

Higher cohomology of divisors on a projective variety

Tommaso de Fernex · Alex Küronya ·
Robert Lazarsfeld

Received: 7 January 2006 / Revised: 12 May 2006 /
Published online: 5 September 2006
© Springer-Verlag 2006

Mathematics Subject Classification 14C20 · 14F99

1 Introduction

The purpose of this paper is to study the growth of higher cohomology of line bundles on a projective variety.

Let X be an irreducible complex projective variety of dimension $d \geq 1$, and let L be a Cartier divisor on X . It is elementary and well-known that the dimensions of the cohomology groups $H^i(X, \mathcal{O}_X(mL))$ grow at most like m^d , i.e.

$$\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL)) = O(m^d) \quad \text{for all } i \geq 0$$

T. de Fernex (✉)
Institute for Advanced Study, School of Mathematics, 1 Einstein Drive,
Princeton, NJ 08540, USA

Present Address:
T. de Fernex
Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City,
UT 84112-0090, USA
e-mail: deferfex@math.utah.edu

A. Küronya
Mathematical Institute, Budapest University of Technology and Economics, P.O. Box 91,
Budapest 1152, Hungary

A. Küronya
Campus Essen, Universität Duisburg-Essen, FB 6 Mathematik, 45117 Essen, Germany
e-mail: alex.kuronya@math.bme.hu

R. Lazarsfeld
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
e-mail: rlaz@umich.edu

(cf. [10, 1.2.33]). It is natural to ask when one of these actually has maximal growth, i.e. when $h^i(X, \mathcal{O}_X(mL)) \geq Cm^d$ for some positive constant $C > 0$ and arbitrarily large m . For $i = 0$ this happens by definition exactly when L is big, and the geometry of big classes is fairly well understood. Here we focus on the question of when one or more of the higher cohomology groups grows maximally.

If L is ample, or merely nef, then of course $h^i(X, \mathcal{O}_X(mL)) = o(m^d)$ for $i > 0$. In general the converse is false: for instance, it can happen that $H^i(X, \mathcal{O}_X(mL)) = 0$ for $m > 0$ and all i even if L is not nef (or, for that matter, pseudoeffective). However our main result shows that if one considers also small perturbations of the divisor in question, then in fact the maximal growth of higher cohomology characterizes non-ample divisors:

Theorem A *Fix any very ample divisor A on X . If L is not ample, then for sufficiently small rational numbers $t > 0$, at least one of the higher cohomology groups of suitable multiples of $L - tA$ has maximal growth. More precisely, there is an index $i > 0$ such that for any sufficiently small $t > 0$,*

$$\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(m(L - tA))) \geq C \cdot m^d$$

for some constant $C = C(L, A, t) > 0$ and arbitrarily large values of m clearing the denominator of t .

In other words, a divisor L is ample if and only if

$$h^i(X, \mathcal{O}_X(m(L - tA))) = o(m^d) \quad \text{when } i > 0$$

for all small t and suitably divisible m . We remark that the essential content of the theorem is the maximal growth of higher cohomology when L is big but not ample.

One can get a more picturesque statement by introducing asymptotic invariants of line bundles. As above let X be an irreducible complex projective variety of dimension d , and let L be a Cartier divisor on X . Define

$$\widehat{h}^i(X, L) \stackrel{\text{def}}{=} \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL))}{m^d/d!}.$$

The definition extends in the natural manner to \mathbb{Q} -divisors. When $i = 0$ this is the volume $\text{vol}_X(L)$ of L , which has been the focus of considerable attention in recent years [2, 3, 7, 10]. The higher cohomology functions were introduced and studied by the second author in [9]. It was established there that $\widehat{h}^i(L)$ depends only on the numerical equivalence class of a \mathbb{Q} -divisor L , and that it uniquely determines a continuous function

$$\widehat{h}^i = \widehat{h}^i(X, \cdot): N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

on the real Néron–Severi space of X . This generalizes the corresponding results for vol_X proved by the third author in [10, 2.2.C]. When X is a toric variety, these functions were studied in [8]. We refer to [6] for a survey of the circle of ideas surrounding asymptotic invariants.

Theorem A then implies:

Corollary B *A class $\xi_0 \in N^1(X)_{\mathbb{R}}$ is ample if and only if*

$$\widehat{h}^i(\xi) = 0$$

for all $i > 0$ and all $\xi \in N^1(X)_{\mathbb{R}}$ in a small neighborhood of ξ_0 .

One can see this as an asymptotic analogue of Serre’s criterion for amplitude, and this was the motivation for our work on these questions. In the toric case, a somewhat stronger statement appears in [8].

The proof of the theorem combines some algebraic constructions from [9] with geometric facts about big line bundles that fail to be ample. Specifically, choose a collection of very general divisors $E_1, \dots, E_p \in |A|$, and assume for a contradiction that L is not nef, but that

$$\widehat{h}^i(X, L - tA) = 0$$

for $i > 0$ and small positive t . The first point is to show that this vanishing descends to the divisors E_α . By induction it follows each of the restrictions $L|_{E_\alpha}$ is ample, and we can assume furthermore that they are all very ample. Now consider the complex

$$H^0(\mathcal{O}_X(mL)) \xrightarrow{v} \bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL)) \xrightarrow{u} \bigoplus^{\binom{p}{2}} H^0(\mathcal{O}_{E_\alpha \cap E_\beta}(mL)), \quad (*)$$

the maps u and v being determined by restriction. The cohomology of this injects into $H^1(X, \mathcal{O}_X(mL - pA))$. On the other hand, using the assumption that L fails to be nef, we show that one can arrange things so that there is a non-trivial ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, vanishing on a set of dimension ≥ 1 , such that the base ideal of $|mL|$ grows like \mathfrak{a}^m , i.e.

$$\mathfrak{b}(|mL|) \subseteq \mathfrak{a}^m.$$

Therefore the image of v is contained in the subgroup

$$\bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^m) \subseteq \bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL)),$$

and this leads to a lower bound on the dimension of $\ker(u)/\text{im}(v)$. In fact, a dimension count shows that if $p \sim m\delta$ for $0 < \delta \ll 1$, then $h^1(mL - pA)$ will grow like m^d , which produces the required contradiction.

These issues are closely related to the paper [4] of Demailly, Peternell and Schneider.¹ Working on a smooth projective variety X , these authors study in effect the vanishing of the groups $H^i(X, \mathcal{O}_X(m(L - tA)))$ for small t and large divisible m . It would be interesting to understand more precisely the circumstances under which the sub-maximal growth of cohomology studied here forces the actual vanishing of cohomology.

Concerning the organization of the paper, we start in Sect. 3 with a lemma about base ideals of big linear series. The main result appears in Sect. 4.

The first author would like to thank Eckart Viehweg for support during his visit at the Universität Duisburg-Essen. We are grateful to Lawrence Ein and Sam Payne for useful discussions.

2 Conventions and background

2.1

We work throughout over the complex numbers. A *variety* is a reduced and irreducible scheme, and we always deal with closed points.

2.2

We follow the conventions of [10, Chap. 1] concerning divisors on a projective variety X . Thus a *divisor* on X means a Cartier divisor. A \mathbb{Q} - or \mathbb{R} -divisor indicates an element of

$$\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes \mathbb{Q} \quad \text{or} \quad \mathrm{Div}_{\mathbb{R}}(X) = \mathrm{Div}(X) \otimes \mathbb{R}.$$

$N^1(X)$ is the Néron–Severi group of numerical equivalence classes of divisors, while $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ denote the corresponding groups for \mathbb{Q} - and \mathbb{R} -divisors.

2.3

Given a projective variety X of dimension d , and a divisor L on X , we set

$$\widehat{h}^i(X, L) = \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL))}{m^d/d!}. \quad (1)$$

¹ We thank the referee for calling [4] to our attention. In an earlier version of the present paper, we included a criterion, involving amplitude of restrictions, for certain of the \widehat{h}^i to vanish. However the statement we gave also follows from the results of [4] concerning “flag positivity,” so we have removed it.

It is established in [9, Proposition 5.15, Theorem 5.1] that this depends only on the numerical equivalence class of L , and that it satisfies the homogeneity

$$\widehat{h}^i(pL) = p^d \cdot \widehat{h}^i(L). \quad (2)$$

This allows one to define $\widehat{h}^i(L)$ for an arbitrary \mathbb{Q} -divisor by clearing denominators, giving rise to a function $\widehat{h}^i : N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}$. The main result of [9] is that this extends uniquely to a continuous function

$$\widehat{h}^i : N^1(X)_{\mathbb{R}} \rightarrow \mathbb{R}$$

satisfying the same homogeneity property ([9, Theorem 5.1]). When L is an \mathbb{R} -divisor, we typically write $\widehat{h}^i(L)$ to denote the value of this function on the numerical equivalence class of L . The \widehat{h}^i are called asymptotic cohomological functions in [9], although we occasionally use some slight variants of this terminology. Observe finally that the homogeneity (2) implies that if L is a Cartier divisor, then

$$\widehat{h}^i(X, L) = \frac{\widehat{h}^i(X, pL)}{p^d} = \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(pmL))}{(pm)^d/d!}. \quad (3)$$

for every fixed integer $p > 0$. The analogous statement holds when L is a \mathbb{Q} -divisor, provided that p is sufficiently divisible to clear the denominators of L .

3 A lemma on base-loci

This brief section is devoted to the proof of a useful fact concerning the base-ideals of linear systems on a normal variety. The result in question asserts that the base-ideals associated to multiples of a divisor which is not nef grow at least like powers of the ideal of a curve. On a smooth variety this fact, which is a small elaboration of [10, Corollary 11.2.13], is well-known to experts. The main point here is to extend the statement to normal varieties.²

Proposition 3.1 *Let D be a divisor on a normal projective variety V , and denote by*

$$\mathfrak{b}(|\ell D|) \subseteq \mathcal{O}_V$$

the base-ideal of the indicated linear series. Assume that D is not nef. Then there exist positive integers q and c , and an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_V$ vanishing on a set of

² We thank Mircea Mustața for pointing out that in a first version of this paper, this reduction contained an error.

dimension ≥ 1 , such that

$$\mathfrak{b}(|mqD|) \subseteq \mathfrak{a}^{m-c}$$

for all $m > c$.

Proof The assertion being trivial otherwise, we may suppose that D has non-negative Kodaira–Iitaka dimension $\kappa(V, D) \geq 0$. Since D is not nef, there exists an irreducible curve $Z \subset V$ intersecting D negatively. We will show that one can take $\mathfrak{a} = I_Z \subset \mathcal{O}_V$ to be the ideal sheaf of Z .

Let $\mu : V' \rightarrow V$ be a resolution of singularities, and set $D' = \mu^*D$. We may suppose that

$$\mathfrak{a} \cdot \mathcal{O}_{V'} = \mathcal{O}_{V'}(-E),$$

where E is an effective divisor on V' with simple normal crossing support. Note that the projection formula implies that D' intersects negatively every curve $C \subset V'$ that dominates Z .

Given $q \geq 1$, consider the asymptotic multiplier ideal sheaf

$$\mathcal{J}(V', \|qD'\|) = \mathcal{J}(\|qD'\|) \subseteq \mathcal{O}_{V'}.$$

Since D' has non-negative Kodaira–Iitaka dimension, there exists a fixed divisor A on V' such that

$$\mathcal{O}_{V'}(qD' + A) \otimes \mathcal{J}(\|qD'\|)$$

is globally generated for every q ; this follows from [10, Corollary 11.2.13] by taking $A = K_{V'} + (\dim V' + 1)H$, where H is any very ample divisor on V' . If $C \subset V'$ is an irreducible curve that is not contained in the zero locus of $\mathcal{J}(\|qD'\|)$, then one has

$$((qD' + A) \cdot C) \geq 0.$$

We conclude that if $C \subseteq V'$ is any curve dominating Z , then the ideal $\mathcal{J}(\|qD'\|)$ must vanish along C for $q > -(A \cdot C)/(D' \cdot C)$.

Lemma 3.2 *There exists $q > 0$ such that $\mathcal{J}(\|qD'\|) \subseteq \mathcal{O}_{V'}(-E)$.*

Proof We observe that it is enough to prove that

$$\mathcal{J}(\|kD'\|) \subseteq \mathcal{O}_{V'}(-E_{\text{red}}) \quad \text{for some } k > 0. \quad (4)$$

Indeed, assuming this, if k is as in (4) and a is the largest coefficient appearing in the divisor E , then by the subadditivity theorem for multiplier ideals ([10, Corollary 11.2.4]) we have

$$\mathcal{J}(\|akD'\|) \subseteq \mathcal{J}(\|kD'\|)^a \subseteq \mathcal{O}_{V'}(-aE_{\text{red}}) \subseteq \mathcal{O}_{V'}(-E).$$

Hence we can take $q = ak$.

Turning to (4), note to begin with that for some k_0 , the ideal sheaf $\mathcal{J}(\|k_0 D'\|)$ vanishes along all irreducible components of E that dominate Z . In fact, suppose that F is such a component, and let $C \subset F$ be a general complete intersection curve that surjects onto Z . Then as we have seen, $\mathcal{J}(\|qD'\|)$ vanishes along C for $q \gg 0$. So this ideal must in fact vanish on all of F .

Now since V is normal, $E = \mu^{-1}(Z)$ is connected. Therefore, in order to prove (4), it suffices to prove the following claim: If E_i is an irreducible component of E mapping to a point of Z , and if E_i meets another irreducible component E_j of E along which $\mathcal{J}(\|k_j D'\|)$ is known to vanish for some $k_j > 0$, then there is a $k_i \geq k_j$ such that $\mathcal{J}(\|k_i D'\|)$ vanishes along E_i . (Note that then $\mathcal{J}(\|k_i D'\|)$ also vanishes along E_j since $\mathcal{J}(\|\ell D'\|) \subseteq \mathcal{J}(\|m D'\|)$ whenever $\ell > m$.)

So, let E_i and E_j be as in this scenario. Since $\mathcal{J}(\|k_j D'\|)$ vanishes on E_j , we have

$$\mathcal{J}(\|mk_j D'\|) \subseteq \mathcal{J}(\|k_j D'\|)^m \subseteq \mathcal{O}_{V'}(-mE_j) \quad \text{for all } m > 0$$

thanks again to the subadditivity theorem. Now take a general complete intersection curve $C \subseteq E_i$ that meets E_j in at least one point P , which we may assume to be a smooth point of C . As above, we know that

$$\mathcal{O}_{V'}(mk_j D' + A) \otimes \mathcal{J}(\|mk_j D'\|)$$

is globally generated for $m > 0$. On the other hand, $\mathcal{O}_C(D') = \mathcal{O}_C$ since C maps to a point in V , and hence

$$\left(\mathcal{O}_{V'}(mk_j D' + A) \otimes \mathcal{J}(\|mk_j D'\|) \right) \cdot \mathcal{O}_C \subseteq \mathcal{O}_C(A|_C - mP).$$

Therefore, assuming that $\mathcal{J}(\|mk_j D'\|)$ does not vanish along C , it follows that $(A \cdot C) \geq m$. Thus if $k_i \geq k_j \cdot ((A \cdot C) + 1)$, then in fact $\mathcal{J}(\|k_i D'\|)$ must vanish along E_i . This completes the proof of the lemma. \square

We now complete the Proof of the Proposition. Note that $H^0(V, \mathcal{O}_V(\ell D)) = H^0(V', \mathcal{O}_{V'}(\ell D'))$ by normality, and in particular $\mathfrak{b}(\|\ell D'\|) = \mathfrak{b}(\|\ell D\|) \cdot \mathcal{O}_{V'}$. On the other hand, we have

$$\mathfrak{b}(\|mqD'\|) \subseteq \mathcal{J}(\|mqD'\|) \subseteq \mathcal{J}(\|qD'\|)^m \subseteq \mathcal{O}_{V'}(-mE)$$

by [10, Corollary 11.2.4] and Lemma 3.2. Therefore

$$\mathfrak{b}(\|mqD'\|) \subseteq \mu_* \mathcal{O}_{V'}(-mE) = \overline{\mathfrak{a}^m},$$

where as usual $\overline{\mathfrak{a}^m}$ denotes the integral closure of the ideal in question. But quite generally, given any ideal $\mathfrak{a} \subseteq \mathcal{O}_V$, there exists an integer $c > 0$ such that $\overline{\mathfrak{a}^{k+1}} = \mathfrak{a} \cdot \overline{\mathfrak{a}^k}$ for $k \geq c$ (cf. [10, (**)] on p. 218) and hence $\overline{\mathfrak{a}^m} \subseteq \mathfrak{a}^{m-c}$ for $m \geq c$. \square

Remark 3.3 We do not know whether the normality hypothesis is essential.

4 A characterization of ample divisors

This section is devoted to the statement and proof of our main result.

We start by fixing notation. In what follows, X will be a projective variety of dimension d over the complex numbers, and L will denote a Cartier divisor on X .

If the divisor L is ample, then the functions \widehat{h}^i vanish identically in a neighborhood of $[L]$ in $N^1(X)_{\mathbb{R}}$ for every $i \geq 1$; this follows easily from Serre vanishing, the continuity of the functions \widehat{h}^i , and the fact that the ample cone is open inside $N^1(X)_{\mathbb{R}}$. In particular these functions vanish at $L - tA$ for every very ample divisor A and every sufficiently small $t \geq 0$. The main result of this section is that this property characterizes amplitude.

Theorem 4.1 *Let X be a projective variety, and let L be a Cartier divisor on X . Assume that there exists a very ample divisor A on X and a number $\varepsilon > 0$ such that*

$$\widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, \quad 0 \leq t < \varepsilon.$$

Then L is ample.

Theorem A from the Introduction follows immediately. We will deduce Corollary B at the end of the section.

We now begin working towards the proof of Theorem 4.1. First of all, in order to eventually be able to apply Proposition 3.1, we reduce to the situation when the variety X is normal.

Lemma 4.2 *Assume that Theorem 4.1 holds for normal projective varieties. Then it holds in general.*

Proof Let X be an arbitrary projective variety, and suppose that L and A are divisors on X satisfying the assumptions of Theorem 4.1, so that L is Cartier, A is ample, and there exists an $\varepsilon > 0$ such that $\widehat{h}^i(X, L - tA) = 0$ for all $i > 0$ and $0 \leq t < \varepsilon$. Consider the normalization $\nu : \widetilde{X} \rightarrow X$ of X . Since ν is a finite morphism, ν^*A is ample. Moreover

$$\widehat{h}^i(\widetilde{X}, \nu^*(L - tA)) = \widehat{h}^i(X, L - tA).$$

thanks to the birational invariance of higher cohomology functions ([9, Proposition 2.9]). Assuming the theorem for normal varieties we conclude that ν^*L is ample, and hence L is ample as well. \square

So we henceforth assume that X is normal. The plan of the proof is now to study the \widehat{h}^i via restrictions to divisors and use induction on dimension. Specifically, choose a sequence of very general divisors

$$E_1, E_2, \dots \in |A|.$$

Given $m, p > 0$ we take the first p of the E_α and form the complex $K_{m,p}^\bullet$:

$$\mathcal{O}_X(mL) \longrightarrow \bigoplus^p \mathcal{O}_{E_\alpha}(mL) \longrightarrow \bigoplus^{\binom{p}{2}} \mathcal{O}_{E_\alpha \cap E_\beta}(mL) \longrightarrow \cdots, \quad (5)$$

obtained as a twist of the p -fold tensor product of the one-step complexes $\mathcal{O}_X \longrightarrow \mathcal{O}_{E_\alpha}$. Because it will be important to keep track of the number of summands, we denote by

$$\bigoplus^{\binom{p}{i}} \mathcal{O}_{E_{\alpha_1} \cap \cdots \cap E_{\alpha_i}}$$

the direct sum of the sheaves $\mathcal{O}_{E_{\alpha_1} \cap \cdots \cap E_{\alpha_i}}$ over all choices of i increasing indices. It is established in [9, Corollary 4.2] that $K_{m,p}^\bullet$ is acyclic, and hence resolves $\mathcal{O}_X(mL - pA)$. In particular,

$$H^r(X, \mathcal{O}_X(mL - pA)) = \mathbb{H}^r(K_{m,p}^\bullet). \quad (6)$$

The hypercohomology group on the right in (6) is in turn computed by a first-quadrant spectral sequence with

$$E_1^{i,j} = \begin{cases} H^j(\mathcal{O}_X(mL)) & i = 0 \\ \bigoplus^{\binom{p}{i}} H^j(\mathcal{O}_{E_{\alpha_1} \cap \cdots \cap E_{\alpha_i}}(mL)) & i > 0. \end{cases} \quad (7)$$

As in [10, 2.2.37] or [9, Sect. 5] we may – and do – assume that the dimensions of all the groups appearing on the right in (7) are independent of the particular divisors E_α that occur. We will write these dimensions as

$$h^j(\mathcal{O}_{E_1}(mL)), \quad h^j(\mathcal{O}_{E_1 \cap E_2}(mL))$$

and so on.

The first point is to show that the vanishing hypothesis of the theorem descends to very general divisors in $|A|$.

Lemma 4.3 *Keeping notation as in Theorem 4.1, assume that there is a positive real number $\varepsilon > 0$ such that*

$$\widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, \quad 0 \leq t < \varepsilon. \quad (8)$$

Let $E \in |A|$ be a very general divisor. Then

$$\widehat{h}^i(E, (L - tA)|_E) = 0 \quad \text{for all } i > 0, \quad 0 \leq t < \varepsilon. \quad (9)$$

Proof Assuming (8), it's enough to prove

$$\widehat{h}^i(E, L_E) = 0 \quad \text{for all } i > 0. \quad (10)$$

For then the more general statement (9) follows (using the homogeneity and continuity of the higher cohomology functions on X and on E) upon replacing L by $L - \delta A$ for a rational number $0 < \delta < \varepsilon$.

Suppose then that (10) fails, and consider the complex $K_{m,p}^\bullet$. We compute a lower bound on the dimension of the group $E_\infty^{1,i}$ in the hypercohomology spectral sequence. Specifically, by looking at the possible maps coming into and going out from the $E_r^{1,i}$, one sees that

$$\begin{aligned} h^{i+1}(X, mL - pA) + h^i(X, mL) &\geq p \cdot h^i(\mathcal{O}_{E_1}(mL)) - \binom{p}{2} \cdot h^i(\mathcal{O}_{E_1 \cap E_2}(mL)) \\ &\quad - \binom{p}{3} \cdot h^{i-1}(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)) - \dots \end{aligned}$$

Now we can find some fixed constant $C_1 > 0$ such that for all $m \gg 0$:

$$\begin{aligned} h^i(\mathcal{O}_{E_1 \cap E_2}(mL)) &\leq C_1 \cdot m^{d-2}, \\ h^{i-1}(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)) &\leq C_1 \cdot m^{d-3}, \text{ etc.} \end{aligned}$$

Moreover, since we are assuming for a contradiction that $\widehat{h}^i(E, L_E) > 0$, we can find a constant $C_2 > 0$, together with a sequence of arbitrarily large integers m , such that

$$h^i(\mathcal{O}_{E_1}(mL)) \geq C_2 \cdot m^{d-1}. \quad (11)$$

Putting this together, we find that there are arbitrarily large integers m such that

$$\begin{aligned} h^{i+1}(X, mL - pA) + h^i(X, mL) \\ \geq C_3 \cdot (pm^{d-1} - p^2m^{d-2} - p^3m^{d-3} - \dots) \end{aligned} \quad (12)$$

for suitable $C_3 > 0$. Note that this constant C_3 is independent of p . At this point, we fix a very small rational number $0 < \delta \ll 1$. By the homogeneity of \widehat{h}^i on E_1 , we can assume that the sequence of arbitrarily large values of m for which (11) and (12) hold is taken among multiples of the denominator of δ (see 0.3). Then, restricting m to this sequence and taking $p = \delta m$, the first term on the RHS of (12) dominates provided that δ is sufficiently small. Hence

$$h^{i+1}(X, mL - pA) + h^i(X, mL) \geq C_4 \cdot \delta m^d$$

for a sequence of arbitrarily large m , and some $C_4 > 0$. But this implies that

$$\widehat{h}^{i+1}(X, L - \delta A) + \widehat{h}^i(X, L) > 0,$$

contradicting the hypothesis. \square

Proof of Theorem 4.1 By Lemma 4.2, we can suppose without loss of generality that X is normal. We assume that there exists $\varepsilon > 0$ such that

$$\widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, 0 \leq t < \varepsilon, \quad (13)$$

but that L is not ample, and we'll aim to get a contradiction.

Note that the Theorem fails for L if and only if it fails for integral multiples of $L - \delta A$ when $0 < \delta \ll 1$. So we can suppose that we have a non-nef divisor L satisfying (13).

Let $E \in |A|$ be one of the very general divisors fixed at the outset. Thanks to Lemma 4.3, we can assume by induction on dimension that $\mathcal{O}_E(L)$ is ample. Replacing L again by a multiple, we can suppose in addition that $\mathcal{O}_E(L)$ is very ample with vanishing higher cohomology. This combination being an open condition in families, we can further assume that $\mathcal{O}_{E_\alpha}(L)$ is very ample for each of the E_α .

As above, form the complex $K_{m,p}^\bullet$, and consider in particular the beginning of the bottom row of the spectral sequence (7):

$$H^0(\mathcal{O}_X(mL)) \xrightarrow{v_{m,p}} \bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL)) \xrightarrow{u_{m,p}} \bigoplus^{\binom{p}{2}} H^0(\mathcal{O}_{E_\alpha \cap E_\beta}(mL)).$$

There is a natural injection

$$\frac{\ker(u_{m,p})}{\text{im}(v_{m,p})} \subseteq H^1(X, mL - pA),$$

and the plan is to estimate from below the dimension of this subspace.

As in the proof of Lemma 4.3 there is a uniform bound having the shape

$$h^0(\mathcal{O}_{E_1 \cap E_2}(mL)) \leq C_1 \cdot m^{d-2}.$$

Therefore, considering $\ker(u_{p,m})$ as a subspace of $\bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL))$, one has

$$\text{codim } \ker(u_{m,p}) \leq C_2 \cdot p^2 m^{d-2} \quad (14)$$

for some $C_2 > 0$ and all $m \gg 0$.

By Proposition 3.1, after possibly replacing L by a suitable multiple, we can find an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ vanishing on a set of dimension ≥ 1 , together with an integer $c \geq 0$ such that

$$\mathfrak{b}(|mL|) \subseteq \mathfrak{a}^{m-c} \quad \text{for all } m > c.$$

Then $v_{m,p}$ admits a factorization

$$\begin{array}{ccc}
 H^0(X, \mathcal{O}_X(mL) \otimes \mathfrak{a}^{m-c}) & \xrightarrow{v'_{m,p}} & \bigoplus^p H^0(E_\alpha, \mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \\
 \parallel & & \downarrow \\
 H^0(X, \mathcal{O}_X(mL)) & \xrightarrow{v_{m,p}} & \bigoplus^p H^0(E_\alpha, \mathcal{O}_{E_\alpha}(mL))
 \end{array}$$

We claim that there is a constant $C_3 > 0$ such that for all $m \gg 0$:

$$H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \quad \text{has codimension} \geq C_3 \cdot m^{d-1} \quad \text{in } H^0(\mathcal{O}_{E_\alpha}(mL)). \tag{15}$$

Granting this, we have

$$\dim \frac{\ker(u_{m,p})}{\text{im}(v_{m,p})} \geq C_4 \cdot (pm^{d-1} - p^2m^{d-2}) \tag{16}$$

for some constant $C_4 > 0$ and all $m \gg 0$. Once again fixing $0 < \delta \ll 1$, limiting m to multiples of the denominator of δ , and setting $p = \delta m$, one finds that

$$h^1(X, \mathcal{O}_X(m(L - \delta A))) \geq C_5 \cdot \delta m^d$$

for m large enough. This implies that $\widehat{h}^1(X, L - \delta A) > 0$, giving the required contradiction.

It remains to prove (15). Choose any point $x = x_\alpha \in \text{Zeroes}(\mathfrak{a}) \cap E_\alpha$, and write $\mathfrak{m}_x \subseteq \mathcal{O}_{E_\alpha}$ for its maximal ideal: here we use that $\dim \text{Zeroes}(\mathfrak{a}) \geq 1$ to know that such a point exists. Since

$$H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \subseteq H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c}),$$

it is enough to bound the codimension of $H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c})$ in $H^0(\mathcal{O}_{E_\alpha}(mL))$. But since $\mathcal{O}_{E_\alpha}(L)$ is very ample, it follows that $\mathcal{O}_{E_\alpha}(mL)$ separates $(m - c)$ -jets at the point x . The dimension of the space of $(m - c)$ -jets at a point of a possibly singular variety is no smaller than the dimension of the space of $(m - c)$ -jets at a smooth point of a variety of the same dimension, and thus we have

$$\text{codim } H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c}) \geq \binom{m - c + d}{d - 1},$$

as required. □

Finally, we give the proof of Corollary B from the Introduction.

Proof of Corollary B Consider the following three statements concerning a Cartier divisor L on X :

- (a) L is ample.
 (b) For all $i > 0$ the function \widehat{h}^i vanishes in a neighborhood of $[L]$.
 (c) $\widehat{h}^i(L - tA) = 0$ for some ample divisor A and all $i > 0$ and $0 \leq t \ll 1$.

We have (a) \Rightarrow (b) by Serre's vanishing and the continuity of \widehat{h}^i , the implication (b) \Rightarrow (c) is obvious, and Theorem 4.1 yields (c) \Rightarrow (a). Therefore (a) \Leftrightarrow (b), which is the content of the Corollary. \square

Acknowledgements Research of the first author was partially supported by NSF grants DMS 0111298 and DMS 0548325, and by MIUR National Research Project "Geometry on Algebraic Varieties" (Cofin 2004). Research of the second author was partially supported by the Leibniz program of the Deutsche Forschungsgemeinschaft and the OTKA grant T042481 of the Hungarian Academy of Sciences. Research of the third author was partially supported by NSF grant DMS 0139713.

References

1. Bauer, T., Küronya, A., Szemberg, T.: Zariski chambers, volumes and stable base loci. *J. für die reine und angewandte Math.* 576, 209–233 (2004)
2. Boucksom, S.: On the volume of a line bundle. *Internat. J. Math.* 13(10), 1043–1063 (2002)
3. Demailly, J.-P., Ein, L., Lazarsfeld, R.: A subadditivity property of multiplier ideals. *Mich. Math. J.* 48, 137–156 (2000)
4. Demailly, J.-P., Peternell, T., Schneider, M.: Holomorphic line bundles with partially vanishing cohomology. In: *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, Isreal Mathematical Conference Proceedings*, pp. 165–198 (1996)
5. Ein, L., Lazarsfeld, R., Mustață, M., Nakamaye, M., Popa, M.: Asymptotic invariants of base loci, to appear in the *Annales de l'Institut Fourier*; math.AG/0308116
6. Ein, L., Lazarsfeld, R., Mustață, M., Nakamaye, M., Popa, M.: Asymptotic invariants of line bundles, to appear in the *Pure Appl. Math. Q.* math.AG/0505054
7. Fujita, T.: Approximating Zariski decomposition of big line bundles. *Kodai Math. J.* 17(1), 1–3 (1994)
8. Hering, M., Küronya, A., Payne, S.: Asymptotic cohomological functions of toric divisors, to appear in *Adv. Math.* math.AG/0501104
9. Küronya, A.: Asymptotic cohomological functions on projective varieties, to appear in the *Am. J. Math.* math.AG/0501491
10. Lazarsfeld, R.: Positivity in algebraic geometry I–II. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vols. 48–49, Springer, Berlin Heidelberg New York (2004)