Linkage of General Curves of Large Degree

by

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Introduction.

Our purpose is to describe the liaison class of a general curve in $\mathbb{P}^3$ of degree much larger than its genus. In particular, we prove a conjecture of Joe Harris ([H], p.80) to the effect that such a curve can be linked only to curves of larger degree and genus.

Recall that two curves $X,Y \subseteq \mathbb{P}^3$ are directly linked if X is residual to Y in the complete intersection of two surfaces; they are linked if the one can be obtained from the other by a succession of direct linkages. Classically, linkage was seen as a method for producing interesting examples of space curves starting from simpler ones. Later work on linkage-or liaison, as it is also called-has largely focused on the equivalence relation it generates. Apéry [A] and Gaeta [G] proved that a curve $X \subseteq \mathbb{P}^3$ is linked to a complete intersection if and only if it is projectively Cohen-Macaulay; the analogous statement in higher dimensions was proved by Peskine and Szpiro [P-S]. The theorem of Apéry and Gaeta was generalized by the second author, who studied the deficiency module

$$M(X) = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_X(n))$$

of $X \subseteq \mathbb{P}^3$, a finite module over the homogeneous coordinate ring $S = k[T_0,T_1,T_2,T_3]$. Specifically, it was shown in [R] that two curves $X,Y \subseteq \mathbb{P}^3$ are linked if and only if the module $M(X)$ of X coincides up to grading with either the module $M(Y)$ of Y or its dual $M(Y)^{\vee}$. Moreover, any finite S-module M arises as the deficiency module of some curve in $\mathbb{P}^3$. Thus one has an essentially complete picture, from a cohomological point of view, of the various liaison equivalence classes that can occur for curves in $\mathbb{P}^3$.

It is natural to ask, however, for a clearer geometric understanding of the curves that exist within a given liaison class. In the present paper, we consider the linkage class of a general smooth irreducible curve $X \subseteq \mathbb{P}^3$ of sufficiently large degree. Our main result (§3) states that if Y is any curve linked to X, other than X itself, then $\deg(Y) > \deg(X)$ and $p_a(Y) > p_a(X)$. Somewhat more precisely, we distinguish between <u>even linkage</u>-i.e., liaison involving an even number of direct linkages-and <u>odd linkage</u>, defined similarly. We show that if Y is evenly linked to X, then it is a deformation of the curve obtained by taking the union of X and certain complete intersection curves. If Y is oddly linked to X, then it arises in an analogous manner from the curve Z directly linked to X by irreducible surfaces of lowest possible degree.

The questions we consider here were first raised by J. Harris (cf [H]). A priori, one could hope-as some of the classical geometers apparently did-that techniques of liaison could be used to study space curves inductively, by linking a given curve to a (possibly very special) curve of lower degree or genus. Believing that at least for general curves such an approach is fundamentally flawed, Harris suggested that a general curve should in various senses be minimal in its liaison class. Our results may be seen, then, as giving additional support (if any is needed) to the philosophy that there is no easy way to get one's hands on a "general" curve. Some suggestive results in the direction of Harris's conjectures were obtained for lines and rational curves by Migliore [M]; at least indirectly these have contributed substantially to the present paper, as has work of Schwartau [S].

Most of our results are stated for an arbitrary curve $X \subseteq \mathbb{P}^3$ subject only to the condition that it not lie on any surfaces of degree e+4 or less, e being the largest integer such that $h^1(X,\mathcal{O}_X(e)) \neq 0$. The generality assumption is used only in §3 to keep the curves in question off surfaces of low degree. It seems likely that similar results hold for curves $X \subseteq \mathbb{P}^3$ general in the sense of Brill-Noether theory. What is missing is even a weak approximation to the maximal rank conjecture (cf [H], p.79).

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§0. Notation and Conventions.

(0.1). We work over an algebraically closed field k of arbitrary characteristic. A <u>curve</u> $X \subseteq \mathbb{P}^3$ is a subscheme of pure dimension one, without embedded points. Thus X is (locally) Cohen-Macaulay. I_X is the ideal sheaf of X, and I(X) its homogeneous ideal.

(0.2). If F is a coherent sheaf on $\mathbb{P}^3$, we let

$$H^i_*(\mathbb{P}^3, F) = \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^3, F(n)),$$

so that $H^i_*(\mathbb{P}^3, F)$ is a graded module over the homogeneous coordinate ring S. We write simply $\mathcal{O}$ for the structure sheaf $\mathcal{O}_{\mathbb{P}^3}$.

(0.3). Given a curve $X \subseteq \mathbb{P}^3$, we say that X lies on a surface F if $F \in I(X)$. If $F,G \in I(X)$ meet properly, then they link X to a curve Y whose scheme structure is determined as in [P-S]. If $0 \to P \to N \to \mathcal{O} \to \mathcal{O}_X \to 0$ is a locally free resolution of $\mathcal{O}_X$, with $H^1_*(\mathbb{P}^3,P) = 0$, then $\mathcal{O}_Y$ has a resolution

(0.4) $\quad 0 \longrightarrow N^\vee(-f-g) \longrightarrow \mathcal{O}(-f) \oplus \mathcal{O}(-g) \oplus P^\vee(-f-g) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Y \longrightarrow 0$

where f and g denote respectively the degrees of F and G ([P-S], Propn. 2.5).

§1. Curves minimal in their even liaison class.

Given a curve $X \subseteq \mathbb{P}^3$, set

$$e(X) = \max\left\{n \mid H^1(X, \mathcal{O}_X(n)) \neq 0\right\}.$$

Our goal in this section and the next is to show that if X does not lie on any surfaces of degree $e(X) + 4$ or less, then X is in various senses minimal in its even liaison class. For example, we will see that X has smaller degree and arithmetic genus than any other curve to which it is evenly linked (Corollary 1.5). The basic idea is that given any two evenly linked curves $X,Y \subseteq \mathbb{P}^3$, there exist vector bundle maps

$$\bigoplus_{i=1}^{r} \mathcal{O}(-a_i) \xrightarrow{u} E \quad \text{and} \quad \bigoplus_{i=1}^{r} \mathcal{O}(-b_i) \xrightarrow{v} E$$

which drop rank respectively on X and Y. The crucial fact is that if X lies on no surfaces of degree $\leq e(X) + 4$, then $b_i \geq a_i$, and at least one inequality is strict if $X \neq Y$ (Lemma 1.2). This allows us to compare the numerical invariants of X to those of Y, and to describe geometrically how Y is obtained from X. Under mild additional hypotheses, analogous statements can be made for odd linkage (Proposition 1.6). We remark that the importance of the integer e(X) in questions of liaison was known already to Gaeta.

We start by recalling a useful representation of a given curve as a determinantal locus.

Lemma 1.1. *Let $X \subseteq \mathbb{P}^3$ be a curve. Then there is an exact sequence*

$$0 \longrightarrow P \xrightarrow{u} N \longrightarrow \mathcal{I}_X \longrightarrow 0,$$

where N *is a vector bundle, with* $H^2_*(\mathbb{P}^3, N) = 0$, *and* P *is a direct sum of line bundles of degrees* $\geq -e(X) - 4$.

Proof. We use a construction similar to one used by Sernese [Se] (cf also [GLP] §2). Consider the graded S-algebra $R = H^0_*(\mathbb{P}^3, \mathcal{O}_X)$. Since $R_n = 0$ for $n<<0$, R necessarily has a minimal generator in degree zero, which we may take to be the identity. The sheafification of a minimal free S-resolution of R therefore takes the form

$$0 \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{\begin{pmatrix} p_1 \\ \phi \end{pmatrix}} P_0 \oplus \mathcal{O} \xrightarrow{(p_0, \varepsilon)} \mathcal{O}_X \longrightarrow 0,$$

where each P_i is a direct sum of line bundles, and $\varepsilon : \mathcal{O} \longrightarrow \mathcal{O}_X$ is the natural map. One then obtains, using the snake lemma, the following commutative diagram of exact sequences of sheaves:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O} \arrow[r, "\varepsilon"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_2 \arrow[r, "p_1"] \arrow[d] & P_1 \arrow[r, "\begin{pmatrix} p_1 \\ \phi \end{pmatrix}"] \arrow[d, equal] & P_0 \oplus \mathcal{O} \arrow[r] \arrow[d, "{(1,0)}"] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r] \arrow[d] & 0 & \\
 & I_X \arrow[d] & & 0 & & \\
 & 0 & & & &
\end{tikzcd}$$

Here N is of course defined as the kernel of p_1; the vanishing of $H^2_*(\mathbb{P}^3, N)$ follows from the vanishings of $H^1_*(\mathbb{P}^3, P_0)$ and $H^2_*(\mathbb{P}^3, P_1)$. By duality one has $\mathrm{Ext}^2_S(R, S(-4)) = H^1_*(\mathbb{P}^3, \mathcal{O}_X)^v$, and hence all the summands of P_2 have degrees $\geq -e(X) - 4$. Taking $P = P_2$, the lemma follows. ■

<u>Remark</u>. Keeping the notation of the previous proof, observe that $M(X) = H^1_*(\mathbb{P}^3, N)$. Hence the map induced by p_1 on global sections gives a presentation of the deficiency module of X:

$$H^0_*(\mathbb{P}^3, P_1) \longrightarrow H^0_*(\mathbb{P}^3, P_0) \longrightarrow M(X) \longrightarrow 0$$

Moreover, if X lies on no surface of degree $e(X) + 3$, so that $H^0(\mathbb{P}^3, N(t)) = H^0(\mathbb{P}^3, P_2(t))$ for $t \leq e(X) + 3$, then this is actually a minimal presentation of $M(X)$. In practice, this yields a convenient method for computing a presentation of $M(X)$-or at least determining the number of generators and relations in each degree-in concrete examples.

For instance, suppose that X is the disjoint union of d lines $L_i = \{A_i=B_i=0\}$, where A_i and B_i are linear forms. Then $H^0_*(\mathbb{P}^3, \mathcal{O}_X)$ is resolved by taking the direct sum of the Koszul complexes formed from A_i and B_i. Hence $M(X)$ can be described as the S-module given by generators $e_1,\ldots,e_d$ in degree 0, subject to the relations

$$\sum_{i=1}^{d} e_i = 0$$

$$A_i \cdot e_i = 0, \quad B_i \cdot e_i = 0 \qquad (1 \leq i \leq d).$$

Using the first relation to eliminate one of the generators, one obtains a minimal presentation having the form $S(-1)^{2d} \longrightarrow S^{d-1} \longrightarrow M(X) \longrightarrow 0$. The disjoint union of d complete intersections is treated almost identically. Similarly, the deficiency module of a smooth rational curve $X \subseteq \mathbb{P}^3$ of degree $d \geq 4$ and not on a quadric surface has a minimal presentation of the form $S(-2)^{2d-3} \longrightarrow S^{d-3}(-1) \longrightarrow M(X) \longrightarrow 0$. We refer to Migliore [M] for a more geometric discussion of the linkage properties of lines and rational curves.

Now suppose that $Y \subseteq \mathbb{P}^3$ is evenly linked to X. Then we may repeatedly apply (0.4) to the exact sequence of (1.1) to obtain an exact sequence

$$(*) \qquad 0 \longrightarrow B \longrightarrow N \oplus F \longrightarrow \mathcal{I}_Y(\delta) \longrightarrow 0$$

for some $\delta \in \mathbb{Z}$, where B and F are direct sums of line bundles. On the other hand, if $A = P \oplus F$, then of course we have

$$(**) \qquad 0 \longrightarrow A \xrightarrow{u \oplus 1} N \oplus F \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Our main technical lemma allows us to compare the degrees of the summands of A and B provided that X lies on no surfaces of degree $e(X) + 3$ or less. Specifically, in the next section we shall prove

Lemma 1.2. _In the notation just introduced, write_

$$A = \bigoplus_{i=1}^{r} \mathcal{O}(-a_i), \quad \underline{\text{with}} \quad a_1 \leq a_2 \leq \ldots \leq a_r$$

and

$$B = \bigoplus_{i=1}^{r} \mathcal{O}(-b_i), \quad \underline{\text{with}} \quad b_1 \leq b_2 \leq \ldots \leq b_r.$$

_If_ X _lies on no surfaces of degree_ $e(X) + 3$, _then_

$$b_i \geq a_i \quad \text{for all} \quad 1 \leq i \leq r.$$

If moreover X _lies on no surfaces of degree_ $e(X) + 4$, _and if_ $Y \neq X$, _then_ $b_i > a_i$ _for at least one index_ i.

Note that the integer δ in (*) is just the sum $\sum_{i=1}^{r} (b_i - a_i)$. Observe also that at least when $F = 0$ so that $A = P$, the lemma is highly plausible: for then the hypothesis on X implies that the free submodule $H^0_*(\mathbb{P}^3, P) \subseteq H^0_*(\mathbb{P}^3, N)$ consists of the lowest degree generators of $H^0_*(\mathbb{P}^3, N)$.

Before proceeding, we note the amusing

Corollary 1.3. _Let_ $X \subseteq \mathbb{P}^3$ _be a curve not lying on any surface of_ degree $e(X) + 4$ or less. Then X is the only curve whose deficiency module is isomorphic to $M(X)$ (with the given grading), and for $n > 0$, $M(X)(n)$ cannot be realized as the deficiency module of any curve. In particular, X is determined by its module.

Proof. If $Y \subseteq \mathbb{P}^3$ is a curve whose module coincides with $M(X)$ up to grading, then Y is evenly linked to X (by [R]), and we have

$$M(Y) = H^1_*(\mathbb{P}^3, N)(-\delta) = M(X)(-\delta),$$

δ being the integer introduced above. But under the hypothesis on X, Lemma 1.2 asserts that $\delta > 0$ unless $Y = X$. ∎

Remarks.

(1) By contrast, using a construction which we shall review below, Schwartau [S] has shown that given any curve $X \subseteq \mathbb{P}^3$, and any $n < 0$, there exist infinitely many curves $Y \subseteq \mathbb{P}^3$ with $M(Y) = M(X)(n)$.

(2) We shall check in §3 that the hypothesis of the corollary is satisfied when X is a general curve of sufficiently large degree. In this form, the result had been conjectured by J. Harris ([H], p.80). The last statement of the corollary was established by Migliore [M] when X is a union of lines.

(3) At least in a special case, there is a simple geometric argument showing that the hypothesis on X is necessary for the validity of the result. Specifically, suppose that X is reduced, and lies on a smooth surface $S \subseteq \mathbb{P}^3$ of degree $f + 4 \leq e(X) + 4$. Then one has an exact sequence

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\cdot X} \mathcal{O}_S(X) \longrightarrow \omega_X(-f\cdot H) \longrightarrow 0,$$

where H denotes the hyperplane divisor class on S. Since $f \leq e(X)$, duality shows that $H^0(X, \omega_X(-f \cdot H)) \neq 0$. In view of the vanishing of $H^1(S, \mathcal{O}_S)$, it follows that X moves in a non-trivial linear system on S. But any two curves in such a linear system are evenly linked, and their deficiency modules are isomorphic (with the same grading).

Our next object is to interpret geometrically the conclusion of Lemma 1.2. To this end, we need first to describe certain "basic double linkages."

Given a curve $X \subseteq \mathbb{P}^3$, let F be a surface of degree f containing X, and choose any surface H of degree $h \geq 1$ meeting F properly. If G is a general surface through X of sufficiently large degree, we may use F and G to link X to a curve X^*, and then use F and $G \cdot H$ to link X^* to a curve Y. Y does not depend on the surface G, and we will say that it is obtained from X by a <u>basic double linkage</u> using F and H. This is a special case of the construction of liaison addition introduced by Schwartau [S]. Set theoretically, Y is the union of X and the complete intersection of F and H. Evidently $\deg(Y) = \deg(X) + f \cdot h$, and it follows from ([S], p.91) that $p_a(Y) - p_a(X) = hf \cdot (h+f-4)/2 + h \cdot \deg(X)$. Observe that this difference is always non-negative, and is strictly positive unless X is a line, and $h = f = 1$.

The geometric meaning of Lemma 1.2 is summarized in

<u>Proposition 1.4.</u> <u>Let</u> $X \subseteq \mathbb{P}^3$ <u>be a curve not contained in any surface of degree</u> $\leqslant e(X) + 3$, <u>and let</u> $Y \subseteq \mathbb{P}^3$ <u>be any curve evenly linked to</u> X. <u>Then there exists a sequence of curves</u>

$$X = X_1, X_2, \ldots, X_{m-1}, X_m$$

<u>such that</u> X_{i+1} <u>is obtained from</u> X_i <u>by a basic double linkage using suitable surfaces</u> $F_i \in I(X_i)$ <u>and</u> H_i, <u>and such that</u> Y <u>is a deformation of</u> X_m <u>through curves having a fixed deficiency module. Moreover if</u> X <u>lies on no surface of degree</u> $e(X) + 4$, <u>and if</u> $Y \neq X$, <u>then at least one non-trivial basic double linkage must occur</u>.

The statement of the Proposition is illustrated in Figure 1. Observe that, conversely, any curve Y obtained from X as indicated is evenly linked to X (by virtue of [R]).

<u>Proof</u>. Under the hypothesis on X, the assertion of Lemma 1.2 is that X and Y arise via exact sequences

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}(-a_i) \xrightarrow{u} E \longrightarrow \mathcal{I}_X \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}(-b_i) \xrightarrow{v} E \longrightarrow \mathcal{I}_Y(\delta) \longrightarrow 0,$$

where E is a vector bundle of rank r+1, with $H^2_*(\mathbb{P}^3,E) = 0$, and

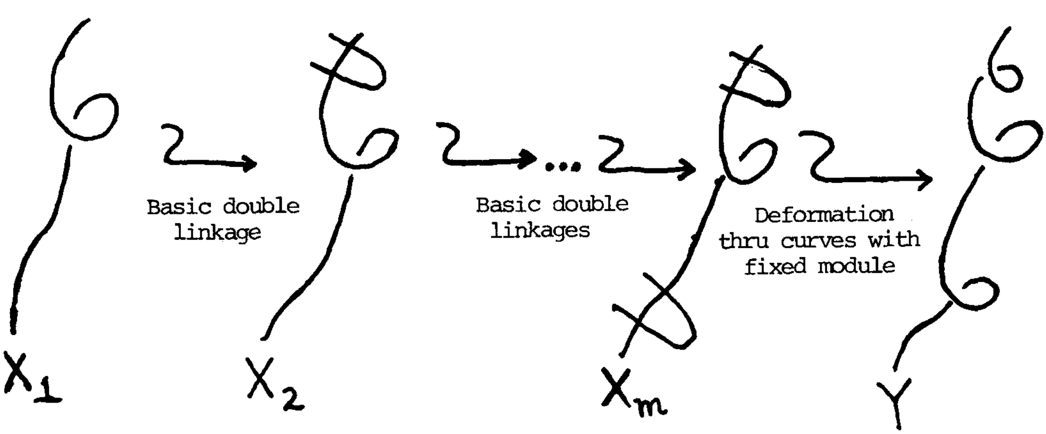

Figure 1

$$\delta_i = b_i - a_i \geq 0 \text{ for all } 1 \leq i \leq r.$$

We argue by descending induction on $\delta = \sum \delta_i$ that this set-up implies the assertion of the Proposition.

Suppose first that $\delta = 0$, so that $a_i = b_i$ for $1 \leq i \leq r$. Given $t \in k$, let

$$w_t = tu + (1-t)v \in \mathrm{Hom}(\bigoplus \mathcal{O}(-a_i), E).$$

Then for general $t \in k$, the vector bundle map w_t drops rank along a curve X_t, and it is elementary that the curves X_t fit together to form a flat family of subschemes of $\mathbb{P}^3$, parametrized by a Zariski open set $U \subseteq \mathbb{A}^1$ containing 0 and 1. For $t \in U$ one has

$$M(X_t) = H^1_*(\mathbb{P}^3, E),$$

and so Y is a deformation of X through curves with fixed deficiency module.

Assuming then that $\delta > 0$, let u_i and v_i be the i^{th} components of u and v respectively, and denote by s_i the image of v_i in $\mathrm{Hom}(\mathcal{O}(-b_i), \mathcal{I}_X)$:

$$\begin{tikzcd} & & \mathcal{O}(-b_i) \arrow[d, "v_i"] \arrow[dr, "s_i"] & & \\ 0 \arrow[r] & \bigoplus_{i=1}^{r} \mathcal{O}(-a_i) \arrow[r, "u=(u_1,\ldots,u_r)"] & E \arrow[r] & \mathcal{I}_X \arrow[r] & 0. \end{tikzcd}$$

As before we suppose that the integers $\{a_i\}$ and $\{b_i\}$ are non-decreasing in i.

Let $\ell \in [1,r]$ be the largest integer such that $\delta_\ell > 0$. Re-indexing the $\{b_i\}$ if necessary, we may assume that either $\ell = r$ or $b_{\ell+1} > b_\ell$. We assert that for some index $j \leq \ell$, the section $s_j \in H^0(\mathbb{P}^3, \mathcal{I}_X(b_j))$ is non-zero. In fact, one has $a_i = b_i > b_\ell$ for $i > \ell$, and if $s_j = 0$ for all $j \leq \ell$, then the first ℓ components of v would factor through those of u:

$$\begin{tikzcd} & \bigoplus\limits_{i=1}^{\ell} \mathcal{O}(-b_i) \arrow[dl, dashed, "\phi"'] \arrow[d, "(v_1,\ldots,v_\ell)"] \\ \bigoplus\limits_{i=1}^{\ell} \mathcal{O}(-a_i) \arrow[r, "(u_1,\ldots,u_\ell)"'] & E \end{tikzcd}$$

But $\sum_{i=1}^{\ell} a_i < \sum_{i=1}^{\ell} b_i$ since $\delta > 0$, and so the map ϕ, and hence also v, would drop rank along a surface, whereas in reality v drops rank exactly on the curve Y. Hence $s_j \neq 0$ for some $j \leq \ell$, as claimed.

If Q is a general form of degree $b_\ell - b_j$, then the section

$$F = s_\ell + Qs_j \in H^0(\mathbb{P}^3, \mathcal{I}_X(b_\ell))$$

is non-zero. Let X_2 be the curve obtained from $X = X_1$ by a basic double linkage using F and a general surface H of degree δ_ℓ. Then by (0.4), X_2 and Y (trivially) are realized via

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}(-a_i) \oplus \mathcal{O}(-a_\ell-\delta_\ell) \longrightarrow E \oplus \mathcal{O}(-a_\ell) \longrightarrow \mathcal{I}_{X_2}(\delta_\ell) \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}(-a_i-\delta_i) \oplus \mathcal{O}(-a_\ell) \longrightarrow E \oplus \mathcal{O}(-a_\ell) \longrightarrow \mathcal{I}_Y(\delta) \longrightarrow 0.$$

These sequences satisfy the conditions stated at the beginning of the proof, hence the existence of the desired sequence of curves follows by induction. Finally, the last assertion of the proposition follows from the fact that if X lies on no surfaces of degree $e(X) + 4$, and if $X \neq Y$, then at least one δ_i is non-zero. ∎

Corollary 1.5. *If X lies on no surface of degree $e(X) + 4$ or less, and Y is evenly linked to X, then either $X = Y$ or $\deg(Y) > \deg(X)$ and $p_a(Y) > p_a(X)$.* ∎

We now show that under a small additional hypothesis, a similar picture applies to odd linkage:

Proposition 1.6. *Let $X \subseteq \mathbb{P}^3$ be a curve not lying on any surface of degree $e(X) + 3$ or less. Choose a system of minimal generators*

$$F_i \in I(X) \quad (1 \leq i \leq \ell)$$

of the homogeneous ideal of X, with $c_i = \deg(F_i)$ non-decreasing in i. Assume that F_1 and F_2 meet properly, so that they link X to a curve Z. Then Z does not lie on any surface of degree $e(Z) + 3$ or less.

Remarks.

(1) If X is reduced and irreducible, it is automatic that F_1 and F_2 meet properly.

(2) The curve Z may depend on the choice of the surfaces F_1 and F_2. However if Z' is obtained via F_1' and F_2', then Z' is a deformation of Z through curves with fixed deficiency module (thanks to (1.4)).

Proof. Consider a minimal free resolution of $H^0_*(\mathbb{P}^3, N)$, where N is the vector bundle constructed from X in Lemma 1.1.

Such a resolution has length 2, and sheafifying yields an exact sequence which we use to define a bundle E as shown:

$$(1.7)\qquad \begin{tikzcd}[column sep=small] 0 \arrow[r] & L_2 \arrow[r] & L_1 \arrow[rr] \arrow[dr] & & L_0 \arrow[r] & N \arrow[r] & 0 \\ & & & E \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Thus each L_i is a sum of line bundles, and in particular $H^1_*(\mathbb{P}^3, E) = 0$. On the other hand, we have from (1.1) an exact sequence

$$0 \longrightarrow P \longrightarrow N \longrightarrow \mathcal{I}_X \longrightarrow 0,$$

where

$$P = \bigoplus_{i=1}^{s} \mathcal{O}(-d_j), \text{ with all } d_j \le e(X) + 4.$$

Observe that our hypothesis on X implies that

$$(*) \qquad d_j \le c_i \quad (1 \le j \le s,\ \ 1 \le i \le \ell).$$

It follows in particular from (*) that any minimal generator of $H^0_*(\mathbb{P}^3, P)$ must be a minimal generator of $H^0_*(\mathbb{P}^3, N)$. Therefore

$$L_0 = \bigoplus_{i=1}^{\ell} \mathcal{O}(-c_i) \oplus P,$$

and then one sees that the bundle E defined above is isomorphic to the kernel of the natural map $\mathcal{O}(-c_i) \longrightarrow \mathcal{I}_X$:

$$(**) \qquad 0 \longrightarrow E \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}(-c_i) \xrightarrow[(F_1,\ldots,F_\ell)]{} \mathcal{I}_X \longrightarrow 0$$

We next use (0.3) to read off from (**) a locally free resolution of $\mathcal{I}_Z$. Cancelling redundant terms, one finds:

$$(***) \qquad 0 \longrightarrow \bigoplus_{i=3}^{\ell} \mathcal{O}(c_i-c_1-c_2) \longrightarrow E^{\vee}(-c_1-c_2) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

Since $H^2_*(\mathbb{P}^3, E^{\vee}) = 0$, it follows that $H^1_*(\mathbb{P}^3, \mathcal{O}_Z)$ $(=H^2_*(\mathbb{P}^3, \mathcal{I}_Z))$ injects into

$$\bigoplus_{i=3}^{\ell} H^3_*(\mathbb{P}^3, \mathcal{O}(c_i-c_1-c_2)).$$

Recalling that the c_i are non-decreasing, we deduce that

$$e(Z) \leq c_1 + c_2 - (c_3+4) \leq c_1 - 4.$$

On the other hand $H^1_*(\mathbb{P}^3, N^{\vee}) = 0$, so it follows from (1.7) that $H^0_*(\mathbb{P}^3, L_0^{\vee}(-c_1-c_2))$ surjects onto $H^0_*(\mathbb{P}^3, E^{\vee}(-c_1-c_2))$, and hence, thanks to (***), onto $H^0_*(\mathbb{P}^3, \mathcal{I}_Z)$. Therefore, generators of the homogeneous ideal $I(Z) = H^0_*(\mathbb{P}^3, \mathcal{I}_Z)$ can occur only in degrees

$$c_1\ ,\ c_2\ ,\ \text{and}\ c_1 + c_2 - d_j \qquad (1 \leq j \leq s).$$

But $c_1 + c_2 - d_j \geq c_2 \geq c_1$ by (*), so $H^0_*(\mathbb{P}^3, \mathcal{I}_Z)$ vanishes in degrees $c_1 - 1$ and less. Since $e(Z) + 3 \leq c_1 - 1$, this proves the Proposition. ∎

Corollary 1.8. Let X and Z be as in the statement of Proposition 1.6, and let Y be any curve oddly linked to X. Then Y is obtained from Z by a succession of basic double linkages, and then a deformation, as described in (1.4). In particular, $\deg(Y) \geq \deg(Z)$, and $p_a(Y) \geq p_a(Z)$. ∎

§2. Proof of Lemma 1.2.

We keep the notation introduced in §1, so that we have exact sequences

$$0 \longrightarrow P \xrightarrow{u} N \longrightarrow \mathcal{I}_X \longrightarrow 0$$

and

$$0 \longrightarrow B \xrightarrow{v} N \oplus F \longrightarrow \mathcal{I}_Y(\delta) \longrightarrow 0.$$

Here B,F and P are direct sums of line bundles, every summand of P having degree $\geq -f$, where $f = e(X) + 4$. We assume that X lies on no surface of degree $e(X) + 3$ or less, i.e., that

(2.1) $$H^0(\mathbb{P}^3, P(t)) \xrightarrow{\simeq} H^0(\mathbb{P}^3, N(t)) \quad \text{for } t \leq f - 1.$$

Note that this implies that N cannot itself split as a sum of line bundles (at least if $X \neq \emptyset$). We wish to compare the degrees of the summands of $A = P \oplus F$ to those of B.

To begin with, we rephrase the desired statement (1.2). If H is any bundle on $\mathbb{P}^3$ splitting as a direct sum of line bundles, and ℓ is any integer, set

$$H_{\geq \ell} = \bigoplus (\text{summands of } H \text{ of degree } \geq \ell),$$

and define $H_{<\ell}$ similarly,* so that $H = H_{<\ell} \oplus H_{\geq \ell}$. A moment's thought shows that the first statement of Lemma 1.2 is equivalent to

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*More formally, for any $\ell \in \mathbb{Z}$ there is a natural map

$$H^0(\mathbb{P}^3, H(-\ell)) \otimes_k \mathcal{O}(\ell) \longrightarrow H.$$

$H_{\geq \ell}$ is its image, and $H_{<\ell} = H/H_{\geq \ell}$.

(2.2) <u>For every</u> $\ell \in \mathbb{Z}$,

$$\mathrm{rk}(P_{\geq \ell} \oplus F_{\geq \ell}) \geq \mathrm{rk}(B_{\geq \ell}).$$

<u>Proof of (2.2)</u>. We proceed in two cases: (i) $\ell \leq -f$; and (ii) $\ell > -f$.

<u>Case (i)</u>: $\ell \leq -f$.

We have $B = B_{\geq \ell} \oplus B_{< \ell}$ and $F = F_{\geq \ell} \oplus F_{< \ell}$, and the given map $v \in \mathrm{Hom}(B, N \oplus F)$ is an injection of sheaves. Since $\mathrm{Hom}(B_{\geq \ell}, F_{<\ell}) = 0$, we get an injection $B_{\geq \ell} \hookrightarrow N \oplus F_{\geq \ell}$. Hence

$$(*) \qquad \mathrm{rk}(B_{\geq \ell}) \leq \mathrm{rk}(N) + \mathrm{rk}(F_{\geq \ell})$$

Now suppose that equality holds in (*). Then $B_{\geq \ell}$ and $N \oplus F_{\geq \ell}$ are vector bundles of the same rank, and since N is not a sum of line bundles, the map $B_{\geq \ell} \to N \oplus F_{\geq \ell}$ (thought of as a homomorphism of vector bundles) must drop rank along a surface. Hence so too must the given homomorphism v. But v in fact drops rank only along the curve Y, and therefore strict inequality must hold in (*), i.e.,

$$(**) \qquad \mathrm{rk}(B_{\geq \ell}) \leq \mathrm{rk}(N) - 1 + \mathrm{rk}(F_{\geq \ell}).$$

Finally, since $\ell \leq -f$, one has $P_{\geq \ell} = P$. But $\mathrm{rk}(P) = \mathrm{rk}(N) - 1$, so (2.2) follows from (**) in the case at hand.

<u>Case (ii)</u>: $\ell > -f$.

Let $N_{< \ell}$ denote the cokernel of the composition of the natural inclusion $P_{\geq \ell} \hookrightarrow P$ with the given map $u: P \to N$:

$$0 \longrightarrow P_{\geq \ell} \longrightarrow N \longrightarrow N_{< \ell} \longrightarrow 0$$

In view of (2.1) and the hypothesis $\ell > -f$, one has

$$H^0(\mathbb{P}^3, P_{\geq \ell}(t)) \xrightarrow{\cong} H^0(\mathbb{P}^3, P(t)) \xrightarrow{\cong} H^0(\mathbb{P}^3, N(t))$$

when $t \leq - \ell$ ($\leq f-1$). Hence

$$(*) \qquad H^0(\mathbb{P}^3, N_{<\ell}(t)) = 0 \quad \text{for any } t \leq - \ell.$$

Consider now the diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & B_{\geq \ell} \oplus B_{<\ell} \arrow[d, "v"] & & \\ 0 \arrow[r] & P_{\geq \ell} \oplus F_{\geq \ell} \arrow[r] & N \oplus F \arrow[r] & N_{<\ell} \oplus F_{<\ell} \arrow[r] & 0, \end{tikzcd}$$

where horizontally we've just formed the sum of two exact sequences. Evidently $\mathrm{Hom}(B_{\geq \ell}, F_{<\ell}) = 0$, and it follows from (*) that similarly

$$\mathrm{Hom}(B_{\geq \ell}, N_{<\ell}) = 0.$$

Hence v induces an injection of sheaves $B_{\geq \ell} \hookrightarrow P_{\geq \ell} \oplus F_{\geq \ell}$, and taking ranks gives (2.2) when $\ell > - f$. ∎

To complete the proof of Lemma 1.2, it remains to show that if X lies on no surfaces of degrees $f = e(X) + 4$ or less, and if $B = P \oplus F$, so that one has an exact sequence

$$0 \longrightarrow P \oplus F \xrightarrow{v} N \oplus F \longrightarrow I_Y \longrightarrow 0,$$

then $Y = X$. To this end, observe that v gives rise via the decomposition $F = F_{\geq -f} \oplus F_{<-f}$ to a homomorphism

$$\alpha : F_{<-f} \longrightarrow F_{<-f} \ ,$$

and since $\mathrm{Hom}(P \oplus F_{>-f}, F_{<-f}) = 0$ for reasons of degree, coker α is a quotient of coker $v = I_Y$. Since in any event coker α is locally free, this implies that α is an isomorphism. Hence α splits off from v, i.e., I_Y arises as the cokernel of an injective sheaf homomorphism $w: P \oplus F_{\geq -f} \longrightarrow N \oplus F_{\geq -f}$. But then consider the diagram

(*)
$$\begin{tikzcd}
 & & P \oplus F_{\geq -f} \arrow[d, "w"] \arrow[dl, dashed, "\cong"'] & & \\
0 \arrow[r] & P \oplus F_{\geq -f} \arrow[r, "u \oplus 1"] & N \oplus F_{\geq -f} \arrow[r] & I_X \arrow[r] & 0.
\end{tikzcd}$$

Since $H^0(\mathbb{P}^3, I_X(t)) = 0$ for $t \leq f$, (*) shows that w factors through an isomorphism as indicated, whence $X = Y$.

The proof of Lemma 1.2 is now complete.

§3. The liaison class of a general curve of large degree.

We now apply the results of §1 to a smooth irreducible curve $X \subseteq \mathbb{P}^3$ of genus g and degree $d >> g$. To begin with, we assume that $d \geq 2g - 1$, so that $e(X) \leq 0$. Moreover, when $d \geq 2g - 1$ the family of all such curves is irreducible, so it makes sense to speak of the properties enjoyed by a general smooth curve of genus g and degree d.

All that is needed at this point is to bound from below the degrees of the surfaces on which X lies. For our purposes, the following elementary estimate is sufficient:

Lemma 3.1. *Fix an integer $g \geq 0$. Then there exists a constant $C(g) \geq 2g-1$ such that a sufficiently general curve of genus g and*

<u>degree</u> $d \geq C(g)$ <u>lies on no surfaces of degree</u> $\sqrt{5d}$ <u>or less</u>.

<u>Proof</u>. Recall first of all that Hirschowitz [Hi] has shown that a general rational curve $D \subseteq \mathbb{P}^3$ of degree f lies on a surface of degree n if and only if

$$\binom{n+3}{3} > nf + 1,$$

i.e.,

$$n > -3 + \sqrt{6f - 2}.$$

Now let $C \subseteq \mathbb{P}^3$ be any smooth curve of genus g and degree $d_0 \geq 2g-1$. Choose a smooth rational curve $D \subseteq \mathbb{P}^3$ of degree f for which Hirschowitz's theorem holds. Translate D by an automorphism of $\mathbb{P}^3$ so that it meets C at a single point with distinct tangents, and let

$$X_0 = C \cup D.$$

Thus X_0 has degree $d = d_0 + f$ and arithmetic genus g, and by construction X_0 lies on no surfaces of degree $\leq \sqrt{6(d-d_0) - 2} - 3$. But X_0 moves in an irreducible flat family of curves in $\mathbb{P}^3$ whose general member is smooth (cf [T]). Therefore a generic smooth curve of degree d and genus g lies on no surfaces of degree $\leq \sqrt{6(d-d_0) - 2} - 3$, and letting $f \to \infty$ the lemma follows. ∎

Thus all the results of §1 apply in the case at hand, and in summary we have established

<u>Theorem 3.2</u>. <u>Let</u> $X \subseteq \mathbb{P}^3$ <u>be a general smooth irreducible curve of genus</u> g <u>and degree</u> $d \gg g$, <u>and let</u> Z <u>be the curve directly linked to</u> X <u>by two irreducible surfaces of lowest degree through</u> X. <u>(Thus</u>

for $d>C(g)$, Z has degree $> 4d$ and arithmetic genus $\geqslant g + 3d(\sqrt{5d}-2)$.)
Then:

(a) X is the only curve with deficiency module $M(X)$ (with the given grading), and for $n > 0$ there is no curve with module $M(X)(n)$.

(b) If $Y \subseteq \mathbb{P}^3$ is evenly linked to X, then Y is a deformation, through curves with fixed deficiency module, of a curve obtained from X by a sequence of basic double linkages. In particular, if $Y \neq X$, then

$$\deg(Y) > \deg(X) \quad \text{and} \quad p_a(Y) > p_a(X).$$

(c) If $Y \subseteq \mathbb{P}^3$ is oddly linked to X, then Y is a deformation of a curve obtained from Z by a sequence of basic double linkages, and in particular

$$\deg(Y) \geq \deg(Z) > \deg(X)$$

and

$$p_a(Y) \geq p_a(Z) > p_a(X). \quad \blacksquare$$

Remark. In case (a) one has the estimates

$$\deg(Y) \geq d + \sqrt{5d}$$

$$p_a(Y) \geq g + (7d - 3\sqrt{5d})/2.$$

One can replace $\sqrt{5d}$ in the lemma by $\sqrt{(6-\varepsilon)d}$ $(\varepsilon>0)$, and this leads to somewhat sharper bounds on the degree and genus of Y.

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