A remark on projective embeddings of varieties with non-negative cotangent bundles

Lawrence Ein*, Bo Ilic** and Robert Lazarsfeld***

Introduction

The purpose of this note is to establish an elementary but somewhat unexpected lower bound on the degrees of projective embeddings of varieties with numerically effective cotangent bundles, including in particular quotients of bounded domains.

In recent years, there has been interest in understanding the geometry of complex projective varieties whose tangent or cotangent bundles satisfy various positivity properties. In this note, we shall be concerned with smooth complex projective varieties X satisfying the following non-negativity property:

(NCB). *The cotangent bundle Ω_X^1 of X is numerically effective (nef).*

By definition, the condition means that the Serre line bundle $\mathcal{O}_{\mathbf{P}(\Omega_X^1)}(1)$ on the projectivization $\mathbf{P}(\Omega_X^1)$ is numerically effective, or equivalently that for any non-constant map $\nu : C \longrightarrow X$ from a smooth curve C to X, any quotient bundle of $\nu^*\Omega_X^1$ has non-negative degree. Property (NCB) is satisfied, for example, by smooth subvarieties of abelian varieties, by varieties uniformized by the ball or other irreducible Hermitian symmetric spaces (cf. [Mok], §1), and by products and submanifolds thereof. More generally, a very nice theorem of Kratz ([Kratz], Theorem 2) states that any projective manifold whose universal covering is a bounded domain in $\mathbf{C}^n$ satisfies (NCB).

Our result is that if X satisfies (NCB), then the degree of X in any projective embedding must grow essentially exponentially in the dimension of X. Specifically, given a positive integer n, define

$$\delta(n) = 2^{[\sqrt{n}]},$$

where as usual $[x]$ denotes the integer part of x.

Theorem. *Let X be a smooth projective variety of dimension n which satisfies Property (NCB), and let*

$$f : X \longrightarrow \mathbf{P}^n$$

be any finite surjective mapping. Then $\deg(f) \geq \delta(n)$. *In particular, the degree of X in any projective embedding* $X \subset \mathbf{P}^r$ *must be at least* $\delta(n)$.

We suspect that these statements are not optimal, and that there should be genuinely exponential, or even factorial, bounds on the degree. It would be interesting to prove results along these lines. More philosophically, these results suggest that the complexity of the projective geometry associated to varieties satisfying (NCB) grows exponentially with their dimension. It would be interesting to know if one could make this viewpoint precise, and whether it has any other manifestations.

The proof of the theorem requires only a few lines, and in fact the two ingredients that enter into the argument are at least implicitly quite well known. One simply notes that the hypothesis (NCB) forces the presence of points where the derivative of f drops rank substantially, and that this in turn leads to a lower bound on $\deg(f)$. Nonetheless, the conclusion came as something of a surprise to us: while linear bounds on the degree are very familiar (e.g. [GL], Theorem 2), the existence of essentially exponential statements seems to have been overlooked.

The third author had the opportunity to discuss some of these matters with Michael Schneider about a year before his death, and as always Michael was enthusiastic and encouraging. We hope therefore that the present note might not be out of place in this volume dedicated to his memory. Schneider contributed a lot to algebraic geometry on both a personal and a professional level, and he will be greatly missed.

The proof of the main result occupies in §1. Some applications and variants appear in §2. We are grateful to D. Burns and N. Mok for some valuable discussions. We particularly wish to thank Mohan Ramachandran for informing us of Kratz's theorem.

1. Proof of the theorem

We start with a lemma on degrees and singularities of branched coverings. It was suggested by some examples of Flenner and Ran alluded to in [Ran 2].

Lemma 1.1. *Let* $f : X \longrightarrow Y$ *be a finite surjective map of smooth complex varieties of dimension n. Fix a point* $x \in X$, *let* $y = f(x) \in Y$, *and denote by* $e_f(x)$ *the local degree of f at x, i.e. the multiplicity of x in its fibre* $f^{-1}f(x)$. *Suppose that derivative* $df_x : T_x X \longrightarrow T_y Y$ *of f at x has rank* $n - k$. *Then* $e_f(x) \geq 2^k$, *and consequently* $\deg(f) \geq 2^k$.

Proof. By hypothesis, the co-derivative $df_x^* : T_y^* Y \longrightarrow T_x^* X$ has a k-dimensional kernel. Denoting by $m_x \subset \mathcal{O}_x X$ and $m_y \subset \mathcal{O}_y Y$ the maximal ideals of x and y

respectively, we can therefore choose a system of parameters $u_1, \ldots, u_n \in m_y$ in such a way that $f^* u_1, \ldots, f^* u_k \in m_x^2$. Now

$$e_f(x) = \dim_{\mathbf{C}} \mathcal{O}_x X / f^* m_y,$$

i.e. $e_f(x)$ is alternatively the intersection multiplicity at x of the (germs of) divisors defined by the $f^* u_i$. On the other hand, it is well known (cf [Fult, 12.4]) that this intersection multiplicity is at least the product of the multiplicities $\mathrm{ord}_x f^* u_i$ of the individual divisors. Since by construction $\mathrm{ord}_x f^* u_i \geq 2$ for $1 \leq i \leq k$, the stated lower bound on $e_f(x)$ follows. The inequality on $\deg(f)$ is then a consequence the fact that for fixed $y \in Y$,

$$\sum_{f(x)=y} e_f(x) = \deg(f). \qquad \square$$

The plan is to apply the lemma to branched coverings of projective space. The following well-known fact, which we include for the convenience of the reader, will let us apply theorems on degeneracy loci to guarantee the existence of singularities.

Lemma 1.2. *Let X be a projective variety, and let E and F be vector bundles on X. If E is nef and F is ample, then $E \otimes F$ is ample.*

Sketch of Proof. The statement is a consequence of Kleiman's criterion (cf. [Hart]) that the nef cone is the closure of the ample cone, and the argument is most easily stated using the language of vector bundles twisted by **Q**-divisors, as in [Myka]. First, one verifies the statement when F is a line bundle, or more generally an ample Q-divisor: we leave this to the reader. Next, fix an ample line bundle H on X. Since E is nef, it follows that $E(\frac{1}{N}H)$ is ample for any $N > 0$, and since F is ample, $F(-\frac{1}{N}H)$ is ample for $N \gg 0$. Therefore $E \otimes F = E(\frac{1}{N}H) \otimes F(-\frac{1}{N}H)$ is ample. $\qquad \square$

Now we turn to the

Proof of the theorem. Assume that X is smooth projective variety of dimension n whose cotangent bundle Ω_X^1 is nef, and suppose given a branched covering $f : X \longrightarrow \mathbf{P}^n$. Let

$$S_i(f) = \{x \in X \mid \mathrm{rank}\, df_x \leq n - i\}.$$

This is an algebraic subset of X whose expected dimension is $n - i^2$ (cf. [Fult], Chapter 14). In particular, setting $k = [\sqrt{n}]$, $S_k(f)$ has non-negative postulated dimension. The asserted bound on $\deg(f)$ will follow from Lemma 1.1 as soon as we show that $S_k(f) \neq \emptyset$. But this is a consequence of [FL1] or [L, §2] or [FL2]. In fact, since the tangent bundle $T\mathbf{P}^n$ (and hence also $f^* T\mathbf{P}^n$) is ample, the hypothesis (NCB) implies by Lemma 1.2 that $\Omega_X^1 \otimes f^* T\mathbf{P}^n$ is an ample vector bundle on X. The cited results then guarantee that the vector bundle map $df : TX \longrightarrow f^* T\mathbf{P}^n$ must actually drop rank whenever it is dimensionally predicted to do so. Finally, given an embedding

$X \subset \mathbf{P}$ of X into some projective space, we get by projection a branched covering $f : X \longrightarrow \mathbf{P}^n$ whose degree is the degree of X in $\mathbf{P}$, and so $\deg(X) \geq \delta(n)$.   □

Remark. Given a smooth variety X with nef cotangent bundle, and an ample line bundle L on X which is generated by its global sections, the theorem is equivalent to the assertion that $\int c_1(L)^n \geq \delta(n)$. It is perhaps worth noting that this bound can fail if L is not globally generated. For example, fixing n, let C be a smooth curve of genus $g \gg n$ which carries no g_n^1, and let $X = \mathrm{Sym}^n(C)$ be the nth symmetric product of C. The Abel–Jacobi map $X \longrightarrow \mathrm{Jac}^n(C)$ is an embedding, so X satisfies (NCB). On the other hand, upon choosing a base-point $P \in C$, $\mathrm{Sym}^{n-1}(C)$ embeds as a divisor in X (via $D \mapsto D + P$), and the corresponding line bundle $L = \mathcal{O}_X(\mathrm{Sym}^{n-1}(C))$ is ample (cf. [FL1, §2]). But $\int_X c_1(L)^n = 1$, as one sees from the fact that there is a unique effective divisor of degree n containing n given points of C.

2. Applications and variants

We begin with a simple application of the theorem:

Corollary 2.1. *Let A be an abelian variety of dimension m, and let $X \subset A$ be a smooth subvariety of dimension n. Assume that X is of general type. Then the top self-intersection of the canonical bundle of X satisfies the inequality:*

$$\int c_1(\mathcal{O}_X(K_X))^n \geq \delta(n).$$

Proof. The embedding $X \subset A$ gives rise to a Gauss mapping $\gamma : X \longrightarrow \mathbf{G}$ of X into the Grassmannian $\mathbf{G} = \mathbf{G}(n, m)$ of n-dimensional subspaces of $T_0 A$, which is generically finite since X is of general type (cf. [Mori, §3]). A theorem of Ran [Ran1] implies that then γ is actually finite. On the other hand, the Plücker line bundle $\mathcal{O}_{\mathbf{G}}(1)$ on $\mathbf{G}$ pulls back to the canonical bundle on X. Therefore the canonical bundle $\mathcal{O}_X(K_X)$ is ample and globally generated. But X – like any submanifold of A – satisfies Property (NCB), and the desired inequality then follows from the theorem.   □

We next prove a variant of the theorem for certain smooth subvarieties of projective space:

Proposition 2.2. *Let $X \subset \mathbf{P}^{n+e} = \mathbf{P}$ be a smooth subvariety of projective space having dimension n and codimension e, and denote by $N = N_{X/\mathbf{P}}$ the normal bundle to X in $\mathbf{P}$. If $N(-1)$ is ample, then*

$$\deg(X) \geq \min\left\{2^e, \delta(n)\right\}.$$

Recall that the hypothesis on $N(-1)$ is equivalent to requiring that every hyperplane tangent to X be tangent at only finitely many points. Note that we do not assume here that X satisfies (NCB). Observe also that if $e^2 \leq n$, then the stated bound $\deg(X) \geq 2^e$ is best possible for a complete intersection of quadrics.

Proof of Proposition 2.2. Fix a linear space L^{e-1} disjoint from X, and project from L to get a finite mapping $f : X \longrightarrow \mathbf{P}^n$. Setting $k = \min\{e, [\sqrt{n}]\}$, we will show that the singularity locus $S_k(f)$ appearing in the proof of the theorem is non-empty, and then the result will follow as above from Lemma 1.1. To this end, recalling that $\mathbf{P}^{n+e} - L$ is the total space of $\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus e}$, one finds the exact sequence of bundles on X:

$$0 \longrightarrow \mathcal{O}_X(1)^{\oplus e} \longrightarrow T\mathbf{P}^{n+e}|X \longrightarrow f^*T\mathbf{P}^n \longrightarrow 0.$$

Combining this with the sequence

$$0 \longrightarrow TX \longrightarrow T\mathbf{P}^{n+e}|X \longrightarrow N_{X/\mathbf{P}} \longrightarrow 0,$$

we arrive at a mapping of vector bundles

$$u : \mathcal{O}_X(1)^{\oplus e} \longrightarrow N_{X/\mathbf{P}}$$

on X whose degeneracy loci are the same as the degeneracy loci of the derivative $df : TX \longrightarrow f^*T\mathbf{P}^n$. Since the bundle $N_{X/\mathbf{P}}(-1)$ is ample, the results cited in the proof of the theorem imply that $S_k(u) \neq \emptyset$, as desired. □

Exercise 2.3. Suppose that $X \subset \mathbf{P}^{n+e}$ is a smooth subvariety having the property that for some $x \in X$ the embedded tangent space $T_x X \subset \mathbf{P}^{n+e}$ meets X at only finitely many points (so that in particular $e \geq n$). Then $\deg(X) \geq 2^n$.

An argument similar to the one proving Proposition 2.2 also leads to the following generalization of the Main Theorem:

Proposition 2.4. *Let X be a smooth variety of dimension n satisfying (NCB), and let E be an ample vector bundle of rank e on X which is generated by its global sections. Then*

$$\int_X s_n(E) \geq \min\left\{2^n, \delta(n + e - 1)\right\},$$

where $s_n(E)$ denotes the nth Segre class of E.

Outline of proof. In brief, consider the projective bundle $\pi : \mathbf{P}(E) \longrightarrow X$, and fix a general subspace $V \subset H^0(X, E)$ of dimension $n + e$ generating E. This gives rise to a finite mapping $f : \mathbf{P}(E) \longrightarrow \mathbf{P}(V) = \mathbf{P}^{n+e-1}$ whose degree is equal to $\int s_n(E)$. Setting $k = \min\{n, [\sqrt{n + e - 1}]\}$, it is enough as above to show that the singularity locus $S_k(f)$ is non-empty. To this end, let M be the vector bundle of rank n on X

defined by the exact sequence

(*) $0 \longrightarrow M \longrightarrow V \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow E \longrightarrow 0,$

the homomorphism on the right being the canonical evaluation map. Now f factors through the embedding $\mathbf{P}(E) \subset \mathbf{P}(V \otimes_{\mathbf{C}} \mathcal{O}_X) = X \times \mathbf{P}(V)$ determined by (*), and as in the proof of the proposition, the degeneracy loci of df coincide with those of the resulting vector bundle map

$$u : \pi^* TX \longrightarrow N_{\mathbf{P}(E)/X \times \mathbf{P}(V)}.$$

But $\mathbf{P}(E)$ is cut out in $X \times \mathbf{P}(V)$ by a section of $pr_1^* M^* \otimes pr_2^* \mathcal{O}_{\mathbf{P}(V)}(1)$, and consequently $N_{\mathbf{P}(E)/X \times \mathbf{P}(V)} = \pi^* M^* \otimes \mathcal{O}_{\mathbf{P}(E)}(1)$, which by Lemma 1.2 is ample thanks to the amplitude of E and the fact that M^* is globally generated. As $\pi^* \Omega_X^1$ is nef by assumption, it follows that $\pi^* \Omega_X^1 \otimes (\pi^* M^* \otimes \mathcal{O}_{\mathbf{P}(E)}(1))$ is ample. But then [FL1] or the other references cited above guarantee that u must actually drop rank whenever it is dimensionally predicted to do so.   □

Remark 2.5. The inequalities established in this note all spring via Lemma 1.1 from producing singularities of a branched covering of projective space. It would be interesting to know whether one can recover or improve these statements by applying positivity theorems to some well-chosen Chern class calculations. It is natural to wonder in particular whether the inequalities of [BSS] might not be relevant here.

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