STABILITY AND RESTRICTIONS OF PICARD BUNDLES, WITH AN APPLICATIONS TO THE NORMAL BUNDLES OF ELLIPTIC CURVES

by

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Introduction.

Let C be a smooth irreducible projective curve of genus $g \geq 1$, and for each integer d let $J_d(C)$ be the Jacobian of C, which we view as parametrizing all line bundles on C of degree d. Denote by L_t the bundle on C corresponding to the point $t \in J_d(C)$. Provided that $d \geq 2g-1$, the vector spaces $H^0(C,L_t)$ fit together to form the fibres of a vector bundle P_d on $J_d(C)$, of rank $d+1-g$, called the degree d *Picard bundle* (defined by this description up tp tensoring by line bundles on $J_d(C)$). These bundles have been the focus of considerable study in recent years, notably by Kempf and Mukai ([K1], [K2], [K3], [M]). To better understand their geometry, it is natural to ask whether P_d is stable with respect to the canonical principal polarization of $J_d(C)$. Kempf [K1] shows that this is indeed the case for the first bundle P_{2g-1}. The main purpose of this note is to complete Kempf's result by proving the following

Theorem. *For every* $d \geq 2g$, *the Picard bundle* P_d *is stable with respect to the polarization on* $J_d(C)$ *defined by the theta divisor* $\Theta_C \subset J_d(C)$.

For g = 2, this was established by Umemura [U]. As in [K1], the proof depends on analyzing the restriction of P_d to C. We show that the restriction of P_d to both $C \subset J_d(C)$ and $(-C) \subset J_d(C)$ are stable; either of these statements implies the result. In the hope that the techniques involved may find other uses in the future, we give rather different arguments for the stability of each of these restrictions.

The Theorem leads to a quick proof of the semi-stability of the normal bundles to an elliptic curve embedded by a complete linear series. More precisely, suppose that X is a compact Riemann surface of genus 1. Let L be a line bundle of degree d on X, and denote by $P^i(L)$ the bundle of i^{th} order principal parts of L, so that $P^i(L)$ has rank i+1. The global sections of L lift canonically to sections of $P^i(L)$, and they surject when $i \leq d-2$. In this case we define a vector bundle $R^i(L)$ by the exact sequence

$$0 \longrightarrow R^i(L) \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow P^i(L) \longrightarrow 0.$$

Thus $R^1(L) = N^* \otimes L$, where N is the normal bundle to X in $\mathbb{P}H^0(L)$, and in general we think of the $R^i(L)$ as higher-order conormal bundles of X. Observing that $R^i(L)$ is essentially the pull-back of a Picard bundle under an étale morphism $X \to X = J_{d-i-1}(X)$, we deduce in §4 the

Corollary. *Provided that* $\deg(L) \geq i+2$, *the bundle* $R^i(L)$ *is semi-stable.*

When i=1 the result is due to Ellingsrud (although by a more involved argument). The general case answers a question of Dolgachev.

The theorem and its corollary give rise to some interesting open problems. First, it follows by well known results of Donaldson and Uhlenbeck-Yau that P_d, like any stable bundle, carries a Hermitian-Einstein metric. The question, suggested by Narasimhan, is whether one can construct

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these metrics explicity. The second problem concerns a characterization of the Picard bundles. Mukai [M] proves that when $g=2$, P_d is (up to twists and translations) the only stable bundle on $J_d(C)$ with the appropriate Chern classes. Is there an analogous result in higher genus? Mukai [M] and Kempf [K3] have shown that if C is non-hyperelliptic, then in any event a small deformation of P_d is again (a twist of a translate of) a Picard bundle. Kempf [K2] has also given some other characterizations of Picard bundles. Finally, if L is a line bundle of degree $d\geq 2g+i$ on any curve C of genus g, one may define higher conormal bundles $R^i(L)$ as above. We conjecture that $R^i(L)$ is always semi-stable for $d >> 0$. Some evidence in this direction appears in §4.

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§ 1. Restrictions of Picard Bundles

We start with some notation. Throughout, C denotes a smooth irreducible projective curve of genus $g\geq 1$ defined over an algebraically closed field of arbitrary characteristic, and $x_0\in C$ is a fixed base-point. We denote by $J_d(C)$ the Picard variety parametrizing line bundles of degree d on C, and we write $[L]\in J_d(C)$ for the point corresponding to a bundle L. Finally, let U_d be the universal bundle on $P_d(C)\times C$, normalized so that U_d is trivial on $J_d(C)\times\{x_0\}$. Thus if $\pi:J_d(C)\times C \to J_d(C)$ is the projection, and if $[L]\in J_d(C)$ is an arbitrary point, then $U_d|\pi^{-1}([L]) \simeq L$. The degree d Picard sheaf on $J_d(C)$ is defined by

$$P_d = \pi_*(U_d).$$

It follows from the base-change theorem and Riemann Roch that if $d\geq 2g-1$, then P_d is actually a vector bundle on $J_d(C)$, with $\mathrm{rk}(P_d)=d+1-g$.

Next, suppose given line bundles

$$A\in J_{d-1}(C) \quad \text{and} \quad B\in J_{d+1}(C).$$

Define embeddings

$$u_A:C \longrightarrow J_d(C) \quad \text{and} \quad v_B:C \longrightarrow J_d(C)$$

via

$$u_A(x)=[A(x)], \qquad v_B(x)=[B(-x)],$$

where as customary $A(x)=A\otimes\mathcal{O}_C(x)$ and $B(-x)=B\otimes\mathcal{O}_C(-x)$. We denote by $C_A\subset J_d(C)$ and $C_B\subset J_d(C)$ the images of u_A and v_B respectively. Observe that if $A, A'\in J_{d-1}(C)$ are two line bundles of degree d-1, then $C_{A'}$ is a translate of C_A (and similarly for C_B and $C_{B'}$).

Lemma 1.1. *If* $d\geq 2g-1$, *then one has canonical isomorphisms*

$$(u_A)^*(P_d) = p_*(q^*A\otimes\mathcal{O}_{C\times C}(\Delta))\otimes\mathcal{O}_C(-x_0)$$

and

$$(v_B)^*(P_d) = p_*(q^*B\otimes\mathcal{O}_{C\times C}(-\Delta))\otimes\mathcal{O}_C(x_0),$$

where $p:C\times C \longrightarrow C$ *and* $q:C\times C \longrightarrow C$ *denote the first and second projection respectively, and* $\Delta\subset C\times C$ *is the diagonal.*

Proof. Taking into account the normalization of U_d, one sees fibrewise that

$$(u_A \times 1_C)^*(U_d) = q^*(A) \otimes \mathcal{O}_{C\times C}(\Delta) \otimes p^*(\mathcal{O}_C(-x_0))$$

and

$$(v_B \times 1_C)^*(U_d) = q^*(B) \otimes \mathcal{O}_{C\times C}(-\Delta) \otimes p^*(\mathcal{O}_C(x_0)).$$

The lemma then follows from the theorem on cohomology and base-change. ∎

Let $\Theta \subset J_d(C)$ denote the canonical principal polarization. Recall that the *slope* (with respect to Θ) of a torsion-free sheaf F on $J_d(C)$ is the rational number

$$\mu(F) = \frac{c_1(F).[\Theta]^{g-1}}{rk(F)} .$$

By definition, a vector bundle P is *stable* [resp. *semi-stable*] with respect to Θ if $\mu(F)<\mu(P)$ [resp. $\mu(F)\leq\mu(P)$] for every non-zero torsion free subsheaf $F\subset P$ with rank(F)<rank(P). Similarly, if V is a bundle on C, then $\mu(V)=\deg(V)/rk(V)$, and V is stable [resp. semi-stable] if $\mu(W)<\mu(V)$ [resp. $\mu(W)\leq\mu(V)$] for all sub-bundles $W\subset V$ with rk(W)<rk(V). (It is equivalent to demand the reverse inequalities on quotients.) As in [K1], the next point to observe is

Lemma 1.2. *Fix* $d\geq 2g$. *Then the stability of* P_d *is implied by either the stability of* $u_A^*(P_d)$ *for general* $[A]\in J_{d-1}(C)$, *or by the stability of* $v_B^*(P_d)$ *for general* $[B]\in J_{d+1}(C)$.

Proof. (Compare [K1]). Working in the ring $Num(J_d(C))$ of cycles on $J_d(C)$ modulo numerical equivalence, recall that $[C_A]=[\Theta]^{g-1}/(g-1)!$ c.f. [F, pp. 256-257] or [ACGH]. Hence the stability of P_d is equivalent to the assertion that

$$(*)\qquad \frac{[C_A].c_1(F)}{rk\ F} < \frac{[C_A].c_1(P_d)}{rk\ P_d}$$

for every torsion free $F\subset P_d$ with $rk(F)<rk(P_d)$. On the other hand, F is locally free outside a set $Z\subset J_d(C)$ of codimension ≥ 2, and we may assume that in fact F is sub-bundle of P_d outside Z. It follows by a dimension count that F is locally free in a neighborhood of $C_A\subset J_d(C)$ for sufficienty general $[A]\in J_{d-1}(C)$, and that $F|C_A$ sits as a sub-bundle of $P_d|C_A$. But this being so, (*) is implied by the stability of $u_A^*(P_d)$ for general A. The same argument proves the statement for $v_B^*(P_d)$ upon observing that if $-1 : J_d(C) \to J_d(C)$ denotes multiplication by -1, then $(-1)^*[\Theta]=[\Theta]$ in $Num(J_d(C))$, and hence $[C_B]=(-1)^*[C_A]=[\Theta]^{g-1}/(g-1)!$. ∎

In view of Lemma 1.2, the issue is to understand something about the bundles appearing in Lemma 1.1. To this end, suppose that L is a non-special line bundle on C, generated by its global sections. Define bundles M_L and E_L on C by

$$E_L = p_*(q^*L\otimes\mathcal{O}_{C\times C}(\Delta))$$

and

$$M_L = p_*(q^*L\otimes\mathcal{O}_{C\times C}(-\Delta)).$$

Starting with the sequence $0 \to q^*L\otimes\mathcal{O}_{C\times C}(-\Delta) \to q^*L \to L\otimes\mathcal{O}_\Delta \to 0$ and taking direct imags, one finds that M_L sits in an exact sequence

$$(1.3)\qquad 0 \longrightarrow M_L \longrightarrow H^0(L)\otimes_{\mathbb{C}}\mathcal{O}_C \longrightarrow L \longrightarrow 0,$$

the homomorphism on the right being the canonical evaluation map. This bundle -- which controls the syzygies of L -- is quite well understood (c.f. [GL], [PR], or [L, §1]). As for E_L, we obtain analogously the exact sequence

(1.4) $$0 \longrightarrow H^0(L)\otimes\mathcal{O}_C \longrightarrow E_L \longrightarrow L\otimes\theta_C \longrightarrow 0,$$

where θ_C denotes the tangent bundle to C. The extension class of (1.4) is given by an element $e_L \in H^0(L)\otimes H^1(L^*\otimes\omega_C) \simeq H^0(L)\otimes H^0(L)^*$, and we will see in §2 that up to scalars e_L=id. In any event, putting together Lemmas 1.1 and 1.2, and noting that tensoring by a line bundle does not affect stability, we see that the Theorem stated in the introduction follows from

Proposition 1.5. *If* $\deg(L)\geq 2g-1$, *then* E_L *is stable, and if* $\deg(L)\geq 2g+1$, *then* M_L *is stable.*

We prove the first statement in §2, while the stability of M_L occupies §3.

§ 2. Stability of E_L

Throughout this section, L denotes a non-special line bundle of degree d on C, generated by its global sections. As above we put $E_L=p_*(q^*L\otimes\mathcal{O}_{C\times C}(\Delta))$, where $p:C\times C\to C$ and $q:C\times C\to C$ are the two projections. Taking direct images of $0 \to q^*L \to q^*L\otimes\mathcal{O}_{C\times C}(\Delta) \to L\otimes\mathcal{O}_\Delta(\Delta) \to 0$ yields the basic exact sequence

(2.1) $$0 \longrightarrow H^0(L)\otimes\mathcal{O}_C \longrightarrow E_L \longrightarrow L\otimes\theta_C \longrightarrow 0,$$

θ_C being the tangent bundle to C. Our purpose it to prove

Proposition 2.2. *If* $d>2g-2$ [*resp. if* $d\geq 2g-2$] *then* E_L *is stable* [*resp. is semi-stable*].

We start with several lemmas.

Lemma 2.3. *Using Serre duality to make the identification* $H^1(L^*\otimes\omega_C)=H^0(L)^*$, *the extension class* $e\in H^0(L)\otimes H^1(L^*\otimes\omega_C)$ *defining* (2.1) *is given by a non-zero scalar multiple of the identity* $\mathrm{id}\in H^0(L)\otimes H^0(L)^*$. *In particular,* $H^0((E_L)^*)=0$.

Proof. The second statement follows easily from the first. Consider the sequence $0 \to q^*L\otimes p^*(\omega_C\otimes L^*) \to q^*L\otimes p^*(\omega_C\otimes L^*)(\Delta) \to \mathcal{O}_\Delta \to 0$. Then e is the image of $1\in H^0(\mathcal{O}_\Delta)$ in $H^0(L)\otimes H^1(\omega_C\otimes L^*)$, i.e. the kernel of $H^0(L)\otimes H^1(\omega_C\otimes L^*) \to H^1(q^*L\otimes p^*(\omega_C\otimes L^*)(\Delta))$. Now compute this latter map by taking direct images under q: using duality for q it follows that e spans the kernel of the map induced on global sections by $q_*(p^*L)^*\otimes L = =H^0(L)^*\otimes L \to q_*(p^*L(-\Delta))^*\otimes L=(M_L)^*\otimes L$. But we recognize this homomorphism as a piece of the Euler sequence, and the assertion follows. ■

Lemma 2.4. (Compare [PR] and [B]). *Let* V *be a globally generated vector bundle on* C, *with no trivial summands (i.e. with* $h^0(V^*)=0$). *Then* $\mu(V)>1$.

Proof. Suppose that V has rank r and degree n. Choosing (r+1) general sections of V, we construct an exact sequence

(*) $$0 \longrightarrow V^* \longrightarrow \mathcal{O}^{r+1} \longrightarrow \det V \longrightarrow 0,$$

and since $h^0(V^*)=0$ it follows that $h^0(\det V)\geq r+1$. If det V is special, then Clifford's theorem applies to yield $n=\deg(\det(V))\geq 2(h^0(\det V)-1)\geq 2r$, and so $\mu(V)=n/r\geq 2$ in this case. On the other hand, if det V is non-special then $r \le h^0(\det V)-1=n-g$ by Riemann Roch, and hence $\mu(V) \ge 1+(g/r)$. ■

Lemma 2.5. *Consider an exact sequence*

$$0 \longrightarrow T \longrightarrow V \longrightarrow \tau \longrightarrow 0$$

of sheaves on C, *where* $T=\mathcal{O}^r$ *is a trivial bundle of rank* r, *and* τ *is a torsion sheaf supported on a finite set. If* $\mathrm{length}(\tau)<r$, *then* $h^0(V^*)\neq 0$, *i.e.* V *has a trivial summand.*

Proof. Dualizing the given sequence yields $0 \to V^* \to T^* \to \mathcal{E}xt^1(\tau,\mathcal{O}_C) \to 0$, and $\mathrm{length}(\mathcal{E}xt^1(\tau,\mathcal{O}_C))=\mathrm{length}(\tau)$. The assertion follows. ■

Proof of Proposition 2.2. When $d=\deg(L)=2g-2$ the semi-stability of E_L is clear from (2.1), so we assume $d\ge 2g-1$. Then $\mu(E_L)<1$. Suppose now that E_L fails to be stable. Then there exists a stable quotient sheaf G of E_L with $\mu(G)\le\mu(E_L)<1$. Letting F be the image of the composition $H^0(L)\otimes\mathcal{O}_C \to E_L \to G$, the situation is summarized in the following diagram, which defines a sheaf τ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(L)\otimes\mathcal{O}_C & \longrightarrow & E_L & \longrightarrow & L\otimes\mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & \tau & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that τ -- being a quotient of $L\otimes\theta_C$ -- is either a torsion sheaf or isomorphic to $L\otimes\theta_C$. In particular, $F\neq 0$: for otherwise $G=\tau=L\otimes\theta_C$, but $L\otimes\theta_C$ doesn't destabilize E_L when $d\ge 2g-1$.

We assert that F is trivial. In fact, since F is generated by its global sections one can write $F=F_1\oplus F_2$, where F_1 is trivial and F_2 has no trivial summands. Thus F_2 is a sub-sheaf of G. But if $F_2\neq 0$, then $\mu(F_2)>1$ by Lemma 2.4. This contradicts the stability of G and hence $F=F_1$ is trivial as claimed.

If $\tau=L\otimes\theta_C$ then $\mu(G)>\mu(E_L)$ by a direct computation, so we may assume that τ is a torsion sheaf. If $\mathrm{length}(\tau)\ge\mathrm{rank}(G)$, then again $\mu(G)\ge 1>\mu(E_L)$. So there remains only the possibility that $\mathrm{length}(\tau)<\mathrm{rank}(G)$. But then $h^0((E_L)^*)\neq 0$ thanks to Lemma 2.5, and this contradicts Lemma 2.3. This complete the proof of the Proposition. ■

§ 3. Cohomological Stability of M_L

Let L be a globally generated line bundle on C, and define M_L as at the end of §1. The stability of M_L when $\deg(L)\ge 2g+1$ follows almost immediately from the proof of Lemma 2.4. Indeed, an argument along these lines was given with M. Green some years ago, and Paranjape and Ramanan [PR] independently used such an approach to prove the stability of M_Ω when C is non-hyperelliptic. However, in response to a question of Kempf, we will give an alternative cohomological proof. We start with a

Definition 3.1. Let V be a vector bundle on C. We say that V is *cohomologically stable* [resp. *cohomologically semistable*] if for every line bundle A of degree a, and for every integer $t<\mathrm{rk}(V)$, one has

$$H^0(\Lambda^t V\otimes A^*)=0 \qquad \text{whenever} \qquad a\geq t.\mu(V) \qquad [\text{resp. when } a>t.\mu(V)\ ].$$

Note that cohomological stability indeed implies stability in the usual sense. In fact, a proper subbundle $T\subset V$ of degree a and rank t determines an inclusion $A=_{\mathrm{def}}\Lambda^t T\subset\Lambda^t V$, and hence a non-zero section of $\Lambda^t V\otimes A^*$. The condition in the definition then implies that $\mu(T)<\mu(V)$. In characteristic zero any exterior power of a semistable bundle is semistable, and it follows that in this case cohomological semistability is equivalent to semistability.

Proposition 3.2. *If* $\deg(L)\geq 2g+1$ [*resp.* $\deg(L)\geq 2g$] *then* M_L *is cohomologically stable* [*resp. cohomologically semistable*].

Proof. We assume $d\geq 2g+1$, the other case being almost identical. Keeping notation as in the definition, we must prove that $H^0(\Lambda^t M_L\otimes A^*)=0$ whenever

$$(*) \qquad \frac{a}{t}\geq\mu(M_L)=-1-\frac{g}{d-g}>-2.$$

We use what is by now a standard filtration argument, as in [GLP, p.498], [GL], or [L]. Specifically, set $r=r(L)=d-g$, and choose general points $x_1,\dots,x_{r-1}\in C$. Then (c.f. [L,§1.4]) there is an exact sequence

$$0\longrightarrow L^*(x_1+\dots+x_{r-1})\longrightarrow M_L\longrightarrow\bigoplus^{r-1}\mathcal{O}_C(-x_i)\longrightarrow 0.$$

Put $D=D_{r-1}=x_1+\dots+x_{r-1}$. Taking exterior powers yields

$$0\longrightarrow\Lambda^{t-1}\{\bigoplus^{r-1}\mathcal{O}_C(-x_i)\}\otimes L^*(D)\longrightarrow\Lambda^t M_L\longrightarrow\Lambda^t\bigoplus^{r-1}\mathcal{O}_C(-x_i)\longrightarrow 0.$$

One deduces from this that $H^0(\Lambda^t M_L\otimes A^*)=0$ so long as:

(i). $H^0(A^*(-D_r))=0$ for a general effective divisor D_r of degree t,

and

(ii). $H^0(A^*\otimes L^*(D_{r-t}))=0$ for a general effective divisor D_{r-t} of degree $r-t=d-g-t$.

The line bundle appearing in (i) has degree $-a-t$, and we have t degrees of freedom in choosing it. So provided that $-a-t<g$, the desired vanishing will follow if $t>-(a+t)$. But both of these inequalities are consequences of (*). Similarly, for (ii) it is enough that $\deg(A^*\otimes L^*(D_{r-t}))=-a-t-g<0$. ■

Remark. If E is a globally generated vector bundle on C, one can use the canonical sequence $0\to M_E\to H^0(E)\otimes\mathcal{O}_C\to E\to 0$ to define a bundle M_E on C. Butler [B] has generalized Proposition 3.2 by proving that M_E is stable provided that E is stable and $\mu(E)>2g$. He applies this to obtain interesting surjectivity theorems for the multiplication maps $H^0(E)\otimes H^0(F)\to H^0(E\otimes F)$ on sections of stable bundles, and to prove a conjecture of Kempf concerning the syzygies of the homogeneous coordinate rings of curves. He also studies the stability of M_L for line bundles L with $\deg(L)\leq 2g$. The referee informs us that the stability of M_L has also been investigated by Paranjape in his 1989 thesis.

§ 4. Poly-stability of Normal Bundles to Complete Linear Series on an Elliptic Curve

A number of authors have considered the stability of the normal bundles to space cures (c.f. [GS], [EV], [EL],[Hu], or [Ha]), but in general the situation seems rather complicated. However as a very simple application of our main theorem, we show that for linearly normal embeddings of elliptic curves in characteristic zero, one obtains a fairly clean picture.

Let X be a compact Riemann surface of genus 1, and let L be a line bundle of degree d on X. Fix $i \leq d-2$, and let $R^i(L)$ be the rank d-i-1 vector bundle on X defined by

$$R^i(L)=p_*(q^*L\otimes\mathcal{O}_{X\times X}(-(i+1)\Delta)),$$

where as above $p,q:X\times X \to X$ denote the two projections. As noted in the introduction, these higher conormal bundles fit into exact sequences

$$0 \longrightarrow R^i(L) \longrightarrow H^0(L)\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow P^i(L) \longrightarrow 0.$$

Theorem 4.1. *The bundle* $R^i(L)$ *is poly-stable, i.e. it is a direct sum of stable bundles of the same slope. In particular,* $R^i(L)$ *is semi-stable.*

Proof. The line bundle $\mathcal{L}=q^*L\otimes\mathcal{O}_{X\times X}(-(i+1)\Delta)$ defines a family of degree d-i-1 line bundles on X parametrized by X. This induces a finite surjective (and hence étale) classifying morphism

$$f:X \longrightarrow J_{d-i-1}(X)=X$$

with the property that $\mathcal{L}=(1\times f)^*(U_{d-i-1})\otimes p^*\eta$ for some bundle η on X, where as in §1 U_d denotes the Poincaré bundle on $X\times X$. It follows from the base-change theorem that $R^i(L)=f^*(P_{d-i-1})\otimes\eta$, and hence $R^i(L)$ is a twist of the pull-back of a stable bundle P under an étale covering. But such a pull-back is automatically polystable. (The semi-stability of f^*P follows by a standard descent argument from the uniqueness of a maximal destabilizing sub-bundle. The stronger assertion that f^*P is actually poly-stable ia a consequence of the characterization of such bundles as those having an Hermitian-Einstein connection: alternatively, in the case at hand one could give a more direct elementary argument.) ■

Theorem 4.1 suggests that unlike the situation for incomplete linear series, the normal bundles of curves embedded by a complete linear series of sufficiently large degree behave in a uniform manner:

Conjecture 4.2. *There is an integer* $d(g,i)$ *such that if* C *is any curve of genus* g *(say in characteristic zero), then the conormal bundle* $R^i(L)$ *defined as above is semi-stable for any line bundle* L *of degree* $d\geq d(g,i)$.

One can use Proposition 3.2 to show that in any event $R^i(L)$ cannot be "too unstable" for $d>>0$. In fact, to fix ideas let L be a line bundle of degree $d\geq 2g+1$, and consider $R(L^2)=R^1(L^2)$, which has slope $-2-4g/(4d-g-1)$. One may identify the fibre of M_L at a point $p\in C$ with the vector space $H^0(L(-p))$, and similarly the fibre of $R(L^2)$ at p is $H^0(L^2(-2p))$. Then the canonical map

$$H^0(L(-p))\otimes H^0(L(-p)) \longrightarrow H^0(L^2(-2p))$$

globalizes to a vector bundle homomorphism $M_L\otimes M_L \to R(L^2)$ which is surjective for $d\geq 2g+2$. But in characteristic zero the tensor product of two stable bundles is semi-stable, and hence $M_L\otimes M_L$ is semi-stable, of slope $2\mu(M_L)=-2-4g/(4d-2g)$. In particular, any quotient of $R(L^2)$ has slope $\geq$-2-4g/(4d-2g). Unfortunately, when g$\geq$2 this falls slightly short of proving the semi-stability of $R(L^2)$.

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