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On a Theorem of Castelnuovo, and the Equations Defining Space Curves

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Introduction

Consider a reduced irreducible curve

$$X \subseteq \mathbb{P}^r \qquad (r \geqq 3)$$

of degree d, not contained in any hyperplane. For a given integer $n \geqq 0$, it is natural to ask whether X enjoys one or more of the following properties:

(A_n) The line bundle $\mathcal{O}_X(n)$ is non-special.

(B_n) Hypersurfaces of degree n trace out a *complete* linear system on X, i.e., the homomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(X, \mathcal{O}_X(n))$$

is surjective.

(C_n) X is cut out in $\mathbb{P}^r$ by hypersurfaces of degree n, and the homogeneous ideal of X is generated in degrees $\geqq n$ by its component of degree n.

It is of course classical that each of these conditions is satisfied for all sufficiently large n. But one would like to have an *explicit* bound on how large n must be, and to understand the extremal examples.

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The fundamental work in this direction was carried out by Castelnuovo [3]. For property (A), using the argument leading to his celebrated bound on the genus of a space curve, Castelnuovo obtained complete results (cf. [20], or [6]; when $r=3$ a considerable refinement was stated earlier by Halphen [9], and recently proved by the first and third authors [7] and by Hartshorne [10]). The present paper is chiefly concerned with property (B). Castelnuovo proved that for smooth curves, at least, (B_n) holds for $n \geqq d-2$, and he suggested that an irrational curve satisfies (B_{d-3}). Examples show that Castelnuovo's estimate is optimal for curves in $\mathbb{P}^3$. But given $X \subseteq \mathbb{P}^r$ it is natural to expect that property (B_n) holds in fact for $n \geqq d+1-r$. Our purpose here is to complete Castelnuovo's theorem by showing that this is indeed the case, and by describing the examples on the boundary.

Our main result is the following

Theorem. *Let $X \subseteq \mathbb{P}^r$ be a reduced irreducible curve of degree d, not contained in any hyperplane. Then:*

(i) *Property (B_n) is satisfied for all $n \geqq d+1-r$.*

(ii) *Property (B_{d-r}) fails if and only if X is smooth and rational, and either $d=r+1$, or $d>r+1$ and X has a $(d+2-r)$-secant line.*

Note that it is not required in (i) that X be smooth. The first and third authors had previously extended the estimate given by Castelnuovo to possibly singular curves. They had also proved Castelnuovo's assertion that any extremal example in $\mathbb{P}^3$ is rational (unpublished). Special cases of (i) were proved by Jongmans [12] via a reduction to Castelnuovo's results, and by Meadows [13] for certain rational curves. A modern exposition of Castelnuovo's theorem is given by Szpiro in his notes [20].

It is known by work of Mumford [14], who attributes the idea to Castelnuovo, that if a curve satisfies (A_{n_0-2}) and (B_{n_0-1}) for some $n_0 \geqq 0$, then (C_n) holds for all $n \geqq n_0$. It follows from Castelnuovo's results for property (A), or from the proofs below, that (A_{d-r}) always holds. The cases where (A_{d-1-r}) or (B_{d-r}) fail are easily analyzed separately, and we obtain the

Corollary. *Let $X \subseteq \mathbb{P}^r$ $(r \geqq 3)$ be a reduced, irreducible curve of degree d, not contained in any hyperplane. Then:*

(i) *Property (C_n) holds for all $n \geqq d+2-r$.*

(ii) *Property (C_{d+1-r}) fails if and only if X is a smooth rational curve having a $(d+2-r)$-secant line.*

The equations defining space curves have been studied by several authors, notably Mumford [15], Saint-Donat [18], and Arbarello-Sernese [1]. Our viewpoint differs somewhat from theirs, however, in that we are forced to deal with embeddings defined by possibly incomplete linear systems.

Unlike Castelnuovo's arguments, which are geometric in nature, our proofs are essentially cohomological. They depend on a simple but somewhat surprising technique. Roughly speaking, the method is to "resolve" the ideal sheaf $\mathscr{I}_{X/\mathbb{P}^r}$ (or something related) by a complex with generally non-trivial homology supported on X. For example, to prove the first statement of the Theorem, we use a Beilinson-type construction to express X as the locus where a matrix of linear forms drops rank, and then take the corresponding Eagon-Northcott complex. By allowing non-trivial homology, one arrives at complexes much simpler than those that would be required for an actual resolution of the ideal sheaf. Happily, as long as the complex is exact away from a one-dimensional set, one is still able to read off the desired vanishings. We hope that this technique may find other applications in the study of space curves.

An explicit estimate for the regularity of an arbitrary ideal sheaf on $\mathbb{P}^r$ has been obtained by Gotzmann [5] and generalized by Bayer [2] to any coherent sheaf. Gotzmann's bound depends only on the Hilbert polynomial of the scheme in question, however, and as one would expect the numbers that it gives are generally far from optimal for reduced curves.

Our exposition proceeds in three stages. First we establish the regularity assertions (i) of the Theorem and Corollary (§1). We next show (§2) by a similar but independent argument that an *irrational* curve satisfies (B_{d-r}) and (C_{d+1-r}), so that any extremal example must be rational. For rational curves, a refinement of the proof in §1 then gives the classification statements (ii) (§3). Strictly speaking, the first assertions of the Theorem and its Corollary are consequences of the enumeration of the extremal examples, and it would have been possible to organize the presentation in such a way as to avoid at least the last lemma of §1. However the regularity results (i) are substantially easier to prove than the classification statements (ii), and it seemed to us worthwhile to treat them directly. Finally we discuss in §4 some open problems.

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§0. Notation and Conventions

(0.1) We work over an algebraically closed field k of arbitrary characteristic.

(0.2) Unless otherwise stated, a *curve* is a *reduced* and *irreducible*, but possibly singular, projective variety of dimension one. Recall that a curve $X \subseteq \mathbb{P}^r$ is *non-degenerate* if it is not contained in a hyperplane. Given a variety $V \subseteq \mathbb{P}^r$, we denote by $\mathscr{I}_V$ the ideal sheaf of V in $\mathbb{P}^r$; if $X \subseteq V$ is a subvariety, we indicate the ideal sheaf of X in V by $\mathscr{I}_{X/V}$. For a coherent sheaf $\mathscr{F}$ on V, $\mathscr{F}(n)$ denotes as usual $\mathscr{F} \otimes \mathcal{O}_{\mathbb{P}^r}(n)$, and following common usage we let $h^i(V, \mathscr{F}) = \dim H^i(V, \mathscr{F})$.

(0.3) Consider a generically surjective homomorphism

$$u\colon E \to F$$

of vector bundles of ranks e and f on a smooth variety. Associated to u are several *Eagon-Northcott complexes* (cf. [4, 19, or 8]), of which we shall need the following two:

$$(0.4) \qquad 0 \to \Lambda^e E \otimes S^{e-f}(F)^* \to \ldots \to \Lambda^{f+1} E \otimes F^* \to \Lambda^f E \xrightarrow{\Lambda^f u} \Lambda^f F \to 0$$

$$(0.5) \qquad 0 \to \Lambda^e E \otimes S^{e-f-1}(F)^* \to \ldots \to \Lambda^{f+2} E \otimes F^* \to \Lambda^{f+1} E$$
$$\to E \otimes \Lambda^f F \xrightarrow{u \otimes 1} F \otimes \Lambda^f F \to 0.$$

The basic fact for our purposes is that *these complexes are exact away from the support of* coker u ([4, 19, 8]). (In general they aren't acyclic unless Supp(coker u) has the expected codimension $e-f+1$.)

§1. A Regularity Theorem

It will be convenient to phrase our results in terms of the regularity of the ideal sheaf. Recall ([14], Lecture 14) that a coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$ is said to be *n-regular* if $H^i(\mathbb{P}^r, \mathscr{F}(n-i))=0$ for $i>0$. The usefulness of this concept lies in the fact that if $\mathscr{F}$ is n-regular, then:

(i) $\mathscr{F}(n)$ is generated by its global sections,

(ii) the maps

$$H^0(\mathscr{F}(n)) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(l)) \to H^0(\mathscr{F}(n+l))$$

are surjective for $l \geqq 0$, and

(iii) $\mathscr{F}$ is $(n+1)$-regular.

([14], loc. cit.). We will say that a curve $X \subseteq \mathbb{P}^r$ is n-regular if its ideal sheaf $\mathscr{I}_X$ is, and n-irregular otherwise. Thus for $n \geqq 0$, X is n-regular if and only if properties (A_{n-2}) and (B_{n-1}) of the Introduction are satisfied, in which case (C_n) holds by facts (i) and (ii), as do $(A_{n'-2})$, $(B_{n'-1})$ and $(C_{n'})$ for any $n' \geqq n$ by (iii). Hence the first assertions of the results stated in the Introduction are consequences of

Theorem 1.1. *Let $X \subseteq \mathbb{P}^r$ be a (reduced and irreducible) non-degenerate curve of degree d. Then X is $(d+2-r)$-regular.*

The key to Theorem 1.1 is the following result, which allows one to estimate the regularity of a curve in terms of a vanishing on its normalization.

Proposition 1.2. *Let $X \subseteq \mathbb{P}^r$ be a reduced (but possibly reducible) curve, with normalization $\tilde{X}$, and let $p\colon \tilde{X} \to \mathbb{P}^r$ denote the natural map. Set*

$$M = p^* \Omega^1_{\mathbb{P}^r}(1),$$

and suppose that A is a line bundle on $\tilde{X}$ such that

$$H^1(\tilde{X}, \Lambda^2 M \otimes A) = 0.$$

Then $X \subseteq \mathbb{P}^r$ is $h^0(\tilde{X}, A)$-regular.

Proof. The first step is to show that there is an exact sequence

$$(1.3)\qquad H^0(\tilde{X}, M\otimes A)\otimes_k \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{u} H^0(\tilde{X}, A)\otimes_k \mathcal{O}_{\mathbb{P}^r} \to p_* A \to 0$$

of sheaves on $\mathbb{P}^r$. This follows readily from an evident variation of the Beilinson spectral sequence (cf. [16], II.3.1). For later reference, and for the benefit of the reader not versed in such matters, we may give an elementary and self-contained (but equivalent) derivation as follows. Setting $\mathcal{O}_{\tilde{X}}(1) = p^*\mathcal{O}_{\mathbb{P}^r}(1)$, the morphism $p: \tilde{X} \to \mathbb{P}^r$ is determined by a subspace $V \subseteq H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(1))$ of dimension $r+1$. This gives rise to an exact sequence

$$(1.4)\qquad 0 \to M \to V_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}(1) \to 0$$

of vector bundles on $\tilde{X}$, where $V_{\tilde{X}} = V\otimes_k \mathcal{O}_{\tilde{X}}$. Denote by π and f the projections of $\tilde{X}\times\mathbb{P}^r$ onto $\tilde{X}$ and $\mathbb{P}^r$ respectively. On $\tilde{X}\times\mathbb{P}^r$ there are vector bundle homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \pi^* M \arrow[r] \arrow[dr, "s"'] & \pi^* V_{\tilde{X}} \arrow[d] \\ & & f^*\mathcal{O}_{\mathbb{P}^r}(1), \end{tikzcd}$$

and the graph $\Gamma \subseteq \tilde{X}\times\mathbb{P}^r$ of p is defined scheme-theoretically by the vanishing of s. $\tilde{X}\times\mathbb{P}^r$ being smooth, one obtains a Koszul resolution of $\mathcal{O}_\Gamma$, and twisting by $\pi^* A$ gives the following resolution of $\mathcal{O}_\Gamma\otimes\pi^* A$:

$$(1.5)\qquad \begin{tikzcd}[column sep=tiny] & & & 0 \arrow[dr] & & 0 & & \\ & & & & \mathcal{F}_0 \arrow[ur] \arrow[dr] & & & \\ \ldots \arrow[r] & \pi^*(\Lambda^2 M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-2) \arrow[rr] \arrow[dr] & & \pi^*(M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-1) \arrow[ur] \arrow[rr] & & \pi^* A \arrow[r] & \mathcal{O}_\Gamma\otimes\pi^* A \arrow[r] & 0. \\ & & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & & & \\ & 0 \arrow[ur] & & 0 & & & & \end{tikzcd}$$

The sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$ are defined as indicated. By Künneth,

$$R^q f_*(\pi^*(\Lambda^p M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-p)) = H^q(\tilde{X}, \Lambda^p M\otimes A)\otimes_k \mathcal{O}_{\mathbb{P}^r}(-p).$$

Therefore

$$R^1 f_* \mathcal{F}_1 = 0$$

thanks to the vanishing of $H^1(\tilde{X}, \Lambda^2 M\otimes A)$ and the fact that $R^1 f_*$ is right exact since f has fibre dimension one. Taking direct images in (1.5) one then finds the exact sequence

$$f_*(\pi^*(M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \xrightarrow{u} f_*(\pi^* A) \to f_*(\mathcal{O}_\Gamma \otimes \pi^* A)$$

$$\xrightarrow{\delta} R^1 f_*(\pi^*(M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-1))$$

of sheaves on $\mathbb{P}^r$. But $f_*(\mathcal{O}_\Gamma\otimes\pi^* A)=p_* A$ is a torsion $\mathcal{O}_{\mathbb{P}^r}$-module, whereas $R^1 f_*(\pi^*(M\otimes A)\otimes f^*\mathcal{O}_{\mathbb{P}^r}(-1))$ is locally free, and so $\delta=0$. Thus one arrives finally at the exact sequence (1.3). We set

$$n_0 = h^0(\tilde{X}, A).$$

Let $\mathcal{J}\subseteq\mathcal{O}_{\mathbb{P}^r}$ denote the zeroth Fitting ideal sheaf of $p_* A$ computed from (1.3), i.e., the image of $\Lambda^{n_0} u$. Since $p_* A$ is supported on X, the subscheme defined by $\mathcal{J}$ coincides set-theoretically with X, and hence

$$\mathcal{J}\subseteq\mathcal{I}_X$$

because X is reduced. Moreover as the Fitting ideals of a module are independent of the presentation used to compute them, $\mathcal{I}_X/\mathcal{J}$ is supported in the finitely many points of $\mathbb{P}^r$ at which $p_* A$ fails to be locally isomorphic to $\mathcal{O}_X$ (i.e., the singular points of X). Thus $H^i(\mathbb{P}^r, \mathcal{I}_X(m))$ is a quotient of $H^i(\mathbb{P}^r, \mathcal{J}(m))$ for all $i>0$ and $m\in\mathbb{Z}$, so it suffices to prove that $\mathcal{J}$ is n_0-regular.

Consider to this end the Eagon-Northcott complex (0.4) constructed from u. It takes the form

$$\ldots\to\mathcal{O}_{\mathbb{P}^r}^{M_{r-1}}(-n_0+1-r)\to\ldots\to\mathcal{O}_{\mathbb{P}^r}^{M_1}(-n_0-1)\to\mathcal{O}_{\mathbb{P}^r}^{M_0}(-n_0) \xrightarrow{\varepsilon} \mathcal{J}\to 0,$$

where $\varepsilon=\Lambda^{n_0}u$ is surjective. This complex is exact off X, and we observe that twisting by $\mathcal{O}_{\mathbb{P}^r}(n_0-m)$ for some $1\leqq m\leqq r$ kills the higher cohomology of the first $r+1-m$ locally free terms from the right. Thus the following Lemma applies to prove the n_0-regularity of $\mathcal{J}$. ∎

Lemma 1.6. *Let*

$$\mathcal{L}.: \ldots\to\mathcal{L}_{r-1}\to\ldots\to\mathcal{L}_1\to\mathcal{L}_0\xrightarrow{\varepsilon}\mathcal{J}\to 0$$

be a complex of coherent sheaves on a projective variety V of dimension r, with ε surjective. Assume that

(a) *$\mathcal{L}.$ is exact away from a set of dimension $\leqq 1$,*

and

(b) *For a given integer $1\leqq m\leqq r$, one has*

$$H^i(V,\mathcal{L}_0)=\ldots=H^i(V,\mathcal{L}_{r-m})=0 \quad \textit{for } i>0.$$

Then $H^i(V,\mathcal{J})=0$ for $i\geqq m$.

Proof. Hypothesis (a) guarantees that the homology sheaves $\mathcal{H}_j$ of $\mathcal{L}.$ have vanishing cohomology in degrees $\geqq 2$:

$$H^i(V, \mathscr{H}_j) = 0 \quad \text{for } i \geqq 2 \quad \text{and all} \quad j.$$

With this in mind, the Lemma is most simply proved by chopping $\mathscr{L}.$ into short exact sequences in the usual way, and chasing through the resulting diagram. ■

Remark. Note for later reference that Proposition 1.2 remains valid for any line bundle A for which the sheaf $R^1 f_* \mathscr{F}_1$ occurring in the proof is supported on a set of dimension $\leqq 0$. For then one still has $\mathscr{J} \subseteq \mathscr{I}_X$, while $\mathscr{I}_X/\mathscr{J}$ and coker u continue to be supported in sets of dimensions zero and one respectively.

Theorem 1.1 now follows from

Lemma 1.7. *Let $\tilde{X}$ be a smooth irreducible curve of genus g, and let $p\colon \tilde{X} \to \mathbb{P}^r$ be a morphism of degree d (i.e., $\deg p^* \mathcal{O}_{\mathbb{P}^r}(1) = d$). Assume that $\tilde{X}$ does not map into a hyperplane, and set $M = p^* \Omega^1_{\mathbb{P}^r}(1)$. Then there exists a line bundle A on $\tilde{X}$, with $h^0(\tilde{X}, A) = d + 2 - r$, such that $H^1(\tilde{X}, \Lambda^2 M \otimes A) = 0$.*

Proof. We assert that M admits a filtration

$$M = F^1 \supseteq F^2 \supseteq \ldots \supseteq F^r \supseteq F^{r+1} = 0 \tag{1.8}$$

by vector bundles such that each of the quotients $L_i = F^i/F^{i+1}$ is a line bundle of strictly negative degree. Indeed, $H^0(\tilde{X}, M) = 0$ by (1.4) since $\tilde{X}$ does not map to a hyperplane, and in particular no non-zero sub-bundle of M is trivial. The existence of the desired filtration is then a consequence of the elementary observation that if a non-trivial vector bundle F on a smooth curve is a sub-bundle of a trivial bundle, then F has a line bundle quotient of negative degree.

In order that $H^1(\tilde{X}, \Lambda^2 M \otimes A) = 0$, it suffices that

$$H^1(\tilde{X}, L_i \otimes L_j \otimes A) = 0 \qquad \text{for all } 1 \leqq i < j \leqq r. \tag{$*$}$$

Since M has degree $-d$ and all the L_i have degree < 0,

$$\deg(L_i \otimes L_j) \geqq r - 2 - d$$

for any $1 \leqq i < j \leqq r$. But a generic line bundle of degree $\geqq g - 1$ is non-special, so $(*)$ will be satisfied if A is a sufficiently general line bundle of degree $g + d + 1 - r$. Moreover $p(\tilde{X})$ being non-degenerate one has $d \geqq r$, so we may suppose in addition that $H^1(\tilde{X}, A) = 0$, in which case $h^0(\tilde{X}, A) = d + 2 - r$. ■

Remarks. (1) If $X \subseteq \mathbb{P}^r$ is reduced but possibly reducible, a variant of Theorem 1.1 holds. Specifically, suppose that X has irreducible components X_i of degree d_i, and that X_i spans a $\mathbb{P}^{r_i} \subseteq \mathbb{P}^r$. Set

$$m_i = \begin{cases} d_i + 2 - r_i & \text{if} \quad d_i \geqq 2 \\ 1 \quad \text{if} \quad d_i = 1 & \text{(i.e., if } X_i \text{ is a line).} \end{cases}$$

Applying a slight modification of (1.7) component by component on $\tilde{X}$, one finds from Proposition 1.2 that X is (Σm_i)-regular.

For example, suppose that $X \subseteq \mathbb{P}^r$ consists of d straight lines. Then X is d-regular. In general this is optimal, for if there is a line $L \nsubseteq X$ meeting each of the components of X at distinct points, then X is not cut out by hypersurfaces of degree $d-1$ (compare §3). By contrast, when X is the union of d *generic* lines, Hartshorne and Hirschowitz [11] have shown that the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(X, \mathcal{O}_X(n))$$

has maximal rank for all n, so that generically one has a much better regularity result.

(2) In the situation of Lemma 1.7 it is amusing to vary the filtration (1.8) while twisting by a fixed line bundle. Suppose, for example, that $X \subseteq \mathbb{P}^r$ is a smooth non-degenerate curve of degree d and genus g. At least in characteristic zero, so that not every secant line is multi-secant, one sees that for almost all choices of $r-1$ points $P_1, \ldots, P_{r-1} \in X$ there is a filtration (1.8) with

$$F^i/F^{i+1} = \begin{cases} \mathcal{O}_X(-P_i) & \text{if } 1 \leqq i \leqq r-1 \\ \mathcal{O}_X(-1) \otimes \mathcal{O}_X(\Sigma P_j) & \text{if } i = r \end{cases}$$

(project from the P_i). In particular, by taking the P_i sufficiently generally, it follows that $H^1(X, M(1)) = 0$ provided that $r-1 \geqq g$. This in turn implies that if $r \geqq g+1$, then the natural map

$$(*) \qquad H^1(\mathbb{P}^r, \mathscr{I}_X(1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^1(\mathbb{P}^r, \mathscr{I}_X(n+1))$$

is *surjective* for $n \geqq 0$. So for example if $X \subseteq \mathbb{P}^3$ has genus two or less, then the Horrocks-Hartshorne-Rao module $\oplus H^1(\mathbb{P}^3, \mathscr{I}_X(n))$ of X (cf. [17]) is generated by its degree one piece. Similarly, if $X \subseteq \mathbb{P}^r$ is defined by the *complete* linear system associated to a line bundle of degree $d \geqq 2g+1$, then $r = d-g \geqq g+1$, and so we recover from $(*)$ Mumford's theorem [15] that X is projectively normal.

§ 2. A Rationality Theorem

We begin in this section the classification of all curves for which Theorem 1.1 is optimal, i.e., all $(d+1-r)$-irregular curves. The final list is summarized in a table at the end of §3. Our immediate goal is to show that with one exception such a curve must be rational:

Theorem 2.1. *Let*

$$X \subseteq \mathbb{P}^r \qquad (r \geqq 3)$$

be a (reduced, irreducible) non-degenerate curve of degree d and geometric genus g. If $g \geqq 1$, then X is $(d+1-r)$-regular unless it is an elliptic normal curve.

Remark. If X is elliptic normal, so that $d = r+1$, then evidently (A_0) fails. On the other hand, X is projectively normal, and property (C_2) holds provided that $r \geqq 3$ (cf. (2.3) below).

Unfortunately, there are a few examples where there does not exist a line bundle A with $h^0(A) \leqq d+1-r$ satisfying the hypotheses of Proposition 1.2 (e.g. an elliptic quintic in $\mathbb{P}^3$). It is simplest for Theorem 2.1 to argue directly. As in § 1, let $\tilde{X}$ be the normalization of X, so that $\tilde{X}$ has genus $g \geqq 1$. Denote by p: $\tilde{X} \to \mathbb{P}^r$ the natural map, and set $\mathcal{O}_{\tilde{X}}(1) = p^* \mathcal{O}_{\mathbb{P}^r}(1)$.

Lemma 2.2. *There exists a line bundle A on $\tilde{X}$ of degree $g-1$ such that $h^0(A) = 1$. For any such line bundle, and any $n > 0$,*

$$h^0(A(-n)) = 0 \quad \textit{and} \quad h^1(A(n)) = 0.$$

Proof. A sufficiently general effective divisor of any degree $\leqq g$ corresponds to line bundle A with $h^0(A) = 1$. A non-zero section of $A(-n)$ would give rise to an inclusion $0 \to \mathcal{O}_{\tilde{X}}(n) \to A$, but since $h^0(\mathcal{O}_{\tilde{X}}(n)) > 1$ when $n > 0$ this is absurd. When A has degree $g-1$, $h^0(\Omega^1_{\tilde{X}} \otimes A^*) = 1$ and the same argument shows that $h^1(A(n)) = 0$ for positive n. ■

Fix a line bundle A as in the Lemma, and set

$$E = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^r, p_* A(n)) \qquad \Big[= \bigoplus_{n \in \mathbb{Z}} H^0(\tilde{X}, A \otimes \mathcal{O}_{\tilde{X}}(n))\Big],$$

so that E is in the natural way a graded module over the homogeneous coordinate ring $S = k[T_0, \ldots, T_r]$.

Lemma 2.3. *E admits a minimal free resolution of length $r-1$ having the following numerical type:*

$$\begin{aligned} 0 \to S(-r-1) \oplus S^{d-r-1}(-r) \to S^{n_{r-2}}(-r+1) \to \ldots \\ \ldots \to S^{n_1}(-2) \to S^{d-r-1}(-1) \oplus S \to E \to 0. \end{aligned}$$

Proof. E is a Cohen-Macaulay module of dimension two, so in any event has a free resolution of length $r-1$.

By Lemma 2.2, E has one generator, say e, in degree zero and none in negative degrees. Since $X \subseteq \mathbb{P}^r$ is non-degenerate, there are no relations of linear dependence among the elements $T_0 \cdot e, \ldots, T_r \cdot e \in E_1$. Hence by Lemma 2.2 and Riemann-Roch, we require $d-r-1$ new generators in degree one. Similarly, it follows from duality that $\mathrm{Ext}_S^{r-1}(E, S(-r-1))$ vanishes in negative degrees, has one generator in degree zero, and $d-r-1$ in degree one. Therefore a minimal resolution of E must be of the form

$$\begin{aligned} 0 \to S(-r-1) \oplus S^{d-r-1}(-r) \oplus \bigoplus_{j_{r-1}} S(-c_{r-1, j_{r-1}}) \to \bigoplus_{j_{r-2}} S(-c_{r-2, j_{r-2}}) \to \ldots \\ \ldots \to \bigoplus_{j_1} S(-c_{1, j_1}) \to \bigoplus_{j_0} S(-c_{0, j_0}) \oplus S^{d-r-1}(-1) \oplus S \to E \to 0 \end{aligned}$$

for suitable integers c_{k, j_k}. Furthermore, by minimality the numbers

$$u_k = \min_{j_k} \{c_{k, j_k}\} \quad \text{and} \quad v_k = \max_{j_k} \{c_{k, j_k}\}$$

are strictly increasing for $1 \leqq k \leqq r-2$. But in view of the remarks above, the first module of syzygies of E vanishes below degree two, i.e., $u_1 \geqq 2$, and similarly $v_{r-2} \leqq r-1$. Hence $u_k = v_k = k+1$ for $1 \leqq k \leqq r-2$, and by the same token the terms $\bigoplus S(-c_{0,j_0})$ and $\bigoplus S(-c_{r-1,j_{r-1}})$ do not actually appear. ∎

Proof of Theorem 2.1. If X is elliptic normal there is nothing to prove. Assuming that X is neither rational nor elliptic normal, we have $d-r \geqq 2$.

Consider (for $r \geqq 3$) the resolution of $p_* A$ obtained from Lemma 2.3 by sheafifying:

$$\ldots \to \mathcal{O}_{\mathbb{P}^r}^{n_1}(-2) \xrightarrow{v} \mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1) \oplus \mathcal{O}_{\mathbb{P}^r} \to p_* A \to 0.$$

The non-zero section of $p_* A$ gives rise to a homomorphism $0 \to \mathcal{O}_X \xrightarrow{s} p_* A$, and if u denotes the composition of v with the projection $\mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1) \oplus \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1)$, then one obtains the following commutative diagram of exact sequences of sheaves on $\mathbb{P}^r$:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O}_{\mathbb{P}^r} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^r}^{n_1}(-2) \arrow[r, "v"] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1) \oplus \mathcal{O}_{\mathbb{P}^r} \arrow[r] \arrow[d] & p_* A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & \mathcal{O}_{\mathbb{P}^r}^{n_1}(-2) \arrow[r, "u"] & \mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1) \arrow[r] \arrow[d] & \operatorname{coker} s \arrow[r] \arrow[d] & 0. \\
 & & & 0 & 0 &
\end{tikzcd}$$

Here K and N are of course defined as the kernels of v and u respectively. Note that $\operatorname{coker} u = \operatorname{coker} s$ is supported in a finite set.

The snake lemma shows that $N/K \cong \mathcal{I}_X$. Hence to prove the $(d+1-r)$-regularity of $\mathcal{I}_X$, it suffices to establish:

(i) K is $(d+2-r)$-regular;

and

(ii) N is $(d+1-r)$-regular.

Assertion (i) is clear: $H^1(\mathbb{P}^r, K(n)) = H^2(\mathbb{P}^r, K(n)) = 0$ for all $n \in \mathbb{Z}$ by construction, $H^3(\mathbb{P}^r, K(d-1-r)) = H^1(\mathbb{P}^r, p_* A(d-1-r)) = 0$ by Lemma 2.2 since $d-1-r>0$, and the remaining vanishings follow from the fact that $p_* A$ is supported on a curve.

Turning to (ii), consider the Eagon-Northcott complex (0.5) constructed from u. Twisting by $\mathcal{O}_{\mathbb{P}^r}(d-r-1)$, it takes the form

$$(*) \quad \ldots \to \mathcal{O}_{\mathbb{P}^r}^{M_{r-1}}(-d) \to \ldots \to \mathcal{O}_{\mathbb{P}^r}^{M_0}(-d-1+r) \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^r}^{n_1}(-2) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^r}^{d-r-1}(-1).$$

Thus $\mathrm{Im}\,\varepsilon$ is a subsheaf of N, and $N/\mathrm{Im}\,\varepsilon$ – like all the homology of $(*)$ – is supported on the zero-dimensional set $\mathrm{Supp}(\mathrm{coker}\,u)$. So it suffices to verify the $(d+1-r)$-regularity of $\mathrm{Im}\,\varepsilon$. But this follows from Lemma 1.6. ■

Remarks. (1) If $X \subseteq \mathbb{P}^r$ is rational (i.e., $\tilde{X} = \mathbb{P}^1$) but singular, then X satisfies (B_{d-r}), and is $(d+1-r)$-regular provided that $d > r+1$. Indeed, we may construct a partial desingularization $X' \to X$ of X such that X' has a single simple node or cusp. Thus X' has arithmetic genus one, and $\omega_{X'} = \mathcal{O}_{X'}$. Then the previous arguments go through with $A = \mathcal{O}_{X'}$ and $E = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^r, p'_* A(n))$, where $p'\colon X' \to \mathbb{P}^r$ denotes the natural map.

(2) A similar approach can be used to prove the $(d-1)$-regularity of a rational curve $X \subseteq \mathbb{P}^r$ (resolve $p_* \mathcal{O}_{\mathbb{P}^1}(d-2)$, as in § 1).

§ 3. The Existence of Secant Lines

The present section is devoted to the analysis of $(d+1-r)$-irregular rational curves, completing the proof of the results stated in the Introduction.

We begin with some remarks on secants. Let $X \subseteq \mathbb{P}^r$ $(r \geqq 3)$ be a non-degenerate curve of degree d. One says that a linear space $L \subset \mathbb{P}^r$ is *n-secant* to X if

$$\dim_k(\mathcal{O}_{\mathbb{P}^r}/\mathcal{I}_X + \mathcal{I}_L) \geqq n.$$

If X has an n-secant *line*, then evidently X cannot be cut out by hypersurfaces of degree $n-1$. In particular, any non-degenerate $X \subseteq \mathbb{P}^r$ with a $(d+2-r)$-secant line L is $(d+1-r)$-irregular. Such a curve is necessarily rational (clearly) and smooth (e.g., by Remark 1 at the end of § 2). Examples exist for any $d \geqq r \geqq 3$.

Our object now is to show that (almost) all $(d+1-r)$-irregular rational curves arise in this manner:

Theorem 3.1. *Let $X \subseteq \mathbb{P}^r$ $(r \geqq 3)$ be a non-degenerate curve of degree d, and assume that X is rational (i.e., that its normalization $\tilde{X}$ is $\mathbb{P}^1$). Then X fails to be $(d+1-r)$-regular if and only if either:*

(i) $d = r$, *i.e., X is a rational normal curve,*

(ii) $d = r+1$,

or

(iii) $d > r+1$, *and X has a $(d+2-r)$-secant line.*

Remark. The case $d = r+1$ is exceptional. If X is smooth, then clearly property (B_1) fails whether or not X has a trisecant line. On the other hand, if X is singular, then (B_1) holds but $h^1(X, \mathcal{O}_X) \neq 0$, i.e., (A_0) fails. We will see, however, that (C_2) is satisfied unless X has a trisecant.

Proof. In view of what has already been said, it remains only to prove the existence of a $(d+2-r)$-secant line when $d > r+1$ and X is $(d+1-r)$-irregular. Moreover by the first remark at the end of the previous section we may assume that X is smooth. We use the notation introduced in the proof of

Proposition 1.2: in particular, $p\colon \mathbb{P}^1=\tilde{X}\to\mathbb{P}^r$ denotes the evident map, and $M=p^*\Omega^1_{\mathbb{P}^r}(1)$.

We assert to begin with that the decomposition of M into a direct sum of line bundles must take one of the following forms:

(1) $$M=\mathcal{O}^{r-2}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-a)\oplus\mathcal{O}_{\mathbb{P}^1}(-b)\qquad (a,b\geqq 2),$$

or

(2) $$M=\mathcal{O}^{r-1}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-d-1+r).$$

Indeed, X being non-degenerate, all the summands of M have negative degree. On the other hand, $H^1(\mathbb{P}^1,\Lambda^2 M\otimes\mathcal{O}_{\mathbb{P}^1}(d-r))\neq 0$ by Proposition 1.2 since X is $(d+1-r)$-irregular. Recalling that M has degree $-d$, it follows that M must contain $\mathcal{O}_{\mathbb{P}^1}(-1)$ as a summand at least $r-2$ times, as claimed. We treat the two cases (1) and (2) separately.

Case (1). Set $A=\mathcal{O}_{\mathbb{P}^1}(d-r)$ and consider the diagram (1.5) arising in the proof of Proposition 1.2. Taking direct images as in that proof, and using the decomposition (1) of M, one finds the following diagram of sheaves on $\mathbb{P}^r$, whose top row is exact:

(*) $$\begin{tikzcd} H^1(\mathbb{P}^1,\Lambda^3 M\otimes A)\otimes_k\mathcal{O}_{\mathbb{P}^r}(-3) \arrow[r, "v"] & H^1(\mathbb{P}^1,\Lambda^2 M\otimes A)\otimes_k\mathcal{O}_{\mathbb{P}^r}(-2) \arrow[r] & R^1f_*\mathcal{F}_1 \arrow[r] & 0. \\ H^1(\mathbb{P}^1,\mathcal{O}^{r-2}_{\mathbb{P}}(-a-b-1)\otimes A)\otimes_k\mathcal{O}_{\mathbb{P}^r}(-3) \arrow[u, hook] \arrow[ur, "w"'] & & & \end{tikzcd}$$

Observe that $R^1f_*\mathcal{F}_1$ must be supported on a set of dimension at least one; otherwise, as noted following the proof of Lemma 1.6, the arguments of § 1 would apply to give the $h^0(\mathbb{P}^1,A)$-regularity of X. The goal now is to show that

$$L\underset{\text{def}}{=}\operatorname{Supp}(R^1f_*\mathcal{F}_1)$$

is a $(d+2-r)$-secant line to X.

To this end we analyze the map w in (*). Use the decomposition (1) of M and (1.4) to construct the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^{r-2}_{\mathbb{P}^1}(-1) \arrow[r] \arrow[d, hook] & V_{\mathbb{P}^1} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & M \arrow[r] & V_{\mathbb{P}^1} \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(d) \arrow[r] & 0 \end{tikzcd}$$

defining a vector bundle E of rank three and degree $r-2$ on $\mathbb{P}^1$. Then $\mathbb{P}(E)$ $[=\operatorname{Proj}(\operatorname{Sym}(E))]$ embeds naturally in $\mathbb{P}(V_{\mathbb{P}^1})=\mathbb{P}^1\times\mathbb{P}^r$, and the graph Γ of p sits in $\mathbb{P}(E)$ via the quotient $E\to\mathcal{O}_{\mathbb{P}^1}(d)\to 0$. Comparing the Koszul complex (1.5) with the evident Koszul resolution of $\mathcal{O}_{\mathbb{P}(E)}$ on $\mathbb{P}^1\times\mathbb{P}^r$, and bearing in mind the splitting (1) of M, one finds that

$$\operatorname{coker} w=R^1f_*(\mathcal{O}_{\mathbb{P}(E)}\otimes N),$$

where N is a line bundle on $\mathbb{P}^1\times\mathbb{P}^r$. In particular, $\operatorname{coker} w$ is supported in the locus on $\mathbb{P}^r$ over which the projection $\mathbb{P}(E)\to\mathbb{P}^r$ fails to be finite. This is in any event a linear space, of dimension one less than the number of trivial summands of E. But E is non-trivial for reasons of degree (when $r \geqq 3$), and so $\operatorname{Supp}(\operatorname{coker} w)$ has dimension $\leqq 1$. On the other hand, $L \subseteq \operatorname{Supp}(\operatorname{coker} w)$ and $\dim L \geqq 1$. Therefore $L = \operatorname{Supp}(\operatorname{coker} w)$ is a line, and $E = \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(r-2)$; L is the image of the divisor $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}) \subseteq \mathbb{P}(E)$ under projection to $\mathbb{P}^r$:

$$\begin{tikzcd}[column sep=small] \mathbb{P}^1 = \Gamma \arrow[d] & \subseteq & \mathbb{P}(E) \arrow[d] \arrow[dr, phantom, "\square"] & \supseteq & \mathbb{P}(\mathcal{O}\oplus\mathcal{O}) = Y \arrow[d] \\ X & \subseteq & \mathbb{P}^r & \supseteq & L \end{tikzcd}$$

But Y intersects Γ in a divisor of degree $d+2-r$. Since X is smooth by assumption, it follows that L is $(d+2-r)$-secant to X, as desired.[1]

Case (2). Much as above, define E to be the cokernel of the composition

$$\mathcal{O}_{\mathbb{P}^1}^{r-1}(-1) \hookrightarrow M \hookrightarrow V_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}^{r+1},$$

so that E is a vector bundle of rank two and degree $r-1$. As before, $\mathbb{P}(E)$ embeds in $\mathbb{P}^1 \times \mathbb{P}^r$, and the graph Γ of p sits in $\mathbb{P}(E)$. Denote by $S \subseteq \mathbb{P}^r$ the rational normal scroll arising as the image of the projection $\mathbb{P}(E) \to \mathbb{P}^r$, and let $t\colon \mathbb{P}(E) \to \mathbb{P}^1$ be the bundle map. Note that $\int c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^2 = r-1$, and that $\Gamma \subseteq \mathbb{P}(E)$ is the divisor of a section of $t^* \mathcal{O}_{\mathbb{P}^1}(d+1-r) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$.

Consider the decomposition of E as a sum of line bundles:

$$E = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

with

$$0 \leqq a \leqq b, \quad a+b = r-1.$$

If $a=0$, then S is a cone and X is singular when $d \geqq r+1$. So we may assume that $a \geqq 1$, in which case $S \cong \mathbb{P}(E)$ and X is smooth. The vanishing of $H^1(X, \mathcal{O}_X(d-r-1))$ is then automatic if $d \geqq r+1$, and we may suppose that $H^1(\mathbb{P}^r, \mathcal{I}_X(d-r)) \neq 0$. Recalling that S is projectively normal, it follows that $H^1(S, \mathcal{I}_{X/S}(d-r)) \neq 0$. But $\mathcal{I}_{X/S}(d-r) = t^* \mathcal{O}_{\mathbb{P}^1}(r-1-d) \otimes \mathcal{O}_{\mathbb{P}(E)}(d-r-1)$, and so

$$t_*(\mathcal{I}_{X/S}(d-r)) = \operatorname{Sym}^{d-r-1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \otimes \mathcal{O}_{\mathbb{P}^1}(r-1-d).$$

Then $\mathcal{I}_{X/S}(d-r)$ has non-vanishing H^1 if and only if

$$(d-r-1)(a-1) \leqq 0.$$

Provided that $d > r+1$, this forces $a=1$. But when $a=1$, the line bundle $t^* \mathcal{O}_{\mathbb{P}^1}(2-r) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ has a section whose divisor L is a line in $\mathbb{P}^r$. Computing intersection numbers on S, one has

$$\#(\Gamma \cdot L) = d+2-r,$$

so L is $(d+2-r)$-secant to X, and we are done. ∎

---

[1] One may verify that in Case (1),

$$h^1(\mathbb{P}^r, \mathcal{I}_X(d-r)) = 1$$

The theorem stated in the Introduction is now proved.

Remarks. (1) In case (2), the computation just completed shows that if $h^1(\mathbb{P}^r, \mathcal{I}_X(d-r)) \neq 0$, then $h^1(\mathbb{P}^r, \mathcal{I}_X(d-r)) = 1$ unless $r=3$, in which case X lies on the smooth quadric $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = S$ and $h^1(\mathbb{P}^r, \mathcal{I}_X(d-r)) = d-3$.

(2) For the Corollary in the Introduction, it remains to analyze property (C) for a rational curve $X \subseteq \mathbb{P}^r$ of degree $d=r+1$. Keeping the notation of the previous proof, we are necessarily in case (2), and there are three possibilities:

(i) $a=0$ ($\Leftrightarrow X$ is singular)
(ii) $a=1$ ($\Leftrightarrow X$ has a trisecant line)
(iii) $a>1$ ($\Leftrightarrow X$ is smooth and has no trisecant).

We assert that (C_2) holds in cases (i) and (iii); it obviously fails when X has a trisecant line. In fact, since the scroll S is itself 2-regular, it is equivalent to verify that the homomorphism

$$H^0(S, \mathcal{I}_{X/S}(2)) \otimes H^0(S, \mathcal{O}_S(n)) \to H^0(S, \mathcal{I}_{X/S}(n+2))$$

is surjective for $n \geqq 0$. One has

$$H^0(S, \mathcal{I}_{X/S}(n)) = H^0(\mathbb{P}^1, \mathrm{Sym}^{n-1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$$
$$H^0(S, \mathcal{O}_S(n)) = H^0(\mathbb{P}^1, \mathrm{Sym}^n(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))),$$

and writing R for the graded polynomial ring in two variables, the question is in turn equivalent to the surjectivity of the evident map

$$(R_{a-2} \oplus R_{b-2}) \otimes \mathrm{Sym}^n(R_a \oplus R_b) \to \bigoplus_{l=0}^{n+1} R_{(n+1-l)a+lb-2}.$$

But this is clearly surjective if $a=0$ or $a>1$ (whereas when $a=1$ the $l=0$ term on the right is not in the image).

It is amusing to tabulate the data we have collected. The chart on the preceeding page shows the various possibilities for a $(d+1-r)$-irregular non-degenerate curve of degree d in $\mathbb{P}^r$.

Table of All Non-degenerate $(d+1-r)$-Irregular Curves $X \subseteq \mathbb{P}^r$ $(r \geqq 3)$ of Degree d

			C_{d+1-r}	B_{d-r}	A_{d-1-r}
$d=r$: X rational normal			No	0	$d-1$
$d=r+1$: X elliptic normal or rational					
X elliptic normal			Yes	0	1
X rational, singular			Yes	0	1
X rational, smooth		$\exists$ tri-sec line	No	1	0
		$\nexists$ tri-sec line	Yes	1	0
$d>r+1$: X rational, smooth, with a $(d+2-r)$-secant line					
$r=3$	$X \subseteq$ smooth quadric		No	$d-3$	0
	$X \nsubseteq$ smooth quadric		No	1	0
$r \geqq 4$			No	1	0

The first column of data indicates whether or not property (C_{d+1-r}) is satisfied. The entries in the columns B_{d-r} and A_{d-1-r} give respectively the dimensions of the groups $H^1(\mathbb{P}^r, \mathcal{I}_X(d-r))$ and $H^2(\mathbb{P}^r, \mathcal{I}_X(d-1-r))$ (these being the superabundances measuring the failure of the corresponding property)

§ 4. Open Problems

We discuss in conclusion some open questions.

(1) The fact that the extremal examples in the main theorem and its corollary are smooth rational curves suggests that one should have progressively stronger regularity estimates as the curve $X \subseteq \mathbb{P}^r$ becomes in some sense increasingly complex. The most naive hope might be for an estimate in terms of the genus g of X, but even for curves on a smooth quadric surface there is no non-decreasing function $f(g)$, going to infinity with g, for which property (B_n) holds for $n \geqq d+1-r-f(g)$. A more promising invariant is the integer

$$e(X) \underset{\text{def}}{=} \max\{n \mid H^1(X, \mathcal{O}_X(n)) \neq 0\},$$

so that for example $e(X) = -1$ if and only if X is smooth and rational.

Conjecture. *For a non-degenerate curve $X \subseteq \mathbb{P}^r$ of degree d, property (B_n) is satisfied for*

$$n \geqq d - r - e(X).$$

When $r=3$ the conjecture has been proved by the first and third authors (to appear), to whom it is due. The conjecture would imply Theorems 1.1 and 2.1.

(2) The thrust of our classification results is that the failure of a curve $X \subseteq \mathbb{P}^r$ to satisfy (B_{d-r}) is accounted for by the presence of a $(d+2-r)$-secant line. We may ask whether the same phenomenon persists for curves for which (B_n) fails provided that n is not too small. Specifically, one might venture the

Conjecture. *For $n \geqq \frac{2d}{3} - (r-3)$, (B_n) fails (essentially) if and only if X has an $(n+2)$-secant line.*

The equivocation is to allow for the possibility of minor exceptions such as the situation with rational curves of degree $d = r+1$. The conjecture has been verified for $X \subseteq \mathbb{P}^3$ and $n \geqq d-4$. We refer the reader to the survey [21] for further discussion.

(3) Practically nothing is known in the way of reasonably sharp explicit regularity results for varieties of dimension two or more. It would be premature to suggest any conjectures, but a number of possibilities present themselves. As Eisenbud among others has remarked, an extremely optimistic hope might be that if $X \subseteq \mathbb{P}^r$ is a reduced, irreducible, non-degenerate variety of dimension m and degree d, then perhaps X satisfies (B_n) for $n \geqq d+m-r$, and hence is $(d+m+1-r)$-regular. This is checked for certain projections of rational scrolls by Meadows [13], and it would be instructive to work out other examples. Even substantially weaker results and estimates would be of interest. For example,. it is elementary that a variety $X \subseteq \mathbb{P}^r$ as above is cut out by hypersurfaces of degree d, at least if it is smooth (cf. [15], proof of Theorem 1). Is X in fact d-regular? Another interesting problem, suggested by Rao, is to generalize Castelnuovo's geometric argument [3] to varieties of dimension two or more. Finally, in thinking about how the techniques above might generalize,

one is led to ask the following question: given a homomorphism

$$u\colon \mathcal{O}_{\mathbb{P}^r}(-1)^N \to \mathcal{O}_{\mathbb{P}^r}^a,$$

is $\mathcal{J} = F^0(u) \subseteq \mathcal{O}_{\mathbb{P}^r}$ a-regular? (This would imply Eisenbud's estimate for smooth varieties ruled over a curve.)

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Note Added in Proof

We would like to call the reader's attention to very interesting work by M. Green (Koszul cohomology and the geometry of projective varieties, to appear), which among other things generalizes and clarifiés some of the results of [15].