $n=8$: $X=\mathbb{G}(1,5)\subseteq\mathbb{P}^{14}$, *the Plücker embedding of the Grassmannian* $\mathbb{G}(1,5)$.

(4) $n=16$: $X=E\subseteq\mathbb{P}^{26}$, *the „$E_6$-variety."*

The non-classical example $E\subseteq\mathbb{P}^{26}$, which was pointed out to Zak by the first author (R. L.), will be discussed briefly in § 1 below. The case $n=4$ of the theorem was proved independently by Fujita-Roberts [F R], who also obtained restrictions on the possible dimensions of Severi varieties. Fujita [F] gives additional results in the near-extremal case.

Our object here is to describe Zak's beautiful proof of the classification theorem. In § 1 we go over some preliminary material (mostly without proof), and explain why the four varieties listed in the statement of the theorem—which we call the four *standard* Severi varieties—are in fact Severi varieties. The remaining sections of these notes correspond to the three principal stages of Zak's proof. Given a Severi variety $X\subseteq\mathbb{P}^m$, Zak's first step is to analyze the quadrics on $X$ (§ 2). By studying the maximal linear spaces on these quadrics, Zak then shows that $n=\dim X=2$, 4, 8 or 16 (§ 3). Finally, a case-by-case argument proves the theorem (§ 4). Zak's proof is for the most part elementary, but it is quite long, and we do not pretend to have included all details. Nonetheless, a dedicated reader should have little problem filling the gaps, and in any event we trust that Zak will eventually publish a full account.

A word or two is in order on the larger context of Zak's work. Results of Barth *et. al.* in the early 1970's suggested that it is very difficult to find examples of smooth subvarieties $X\subseteq\mathbb{P}^m$, other than complete intersections, with codim $X$ $\ll$ dim $X$, and the suspicion arose that no such examples can exist (cf. [H] for a survey). Specifically, one has Hartshorne's

**Conjecture on complete intersections:**

*If $X\subseteq\mathbb{P}^m$ is a smooth projective variety with*

$\dim X > 2$ codim (X),

*then X must be a complete intersection.*

This conjecture has sparked an enormous amount of work over the past decade. For example, in the codimension two case it is equivalent to the problem of whether any rank two vector bundle on $\mathbb{P}^n$ $(n \geqq 7)$ must split as a sum of line bundles, a question which seems to have been partly responsible for the tremendous activity in the field of vector bundles on projective space. Work by Barth and Van de Ven is described in the former's contribution to the 1974 ICM; see also [FL].

Complete intersections have the property that they are *linearly normal*, meaning that the given embedding $X \subseteq \mathbb{P}^m$ does not arise in a non-trivial way by projection from an embedding $X \hookrightarrow \mathbb{P}^{m+1}$. Motivated by his conjecture on complete intersections, Hartshorne [H] suggested that a smooth variety

$$X \subseteq \mathbb{P}^r$$

of dimension $n$ must be linearly normal if $3n > 2(r-1)$. This is precisely what Zak proved in the theorem on linear normality stated above. The problem of classifying the extremal examples had been posed by Hartshorne in the same paper.

In a number of letters to J. Roberts and the first author over the period July 1980—December 1981 Zak stated that he had had new ideas on Hartshorne's conjecture on complete intersections, which he felt would prove at least the codimension two and three cases, and lead to a classification of all examples on the boundary. However, perhaps understandably, Zak has been unwilling to provide any details, and as of December 1981 there were still "some lemmas that I [Zak] haven't seriously tried to prove". In the most recent communication of which we are aware (letter to J. Roberts dated 15 September 1982) no mention is made of the complete intersection problem. Our sense is that Zak has developed a totally new approach to the conjecture, but that is has not yet achieved fruition. In view of the tremendous importance of the problem, even in the codimension two case, we hope very much that Zak will publish at least some partial results, and clarify the status of his work, in the not-too-distant future.

**Acknowledgements.** We are grateful to Joel Roberts for numerous helpful comments on an early version of these notes. The authors lectured on this material at the DMV Seminar at Düsseldorf in September, 1982, and we wish to thank G. Fischer, who organized the seminar, and all the participants for their suggestions and enthusiasm.

## §1 Preliminaries; the four standard Severi varieties

Let $X \subseteq \mathbb{P}^m$ be a smooth non-degenerate projective variety of dimension $n$. Recall that the *secant variety*

$$\mathrm{Sec}(X) \subseteq \mathbb{P}^m$$

of $X$ is by definition the union of all secant lines to $X$ and their limits (i.e., tangent lines). Given a point $P \in \mathbb{P}^m - X$, the projection $\pi_P\colon X \to \mathbb{P}^{m-1}$ is an embedding if and only if $P \notin \mathrm{Sec}(X)$. Thus $X$ is a Severi variety if and only if $m = 3/2n+2$ and $\dim \mathrm{Sec}(X) \leq m-1$. If $X$ is a Severi variety, then by Zak's theorem on linear normality $\mathrm{Sec}(X)$ must be a hypersurface in $\mathbb{P}^m$.

A naive dimension count shows that the postulated dimension of the secant variety $\mathrm{Sec}(X)$ is $2n+1$: it requires $n$ parameters to specify each of two points on $X$, and one parameter to specify a point on the line joining them. The difference

$$(2n+1) - \dim \mathrm{Sec}(X)$$

is the dimension of the family of secant lines passing through a general point of $\mathrm{Sec}(X)$. More formally, for $P \in \mathrm{Sec}(X) - X$, consider the *secant locus* $Q_P \subseteq X$:

$$Q_P = \left\{ x \in X \,\middle|\, \begin{array}{l} P \text{ lies on a secant or} \\ \text{tangent line to } X \text{ passing} \\ \text{through } x. \end{array} \right\} \tag{1.1}$$

(Figure 1). Then $\dim \mathrm{Sec}(X) = (2n+1) - \dim(Q_P)$ for generic $P \in \mathrm{Sec}(X)$.

## §1a The standard examples

For each of the four varieties listed in the statement of Zak's classification theorem, one has

(1.2) *For every $P \in \mathrm{Sec}(X) - X$, $Q_P$ is a smooth quadric of dimension $n/2$.*

It follows that in each case $\dim \mathrm{Sec}(X) = 3/2n+1 = m-1$, so that each of the examples is, in fact, a Severi variety.

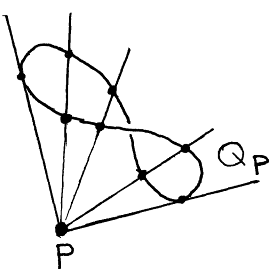



Figure 1

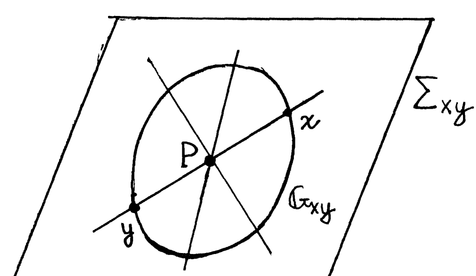



Figure 2

For the three classical examples, (1.2) is easily verified by hand (cf. [H]). Consider, for example, the Grassmannian $X = \mathbb{G}(1,5) \subseteq \mathbb{P}^{14}$. A pair of points $x, y \in X$ corresponds to two lines $l_x, l_y \subseteq \mathbb{P}^5$, and provided that $\overline{xy} \nsubseteq X \subseteq \mathbb{P}^{14}$ the corresponding lines span a 3-plane:

$$\mathbb{P}^3_{xy} = \mathrm{Span}(l_x, l_y) \subseteq \mathbb{P}^5 . \tag{$*$}$$

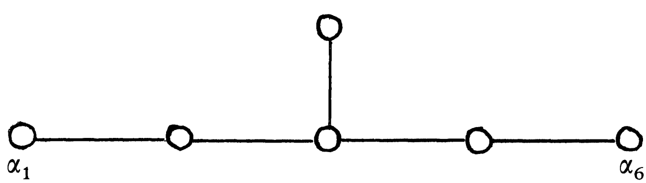
Let $\mathbb{G}_{xy} \subseteq X$ denote the four-dimensional Grassmannian of lines in $\mathbb{P}^3_{xy}$. The Plücker embedding $\mathbb{G}(1,5) \hookrightarrow \mathbb{P}^{14}$ of $\mathbb{G}(1,5)$ restricts to the Plücker embedding of $\mathbb{G}_{xy}$, and hence $\mathbb{G}_{xy}$ is realized in $\mathbb{P}^{14}$ as a smooth quadric hypersurface in a linear space $\Sigma_{xy}$ of dimension 5. Then for any point $P \in \overline{xy}$, $P \notin X$ (in fact for any $P \in \Sigma_{xy} - \mathbb{G}_{xy} = \Sigma_{xy} - X$) one has

$$Q_P = \mathbb{G}_{xy}$$

(Figure 2). This verifies (1.2) in the case $n=8$; the cases $n=2$ and $n=4$ are similar.

## §1b The $\mathbf{E}_6$-variety $\mathbf{E} \subseteq \mathbb{P}^{26}$

There are various heuristic geometric arguments for the existence of $E$. The quickest way actually to construct the example, however, is to use representation theory. Specifically, let $G$ be a simply connected algebraic group of type $E_6$, and let $V$ be the irreducible $G$-module whose highest weight $\lambda$ is dual under the Killing form to the root $\alpha_6$:




$E_6$.

(One could as well work with $\alpha_1$.) The variety in question arises as the projectivized orbit of a highest weight vector $0 \neq v_\lambda \in V_\lambda$:

$$E = \mathbb{P}(G \cdot v_\lambda) \subseteq \mathbb{P}(V) = \mathbb{P}^{26} .$$

One has

(1.3). *$E$ is a sixteen dimensional variety whose secant variety is a (cubic) hypersurface.*

Thus $E$ is indeed a Severi variety.

The geometry of the $E_6$-module $V$ has been described in another context by Kempf [K], who showed in particular that $\dim E = 16$. We sketch in (1.4) below an ad hoc verification that $\mathrm{Sec}(E)$ has dimension 25. Zak [Z3] has made a systematic study of the projective geometry of the varieties arising from irreducible representations of semi-simple algebraic groups. In particular, he verifies (1.3), and shows that $E$ satisfies (1.2), without any explicit calculations other than those found in the literature. We refer the reader to Zak's addendum to these notes.

(1.4) *Sketch of proof of* (1.3). The easiest—but probably least revealing—approach to (1.3) is simply to make explicit infinitesimal computations. To begin with, one writes out the 27 weights of $V$; it is convenient here to use the explicit description of the root system $E_6$ given by Demazure [D]. Noting that the embedded tangent space to $E$ at $x_\lambda = [v_\lambda] \in E$ is given by

$$T_{x_\lambda} E = \mathbb{P}(\mathfrak{g} \cdot v_\lambda) \subseteq \mathbb{P}(V)$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, an explicit computation shows that $\dim E = 16$. (One simply has to count the number of positive roots $\alpha$ for which $\lambda - \alpha$ is again a weight of $V$.) To check that $\dim \mathrm{Sec}(E) = 25$, one can use Terracini's lemma (cf. § 1 e below), which asserts in the case at hand that

$$\dim \mathrm{Sec}(E) = 32 - \dim(T_x E \cap T_y E) \tag{$*$}$$

for generic points $x, y \in E$. If now $0 \neq v_\mu \in V_\mu$ is a weight vector of lowest weight $\mu$ (i.e., $\mu - \alpha$ is not a weight of $V$ for any positive root $\alpha$), then in the first place $\mu$ is in the orbit of $\lambda$ under the Weyl group, and hence $v_\mu$ is in the $G$-orbit of $v_\lambda$. Moreover, the $G$-orbit of $(x_\lambda, x_\mu)$ is dense in $X \times X$, so $(*)$ applies to the pair $(x_\lambda, x_\mu)$, and one concludes with another explicit computation.

## § 1 c An alternative construction of the standard Severi varieties

For the purposes of Zak's proof of the classification theorem, it is necessary to construct an explicit birational correspondence between each of the four standard examples and a projective space of the appropriate dimension. This is given by:

(1.5) *In each of cases* $n = 2, 4, 8$ *and* 16, *there is a smooth variety* $Y \subseteq \mathbb{P}^n$ *such that the linear system of quadrics on* $\mathbb{P}^n$ *through* $Y$ *defines a rational map*

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^{3/2n+2},$$

*mapping* $\mathbb{P}^n$ *birationally onto the standard Severi variety of dimension* $n$.

*The variety* $Y$ *in each of the four cases is*:

$n=2.\quad Y=\phi$
$n=4.\quad Y=\mathbb{P}^1 \coprod \mathbb{P}^1$, *two skew lines*

$n=8.\quad Y=\mathbb{P}^1\times\mathbb{P}^3$, *the Segre variety*

$n=16.\quad Y=S$, *the* 10-*dimensional spinor variety parametrizing one of the two families of* 4-*planes on a smooth quadric of dimension* 8.

(We will say more about the spinor variety $S$ below.) We note that in each of the cases $n=4$, 8 and 16 the variety $Y\subseteq\mathbb{P}^n$ lies in a hyperplane.

Letting $\pi:\tilde{X}\longrightarrow\mathbb{P}^n$ denote the blowing up of $\mathbb{P}^n$ along $Y$, we may rephrase the assertion of (1.5) by saying that $\tilde{X}$ maps to $\mathbb{P}^{3/2n+2}$, birationally onto the standard example of dimension $n$:

$$\begin{array}{ccc} & & \mathbb{P}^{3/2n+2} \\ & & \cup\!\mid \\ & \tilde{X} & \xrightarrow{f} X \\ & \Big\downarrow{\scriptstyle \pi=bl_Y} & \\ Y\subseteq\mathbb{P}^{n-1} & \subseteq\mathbb{P}^n & \end{array} \tag{1.6}$$

In fact, $\tilde{X}$ is the blowing up of $X$ along one of the quadrics $Q_P$. The culmination of Zak's classification is to recover the diagram (1.6) starting from an "unknown" Severi variety $X$ (§4).

The case $n=2$ of (1.5) is trivial. The cases $n=4$ and $n=8$ were known classically, and can be treated by hand. Zak uses representation theory to verify (1.5) for the $E_6$-variety.

**Remark.** J. Roberts and independently T. Banchoff have found another, very suggestive, approach to the four standard Severi varieties. Specifically, let $k$ denote one of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and consider $\mathbb{P}^2(k)$ as a variety over $\mathbb{R}$. Then its complexification $X=\mathbb{P}^2(k)\otimes_{\mathbb{R}}\mathbb{C}$ admits an embedding into a complex projective space of the appropriate dimension, and one obtains in this way each of the four standard varieties. We hope that Roberts and/or Banchoff will soon publish the details of this construction. It would be marvelous to exploit this approach to give a new proof of Zak's theorem. Conversely, D. Eisenbud and W. Fulton have asked whether one could use some of Zak's ideas to give an algebro-geometric proof of the topological theorem that there exist only four real division algebras.

## §1d Linear spaces on smooth quadrics; spinor varieties

We recall for later reference the basic facts about linear spaces of maximal dimension on a smooth quadric hypersurface.

$$Q \subseteq \mathbb{P}^{2k+1}$$

of dimension $2k$. (The situation on quadrics of odd dimension is slightly different, and will not be needed.)

(1) The linear spaces on $Q$ of maximal dimension have dimension $k$.

(2) There are two disjoint families of $k$-planes on $Q$, each parametrized by a smooth irreducible projective variety $S_k$ of dimension $k(k+1)/2$. Two $k$-planes

$$\Lambda, \Lambda' \subseteq Q$$

are in the same family if and only if

$$\dim(\Lambda \cap \Lambda') \equiv k \pmod 2.$$

(cf. [G-H], Chapter 6.) We call $S_k$ a *spinor variety.*

**Examples.**

$k=1$: $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and the two families of lines are exactly the two rulings of $Q$. Thus $S_1 \cong \mathbb{P}^1$.

$k=2$: $Q = \mathbb{G}(1,3)$, the Grassmannian of lines in $\mathbb{P}^3$. Any 2-plane on $Q$ must be one of the Schubert cycles

$$\sigma(P) = \{l \subseteq \mathbb{P}^3 \mid l \ni \text{fixed } pt\ P\}$$

or

$$\sigma(H) = \{l \subseteq \mathbb{P}^3 \mid l \subseteq \text{fixed plane } H\}.$$

Letting $P$ vary over $\mathbb{P}^3$ or $H$ over $\mathbb{P}^{3*}$, we see that $S_2 \cong \mathbb{P}^3$.

$k=3$: Here it turns out that $S_3$ is itself isomorphic to a smooth quadric of dimension six (cf. [Z3]).

$k=4$: $S_4 = S$ is the ten-dimensional spinor variety appearing in (1.5).

One of the more interesting features of the spinor variety $S_k$ is its projective embeddings. By its very definition, $S_k$ admits an embedding in the Grassmannian of $k$-planes in $\mathbb{P}^{2k+1}$:

$$S_k \hookrightarrow \mathbb{G}(k, 2k+1) \subseteq \mathbb{P}^{\binom{2k+2}{k+1}-1} = \mathbb{P}.$$

However as illustrated already in the case $k=1$, $\mathcal{O}_{\mathbb{P}}(1)|S_k$ is divisible by two in $\mathrm{Pic}(S_k)$, and indeed one has:

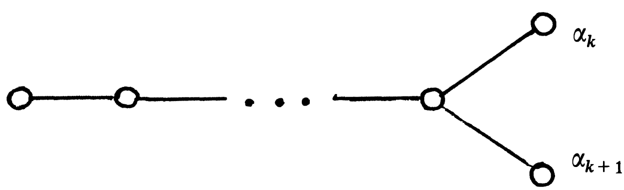
(1.7) $\mathrm{Pic}(S_k) \cong \mathbb{Z}$, *and the positive generator* $\mathcal{O}_{S_k}(1)$ *defines an embedding*

$$S_k \hookrightarrow \mathbb{P}^{2^k-1}.$$

*Sketch of proof of* (1.7). The spinor variety $S_k$ is isomorphic to

$$G/P,$$

where $G$ is the simply connected covering of the orthogonal group $SO(2k+2, \mathbb{C})$, and $P$ is either of the maximal parabolic subgroups associated to the roots $\alpha_k$ or $\alpha_{k+1}$ in $D_{k+1}$:




$D_{k+1}$.

(cf. [Z 3].) One has $\mathrm{Pic}(S_k) \cong \mathbb{Z}$ since $P$ is maximal. If $\lambda_k$, $\lambda_{k+1}$ are the weights dual to $\alpha_k$, $\alpha_{k+1}$ respectively, then the corresponding representations of $G$—which are the two spinor representations—each have dimension $2^k$ (cf. [*S*, p. 116]), which proves the second assertion of (1.7). The fact that the embedding

$$S_k \hookrightarrow \mathbb{P}^{\binom{2k+2}{k+1}-1}$$

is defined by $\mathcal{O}_{S_k}(2)$ is a reflection of a well-known fact about the representations of $G$. Viz, if $V$ is the canonical $SO(2k+2, \mathbb{C})$-module of dimension $2k+2$, then $\Lambda^{k+1} V$ splits as the direct sum of two irreducible representations $\Lambda^-$ and $\Lambda^+$ (corresponding to the two families of $k$-planes on $Q$), where $\Lambda^-$ and $\Lambda^+$ are the representations with highest weights $2\lambda_k$ and $2\lambda_{k+1}$ respectively (cf. [*S*, pp. 133—140]).

**Remark.** The 10-dimensional spinor variety $S \subseteq \mathbb{P}^{15}$ is, to the best of our knowledge, the largest known example of a non-complete intersection on the boundary of Hartshorne's conjecture. The projective geometry of this variety is very interesting (e.g., it is isomorphic to its dual variety); as W. Fulton points out, this variety was studied classically by Room [R].

## §1e Terracini's lemma; Zak's theorem on tangencies

We state for the reader's convenience two results which will be needed in the sequel. We refer to [FL, §7] for proofs and further references.

**Terracini's lemma.** *Let $X \subseteq \mathbb{P}^m$ be a smooth projective variety, and choose points $x, y \in X$, $P \in \overline{xy} - X$. Then*

$$T_P \operatorname{Sec}(X) \supseteq \operatorname{Span}(T_x X, T_y X), \tag{$*$}$$

*and for general $x$, $y$ and $P$ equality holds in* $(*)$.

The tangent spaces referred to in $(*)$ are embedded Zariski tangent spaces in $\mathbb{P}^m$, as are those in:

**Zak's theorem on tangencies.** *Let $X \subseteq \mathbb{P}^m$ be a smooth non-degenerate variety of dimension $n$. Fix a $k$-plane*

$$L \subseteq \mathbb{P}^m \quad (n \leqq k \leqq m-1),$$

*and set*

$$Z_L = \{x \in X \mid T_x X \subseteq L\}.$$

*Then* $\dim Z_L \leqq k - n$.

## §1f Some open problems

It emerges from Zak's classification theorem that every Severi variety is homogeneous. As the reader will see, an "unknown" Severi variety $X \subseteq \mathbb{P}^m$ comes to look more and more homogeneous throughout the course of Zak's proof, but it is not until one has the complete list that one actually knows $X$ to be homogeneous. Is there some way of proving *a priori* that any Severi variety is homogeneous? This could lead to a great simplification of the proof of Zak's theorem, for it is an easy matter to check using representation theory that the four standard examples are the only Severi varieties of the form $G/P$, where $G$ is a semi-simple algebraic group and $P \subseteq G$ is a parabolic subgroup (cf. [Z3]).

A second, very interesting, problem is to construct projective varieties of large "secant deficiency". Specifically, consider a smooth projective variety

$$X \subseteq \mathbb{P}^m$$

of dimension $n$ not contained in a hyperplane, with $\dim \operatorname{Sec}(X) < m$. Define the *secant deficiency* of $X$ to be the integer

$$\delta = 2n + 1 - \dim \operatorname{Sec}(X).$$

(The notion is uninteresting if $\mathrm{Sec}(X) = \mathbb{P}^m$.) For instance, just as in § 1a above one sees that the Segre varieties

$$\mathbb{P}^a \times \mathbb{P}^b \subseteq \mathbb{P}^{ab+a+b} \quad (a, b \geqq 2)$$

have deficiency $\delta = 2$, while the Grassmannians

$$\mathbb{G}(1, k) \subseteq \mathbb{P}^{\binom{k+1}{2}-1} \quad (k \geqq 5)$$

have $\delta = 4$. We pose the

**Problem.** *Do there exist smooth projective varieties with arbitrarily large secant deficiency?*

The $E_6$-variety $E \subseteq \mathbb{P}^{26}$ has secant deficiency $\delta = 8$, and we know of no examples where $\delta \geqq 9$. It seems to us rather incredible that there could exist an absolute bound on the secant deficiency of a smooth projective variety, but it also seems to be fairly difficult to construct examples with large deficiency.

## § 2 Quadrics on a Severi variety

We now consider an arbitrary Severi variety

$$X \subseteq \mathbb{P}^m \quad \left(m = \frac{3}{2} n + 2\right)$$

of dimension $n$. Thus $X$ is a smooth non-degenerate variety whose secant variety is a hypersurface. As in § 1, for $P \in \mathrm{Sec}(X) - X$ let

$$Q_P = \{x \in X \mid \overline{xP} \text{ a secant or tangent line to } X\},$$

and put

$$\Sigma_P = \{R \in \mathbb{P}^m \mid R \in \overline{xP}, \quad \text{with} \quad x \in Q_P\}.$$

Thus $\Sigma_P$ is a cone over $Q_P$ with vertex $P$. The first main step in Zak's analysis is to recover the picture described in § 1a:

**Theorem 2.1.** *For any* $P \in \mathrm{Sec}(X) - X$:

a) *$Q_P$ is a smooth quadric of dimension $n/2$, and $\Sigma_P$ is a linear space of dimension $n/2 + 1$.*

b) $\Sigma_P \cap X = Q_P$

c) *Given $P' \in \mathrm{Sec}(X) - X$, one has $Q_{P'} = Q_P$ if and only if $P' \in \Sigma_P$.*

The proof of Theorem 2.1 proceeds in three stages. First, as in Fujita-Roberts [FR], one shows that the statement holds for a generic point $P \in \mathrm{Sec}(X) - X$. One then deduces that $\mathrm{Sec}(X)$ is a cubic hypersurface, *singular only along* $X$. Finally, one checks that the assertion holds for all $P \in \mathrm{Sec}(X) - X$.

Turning to the details, let $\widetilde{X \times X}$ denote the blowing up of $X \times X$ along the diagonal. Thus any point $a \in \widetilde{X \times X}$ determines in the evident way a line $l_a \subseteq \mathbb{P}^m$, and one has a natural $\mathbb{P}^1$ bundle

$$S_X \to \widetilde{X \times X}$$

whose fibre over $a$ is the line $l_a$. $S_X$ maps to $\mathbb{P}^m$ and its image is precisely $\mathrm{Sec}(X)$:

$$\begin{tikzcd} S_X \arrow[r] \arrow[d, "\varphi"'] \arrow[rrr, bend right, "\pi"'] & \widetilde{X \times X} \arrow[r, "bl_\Delta"] & X \times X \arrow[r, "pr_1"] & X \\ \mathbb{P}^m \supseteq \mathrm{Sec}(X) & & & \end{tikzcd} \tag{2.2}$$

Note that in (2.2), $Q_P = \pi(\varphi^{-1}(P))$.

**Proposition 2.3.** *For general* $P \in \mathrm{Sec}(X) - X$, $Q_P$ *is a smooth quadric of dimension* $n/2$, *and* $\Sigma_P$ *is a linear space of dimension* $n/2+1$.

We refer to [F-R] for a detailed proof. Zak gives a different argument, based on his theorem on tangencies (§ 1 e). Roughly speaking, Zak's idea is to choose general points $x, y \in X$, and $P \in \overline{xy} - X$, such that $P$ is a smooth point of $\mathrm{Sec}(X)$, and such that the line $\overline{xy}$ is not trisecant to $X$. Thus $\dim Q_P = n/2$, and by Terracini's lemma (§ 1 e) one may assume that

$$H =_{\mathrm{def}} T_P \,\mathrm{Sec}(X) = \mathrm{Span}(T_x X, T_y X). \tag{$*$}$$

The point now is to show that $\dim(\mathrm{Sec}(Q_P)) = n/2 + 1$. To this end, assuming for simplicity of exposition that $Q_P$ is irreducible, we observe that $(*)$ holds for any two general points $x, y \in Q_P$, whether or not $P \in \overline{xy}$. Thus for general $P' \in \mathrm{Sec}(Q_P)$, one has

$$T_{P'} \,\mathrm{Sec}(X) = H.$$

Hence, again by Terracini, there exists a dense open set $U \subseteq \mathrm{Sec}(Q_P)$ such that

$$Q_{P'} \subseteq Z_H =_{\mathrm{def}} \{x \in X \mid T_x X \subseteq H\} \quad \forall\, P' \in U.$$

On the other hand, by Zak's theorem on tangencies, applied to a projection $\pi : X$

$\hookrightarrow \mathbb{P}^{3/2n+1}$, one has

$$\dim Z_H \leqq n/2.$$

Since $\dim(Q_{P'}) \geqq n/2$ for all $P' \in U$, it follows that after possibly shrinking $U$, all the $Q_{P'}$ coincide for $P' \in U$. If $Q$ denotes the common secant locus, then evidently $Q \supseteq Q_P$. Since for general $x, y \in Q_P$ the line $\overline{xy}$ is not a trisecant, the upshot is that a general point $P' \in \operatorname{Sec}(Q_P)$ lies on at least $\infty^{n/2}$ secant lines to $Q_P$, and so $\dim \operatorname{Sec}(Q_P) \leqq n/2+1$. But this implies that $\operatorname{Sec}(Q_P)$ is a linear space: in fact, if $V \subseteq \mathbb{P}^N$ is any irreducible projective variety of dimension $k$ such that $\dim \operatorname{Sec}(V) = k+1$, then by cutting down to the case $k=1$ one sees that $\operatorname{Sec}(V)$ is a linear space. It follows that $Q_P$ must be a quadric; for otherwise $\overline{xy}$ would be a trisecant line for every $x, y \in Q_P$. Hence $\Sigma_P$ is a linear space of dimension $n/2+1$. Finally, the nonsingularity of $Q_P$ for general $P$ is deduced from the generic smoothness of the map $\varphi$ in (2.2) (cf. [F-R]).

**Corollary 2.4.** *Fix a general*) point* $P \in \operatorname{Sec}(X) - X$. *Then*

$$\Sigma_P \cap X = Q_P,$$

*and* $Q_P = Q_{P'}$ *for general* $P' \in \Sigma_P - Q_P$.

*Proof.* Since $Q_P$ is a hypersurface in $\Sigma_P$, clearly any point $x \in \Sigma_P \cap X$ lies on $Q_P$. By the same token, $Q_P \subseteq Q_{P'}$ for any $P' \in \Sigma_P - Q_P$; but for generic $P'$, $Q_{P'}$ is a quadric. □

**Remark.** We shall see below that most of (2.3) and (2.4) hold for any $P \in \operatorname{Sec}(X) - X$ which is a smooth point of $\operatorname{Sec}(X)$. Thus it is essential to control the singularities of $\operatorname{Sec}(X)$.

We come now to a basic technical result:

**Proposition 2.5.** *Fix a general smooth point* $P \in \operatorname{Sec}(X) - X$, *and let* $Q$ *denote the quadric* $Q_P$. *Then*

$$\operatorname{Sec}(X) = S(Q, X),$$

*where* $S(Q, X)$ *is the join of* $Q$ *and* $X$.

*Proof.* Putting $T(Q, X) = \bigcup\limits_{x \in Q} T_x X$, it follows in a standard way from the connectedness theorem of Fulton and Hansen that either $S(Q, X) = T(Q, X)$, or

*) We adopt the convention that a "general" point $P$ is supposed to satisfy all the properties previously shown to hold generically. In particular, in (2.4) $Q_P$ is assumed to be a quadric.

else both $S(Q, X)$ and $T(Q, X)$ attain the expected dimensions $3/2n+1$ and $3/2n$ respectively. So it suffices to prove that $T(Q, X) \neq S(Q, X)$. But by Terracini's lemma, $T(Q, X) \subseteq T_P \operatorname{Sec}(X)$. Since $X \subseteq S(Q, X)$, and $X$ is not contained in the hyperplane $T_P \operatorname{Sec}(X)$, the proposition follows. □

**Corollary 2.6.** *Fix a general point $P \in \operatorname{Sec}(X) - X$. Then $Q_{P'}$ meets $Q_P$ for any $P' \in \operatorname{Sec}(X) - X$.* □

We want next to show that $Q_P \cap Q_{P'}$ consists of exactly one point for generic $P, P' \in \operatorname{Sec}(X) - X$. To this end, we start with some remarks on planes containing $\Sigma_P$.

Fix a general point $P \in \operatorname{Sec}(X) - X$, and let $M$ be any $(n/2+2)$-plane through $\Sigma_P$, with $M \nsubseteq T_P \operatorname{Sec}(X)$. Thus $M$ meets $\operatorname{Sec}(X)$ transversely at $P$, and hence

$$S_M =_{\text{def}} \operatorname{Sec}(X) \cap M$$

contains $\Sigma_P$ as an irreducible component. For reasons of degree, $S_M$ must contain other components as well.

**Lemma 2.7.** (i) *Besides $Q_P$, $M \cap X$ contains only finitely many points*

$$x_1, \ldots, x_r \in X \quad (r \geqq 1).$$

(ii) *Besides $\Sigma_P$, $S_M$ consists precisely of the cones*

$$S(x_i, Q_P) \quad (1 \leqq i \leqq r).$$

*Proof.* It follows from (2.5) that $S_M = S(M \cap X, Q_P)$. If $(M \cap X) - Q_P$ contained at least a curve, this join would have dimension $\geqq \frac{n}{2}+2$, i.e. would fill up $M$; this proves (i). As for (ii), obviously $S(x_i, Q_P) \subseteq S_M$, and the reverse inclusion again follows from (2.5). □

The assertion of (2.7) is pictured schematically in Figure 3.

**Corollary 2.8.** *For almost all points $P, P' \in \operatorname{Sec}(X) - X$, $Q_P \cap Q_{P'}$ consists of a single point.*

*Proof.* Fixing $P$, $M$ as above, the assertion is true for any $P' \in S_M$ lying on only one of the cones appearing in Lemma 2.7, and this implies the corollary. □

We now show that in the situation of Lemma 2.7, one has $r=1$:

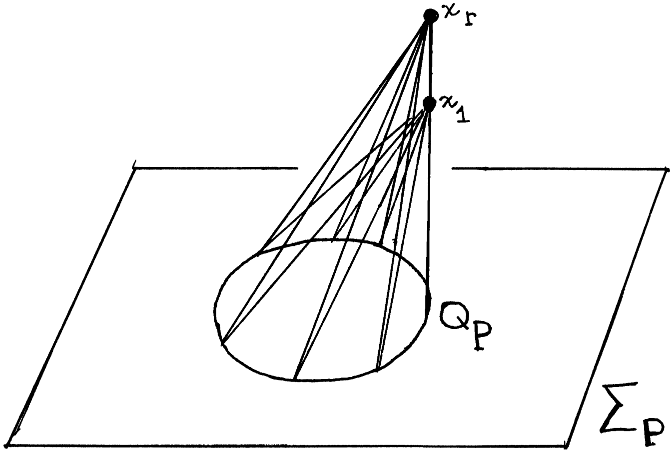



Figure 3

**Proposition 2.9.** Sec$(X)$ *is a cubic hypersurface.*

*Sketch of proof.* Let $P$, $P' \in \operatorname{Sec}(X) - X$ be two general points, and assume that

$$P \notin T_{P'} \operatorname{Sec}(X) \quad \text{and} \quad P' \notin T_P \operatorname{Sec}(X). \tag{$*$}$$

We observe first that $P$, $P'$ canonically determine a third point

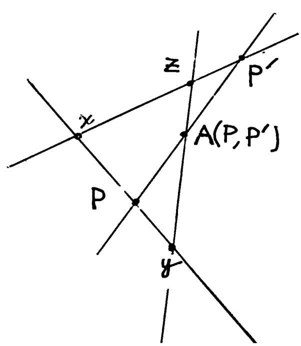
$$A(P, P') \in \operatorname{Sec}(X) - X$$

on the line $\overline{P, P'}$. In fact, let $x = x(P, P') = Q_P \cap Q_{P'}$ (2.8). By $(*)$, neither $P$ nor $P'$ lie on a tangent line to $X$ at $x$. Hence since $x \in Q_P$ there exists some point $y = y(P, P') \in \overline{xP} \cap X$ $(y \neq x)$. For general $P$, there is moreover only one such $y$. Similarly, there is a unique point $z \in X$, $z \neq x$, such that

$$P' \in \overline{xz}.$$

We then set

$$A(P, P') = \overline{yz} \cap \overline{PP'}.$$



Now for generic $P, P' \in \mathrm{Sec}(X) - X$, the line $\overline{PP'}$ meets $\mathrm{Sec}(X)$ transversely. If $\deg(\mathrm{Sec}(X)) \geqq 4$, $\overline{PP'}$ would meet $\mathrm{Sec}(X)$ at $P$, $P'$ plus at least two other points. We have singled out one of these—$A(P, P')$—canonically, and since $\mathrm{Sec}(X)$ is irreducible a standard monodromy argument gives a contradiction. □

Proposition 2.9 gives good control over $\mathrm{Sec}(X)$ locally. Specifically, fix a general point $P \in \mathrm{Sec}\,(X) - X$, and consider *any* point

$$P' \in U_P =_{\mathrm{def}} \mathrm{Sec}(X) - T_P \mathrm{Sec}(X).$$

Then Lemma 2.7 applies to the $(n/2+2)$-plane $M = S(\Sigma_P, P')$. By (2.9) one must have $r=1$ in that lemma, and it follows that

$$\mathrm{Sec}(X) \textit{ is smooth at any point } P' \in U_P - X. \tag{2.10}$$

Moreover,

$$Q_{P'} \textit{ has pure dimension } n/2 \textit{ for any } P' \in U_P - X, \tag{2.11}$$

since, as in Zak's proof of Proposition 2.3,

$$Q_{P'} \subseteq \{x \in X \mid T_x X \subseteq T_{P'} \mathrm{Sec}\}$$

and the set on the right has dimension $\leqq n/2$. Returning to the diagram (2.2), since $S_X$ is smooth it follows from (2.10) and (2.11) that the map $\varphi$ is flat over $U_P - X$. A simple argument then shows that

$$Q_{P'} \textit{ is a quadric for all } P' \in U_P - X. \tag{2.12}$$

We next wish to show that (2.10), (2.11), and (2.12) hold for all $P' \in \mathrm{Sec}(X) - X$. To this end it evidently suffices to prove

**Lemma 2.13.** *For any point $R \in \mathrm{Sec}(X)$, there is a dense open set of smooth points $P' \in \mathrm{Sec}(X) - X$ such that*

$$R \notin T_{P'} \mathrm{Sec}(X).$$

*Proof.* Pick a general point $P \in \mathrm{Sec}(X) - X$ such that $Q = Q_P$ is a smooth quadric with $\mathrm{Sec}(X) = S(Q, X)$. Computing tangent spaces to $S(Q, X)$ à la Terracini, one argues that it suffices to show that for a sufficiently general point $x \in X$, and general points $y_1, \ldots, y_N \in Q$,

$$\bigcap_{i=1}^{N} S(T_{y_i} Q, T_x X) = T_x X. \tag{$*$}$$

For $(*)$, in turn, it suffices to show that $T_x X \cap \Sigma_P = \phi$ for general $x \in X$, for then

$$\bigcap_{i=1}^{N} S(T_{y_i}Q, T_xX) = S\left(\bigcap_{i=1}^{N} T_{y_i}Q, T_xX\right),$$

and we can certainly arrange that $\cap\, T_{y_i}Q = \phi$. But it follows from Lemma 2.7 that projection from $\Sigma_P$ defines a dominant map $\pi: X - Q \to \mathbb{P}^n$, and so $T_xX \cap \Sigma_P = \phi$ for general $x \in X$ by the theorem on generic smoothness. □

To recapitulate, we have so far proved

(2.14) *For any $P \in \mathrm{Sec}(X) - X$, $\mathrm{Sec}(X)$ is smooth at $P$, $Q_P$ is a quadric of dimension $n/2$, and hence $\Sigma_P$ is a linear space of dimension $n/2+1$. Furthermore, $\Sigma_P \cap X = Q_P$, and for any $P' \in \Sigma_P - Q_P$ one has $Q_{P'} = Q_P$.*

(The last sentence follows from the first as in the proof of Corollary (2.4).) Hence Theorem 2.1 is a consequence of

**Lemma 2.15.** *For any $x \in X$, and any quadric $Q_P$ through $x$ $(P \in \mathrm{Sec}(X) - X)$, $Q_P$ is smooth at $x$.*

*Proof.* Choose a general point $P' \in \mathrm{Sec}(X) - X$ such that $x \notin T_{P'}\mathrm{Sec}(X)$ (Lemma 2.13), and let $M = \mathrm{Span}(\Sigma_{P'}, x)$, so that we are in the situation of Lemma 2.7. In particular, $x$ is an isolated point of $M \cap X$. Given $Q_P \ni x$, $Q_P$ meets $Q_{P'}$, say at $y \in Q_{P'}$ (2.6). The line $\overline{xy}$ meets $Q_P$ at exactly the two distinct points $x$ and $y$; in particular, $x$ cannot be a singular point of $Q_P$. □

(See Figure 4.)

Theorem 2.1 is now proved.

In the sequel, we shall make extensive use of

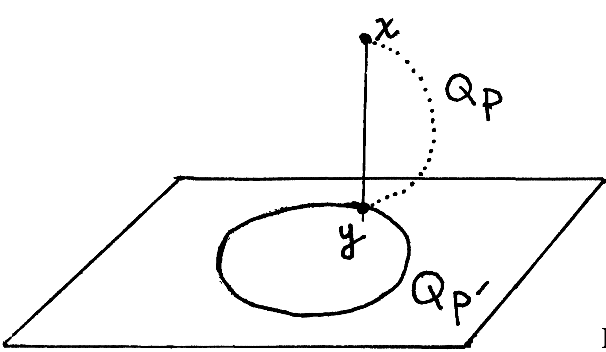



Figure 4

**Corollary 2.16.** *Given any two points $P, P' \in \mathrm{Sec}(X) - X$, the intersection*

$$Q_P \cap Q_{P'}$$

*is a linear space unless $Q_P = Q_{P'}$.*

*Proof.* Given distinct points $x, y \in Q_P \cap Q_{P'}$, it suffices to show that $\overline{xy} \subseteq Q_P \cap Q_{P'}$ unless $Q_P = Q_{P'}$. Now in any event, $\overline{xy} \subseteq \Sigma_P \cap \Sigma_{P'}$. If there exists a point $R \in \overline{xy}$ with $R \notin Q_P \cap Q_{P'}$, then $R$ lies in neither $Q_P$ nor $Q_{P'}$: for otherwise one obtains a contradiction to statement (b) of Theorem 2.1. But then

$$Q_P = Q_R = Q_{P'}$$

by two applications of (2.1c). □

**Remark 2.17.** The last two results show that for fixed $x \in X$, the set of all secant quadrics $Q_P$ $(P \in \mathrm{Sec}(X) - X)$ passing through $x$ is itself naturally parametrized by a smooth quadric $Q'$ of dimension $n/2$. In fact, as in the proof of Lemma 2.15 choose any $Q' = Q_{P'}$ such that $x \notin T_{P'} \mathrm{Sec}(X)$. Given $Q_P \ni x$, the intersection $Q_P \cap Q'$ has dimension zero (by (2.7)) and so consists of a single point thanks to (2.16). In the other direction, given any point $y \in Q'$, choose some $R \in \overline{xy} - X$ and associate to $y$ the quadric $Q_R$.

## § 3 Dimensions of Severi varieties

The second main step in Zak's classification is to show that there are only four possibilities for the dimension of a Severi variety:

**Theorem 3.1.** *Let $X \subseteq \mathbb{P}^m$ be a Severi variety. Then*

$$n = \dim X = 2, 4, 8 \quad \textit{or} \quad 16.$$

We may—and do—assume that $n \geqq 4$. Fix for the rest of this section a point

$$x \in X$$

and two secant quadrics

$$Q_1 = Q_{P_1} \quad \text{and} \quad Q_2 = Q_{P_2} \ (P_i \in \mathrm{Sec}(X) - X)$$

meeting at $x$ and nowhere else. Zak's beautiful idea is to estimate in different ways the dimension of the family of secant quadrics $Q_P$ passing through $x$ and meeting each $Q_i$ in positive dimension. These estimates lead to inequalities on the dimension $n$ of $X$ which ultimately imply the theorem.

In the course of the proof, it becomes necessary to produce secant quadrics meeting both $Q_1$ and $Q_2$ in positive dimension. To this end, Zak introduces the following basic construction. Let

$$C_i = T_x Q_i \cap Q_i \quad (i = 1, 2).$$

Thus $C_i$ is a cone, with vertex $x$, over a smooth quadric of dimension $n/2-2$. Denote by

$$S(C_1, C_2)$$

the joint of $C_1$ and $C_2$. Then, evidently,

(3.2) *For any point $P \in S(C_1, C_2)$, $P \notin X$, one has*
$\dim(Q_P \cap Q_i) \geqq 1 \quad (i=1, 2)$.

(See Figure 5.) Recall that then $Q_P \cap Q_i$ is a linear space (Corollary 2.16).

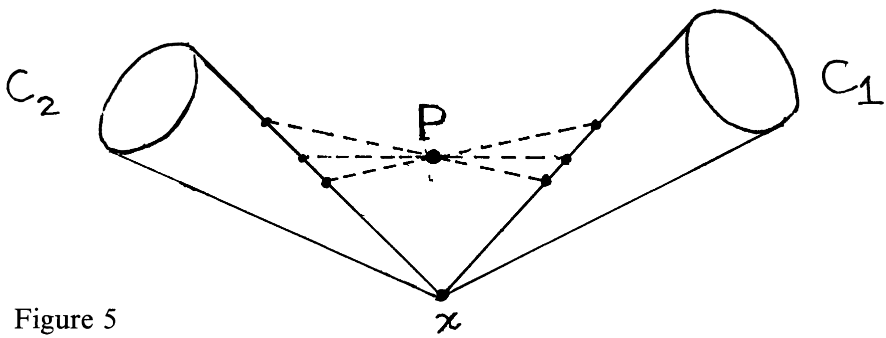



Figure 5

Finally, observe that

$$\dim S(C_1, C_2) = 2(n/2-1) = n-2,$$

and that $S(C_1, C_2)$ is irreducible if each $C_i$ is, i.e. if $n \geqq 6$.

**Lemma 3.3.** *Fix two points $P, P' \in S(C_1, C_2) - X$, and put*

$$L_i = Q_P \cap Q_i \quad \textit{for} \quad i=1 \textit{ and } 2.$$

*Then $Q_{P'} = Q_P$ if and only if $P' \in S(L_1, L_2)$, where $S(L_1, L_2)$ denotes the join of $L_1$ and $L_2$.*

*Proof.* If $P' \in S(L_1, L_2)$, then $P' \in \Sigma_P$ and hence $Q_{P'} = Q_P$ (Theorem 2.1). Conversely, if say $P' \in \overline{y_1 y_2}$ with $y_i \in C_i$, and if $Q_{P'} = Q_P$, then

$$y_i \in Q_i \cap Q_{P'} = Q_i \cap Q_P \quad (i=1, 2),$$

and hence $y_i \in L_i$. $\square$

**Lemma 3.4.** *The family of all secant quadrics through $x$ meeting both $Q_1$ and $Q_2$ in positive dimension has dimension $\leqq n/2-2$.*

*Proof.* This is guggested by a naive parameter count. In fact,

$$\dim\{Q_P \mid Q_P \ni x\} = n/2$$

(Remark 2.17), whereas a general secant quadric $Q_P$ through $x$ meets each $Q_i$ only at $x$. Hence one expects at least two conditions to be imposed by the requirement that $Q_P$ meet both $Q_1$ and $Q_2$ in positive dimension. □

A possibly more convincing formulation of this argument makes use of the quadric $Q'$ parametrizing all secant quadrics through $x$ (2.17). The quadric $Q_i \ni x$ corresponds to a point $y_i \in Q'$, and the reader may find it amusing to verify

(3.5) *If $Q_P \ni x$ is a secant quadric parametrized by a point $y \in Q'$, and if*
$$\dim(Q_P \cap Q_i) \geqq 1,$$
*then* $y \in T_{y_i}Q'$.

(It ultimately emerges that the converse holds as well.) Lemma 3.4 follows immediately.

One now obtains already a first restriction on the dimension of $X$:

**Proposition 3.6.** *Assume that* $n = \dim X \geqq 6$. *Then*

$$n \equiv 0 \pmod 4$$

*and*

$$\dim(Q_P \cap Q_i) = n/4 \quad \textit{for all} \quad P \in S(C_1, C_2) - X.$$

*Proof.* Since $n \geqq 6$, $S(C_1, C_2)$ is irreducible. We may hence set

$$\alpha_i = \dim(Q_P \cap Q_i) \quad (i = 1, 2)$$

for a general point $P \in S(C_1, C_2) - X$. Then, for general $P$, the joins in Lemma 3.3 have dimension $\alpha_1 + \alpha_2$. Hence the family of distinct quadrics $Q_P$ obtained as $P$ varies over $S(C_1, C_2) - X$ has dimension $(n-2) - (\alpha_1 + \alpha_2)$. Therefore, by Lemma 3.4,

$$(n-2) - (\alpha_1 + \alpha_2) \leqq \frac{n}{2} - 2,$$

i.e.

$$\alpha_1 + \alpha_2 \geqq \frac{n}{2}. \tag{$*$}$$

If $n=4k+2$, then $\alpha_i \leqq k$ $(i=1,2)$, since the largest linear spaces on a smooth quadric of dimension $2k+1$ have dimension $k$. But this leads to the inequality

$$2k+1 \leqq \alpha_1+\alpha_2 \leqq 2k,$$

a contradiction. Hence $n=4k$, and one finds from $(*)$ that $\alpha_1=\alpha_2=k$. Finally, since in any event $\dim(Q_P \cap Q_i) \leqq k$, it follows by semi-continuity that $\dim(Q_P \cap Q_i)=k$ for any $P \in S(C_1, C_2)-X$. □

**Remarks.** (1) It follows from the proof that the inequality in Lemma 3.4 must actually be an equality (provided that $n>4$). On the other hand, returning to the situation of (3.5), let $y_1, y_2 \in Q'$ be the points parametrizing the fixed quadrics $Q_1$ and $Q_2$. One now sees that if $y \in Q'$ corresponds to a secant quadric $Q_P$, then $\dim(Q_P \cap Q_i)>0$ for $i=1$ and 2 iff $y \in T_{y_1}Q' \cap T_{y_2}Q'$. Therefore, by fixing $y_1$ and varying $y_2$ (or vice-versa) one deduces

$$\dim(Q_P \cap Q_i)>0 \Leftrightarrow y \in T_{y_i}Q'. \qquad (i=1 \text{ or } 2)$$

In particular, the set of all such $Q_P$ is irreducible if $n>4$.

(2) The congruence $n \equiv 0 \pmod 4$, as well as the assertion of Corollary 3.7 below, were obtained also by Fujita-Roberts [FR], using completely different, less elementary, methods.

**Corollary 3.7.** *If* $n \geqq 8$, *then* $n \equiv 0 \pmod 8$.

*Proof.* Pick $Q_P$ meeting each $Q_i$ $(i=1,2)$ in a linear space $L_i$ of dimension $n/4$. Then $L_1 \cap L_2=\{x\}$. Hence by the results quoted in § 1 d, it suffices to show that $L_1$ and $L_2$ lie in the same family of $(n/4)$-planes on $Q_P$. But in fact by the Proposition and Remark 1 above, the family of secant quadrics

$$\left\{ Q_{P'} \;\middle|\; \begin{array}{l} Q_{P'} \ni x \\ Q_{P'} \cap Q_P \text{ an } (n/4)\text{-plane} \end{array} \right\}$$

is irreducible if $n>4$. □

We now arrive at the most interesting part of Zak's argument, namely the bound $n \leqq 16$. Recall from § 1 d that the quadric $Q_1$ contains two families of $(n/4)$-planes. Since $S(C_1, C_2)-X$ is irreducible if $n>4$, it is meaningful to make the following

**Definition 3.8.** Assuming $n \geqq 8$, let $\mathfrak{F}(Q_1)$ denote the family of $(n/4)$-planes on $Q_1$ containing the intersections

$$\{Q_1 \cap Q_P\}_{P \in S(C_1, C_2) - X},$$

and let $\mathfrak{F}'(Q_1)$ be the other family. Write

$$\mathfrak{F}_x(Q_1), \ \mathfrak{F}'_x(Q_1)$$

for those $(n/4)$-planes in each family which pass through $x$.

**Example.** Consider $X = \mathbb{G}(1,5) \subseteq \mathbb{P}^{14}$. The secant quadric $Q_1$ is obtained as the Schubert cycle $\{\text{lines } l \subseteq L^3\}$ for some 3-plane $L^3 \subseteq \mathbb{P}^5$. The 2-planes in $\mathfrak{F}(Q_1)$ are the Schubert cycles $\{\text{lines } l \subseteq H^2\}$ for some 2-plane $H^2 \subseteq L^3$, while those in $\mathfrak{F}'(Q_1)$ are the Schubert cycles $\{l \subseteq L \mid l \ni O\}$ for some fixed point $O \in L^3$. Note that a 2-plane in the $\mathfrak{F}$-family does not lie on any larger linear space in $X$, whereas any 2-plane in the $\mathfrak{F}'$-family lies on a unique 4-plane in $X$ (namely $\{l \subseteq \mathbb{P}^5 \mid l \ni O\}$).

*We henceforth assume* $n \geqq 8$. Zak's crucial observation now is

**Proposition 3.9.** *For any* $\frac{n}{4}$*-plane* $\Lambda \subseteq Q_1$ *corresponding to a point* $[\Lambda] \in \mathfrak{F}_x(Q_1)$, *there exists a point* $P \in S(C_1, C_2) - X$ *such that*

$$(Q_P \cap Q_1) = \Lambda .$$

Note that Theorem 3.1 follows at once. For in any event

$$\dim \mathfrak{F}_x(Q_1) = \frac{1}{2}\left(\frac{n}{4}\right)\left(\frac{n}{4} - 1\right),$$

and combining the Proposition with Lemma 3.4 yields the inequality

$$n/2 - 2 \geqq \frac{1}{2}\left(\frac{n}{4}\right)\left(\frac{n}{4} - 1\right),$$

i.e.

$$(n-4)\,(n-16) \leqq 0 .$$

Hence $n \leqq 16$!

For use in the proof of (3.9), and elsewhere, we record the following *elementary observation*:

(3.10) *If* $M$, $N \subseteq X$ *are two linear spaces of dimension* $a$, *and if*

$$\dim(M \cap N) = a - 1,$$

*then for any* $P \in \operatorname{Span}(M, N) - X$, *one has*

$M, N \subseteq Q_P$.

Indeed, if $m \in M$ is any point, the line $\overline{mP}$ must meet $N$. In practice, (3.10) is used either to construct secant quadrics containing a given plane $N$, or to show that the span of two linear spaces must lie in $X$.

*Proof of Proposition* 3.9. Fix a point $P \in S(C_1, C_2) - X$, and set:

$$\Lambda_i = Q_P \cap Q_i \quad (i=1,2)$$
$$L = \Lambda_1 \cap \Lambda$$
$$\alpha = \dim L.$$

Arguing by induction on $\alpha$, we may assume that $\alpha < n/4$, and it suffices to construct a point $P' \in S(C_1, C_2) - X$ such that

$$\dim(\Lambda \cap Q_{P'}) > \alpha.$$

Fix any point

$$x_0 \in \Lambda - L.$$

We assert:

(3.11) *There exists a point* $y_0 \in \Lambda_2$ *such that*:

a) $\operatorname{Span}(y_0, L) \subseteq X$

*and*

b) $\overline{x_0 y_0} \nsubseteq X$.

(See Figure 6.) Granting (3.11), consider any point $P' \in \overline{x_0 y_0} - X$. Then clearly $P' \in S(C_1, C_2)$. Furthermore, applying (3.10) with $M = \operatorname{Span}(y_0, L)$ and $N = \operatorname{Span}(x_0, L)$, one has $N \subseteq Q_{P'}$. Thus $\dim(Q_{P'} \cap \Lambda) > \alpha$, so it remains only to prove (3.11).

To this end, observe to begin with that

(3.12) $A(x_0) =_{\text{def}} \{y \in Q_P \mid \overline{x_0 y} \subseteq X\}$

*is a* **linear** *subspace of* $Q_P$.

Indeed, given $y, y' \in A(x_0)$, suppose there were to exist a point $P' \in \overline{y, y'}$, $P' \notin Q_P$.

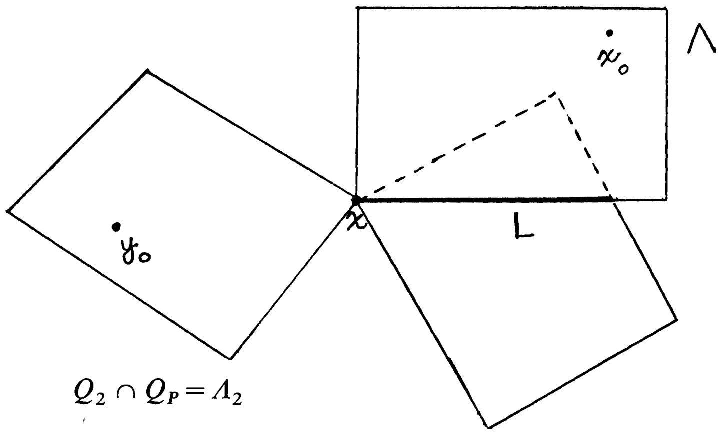



Figure 6

Then $P' \notin X$, since in any event $P' \in \Sigma_P$. But each of the lines $\overline{x_0 y}$ and $\overline{x_0 y'}$ lie in $X$, and hence by (3.10) they would lie also in $Q_{P'}$. But $Q_{P'} = Q_P$ (Theorem 2.1 c), and so $x_0 \notin Q_P$, a contradiction.

Note next that since $\Lambda$ is in the family $\mathfrak{F}(Q_1)$, and since $n/4 \equiv 0 \pmod 2$, we have (cf. § 1 d)

$$\alpha = \dim(\Lambda \cap \Lambda_1) \equiv 0 \pmod 2,$$

whence

$$n/4 - \alpha \geqq 2.$$

But this implies:

(3.13) *There exists a linear space* $L_2 \subseteq \Lambda_2$ *of dimension* $\geqq 2$ *such that*

$$\mathrm{Span}(L_2, L) \subseteq Q_P.$$

In fact, if $Q$ is any smooth quadric of dimension $2k$, and if $L$ and $\Lambda_2$ are linear spaces on $Q$ of dimensions $\alpha \leqq k-2$ and $k$ respectively, with $L \cap \Lambda_2 = \{x\}$, then there exists a plane $L_2 \subseteq \Lambda$ of dimension $\geqq 2$ such that $\mathrm{Span}(L_2, L) \subseteq Q$. (Hint: if $\bar{Q}$ is the intersection of $Q$ with $\mathrm{Span}(L, \Lambda_2)$, show that $\dim(\mathrm{Sing}\,\bar{Q} \cap \Lambda_2) \geqq 2$.)

Similarly:

(3.14) *There exists an* $\left(\frac{n}{4} - 1\right)$*-plane* $L_1 \subseteq \Lambda_1$ *such that*

$$\mathrm{Span}(L_1, x_0) \subseteq Q_1 \subseteq X.$$

But now (3.11) follows. For since $Q_P$ is smooth, the linear space $A(x_0)$ has dimension $\leqq n/4$. On the other hand, $\dim(A(x_0) \cap \Lambda_1) \geqq n/4 - 1$ by (3.14), so $\dim(A(x_0) \cap \Lambda_2) \leqq 1$. In particular $L_2 \nsubseteq A(x_0)$, so we simply choose any point

$$y_0 \in L_2 - A(x_0). \quad \square$$

This completes the proof of Proposition 3.9, and hence of Theorem 3.1.

We conclude this section with a result needed later on linear spaces in the family $\mathfrak{F}'(Q_1)$.

**Proposition 3.15.** *Assume $n \geqq 8$, and let $l \subseteq X$ be a line through $x$ not lying in $Q_1$. Then there exists an $n/4$-plane $\Lambda' \subseteq Q_1$, with $[\Lambda'] \in \mathfrak{F}'(Q_1)$, such that the join*

$$S(\Lambda', l) \subseteq X.$$

*Proof.* Choose any $P \in S(C_1, l) - X$, and set

$$\Lambda = Q_P \cap Q_1 .$$

Then as in (3.14) there is an $(n/4-1)$-plane $H \subseteq \Lambda$ such that $\bar{\Lambda} =_{\text{def}} S(H, l) \subseteq Q_P$. Then on $Q_1$ there exists an $n/4$-plane $\Lambda'$ such that $\Lambda \cap \Lambda' = H$. Note that $[\Lambda'] \in \mathfrak{F}'(Q_1)$ (since it meets $\Lambda$ in codimension one). But then

$$S(\Lambda', \bar{\Lambda}) \subseteq X;$$

for if there were a point $R \in S(\Lambda', \bar{\Lambda}) - X$, then one would have $\Lambda' = Q_R \cap Q_1$ by (3.10), which is impossible since $\Lambda' \in \mathfrak{F}'(Q_1)$. $\square$

## §4 The classification of Severi varieties

Let

$$X \subseteq \mathbb{P}^{3/2n+2}$$

be a Severi variety of one of the four possible dimensions $n = 2$, 4, 8 or 16, and fix a secant quadric

$$Q = Q_P$$

spanning the $\left(\dfrac{n}{2}+1\right)$-plane $\Sigma = \Sigma_P$. Following one of the classical approaches to the case $n=2$, Zak considers the projection of $X$ from $\Sigma$ to a complementary $\mathbb{P}^n$. More formally, let $\tilde{X}$ denote the blowing-up of $X$ along $Q$. Then $\tilde{X}$ maps to $\mathbb{P}^n$

in a natural way, and one has the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f=bl_Q} & X \subseteq \mathbb{P}^{3/2n+2} \\ {\scriptstyle \pi}\big\downarrow & & \\ \mathbb{P}^n & & \end{array} \tag{4.1}$$

The main point in the final stage of Zak's classification is now to prove the following

**Theorem 4.2.** *The projection $\pi$ realizes $\tilde{X}$ as the blowing-up of $\mathbb{P}^n$ along a copy of the variety $Y \subseteq \mathbb{P}^n$ listed in* § 1 c.

It is then a small matter to check that $X$ is, in fact, one of the four standard examples.

We assume henceforth that $n \geqq 4$, the case $n=2$ being at this point clear. As the proof of Theorem 4.2 is rather long, we begin by explaining briefly the central geometric picture. Let $H = T_P \operatorname{Sec}(X)$. By (2.7) and (2.9) the projection $\pi_\Sigma$ is an isomorphism on $X - H$; thus the real issue is to understand $\pi_\Sigma$ on $X \cap H$. To this end, we choose $x^* \in X$, $x^* \notin H$, and we think of $\pi_\Sigma$ as mapping onto $T_{x^*}X = \mathbb{P}^n$. Setting $Z = T_{x^*}X \cap X \cap H$, the basic fact is this:

*For each $y \in Z$, there is a unique $n/4$-plane $\Lambda_y \subseteq Q$—with $[\Lambda_y] \in \mathfrak{F}'(Q)$* [cf. (3.8)] *if $n \geqq 8$—such that the spans*

$$M_y = S(\Lambda_y, y)$$

*lie in $X$. As $y$ varies over $Z$, these sweep out the exceptional divisor of $\pi$.*

Along the way, it will emerge that $Z$ is isomorphic to the appropriate $Y \subseteq \mathbb{P}^n$ of § 1.

Turning now to details, for $x \in X$ let

$$Z_x^* = T_x X \cap X.$$

Observe to begin with that

$$Z_x^* \text{ is a cone with vertex } x. \tag{4.3}$$

Indeed, it suffices to show that $X$ is cut out by quadrics, at least set-theoretically. But this is a consequence of the fact that $\operatorname{Sec}(X)$ is a cubic hypersurface singular exactly along $X$.

Denote by $Z_x$ the base of the cone $Z_x^*$: we may realize $Z_x$ as the intersection of $Z_x^* \subseteq T_x X$ with a hyperplane $H$ not passing through $x$.

**Lemma 4.4.** $Z_x^* = \bigcup_{Q_P \ni x} (C_x Q_P)$,

*where* $C_x Q_P = T_x Q_P \cap Q_P$, *the union taken over all secant quadrics* $Q_P$ *passing through* $x$.

*Proof.* A simple argument shows that $Z_x^*$ is not a linear space, and it follows that for almost every point $z \in Z_x$ there exists a point $z' \in Z_x$ such that $\overline{z, z'} \nsubseteq Z_x$. If $P$ is a point in the 2-plane $\overline{x, z, z'}$, with $P \notin X$, then each of the lines $\overline{xz}$ and $\overline{xz'}$ lies $C_x(Q_P)$. In short,

$$\bigcup_{Q_P \ni x} C_x(Q_P)$$

contains a dense open subset of $Z_x^*$. But $\bigcup C_x(Q_P)$ is Zariski-closed in $Z_x^*$, so the two coincide. □

We now describe the varieties $Z_x$ explicitly in the cases $n=4$ and $n=8$.

**Proposition 4.5.** *If* $n=4$, *then* $Z_x$ *consists of two skew lines in* $\mathbb{P}^3 = T_x X \cap H$ *for every* $x \in X$.

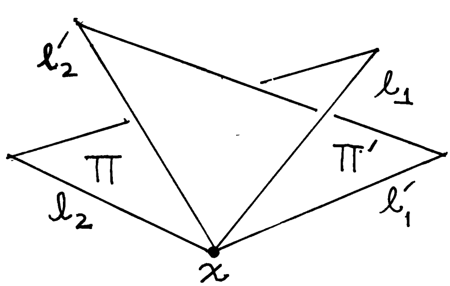
*Proof.* As in § 3, fix any two secant quadrics $Q_i$ $(i=1, 2)$ through $x$, meeting only at $x$, and denote by $C_i$ the corresponding cones. Thus each $C_i$ consists of two lines through $x$, and we observe that by Theorem 2.1 (b) the 2-plane spanned by $C_i$ cannot be contained in $X$.

Consider now a line $l_1$ of $C_1$ and $l_2$ of $C_2$. If the join $S(l_1, l_2) \nsubseteq X$, then taking $P \in S(l_1, l_2) - X$ gives a secant quadric $Q_P$ meeting each $Q_1$ in dimension $> 0$. On the other hand, according to (3.5) the secant quadrics through $x$ meeting each $Q_i$ in positive dimension are parametrized by some subset of a smooth quadric of dimension $n/2 - 2$, i. e. in the case at hand by two points. Hence of the four possible joins

$$S(l_1, l_2) \quad (l_1 \subseteq C_1, l_2 \subseteq C_2)$$

two at least must lie in $X$. However neither $S(l_1, C_2)$ nor $S(l_2, C_1)$ can be contained in $X$: for otherwise one could use (3.10) to construct a secant quadric $Q_P$ containing the 2-plane $S(l_1, l_2)$, which is impossible since $Q_P$ is smooth.

The upshot is that the components of $C_1$ and $C_2$ pair up in such a way that the joins of the corresponding lines lie in $X$:



One thus obtains a pair of 2-planes in $X$—call them $\Pi$ and $\Pi'$—meeting (exactly) at $x$. It remains to show that $\Pi \cup \Pi' = Z_x^*$.

To this end fix a line $l_1 \subseteq C_1$—say with $l_1 \subseteq \Pi$—and a line $n' \subseteq \Pi'$. We assert that the span $S(l, n') \nsubseteq X$; for otherwise one obtains as above a contradiction by applying (3.10) to the planes $\Pi'$ and $S(l_1, n')$. Hence taking $P \in S(l_1, n') - X$ we obtain a secant quadric with $C_x(Q_P) = l \cup n'$. Now fixing $n'$ and letting $l_1$ vary in $\Pi$ it follows that for any lines $n \subseteq \Pi$ and $n' \subseteq \Pi'$ there exists a secant quadric $Q_P$ through $x$ such that $n \cup n' = C_x(Q_P)$. One thus obtains an irreducible two-dimensional family of secant quadrics $Q_P$ through $x$ with

$$C_x(Q_P) \subseteq \Pi \cup \Pi'. \tag{$*$}$$

But the family of all secant quadrics through $x$ is itself irreducible of dimension two (2.17), so $(*)$ holds for every $Q_P \ni x$, and hence $Z_x^* \subseteq \Pi \cup \Pi'$ by (4.4). $\square$

**Remark.** As Zak points out, this Proposition can already be used to give a quick proof of the classification theorem in the case $n=4$. Keeping notation as in the proof, the idea in brief is as follows. Since the proposition holds for every $x \in X$, one must obtain for each $x \in \Pi$ a second 2-plane $\Pi'(x) \subseteq X$ through $x$. One checks that the $\Pi'(x)$ are disjoint, and hence are the fibres of a map $X \to \Pi$. One defines a morphism $X \to \Pi'$ similarly. The resulting map

$$X \to \Pi \times \Pi'$$

is the desired isomorphism.

**Proposition 4.6.** *When $n=8$, $Z_x$ is the Segre variety*

$$\mathbb{P}^1 \times \mathbb{P}^3 \subseteq \mathbb{P}^7 = T_x X \cap H$$

*for any $x \in X$.*

*Sketch of proof.* The argument is similar to the one just completed. We indicate the main steps.

We start with an elementary construction in projective geometry. Consider two disjoint $\mathbb{P}^3$'s in $\mathbb{P}^7$ and a smooth quadric $B_i$ $(i=1, 2)$ in each. Speaking of the two rulings of each $B_i$ as "horizontal" and "vertical" lines, we suppose given an isomorphism between a vertical line in $B_1$ and a vertical line in $B_2$. Such an isomorphism arises, for example, if one has a third smooth quadric $B$ meeting each $B_i$ in a vertical line. (Figure 7.) The isomorphism of vertical lines determines a natural bijective correspondence between the horizontal lines of $B_1$ and those of $B_2$. We may then consider the union of the $\infty^1$ three-planes spanned by corresponding horizontal lines. The resulting variety is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7$.

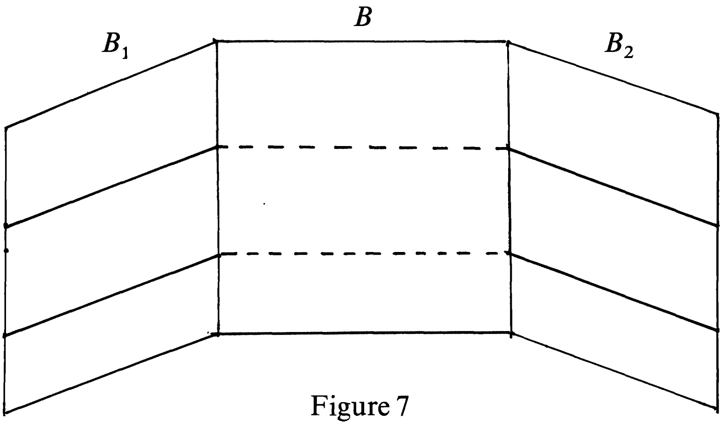



Figure 7

Now choose two secant quadrics $Q_1, Q_2$ through $x$, as in § 3, and a third quadric $Q = Q_P$ meeting each $Q_i$ in a $\mathbb{P}^2$. Put

$$C_i = C_x(Q_i) \quad (i = 1, 2), \quad C_x = C_x(Q),$$

and let $B_1$, $B_2$ and $B$ denote the bases of these cones, which we may think of as living in a $\mathbb{P}^7 \subseteq \mathbb{P}^8 = T_x X$. We are thus in the situation just described, and the first point to check is that the corresponding Segre variety lies in $Z_x$. To this end, take a horizontal line $l_1 \subseteq B_1$ and the corresponding line $l \subseteq B$. Denoting by $\Lambda_1$ and $\Lambda$ the resulting 2-planes through $x$, observe that

$$\Lambda_1 \in \mathfrak{F}'_x(Q_1)$$

and

$$\Lambda \in \mathfrak{F}'_x(Q). \qquad \text{(See Figure 8.)}$$

Then $M_1 =_{\text{def}} S(\Lambda_1, \Lambda) \subseteq X$, for otherwise (3.10) would give a point $P' \in \mathrm{Sec}(X) - X$ with $\Lambda_1 = Q_P \cap Q_1$, which in turn would mean that $\Lambda_1 \in \mathfrak{F}_x(Q_1)$. If $l_2 \subseteq B_2$ is the horizontal line meeting $l$, and $\Lambda_2 \subseteq C_2$ the corresponding 2-plane, then

$$M_2 =_{\text{def}} S(\Lambda_2, \Lambda) \subseteq X.$$

Another application of (3.10) now shows that $S(M_1, M_2) \subseteq X$. Thus $S(l_1, l_2) \subseteq Z_x$. Varying $l_1$ one thus obtains the desired copy of $\mathbb{P}^1 \times \mathbb{P}^3$ in $Z_x$. We leave it to the reader to verify that in fact $Z_x = \mathbb{P}^1 \times \mathbb{P}^3$. $\square$

We will treat the case $n = 16$ later, by an indirect argument.

We now start a more detailed analysis of the map $\pi$ in (4.1). To this end, fix any point $x^* \in X$, and fix $P \in \mathrm{Sec}(X) - X$ such that

$$x^* \notin H =_{\text{def}} T_P \,\mathrm{Sec}(X). \tag{$*$}$$

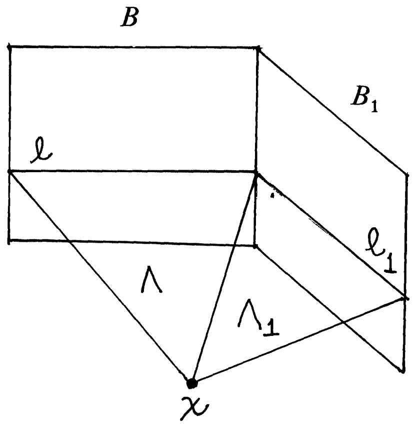



Figure 8

Putting $Q=Q_P$ and $\Sigma=\Sigma_P$, it follows from (2.7) and (2.9) that

$$\text{the map } \pi \text{ in (4.1) is an isomorphism over } \mathbb{P}^n - \mathbb{P}^n \cap H. \qquad (*)$$

In particular, $(*)$ implies that $T_{x^*}X \cap \Sigma = \phi$. We may then think of the projection $\pi_\Sigma$ from $\Sigma$ as a birational map

$$\pi_\Sigma : X - Q \to T_{x^*}X = \mathbb{P}^n.$$

We set $L = T_{x^*}X \cap H$. (See Figure 9.)

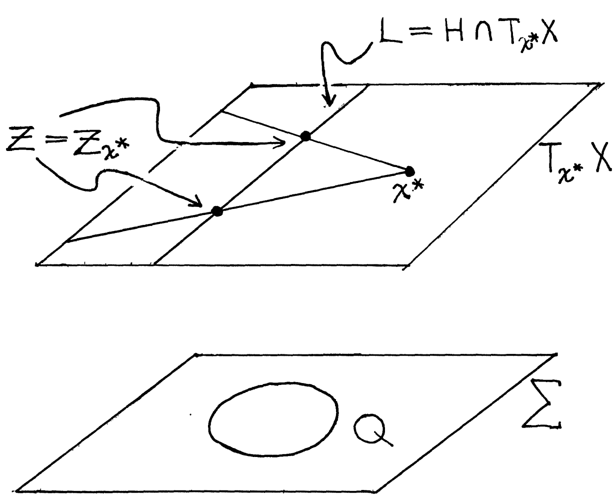



Figure 9

Let $S$ be the spinor variety parametrizing $n/4$-planes $\Lambda \in \mathfrak{F}'(Q)$ if $n=8$ or 16, or simply lines in $Q$ if $n=4$. Thus

$$\begin{aligned} &S=\mathbb{P}^1 \coprod \mathbb{P}^1 \quad \text{if} \quad n=4\\ &S=\mathbb{P}^3 \qquad\qquad \text{if} \quad n=8\\ &S=10\text{-dimensional spinor variety} \quad \text{if} \quad n=16. \end{aligned}$$

Leeting $Z=Z_{x^*}=Z^*_{x^*} \cap L$, we consider the basic correspondence

$$\begin{tikzcd}[column sep=small] & Z \times S \supseteq W=\{(y, [\Lambda]) \mid S(y, \Lambda) \subseteq X\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Z & & S \end{tikzcd} \tag{4.7}$$

**Proposition 4.8.** (i) *The map $p$ is bijective.*
(ii) *The fibres of $q$ are linear subspaces of $X$.*

*Proof.* (i) We start with the surjectivity of $p$: given $y \in Z$ we must exhibit $\Lambda \subseteq Q$, corresponding to $[\Lambda] \in S$, such that $S(y, \Lambda) \subseteq X$. Note to begin with that $\operatorname{Sec}(X) \cap H=T(X, Q)$ (cf. proof of Proposition 2.5), so $y \in T_x X$ for some $x \in Q$. By (4.3) applied at $x \in X$, the line $\overline{xy}$ then lies in $X$. When $n \geqq 8$, (3.15) gives the desired $\Lambda \subseteq Q$. When $n=4$, the existence of $\Lambda$ follows from Proposition 4.5.

To show that $p$ is one-to-one, suppose to the contrary that for some $y \in Z$ there are distinct planes $\Lambda_1, \Lambda_2 \subseteq Q$—corresponding to points in $S$—such that $S(\Lambda_i, y) \subseteq X$ for $i=1, 2$. We consider first the case when $\Lambda_1$ and $\Lambda_2$ meet along a non-empty linear space $L$; this is automatic when $n=8$ or 16, since then $\dim(\Lambda_1 \cap \Lambda_2) \equiv 0 \pmod 2$. Choose points $a \in \Lambda_1$, $b \in \Lambda_2$, with $a, b \notin L$, such that $\overline{ab} \nsubseteq Q$. Then $\overline{ab} \nsubseteq X$ (Theorem 2.1), so there exists a point $R \in \overline{ab}-X$. Applying (3.10) to $\operatorname{Span}(L, a, y)$ and $\operatorname{Span}(L, b, y)$ one finds that $y \in Q_R$. But $Q_R=Q$, and $y \notin \Sigma$, a contradiction. The possibility that $n=4$, and that $\Lambda_1$, $\Lambda_2$ are two skew lines, may be handled by (4.5).

(ii) If $y_1, y_2 \in Z$ are distinct points such that $S(y_1, \Lambda) \subseteq X$ and $S(y_2, \Lambda) \subseteq X$, then $S(\overline{y_1 y_2}, \Lambda) \subseteq X$ by (3.10), i.e. $\overline{y_1 y_2} \subseteq q^{-1}([\Lambda])$. $\square$

In the cases $n=4$ and 8 we know already that $Z$ is smooth, and it follows that *$p$ is an isomorphism*. (When $n=4$, $q$ is also an isomorphism; when $n=8$, $q$ is the projection of $\mathbb{P}^1 \times \mathbb{P}^3$ onto its second factor.) In the last case, one has:

**Proposition 4.9.** *When $n=16$, $p$ and $q$ are isomorphisms.*

*Proof.* Note to begin with that the irreducible variety $Z$ is a section of $X$ by a linear space $L$ of codimension $26-15=11$ in $\mathbb{P}^{26}$. Hence $H^i(X, Z; \mathbb{Z})=0$ when $i \leqq 16-11=5$ by the Lefschetz theorem. Moreover $H^i(\mathbb{P}^{26}, X; \mathbb{Z})=0$ for $i \leqq 2$ by Barth and Larsen (cf. [FL, § 9]). Hence $H^2(Z; \mathbb{Z})=\mathbb{Z}$, and if $\mathcal{O}_Z(1)$ is the line bundle defining the embedding $Z \subseteq \mathbb{P}^{15} = L$, then $c_1(\mathcal{O}_Z(1))$ is a generator of $H^2(Z; \mathbb{Z}) = H^2(W; \mathbb{Z})$. It now follows that $q$ is finite: in fact, any surjective morphism $V \to W$ between irreducible projective varieties of dimension $\geqq 1$ must be finite if $H^2(V; \mathbb{Z}) = \mathbb{Z}$, and certainly im $q$ is not a point. On the other hand, an elementary argument using (4.4) and (3.5) shows that in any event $\dim Z = \dim W \geqq 3/4n - 2 = 10$. Thus $q$ is surjective and so by (4.8 ii) is an isomorphism. Identifying $S$ with $W$, $p: S \to Z \subseteq \mathbb{P}^{15}$ is a *one-to-one* map to $\mathbb{P}^{15}$ defined by a positive generator of $\mathrm{Pic}(S) = H^2(S, \mathbb{Z}) = \mathbb{Z}$. But the only such is the projectively normal embedding $S \hookrightarrow \mathbb{P}^{15}$ (1.7). □

*Proof of Theorem* 4.2. In each of the cases $n = 4$, 8 and 16, $p$ is an isomorphism and $Z \subseteq T_{x^*}X \cap H \subseteq T_{x^*}X = \mathbb{P}^n$ coincides with the appropriate variety $Y$ listed in § 1c. Given $y \in Z$, let $\Lambda_y$ denote the $n/4$-plane in $Q$ corresponding to the point $q(p^{-1}(y)) \in S$, so that

$$M_y =_{\mathrm{def}} S(y, \Lambda_y) \subseteq X.$$

Observe that for distinct $y, y' \in Z$, the planes $M_y$ and $M_{y'}$ can meet only in $Q$. A simple argument now suffices to verify

(*) Referring to diagram (4.1), the fundamental locus of $\pi$ is exactly the smooth subvariety $Z \subseteq \mathbb{P}^n = T_{x^*}X$ and $\pi^{-1}(Z)$ is the $\mathbb{P}^{n/4+1}$-bundle over $Z$ swept out by the planes $M_y$ as $y$ varies over $Z$.

But now the theorem follows from a result of Aeppli [A] to the effect that if $\pi: \tilde{X} \to P$ is a proper birational morphism between smooth varieties such that the fundamental locus $Z \subseteq P$ of $\pi$ is smooth, and also the exceptional set $F = \pi^{-1}(Z) \subseteq \tilde{X}$ is smooth, then $\pi$ is blowing up of $P$ along $Z$. □

Finally, to complete the proof of the classification theorem, it suffices to show that the rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{3/2n+2}$ arising from (4.1) is defined by the (necessarily complete) linear system of quadrics on $\mathbb{P}^n$ passing through $Y$. Since $\pi$ is the blowing up of $Y$, the map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{3/2n+2}$ is in any event defined by a linear system of hypersurfaces of degree $k$ passing $r$ times through $Y$. Hence it suffices to show that $k = 2$, i.e. that the transform of a line $l \subseteq \mathbb{P}^n$ disjoint from $Y$ is a conic curve in $\mathbb{P}^{3/2n+2}$. But this is clear, for given $x \in Q$ choose a smooth conic curve $C \subseteq X$ meeting $Q$ exactly at $x$, with $T_xX \cap C = \{x\}$. Under the projection from $\Sigma$, $C$ maps to the desired line $l \subseteq \mathbb{P}^n$, and we are done.

## References


[A] Aeppli, A., *Reguläre Modifikation komplexer Mannigfaltigkeiten regulär verzweigte Überlagerungen*, Comm. Math. Helv. **33** (1959), pp. 1—22.

[D] Demazure, M., *Surfaces de del Pezzo*, II in M. Demazure, H. Pinkham and B. Tessier, eds., *Séminaire sur les singularities des Surfaces*, Lect. Notes in Math., No. **777** (1980).

[FL] Fulton, W. and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, in Algebraic Geometry (Proceedings), Lect. Notes in Math., No. **862**, pp. 26—92.

[FR] Fujita, T. and J. Roberts, *Varieties with small secant varieties: the extremal case*, Am. J. Math. **103**, pp. 953—976.

[F] Fujita, T., *Projective threefolds with small secant varieties*, Sci. papers of the college of general education, Univ. of Tokyo, **32**, (1982), pp. 33—46.

[GH] Griffiths, P. and J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience, 1978.

[H] Hartshorne, R., *Varieties of small codimension in projective space*, Bull. A.M.S. **80** (1974), pp. 1017—1032.

[K] Kempf, G., *On the collapsing of homogeneous bundles*, Inven. Math. **37** (1976), pp. 229—239.

[R] Room, T., *A synthesis of the Clifford matrices and its generalization*, Am. J. Math. **74** (1952), pp. 967—984.

[S] Samelson, H., *Notes on Lie Algebras*, Van Nostrand Reinhold, 1969.

[Z1] Zak, F. L., *Projections of algebraic varieties*, Math. USSR Sbornik, **44** (1983), pp. 535—544.

[Z2] Zak, F. L., Letters to J. Roberts; 11 July 1980; 26 August 1980; 15 September 1982.

[Z3] Zak, F. L., Letters to R. Lazarsfeld; 12 November 1981; 19 November 1981; 30 November 1981; 10 December 1981.


## ADDENDUM by F. L. Zak

### Varieties of small codimension arising from group actions
(excerpts from [Z3])

Here follow excerpts from [Z3] in which I show how to use representation theory in the study of geometric properties of some important projective varieties of small codimension. In particular, I recover in the group-theoretic term all constructions in the preceding notes by Lazarsfeld and Van de Ven of the four standard Severi varieties. Of course, there are other methods to prove the results stated in §1.b and §1.c of the above notes (for example, 1.5 has been proved by C. Segre for $n=4$ and J. Semple for $n=8$, and a similar result for the variety $S$ is due to P. Heymans), but I think it worthwhile to have a unified approach to all the examples.

I am greatly indebted to R. Lazarsfeld who wrote me about his discovery of the Severi variety $E$. I am also grateful to V. L. Popov for useful discussion of representation theory. Finally, I wish to thank Prof. Fischer who suggested to include these excerpts in the present volume.

... Let $G$ be a linear algebraic group over an algebraically closed field $k$ acting on a vector space $V$, $\dim V=N+1$. We are interested in invariant subvarieties $K\subset V$ corresponding to nonsingular projective subvarieties in $\mathbb{P}^N$

$= \mathbb{P}(V)$. Thus $K$ must be a cone, and to guarantee the nonsingularity it is natural to assume that there exists an orbit $Gv$, $v \in K$ such that $\overline{K} = \overline{Gv} = Gv \cup 0$, where 0 is the origin in $V$.

In V. L. Popov's note "On Hilbert's theorem on invariants" (Dokl. AN SSSR 249:3 (1979), 551—555; Engl. transl.: Math. USSR Doklady) it is shown that in this case the stabilizer $H$ of the point $v$ contains a maximal unipotent subgroup of the group $G$ (cf. n° 4, cor. 2; in our case this simply reflects the fact that each parabolic subgroup contains a Borel subgroup). In particular, $H$ contains the unipotent radical of $G$, and without loss of generality we may assume that $G$ is *reductive*. Furthermore, since we are interested only in the projective variety corresponding to $Gv$, we may assume that $G$ is *semisimple*.

Fixing a Borel subgroup containing the above maximal unipotent subgroup, we can write $v = v_1 + \cdots + v_k$, where $bv_1 = \Lambda_i(b)v_i$ for $b \in B$, $\Lambda_i$ is the highest weight of the restriction of the action of $G$ on an invariant subspace $V_i \subset V$, and $v_i$ is (by definition) the highest weight vector (primitive element) in $V_i$. Clearly $\overline{Gv} \subset \bigoplus_{i=1}^{k} V_i$, and without loss of generality we may assume that $V = \bigoplus_{i=1}^{k} V_i$. Since $\overline{Gv}$ contains only two orbits, it follows that all $\Lambda_i$ lie in an affine hyperplane. Therefore we may assume that $k=1$ and $Gv$ is the orbit of the highest weight vector of an *irreducible* representation of a semisimple group $G$. In particular, it easily follows that the smooth projective variety $X = Gv/k^* \subset \mathbb{P}^N$ is rational.

Let $\Lambda$ (resp. $M$) be the highest (resp. lowest) weight of this representation, let $v_\Lambda$ (resp. $v_M$) be the corresponding weight vector, and let $\mathfrak{g}$ be the Lie algebra of $G$. Let $TX$ (resp. $SX$) be the variety of tangents (resp. the secant variety) of the projective variety $X$. Clearly $TX$ corresponds to the affine cone $G\mathfrak{g}v_\Lambda \subset V$. Furthermore, if $P_\Lambda \subset G$ is the (parabolic) stabilizer of the line $kv_\Lambda$ (or, which is the same, of the point $x_\Lambda \in X$ corresponding to $v_\Lambda$), then it is easy to verify that the stabilizer of $\mathfrak{g}v_\Lambda$ in $G$ coincides with $P_\Lambda$ (to do this, one can use the theorem on tangencies). Similarly, let $N_\Lambda \subset V^*$ be the subspace of points corresponding to hyperplanes passing through $\mathfrak{g}v_\Lambda$ (the "normal" subspace). Then the projective variety corresponding to the cone $GN_\Lambda$ is just the dual variety $X^* \subset (\mathbb{P}^N)^*$ (here $G$ acts on $V^*$ via the contragredient representation). The stabilizer of $N_\Lambda$ clearly coincides with the stabilizer of $\mathfrak{g}v_\Lambda$. Now, although the action of $P_\Lambda$ on $\mathfrak{g}v_\Lambda$ and $N_\Lambda$ is not completely reducible and one cannot directly apply Kempf's theorem 0 on p. 229 in Inv. Math. 37:3 (1976), it seems that the same methods show that $TX$ and $X^*$ (as well as $X$) are (arithmetically) Cohen-Macaulay varieties (it is also clear that $TX$ and $X^*$ are unirational; probably, they are even rational—do you know of anybody has considered these normality and rationality questions?).

If $X$ is a complete intersection, then $X^*$ is the image of a (smooth) projective bundle under a *finite* birational map. If $X^*$ is normal, that means that $X^*$ is a nonsingular hypersurface in $(\mathbb{P}^N)^*$ which at the same time is a projective bundle over $X$ with fibers $\mathbb{P}^{N-n-1} \subset (\mathbb{P}^N)^*$, where $n = \dim X$. From this it clearly follows that $N-n-1 \leqq 0$, i.e. either $X = \mathbb{P}^N$ or $X$ is a hypersurface in $\mathbb{P}^N$. In the last case it is immediate that $X$ is in fact a quadric. Thus there are three possibilities: a) $G$ acts transitively on $V^* = V \setminus 0$ and $X = \mathbb{P}^N$, so that the only invariants are constants; b) $X$ is a nonsingular quadric, and the ring of invariants coincides with $k[Q]$, where $Q = 0$ is the equation of $X$; c) $X$ is not a complete intersection (of course, all these possibilities actually occur). Certainly, there are other, less elegant but more "reliable" methods of proving that $X$ usually is not a complete intersection and $X^*$ and $TX$ are (arithmetically) Cohen-Macaulay varieties (in our examples they will be determinantal varieties).

Next we notice that, although the orbit $Gv_\Lambda$ does not necessarily contain all weight vectors, $v_M \in Gv_\Lambda$ because $M = w_0(\Lambda)$, where $w_0$ is the involution in the Weyl group $W$ of the group $G$ which transforms the positive Weyl chamber into the negative one (cf. Bourbaki, Groupes et algèbres de Lie, ch. VI, § 1, n° 6, cor. 3), so that we may assume that $w_0$ lies in the normalizer of the maximal torus of $G$. Let $x_M$ be the point in $X$ corresponding to $v_M$, and let $P_M$ be the stabilizer of $x_M$. Consider the orbit of the point $x_\Lambda \times x_M \in X \times X$ under the natural action of $G$ on $X \times X$. Then the stabilizer of $x_\Lambda \times x_M$ is $P_\Lambda \cap P_M$, and

$$\begin{aligned}\dim G(x_\Lambda + x_M) &= \dim G - \dim(P_\Lambda \cap P_M) = \\ &= (\dim G - \dim P_\Lambda) + \big(\dim P_\Lambda - \dim(P_\Lambda \cap P_M)\big) = \\ &= \dim X + \big(\dim(P_\Lambda \cdot P_M) - \dim P_M\big) = 2 \dim X\end{aligned}$$

since $\dim(P_\Lambda \circ P_M) = \dim G$ because $P_\Lambda$ contains the "upper" and $P_M$ contains the "lower" Borel subgroup of $G$ (cf. A. Borel, Linear algebraic groups, ch. IV, *theorem* 14.1). Thus the orbit $G(x_\Lambda \times x_M)$ is dense in $X \times X$ and $SX = \overline{G \cdot [x_\Lambda, x_M]}$, where $[x_\Lambda, x_M]$ denotes the secant joining $x_\Lambda$ with $x_M$ and the bar denotes projective closure.

Let $U$ be the plane in $V$ generated by $v_\Lambda$ and $v_M$, let $\mathfrak{N}$ be the cone of nullforms in $V$ (recall that $\mathfrak{N}$ is the subset of $V$ defined by vanishing of all non-constant $G$-invariant polynomials), and let $Z \subset \mathbb{P}^N$ be the projective variety corresponding to $\mathfrak{N}$. Clearly $X \subset Z$. Consider the action of the maximal torus $T \subset G$ on $U$. Let $v = \alpha v_\Lambda + \beta v_M \in U$, $\alpha, \beta \neq 0$. Then $Tv \subset U$ and there are two possibilities: either $\Lambda + M \neq 0$ and $\dim(Tv) = 2$ or $\Lambda + M = 0$ and $\dim(Tv) = 1$. In the first case $\overline{Gv} \supset \overline{Tv} \ni 0$, and therefore $\overline{GU} = \overline{Gv} \subset \mathfrak{N}$ and $SX \subset Z$. In the second case $\overline{GU} = \overline{G(kv)}$; examples show that in this case $SX$ may lie or not lie in $Z$. Anyhow, this consideration shows that if $X$ cannot be isomorphically projected to $\mathbb{P}^{N-1}$, then either $\mathfrak{N} = V$, i.e. $I_G[V] = k$, where $I_G[V]$ is the algebra of polynomials on $V$ invariant with respect to the action of $G$, or $\overline{Gv} \not\subset \mathfrak{N}$ is a hypersurface in $V$ and, since $\dim \mathfrak{N} \geqq \dim \overline{Gv}$, $I_G[V] = k[F]$, where $F$ generates the ideal of $\mathfrak{N}$ (it should be noted that all representations with these properties have been classified). The involution $w_0$ for simple Lie groups is given in the tables at the end of the Bourbaki book. In particular, $w_0 = -1$ (and hence $\Lambda + M = 0$ for *all* representations) iff $G$ has one of the following types:

$$A_1,\ B_r (r \geqq 2),\ C_r (r \geqq 2),\ D_{2l} (l \geqq 2),\ E_7,\ E_8,\ F_4,\ G_2.$$

Now we are sufficiently equipped to consider concrete examples. Suppose first that $G$ is not simple. Without loss of generality we may assume that $G = G_1 \times \cdots \times G_d$, $V = V_1 \otimes \cdots \otimes V_d$ and $G \to \mathrm{Aut}(V)$ is a tensor product of irreducible representations $G_i \to \mathrm{Aut}(V_i)$ with highest weight $\Lambda_i$ and primitive element $v_i$. It is clear that $v = v_1 \otimes \cdots \otimes v_d$ is a primitive element of our representation and the corresponding highest weight is $\Lambda_1 + \cdots + \Lambda_d$. Let $X_i$ be the projective variety of dimension $n_i$ in $\mathbb{P}^{N_i} = \mathbb{P}(V_i)$ corresponding to $G_i v_i$. Then it is easy to see that the $n$-dimensional projective variety $X$ in $\mathbb{P}^N = \mathbb{P}(V)$ corresponding to $Gv$ is the Segre embedding of $X_1 \times \cdots \times X_d$.

In what follows I'll try to describe varieties corresponding to orbits of highest weight vectors.

Since we are primarily (although not exclusively!) interested in varieties of "small" codimension, in this letter I'll focus attention on the case when $N \leqq 2n$. Suppose first that $d > 1$, so that $n = n_1 + \cdots + n_d$, $N = (N_1 + 1) \cdot \ldots \cdot (N_d + 1) - 1$. Elementary computations show that $d \leqq 2$, and if $d = 2$, then either $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ or $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ (these varieties correspond to the standard representations of the groups $SL_2 \times SL_{n+1}$ and $SL_3 \times SL_3$ respectively). In the first case $\mathfrak{N} = V$ and $SX = \mathbb{P}^{2n+1}$. In the second case the above theory yields $SX \subset Z$, and it is easy to see that $I_G[V] = k[\det]$. Thus $SX$ is a cubic hypersurface in $\mathbb{P}^8$ (corresponding to the set of nonzero matrices of rank $\leqq 2$), and Sing $SX = X$ ($X$ corresponds to the set of matrices of rank 1). Furthermore, from the above discussion it follows that $(SX)^* \cong X$, $X^* \cong SX$. It remains to consider representations of *simple* Lie groups. Since the stabilizer $P_\Lambda$ of the primitive element $x_\Lambda$ contains a Borel subgroup $B \subset G$,

$$n = \dim X = \dim G - \dim P_\Lambda \leqq \dim G - \dim B = \frac{1}{2}(\dim G - rkG).$$

Therefore $2n + rkG \leqq \dim G$, and if $N + 1 = \dim V \geqq \dim G$, then $N \geqq 2n + (rkG - 1)$, so that either $X$ is a quadric in $\mathbb{P}^2$ ($G = SL_2$ and the representation is the $2^d$ symmetric power of the standard representation of $SL_2$ in $k^2$) or $X$ does not satisfy the dimensionality condition. Thus from the dimensionality condition it follows that the maximal dimension of orbits in our representation is less than $\dim G$, and all such representations have been classified in the paper by Elašvili in Funkc. Anal. Priloz. 6:1 (1972), 51—62 (English transl.: Funct. Anal. Applic...). So we only need to consider (using the above theory) Elašvili's tables and to pick those varieties that present interest in our context, i.e. either have small codimension or have small secant varieties.

The results are as follows ($\varphi_i$ denotes the $i$-th fundamental weight in the notation of Bourbaki; we do not list those representations for which $X$ is a projective space or a quadric).

1) $G=SL_{r+1}(A_r)$.

a) $\Lambda=2\varphi_1$ (or $2\varphi_r$), $r=2$, $n=2$, $N=5$, $X=v_2(\mathbb{P}^2)$

(in this case $N=2n+1$, but $I$ include it for the sake of completeness). By the above, $SX\subset Z$, and it is clear that $V$ is identified with the space of symmetric $3\times 3$ matrices and $I_G[V]=k[\det]$. Thus $SX$ is a cubic hypersurface in $\mathbb{P}^5$ (corresponding to the punctured cone of nonzero symmetric matrices of rank $\leqq 2$), and $\operatorname{Sing} SX=X$ ($X$ corresponds to the punctured cone of symmetric matrices of rank 1). As above, we see that $(SX)^*\cong X$ and $X^*\cong SX$;

b$_1$) $\Lambda=\varphi_2$ (or $\varphi_{r-1}$), $r=4$, $n=6$, $N=9$, $X=G(4,1)$.

Here $\mathfrak{N}=V$, $SX=\mathbb{P}^9$, and there are only three orbits: 0, the punctured cone over $X$ (the set of (nonzero) decomposable 2-vectors) and the set of indecomposable 2-vectors (each of which has rank 4). From the above it follows that $X^*\cong X$;

b$_2$) $\Lambda=\varphi_2$ (or $\varphi_{r-1}$), $r=5$, $n=8$, $N=14$, $X=G(5,1)$.

In this case $\mathfrak{N}$ is the cone of bivectors of rank $\leqq 4$, and in $\mathfrak{N}$ there are 3 orbits: 0, the cone over $X$ (the set of (nonzero) decomposable 2-vectors), and the set of 2-vectors of rank 4. Furthermore, $I_G[V]=k[Pf]$, where $Pf$ denotes the Pfaffian, and therefore $SX$ is a cubic hypersurface in $\mathbb{P}^{14}$. As above, we see that

$$\operatorname{Sing} SX=X,\quad (SX)^*\cong X,\quad X^*\cong SX;$$

b$_3$) $\Lambda=\varphi_2$ (or $\varphi_{r-1}$), $r=6$, $n=10$, $N=20$, $X=G(6,1)$.

This example is similar, but less interesting. Here $\mathfrak{N}=V$ and there are only four orbits: 0, the set of bivectors of rank 2, the set of bivectors of rank 4, and the set of bivectors of rank 6. The cone corresponding to $SX$ consists of all bivectors of rank $\leqq 4$, $SX$ is an (arithmetically) Cohen-Macaulay variety, $\dim(SX)=17$, $\operatorname{Sing} SX=X$, $(SX)^*\cong X$, $X^*\cong SX$.

2) $G=\mathrm{Spin}_9(B_4)$, $\Lambda=\varphi_4$, $n=10$, $N=15$, $I_G[V]=k[Q]$, where $Q$ is a nonsingular quadratic form, $SX=\mathbb{P}^{15}$. It is easy to verify that in $\mathfrak{N}$ there are three orbits: 0, the punctured cone over $X$ (the set of "pure" spinors), and the set of "impure" spinors from $\mathfrak{N}$. Here $X$ is the same variety that arises in connection with the spinor representation of $\mathrm{Spin}_{10}$; it will be discussed in more detail below.

3) $G=\mathrm{Sp}_{2r}(C_r)$, $\Lambda=\varphi_2$. Consider $\mathrm{Sp}_{2r}$ as a subgroup of $SL_{2r}$ and restrict the standard representation of $SL_{2r}$ on the space of alternating $2r\times 2r$ matrices to $Sp_{2r}$. Then the resulting representation is a direct sum of the representation of $Sp_{2r}$ with highest weight $\Lambda=\varphi_2$ in a hyperplane of $\Lambda^2(k^{2r})$ and the trivial representation in the one-dimensional subspace of $\Lambda^2(k^{2r})$ generated by the standard skew-symmetric matrix $\begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}$. It is easy to verify that the orbit of the primitive element in this representation of $Sp_{2r}$ is the intersection of the hyperplane and the orbit of the primitive element in the representation of $SL_{2r}$ in $\Lambda^2(k^{2r})$. Furthermore, $I_G[V]$ is a free algebra generated by $(r-1)$ forms $F_2, \ldots, F_r$, $\deg F_i=i$ (the coefficients of the "characteristic Pfaffian polynomial"). The only case satisfying the dimensionality condition $\bigl(2(4r-5)\geqq 2r^2-r-2\bigr)$ is $r=3$ (for $r=2$ $X$ is a quadric in $\mathbb{P}^4$). In this case $n=7$, $N=13$, $X$ is a hyperplane section of the variety $G(5,1)$ considered in 1), $\mathrm{b}_2$), $SX$ is a hyperplane section of $SG(5,1)$, and $\dim SX=12$. If $D(F_2,F_3)=4F_2^3-27F_3^2$ is the discriminant of the Pfaffian characteristic polynomial and $Z_{ab}=(F_2,F_3)^{-1}(a,b)$, then $Z_{ab}$ is an orbit of dimension 12 provided $D(a,b)\neq 0$, $Z_{ab}$ consists of two orbits of dimensions 8 and 12 if $D(a,b)=0$ but $ab\neq 0$, and $Z_{00}$ consists of three orbits of dimensions 0, 8, and 12. Using this orbit decomposition, it is easy to describe the secant variety and the duals of $X$ and $SX$.

4) $G=\mathrm{Spin}_{2r}(D_r)$, $r=5$, $\Lambda=\varphi_5$ (or $\varphi_4$), $n=10$, $N=15$. In this case $I_G[V]=k$, $\mathfrak{N}=V$, and there are only three orbits: 0, the punctured cone over $X$ (the set of nonzero "pure" spinors), and the set of impure spinors. Clearly $SX=\mathbb{P}^{15}$ and $X^*\cong X$. This example will be considered in more detail below. It should be noted that "pure" spinors are in the same relation to linear subspaces on quadrics as decomposable polyvectors are to linear subspaces in projective spaces. Like the Grassmanians, the varieties of pure spinors are defined by quadratic equations.

5) $G=E_6$, $\Lambda=\varphi_1$ (or $\varphi_6$), $n=16$, $N=26$, $I_G[V]=k[F]$, where $F$ is a cubic form (the exact form of $F$ will be given in 6)), and in $\mathfrak{N}$ there are only three orbits: 0, the punctured cone over $X$ (the primitive idempotents; cf. 6)), and $\mathfrak{N}\setminus$(the cone over $X$). Thus $SX$ is a cubic hypersurface in $\mathbb{P}^{26}$, $X$ is a Severi variety, $\mathrm{Sing}(SX)=X$, $(SX)^*\cong X$ and $X^*\cong SX$. Later this example will be considered in more detail. It is interesting that defining equations of $X$ and $SX$ were written down by E. Cartan in ch. VIII, § 8, n° 5 of his thèse "Sur la structure des groupes de transformations finis et continus" published in 1894 (cf. also Oeuvres complètes, partie I, vol. 1, Paris 1952). However to treat the next case it seems necessary to assume the modern point of view (representations on the exceptional Jordan algebra; cf. 6)) since the connection between the simplest representations of $E_6$ and $F_4$ does not seem evident from Cartan's equations in op. cit., ch. VIII, § 8, n° 8 (it also seems that Cartan has found only one of the two invariants in the case of $F_4$).

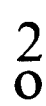
6) $G = F_4$, $\Lambda = \varphi_4$, $n = 15$, $N = 25$. Suppose that char $k \neq 2, 3$, and let $V$ be the 27-dimensional exceptional simple Jordan algebra of $3 \times 3$ Hermitian matrices over the Cayley numbers (this algebra is not associative; multiplication is given by the formula $v \circ w = \frac{1}{2}(vw + wv)$). The trace defines a quadratic form $Q$ on $V(Q(v) = Tr(v \circ v) = Tr(v^2))$, and the determinant defines a cubic form det on $V$. It is known that $E_6$ is the group of linear transformations of $V$ preserving det and $F_4$ is the subgroup of $E_6$ preserving the unit element $e$ (so that $F_4$ is the group of automorphisms of the *algebra* $V$ (this description is due to Chevalley, Schafer, Springer, and Jacobson)). Clearly this representation of $F_4$ splits into a direct sum of two representations: the trivial representation in the one-dimensional subspace generated by $e$ and the irreducible representation induced in the subspace $e^\perp \subset V$ (it should be noted that $F_4$ preserves not only det, but the quadratic form $Q$ as well; $e^\perp$ consists of all matrices from $V$ whose trace is equal to zero, $\dim e^\perp = 26$). Here $X$ and $SX$ are hyperplane section of the corresponding varieties in example 5). The ring of invariants is a free $k$-algebra generated by $F_2 = Q$ and $F_3 = \det$. Furthermore, $x^3 - Q(x)x - \det x \cdot e = 0$ for all $x$ in $e^\perp$, and we see that this example is very similar to 3). In particular, in the notation of 3), we see that $Z_{ab}$ is an orbit of dimension 24 provided $D(a, b) \neq 0$, $Z_{ab}$ consists of two orbits of dimensions 16 and 24 if $D(a, b) = 0$ but $ab \neq 0$, and $Z_{00}$ consists of three orbits of dimensions 0, 16, and 24. Using this orbit decomposition, it is easy to give a detailed description of $SX$, $(SX)^*$, and $X^*$.

The remaining groups do not provide interesting examples (in particular, $G_2$ gives a quadric in $\mathbb{P}^6$ and $E_7$ a 27-dimensional subvariety $X \subset \mathbb{P}^{55}$ with $SX = \mathbb{P}^{55}$).

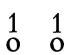
Summing up, we see that representations of algebraic groups yield four examples of Severi varieties, namely $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, $G(5,1) \subset \mathbb{P}^{14}$, and the variety $E^{16} \subset \mathbb{P}^{26}$ described in 5), and two examples of "Hartshorne varieties" (i.e. varieties "on the boundary" of Hartshorne's conjecture on complete intersections), namely $G(4,1) \subset \mathbb{P}^9$ and the variety $S^{10} \subset \mathbb{P}^{15}$ described in 2) and 4). All these varieties are rational homogeneous Cohen-Macaulay varieties defined by quadratic equations. The secant varieties of the above Severi varieties are rational cubic hypersurfaces (defined by some kind of "determinantal" equation), and if $X$ is a Severi variety of the above type, then $\operatorname{Sing} SX = X$, $(SX)^* \cong X$, and $X^* \cong SX$ (most of these properties for general Severi varieties were already indicated in my letter to Roberts; in view of this letter, $\deg SX = \operatorname{mult}_x SX + 1$ for all $x \in X$ and $\deg SX = 3$ (the minimal possible degree for a secant variety unless it coincides with the ambient space) simply means that all points of $X$ have multiplicity 2 on $SX$ (the minimal possible multiplicity since in general $Y \subset \operatorname{Sing} SY$ unless $SY$ is the ambient space)). For the Hartshorne varieties listed above $SX = \mathbb{P}^N$ and $X \cong X^*$.

It should also be noted that hyperplane section of the above Severi varieties also correspond to the orbits of highest weight vectors of representations of simple Lie groups. Namely, a section of 5) is 6), a section of 1), $b_2$) is 3), a section of $\mathbb{P}^2 \times \mathbb{P}^2$ corresponds to the adjoint representation of $SL_3$ in $k^8$, and a section of 1 a) corresponds to the adjoint representation of $SL_2$ in $k^3$.

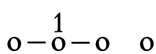
Furthermore, an $n$-dimensional Severi variety is ruled by $n/2$-dimensional quadrics, and an $n$-dimensional Hartshorne variety is ruled by $(n/2-1)$-dimensional projective spaces. In the above examples, these rulings not only come naturally from the dual varieties, but can be explicitly constructed with the help of representation theory. In fact, along with the parabolic subgroup $P_\Lambda$ containing the Borel subgroup $B$, there exists another parabolic subgroup containing $B$, namely the parabolic subgroup $P_{-M}$ corresponding to the highest weight vector of the contragredient representation (the corresponding highest weight is equal to $-M$). It is clear that $\dim P_\Lambda = \dim P_{-M}$; moreover, $G/P_\Lambda \cong G/P_{-M}$ (this situation for the "simplest" representations (in particular, for the 27-dimensional representation of $E_6$) was described by E. Cartan, op. cit. ch. VIII, § 8, n° 11). In the case of Severi and Hartshorne varieties $\Lambda + M \neq 0$, $P_\Lambda \neq P_{-M}$. Consider the orbit of the point $x_\Lambda$ under the action of $P_{-M}$. Let $H_{-M}$ be the semisimple quotient of $P_{-M}$ (the corresponding Dynkin diagram is obtained from the Dynkin diagram of $G$ by deleting the vertices at which $M \neq 0$). Then $P_{-M} x_\Lambda = H_{-M} x_\Lambda$ (the action of $H_{-M}$ corresponds to the restriction of $\Lambda$ to the maximal torus of $H_{-M}$). The representations of $H_{-M}$ for the above Severi and Hartshorne varieties can be pictured as follows:

$v_2(\mathbb{P}^2)$: 

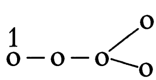
$(A_1, 2\varphi_1)$;

$\mathbb{P}^2 \times \mathbb{P}^2$: 

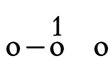
$(A_1 \oplus A_1, \varphi_1 \oplus \varphi_1)$;

$G(5,1)$: 

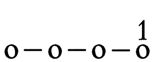
$(A_3 \oplus A_1, \varphi_2 \oplus 0)$;

$E$: 

$(D_5, \varphi_1)$

$G(4,1)$: 

$(A_2 \oplus A_1, \varphi_2 \oplus 0)$;

$S$: 

$(A_4, \varphi_4)$.

It is clear that in the first four examples (Severi varieties) the orbit of $x_\Lambda$ under the action of $H_{-M}$ is a nonsingular quadric $Q_\Lambda$, $\dim Q_\Lambda = \frac{1}{2} n$, and in the last two examples (Hartshorne varieties) the orbit of $x_\Lambda$ is a projective space $\mathbb{P}_\Lambda$, $\dim \mathbb{P}_\Lambda = \frac{1}{2} n - 1$ (where $n = \dim X$, $X = G x_\Lambda$). Furthermore, in the case of Severi varieties $P_\Lambda Q_\Lambda = P_\Lambda P_{-M} x_\Lambda = X$, so that two points of $X$ can always be joined by a quadric of the form $gQ_\Lambda (g \in G)$. The situation here is similar to the one described on p. 235 of the paper by Kempf. In particular, we get a resolution of singularities $G \times^{P_{-M}} \{Q_\Lambda\}$ of $SX$ which is a projective bundle of rank $\frac{1}{2}n+1$ over $X' \cong X$ (here $X'$ corresponds to the orbit of the highest weight vector of the contragredient representation, $X' \cong G/P_{-M}$). Furthermore, $P_\Lambda \{Q_\Lambda\} = S_{x_\Lambda} X$, its resolution of singularities is the projective bundle $P_\Lambda \times^{P_\Lambda \cap P_{-M}} \{Q_\Lambda\}$ of rank $\frac{1}{2}n+1$ (over $P_\Lambda / P_\Lambda \cap P_{-M}) = Q_{-M}$,

and $P_\Lambda \times^{P_\Lambda \cap P_{-M}} Q_\Lambda$

(which is a bundle over $Q_{-M}^{\frac{1}{2}n}$ with fiber $Q_\Lambda^{\frac{1}{2}n}$) maps birationally onto $X$.

For Hartshorne varieties $P_\Lambda \mathbb{P}_\Lambda \neq X$, and $P_\Lambda \mathbb{P}_\Lambda$ is the image of $P_\Lambda \times^{P_\Lambda \cap P_{-M}} \mathbb{P}_\Lambda$ which is a projective bundle of rank $\frac{1}{2}n-1$ over $\mathbb{P}^{\frac{1}{2}n-1}$. The map $P_\Lambda \times^{P_\Lambda \cap P_{-M}} \mathbb{P}_\Lambda \longrightarrow P_\Lambda \mathbb{P}_\Lambda$ is birational for $G(4,1)$ and has generic fiber $\mathbb{P}^1$ for $S$. This information about $P_\Lambda \mathbb{P}_\Lambda$ is important because it gives a description of $T_{X,x_\Lambda} \cap X$. As a matter of fact, one can show that $T_{X,x_\Lambda} \cap X$ is a cone with vertex $x_\Lambda$ over $\mathbb{P}^1 \times \mathbb{P}^2$ (if $X = G(4,1)$) or over $G(4,1)$ (if $X = S$). Similarly, for Severi varieties $T_{X,x_\Lambda} \cap X$ is a cone with vertex $x_\Lambda$ over $\emptyset$ (if $X = v_2(\mathbb{P}^2)$), over $\mathbb{P}^1 \sqcup \mathbb{P}^1$ (if $X = \mathbb{P}^2 \times \mathbb{P}^2$), over $\mathbb{P}^1 \times \mathbb{P}^3$ (if $X = G(5,1)$), or over $S$ (if $X = E$). This will be clarified below.

In the case of Hartshorne varieties there exists another ruling by quadrics of dimension $\frac{1}{2}n+1$. To obtain this ruling, consider the orbit of $x_\Lambda$ under the action of the (maximal) subgroup $P_e \subset G$ whose semisimple quotient corresponds to the following Dynkin diagram:

$G(4,1)$: $\circ - \overset{1}{\circ} - \circ$; $\quad$ $S$: $\circ - \circ \!\!<\!\! \begin{smallmatrix} \overset{1}{\circ} \\ \circ \end{smallmatrix}$.

It is clear that the orbit of $x_\Lambda$ under this action is a quadric $Q_e$ of dimension $\frac{1}{2}n+1$. According to Kempf, we obtain a birational map $G \times^{P_e} \{Q_e\} \longrightarrow \mathbb{P}^N$, where $G/P_e$ is $\mathbb{P}^4$ for $G(4,1)$ and $Q^8$ for $S$, $P_\Lambda Q_e = P_\Lambda P_e x_\Lambda = X$, and $P_\Lambda +^{P_\Lambda \cap P_e} Q_e$ maps birationally on to $X(P_\Lambda / P_\Lambda \cap P_e \cong \mathbb{P}^{\frac{1}{2}n-1})$. Thus any two points of the above Hartshorne varieties can be joined by an $\left(\frac{1}{2}n+1\right)$-dimensional quadric.

Finally, for reference purposes, we mention that a Severi variety (of the above type) of dimension $n$ is defined by $3(n+2)/2$ quadratic equations, $G(4,1)$ is defined by 5 quadratic equations, and $S$ is defined by 10 quadratic equations.

Let $H_A$ denote the semisimple quotient of the parabolic subgroup $P_A$. Then, restricting the representation of $G$ on $H_A$, we obtain a representation of $H_A$ on $V$. This representation is always reducible; in fact, the tangent space $\mathfrak{g}x_A$ is obviously stable with respect to the action of $H_A$; on the other hand, by definition, $H_A(kv_A)=kv_A$. Thus the $H_A$-module $V$ splits as follows: $V=kv_A \oplus V^t \oplus V^n$, where $kv_A \oplus V^t$ is the tangent space to $Gv_A$ at $v_A$ and $V^n$ is the "normal" space. Writing out the weights of the representation of $G$ and restricting them to the maximal torus of $H_A$, one can find the explicit form of these representations of $H_A$ (the situation in the above examples is so simple that one can avoid any computations). The results are as follows ($R$ denotes the representation of $H_A$ on $V$, 0 denotes the trivial representation, and $R(\psi)$ denotes the irreducible representation of $H_A$ corresponding to a weight $\psi$ of $H_A$; in the cases when $H_A$ is simple $\varphi_i$ denotes the $i$-th fundamental weight of $H_A$; if $H_A$ is not simple, it is a product of two simple groups, the fundamental weights of the first of them are denoted by $\varphi_i$ and of the second one by $\varphi_i'$; the first summand in the following formulae corresponds to $kv_A$, the second summand to $V^t$, and the third one to $V^n$).

(i) $X=v_2(\mathbb{P}^2)$, $H_A=SL_2$, $R=0\oplus R(\varphi_1)\oplus R(2\varphi_1)$;

(ii) $X=\mathbb{P}^2\times\mathbb{P}^2$, $H_A=SL_2\times SL_2$, $R=0\oplus[R(\varphi_1)\oplus R(\varphi_1')]\oplus R(\varphi_1+\varphi_1')$;

(iii) $X=G(4,1)$, $H_A=SL_2\times SL_3$, $R=0\oplus R(\varphi_1+\varphi_1')\oplus R(\varphi_2')$;

(iv) $X=G(5,1)$, $H_A=SL_2\times SL_4$, $R=0\oplus R(\varphi_1+\varphi_1')\oplus R(\varphi_2')$;

(v) $X=S$, $H_A=SL_5$, $R=0\oplus R(\varphi_3)\oplus R(\varphi_1)$;

(vi) $X=E$, $H_A=\mathrm{Spin}_{10}$, $R=0\oplus R(\varphi_5)\oplus R(\varphi_1)$.

It is clear that $Gv_A\cap(V^t\oplus V^n)$ is invariant with respect to $H_A$, and the stabilizer of an arbitrary point of $Gv_A\cap(V^t\oplus V^n)$ contains a maximal unipotent subgroup of $H_A$. Thus we only need to consider the orbits of sums of (some of) the highest weight vectors of the irreducible representations of $H_A$ making up $V^t$ and $V^n$ (with respect to suitable Borel subgroups). It immediately follows that $X\cap((V^t\oplus V^n)\setminus 0/k^*)$ is a singular hyperplane section of $X$, and its singular locus coincides with $X\cap(V^n\setminus 0/k^*)$. Moreover, it is clear that $v_M$ is a lowest weight vector of the representation of $H_A$ in $V^n$, so that $X\cap(V^n\setminus 0/k^*)$ corresponds to the orbit of a highest weight vector of the representation of $H_A$ in $V^n$; $X\cap(V^n\setminus 0/k^*)=P_{-A}x_M=w_0P_{-M}w_0x_M=w_0P_{-M}x_A$ and is either the $\frac{1}{2}n$-dimensional quadric $Q_M=w_0Q_A$ (in the case of Severi varieties) or the $\left(\frac{1}{2}n-1\right)$-dimensional projective space $\mathbb{P}_M=w_0\mathbb{P}_A$ (in the case of Hartshorne varieties).

Now we project $X$ to $\mathbb{P}^n$ from the projective subspace corresponding to $V^n$ (it has dimension $\frac{1}{2}n+1$ for Severi varieties and dimension $\frac{1}{2}n-1$ for Hartshorne varieties). This projection $\pi$ is a rational map onto $\mathbb{P}^n$, and it is convenient to regard $\mathbb{P}^n$ as the projective space corresponding to $kv_\Lambda \oplus V^t$ (i.e. the tangent space to $X$ at $x_\Lambda$). Let $\mathbb{P}^{n-1}$ be the projective subspace of $\mathbb{P}^n$ corresponding to $V^t$, and let $Y \subset \mathbb{P}^{n-1}$ be the subvariety corresponding to the cone $Gv_\Lambda \cap V^t$. It is clear that the hyperplanes passing through the center of our projection are mapped onto hyperplanes in $\mathbb{P}^n$; in particular, the hyperplane corresponding to $V^t \oplus V^n$ is mapped onto $\mathbb{P}^{n-1}$, and the hyperplane section $\tilde{Y} \subset X$ corresponding to $Gv_\Lambda \cap (V^t \oplus V^n)$ is mapped onto $Y$. Clearly the projection $Gv_\Lambda \to kv_\Lambda \oplus V^t$ is a map of $H_\Lambda$-spaces, and its fibers over points belonging to the same orbit are isomorphic to each other. In particular, the generic fiber of $\pi$ is a single point (to see this without computations one can use the purity of branch locus). More precisely, $\pi$ defines an isomorphism between $X \setminus \tilde{Y}$ and $\mathbb{P}^n \setminus Y$, and $\pi|_{\tilde{Y}}: \tilde{Y} \to Y$ is a rational fiber bundle whose fibers are projective spaces (for example, if $X=E$, then the fibers are $\mathbb{P}^5$'s (corresponding to the Dynkin subdiagram $A_5 \subset E_6$) intersecting the quadric $Q^8_M$ lying in the center of the projection $\pi$ along the various $\mathbb{P}^4 \subset Q^8_M$ belonging to one and the same family (these $\mathbb{P}^4$'s are parametrized by the spinor variety $Y=S$)). One can use Kempf's method to obtain a resolution of singularities of $\tilde{Y}$ ($\tilde{Y}$ may be viewed as a generalization of the Schubert variety of codimension 1).

From the above description of $\tilde{Y}$ (or the fact that $T_{X,x} \cap X$ is a cone over $Y$ with vertex $x$ for all $x \in X$) it follows that if $L \subset \mathbb{P}^N$ is a hyperplane which does not contain the center of our projection $\pi$, then $\pi$ maps the hyperplane section $X_L = X \cap L$ onto a hypersurface in $\mathbb{P}^n$ passing through $Y$. Suppose that the birational map $\sigma = (\pi|_X)^{-1}: \mathbb{P}^n \to X \hookrightarrow \mathbb{P}^N$ is given by a linear system $\mathscr{L}$ without fixed components (the members of $\mathscr{L}$ are the preimages of the hyperplanes in $\mathbb{P}^N$ with respect to the map $\sigma$). As we have already seen, the fundamental subset of $\mathscr{L}$ coincides with $Y$. Furthermore, $\mathscr{L}$ is a *complete* linear system of hypersurfaces of a fixed degree in $\mathbb{P}^n$ passing through $Y$, since otherwise $X$ wouldn't be linearly normal. Now it immediately follows that $\deg \mathscr{L} = 2$, i.e. $X$ is defined by the linear system of quadrics in $\mathbb{P}^n$ passing through $Y$. In other words, $X$ is obtained by projecting $v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{1}{2}n(n+3)}$ from the smallest linear subspace containing $Y$ (recall that $Y$ is defined by quadratic equations). This generalizes the results of Segre, Todd, and Semple cited in my letter to Roberts.

It should be noted that $Y$ is obtained from $\mathbb{P}^{\dim Y}$ by a similar procedure; this is clear except in the case of $G(r,1)$, where $Y = \mathbb{P}^1 \times \mathbb{P}^{r-2}$. Proceeding as above, we see that $Y$ is the variety corresponding to the orbit of a highest weight vector of the representation $R(\varphi_1 + \varphi_1')$ of the group $SL_2 \times SL_{r-1}$, $H_\Lambda = SL_{r-2}$, $V^t = kv \oplus V^{t'}$, where $v \in V^t$, $V^{t'}$ and $V^n$ are isomorphic to the standard representation of $SL_{r-2}$ in $k^{r-2}$, and the singular locus of the hyperplane section corresponding to $V^t \oplus V^n$ is $\mathbb{P}^{r-3} = \mathbb{P}(V^n)$. Projecting $Y$ to $\mathbb{P}^{r-1}$ from this $\mathbb{P}^{r-3}$

and reasoning as above, we see that $Y$ is the image of $\mathbb{P}^{r-1}$ under the rational map defined by the complete linear system of quadrics in $\mathbb{P}^{r-1}$ passing through $Y' \subset \mathbb{P}^{r-2}$, where $Y'$ is a union of the point corresponding to $kv$ and the $(r-3)$-dimensional projective space corresponding to $V^{t'}$ (the point does not lie in the projective space).

This is the method of projecting sketched in my letter to Roberts. But of course one can reverse this argument as suggested in your letter. Namely, start with the projective space $T_{X,x_A}$ and use the following formulae ($S^2$ denotes the second symmetric power):

(i) $H_A = SL_2 : S^2(0 \oplus R(\varphi_1)) \cong 0 \oplus R(\varphi_1)^2 \oplus R(2\varphi_1)^3$;

(ii) $H_A = SL_2 \times SL_2 : S^2(0 \oplus [R(\varphi_1) \oplus R(\varphi_1')]) \cong$
$0 \oplus [R(\varphi_1) \oplus R(\varphi_1')]^4 \oplus R(\varphi_1 + \varphi_1') \oplus [R(2\varphi_1) \oplus R(2\varphi_1')]^6$;

(iii) $H_A = SL_2 \times SL_3 : S^2(0 \oplus R(\varphi_1 + \varphi_1')) \cong$
$0 \oplus R(\varphi_1 + \varphi_1')^6 \oplus R(\varphi_2')^3 \oplus R(2\varphi_1 + 2\varphi_1')^{18}$;

(iv) $H_A = SL_2 \times SL_4 : S^2(0 \oplus R(\varphi_1 + \varphi_1')) \cong$
$0 \oplus R(\varphi_1 + \varphi_1')^8 \oplus R(\varphi_2')^6 \oplus R(2\varphi_1 + 2\varphi_1')^{30}$;

(v) $H_A = SL_5,\ S^2(0 \oplus R(\varphi_3)) \cong$
$0 \oplus R(\varphi_3)^{10} \oplus R(\varphi_1)^5 \oplus R(2\varphi_3)^{50}$;

(vi) $H_A = \mathrm{Spin}_{10},\ S^2(0 \oplus R(\varphi_5)) \cong$
$0 \oplus R(\varphi_5)^{16} \oplus R(\varphi_1)^{10} + R(2\varphi_5)^{126}$.

Looking at these formulae, it is easy to observe that the first summand corresponds to $kv_A$, the second one to $V^t$, and the third one to $V^n$ (compare with the formulae above), and mapping $T_{X,x_A}$ by the linear system of quadrics passing through $Y$ we obtain a map of $\mathbb{P}^n$ to the projective space corresponding to the sum of the first three summands. Clearly the corresponding map $kv_A \oplus V^t \to kv_A \oplus V^t \oplus V^n$ is a map of $H_A$-spaces, and since the orbit decomposition in the above cases is described quite explicitly, it is easy to verify that the closure of the image of this map coincides with $Gv_A$ (consider the cone over the $H_A$-orbit of the vector $v_A + v$, where $v$ is the sum of highest weight vectors of the irreducible representations of $H_A$ making up $V^t$ and $V^n$). This completes the study of the examples of Hartshorne and Severi varieties arising from representations of groups.