Local syzygies of multiplier ideals

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Introduction

The purpose of this note is to prove an elementary but somewhat surprising result concerning the syzygies of multiplier ideals. It follows in particular that in dimensions three or higher, multiplier ideals are very special among all integrally closed ideals.

Let X be a smooth complex algebraic variety of dimension d, and let $\mathfrak{b} \subseteq \mathcal{O}_X$ be an ideal sheaf. Given a rational or real number $c > 0$ one can construct the *multiplier ideal*

$$\mathcal{J}(\mathfrak{b}^c) = \mathcal{J}(X, \mathfrak{b}^c) \subseteq \mathcal{O}_X$$

of $\mathfrak{b}$ with weighting coefficient c. This is a new ideal on X that measures in a somewhat subtle manner the singularities of functions $f \in \mathfrak{b}$. In recent years, multiplier ideals have found many applications in local and global algebraic geometry (cf. [2], [1], [4], [13], [5], [14], [16], [7], [8], [3], [9]).

Because of their importance, there has been some interest in trying to understand how general or special multiplier ideals may be among all ideal sheaves. Multiplier ideals are always integrally closed, but up to now they have not been known to satisfy any other local properties. In fact, Favre-Jonsson [6] and Lipman-Watanabe [11] proved that in dimension $d = 2$, every integrally closed ideal can locally be realized as a multiplier ideal.

The first examples of integrally closed non-multiplier ideals in dimension ≥ 3 were discovered by the second author, who recently gave some quite delicate geometric arguments to show that the ideal of a suitable number of general lines through the origin in $\mathbf{C}^3$ couldn't arise as a multiplier ideal. However the construction didn't pinpoint any general features of multiplier ideals that might be violated: rather the idea was to follow a potential

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resolution of singularities of the data with enough care that one could eventually get a contradiction.[1]

Our main result shows that multiplier ideals satisfy some possibly unexpected properties of an algebraic nature. In the following, we work in the local ring $(\mathcal{O}, \mathfrak{m})$ of X at a point $x \in X$, and as above $d = \dim X$.

Theorem A. *Let* $\mathcal{J} = \mathcal{J}(\mathfrak{b}^c)_x \subseteq \mathcal{O}$ *be* (*the germ at x of*) *any multiplier ideal. If* $p \geq 1$, *then no minimal p^{th} syzygy of $\mathcal{J}$ vanishes modulo* $\mathfrak{m}^{d+1-p}$.

Let us explain the statement more precisely. For the case $p = 1$, fix minimal generators $f_1, \ldots, f_b \in \mathcal{J}$, and let $g_1, \ldots, g_b \in \mathfrak{m}$ be functions giving a minimal syzygy

$$\sum g_i f_i = 0$$

among the f_i. Then the claim is that

$$\operatorname{ord}_x(g_i) \leq d - 1$$

for at least one index i. In general, consider a minimal free resolution

$$\cdots \xrightarrow{u_3} \mathcal{O}^{b_2} \xrightarrow{u_2} \mathcal{O}^{b_1} \xrightarrow{u_1} \mathcal{O}^{b_0} \longrightarrow \mathcal{J} \longrightarrow 0$$

of $\mathcal{J}$, where each u_p is a matrix of elements in $\mathfrak{m}$ whose columns minimally generate the module of pth syzygies of $\mathcal{J}$. The assertion of the theorem is that no column of u_p (or any **C**-linear combination thereof) can consist entirely of functions vanishing to order $\geq d + 1 - p$ at x. Equivalently, no minimal generator of the pth syzygy module

$$\operatorname{Syz}_p(\mathcal{J}) =_{\text{def}} \operatorname{Im}(u_p) \subseteq \mathcal{O}^{b_{p-1}}$$

of $\mathcal{J}$ lies in $\mathfrak{m}^{d+1-p} \cdot \mathcal{O}^{b_{p-1}}$.

The theorem implies that if $d \geq 3$, then many integrally closed ideals cannot arise as multiplier ideals. For example consider $2 \leq m \leq d - 1$ functions

$$f_1, \ldots, f_m \in \mathcal{O}$$

vanishing to order $\geq d$ at x. If the f_i are chosen generally, then the complete intersection ideal $\mathcal{I} = (f_1, \ldots, f_m)$ that they generate will be radical, hence integrally closed. On the other hand, the Koszul syzygies among the f_i violate the condition in Theorem A, and hence $\mathcal{I}$ is not a multiplier ideal. If $d \geq 3$ a modification of this construction (Example 2.2) yields $\mathfrak{m}$-primary integrally closed ideals having a syzygy vanishing to high order. The theorem is optimal when $p = 1$ (Example 2.3), but by taking into account all the u_i we give in Sect. 2 an extension of Theorem A that is generally stronger when $p \geq 2$. Note however that there aren't any restrictions on the

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[1] We remark that these examples are in fact not covered by Theorem A.

order of vanishing of *generators* of a multiplier ideal, since for instance all powers of $\mathfrak{m}$ occur as multiplier ideals.

Theorem A follows from a more technical statement involving the vanishing of a map on Koszul cohomology groups. Specifically, let $h_1, \ldots , h_r \in \mathfrak{m}$ be any collection of non-zero elements generating an ideal $\mathfrak{a} \subseteq \mathcal{O}$, and let $K_\bullet(h_1, \ldots , h_r)$ be the Koszul complex on the h_i. We prove:

Theorem B. *For every $0 \le p \le r$, the natural map*

$$H_p\big(K_\bullet(h_1, \ldots , h_r) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)\big) \longrightarrow H_p\big(K_\bullet(h_1, \ldots , h_r) \otimes \mathcal{J}(\mathfrak{b}^c)\big)$$

vanishes.

Now fix generators $z_1, \ldots , z_d \in \mathfrak{m}$, and write $\mathbf{C} = \mathcal{O}/\mathfrak{m}$ for the residue field at x, viewed as an $\mathcal{O}$-module. Taking $r = d$ and $h_i = z_i$, the theorem implies

Corollary C. *The natural maps*

$$\mathrm{Tor}_p(\mathfrak{m}^{d-p}\mathcal{J}, \mathbf{C}) \longrightarrow \mathrm{Tor}_p(\mathcal{J}, \mathbf{C})$$

vanish for all $0 \le p \le d$.

Theorem A is deduced from this statement. As for Theorem B, the proof is simply to note that an exact "Skoda complex" [9, Section 9.6.C] sits inbetween the two Koszul complexes in question.

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1. Proofs

In this section we prove the results stated in the Introduction.

Proof of Theorem B. Write $K_\bullet(h) = K_\bullet(h_1, \ldots , h_r)$ for the Koszul complex on the h_i, and recall that $\mathfrak{a} = (h_1, \ldots , h_r)$ is the ideal they generate. This Koszul complex contains as a subcomplex the *Skoda complex* appearing in [9, Section 9.6.C]:

(Skod$_\bullet$)

$$\cdots \longrightarrow \mathcal{O}^{\binom{r}{2}} \otimes \mathcal{J}(\mathfrak{a}^{r-2}\mathfrak{b}^c) \longrightarrow \mathcal{O}^r \otimes \mathcal{J}(\mathfrak{a}^{r-1}\mathfrak{b}^c) \longrightarrow \mathcal{J}(\mathfrak{a}^r\mathfrak{b}^c) \longrightarrow 0 \, .$$

The fact that Skod$_\bullet$ is a subcomplex of $K_\bullet(h)$ follows from its construction in [9], but more concretely one can think of Skod$_\bullet$ as being the subcomplex of $K_\bullet(h)$ arising by virtue of the inclusions $\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\ell-1}\mathfrak{b}^c) \subseteq \mathcal{J}(\mathfrak{a}^\ell\mathfrak{b}^c)$ deduced from equation (*) below. The basic fact for our purposes is that Skod$_\bullet$ is *exact* [9, Theorem 9.6.36]: this is an elementary consequence of the local vanishing theorems for multiplier ideals.

Recall next that for any ideals $\mathfrak{q}, \mathfrak{b} \subseteq \mathcal{O}$, one has $\mathfrak{q} \cdot \mathcal{J}(\mathfrak{b}^c) \subseteq \mathcal{J}(\mathfrak{q}\mathfrak{b}^c)$. In the case at hand, if we fix ℓ this gives inclusions

$$(*) \qquad \mathfrak{a}^\ell \mathcal{J}(\mathfrak{b}^c) \subseteq \mathcal{J}(\mathfrak{a}^\ell \mathfrak{b}^c) \subseteq \mathcal{J}(\mathfrak{a}^{\ell-i}\mathfrak{b}^c) \subseteq \mathcal{J}(\mathfrak{b}^c)$$

for any $i \leq \ell$. It follows in the first place that $\mathrm{Skod}_\bullet$ is a subcomplex of $K_\bullet(h) \otimes \mathcal{J}(\mathfrak{b}^c)$. Assuming for the time being that $p \geq 1$, let

$$\mathrm{Trunc}_p = \mathrm{Trunc}_p(K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)) \subseteq K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)$$

be the pth truncation of $K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^p)$, i.e. the complex obtained from $K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)$ by replacing the term $\mathcal{O}^{\binom{r}{p-1}} \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)$ by the image I_{p-1} of the incoming map, and the lower terms by 0. Then it also follows from (*) that Trunc_p is actually a subcomplex of $\mathrm{Skod}_\bullet$. Thus all told we have inclusions

$$(**) \qquad \mathrm{Trunc}_p(K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)) \subseteq \mathrm{Skod}_\bullet \subseteq K_\bullet(h) \otimes \mathcal{J}(\mathfrak{b}^c),$$

which are pictured concretely in the following commutative diagram.

$$
\begin{array}{ccccccc}
\cdots \to & \mathcal{O}^{\binom{r}{p+1}} \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c) & \to & \mathcal{O}^{\binom{r}{p}} \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c) & \longrightarrow & I_{p-1} & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & \mathcal{O}^{\binom{r}{p+1}} \otimes \mathcal{J}(\mathfrak{a}^{r-p-1}\mathfrak{b}^c) & \to & \mathcal{O}^{\binom{r}{p}} \otimes \mathcal{J}(\mathfrak{a}^{r-p}\mathfrak{b}^c) & \to & \mathcal{O}^{\binom{r}{p-1}} \otimes \mathcal{J}(\mathfrak{a}^{r-p+1}\mathfrak{b}^c) & \to \cdots \\
& \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & \mathcal{O}^{\binom{r}{p+1}} \otimes \mathcal{J}(\mathfrak{b}^c) & \longrightarrow & \mathcal{O}^{\binom{r}{p}} \otimes \mathcal{J}(\mathfrak{b}^c) & \longrightarrow & \mathcal{O}^{\binom{r}{p-1}} \otimes \mathcal{J}(\mathfrak{b}^c) & \to \cdots
\end{array}
$$

Theorem B now follows at once from the exactness of the Skoda complex. Indeed,

$$H_p(K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)) = H_p\big(\mathrm{Trunc}_p(K_\bullet(h) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^p))\big)$$

and the map appearing in Theorem B is induced by the inclusion of the two outer complexes in (**). But this inclusion factors through the exact complex $\mathrm{Skod}_\bullet$, and so the map on homology vanishes. When $p = 0$ the same argument works ignoring the truncations (although the statement is tautologous when $p = 0$). $\qquad\square$

Proof of Corollary C. Denote by $K_\bullet = K_\bullet(z_1, \dots, z_d)$ the Koszul complex on the generators $z_1, \dots, z_d \in \mathfrak{m}$. Then

$$\mathrm{Tor}_p(\mathfrak{m}^{d-p}\mathcal{J}(\mathfrak{b}^c), \mathbf{C}) = H_p(K_\bullet \otimes \mathfrak{m}^{d-p}\mathcal{J}(\mathfrak{b}^p))$$
$$\mathrm{Tor}_p(\mathcal{J}(\mathfrak{b}^c), \mathbf{C}) = H_p(K_\bullet \otimes \mathcal{J}(\mathfrak{b}^c)),$$

and so the assertion is a special case of Theorem B. $\qquad\square$

Turning to Theorem A, consider any ideal $\mathcal{I} \subseteq \mathcal{O}$, and choose a minimal free resolution $R_\bullet$ of $\mathcal{I}$:

$$(1) \qquad \cdots \xrightarrow{u_3} R_2 \xrightarrow{u_2} R_1 \xrightarrow{u_1} R_0 \xrightarrow{\pi} \mathcal{I} \longrightarrow 0,$$

where $R_i = \mathcal{O}^{b_i}$. Fix $p \geq 1$, and let $e \in R_p = \mathcal{O}^{b_p}$ be a generator, so that e determines a non-zero class in

$$\text{Tor}_p(\mathcal{I}, \mathbf{C}) = H_p(R_\bullet \otimes \mathbf{C}) = \mathbf{C}^{b_p}.$$

In view of Corollary C, Theorem A follows from:

Proposition 1.1. *Suppose that there is an integer $a \geq 2$ such that*

$$(2) \qquad u_p(e) \in \mathfrak{m}^a \cdot R_{p-1} = \mathfrak{m}^a \cdot \mathcal{O}^{b_{p-1}}.$$

Then e represents a class lying in the image of $\text{Tor}_p(\mathfrak{m}^{a-1}\mathcal{I}, \mathbf{C}) \longrightarrow \text{Tor}_p(\mathcal{I}, \mathbf{C})$.

Proof. We propose to work explicitly with the identifications

$$H_p(R_\bullet \otimes \mathbf{C}) = \text{Tor}_p(\mathcal{I}, \mathbf{C}) = H_p(\mathcal{I} \otimes K_\bullet),$$

where as above $K_\bullet = K(z_1, \ldots, z_d)$ is the Koszul complex

$$\cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \longrightarrow \mathbf{C} \longrightarrow 0$$

on generators $z_1, \ldots, z_d \in \mathfrak{m}$, so that $K_i = \mathcal{O}^{\binom{d}{i}}$. Specifically, consider the double complex $R_\bullet \otimes K_\bullet$. Then one has isomorphisms

$$(3) \qquad \begin{aligned} H_p(R_\bullet \otimes \mathbf{C}) &\cong H_p(\text{Tot}(R_\bullet \otimes K_\bullet)) \\ H_p(\text{Tot}(R_\bullet \otimes K_\bullet)) &\cong H_p(\mathcal{I} \otimes K_\bullet). \end{aligned}$$

As explained in [12, Chapter 3.3], the first isomorphism in (3) is obtained by using a "zig-zag" construction to lift a generator $e \in R_p = \mathcal{O}^{b_p}$ to a p-cycle

$$(4) \qquad \widetilde{e} = (e_0, e_1, \ldots, e_p) \in \text{Tot}(R_\bullet \otimes K_\bullet),$$

where $e_i \in R_{p-i} \otimes K_i$ and $e_0 = e \otimes 1 \in R_p \otimes K_0$. The second isomorphism in (3) arises by associating to a p-cycle $\widetilde{f} = (f_0, \ldots, f_p) \in \text{Tot}(R_\bullet \otimes K_\bullet)$ the homology class of the image

$$\overline{f}_p = (\pi \otimes 1)(f_p) \in \mathcal{I} \otimes K_p$$

of $f_p \in R_0 \otimes K_p$. It suffices to prove:

If $u_p(e) \in \mathfrak{m}^a R_{p-1}$, then for $i \geq 1$ one can arrange in (4) that

$$(5) \qquad e_i \in \mathfrak{m}^{a-1}(R_{p-i} \otimes K_i).$$

Indeed, granting this one can take $\overline{e}_p \in \mathfrak{m}^{a-1} \mathcal{I} \otimes K_p$. This is a p-cycle in $\mathfrak{m}^{a-1} \mathcal{I} \otimes K_\bullet$, and it gives the required lifting of e to a class in $\mathrm{Tor}_p(\mathfrak{m}^{a-1} \mathcal{I}, \mathbf{C})$ $= H_p(\mathfrak{m}^{a-1} \mathcal{I} \otimes K_\bullet)$.

As for (5), consider the commutative diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & R_p \otimes K_2 & \xrightarrow{u_p \otimes 1} & R_{p-1} \otimes K_2 & \xrightarrow{u_{p-1} \otimes 1} & R_{p-2} \otimes K_2 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle 1 \otimes \partial_2} & & \downarrow{\scriptstyle 1 \otimes \partial_2} & & \downarrow{\scriptstyle 1 \otimes \partial_2} & \\
\cdots \longrightarrow & R_p \otimes K_1 & \xrightarrow{u_p \otimes 1} & R_{p-1} \otimes K_1 & \xrightarrow{u_{p-1} \otimes 1} & R_{p-2} \otimes K_1 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle 1 \otimes \partial_1} & & \downarrow{\scriptstyle 1 \otimes \partial_1} & & \downarrow{\scriptstyle 1 \otimes \partial_1} & \\
\cdots \longrightarrow & R_p \otimes K_0 & \xrightarrow{u_p \otimes 1} & R_{p-1} \otimes K_0 & \xrightarrow{u_{p-1} \otimes 1} & R_{p-2} \otimes K_0 & \longrightarrow \cdots
\end{array}
$$

One starts by lifting $\pm(u_p \otimes 1)(e \otimes 1) \in R_{p-1} \otimes K_0$ to an element $e_1 \in R_{p-1} \otimes K_1$, i.e. choosing an element $e_1 \in R_{p-1} \otimes K_1$ such that

$$(1 \otimes \partial_1)(e_1) = (-1)^p \cdot (u_p \otimes 1)(e \otimes 1).$$

The hypothesis on $u_p(e)$ implies that the element on the right lies in $\mathfrak{m}^a(R_{p-1} \otimes K_0)$. Since ∂_1 maps $\mathfrak{m}^{a-1} K_1$ onto $\mathfrak{m}^a K_0$, one can take

$$e_1 \in R_{p-1} \otimes \mathfrak{m}^{a-1} K_1 = \mathfrak{m}^{a-1}(R_{p-1} \otimes K_1).$$

The next step is to lift $\mp(u_{p-1} \otimes 1)(e_1)$ to an element $e_2 \in R_{p-2} \otimes K_2$. The minimality of $u_{p-1} : R_{p-1} \longrightarrow R_{p-2}$ implies that

$$(u_{p-1} \otimes 1)(e_1) \in \mathfrak{m}^a(R_{p-2} \otimes K_1) = (R_{p-2} \otimes \mathfrak{m}^a K_1).$$

Thanks to the exactness of $\mathfrak{m}^{a-1} K_{i+1} \longrightarrow \mathfrak{m}^a K_i \longrightarrow \mathfrak{m}^{a+1} K_{i-1}$ for $i \geq 1$, as above we can take

$$e_2 \in R_{p-2} \otimes \mathfrak{m}^{a-1} K_2.$$

Continuing in this manner one arrives eventually at (5), completing the proof. $\qquad\square$

Remark 1.2. A stronger statement is possible if one takes into account the least order of vanishing of any of the non-zero entries in a matrix for u_i in the minimal resolution (1). Specifically, for $1 \leq i \leq p-1$ let $\varepsilon_i = \varepsilon_i(\mathcal{I})$ be the largest integer such that

$$(6) \qquad\qquad \mathrm{Syz}_i(\mathcal{I}) = \mathrm{Im}(u_i) \subseteq \mathfrak{m}^{\varepsilon_i} R_{i-1}.$$

Then with evident modifications, the argument just completed shows that in the situation of Proposition 1.1, $e \in \mathrm{Tor}_p(\mathcal{I}, \mathbf{C})$ lifts to a class in $\mathrm{Tor}_p(\mathfrak{m}^\delta \mathcal{I}, \mathbf{C})$ with

$$\delta = \delta(\mathcal{I}) = (a-1) + (\varepsilon_{p-1} - 1) + (\varepsilon_{p-2} - 1) + \ldots + (\varepsilon_1 - 1). \quad \square$$

2. Examples and complements

This section is devoted to some examples and further information concerning the syzygies of multiplier and other integrally closed ideals. We begin by outlining a construction of $\mathfrak{m}$-primary integrally closed ideals having a minimal first syzygy that vanishes to high order. The starting point is

Lemma 2.1. *Let $J \subseteq \mathbf{C}[x_1, \ldots, x_d]$ be a homogeneous radical ideal, and let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal ideal of polynomials vanishing at the origin. Then for $k \geq 1$ the ideal*

$$J_k =_{\text{def}} J + \mathfrak{m}^k$$

is integrally closed.

Proof. Given a homogeneous polynomial $f \in \overline{J}_k$, we need to show that $f \in J_k$. If $\deg(f) \geq k$ this is trivial, so we can assume that f has degree $a < k$. By definition f satisfies a polynomial

$$f^n + a_1 f^{n-1} + \ldots + a_n = 0, \quad \text{with } a_i \in \left(J + \mathfrak{m}^k\right)^i.$$

We can suppose that each a_i is homogeneous of degree ai, and then by considerations of degree we find that $a_i \in J$. Thus $f^n \in J$, and since J is radical this implies that $f \in J$. □

Example 2.2 (Integrally closed ideal supported at a point with a syzygy vanishing to high order). Keeping notation as in the previous lemma, let

$$J = (f, g) \subseteq \mathbf{C}[x_1, \ldots, x_d]$$

be the complete intersection ideal generated by two general homogeneous polynomials of degree a. Provided that $d \geq 3$ we can suppose that J is radical. Now take $k \gg 0$ and set $I = J + \mathfrak{m}^k$. Thanks to the lemma I is integrally closed, and hence so too is its localization

$$\mathcal{I} \subseteq \mathcal{O} = \mathbf{C}[x_1, \ldots, x_d]_{\mathfrak{m}}.$$

On the other hand, when k is sufficiently large the Koszul syzygy between f and g remains a minimal first syzygy of $\mathcal{I}$. Thus if $d \geq 3$ we have an $\mathfrak{m}$-primary integrally closed ideal with a syzygy vanishing to arbitrary order a at the origin. □

We next give an example to show that the statement in Theorem A is optimal when $p = 1$.

Example 2.3 (A multiplier ideal on the boundary of Theorem A). Let

$$\mathfrak{b} = (f, g) \subseteq \mathbf{C}[x_1, \ldots, x_d]$$

be the complete intersection ideal generated by two general polynomials vanishing to order $d - 1$ at the origin, which we view as defining an ideal

sheaf on $X = \mathbf{C}^d$. As above, provided that $d \geq 3$ we can take $\mathfrak{b}$ to be radical. We claim that

(*) $$\mathfrak{b} = \mathcal{J}(X, \mathfrak{b}^2),$$

so that we have a multiplier ideal with a first syzygy vanishing to maximal possible order $d - 1$ at the origin. To verify (*), let $\mu : X' \longrightarrow X$ be the blow-up of the origin, with exceptional divisor E. By choosing f and g sufficiently generally, we can suppose that

$$\mathfrak{b} \cdot \mathcal{O}_{X'} = \mathfrak{b}' \cdot \mathcal{O}_{X'}(-(d-1)E),$$

where $\mathfrak{b}' \subseteq \mathcal{O}_{X'}$ is the ideal sheaf of a smooth codimension two subvariety meeting E transversely. In particular, $\mathfrak{b}' = \mathcal{J}(X', (\mathfrak{b}')^2)$. On the other hand, the birational transformation rule [9, 9.2.33] gives that

$$\begin{aligned} \mathcal{J}(X, \mathfrak{b}^2) &= \mu_* \big(\mathcal{J}(X', (\mathfrak{b} \cdot \mathcal{O}_{X'})^2) \otimes \mathcal{O}_{X'}(K_{X'/X}) \big) \\ &= \mu_* \big(\mathfrak{b}' \cdot \mathcal{O}_{X'}(-(d-1)E) \big) \\ &= \mathfrak{b}. \qquad \qquad \square \end{aligned}$$

However when $p \geq 2$, an extension of Theorem A generally gives a stronger bound. Specifically, combining Corollary C with Remark 1.2, one arrives at:

Variant 2.4. *In the situation of Theorem A, suppose that $\mathcal{J}$ has a minimal p^{th} syzygy vanishing to order a_p at x, and for $1 \leq i \leq p-1$ denote by*

$$\varepsilon_i = \varepsilon_i(\mathcal{J})$$

the least order of vanishing at x of all non-zero entries in the matrix u_i appearing in the minimal resolution of $\mathcal{J}$. Then

(*) $$a_p + \varepsilon_{p-1} + \ldots + \varepsilon_1 \leq d - 1. \qquad \qquad \square$$

For example, consider when $d = 4$ the complete intersection ideal

$$\mathcal{I} = (f_1, f_2, f_3) \subseteq \mathcal{O}$$

generated by three functions vanishing to order 2 at the origin. Then $a_2 = \varepsilon_1 = 2$, so $\mathcal{I}$ cannot be a multiplier ideal, but this does not follow from the statement of Theorem A alone. We do not know how close (*) comes to being optimal. However the second author constructs in [10] a number of examples lying on the boundary of Corollary C.

Finally, we say a word about adjoint ideals. Let X be a smooth complex variety, and let

$$\mathfrak{a} = \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$$

be the ideal sheaf of an integral divisor on X. Then $\mathcal{J}(X, \mathfrak{a}) = \mathcal{O}_X(-D)$, so that the multiplier ideal of D is uninteresting. On the other hand, assuming that D is reduced, one can define the *adjoint ideal*

$$\mathrm{adj}(D) \subseteq \mathcal{O}_X$$

of D, which does contain significant information about singularities of D (see [9, Section 9.3.E]). One can ask whether these adjont ideals satisfy the same syzygetic conditions as multiplier ideals. It is a consequence of the following proposition that this is indeed the case:

Proposition 2.5. *Given a reduced divisor $D \subseteq X$, there is an ideal sheaf $\mathfrak{b} \subseteq \mathcal{O}_X$ such that*

$$\mathrm{adj}(D) = \mathcal{J}(\mathfrak{b}).$$

One can view this as a converse of [9, Example 9.3.49]. The Proposition grew out of discussions with Karen Smith and Howard Thompson, and we thank them for allowing us to include it here. See [15] for further applications.

Proof. Let $\mu : X' \longrightarrow X$ be a log resolution of D, with

$$\mu^*(D) = D' + F,$$

D' being the proper transform of D. Set $\mathfrak{b} = \mu_* \mathcal{O}_X(-F)$. We leave it to the reader to check that in fact $\mathrm{adj}(D) = \mathcal{J}(\mathfrak{b})$. $\square$

Note (Added September 13, 2006). D. Kim informs us that he has shown that the analogues for analytic multiplier ideals of the Skoda complexes appearing above remain exact. Hence Theorem A is valid also for the analytic multiplier ideals associated to a PSH function.

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