



# Torelli theorems for some Steiner bundles

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## Introduction

A Steiner bundle is a vector bundle  $E$  on projective space  $\mathbf{P}^r$  that sits in a short exact sequence

$$0 \longrightarrow U_1 \otimes \mathcal{O}_{\mathbf{P}^r}(-1) \xrightarrow{\phi} U_0 \otimes \mathcal{O}_{\mathbf{P}^r} \longrightarrow E \longrightarrow 0, \quad (1)$$

where  $U_1$  and  $U_0$  are finite-dimensional vector spaces. These bundles arise in several geometric settings, and by now they are the focus of a substantial literature (c.f. [1, 4, 5, 7, 8, 10, 12, 15, 18, 19]). When  $\mathbf{P}^r = |V|$  is the projective space of one-dimensional subspaces of a vector space  $V$ ,  $\phi$  is given by a linear map

$$\mu : U_1 \otimes V \longrightarrow U_0 \quad (2)$$

having the property that  $\mu(u_1 \otimes v) \neq 0$  for all non-zero  $u_1 \in U_1, v \in V$ . For example, suppose that  $V \subseteq H^0(X, A)$  is a very ample linear series on a smooth complex projective variety  $X$  of dimension  $n$ . If  $B$  is a sufficiently positive line bundle on  $X$ , then the Steiner bundle corresponding to multiplication

$$\mu : H^0(X, B \otimes A^*) \otimes V \longrightarrow H^0(X, B) \quad (3)$$

is the tautological vector bundle  $E_{|V|,B}$  on  $|V|$  whose fibre at  $[s] \in |V|$  is the vector space  $H^0(X, B \otimes \mathcal{O}_{\text{Div}(s)})$ . For  $X = \mathbf{P}^1$  and  $V = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(r))$ , these are known as Schwarzenberger bundles. The bundles  $E_{|V|,B}$  were considered by Arrondo in [4] from a somewhat different perspective.

In their influential paper [8], Dolgachev and Kapranov consider the bundle  $E = \Omega_{\mathbf{P}^r}^1(\log \Sigma H_i)$  of logarithmic forms with poles along a normal-crossing hyperplane arrangement on  $\mathbf{P}^r$ . They show that  $E$  is a Steiner bundle, and they establish moreover that one can recover the arrangement from  $E$  provided that the planes do not osculate a rational normal

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curve. This is the prototype of a Torelli-type statement, asserting that a Steiner bundle determines the geometric data used to construct it. Other results along these lines appear in the papers [2, 4, 6, 13, 17–19].

The purpose of this note is to point out that similar Torelli theorems hold for the tautological bundles  $E_{|V|,B}$  once  $B$  is positive enough. To begin with, we prove

**Theorem A** *Let  $X$  be a smooth projective variety, and let  $V \subseteq H^0(X, A)$  be a very ample linear series. Fix a line bundle  $B$  on  $X$  that satisfies*

$$H^i(X, B \otimes A^{\otimes -(i+1)}) = 0 \text{ for } i > 0.$$

*Then one can recover the embedding  $X \subseteq \mathbf{P}(V)$ , together with the line bundle  $B$ , from the Steiner bundle associated to the multiplication mapping (3).*

In the event that  $V$  is basepoint-free but possibly not very ample, one can recover from  $E$  the image  $\phi_{|V|}(X) \subseteq \mathbf{P}(V)$ . The Theorem gives a partial answer to Question 0.2 of Arrondo’s paper [4].

Observe that if  $B = \mathcal{O}_X(K_X + (n + 2)A)$ , then the hypothesis of the Theorem is satisfied automatically thanks to Kodaira vanishing. It is natural to ask what happens for slightly less positive  $B$ , for instance  $B = \mathcal{O}_X(K_X + (n + 1)A)$ . The example where  $X = X_d \subseteq \mathbf{P}^{n+1}$  is a smooth hypersurface of degree  $d$ ,  $A = \mathcal{O}_X(1)$  and  $V = H^0(X, A)$  is instructive here. In this case  $B = \mathcal{O}_X(d - 1)$ , and therefore the map  $\mu$  in (3) is identified with multiplication

$$H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d - 2)) \otimes H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(1)) \longrightarrow H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d - 1)).$$

of all homogeneous polynomials of the indicated degrees. In particular,  $\mu$ —and hence also  $E_{|V|,B}$ —doesn’t see  $X$ . Amusingly, it turns out that this is the only situation in which Torelli fails for the tautological bundles associated to  $\mathcal{O}_X(K_X + (n + 1)A)$ .

Specifically, we use considerations of Koszul cohomology and Green’s  $K_{p,1}$  Theorem from [14] to prove:

**Theorem B** *Let  $V = H^0(X, A)$  for a very ample divisor  $A$  on  $X$ , and assume that  $A$  does not embed  $X$  as a hypersurface in  $\mathbf{P}^{n+1}$ . (In particular, we suppose that  $(X, A) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ .)*

- (i). *If  $B = \mathcal{O}_X(K_X + (n + 1)A)$ , then  $E_{|V|,B}$  determines  $X$ .*
- (ii). *Assume that  $\deg_A(X) \geq \dim H^0(X, A) + 2 - n$ , and that  $H^1(X, \mathcal{O}_X) = 0$  when  $n \geq 2$ . Then the same conclusion holds for*

$$B = \mathcal{O}_X(K_X + nA)$$

*except when  $\phi_{|A|}(X) \subseteq \mathbf{P}(V)$  lies on an  $(n + 1)$ -fold of minimal degree.*

(In the exceptional case of (ii), the bundles  $E_{|V|,B}$  only depend on the scroll containing  $X$ : see Example 2.5.) Finally, we show that these ideas lead to a new proof of the theorem of Dolgachev and Kapranov.

Our arguments revolve around a strategy pioneered by Vallès [18, 19]. His idea is to study hyperplanes  $|W| \subseteq |V|$  for which the restriction  $E|_{|W|}$  has a trivial quotient, with the aim of recovering  $X \subseteq \mathbf{P}(V)$  as the locus of all such. In §1, we show that Theorem A follows easily from considerations of Castelnuovo–Mumford regularity. For Theorem B, in §2, we use duality to relate the existence of unstable hyperplanes to the non-vanishing of certain Koszul cohomology groups, where Green’s results apply. In the Appendix, we indicate how to recover the result of Dolgachev–Kapranov.

We work throughout over the complex numbers, although this hypothesis isn't needed for Theorem A. Given a vector space  $V$ ,  $\mathbf{P}(V)$  denotes the projective space of one-dimensional quotients of  $V$ , while  $|V| = \mathbf{P}(V^*)$  is used for the space of one-dimensional subspaces. Somewhat sloppily, we take the liberty of freely confounding divisors and line bundles.

### 1 Proof of Theorem A

We start by fleshing out the construction of Steiner bundles indicated in the Introduction. Fix a linear map

$$\mu : U_1 \otimes V \longrightarrow U_0, \tag{1.1}$$

where  $U_1, U_0$  and  $V$  are finite-dimensional complex vector spaces with  $\dim V = r + 1$ . Composing with the canonical inclusion  $\mathcal{O}_{|V|}(-1) \subseteq V \otimes_{\mathbb{C}} \mathcal{O}_{|V|}$  of vector bundles on the projective space  $|V|$ ,  $\mu$  gives rise to a morphism

$$\phi : U_1 \otimes \mathcal{O}_{|V|}(-1) \longrightarrow U_0 \otimes \mathcal{O}_{|V|}$$

of locally free sheaves. Assume now that  $\mu(u_1 \otimes v) \neq 0$  for all non-zero vectors  $u_1 \in U_1$  and  $v \in V$ . Then  $\phi$  is injective of constant rank, and therefore  $E = \text{coker } \phi$  is a vector bundle that sits in an exact sequence

$$0 \longrightarrow U_1 \otimes \mathcal{O}_{|V|}(-1) \xrightarrow{\phi} U_0 \otimes \mathcal{O}_{|V|} \longrightarrow E \longrightarrow 0. \tag{1.2}$$

We will sometimes write  $E = \text{Steiner}(\mu)$  when we wish to emphasize the role of  $\mu$ . It is elementary that conversely every Steiner bundle  $E$  arises in this fashion.

The main example for our purposes are tautological bundles associated to a linear system of divisors. Let  $X$  be a smooth projective variety of dimension  $n$ , and let  $A$  be a very ample (or at least ample and basepoint-free) line bundle on  $X$ . Fix a very ample (or basepoint-free) subspace  $V \subseteq H^0(X, A)$  of dimension  $r + 1$ , and denote by

$$\mathcal{D} \subseteq X \times |V|$$

the universal divisor, consisting of pairs  $(x, [s])$  such that  $s(x) = 0$ . It is realized as the zero-locus of a section of  $A \boxtimes \mathcal{O}_{|V|}(1)$ . Now consider a line bundle  $B$  on  $X$  which is sufficiently positive so that  $H^1(X, B \otimes A^*) = 0$ , and set

$$E_{|V|,B} = \text{pr}_{2,*}(\text{pr}_1^*B \otimes \mathcal{O}_{\mathcal{D}}).$$

This is a vector bundle on  $|V|$  whose fibre at  $[s]$  is identified with the space  $H^0(X, B \otimes \mathcal{O}_{\text{Div}(s)})$  of sections of the restriction of  $B$  to the divisor  $\{s = 0\}$ . Starting with the exact sequence

$$0 \longrightarrow \text{pr}_1^*(B \otimes A^*) \otimes \text{pr}_2^*\mathcal{O}_{|V|}(-1) \xrightarrow{\cdot \mathcal{D}} \text{pr}_1^*B \longrightarrow \text{pr}_1^*B \otimes \mathcal{O}_{\mathcal{D}} \longrightarrow 0$$

on  $X \times |V|$  and pushing forward to  $|V|$ , one finds:

**Lemma 1.1** *Assuming always that  $H^1(X, B \otimes A^*) = 0$ ,  $E_{|V|,B}$  is the Steiner bundle on  $|V|$  determined by the natural multiplication map*

$$H^0(X, B \otimes A^*) \otimes V \longrightarrow H^0(X, B).$$

□

We remark that these statements remain true without change if  $B$  is a vector bundle of higher rank on  $X$ .

Returning to the general setting of (1.1), fix a codimension one subspace  $W \subseteq V$  defining a hyperplane  $|W| \subseteq |V|$ . One says that  $|W|$  is an *unstable plane* for the Steiner bundle  $E$  if the restriction of  $E$  to  $|W|$  has a trivial quotient, i.e. if

$$H^0(|W|, E^*|_{|W|}) \neq 0.$$

Hyperplanes in  $|V|$  correspond to points in  $\mathbf{P}(V)$ , and we define the *Vallès locus* of  $E$  to be the algebraic subset  $\text{Vallès}(E) \subseteq \mathbf{P}(V)$  parameterizing all unstable planes.

The following remark is elementary but crucial:

**Lemma 1.2** *A hyperplane  $W \subseteq V$  corresponds to a point in the Vallès locus of a Steiner bundle  $E = \text{Steiner}(\mu)$  if and only if the restriction*

$$\mu|_{U_1 \otimes W} : U_1 \otimes W \longrightarrow U_0$$

*of  $\mu$  to  $U_1 \otimes W \subseteq U_1 \otimes V$  fails to be surjective.*

**Proof** In fact, one sees using the exact sequence

$$0 \longrightarrow U_1 \otimes \mathcal{O}_{|W|}(-1) \longrightarrow U_0 \otimes \mathcal{O}_{|W|} \longrightarrow E|_{|W|} \longrightarrow 0$$

that  $\text{Hom}(E|_{|W|}, \mathcal{O}_{|W|}) \cong \text{coker}(\mu|_{U_1 \otimes W})^*$ . □

We now move towards the proof of Theorem A. Let  $X$  be a smooth projective variety of dimension  $n$ , and  $A$  a very ample (or ample and globally generated) line bundle on  $X$ . Fix a line bundle  $B$  on  $X$  that satisfies the vanishings

$$H^i(X, B \otimes A^{\otimes -(i+1)}) = 0 \quad \text{for } i > 0. \tag{1.3}$$

This implies that  $B$  is very ample (e.g. by [16, Example 1.8.22] or via the arguments below).

The Theorem is essentially a consequence of the following

**Proposition 1.3** *Keeping the hypotheses just stated, let  $U \subseteq H^0(X, A)$  be a subspace.*

(i). *If  $U$  is basepoint-free, then the multiplication mapping*

$$\mu_U : H^0(X, B \otimes A^*) \otimes U \longrightarrow H^0(X, B)$$

*is surjective.*

(ii). *Suppose that  $U$  generates the sheaf  $A \otimes \mathfrak{m}_x$  of sections of  $A$  vanishing at some point  $x \in X$  (so that in particular every section in  $U$  vanishes at  $x$ ). Then*

$$\text{im}(\mu_U) = H^0(X, B \otimes \mathfrak{m}_x).$$

**Proof** This follows from the theory of Castelnuovo–Mumford regularity (cf [16, Section 1.8]), but for the convenience of the reader we sketch briefly the argument. Assuming  $U$  is basepoint-free, it determines a surjective mapping  $U \otimes A^* \longrightarrow \mathcal{O}_X$  of bundles on  $X$ . The resulting Koszul complex yields a long exact sequence

$$\dots \longrightarrow \Lambda^3 U \otimes A^{\otimes -3} \longrightarrow \Lambda^2 U \otimes A^{\otimes -2} \longrightarrow U \otimes A^* \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of vector bundles on  $X$ . Tensoring through by  $B$  and taking cohomology, the hypothesis (1.3) implies with a diagram chase that the map

$$H^0(X, U \otimes B \otimes A^*) \longrightarrow H^0(X, B)$$

is surjective. This proves (i). In the setting of (ii),  $U \otimes A^*$  maps onto  $\mathfrak{m}_x$ , and now one arrives at a complex having the shape:

$$\dots \longrightarrow \Lambda^3 U \otimes A^{\otimes -3} \longrightarrow \Lambda^2 U \otimes A^{\otimes -2} \longrightarrow U \otimes A^* \xrightarrow{\varepsilon} \mathfrak{m}_x \longrightarrow 0,$$

with  $\varepsilon$  surjective. This complex is not exact, but its homology sheaves are supported at the point  $x$ , and another diagram chase shows that this suffices to conclude the surjectivity of

$$H^0(X, U \otimes B \otimes A^*) \longrightarrow H^0(X, B \otimes \mathfrak{m}_x).$$

We refer to [16, Example B.1.3] for more details. □

**Proof of Theorem A** Assume that  $V \subseteq H^0(X, A)$  is very ample, and that (1.3) holds. We assert to begin with that

$$\text{Vallès}(E_{|V|,B}) = \phi_{|V|}(X) \subseteq \mathbf{P}(V). \tag{*}$$

In fact, by definition points in the image of  $\phi_{|V|}$  correspond to hyperplanes  $W \subset V$  consisting of sections that vanish at a fixed point  $x \in X$ . On the other hand, the preceding Proposition shows that these are precisely the hyperplanes  $W$  for which

$$H^0(X, B \otimes A^*) \otimes W \longrightarrow H^0(X, B)$$

fails to be surjective. So (\*) follows from Lemma 1.2. It remains to show that one can recover the line bundle  $B$  from  $E_{|V|,B}$ . Suppose then that  $W \subseteq V$  is a hyperplane corresponding to a point  $\phi_{|V|}(x) \in \mathbf{P}(V)$ . Proposition 1.3 (ii) and (the proof of) Lemma 1.2 imply that  $E_{|W|}$  has a unique trivial quotient. Via the isomorphism

$$H^0(X, B) = H^0(|W|, E_{|W|})$$

this determines a one-dimensional quotient of  $H^0(B)$ . One verifies that this is  $\phi_{|B|}(x) \in \mathbf{P}H^0(B)$ , and therefore one can reconstruct from  $E$  the embedding defined by  $B$ , as claimed. □

**Remark 1.4** We leave it to the reader to check that the equality

$$\text{Vallès}(E_{|V|,B}) = \phi_{|V|}(X) \subseteq \mathbf{P}(V)$$

still holds assuming only that  $V \subseteq H^0(X, A)$  is basepoint-free. We note also that everything we have said goes through with only evident minor changes if  $B$  is a vector bundle of higher rank, provided of course that (1.3) still holds. □

## 2 Proof of Theorem B

Theorem B draws on some ideas and results concerning Koszul cohomology. We start by recalling the requisite definitions and facts.

Let  $X$  be a smooth complex projective variety of dimension  $n$ ,  $A$  a very ample line bundle on  $X$ , and  $N$  an arbitrary line bundle. Fix a basepoint-free subspace  $U \subseteq H^0(X, A)$ . For every  $p, q \geq 0$  one can form the Koszul-type complex

$$\begin{aligned} \dots &\longrightarrow \Lambda^{p+1} U \otimes H^0(N \otimes A^{\otimes(q-1)}) \longrightarrow \Lambda^p U \\ &\otimes H^0(N \otimes A^{\otimes q}) \longrightarrow \Lambda^{p-1} U \otimes H^0(N \otimes A^{\otimes(q+1)}) \longrightarrow \dots \end{aligned}$$

The cohomology of this complex is denoted by

$$K_{p,q}(X, N; U),$$

the line bundle  $A$  being understood; when  $N = \mathcal{O}_X$  we write simply  $K_{p,q}(X; U)$ . These vector spaces control the minimal free resolution of  $\bigoplus_k H^0(X, N \otimes A^{\otimes k})$  viewed as a graded module over the symmetric algebra  $\text{Sym}(U)$ , but we do not draw on this interpretation. However this is a very rich and well-developed story, and we refer for instance to [3, 11, 14] for introductions.

Theorem B is a simple consequence of several observations and results about these groups. The first follows immediately from the definitions:

**Lemma 2.1** *Keep notation as above.*

- (i). *If  $p > 0$ , then  $K_{p,0}(X; U) = 0$ .*
- (ii). *Suppose that  $W \subseteq U \subseteq H^0(X, A)$  is a codimension one basepoint-free subspace of  $U$ . Then there is a long exact sequence*

$$\begin{aligned} \dots &\longrightarrow K_{p,q-1}(X, N; W) \longrightarrow K_{p,q}(X, N; W) \longrightarrow K_{p,q}(X, N; U) \\ &\longrightarrow K_{p-1,q}(X, N; W) \longrightarrow \dots \end{aligned}$$

**Proof** The map  $\Lambda^p U \longrightarrow \Lambda^{p-1} U \otimes H^0(A)$  is injective provided that  $p > 0$ , and by definition  $K_{p,0}(X; U)$  is its kernel. For (ii), the exact sequence of vector spaces

$$0 \longrightarrow \Lambda^p W \longrightarrow \Lambda^p U \longrightarrow \Lambda^{p-1} W \longrightarrow 0$$

determines a short exact sequence of the complexes computing Koszul cohomology. This gives rise to the stated long exact sequence of cohomology groups. □

The next point is Serre–Grothendieck duality for Koszul groups:

**Proposition 2.2** *Set  $s = \dim |U|$ , and assume that*

$$H^i(X, N \otimes A^{\otimes(q-i)}) = H^i(X, N \otimes A^{\otimes(q-i-1)}) = 0 \text{ for } 0 < i < n. \tag{2.1}$$

*Then there are isomorphisms*

$$K_{p,q}(X, N; U) \cong K_{s-n-p, n+1-q}(X, \omega_X \otimes N^*; U)^*,$$

where  $\omega_X = \mathcal{O}_X(K_X)$  is the canonical bundle of  $X$ .

This is established in [14, Theorem 2.c.6] or [3, Remark 2.26].

The most interesting input to Theorem B is:

**Theorem 2.3** (Green’s  $K_{p,1}$  Theorem) *Set  $V = H^0(X, A)$ , put  $r = \dim |V|$ , and assume that  $\text{deg}_A(X) \geq r - n + 3$ . Then*

$$K_{r-n-1,1}(X; V) \neq 0$$

*if and only if  $\phi_{|A|}(X) \subseteq \mathbf{P}^r$  lies on an  $(n + 1)$ -fold of minimal degree.*

This is [14, Theorem 3.c.1]; see also [3, Chapter 3] for a somewhat different approach.

With these preliminaries out of the way we return to the setting of Theorem B. As before  $X$  is a smooth complex projective variety of dimension  $n$ , and  $A$  is a very ample line bundle on  $X$ . We will work with the full space of sections  $V = H^0(X, A)$ , and following traditional

notation we write  $|A|$  for the corresponding complete linear series, i.e.  $|A| = |H^0(X, A)|$ . We put  $r = \dim |A| = h^0(X, A) - 1$ .

Our goal is to study the Torelli problem for the tautological Steiner bundles  $E_{|A|, B}$  when

$$B = K_X + (n + 1)A \text{ or } B = K_X + nA. \tag{2.2}$$

The first point to observe is that these divisors are sufficiently positive to guarantee that every point of  $X \subseteq \mathbf{P}^r$  gives rise to an unstable plane.

**Lemma 2.4** *Assume that  $(X, A) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ . Then the divisor  $K_X + (n + 1)A$  is very ample and  $K_X + nA$  is basepoint-free. Consequently, with  $B$  as in (2.2) one has*

$$\phi_{|A|}(X) \subseteq \text{Vallès}(E_{|A|, B}) \subseteq \mathbf{P}H^0(X, A).$$

**Proof** The first assertion is established in [9, Lemma 2]. But when  $B$  is globally generated and  $W \subseteq H^0(X, A)$  consists of sections vanishing at some point  $x \in X$ , then

$$H^0(X, B \otimes A^*) \otimes W \longrightarrow H^0(X, B)$$

cannot be surjective. □

**Proof of Theorem B** With  $B$  as in (2.2), we want to check that in fact

$$\phi_{|A|}(X) = \text{Vallès}(E_{|A|, B}).$$

Let  $W \subseteq H^0(X, A)$  be a basepoint-free subspace of codimension one (hence  $\dim_{\mathbf{C}} W = r$ ). Thanks to Lemma 1.2, the issue is to show that multiplication

$$H^0(X, B \otimes A^*) \otimes W \longrightarrow H^0(X, B) \tag{*}$$

is surjective except in the excluded cases.

Suppose first that  $B = \mathcal{O}_X(K_X + (n + 1)A)$  and that (\*) is not surjective. By definition, this means that

$$K_{0, n+1}(X, K_X; W) \neq 0.$$

We now apply Proposition 2.2 with  $N = K_X$  and  $s = r - 1$ . The hypotheses (2.1) follow from Kodaira vanishing, and we conclude that

$$K_{r-n-1, 0}(X; W) \neq 0.$$

But  $r - n - 1 > 0$  since we assume that  $X$  isn't a hypersurface, and we arrive at a contradiction to Lemma 2.1 (i).

The argument when  $B = \mathcal{O}_X(K_X + nA)$  proceeds along similar lines. Assume that (\*) fails to be surjective. Duality applies thanks to Kodaira and the assumption that  $H^1(X, \mathcal{O}_X) = 0$  when  $n \geq 2$ , and therefore  $K_{r-n-1, 1}(X; W) \neq 0$ . Using again that  $K_{r-n-1, 0}(X; W) = 0$ , we conclude from the exact sequence in Lemma 2.1 that  $K_{r-n-1, 1}(X; V) \neq 0$ . But then Green's Theorem 2.3 implies that  $X \subseteq \mathbf{P}^r$  lies on an  $(n + 1)$ -fold of minimal degree. □

**Example 2.5** (Divisors in scrolls) To complete the picture we analyze the exceptional case in (ii) when  $X \subseteq \mathbf{P}^r$  sits as a divisor in an  $(n + 1)$ -fold of minimal degree  $Y \subseteq \mathbf{P}^r$ . For simplicity assume that  $Y$  is smooth, so that  $Y = \mathbf{P}(Q)$  is the total space of an ample vector bundle  $Q$  of rank  $(n + 1)$  on  $\mathbf{P}^1$ . Write  $q = \deg Q$ , and denote by  $H$  and  $F$  respectively the classes of  $\mathcal{O}_{\mathbf{P}(Q)}(1)$  and a fibre, so that  $A = \mathcal{O}_X(H)$ . Then  $X \equiv_{\text{lin}} dH + eF$  for some

integers  $d \geq 2$  and  $e$ . Recalling that  $K_Y \equiv_{\text{lin}} -(n + 1)H + (q - 2)F$ , we see by adjunction that

$$\mathcal{O}_X(K_X + nA) = \mathcal{O}_X((d - 1)H + (e + q - 2)F).$$

The coefficient of  $H$  here being  $< d$ , this implies that

$$\begin{aligned} H^0(X, \mathcal{O}_X(K_X + (n - 1)A)) &= H^0(Y, \mathcal{O}_Y((d - 2)H + (e + q - 2)F)) \\ H^0(X, \mathcal{O}_X(K_X + nA)) &= H^0(Y, \mathcal{O}_Y((d - 1)H + (e + q - 2)F)). \end{aligned}$$

Consequently the Steiner bundle  $E_{|A|,B}$  doesn't vary with  $X$ .

### Appendix A: The theorem of Dolgachev–Kapranov

Let  $X \subseteq \mathbf{P}(V) = \mathbf{P}^r$  be a finite set of  $d \geq r + 1$  points in linear general position, and denote by  $\mathcal{I} = \mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}(V)}$  the ideal sheaf of  $X$ . Green shows in [14, Theorem (3.c.6)] that  $K_{r-2,2}(\mathbf{P}^r, \mathcal{I}; V) \neq 0$  if and only if  $X$  lies on a rational normal curve.<sup>1</sup> Given the arguments from the previous section, it is natural to expect that one can use this to get a new proof of the Torelli-type theorem of Dolgachev and Kapranov from [8] (along with the numerical improvements by Vallès [18]). Inspired by some of the techniques in [14], we indicate here how this goes. For simplicity we assume that  $r \geq 3$ .

Note to begin with that each  $H_*^i(\mathbf{P}(V), \mathcal{I}) =_{\text{def}} \bigoplus_k H^i(\mathbf{P}(V), \mathcal{I}(k))$  is a graded module over the symmetric algebra  $\text{Sym}(V)$ . In particular, there is a natural map

$$H^1(\mathbf{P}(V), \mathcal{I}) \otimes V \longrightarrow H^1(\mathbf{P}(V), \mathcal{I}(1)) \tag{A.1}$$

On the other hand, every point  $x \in X \subseteq \mathbf{P}(V)$  determines a dual hyperplane  $H \subseteq |V|$ , and so  $X$  itself gives rise to a normal crossing hyperplane arrangement  $\Sigma H_i$  on  $|V|$ . One checks that the Dolgachev–Kapranov bundle  $E = \Omega_{|V|}^1(\log \Sigma H_i)$  is the Steiner bundle on  $|V|$  determined by the multiplication map

$$H^1(\mathbf{P}(V), \mathcal{I}(1))^* \otimes V \longrightarrow H^1(\mathbf{P}(V), \mathcal{I})^*,$$

deduced from (A.1). It is elementary that  $X \subseteq \text{Vallès}(E)$ , and we want to verify

**Proposition A.1** *If  $X \subsetneq \text{Vallès}(E)$ , then  $X$  lies on a rational normal curve in  $\mathbf{P}(V)$ .*

Equivalently, fix a subspace  $W \subseteq V$  of codimension one that generates  $\mathcal{O}_X$ . In view of Lemma 1.2, we need to show that if the mapping

$$H^1(\mathbf{P}(V), \mathcal{I}_X(1))^* \otimes W \longrightarrow H^1(\mathbf{P}(V), \mathcal{I}_X)^* \tag{A.2}$$

fails to be surjective, then  $X$  lies on a rational normal curve.

Since  $W \subseteq V$ , each  $H_*^i(\mathbf{P}(V), \mathcal{I})$  has the structure of a  $\text{Sym}(W)$ -module. The first point is that in bounded degrees, one can realize these as the cohomology modules of a sheaf  $\mathcal{J}$  on  $\mathbf{P}(W)$ . Specifically:

**Lemma A.2** *For any suitably large integer  $k_0 \gg 0$  one can construct a coherent sheaf  $\mathcal{J}$  on  $\mathbf{P}(W)$  with the property that for  $i < r - 1 = \dim \mathbf{P}(W)$  there are isomorphisms*

$$H_*^i(\mathbf{P}(V), \mathcal{I})_{\leq k_0} \cong H_*^i(\mathbf{P}(W), \mathcal{J})_{\leq k_0}$$

<sup>1</sup> The statement in [14] actually involves  $K_{r-1,1}$  of the homogeneous coordinate ring of  $X$ , but this is isomorphic to the stated group. See also [3, Lemma 3.29] for another exposition.

in degrees  $\leq k_0$ , and these isomorphisms are compatible with the  $\text{Sym}(W)$ -module structures on both sides.

Granting the Lemma for the time being, we give the

**Proof of Proposition A.1** Tensoring the universal Koszul complex on  $\mathbf{P}(W)$  by  $\mathcal{J}$ , one arrives at a long exact sequence

$$0 \longrightarrow \Lambda^r W \otimes \mathcal{J} \longrightarrow \Lambda^{r-1} W \otimes \mathcal{J}(1) \longrightarrow \Lambda^{r-2} W \otimes \mathcal{J}(2) \longrightarrow \dots \longrightarrow \mathcal{J}(r) \longrightarrow 0$$

of sheaves on  $\mathbf{P}(W)$ . This in turn gives rise to a hypercohomology spectral sequence abutting to zero. The bottom two rows of its  $E_1$  page have the form

$$\begin{aligned} 0 \longrightarrow \Lambda^r W \otimes H^1(\mathcal{J}) &\longrightarrow \Lambda^{r-1} W \otimes H^1(\mathcal{J}(1)) \longrightarrow \Lambda^{r-2} W \otimes H^1(\mathcal{J}(2)) \longrightarrow \Lambda^{r-3} W \otimes H^1(\mathcal{J}(3)) \longrightarrow \\ 0 \longrightarrow \Lambda^r W \otimes H^0(\mathcal{J}) &\longrightarrow \Lambda^{r-1} W \otimes H^0(\mathcal{J}(1)) \longrightarrow \Lambda^{r-2} W \otimes H^0(\mathcal{J}(2)) \longrightarrow \Lambda^{r-3} W \otimes H^0(\mathcal{J}(3)) \longrightarrow \end{aligned}$$

where the cohomology groups are taken on  $\mathbf{P}(W)$ . The assumption (A.2) means (thanks to the Lemma) that the map

$$H^1(\mathcal{I}) = \Lambda^r W \otimes H^1(\mathcal{J}) \longrightarrow \Lambda^{r-1} W \otimes H^1(\mathcal{J}(1)) = W^* \otimes H^1(\mathcal{I}(1))$$

has a non-trivial kernel. This must cancel against the  $E_2$  term coming from the bottom row of the spectral sequence. In other words,

$$K_{r-2,2}(\mathbf{P}(W), \mathcal{J}) \neq 0.$$

But  $K_{r-2,1}(\mathbf{P}(W), \mathcal{J}) = 0$  since  $X$  does not lie on a hyperplane, so by (the analogue of) Lemma 2.1 (ii), we conclude that  $K_{r-2,2}(\mathbf{P}(V), \mathcal{I}) \neq 0$ . Then Green’s theorem applies to put  $X$  on a rational normal curve.  $\square$

**Proof of Lemma A.2** Let  $w \in \mathbf{P}(V)$  be the point corresponding to  $W \subseteq V$ , so that in particular  $w \notin X$ . Projection  $\pi : (\mathbf{P}(V) - \{w\}) \longrightarrow \mathbf{P}(W)$  from  $w$  gives an identification

$$\mathbf{P}(V) - \{w\} \cong \text{Spec}_{\mathbf{P}(W)}(\text{Sym}(\mathcal{O}_{\mathbf{P}(W)} \oplus \mathcal{O}_{\mathbf{P}(W)}(-1))).$$

Then for  $k_0 \gg 0$ , one arrives at a surjective mapping

$$\varepsilon : \text{Sym}^{k_0}(\mathcal{O}_{\mathbf{P}(W)} \oplus \mathcal{O}_{\mathbf{P}(W)}(-1)) \longrightarrow \pi_* \mathcal{O}_X$$

of sheaves on  $\mathbf{P}(W)$ . It suffices to take  $\mathcal{J} = \ker(\varepsilon)$ .  $\square$

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