Some Applications of the Theory of Positive Vector Bundles

by

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Introduction

A considerable body of work has developed over the last few years loosely centered about the notion of positivity in algebraic geometry. On the one hand, numerous results have appeared on what might be called the geometry of projective space, the theme being the often remarkable special properties enjoyed by low codimensional subvarieties of, and mappings to, projective space (cf. [1], [27], [9], [11], [15], [31], [43], [32]). These results depend on the positivity of projective space itself, as manifested for example in various theorems of Bertini type. In another direction, the general theory of positive vector bundles has recently been extended and, more interestingly, applied in various geometric situations (cf. [42], [12], [13], [31], [7]). Bridging these two groups of results, in a class all by itself, one has Mori's far-reaching proof of the Frankel-Hartshorne conjecture ([35], [6]).

Our lectures at the C.I.M.E. conference were largely concerned with the geometry of projective space, and especially with the work of F. L. Zak on linear normality ([43], [32]). In addition, we discussed a recent theorem of Z. Ran [39] related to Hartshorne's conjecture [27] on complete intersections. Most of this material has been surveyed elsewhere (cf. [23], [11], [32]), and we do not propose to duplicate the existing literature here.

The present paper will rather constitute the notes to a course that we _might_ have given at the Acireale conference, focusing on positive vector bundles and their applications. We start (§1) with an elementary overview of the general theory, emphasizing the similarities and differences between the cases of line bundles and vector bundles of higher rank. The remaining sections are devoted to expositions of several previously unpublished proofs and results. In §2 we give a simple topological proof of a theorem guaranteeing that under suitable positivity and dimensional hypotheses a map of vector bundles must drop rank. We then sketch how this may be applied, along the lines of [12], to give a quick proof of (a slight generalization of) a recent theorem of Ghione [16] concerning the existence of special divisors associated to a vector bundle on an algebraic curve. In §3 we use a theorem of Goresky-MacPherson [17] to prove a homotopy Lefschetz-type result for the zero-loci of sections of certain positive vector bundles (Thm. 3.5). We deduce from this the homotopy analogue of the Barth-type theorem for branched coverings of projective space given in [31]. Finally, in §4 we show how Mori's arguments in [35] lead to the proof of an old conjecture of Remmert and Van de Ven, to the effect that if X is a smooth projective variety of dimension ≥ 1 which is the target of a surjective mapping $f : \mathbb{P}^n \to X$, then X is isomorphic to $\mathbb{P}^n$. There are many interesting questions related to this circle of ideas, and the reader will find open problems - some well-known - scattered throughout the paper.

It may seem at this point that the subject of these notes bears little relationship to the theme of the conference, complete intersections. In reality, however, there is an intimate connection. Suppose, for example, that $X \subseteq \mathbb{P}^n$ is the complete intersection of hypersurfaces $F_1,\ldots,F_e$ of positive degrees $d_1,\ldots,d_e$. If we think of F_i as being the zero-locus of a section s_i of the line bundle $\mathcal{O}_{\mathbb{P}^n}(d_i)$, then it is natural to view X as the zero-locus of the section $s = (s_1,\ldots,s_e)$ of the rank e vector bundle $E = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_e)$. But E is the very prototype of a positive, or <u>ample</u>, vector bundle, and in fact most of the basic results about complete intersections (e.g. Lefschetz-type results) are special cases of general results for positive vector bundles (e.g. (1.8) and (3.5) below). In this sense, the theory of ample vector bundles is a natural generalization of the study of complete intersections in projective space.

We work throughout with algebraic varieties over the complex numbers, although the results of §2 remain valid over an arbitrary algebraically closed ground field. If E is a vector bundle on a variety X, we denote by $\mathbb{P}(E)$ the projective bundle of one-dimensional subspaces of E. We shall follow Hartshorne's definition [24] of an ample vector bundle. The reader should be aware that there is a great deal of conflicting terminology in the literature; in particular, ample vector bundles are called "cohomologically positive" in [19] (where "ample" is used in another sense).

I'd like to take this opportunity to express my gratitude to the many Italian mathematicians - and especially to G. Ceresa, L. Chiantini, N. Chiarli, C. Ciliberto, A. Collino, S. Greco, P. Maroscia, and E. Sernesi - who made my stay in Italy a valuable and enjoyable one.

§1. Ample Line Bundles and Ample Vector Bundles.

Our purpose in this section is to give an elementary survey of the general theory of ample vector bundles.

We start by reviewing the basic facts about positivity in the line bundle case. Let X be an irreducible projective variety, and let L be a line bundle on X. Recall that L is <u>very ample</u> if there is a projective embedding

$$X \subseteq \mathbb{P}^N$$

such that L is the restriction to X of the hyperplane line bundle on $\mathbb{P}^N$:

$$L \cong \mathcal{O}_{\mathbb{P}^N}(1)|X.$$

This is perhaps the most appealing notion of positivity from an intuitive point of view, but unfortunately it is technically rather difficult to work with. For example, even when X is a smooth curve, it can be subtle to determine whether or not a given line bundle is very ample - the canonical bundle is a simple case in point.

It is found to be much more convenient to deal instead with a somewhat weaker notion. Specifically, recall that L is <u>ample</u> if $L^{\otimes k}$ is very ample for some $k > 0$. What this definition may lack in intuitive content is made up in the simplicity it yields. For example, if X is a smooth curve, then L is ample if and only if its degree is positive. Ample line bundles behave well functorially: if $f : X \to Y$ is finite (eg. an embedding), and L is an ample line bundle on Y, then f^*L is an ample line bundle on X. When X is smooth, amplitude is equivalent to Kodaira's differential geometric notion of positivity (cf [21]).

There are essentially four basic theorems on ample line bundles. First, one has Serre's cohomological criterion:

(1.1). <u>A line bundle</u> L <u>on</u> X <u>is ample if and only if for every sheaf</u> $\mathfrak{F}$ <u>on</u> X <u>there exists a positive integer</u> $k(\mathfrak{F})$ <u>such that</u>

$$H^i(X, \mathfrak{F} \otimes L^{\otimes k}) = 0$$

<u>for all</u> $i > 0$ <u>and</u> $k \geq k(\mathfrak{F})$.

When X is smooth, one has the more precise:

(1.2). Kodaira Vanishing Theorem: If L is ample, then

$$H^i(X,L^*) = 0$$

for $i \leq \dim X - 1$.

The basic topological fact is given by the

(1.3). Lefschetz Hyperplane Theorem: Assume that X is smooth, and that L is an ample line bundle on X. Let $s \in \Gamma(X,L)$ be a section of L, and let $Z = Z(s)$ be the zero-locus of s. Then

$$\pi_i(X,Z) = 0 \quad \text{for} \quad i \leq \dim X - 1.$$

A variant of (1.3), which holds for arbitrary irreducible X, states that $X - Z$ has the homotopy type of a CW complex of (real) dimension $\leq \dim X$. When $X - Z$ is smooth, this is a well known fact about affine varieties; the result in general was recently established by Goresky-MacPherson [17] and by Hamm [22]. These authors also show that if X is a local complete intersection, then (1.3) itself remains true.

Finally, one has the theorem of Nakai *et al.* which characterizes ample line bundles numerically:

(1.4) A line bundle L on X is ample if and only if for every irreducible subvariety $Y \subseteq X$ the Chern number

$$\int_Y c_1(I)^k$$

is strictly positive, where $k = \dim(Y)$.

We refer to [26, Ch. 1] or [21, Ch. 1] for fuller accounts of the theory of ample line bundles.

In the 1960's, a number of authors - notably Grauert [18], Griffiths [19], and

Hartshorne [24] - undertook to generalize the notion of ampleness to vector bundles of higher rank. One of the goals was to prove analogues for vector bundles of the basic theorems (1.1) - (1.4), and this led initially to a number of competing notions of positivity. (Indeed, the literature of the period is marked by a certain terminological chaos.) With the passage of time, however, it has become clear that the weakest of these definitions is also the most useful. The idea is simply to reduce the definition of amplitude for vector bundles to the case of line bundles.

Suppose, then, that X is an irreducible projective variety, and that E is a vector bundle on X of rank e. Following [24] one defines E to be _ample_ if the Serre line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ on the projective bundle $\mathbb{P}(E^*)$ is ample.* The first indication that this definition is the correct one is that it leads to various desirable formal properties (cf. [24]):

(1.5) (i) _A quotient of an ample vector bundle is ample._

(ii) _A direct sum of vector bundles is ample if and only if each summand is._

(iii) E _is ample if and only if the symmetric power_ $S^k(E)$ _is for some (or all) positive integer(s)_ k. _If_ E _and_ F _are ample, then so is_ $E \otimes F$.

(iv) _If_ $f : X \to Y$ _is a finite map, and if_ E _is an ample vector bundle on_ Y, _then_ f^*E _is an ample vector bundle on_ X. _If_ f _is in addition flat, then the converse holds._

The basic results (1.1) - (1.3) have good analogues for ample vector bundles:

(1.6) _A vector bundle_ E _on_ X _is ample if and only if for every coherent sheaf_ $\mathfrak{F}$ _on_ X _there exists a positive integer_

*The presence of $\mathbb{P}(E^*)$ here, rather than $\mathbb{P}(E)$, may be explained by the observation that if $E = L$ is a line bundle, so that $\mathbb{P}(E) \cong \mathbb{P}(E^*) \cong X$, then $\mathcal{O}_{\mathbb{P}(E^*)}(1) = L$, whereas $\mathcal{O}_{\mathbb{P}(E)}(1) = L^*$.

$k(\mathfrak{F})$ <u>such that</u>

$$H^i(X, \mathfrak{F} \otimes S^k(E)) = 0$$

<u>for</u> $i > 0$ <u>and</u> $k \geq k(\mathfrak{F})$ ([19], [24]).

The analogue of Kodaira's vanishing theorem (1.2) is due to Le Poitier [33]:

(1.7) <u>If</u> X <u>is smooth, and</u> E <u>is ample, then</u>

$$H^i(X, E^*) = 0$$

<u>for</u> $i \leq \dim X - \operatorname{rk} E$.

The strongest general Lefschetz-type result was proved by Sommese in [42]:

(1.8) <u>Assume that</u> X <u>is smooth, and that</u> E <u>is an ample vector bundle on</u> X <u>of rank</u> e. <u>Let</u> $s \in \Gamma(X,E)$ <u>be a section of</u> E, <u>and let</u> $Z = Z(s)$ <u>be the zero-locus of</u> s. <u>Then</u>

$$H^i(X, Z; \mathbb{Z}) = 0$$

<u>for</u> $i \leq \dim X - e$.

Because Sommese's argument deserves to be better known that it is, we give the

<u>Proof</u>. We will show, for arbitrarily singular irreducible X, that

$$(1.9) \qquad H_i(X - Z; \mathbb{Z}) = 0$$

for $i \geq \dim X + e$. When X is smooth, the theorem as stated follows by Lefschetz duality.

Consider then the projective bundle

$$\pi : \mathbb{P}(E^*) \to X.$$

The Serre line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is a quotient of π^*E, and so the given section s determines a section $s^* \in \Gamma(\mathbb{P}(E^*), \mathcal{O}_{\mathbb{P}(E^*)}(1))$:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(E^*)} \arrow[d, "\pi^* s"'] \arrow[dr, dashed, "s^*"] & & \\ \pi^* E \arrow[r] & \mathcal{O}_{\mathbb{P}(E^*)}(1) \arrow[r] & 0. \end{tikzcd}$$

Let $Z^* \subseteq \mathbb{P}(E^*)$ denote the zero-locus of s^*. We may describe Z^* very concretely as follows. Thinking of $\mathbb{P}(E^*)$ as the bundle of hyperplanes in E:

$$\mathbb{P}(E^*) = \{(x,\Lambda)\,|\,\Lambda \subseteq E(x) \text{ a cod 1 subspace}\},$$

a moment's thought shows that

$$Z^* = \{(x,\Lambda)\,|\,s(x) \in \Lambda\}.$$

In particular, the bundle map π restricts to a morphism

$$p : \mathbb{P}(E^*) - Z^* \longrightarrow X - Z,$$

and in fact, p <u>is a</u> $\mathbb{C}^{e-1}$ - <u>bundle</u> (but not, in general, a vector bundle, i.e. p may not section).

On the other hand, since E is ample, Z^* is an ample divisor on $\mathbb{P}(E^*)$. Therefore $\mathbb{P}(E^*) - Z^*$ is an affine variety, of dimension $\dim X + e - 1$, and hence has the homotopy type of a CW complex of (real) dimension $\leq \dim X + e - 1$. In particular, $H_i(\mathbb{P}(E^*) - Z^*; \mathbb{Z}) = 0$ for $i \geq \dim X + e$. But since p is an affine space bundle, this implies that $H_i(X - Z; \mathbb{Z}) = 0$ for $i \geq \dim X + e$, as desired. □

<u>Problem</u>. In the situation of (1.8), is it true that $\pi_i(X,Z) = 0$ for $i \leq \dim X - e$?

Under a stronger positivity condition on E, and a transversality assumption on the section s, Griffiths [19] has proven such a homotopy statement. Another result of this nature is given in §3. (Thm. 3.5).

Observe that there is no genericity or transversality hypothesis on the section s in (1.8). Hence the result gives topological obstructions to expressing a variety set-theoretically as the zero-locus of a section of an ample vector bundle.

For example, consider the Segre variety

$$S = \mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5$$

Since $b_2(S) = 2$ while $b_2(\mathbb{P}^5) = 1$, we conclude that there cannot exist an ample vector bundle E of rank 2 on $\mathbb{P}^5$ with a section vanishing precisely on E. In particular, (1.8) gives an elementary proof of the well known fact that S is not a set-theoretic complete intersection. (Compare the lectures of Forster and Valla in this volume.)

Another result along the lines of (1.8) has been established by Ein [7], who proves a Noether-type theorem on the Picard group of the zero-locus of a generic section of certain ample rank $n-2$ bundles on $\mathbb{P}^n$; he also treats determinantal surfaces. In the same paper, Ein uses the vanishing theorem (1.7) of Le Poitier to give a simple proof of a theorem of Evans and Griffith on the cohomology of vector bundles of small rank on projective space.

Turning to the numerical properties of ample vector bundles, one finds that there are two questions to ask if one hopes to generalize (1.4). First:

A. What are the numerically positive polynomials for ample vector bundles?

(Recall that a homogeneous polynomial $P \in \mathbb{Q}[c_1,\dots,c_e]$ of weighted degree n is <u>numerically positive</u> if for every irreducible projective variety X of dimension n, and for every ample vector bundle E of rank e on X, the Chern number

$$\int_X P(c_1(E),\dots c_e(E))$$

is strictly positive. For example, if $e = \mathrm{rk}(E) = 1$, the positive polynomials are just $\alpha\, c_1^n$ $(\alpha > 0)$.) And secondly:

B. Is there a numberical criterion for ampleness analogous to the theorem of Nakai <u>et</u> <u>al</u>. for line bundles?

Question (B) was the first to be answered. A theorem of Hartshorne [25] states that a vector bundle E on a smooth curve X is ample if and only if every quotient of E has positive degree. For X of dimension ≥ 2, however, simple examples [8] show that there cannot be a numerical criterion, at least in the form suggested by Griffiths [20].

As for question (A), the numerically positive polynomials may be described succinctly - if unrevealingly - as follows. Let $\Lambda(n,e)$ denote the set of partitions of n into a sum of non-negative integers $\leq e$. Given $\lambda \in \Lambda(n,e)$, λ being the partition $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, one forms the so-called Schur polynomial P_λ, defined as the $n \times n$ determinant

$$P_\lambda = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & & \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & & \\ & & \cdots & & \\ & \cdots & & c_{\lambda_n-1} & c_{\lambda_n} \end{vmatrix}$$

where one makes the convention that $c_0 = 1$ and $c_i = 0$ for $i \notin [0,e]$. The P_λ's from a basis for the homogeneous polynomials of weighted degree n, and the result is:

(1.10) $P \in \mathbb{Q}[c_1,\ldots,c_e]$ <u>is numerically positive for ample vector bundles if and only if</u> $P \neq 0$, <u>and</u> P <u>is a non-negative linear combination of the</u> P_λ $(\lambda \in \Lambda(n,e))$.

We refer to [13] for the proof, and for a discussion of earlier work on question (A).

The determinantal definition of the P_λ is evidently rather awkward to deal with. There is a more conceptual approach, which makes (1.10) seem quite natural. For simplicity, we explain this only for bundles generated by their global sections.

Suppose, then, that E is an ample vector bundle of rank e on X^n which is given as a quotient of a trivial bundle of rank m. Then there is a classifying map

$$\varphi : X \longrightarrow G,$$

where $G = G(m-e,m)$ is the Grassmannian of codimension e subspaces of an m-dimensional vector space; the bundle E is recovered as the pull-back of the universal quotient bundle Q on G. Consider now a codimension n cycle $z \in H^{2n}(G)$. Then $\varphi^*(z)$ is a top dimensional cohomology class on X, and one may ask when $\int_X \varphi^*(z) > 0$. It is not hard to show that the ampleness of E implies that this degree is positive whenever z is represented by an <u>effective</u> algebraic cycle. Conversely, if z is not effective, then there exists an X and E so that $\int \varphi^*(z) < 0$. But the cone of effective cycles on the Grassmannian G is well-understood: it is generated by the codimension n Schubert cycles $\{\Omega_\lambda\}_{\lambda \in \Lambda(n,e)}$ (cf. [21]). And it turns out that the cycle Ω_λ represents the cohomology class $P_\lambda(c_1(Q),\ldots,c_e(Q))$. Thus $\varphi^*(\Omega_\lambda) = P_\lambda(c_1(E),\ldots,c_e(E))$, which proves (1.10) for bundles generated by their global sections. In general, one thinks of the classes $P_\lambda(c_1(E),\ldots,c_e(E))$ as representing "virtual" Schubert cycles; up to now, the explicit formula for the P_λ has not proved to be of any particular significance in itself.

<u>Problem</u>. Find a wider class of vector bundles for which the P_λ are numerically positive.

It seems certain that one could weaken the hypothesis of ampleness and yet retain the positivity of the Schur polynomials. For applications, such a strengthening of (1.10) should prove useful. What seems difficult, however, is to find a suitable class of bundles with which to deal. It might well be that this problem is most sensibly attacked only with some particular application in mind.

<u>Problem</u>. Determine whether the following conjecture of Hartshorne [26, III. 4.5] is true or false:

(*) Let M be a smooth variety, and let $X, Y \subseteq M$ be smooth projective subvarieties with ample normal bundles. If $\dim X + \dim Y \geq \dim M$, then X meets Y.

A number of conjectures have appeared suggesting global consequences of ampleness of normal bundles (eg. [26, III. 4.4], [9]). Simple counter-examples dispose of many of these (cf. [14]), although they tend to be true when the ambient space is a rational homogeneous manifold. What's fascinating about Hartshorne's conjecture (*) is that several approaches to the construction of counter-examples seem systematically to fail. Hence it seems likely that the resolution of the conjecture one way or the other could involve some interesting new ideas.

§2. Degeneracy Loci, and a Theorem of Ghione.

A theorem on the non-emptiness of degeneracy loci.

Let X be an irreducible projective variety of dimension n, and let

$$u : E \longrightarrow F$$

be a homomorphism of vector bundles of ranks e and f respectively. A number of interesting geometric problems can be formulated in terms of the degeneracy loci associated to such a map, i.e. the sets

$$D_k(u) \underset{\text{def}}{=} \{x \in X \mid rk(u(x)) \leq k\}$$

Recall that the set $D_k(u)$ is Zariski-closed, and its postulated codimension in X is $(e-k)(f-k)$; if non-empty, its actual codimension is $\leq (e-k)(f-k)$.

It may happen, of course, that $D_k(u)$ is empty even when its expected dimension is non-negative. Our purpose here is to give a simple proof that this cannot occur under suitable positivity hypotheses:

Theorem 2.1. Assume that the vector bundle

$$\mathrm{Hom}(E,F) = E^* \otimes F$$

is ample. If $n \geq (e-k)(f-k)$, *then* $D_k(u)$ *is non-empty.*

The proof below arose in the course of the author's work on [12], where a more elaborate argument was given to show that in fact $D_k(u)$ is connected if $n > (e-k)(f-k)$.

We shall actually prove a slight strengthening of (2.1). Specifically

(2.2). Assume that $\mathrm{Hom}(E,F)$ is ample. Fix an integer ℓ, and let

$$Y \subseteq D_\ell(u)$$

be an irreducible projective variety of dimension $m \geq (e+f) - 2\ell + 1$. Then

$$D_{\ell-1}(u) \cap Y \neq \emptyset.$$

Note that $(e+f) - 2\ell + 1$ is the expected codimension of $D_{\ell-1}(u)$ in $D_\ell(u)$. Theorem 2.1 follows by applying (2.2) successively to each of the varieties in the chain.

$$X = D_r(u) \supseteq D_{r-1}(u) \supseteq \cdots \supseteq D_k(u),$$

where $r = \min(e,f)$. The idea of the proof is to exploit the observation that if the assertion were false, then the kernel and image of u would be vector bundles on Y. This approach has been taken up again in [10].

Proof of (2.2). We assume that $Y \subseteq D_\ell(u)$ is a projective variety of dimension m which does not meet $D_{\ell-1}(u)$; we will show that $m \leq (e+f) - 2\ell$. Evidently we may suppose that $\ell \leq \min(e,f)$, and for simplicity of notation we write E and F for the restrictions of these bundles to Y.

Let $N = \ker(u|Y)$ and $K = \mathrm{im}(u|Y)$. Since u has rank ℓ everywhere on Y, N and K are vector bundles of ranks $e - \ell$ and ℓ respectively. Consider the projective bundle $\pi : \mathbb{P}(E) \longrightarrow Y$. On $\mathbb{P}(E)$ one has the diagram:

$$\begin{tikzcd} & & \mathcal{O}_{\mathbb{P}(E)}(-1) \arrow[d, hook] \arrow[dr, "s"] & & \\ 0 \arrow[r] & \pi^*N \arrow[r] & \pi^*E \arrow[r, "\pi^*u"'] & \pi^*K \arrow[r] & 0, \end{tikzcd}$$

which defines a section $s \in \Gamma(\mathbb{P}(E), \pi^*K \otimes \mathcal{O}_{\mathbb{P}(E)}(1))$ as shown. Note that the zero-locus $Z(s)$ of s is exactly the subvariety $\mathbb{P}(N) \subseteq \mathbb{P}(E)$. The idea is to apply the Lefschetz theorem (1.9) to study $\mathbb{P}(E) - \mathbb{P}(N)$.

To this end, let t denote the composition

$$\mathcal{O}_{\mathbb{P}(E)}(-1) \xrightarrow{s} \pi^*K \hookrightarrow \pi^*F .$$

Then evidently

$$Z(t) = Z(s).$$

On the other hand, we shall show below that

(2.3) If $E^* \otimes F$ is ample on Y then

$$\pi^*F \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$$

is an ample vector bundle on $\mathbb{P}(E)$.

Thus $\mathbb{P}(N)$ is the zero-locus of the section t of the ample vector bundle $\pi^*F \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$. Hence by (1.9):

$$H_i(\mathbb{P}(E) - \mathbb{P}(N)) = 0 \quad \text{if} \quad i \geq (m+e-1) + f.$$

But there is a natural map

$$\begin{tikzcd} \mathbb{P}(E) - \mathbb{P}(N) \arrow[rr, "p"] \arrow[dr, "\pi"'] & & \mathbb{P}(K) \arrow[dl] \\ & Y & \end{tikzcd} ;$$

fibre by fibre, p is just the linear projection centered at $\mathbb{P}(N(y)) \subseteq \mathbb{P}(E(y))$. In particular, p is a $\mathbb{C}^{e-\ell}$ - bundle map, and hence

$$H_i(\mathbb{P}(E) - \mathbb{P}(N)) = H_i(\mathbb{P}(K))$$

for all i. Therefore $H_i(\mathbb{P}(K)) = 0$ for $i \geq m+e+f-1$. But $\mathbb{P}(K)$ is a compact variety, of dimension $m + \ell - 1$, and so $H_{2(m+\ell-1)}(\mathbb{P}(K)) \neq 0$. We conclude that

$$2(m + \ell - 1) < m + e + f - 1,$$

i.e.

$$m \leq e + f - 2\ell,$$

as desired.

It remains to check (2.3), for which we use an argument suggested by W. Fulton. Consider the projectivization $\mathbb{P} = \mathbb{P}(\pi^*F^* \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)) \longrightarrow \mathbb{P}(E)$. We need to show that $\mathcal{O}_{\mathbb{P}}(1)$ is an ample line bundle. But $\mathbb{P}$ is isomorphic to the fibre product $\mathbb{P}(E) \times_X \mathbb{P}(F^*)$, and $\mathcal{O}_{\mathbb{P}}(1)$ is the restriction of the Serre line bundle $\mathcal{O}_{\mathbb{P}(E\otimes F^*)}(1)$ under the Segre embedding

$$\mathbb{P}(E) \times_X \mathbb{P}(F^*) \subseteq \mathbb{P}(E \otimes F^*).$$

Hence $\mathcal{O}_{\mathbb{P}(E\otimes F^*)}(1)$ is ample since $E^* \otimes F$ is. □

Ghione's generalization of the Kempf-Kleiman-Laksov existence theorem.

One of the most famous examples of determinantal loci are the varieties of special divisors on a smooth projective algebraic curve C of genus g. Specifically, let $J = \operatorname{Pic}^0(C)$ be the Jacobian of C, and fix once and for all a base point $P_0 \in C$. One is interested in the set

$$W^r_d(C) = \{x \in J \,|\, h^0(L_x(dP_0)) \geq r + 1\},$$

where L_x is the line bundle of degree 0 on C corresponding to the point $x \in J$. Thus $W^r_d(C)$ parametrizes linear equivalence classes of divisors of degree d moving in a linear system of (projective) dimension $\geq r$.

Let us recall how these varieties of special divisors are realized as determinantal loci. Choose some integer $n \geq \max(d, 2g)$, and $n-d$ points $p_1, \ldots, p_{n-d} \in C$ (say distinct, to fix ideas). Then for each $x \in J$, evaluation at the p_i yields a homomorphism

$$u(x) : H^0(C, L_x(nP_0)) \longrightarrow \bigoplus_{i=1}^{n-d} H^0(C, L_x(nP_0) \otimes \mathcal{O}_{p_i}).$$

As x varies over J, the vector spaces $H^0(C, L_x(nP_0))$ and $\oplus H^0(C, L_x(nP_0) \otimes \mathcal{O}_{p_i})$ fit together to form vector bundles E and F on J, of ranks $n+1-g$ and

$n-d$ respectively. Furthermore, the maps $u(x)$ globalize to a vector bundle homomorphism

$$u : E \longrightarrow F.$$

Since $\ker u(x) = H^0(C, L_x(nP_0 - \Sigma P_i))$, we see that up to translation

$$W^r_d(C) = D_{n-g-r}(u).$$

(Cf. [28], [29] or [12] for details.) It follows in particular that if $W^r_d(C)$ is non-empty, then

$$\dim W^r_d(C) \geq \rho^r_d(C) \underset{\mathrm{def}}{=} g - (r+1)(g-d+r).$$

The celebrated existence theorem of Kempf [28] and Kleiman-Laksov [29] asserts that in fact $W^r_d(C) \neq \emptyset$ provided that $\rho^r_d(C) \geq 0$.

The traditional approach to the Kempf-Kleiman-Laksov theorem is to compute via Porteous' formula the (postulated) fundamental class of $W^r_d(C)$ (or of a closely related variety). This turns out to be non-zero when $\rho \geq 0$, and the theorem follows. This <u>quantitative</u> approach, as we may call it, has the advantage that a formula for $[W^r_d(C)]$, which is useful in enumerative questions, emerges as a by-product. However there is an alternative <u>qualitative</u> approach based on positivity considerations. Specifically, it was shown in [12, §2] that

(2.4) $E^* \otimes F$ <u>is an ample vector bundle on</u> J.

Thus in fact the existence theorem follows from the elementary result (2.1), and this is one of the quickest proofs available.

Ghione [16] has recently proved an interesting generalization of the Kempf-Kleiman-Laksov theorem. Specifically, fix a vector bundle

$$M \text{ of degree } a \text{ and rank } e$$

on C. Then set

$$W^r_d(C,M) = \{x \in J \mid h^0(M(dP_0) \otimes L_x) \geq r+1\}.$$

Thus the classical set $W_d^r(C)$ correspond to taking $M = \mathcal{O}_C$. As before, the loci $W_d^r(C,M)$ may be realized determinantally. To do so, following [16], we fix an integer $n \geq 2g - d$ large enough so that $M^*(nP_0)$ is generated by its global sections. Choosing $e = rk(M)$ general sections gives an exact sequence

$$0 \longrightarrow M \longrightarrow \mathcal{O}_C^e(nP_0) \longrightarrow \tau \longrightarrow 0$$

on C, where τ is a torsion sheaf of length $en - a$. Then for each $x \in J$ we have homomorphisms

$$u(x) : H^0(C, \mathcal{O}_C^e((n+d)P_0 \otimes L_x)) \longrightarrow J^0(C, \tau \otimes L_x(dP_0)),$$

which as before fit together to form a vector bundle map

$$(2.5) \qquad u : E \longrightarrow F,$$

where E and F are now vector bundles on J of ranks $e(n + d + 1 - g)$ and $en - a$ respectively. Then $\ker u(x) = H^0(C, M(dP_0 \otimes L_x)$, so

$$W_d^r(C) = D_{e(n+d+1-g)-(r+1)}(u).$$

In particular, if $W_d^r(C,M) \neq \emptyset$, then

$$\dim W_d^r(C,M) \geq \rho_d^r(C,M) \underset{\text{def}}{=} g - (r+1)(e(g-d+1) + r + 1 - a).$$

Ghione's generalization of the Kempf-Kleiman-Laksov theorem is:

<u>Theorem 2.6</u>. ([16]). <u>If</u> $\rho_d^r(C,M) \geq 0$, <u>then</u> $W_d^r(C,M)$ <u>is non-empty</u>.*

Ghione takes the quantitative approach to Theorem 2.6, and obtains also a formula for $[W_d^r(C,M)]$ valid when $\dim W_d^r(C,M) = \rho_d^r(C,M)$. For Theorem 2.6 the qualitative approach is very much quicker, and essentially involves nothing beyond what was proved in [12].

<u>Proof of (2.6)</u>. Replacing M by $M(dP_0)$, we may as well assume that $d = 0$. It suffices to show that $E^* \otimes F$ is ample, E and F being the vector bundles defined informally in (2.5). Let us start by defining these bundles more precisely.

---

*Ghione assumes that M is general in a suitable sense. However the proof below shows that this is not necessary.

Denote by f and π the projections of $J \times C$ onto J and C respectively. Let $\mathcal{L}$ be the Poincaré line bundle on $J \times C$, normalized so that $\mathcal{L}|J \times \{P_0\} = \mathcal{O}_J$. We take

$$E = f_*(\mathcal{O}_C^e(nP_0)) \otimes \mathcal{L})$$

and

$$F = f_*(\pi^*\tau \otimes \mathcal{L}).$$

The map u arises by taking direct images from the exact sequence

$$0 \longrightarrow \pi^*M \otimes \mathcal{L} \longrightarrow \pi^*(\mathcal{O}_C^e(nP_0)) \otimes \mathcal{L} \longrightarrow \pi^*\tau \otimes \mathcal{L} \longrightarrow 0.$$

Since $E = \bigoplus_{i=1}^{e} E_1$, where $E_1 = f_*(\pi^*(\mathcal{O}_C(nP_0) \otimes \mathcal{L})$, it is enough to show that $E_1^* \otimes F$ is ample. On the other hand, τ - like any torsion sheaf on C - has a filtration whose successive quotients are torsion sheaves of length one, and hence isomorphic to $\mathcal{O}_{P_i}$ for suitable points $P_i \in C$. Therefore F has a filtration whose successive quotients are line bundles of the form

$$f_*(\pi^*\mathcal{O}_{P_i} \otimes \mathcal{L}) \underset{\text{def}}{=} \mathcal{L}_{P_i}.$$

Recalling that an extension of ample vector bundles is ample, we are reduced to proving the amplitude of $E_1^* \otimes \mathcal{L}_{P_i}$. But this is the assertion of Lemma 2.2 of [12]. (The proof in brief: observing that $\mathcal{L}_{P_i}$ is a deformation of $\mathcal{L}_{P_0} = \mathcal{O}_J$, one shows that it suffices to prove that E_1^* is ample. But $\mathbb{P}(E_1) = C_n$, the n^{th} symmetric product of C, and $\mathcal{O}_{\mathbb{P}(E_1)}(1) = \mathcal{O}_{C_n}(C_{n-1})$, C_{n-1} being embedded in C_n via $D \longrightarrow D+P_0$. And it is elementary - eg. by Nakai's criterion - that C_{n-1} is an ample divisor on C_n.) $\square$

Note that by §1 of [12] we conclude also that

(2.7) <u>In the situation of Theorem 2.6, if</u> $\rho_d^r(C,M) > 0$, <u>then</u> $W_d^r(C,M)$ <u>is connected</u>.

Problem. Work out concretely the varieties $W_d^r(C,M)$ for various vector bundles M on curves of low genus.

The question is whether the geometry of C is reflected in the geometry of $W_d^r(C,M)$ as it is in the geometry of $W_d^r(C)$. (cf. [38, Chapt. 1]).

Problem. Are there theorems of Martens-Mumford type ([34], [37]) for $W_d^r(C,M)$, say when M is stable?

The examples of Raynaud [40] show that the cohomological properties of stable vector bundles can be quite subtle.

§3. A Theorem of Barth-Larsen Type on the Homotopy Groups of Branched Coverings of Projective Space.

A celebrated theorem of Barth and Larsen ([1],[2],[4],[30]) asserts that if $X \subseteq \mathbb{P}^{n+e}$ is a smooth variety of dimension n and codimension e, then the maps $\pi_i(X) \longrightarrow \pi_i(\mathbb{P}^{n+e})$ induced by inclusion are bijective for $i \leq n-e$, and surjective if $i = n-e+1$ (cf. also [11], §9). Our goal in this section is to prove an analogue for branched coverings of projective space:

Theorem 3.1. *Let X be an irreducible, non-singular, projective variety of dimension n, and let $f : X \longrightarrow \mathbb{P}^n$ be a finite mapping of degree d. Fix $x \in X$. Then the induced homomorphisms*

$$f_* : \pi_i(X,x) \longrightarrow \pi_i(\mathbb{P}^n, f(x))$$

are bijective for $i \leq n+1-d$, and surjective if $i = n+2-d$.

Corollary 3.2. *In the setting of the theorem, the maps*

$$f_* : H_i(X\,;\mathbb{Z}) \longrightarrow H_i(\mathbb{P}^n : \mathbb{Z})$$

and

$$f^* : H^i(\mathbb{P}^n\,;\mathbb{Z}) \longrightarrow H^i(X\,;\mathbb{Z})$$

are isomorphisms if $i \leq n+1-d$. When $i = n+2-d$, f_* is surjective and f^* is injective. □

It follows for example that if $d \leq n$ then X is simply connected, while if $d \leq n-1$ then $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{P}^n)$. The theorem was announced in [31], where the analogous result for complex cohomology was proved. The material in this section was part of the author's Ph.D. thesis (unpublished).

It is shown in [31, §1] that canonically associated to a branched covering $f : X \longrightarrow \mathbb{P}^n$ satisfying the hypotheses of (3.1), there exists a vector bundle

$$E \longrightarrow \mathbb{P}^n$$

of rank $d-1$ having the property that f factors through an embedding of X in the total space of E. The bundle E may be defined as the dual of the kernel of the trace $\mathrm{Tr}_{X/\mathbb{P}^n} : f_* \mathcal{O}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n}$. The crucial fact for our purposes is that the bundle associated to a branched covering of projective space satisfies the strong positivity property:

(3.3) $E(-1)$ is generated by its global sections, ie. E arises as a quotient of a direct sum $\oplus\, \mathcal{O}_{\mathbb{P}^n}(1)$ of copies of the hyperplane line bundle.

Proof. ([31], §1). According to a theorem of Mumford [36, Lect. 14], it suffices to show that E is (-1) - regular, i.e. that $H^i(\mathbb{P}^n, E(-i-1)) = 0$ for $i > 0$. This is equivalent by duality to the assertion that

$$(*) \qquad H^{n-i}(\mathbb{P}^n, E^*(i-n)) = 0 \quad \text{for} \quad i > 0.$$

It follows from the definition of E that

$$f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \oplus E^*$$

and hence

$$\mathbb{C} = H^0(X, \mathcal{O}_X) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \oplus H^0(\mathbb{P}^n, E^*) = \mathbb{C} \oplus H^0(\mathbb{P}^n, E^*) .$$

Thus $H^0(\mathbb{P}^n, E^*) = 0$ which proves (*) for $i = n$. When $1 \leq i \leq n-1$ we have similarly

$$H^{n-i}(\mathbb{P}^n, E^*(i-n)) = H^{n-i}(\mathbb{P}^n, f_*\mathcal{O}_X(i-n))$$
$$= H^{n-i}(X, f^*\mathcal{O}_{\mathbb{P}^n}(i-n)).$$

But $f^*\mathcal{O}_{\mathbb{P}^n}(i-n)$ is the dual of an ample line bundle, whence $H^{n-i}(X, f^*\mathcal{O}_{\mathbb{P}^n}(i-n)) = 0$ by the Kodaira vanishing theorem. □

Theorem 3.1 is therefore a consequence of

Theorem 3.4. *Let $E \longrightarrow \mathbb{P}^n$ be a vector bundle of rank e satisfying the positivity condition (3.3). Suppose that X is a compact, connected, local complete intersection variety of pure dimension n embedded in the total space of E:*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & E \arrow[d] \\ & \mathbb{P}^n. \end{tikzcd}$$

Define f as shown, and fix $x \in X$. Then

$$f_* : \pi_i(X,x) \longrightarrow \pi_i(\mathbb{P}^n, f(x))$$

is bijective for $i \leq n-e$, and surjective if $i = n-e+1$
(i.e. $\pi_i(E,X,x) = 0$ for $i \leq n-e+1$).

Note that f, being affine and proper, is finite. When E is the direct sum of e copies of the hyperplane line bundle, the theorem is equivalent to the Barth-Larsen theorem for embeddings $X \subseteq \mathbb{P}^{n+e}$ (cf [31, Rmk. 2.4]). We leave it to the reader to formulate the corresponding results for integral homology and cohomology implied by (3.4). Note that the latter in turn imply that if X is smooth, and if $e \leq n-2$, then $f^* : \mathrm{Pic}(\mathbb{P}^n) \longrightarrow \mathrm{Pic}(X)$ is an isomorphism.

Turning to the proof of (3.4), the strategy is to derive from Deligne's generalization [11, §9] of the Fulton-Hansen connectedness theorem, an analogoue for the diagonal embedding $X = \Delta_X \longrightarrow X \times X$. This will imply (3.4) in much the same way that [11, (9.2)] can be used, as W. Fulton remarked, to prove the Barth-Larsen theorem. The one additional ingredient we shall need is the following Lefschetz-type result, which is proved below.

Theorem 3.5. *Let X be a complete, connected, but possibly reducible local complete intersection variety of pure dimension n, and let A be an ample line bundle on X which is generated by its global sections. Suppose that E is a vector bundle of rank e on X having the property that $E \otimes A^*$ is generated by its global sections. Let $s \in \Gamma(X,E)$ be a section of E, and let $Z = Z(s) \subseteq X$ be the zero locus of s. Then, fixing $x \in Z$, one has*

$$\pi_i(X,Z,x) = 0$$

for $i \leq n-e$.

Proof of Theorem 3.4. Put $Y = (f \times f)^{-1}(\Delta_{\mathbb{P}^n})$, so that the diagonal embedding $\delta : X \longrightarrow X \times X$ factors through an embedding of X in Y. The set-up we shall deal with is summarized in the diagram (3.6) below. Each of the three squares is cartesian, and we henceforth make free use of the natural identifications indicated in that diagram. The inclusion $E \hookrightarrow E \oplus E = E \times_{\mathbb{P}^n} E$ is the evident diagonal homomorphism over $\mathbb{P}^n = \Delta_{\mathbb{P}^n}$.

$$(3.6)\qquad \begin{tikzcd}
X = \Delta_X \arrow[r, hook] \arrow[d, hook] & Y \arrow[r, hook] \arrow[d, hook] & X \times X \arrow[d, hook] \\
E = \Delta_E \arrow[r, hook] \arrow[dr] & E \oplus E = E \times_{\mathbb{P}^n} E \arrow[r, hook] \arrow[d] & E \times E \arrow[d] \\
 & \mathbb{P}^n = \Delta_{\mathbb{P}^n} \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^n
\end{tikzcd}$$

Note that Y is a complete, connected, local complete intersection of pure dimension n. Indeed, Y is locally cut out in $X \times X$ by n equations, and maps finitely to $\mathbb{P}^n$. It follows that Y has pure dimension n, and hence is a local complete intersection variety. The connectedness of Y follows, for instance, from Deligne's theorem [11, Thm. 9.2].

Consider first the inclusion $X = \Delta_X \hookrightarrow Y$. We assert that

$$(3.7) \qquad \pi_i(Y,X) = 0$$

for $i \leq n-e$.* Letting h denote the composition $Y \longrightarrow E \oplus E \longrightarrow \mathbb{P}^n$, the point to observe is that X is defined in Y as the zero-locus of a section of h^*E. In fact, the embedding of Y in the total space of $E \oplus E$ determines a tautological section of $h^*(E \oplus E)$, i.e. two sections $s_1\ s_2 \in \Gamma(Y, h^*E)$, and $X = \text{Zeroes}(s_1 - s_2) \subseteq Y$. But the positivity assumption (3.3) on E implies that $h^*E(-1)$ is generated by its global sections, and hence since h is finite (3.7) is a consequence of Theorem 3.5.

On the other hand, Deligne's theorem [11, (9.2)] applies to the inclusion $Y \longrightarrow X \times X$. In the case at hand, the theorem in question states that

$$\pi_i(X \times X, Y) = 0$$

for $i \neq 2$, $i \leq n$, and that if $n \geq 2$ there is an exact sequence

$$(*) \qquad \pi_2(Y) \longrightarrow \pi_2(X \times X) \longrightarrow \mathbb{Z} \longrightarrow \pi_1(Y) \longrightarrow \pi_1(X \times X) \longrightarrow 0$$

Moreover the map to $\mathbb{Z}$ in $(*)$ may be identified with the difference of the homomorphisms

$$(pr_1 \circ (f \times f))_*,\ (pr_2 \circ (f \times f))_* : \pi_2(X \times X) \longrightarrow \pi_2(\mathbb{P}^n) = \mathbb{Z}.$$

Consider now the composition δ_*:

$$\begin{tikzcd} \pi_i(X) \arrow[r] \arrow[rr, bend right, "\delta_*"'] & \pi_i(Y) \arrow[r] & \pi_i(X \times X). \end{tikzcd}$$

This is just the diagonal map, so δ_* is in any event injective. But it follows from (3.7) and Deligne's theorem that δ_* is surjective when $i \leq n-e$, $i \neq 2$, which implies $\pi_i(X) = 0$. This proves Theorem 3.4 in the range $i \leq n-e$, $i \neq 2$. If $i = 2 \leq n-e$, then one obtains the commutative diagram

---

*As we deal exclusively with path-connected spaces, we will henceforth omit base-points.

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(X \times X) & \longrightarrow & \mathbb{Z} & \longrightarrow 0 = \pi_1(X) \\
& \downarrow f_* & & \downarrow f_* \times f_* & & \downarrow \wr\wr & \\
0 \longrightarrow & \pi_2(\mathbb{P}^n) & \longrightarrow & \pi_2(\mathbb{P}^n \times \mathbb{P}^n) & \longrightarrow & \mathbb{Z} & \longrightarrow 0 = \pi_1(\mathbb{P}^n)
\end{array}$$

of exact sequences. Hence $\ker f_* = \ker(f_* \times f_*)$, and $\operatorname{coker} f_* = \operatorname{coker}(f_* \times f_*)$. But this forces $\ker f_* = \operatorname{coker} f_* = 0$, i.e. f_* is an isomorphism on π_2. Finally, the surjectivity of $\pi_{n-e+1}(X) \longrightarrow \pi_{n-e+1}(\mathbb{P}^n)$ is non-trivial only if $n - e = 1$, and we leave this case to the reader. (Hint: the diagram above remains exact on the right.) $\square$

Proof of the Lefschetz-type theorem (3.5).

The strategy will be to reduce the result to the following theorem of Goresky and MacPherson, which one may view as a non-compact strengthening of the classical Lefschetz theorem:

(3.8). *Let Y be a connected local complete intersection variety of pure dimension n, possibly reducible and non-compact, and let*

$$f : Y \longrightarrow \mathbb{P}^m$$

be a finite-to-one morphism. Let $L \subseteq \mathbb{P}^m$ be a linear space of co-dimension d, and denote by L_ε an ε-neighborhood of L with respect to some Riemannian metric on $\mathbb{P}^n$. Then for sufficiently small ε one has

$$\pi_i(Y, f^{-1}(L_\varepsilon)) = 0 \quad \text{for} \quad i \leq n-d.$$

See [17, §4] for an announcement with indications of proof.

Returning to the situation of (3.5), we start with

Lemma 3.9. *Let X be a compact irreducible variety, and A an ample line bundle on X which is generated by its global sections. Let $T \longrightarrow X$ be the direct sum of t copies of A, and denote by $X_0 \longrightarrow T$ the zero section. Suppose that Y*

is a connected, local complete intersection of pure dimension n, *and that* $g : Y \longrightarrow T$ *is a finite (i.e. finite-to-one and proper) map. Then*

$$\pi_i(Y, g^{-1}(X_0)) = 0 \quad \underline{\text{for}} \quad i \leq n - t.$$

Proof. The assumption on A means that there is a finite map $\varphi : X \longrightarrow \mathbb{P}^r$ such that $A = \varphi^* \mathcal{O}_{\mathbb{P}^r}(1)$. Let S denote the direct sum of t copies of $\mathcal{O}_{\mathbb{P}^r}(1)$. In a standard manner, one can represent S as a Zariski open subset of $\mathbb{P}^{r+t}$. Specifically, fix disjoint linear spaces L_0, $L \subseteq \mathbb{P}^{r+t}$, of dimensions r and $t - 1$ respectively. Then $S = \mathbb{P}^{r+t} - L$, the bundle map $S \longrightarrow L_0 = \mathbb{P}^r$ being linear projection from L onto L_0. The natural inclusion $L_0 \subseteq S$ is identified with the zero section. Hence we can realize the bundle T on X as the fibre product $X \times_{\mathbb{P}^r} S$. The projection $\Phi : T \longrightarrow S$ is finite, and $X_0 = \Phi^{-1}(L_0)$:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\square"] & T \arrow[d, "\Phi"] & Y \arrow[l, "g"'] \arrow[dl, "h"] \\ L_0 \arrow[r, hook] & S \subseteq \mathbb{P}^{r+t} & \end{tikzcd}$$

Now let h, as indicated, be the composition $\Phi \circ g : Y \longrightarrow S$, which is finite. We apply the theorem (3.8) of Goresky-MacPherson to h, and to the linear space $L_0 \subseteq S \subseteq \mathbb{P}^{r+t}$. Denoting by L_ε an ε-neighborhood of L_0, we conclude that for sufficiently small ε,

$$\pi_i(Y, h^{-1}(L_\varepsilon)) = 0$$

for $i \leq \dim Y - \operatorname{codim} L_0 = n - t$. But since h is proper, and L_0 is closed, $h^{-1}(L_0)$ is a deformation retract of $h^{-1}(L_\varepsilon)$ when ε is small, and the lemma follows. $\square$

Proof of Theorem 3.5. The hypothesis on E implies that there is a surjective homomorphism $p : T \longrightarrow E$, where T is the direct sum of some number - say t - copies of A. Let $X_s \longrightarrow E$ be the image of the given section $s \in \Gamma(X,E)$, and set $Y = X_s \times_E T$. Denote by $X_0 \longrightarrow T$ and $X \longrightarrow E$ the zero sections:

$$\begin{tikzcd}
& Y \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & X_s \arrow[d, hook] \\
& T \arrow[r, "p"] & E \\
X_0 \arrow[ur, hook] \arrow[r, "\simeq", "p|X_0"'] & X \arrow[ur, hook, "\substack{\text{zero}\\ \text{section}}"'] &
\end{tikzcd}$$

Since p restricts to an isomorphism on zero-sections, we have

$$X_0 \cap Y \xrightarrow{\simeq} X \cap X_s = \left\{ \begin{matrix} \text{zero-locus} \\ Z \quad \text{of} \quad s \end{matrix} \right\}.$$

Bearing in mind that Y is a $\mathbb{C}^{t-e}$-bundle over X_s, on the level of homotopy groups one gets:

$$(*) \qquad \begin{tikzcd}
\pi_i(Y) \arrow[r, "\simeq"] & \pi_i(X_s) \\
\pi_i(X_0 \cap Y) \arrow[u] \arrow[r, "\simeq"] & \pi_i(Z) \arrow[u]
\end{tikzcd}$$

But Y is a connected, local complete intersection variety of pure dimension $n + t - e$, and the inclusion $Y \hookrightarrow T$ is a closed embedding, and in particular finite. Hence by Lemma 3.9, $\pi_i(Y, X_0 \cap Y) = 0$ for $i \leq (n+t-e) - t = n-e$, and the theorem follows from $(*)$. $\square$

We conclude this section with a problem on branched coverings of projective space. A well-known, and elementary, theorem states that if $X \subseteq \mathbb{P}^m$ is a smooth variety of degree 3 and dimension n, and if $n \geq 4$, then X is a hypersurface. This was generalized in [31] to branched coverings: if $f : X \longrightarrow \mathbb{P}^n$ has degree three, and if $n \geq 4$, then f factors through an embedding in the total space of a line bundle on $\mathbb{P}^n$:

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "f"'] & & L \arrow[dl] \\
& \mathbb{P}^n. &
\end{tikzcd}$$

One direction in which the classical results on subvarieties generalize is through the "Babylonian" theorems of Barth and Van de Ven [2], [5]. The problem, which was suggested by W. Fulton, is to generalize the result to branched coverings.

Specifically, suppose given for each $n \geq 1$ a branched covering:

$$f_n : X_n \longrightarrow \mathbb{P}^n .$$

Suppose also that $X_n = f_{n+1}^{-1}(\mathbb{P}^n)$ for a suitable hyperplane $\mathbb{P}^n \subseteq \mathbb{P}^{n+1}$. Then describe X_n explicitly. For instance, is X_n a complete intersection in the total space of a direct sum of line bundles on $\mathbb{P}^n$?

We note that the example at the end of [31] suggests that the possibilities for coverings are more varied than those for subvarities.

§4. A Problem of Remmert and Van de Ven.

One of the most elementary results in algebraic geometry is that any projective variety can be mapped onto some projective space. What's less clear, however, is whether projective space is the only smooth variety that plays this role. Our purpose in this section is to show how Mori's results in [35] lead to a proof that this property does in fact characterize projective space:

Theorem 4.1. Let X be a smooth projective variety of dimension ≥ 1, and let

$$f : \mathbb{P}^n \longrightarrow X$$

be a surjective map. Then $X \cong \mathbb{P}^n$.

This was conjectured by Remmert and Van de Ven (cf. [41]). Note that one cannot assert that f is an isomorphism, for there are non-trivial branched coverings $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ (obtained by projections of Veronese embeddings). Observe also that the non-singularity of X is crucial. In fact, if one drops this hypothesis then one can take $X = \mathbb{P}^n/G$, where G is a finite group acting on $\mathbb{P}^n$. We refer the reader to Demazure's paper [6] for a highly readable account of Mori's theorem.

The proof of (4.1) is an elementary application of results proved (but not stated) by Mori in the course of his spectacular proof of the Frankel-Hartshorne conjecture that projective space is the only projective manifold with ample tangent bundle. Specifically, we shall use two results:

(4.2) Let X be a smooth projective variety of dimension n such that the anti-canonical bundle $\Lambda^n(TX)$ is ample. Then for a generic point $P \in X$, there exists a map

$$u = (\mathbb{P}^1, a) \longrightarrow (X, P),^*$$

birational onto its image, with P a smooth point of $u(\mathbb{P}^1)$, and

$$\int_{\mathbb{P}^1} u^* c_1(X) \leq n + 1.$$

This is essentially proved in §2 of [35]. (cf. Thm. 6). Mori's statement does not mention the possibility of finding a rational curve through a general point, but it was observed by Kollar that this is in fact what a small elaboration of Mori's proof yields. Note that the result implies that if X is as in (4.2), then X is uniruled.

The second theorem we need is:

(4.3) Let X be a smooth projective variety of dimension n, and let

$$u : (\mathbb{P}^1, a) \longrightarrow (X, P)$$

be a map, birational onto its image, with $\int u^* c_1(X) \leq n+1$. Suppose that $P (= u(a))$ is a smooth point of $u(\mathbb{P}^1)$, and that the following is satisfied:

(*) For any morphism

$$v : (\mathbb{P}^1, a) \longrightarrow (X, P)$$

arising as a deformation of u through maps taking a to P, the pull-back $v^* TX$ of the tangent bundle of X is ample.

---

*i.e. u is a map $\mathbb{P}^1 \longrightarrow X$, and $a \in \mathbb{P}^1$ is a point with $u(a) = P$.

Then

$$X \cong \mathbb{P}^n.$$

The condition in (*) is that the maps u and v correspond to points in the same connected component of the scheme $\mathrm{Hom}((\mathbb{P}^1, a), (X,P))$ parametrizing maps $\mathbb{P}^1 \longrightarrow X$ taking a to P. (4.3) is the essence of [35], §3. If one knows that TX is ample, then (*) is automatic, and in fact this, plus the amplitude of $\Lambda^n TX$, is the only way in which Mori uses the ampleness of TX (cf. [35], p. 594).

Proof of Theorem 4.1. Note to begin with that X has dimension n, and that f is finite (hence flat). In fact, projective space does not map to any variety other than a point with any fibres of positive dimension. We observe next that $\Lambda^n TX$ is ample. To check this, it suffices by (1.5 (iv)) to show that $f^*\Lambda^n TX$ is ample. But $f^*\Lambda^n TX = \mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$, and the inclusion $\Lambda^n T\mathbb{P}^n \longrightarrow \Lambda^n f^* TX$ of sheaves shows that $k \geq n+1$. Thus Mori's theorem (4.2) applies.

Denote by $R \subseteq \mathbb{P}^n$ the ramification divisor of f, and by $B = f(R) \subseteq X$ the branch divisor. By (4.2) there exists a map $u = (\mathbb{P}^1, a) \longrightarrow (X,P)$ as in (4.3) with $P \notin B$. To prove the theorem, it then suffices to show:

(4.4) If $w : (\mathbb{P}^1, a) \longrightarrow (X,P)$ is any non-constant map, with $P \notin B$, then w^*TX is ample.

For once (4.4) is known, (4.3) applies to yeild $X \cong \mathbb{P}^n$.

To prove (4.4), choose a smooth irreducible projective curve C fitting into a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\overline{w}"] \arrow[d, "\overline{f}"'] & \mathbb{P}^n \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r, "w"'] & X \end{tikzcd} ;$$

where $\overline{w}$ and $\overline{f}$ are finite. For example, one may take C to be the normalization of an irreducible component of $\mathbb{P}^1 \times_X \mathbb{P}^n$. Observe that since $P = w(a) \notin B$, the image $\bar{w}(C)$ is not contained in R. Since $\bar{f}$ is flat, it suffices by (1.5 (iv)) to show that $\bar{f}^* w^* TX = \bar{w}^* f^* TX$ is an ample vector bundle on C.

But on $\mathbb{P}^n$ one has the exact sequence

$$(*) \qquad 0 \longrightarrow T\mathbb{P}^n \xrightarrow{df} f^*TX \longrightarrow \mathcal{R} \longrightarrow 0$$

of sheaves, where $\mathcal{R}$ is a torsion sheaf supported on the ramification divisor R. Then $\bar{w}^* df : w^*T\mathbb{P}^n \longrightarrow \bar{w}^* f^* TX$ is an isomorphism away from the finite set $\bar{w}^{-1}(R)$, so pulling $(*)$ back by $\bar{w}^*$ expresses $\bar{w}^* f^* TX$ as an extension of the ample vector bundle $\bar{w}^* T\mathbb{P}^n$ by the torsion sheaf $\bar{w}^*\mathcal{R}$. Hence (4.4) is a consequence of

<u>Lemma 4.5. Let C be a smooth irreducible projective curve, E an ample vector bundle on C, and F a vector bundle on C arising as an extension</u>

$$0 \longrightarrow E \longrightarrow F \longrightarrow \tau \longrightarrow 0,$$

<u>where τ is a torsion sheaf. Then F is ample.</u>

<u>Proof</u>. By Hartshorne's numerical criterion [25], it is equivalent to show that any quotient bundle of F has positive degree. Given such a quotient $F \longrightarrow Q \longrightarrow 0$, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & \tau & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & \tau' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Q' is the image of the composition $E \longrightarrow F \longrightarrow Q$. Thus Q' is locally free, and τ' is a torsion sheaf on C, and since E is ample, $\deg Q' > 0$. But

$$\deg Q = \deg Q' + \text{length } (\tau'),$$

so $\deg Q \geq \deg Q'$. □

This completes the proof of Theorem 4.1.

Problem. Does Theorem 4.1 generalize when $\mathbb{P}^n$ is replaced by a homogeneous space G/P, where G is a semi-simple algebraic group, and $P \subseteq G$ is a maximal parabolic subgroup? For instance, if Q is a quadric of dimension ≥ 3, or a Grassmannian, and if $f = Q \longrightarrow X$ is a non-trivial branched covering, with X smooth, is X a projective space?

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