UNIFORM APPROXIMATION OF ABHYANKAR VALUATION IDEALS IN SMOOTH FUNCTION FIELDS

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Abstract. Fix a rank one valuation ν centered at a smooth point x on an algebraic variety over a field of characteristic zero. Assume that ν is Abhyankar, that is, that its rational rank plus its transcendence degree equal the dimension of the variety. Let a_m denote the ideal of elements in the local ring of x whose valuations are at least m. Our main theorem is that there exists $k > 0$ such that a_{mn} is contained in $(a_{m-k})^n$ for all m and n. This can be viewed as a greatly strengthened form of Izumi's Theorem for Abhyankar valuations centered on smooth complex varieties. The proof uses the theory of asymptotic multiplier ideals.

Introduction. In this paper we use the theory of multiplier ideals to show that the valuation ideals of a rank one Abhyankar valuation centered at a smooth point of a complex algebraic variety are approximated, in a quite strong sense, by sequences of powers of fixed ideals.

Let R be an n-dimensional regular local domain essentially of finite type over a ground field k of characteristic zero, and let ν be a rank one valuation centered on R. Recall that this is equivalent to asking that ν be an $\mathbf{R}$-valued valuation on the fraction field K of R, taking nonnegative values on R and positive values on the maximal ideal $\mathfrak{m} \subseteq R$. A theorem of Zariski and Abhyankar (see Theorem 2.1) states that

$$(1) \qquad \text{trans.deg } \nu + \text{rat.rank } \nu \leq \dim K/k,$$

where the *rational rank* of ν is the rank of its value group, while its *transcendence degree* is the maximal dimension of the center ν on some model of K/k. One says that ν is an *Abhyankar valuation* if equality holds in (1), i.e. if

$$\text{trans.deg } \nu + \text{rat.rank } \nu = \dim K/k.$$

Among all the valuations centered on R, these are the ones from which one expects the best behavior. For example, a divisorial valuation—corresponding to the order of vanishing along a prime divisor $E \subseteq Y$ contracting to the closed

point of $X = \mathrm{Spec}\,(R)$ under a proper birational map $Y \longrightarrow X$—is an Abhyankar valuation centered on R having transcendence degree $n-1$ and rational rank 1. At the other extreme, if $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ are any $\mathbf{Q}$-linearly independent positive real numbers, then there exists a unique valuation ν centered on $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ with $\nu(x_i) = \alpha_i$: here trans.deg $\nu = 0$ and rat.rank $\nu = n$.

Abhyankar valuations have been the focus of considerable attention. For example, they are known to admit local uniformization in any characteristic [14], and already when $\dim R = 2$ they involve a great deal of beautiful and intricate geometry [23].

Given a valuation ν as above, denote by $\Phi = \nu(R) \subseteq \mathbf{R}$ the value semigroup of ν on R. For each real number $m \in \Phi$, let

$$\mathfrak{a}_m = \{f \in R \mid \nu(f) \geq m\}$$

denote the ideal of R consisting of all elements of R whose values are at least m. Clearly $\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}$ for every natural number $\ell \in \mathbf{N}$, but typically the inclusion is strict. However our main theorem shows that for Abhyankar valuations these two ideals lie surprisingly close to each other:

THEOREM A. *Let ν be an Abhyankar valuation centered on R. Then there exists a fixed value $k \in \Phi$ such that*

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{m\ell} \subseteq \mathfrak{a}_{m-k}^\ell$$

for all $m \in \Phi$ and all $\ell \in \mathbf{N}$. (We adopt the convention that $\mathfrak{a}_{m-k} = R$ when $m < k$.)

Roughly speaking, the theorem asserts that the valuation ideals $\mathfrak{a}_{m\ell}$ are closely and uniformly approximated by powers of $\mathfrak{a}_m$. It follows from the theorem that there is a fixed nonzero element $\delta \in R$ such that

$$(2) \qquad\qquad\qquad\qquad \delta^\ell \mathfrak{a}_{m\ell} \subseteq \mathfrak{a}_m^\ell$$

for all $m \in \Phi$ and $\ell \in \mathbf{N}$. This points to the heuristic idea that the associated "Rees ring" of the Abhyankar valuation ν, while usually not finitely generated when $n = \dim R > 1$, is "almost" finitely generated. It would be interesting to know if one can make this precise. Both the theorem and (2) can fail for non-Abhyankar valuations: see Remark 2.6.

Theorem A can be interpreted as a strengthened form of a celebrated theorem of Izumi. In the setting of Theorem A, Izumi's result (in a form due to Hübl and Swanson) asserts that there exists an index p such that $\mathfrak{a}_{p\ell} \subseteq \mathfrak{m}^\ell$ for all natural numbers ℓ, where as above $\mathfrak{m}$ is the maximal ideal of R. (Izumi's statement actually deals with less general valuations but more general rings. It is proved

in [12], [21], and [11].) Theorem A says much more: not only is $\mathfrak{a}_{p\ell} \subseteq \mathfrak{m}^\ell$ for all ℓ, but $\mathfrak{a}_{p\ell}$ is contained in the ℓth power of an ideal $\mathfrak{a}_{p-k}$ very close to $\mathfrak{a}_p$. In other words $\mathfrak{a}_{p\ell} \subseteq \mathfrak{a}_{p-k}^\ell$, where not only are the $\mathfrak{a}_{p-k}$'s getting deeper in R as a function of p (whereas $\mathfrak{m}$ stays fixed), but they are getting deeper at the same rate as the $\mathfrak{a}_p$ themselves. The theorem also implies—at least for regular local rings essentially of finite type over a field—the well-known statement of Izumi's Theorem comparing values of two valuations centered at $\mathfrak{m}$; see Corollary 2.5.

In an algebro-geometric context, Theorem A is most interesting for divisorial valuations. More generally, Theorem A can be applied to a composite of valuations to yield the following:

COROLLARY B. *Let D be an effective divisor on a normal variety X and suppose that* $X \xrightarrow{\pi} Y$ *is a proper birational map contracting D to a smooth (but not necessarily closed) point of Y. Then there exists a natural number k such that*

$$\pi_* \mathcal{O}_X(-m\ell D) \subset [\pi_* \mathcal{O}_X(-(m-k)D)]^\ell$$

for all natural numbers ℓ and all $m \geq k$.

Thus even though the algebra

$$\text{(3)} \qquad\qquad \bigoplus_{\ell \in \mathbf{N}} \pi_* \mathcal{O}_X(-m\ell D)$$

is not finitely generated in general, Corollary B gives some measure of control over it. Of course, it is a central problem of birational geometry to understand when such algebras are finitely generated: when (3) is finitely generated, the corresponding projective scheme is the stable image of X under the birational map to projective space over Y given by the linear series $|-mD|$ for $m \gg 0$.

Another viewpoint involves the concept of the *volume* of a rank one valuation centered on a local domain. By definition, the volume of ν on R is

$$\text{vol}_R(\nu) := \limsup_{m \to \infty} \frac{\text{length}(R/\mathfrak{a}_m)}{m^n/n!},$$

where n is the dimension of the local ring R. In case $\mathfrak{a}_m$ is the mth power of a fixed ideal $\mathfrak{a}$, then of course this is simply the multiplicity of $\mathfrak{a}$. But in general the volume is not actually a multiplicity: indeed, it can be an irrational number; see Example 3.5 (iii). However Theorem A implies that the volume of an Abhyankar valuation is approximated arbitrarily well by the multiplicities of the ideals $\mathfrak{a}_m$.

Specifically:

COROLLARY C. *If ν is an Abhyankar valuation as above, then*

$$vol_R(\nu) = \lim_{m \to \infty} \frac{e(\mathfrak{a}_m)}{m^n}.$$

This corollary in turn leads to an interesting upper bound in the spirit of Teissier on the multiplicity of an $\mathfrak{m}$-primary ideal $I \subseteq R$ in terms of the volumes of its Rees valuations $\nu_1, \ldots, \nu_r$. (Recall that the Rees valuations of I are defined by taking the normalized blowup X of I, and writing $I\mathcal{O}_X = \mathcal{O}_X(-e_1E_1 - \cdots - e_rE_r)$ for some prime Weil-divisors E_i of X and some positive integers e_i. Then ν_i is the valuation on the fraction field of R given by order of vanishing along E_i.) Specifically, we prove an inequality of Minkowski-type on the volumes of Abhyankar valuations which implies that the multiplicity of I satisfies

$$e(I)^{1/n} \leq e_1 vol_R(\nu_1)^{1/n} + \cdots + e_r vol_R(\nu_r)^{1/n},$$

where $e_i = \nu_i(I)$, and as above $n = \dim R$. Since an earlier version of this manuscript was written, Mircea Mustata has shown that Corollary C and the related Minkowski-type statements hold for arbitrary valuations (not just Abhyankar) and even much more generally. See [19].

The proof of Theorem A uses the theory of multiplier ideals. One can associate to the valuation ideals $\{\mathfrak{a}_m\}$ a sequence of *asymptotic multiplier ideals* $\{\mathfrak{j}_m\}_{m \in \Phi}$, as defined (in a slightly different setting) in [8]. The general theory—which we review in §1—shows that these satisfy

(4) $$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{m\ell} \subseteq \mathfrak{j}_m^\ell$$

for all $m \in \Phi$ and $\ell \in \mathbf{N}$. The main work of the present paper—which we carry out in §2—is to establish that $\mathfrak{a}_m$ and $\mathfrak{j}_m$ have "bounded difference" in the sense that there is a fixed $\mathfrak{m}$-primary ideal $\mathfrak{d} \subseteq R$ such that

$$\mathfrak{d} \subseteq (\mathfrak{a}_m : \mathfrak{j}_m)$$

for all sufficiently large $m \in \Phi$. Theorem A then follows from (4) upon taking $k = \nu(\mathfrak{d})$. In §3, we discuss the volume of a valuation: we believe that this is an invariant of independent interest. Finally we give in §4 some further applications, including the proof of Corollary B.

The present paper continues the project started in [8] of using ideas from higher dimensional complex geometry to search for possibly unexpected uniform behavior in Noetherian rings. Our techniques rest on resolution of singularities and vanishing theorems, and so are essentially limited to local rings coming from smooth complex varieties. Although one could expect the statements themselves

to remain valid in less restrictive settings, we haven't seriously investigated the extent to which such a generalization is possible. We hope however that the results appearing here will pique the interest of experts and encourage them to put the picture in a broader perspective. We note that the main theorem here was inspired by an attempt to "localize" a result of Fujita concerning the volumes of big line bundles: a proof via multiplier ideals appears in [6] (see also [16, 10.3B] and [4, 14.5]). The reader may consult [4], [22], [20], [13], [5], [16], [7] and the references therein for numerous other recent applications of multiplier ideals to the global study of linear series on a projective variety.

1. The asymptotic multiplier ideal of a graded family. In this section, we recall the notion of a graded family of ideals and examine the elementary properties of graded families of valuation ideals. We also review the construction and basic properties of the asymptotic multiplier ideals introduced in [8]. Because collections of valuation ideals are naturally indexed by semigroups slightly more general than the natural numbers, it is convenient to allow graded families that are indexed by additive subsemigroups of the real numbers. Thus the exposition here is slightly more general than what is stated in [8], but all the proofs are the same.

1.1. Graded families indexed by a semi-group. Let Φ be an additive sub-semigroup of the nonnegative real numbers. Of course, one of the main examples is the semigroup of natural numbers $\mathbf{N}$.

Definition 1.2. Fix a ring R. A *graded family* or *graded system of ideals indexed by* Φ is a collection $\mathfrak{a}_{\bullet} = \{\mathfrak{a}_m\}_{m \in \Phi}$ of ideals of R satisfying

$$(5) \qquad \mathfrak{a}_m \cdot \mathfrak{a}_\ell \subseteq \mathfrak{a}_{m+\ell} \quad \text{for all} \quad m, \ell \in \Phi.$$

To avoid trivialities, we assume also that $\mathfrak{a}_m \neq 0$ for $m \gg 0$.

It is convenient to assume also that $\mathfrak{a}_0 = R$, in which case condition (5) is equivalent to the statement that the R-module

$$\bigoplus_{m \in \Phi} \mathfrak{a}_m$$

has the natural structure of a Φ-graded R-algebra. We refer to this ring as the *Rees algebra of the graded system* $\mathfrak{a}_\bullet$. The theory of asymptotic multiplier ideals is particularly useful when the Rees algebra fails to be finitely generated (or at least is not known to be so). Of course, one can also define graded families, and develop the theory of asymptotic multiplier ideals, for families of coherent ideal sheaves in the structure sheaf of a scheme. But since all the definitions are local in nature this involves no essential differences from the affine setting.

Example 1.3. The following are familiar examples of graded families indexed by the natural numbers.

(i) The simplest example is the collection $\{\mathfrak{a}^m\}$ of powers of a fixed ideal $\mathfrak{a}$. This should be considered a trivial example of a graded family.

(ii) A slightly less trivial example is the collection $\{\overline{\mathfrak{a}^m}\}$ of integral closures of powers of a fixed ideal $\mathfrak{a}$. From the point of view of multiplier ideals, however, this example is no less trivial than the first, since the multiplier ideals are the same. See also Example 3.5 (i).

(iii) The graded family $\{\mathfrak{a}^{(m)}\}$ of symbolic powers of a fixed ideal $\mathfrak{a}$ was the main example treated in [8].

(iv) The collection $\{\mathfrak{b}_m\}$ of defining ideals for the base loci of the complete linear series $|mD|$, where D is a fixed big line bundle on a projective variety X, forms a graded family of ideals in the sheaf of rings $\mathcal{O}_X$. This graded family plays a central role in Kawamata's work on deformation of canonical singularities [13].

1.4. Graded families arising from a valuation. In this paper, we are primarily interested in graded families of valuation ideals for rank one valuations centered on a regular local domain. Let us recall some of the basic terminology and give a few examples. Good general references on valuation theory are [25] and [26].

Let Γ be an ordered Abelian group, written additively. A valuation on a field K with values in Γ is a map of Abelian groups

$$\nu \colon K^* = K \backslash 0 \longrightarrow \Gamma$$

satisfying the additional condition that

$$(6) \qquad \nu(f + g) \geq \min\{\nu(f), \nu(g)\}.$$

Since only the image of ν is of importance, we assume that ν is surjective. An elementary but useful observation is that in fact,

$$(7) \qquad \nu(f + g) = \min\{\nu(f), \nu(g)\} \text{ whenever } \nu(f) \neq \nu(g).$$

The set of all elements of K which have nonnegative values forms of subring R_ν, called the *valuation ring* of ν. The valuation ring of ν is a local ring with maximal ideal consisting of all the positively valued elements of K, but it is not Noetherian in general. One can also define a valuation ring abstractly as a subring of K containing either x or $\frac{1}{x}$ for every $x \in K$. The data of a valuation ring inside a field K is equivalent to the data of a valuation on K (up to order preserving isomorphism of the value group). The *rank* of a valuation is by definition the Krull dimension of the valuation ring R_ν.

In this paper, we consider only *rank one* valuations on function fields. In this case, the ordered group Γ can and will be identified with an ordered subgroup of the real numbers, the field K is assumed to be a finitely generated field extension of some fixed ground field k, and we consider only those valuations vanishing on k. In particular, the valuation ring R_ν is a k-algebra, though not usually finitely generated.

Let ν be a valuation on a function field K/k. The valuative criterion for properness (see [9, p. 101]) ensures that for any complete algebraic variety X over k with function field K (that is, for any *complete model of X*), there is a unique map $\operatorname{Spec} R_\nu \longrightarrow X$. Thus a valuation chooses, in a consistent way, a (not necessarily closed) point on every complete model of K, namely, the image W of the closed point of $\operatorname{Spec} R_\nu$ under this map. The variety W is called the *center of ν on X*—here and elsewhere, we abuse terminology by failing to distinguish between an irreducible variety and its generic point—and the local ring $\mathcal{O}_{W,X}$ is called the *local ring of ν* on X. The defining ideal $\mathcal{I}_W$ of W consists of all local sections of $\mathcal{O}_X$ of positive value.

For a local domain R contained in K, we say that the valuation ν is *centered* on R if ν takes nonnegative values on R and strictly positive values on the maximal ideal of R. Thus the valuation is centered on any of its local rings.

Let ν be a valuation on a function field K/k and let X be any irreducible k-scheme with function field K. For each nonnegative $m \in \mathbf{R}$, the subset

$$\mathfrak{a}_m = \{f \in \mathcal{O}_X \mid \nu(f) \geq m\}$$

forms an ideal sheaf in $\mathcal{O}_X$, called a *ν-valuation ideal* (or just a *valuation ideal* when ν is understood). When it is necessary to emphasize the model, we will write $\mathfrak{a}_m(X)$. Note that if $\pi \colon X \longrightarrow Y$ is a proper birational map between complete models of K, then $\pi_* \mathfrak{a}_m(X) = \mathfrak{a}_m(Y)$.

For any subsemigroup Ψ of $\mathbf{R}$ (eg, $\Psi = \mathbf{N}$ or $\Psi = \mathbf{R}_{\geq 0}$), the collection $\{\mathfrak{a}_m\}_{m \in \Psi}$ forms a graded family of ideals in $\mathcal{O}_X$, called the *graded family of ν on X*. It is natural to index this graded family by Γ, but allowing the indexing set to be $\mathbf{N}$ or $\mathbf{R}$ will sometimes by more convenient. The *value semigroup* $\Phi = \nu(\mathcal{O}_X)$ is the "optimal" indexing set: every ν-valuation ideal appears exactly once as a member this graded family. When X is Noetherian, the value semigroup Φ is well-ordered, meaning that every subset has a minimal element.

PROPOSITION 1.5. *If ν is a rank one valuation on a function field K/k and X is any complete model of K, then the valuation ideals $\mathfrak{a}_m(X)$ are primary to the ideal defining the center of ν on X. If ν is a rank one valuation centered on local domain R, then each of the valuation ideals $\mathfrak{a}_m$ is primary to the maximal ideal of R.*

Proof. Suppose that f and g are local sections of $\mathcal{O}_X$ with $f \notin \mathfrak{a}_m$ but $fg \in \mathfrak{a}_m$. Then $\nu(fg) = \nu(f) + \nu(g) \geq m$, but $\nu(f)$ is strictly less than m. This means that $\nu(g)$ is positive, so some positive multiple, say $k\nu(g)$, exceeds m. Thus g^k is in $\mathfrak{a}_m$, proving that $\mathfrak{a}_m$ is primary. Furthermore, since any element of $\mathcal{I}_W$ has a power in $\mathfrak{a}_m$, the radical of $\mathfrak{a}_m$ is $\mathcal{I}_W$. The argument for the second statement in the proposition is the same.   □

Example 1.6. We give four simple examples of graded families of valuation ideals in the local ring $R = k[x,y]_{(x,y)}$ of the origin in the affine plane. Each arises from a different rank one valuation on the function field $K = k(x,y)$.

(i) Let ν be the valuation given by "order of vanishing at the origin." Explicitly, for a polynomial f, $\nu(f)$ is the degree of the smallest degree nonzero monomial appearing in the unique expression of f as a sum of monomials $x^a y^b$. The value of any rational function $\frac{f}{g}$ where f and g are polynomials is uniquely determined by virtue of ν being a group homomorphism: $\nu(\frac{f}{g}) = \nu(f) - \nu(g)$.

This valuation has value group $\mathbf{Z}$ and value semigroup $\mathbf{N}$ on R. The valuation ideals of ν are powers of the defining ideal of the origin:

$$\mathfrak{a}_m = (x, y)^m.$$

(ii) Let $\pi\colon X \longrightarrow \operatorname{Spec} R$ be any proper birational map from a normal scheme X and let D be any prime divisor of X collapsed to the origin. Let ν_D be the valuation given by "order of vanishing along D." Explicitly, for any $f \in R$, consider the image of $\pi^* f$ in the local ring $\mathcal{O}_{D,X}$ of X at the generic point of D. Then $\nu_D(f)$ is the maximal integer n such that $\pi^*(f)$ is divisible by t^n in $\mathcal{O}_{D,X}$, where t is a uniformizing parameter for the discrete valuation ring $\mathcal{O}_{D,X}$.

This valuation has value group $\mathbf{Z}$, and value semigroup on R a subsemigroup of $\mathbf{N}$. The valuation ideals are

$$\mathfrak{a}_m = \pi_* \mathcal{O}_X(-mD),$$

which in general, can be tricky to understand. In the simple case where π is the blowup of the origin and D is the resulting exceptional divisor, this example recovers Example (i) above.

(iii) Let ν be the valuation on $k(x,y)$ defined by the assignment $\nu(x) = 1$ and $\nu(y) = \pi \in \mathbf{R}$. This uniquely determines a valuation on $k(x,y)$: the value of each monomial $x^a y^b$ is $a + b\pi$, and because 1 and π are $\mathbf{Z}$-independent, distinct monomials have distinct values, so the value of an arbitrary polynomial is determined by (7).

This valuation has value group $\mathbf{Z} + \mathbf{Z}\pi \subset \mathbf{R}$ and value semigroup $\mathbf{N} + \mathbf{N}\pi$ on R. The valuation ideals for ν_π are all monomial ideals

$$\mathfrak{a}_m = (\{x^a y^b \mid a + \pi b \geq m\}).$$

(iv) Let ν be the valuation given by "order of vanishing along the analytic arc $y = e^x - 1$." Explicitly, for a polynomial f, we define $\nu(f)$ to be the highest power of t dividing the image of f under the map

$$k[x, y] \hookrightarrow k[[t]]$$

$$f(x, y) \mapsto f(t, e^t - 1).$$

Here $e^t - 1$ denotes the power series $t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$ (assuming that k has characteristic zero).

This valuation has value group $\mathbf{Z}$ and value semigroup $\mathbf{N}$ on R. Its valuation ideals are given by

$$\mathfrak{a}_m = \left(x^m, y - x - \frac{x^2}{2!} - \cdots - \frac{x^{m-1}}{(m-1)!} \right).$$

The preceding examples are simple examples of rank one valuations centered on the origin of the plane. The first three are *Abhyankar valuations* (see 2.1); all four have finitely generated value groups. By contrast, there are valuations with value group $\mathbf{Q}$ centered on the origin of the plane (see Example 3.15); while more complicated to describe, these are actually "typical" in a certain sense. The classification of valuations centered on the origin of the plane is a beautiful story; see [23]. For further examples of valuations, including higher rank valuations, consult [25] §10, or [26] VI §15.

1.7. Multiplier ideals. We now recall the construction and basic properties of multiplier ideals as well as the asymptotic constructions from [8]. We will give only a few proofs; the rest can be found for instance in [8] or [16].

Consider a scheme X, smooth and essentially of finite type over a field of characteristic zero—mainly we have in mind the case where X is a smooth complex variety or the spectrum of a regular local k-algebra. Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a (coherent) ideal sheaf on X. Given a rational number $c > 0$ we define *multiplier ideals*

$$\mathcal{J}(X; c \cdot \mathfrak{a}) = \mathcal{J}(X; \mathfrak{a}^c) \subseteq \mathcal{O}_X.$$

Intuitively these are ideals with remarkable cohomological properties which reflect in a somewhat subtle manner the singularities of the divisors of functions $f \in \mathfrak{a}$. The construction starts by taking *log resolution* $\mu \colon X' \longrightarrow X$ of $\mathfrak{a}$. Recall

that this means that μ is a projective birational map from a regular scheme X' to X such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$, where F is an effective Cartier divisor on X' with the property that the sum of F and the exceptional divisor of μ has simple normal crossing support. Such resolutions can be constructed (as we are in characteristic zero) by resolving the singularities of the blow-up of $\mathfrak{a}$. We write $K_{X'/X} = K_{X'} - \mu^* K_X$ for the relative canonical divisor of X' over X. Given $\mathfrak{a}$ and $c > 0$ as above, we now define

$$(8) \qquad \mathcal{J}(X; c \cdot \mathfrak{a}) = \mathcal{J}(c \cdot \mathfrak{a}) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cF]).$$

Here cF is viewed as an effective **Q**-divisor on X', and its integer part $[cF]$ is defined by replacing the coefficient of each component by the greatest integer less than or equal to it. This definition is independent of the log resolution μ; see *e.g.* [16]. When X is the affine scheme Spec R, we write also $\mathcal{J}(R, c \cdot \mathfrak{a})$, and when $c = 1$ we write simply $\mathcal{J}(\mathfrak{a})$.

Remark 1.8. Following Lipman [17] it is also possible to define $\mathcal{J}(X; c \cdot \mathfrak{a})$ without referring to a log resolution. For example, when R is a regular local ring,

$$(9) \qquad \mathcal{J}(R; c \cdot \mathfrak{a}) = \bigcap_{\nu} \{ r \in K \mid \nu(r) \geq c\nu(\mathfrak{a}) - \nu(Jac_{R_\nu/R}) \},$$

where the intersection is taken over all valuations of K given by order of vanishing along some prime divisor on a model X' of K, and where $Jac_{R_\nu/R}$ denotes the Jacobian ideal of the extension $R \hookrightarrow R_\nu$. This is a slight modification of Lipman's definition of an adjoint ideal, in which we have allowed for the possibility of a coefficient c; see [17]. This approach makes sense for arbitrary regular Noetherian schemes (not just for classes of schemes that admit a good theory of resolution of singularities as we have assumed here.) However multiplier ideals derive their power from the properties they satisfy, and the important facts—which rest on vanishing theorems—are so far only known over fields of characteristic zero.

Remark 1.9. Multiplier ideals were originally defined analytically, as ideals of germs of holomorphic functions that are L^2-integrable for a certain weighted L^2 space. See e.g. [4]. In this approach they appear as sheaves of multipliers, whence the name.

Among the various properties these ideals satisfy, we will need three in particular. First:

$$(10) \qquad\qquad \mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a})$$

for any ideal $\mathfrak{a}$. This is elementary: it boils down to the fact that the relative canonical bundle $K_{X'/X}$ of a log resolution is effective. More substantially for

any rational $c > 0$ and any $\ell \in \mathbf{N}$ one has the *subadditivity relation*:

$$(11) \qquad \mathcal{J}(\ell c \cdot \mathfrak{a}) \subseteq \mathcal{J}(c \cdot \mathfrak{a})^{\ell}.$$

This is established in [6] using vanishing theorems. The third useful property, which follows easily from the definitions, is the behavior of these ideals under birational maps. Specifically, if $\mu \colon X' \longrightarrow X$ is a proper birational map, with X' regular, and if $\mathfrak{a} \subseteq \mathcal{O}_X$ is any ideal, then

$$(12) \qquad \mathcal{J}(X; \mathfrak{a}) = \mu_*(\mathcal{J}(X'; \mathfrak{a}') \otimes \mathcal{O}_{X'}(K_{X'/X})),$$

$\mathfrak{a}' = \mathfrak{a} \cdot \mathcal{O}_{X'}$ being the pullback of $\mathfrak{a}$ to X'. The reader may consult [16, Part III] for proofs of these and other basic properties of multiplier ideals.

Given now a graded family $\mathfrak{a}_\bullet$ as above and an index $m \in \Phi$, we will construct an *asymptotic multiplier ideal* $\mathfrak{j}_m = \mathfrak{j}_m(\mathfrak{a}_\bullet)$ which reflects the asymptotic properties of all the ideals $\mathfrak{a}_{pm}$ for $p \in \mathbf{N}$. From (5) and (8), it is easy to check that for each $m \in \Phi$, we have

$$(13) \qquad \mathcal{J}(\mathfrak{a}_m) \subseteq \mathcal{J}(\tfrac{1}{p} \cdot \mathfrak{a}_{pm})$$

for all $p \in \mathbf{N}$; see [8, §1]. This, together with the Noetherian property for $\mathcal{O}_X$, implies that the set of ideals

$$\{\mathcal{J}(\tfrac{1}{p} \cdot \mathfrak{a}_{pm})\}_{p \in \mathbf{N}}$$

has a unique maximal element. We then define the *m*th asymptotic multiplier ideal $\mathfrak{j}_m(\mathfrak{a}_\bullet)$ to be this maximal element. In other words,

$$(14) \qquad \mathfrak{j}_m(\mathfrak{a}_\bullet) = \mathcal{J}(\tfrac{1}{p} \cdot \mathfrak{a}_{pm}) \text{ for sufficiently divisible } p \in \mathbf{N}.$$

In fact, it is not necessary to assume that p is sufficiently divisible: assuming that $\mathfrak{a}_m \neq (0)$ for $m \gg 0$, any very large p will do; see [8], Remark following Definition 1.4.

The essential property of these ideals is summarized in the next result, which was established in [8].

THEOREM 1.10. *For any graded system $\mathfrak{a}_\bullet$, any index $m \in \Phi$, and any natural number $\ell \in \mathbf{N}$ one has inclusions:*

$$(15) \qquad \mathfrak{a}_m^{\ell} \subseteq \mathfrak{a}_{m\ell} \subseteq \mathfrak{j}_m^{\ell}.$$

Sketch of proof. The first inclusion is definitional. For the second, note first that $\mathfrak{a}_{m\ell} \subseteq \mathfrak{j}_{m\ell}$: this is easily checked using (10) and (13). So it is enough to show

that $\mathfrak{j}_{m\ell} \subseteq \mathfrak{j}_m^\ell$. For this, choose any $p \gg 0$. Then using the subadditivity relation (11) one finds:

$$\begin{aligned}
\mathfrak{j}_{m\ell} &= \mathcal{J}(\tfrac{1}{p} \cdot \mathfrak{a}_{m\ell p}) \\
&= \mathcal{J}(\tfrac{\ell}{\ell p} \cdot \mathfrak{a}_{m\ell p}) \\
&\subseteq \mathcal{J}(\tfrac{1}{\ell p} \cdot \mathfrak{a}_{m\ell p})^\ell \\
&= \mathfrak{j}_m^\ell,
\end{aligned}$$

as required. $\square$

Remark 1.11. The second inclusion in (15) does not hold in general if one works with the "absolute" multiplier ideal $\mathcal{J}(\mathfrak{a}_m)$ in place of $\mathfrak{j}_m$.

2. Abhyankar valuations. Let ν be a rank one valuation on a function field K/k. There are two basic invariants of ν. The *rational rank* of ν is the dimension of the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$. The *transcendence degree* of ν is the transcendence degree of the residue field of the valuation ring R_ν over k. Equivalently, the transcendence degree is the maximal dimension of the center of ν over all models of K/k. The basic result relating these invariants is the Zariski-Abhyankar inequality [26] [1].

THEOREM 2.1. (The Zariski-Abhyankar Inequality) *For any valuation on a function field K/k*

(16) $$\text{trans.deg } \nu + \text{rat.rank } \nu \le \dim K/k.$$

Furthermore, if equality holds in (16), then the value group Γ is a finitely generated (free) Abelian group. Here, $\dim K/k$ refers to the transcendence degree of K over k, or equivalently, to the dimension of any complete model for K/k.

A valuation satisfying equality in (16) is called an *Abhyankar valuation*. Abhyankar valuations generalize the familiar example of divisorial valuations, that is, valuations given by order of vanishing along some divisor on a normal model of K/k. Note that divisorial valuations have rational rank one and transcendence degree $n-1$, where n is the dimension $\dim K/k$.

Example 2.2. For the valuations in Example 1.6, the rational rank and transcendence degree are easily computed: for (i) and (ii), the rational rank is 1 and the transcendence degree is 1; for (iii), the rational rank is 2 and the transcendence degree is zero; and for (iv), the rational rank is 1 and the transcendence degree is zero.

We now state the main technical result of the present paper:

THEOREM 2.3. *Let ν be an Abhyankar valuation on a function field K/k of characteristic zero. Let R be the local ring of the center of ν on some smooth model of K, and let $\{\mathfrak{a}_m\}_{m\in\Phi}$ be the graded family of ν-valuation ideals in R. Then there exists a nonzero element $\delta \in R$ such that*

$$\delta \cdot \mathfrak{j}_m \subseteq \mathfrak{a}_m$$

for all $m \in \Phi$, where $\{\mathfrak{j}_m\}_{m\in\Phi}$ are the asymptotic multiplier ideals associated to $\mathfrak{a}_\bullet$.

Remark 2.4. For any ν-valuation ideal $\mathfrak{a}$ in domain R and for an arbitrary ideal $\mathfrak{b}$, the colon ideal $(\mathfrak{a} : \mathfrak{b})$ is also a ν-valuation ideal of R; see [26, p. 342]. Thus (taking Φ to be the full value semigroup $\nu(R)$), there is a function $\alpha\colon \Phi \longrightarrow \Phi$ such that

$$(\mathfrak{a}_m : \mathfrak{j}_m) = \mathfrak{a}_{\alpha(m)}.$$

Theorem 2.3 says that for Abhyankar valuations centered on a regular local domain essentially of finite type over a field of characteristic zero, this function is bounded above. In other words, the graded family of ($\mathfrak{m}$-primary) valuation ideals $(\mathfrak{a}_m : \mathfrak{j}_m)$ has a minimal element.

Proof of Theorem A. In view of the preceding Remark, Theorem A from the Introduction follows immediately upon combining Theorems 1.10 and 2.3. Indeed, one simply takes $k = \nu(\delta)$, where δ is the nonzero element of R whose existence is guaranteed by Theorem 2.3. Since $\delta\mathfrak{j}_m \subset \mathfrak{a}_m$ for all m, clearly $\mathfrak{j}_m \subset \mathfrak{a}_{m-k}$ for all m, and so $\mathfrak{a}_m^\ell \subset \mathfrak{j}_m^\ell \subset \mathfrak{a}_{m-k}^\ell$. □

Theorem 2.3 also implies that Izumi's theorem holds for nondivisorial Abhyankar valuations in our setting:

COROLLARY 2.5. (Izumi's Theorem for Abhyankar valuations) *Let ν and w be rank one Abhyankar valuations on a function field K/k of characteristic zero and let $(R, \mathfrak{m})$ be any regular local ring essentially of finite type over k on which both ν and w are centered. Then there exists $C > 0$ such that*

$$(17) \qquad\qquad\qquad \nu(x) \geq Cw(x)$$

for all nonzero elements $x \in R$.

Proof. Without loss of generality, we may assume that the minimal value obtained by w on R is 1. Indeed, because R is Noetherian, w must achieve some minimal value on $R\backslash 0$; now we can simply scale the values of w and ν by this minimal value.

Now we claim that there exists a value p such that

$$(18) \qquad\qquad\qquad \mathfrak{a}_{mp} \subset \mathfrak{b}_m$$

for all $m \in \mathbf{N}$, where $\{\mathfrak{a}_m\}$ (respectively $\{\mathfrak{b}_m\}$) denote valuation ideals of ν (respectively, w). Indeed, note that $\mathfrak{b}_1 = \mathfrak{m}$, and so $\mathfrak{m}^\ell \subset \mathfrak{b}_\ell$ for all $\ell \in \mathbf{N}$. Thus to prove (18), it is enough to show that there exists p such that

$$(19) \qquad\qquad\qquad \mathfrak{a}_{p\ell} \subset \mathfrak{m}^\ell$$

for all ℓ. (In other words, we are reduced to the case where w is the $\mathfrak{m}$-adic valuation on R.) By Theorem 1.10, we see that (19) follows immediately provided that some $\mathfrak{j}_p \subset \mathfrak{m}$. But if all $\mathfrak{j}_p$ are trivial, we have $\delta \in \cap \mathfrak{a}_m = (0)$, contradicting Theorem 2.3.

Finally, (17) follows as in [11, Lemma 1.4] by setting $C = 2p - 1$ (and enlarging p if necessary so that $p \geq 2$). Indeed, suppose on the contrary that there is some $x \in R$ such that $\nu(x) > Cw(x)$. Set $m = w(x) + 1$. Then $x \in \mathfrak{a}_{mp}$, but $x \notin \mathfrak{b}_m$. $\qquad\square$

Remark 2.6. Theorems A and 2.3—and also the Izumi-type statement of Corollary 2.5—can fail for non-Abhyankar valuations. In fact, consider the valuation ν given by order of vanishing along the exponential curve $y = e^x - 1$ (Example 1.6(iv)). Here $\mathfrak{a}_p \subseteq R = \mathbf{C}[x,y]_{(x,y)}$ has colength p. So there cannot exist a nontrivial ideal $\mathfrak{j} \subseteq R$ having the property that $\mathfrak{a}_{m\ell} \subseteq \mathfrak{j}^\ell$ for fixed m and $\ell \gg 0$, since the colength of $\mathfrak{j}^\ell$ would grow quadratically in ℓ. Therefore the inclusion (4) in Theorem A can only hold with $\mathfrak{j}_m = R$ for all m. On the other hand, $\cap_m \mathfrak{a}_m = (0)$, so there cannot exist a fixed nonzero element $\delta \in R$ with $\delta \cdot \mathfrak{j}_m \subseteq \mathfrak{a}_m$ for all m. However for an arbitrary valuation ν, it is possible that the colon ideals $\mathfrak{d}_m = (\mathfrak{a}_m : \mathfrak{j}_m)$ "grow slowly": see Remark 3.14 for a precise statement.

We now prove Theorem 2.3. The outline is this: Lemma 2.7 below guarantees that it is enough to find δ after blowing up, but after a suitable blow up, any Abhyankar valuation is "essentially monomial" by Proposition 2.8, where a direct computation can be carried out.

LEMMA 2.7. *Let $\pi\colon X \longrightarrow Y$ be a proper birational map of smooth varieties over k. Assume that there exists an ideal $\mathfrak{d}' \subseteq \mathcal{O}_X$ such that $\mathfrak{d}' \cdot \mathfrak{j}_m(X) \subset \mathfrak{a}_m(X)$. Then there exists an ideal $\mathfrak{d} \subseteq \mathcal{O}_Y$ such that $\mathfrak{d} \cdot \mathfrak{j}_m(Y) \subseteq \mathfrak{a}_m(Y)$.*

Proof. Let $\mathfrak{d} = \pi_*(\mathfrak{d}'\omega_{X/Y}^{-1})$. Note that both $\mathfrak{d}'$ and $\omega_{X/Y}^{-1}$ are ideals of $\mathcal{O}_X$, so $\mathfrak{d}$ is an ideal of $\mathcal{O}_Y$. Now

$$\mathfrak{d} \cdot \mathfrak{j}_m(Y) = \pi_*(\mathfrak{d}'\omega_{X/Y}^{-1}) \cdot \pi_* \left(\mathcal{J}\left(X, \frac{1}{p} \cdot \mathfrak{a}_{mp}(Y) \cdot \mathcal{O}_X\right) \otimes \omega_{X/Y}\right)$$

where p is sufficiently large. Here we have used the definition of the asymptotic multiplier ideal together with the transformation rule (12) for multiplier ideals under proper birational morphisms. Therefore

$$\mathfrak{d} \cdot \mathfrak{j}_m(Y) \subseteq \pi_* \left(\mathfrak{d}' \omega_{X/Y}^{-1} \cdot \mathcal{J} \left(X, \frac{1}{p} \cdot \mathfrak{a}_{mp}(Y) \mathcal{O}_X \right) \otimes \omega_{X/Y} \right)$$

$$= \pi_* \left(\mathfrak{d}' \cdot \mathcal{J} \left(X, \frac{1}{p} \cdot \mathfrak{a}_{mp}(Y) \cdot \mathcal{O}_X \right) \right).$$

Because $\mathfrak{a}_{mp}(Y)\mathcal{O}_X \subset \mathfrak{a}_{mp}(X)$ for all m and p, the corresponding inclusion holds for the multiplier ideals. Note that the multiplier ideal appearing on the right is that associated to the pull-back of a valuation ideal on Y rather than the corresponding valuation ideal on X. However $\mathfrak{a}_{mp}(Y) \cdot \mathcal{O}_X \subseteq \mathfrak{a}_{mp}(X)$ and consequently for $p \gg 0$:

$$\mathfrak{d}' \cdot \mathcal{J} \left(X, \frac{1}{p} \mathfrak{a}_{mp}(Y) \cdot \mathcal{O}_X \right) \subseteq \mathfrak{d}' \cdot \mathcal{J} \left(X, \frac{1}{p} \mathfrak{a}_{mp}(X) \right) = \mathfrak{d}' \cdot \mathfrak{j}_m(X).$$

But $\mathfrak{d}' \cdot \mathfrak{j}_m(X) \subseteq \mathfrak{a}_m(X)$, and putting these inclusions together we find that

$$\mathfrak{d} \cdot \mathfrak{j}_m(Y) \subseteq \pi_*(\mathfrak{d}' \cdot \mathfrak{j}_m(X)) \subseteq \pi_*(\mathfrak{a}_m(X)) = \mathfrak{a}_m(Y)$$

for all m, as required.   □

The next Proposition is probably well known, at least for valuations of transcendence degree zero. We learned that case from Dale Cutkosky.

PROPOSITION 2.8. *Let ν be a rank one Abhyankar valuation on a function field K/k of characteristic zero. Given any model Y of K/k, there exists a smooth model X dominating Y and a regular system of parameters $x_1, \ldots, x_r$ for the local ring of ν on X such that $\nu(x_1), \ldots, \nu(x_r)$ freely generate the value group Γ.*

Proof. Let r denote the rational rank of ν, so by Theorem 2.1, $\Gamma \cong \mathbf{Z}^r$. Fix $f_1, \ldots, f_r$ in the field K, whose values generate Γ. By replacing f_i by $\frac{1}{f_i}$ if necessary, we can assume all $\nu(f_i) > 0$.

We can write each f_i as a fraction $\frac{a_i}{b_i}$, where the a_i and b_i are regular on some neighborhood of the center of ν on Y. By blowing up the ideals (a_i, b_i), we can make the fractions $\frac{a_i}{b_i}$ regular on some neighborhood of the center. By blowing up further if necessary, we can assume that the dimension of the center is the transcendence degree of ν—which means its codimension equals the rational rank in the presence of the Abhyankar hypothesis. So we have created a model Y' dominating Y where the elements f_i are regular on a neighborhood of the center of ν, and where the codimension of the center is exactly r, the rational rank of Γ.

Now we use embedded resolution of singularities to resolve the hypersurface defined by the product $f_1 f_2 \cdots f_r$ in a neighborhood of the center on Y'. This produces for us a smooth model X dominating Y' such that the pullback of the

hypersurface to this model has simple normal crossing support. In particular, for any closed point x of X, we have

$$f_1 f_2 \cdots f_r = u x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N},$$

where $x_1, \ldots, x_N$ is a regular system of parameters at x, the exponents a_i are natural numbers, and u is a regular function invertible in a neighborhood of x. Because the local rings of X are unique factorizations domains, for each f_i we have

$$f_i = u_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_N^{a_{iN}}$$

for some $a_{ij} \in \mathbf{N}$ and some unit u_i.

In particular, choosing the point x to be in the center W of ν on X, then the elements u_i are also units in the local ring $\mathcal{O}_{W,X}$. Because units in $\mathcal{O}_{W,X}$ have value zero, we see that

$$\nu(f_i) = \sum_{j=1}^{N} a_{ij} \nu(x_j).$$

So clearly, the elements $\nu(x_j)$ generate Γ.

We claim that exactly r of the elements x_j have nonzero value. Indeed, if fewer have nonzero value, then the rank of Γ can not be r. But if more have nonzero value, then there are at least $r+1$ of the parameters $x_1, \ldots, x_{r+1}$ contained in the defining ideal of the center W. This would force W to have codimension greater than r, a contradiction.

Relabeling so that the parameters $x_1, \ldots, x_r$ are those with positive value, note finally that the images of these elements generate the maximal ideal in the local ring of X along W. Indeed, this maximal ideal is generated by the image of the defining ideal $\mathcal{I}_W$ of W, and we have already remarked that $(x_1, \ldots, x_r) \subset \mathcal{I}_W$. But since $x_1, \ldots, x_r$ are part of a regular sequence of parameters in a neighborhood of W, they must generate the maximal ideal after localizing at $\mathcal{I}_W$. Thus the proposition is proved: the elements $x_1, \ldots, x_r$ of K are a regular system of parameters for the local ring $\mathcal{O}_{W,X}$ and the values $\nu(x_1), \ldots, \nu(x_r)$ generate Γ. $\square$

We can now finish the proof of the main theorem.

Proof of Theorem 2.3. By Lemma 2.7 and Proposition 2.8, we can assume that we are in the following situation. The variety X is smooth, and the center W of ν on X is of codimension r equal to the rational rank of ν; furthermore, the local ring R of X along W has a regular system of parameters $x_1, \ldots, x_r$ whose values generate the value group Γ. We wish to prove that there exists $\delta \subset \mathcal{O}_X$ such that $\delta \mathfrak{j}_m \subset \mathfrak{a}_m$. Because $\mathfrak{a}_m$ is primary to $\mathcal{I}_W$ (see Proposition 1.5), it is enough to check this after localizing along the defining ideal of W, so we consider the graded family of valuation ideals $\{\mathfrak{a}_\bullet\}$ in the local ring $(R, \mathfrak{m})$.

Because the values of the parameters $x_1, \ldots, x_r$ are all $\mathbf{Z}$-independent, the ideals $\mathfrak{a}_m$ are generated by "monomials" in $x_1, \ldots, x_r$. Indeed, fix any $m \in \Phi$. By (1.5), some power of the maximal ideal of R, say $\mathfrak{m}^t$, is contained in $\mathfrak{a}_m$. Now consider an arbitrary element f of R (not already in $\mathfrak{m}^t$). Modulo $\mathfrak{m}^t$, f can be written as a sum of monomials in the regular system of parameters $x_1, \ldots, x_r$ with unit coefficients. Because the values of the x_i are independent, each of these monomials has a distinct value, and so the value of f is equal to the value of the unique smallest value monomial in this sum (Cf (7)). So each of the monomial "terms" of f are in $\mathfrak{a}_m$, and $\mathfrak{a}_m$ is generated by monomials in the regular system of parameters for R.

We now claim that this setup is sufficiently close to the standard monomial case so as to be able to apply the computation derived in [10] for the multiplier ideal of a monomial ideal in a polynomial ring. Roughly this reason is this: the ring R is étale over a polynomial ring and the computation of multiplier ideals commutes with étale extension. We justify this carefully in the next paragraph.

Think of the parameters $x_1, \ldots, x_r$ as local sections of $\mathcal{O}_X$ and extend them to a full set of regular parameters in some affine neighborhood U of W on X. The inclusion

$$(20) \qquad k[x_1, \ldots, x_r, x_{r+1}, \ldots, x_n] \hookrightarrow \mathcal{O}_X(U)$$

induces a natural map

$$(21) \qquad U \longrightarrow \mathbf{A}^n$$

consisting of an open immersion followed by a finite map. Localizing at the prime ideal of W and its corresponding contraction to the polynomial ring, we have an inclusion

$$(22) \qquad A = k[x_1, \ldots, x_r, x_{r+1}, \ldots, x_n]_{(x_1, \ldots, x_r)} \hookrightarrow \mathcal{O}_{W,X} = R.$$

Our claim above that $\mathfrak{a}_m$ is a monomial ideal is tantamount to saying that $\mathfrak{a}_m$ is the expansion of a monomial ideal $\mathfrak{a}'_m \subset k[x_1, \ldots, x_n]$. The monomial ideal $\mathfrak{a}'_m$ is itself a valuation ideal for the valuation on $k(x_1, \ldots, x_n)$ obtained by restricting ν to this subfield. Because the maximal ideal of A expands to the maximal ideal of R, the map of rings (22) is étale, which is to say, the morphism (21) is étale is a neighborhood of W [18]. Thus replacing U by a possibly smaller open neighborhood, we can assume the morphism

$$U \longrightarrow \mathbf{A}^n$$

consists of compositions of open immersions with a finite étale map. But the computation of multiplier ideals commutes with pullback under both open immersions

(obvious) and finite étale maps (straightforward; see [16, 9.5E]). So

$$\mathcal{J}(\mathbf{A}^n, c \cdot \mathfrak{a}'_m) \cdot \mathcal{O}_U = \mathcal{J}(U, c \cdot \mathfrak{a}'_m \mathcal{O}_U),$$

and passing to the local ring at W (which, after all, amounts to taking a limit of pullbacks to smaller and smaller affine neighborhoods of W), we see that

$$\mathcal{J}(\mathbf{A}^n, c \cdot \mathfrak{a}'_m)R = \mathcal{J}(R, c \cdot \mathfrak{a}'_m R) = \mathcal{J}(R, c \cdot \mathfrak{a}_m).$$

So to compute the multiplier ideal of $\mathfrak{a}_m$ in the local ring R, it is sufficient to compute the multiplier ideal of the monomial ideal $\mathfrak{a}'_m$ in the polynomial ring $k[x_1, \ldots, x_n]$ and expand to R.

We recall the formula for the multiplier ideal of a monomial ideal from [10]. Let $\mathfrak{a} \subset k[x_1, \ldots, x_n]$ be an ideal generated by monomials and let L denote its lattice of exponents:

$$L = \{(a_1, \ldots, a_n) \mid x_1^{a_1} \cdots x_n^{a_n} \in \mathfrak{a}\}.$$

Then the multiplier ideal $\mathcal{J}(c \cdot \mathfrak{a})$ is the ideal of the polynomial ring generated by those monomials $x_1^{b_1} \cdots x_n^{b_n}$ satisfying

$$(b_1, \ldots, b_n) + (1, \ldots, 1) \in \{\text{hull}(cL)\}^{int},$$

where $\{\text{hull}(cL)\}^{int}$ denotes the interior of the convex hull of the lattice L scaled by the real number c.

In our case, the monomial ideals $\mathfrak{a}'_m$ are generated by

$$\left\{ x_1^{a_1} \cdots x_n^{a_n} \mid \sum a_i \nu(x_i) \geq m \right\},$$

so the multiplier ideals are given by

$$\mathcal{J}(c \cdot \mathfrak{a}'_m) = \left(\left\{ x_1^{b_1} \cdots x_n^{b_n} \mid \sum (b_i + 1)\nu(x_i) > cm \right\} \right).$$

In particular, for any real positive p,

$$\mathcal{J}(c \cdot \mathfrak{a}'_m) = \mathcal{J}\left(\frac{c}{p} \cdot \mathfrak{a}'_{mp} \right)$$

so that the asymptotic multiplier ideals $\mathfrak{j}'_m$ of the graded family $\{\mathfrak{a}'_m\}$ in the polynomial ring satisfy

$$\mathfrak{j}'_m = \mathcal{J}(\mathfrak{a}'_m).$$

Expanding to R, we see that the asymptotic multiplier ideals in R are given by

$$j_m(\mathfrak{a}_\bullet) = \left(\left\{ x_1^{b_1} \cdots x_n^{b_n} \mid \sum (b_i + 1)\nu(x_i) > m \right\} \right).$$

Finally, using this description of the asymptotic multiplier ideals, we observe that the element $\delta = x_1 \cdots x_r$ in R satisfies the condition that

$$\delta j_m \subseteq \mathfrak{a}_m$$

and Theorem 2.3 is proved. □

3. The volume of a graded family of $\mathfrak{m}$-primary ideals. In this section we introduce the volume of a graded family of $\mathfrak{m}$-primary ideals, and compare it to the multiplicities of the individual ideals of the graded system. In particular, we deduce Corollary C from the Introduction as a special case of a general phenomenon.

Definition 3.1. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \Phi}$ be a graded family of $\mathfrak{m}$-primary ideals in a local Noetherian ring $(R, \mathfrak{m})$ of dimension n. The volume of $\mathfrak{a}_\bullet$ is the real number

(23) $$\mathrm{vol}(\mathfrak{a}_\bullet) = \limsup_{m \in \Phi} \frac{\mathrm{length}\,(R/\mathfrak{a}_m)}{m^n/n!}.$$

Remark 3.2. The volume of a graded system is the local analogue of the volume of a big divisor D on a projective variety X of dimension n, which is defined to be

$$\mathrm{vol}(D) = \limsup_m \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

When X is smooth and D is ample, this coincides up to constants with the volume of X determined by any Kähler form representing $c_1(\mathcal{O}_X(D))$, which explains the terminology.

While Definition 3.1 works well for graded systems indexed by the natural numbers $\mathbf{N}$, it does not have very good behavior for arbitrary graded systems $\{\mathfrak{a}_m\}$ indexed by more general semigroups $\Phi \subseteq \mathbf{R}_{\geq 0}$. For example, suppose that $\Phi = \mathbf{N} + \mathbf{N}\sqrt{2}$, and put

$$\mathfrak{a}_{j+k\sqrt{2}} = \mathfrak{b}^j \cdot \mathfrak{c}^k$$

for some fixed ideals $\mathfrak{b}, \mathfrak{c} \subseteq R$. Then the volumes of the two $\mathbf{N}$-graded subsystems $\{\mathfrak{a}_j\}$ and $\{\mathfrak{a}_{k\sqrt{2}}\}$ will not in general coincide. In order to avoid this sort of

pathology, we will henceforth adopt the following:

Convention 3.3. For the remainder of this section, we deal with the graded families arising from ideal filtrations; that is, we work with graded families $\{\mathfrak{a}_m\}_{m\in\Phi}$ which satisfy the additional condition

(24) $\mathfrak{a}_m \subseteq \mathfrak{a}_{m'}$ for any two indices $m, m' \in \Phi$ with $m \geq m'$.

Of course (24) is automatic for the graded families arising from valuations.

Remark 3.4. The volume of a graded family of $\mathfrak{m}$-primary ideals satisfying (24) is finite. Indeed, fix any $m \in \Phi$. Since $\mathfrak{a}_m^{\ell} \subset \mathfrak{a}_{m\ell}$ for every positive integer ℓ, we see that

$$\frac{\text{length}\,(R/\mathfrak{a}_m^{\ell})}{(m\ell)^n} \geq \frac{\text{length}\,(R/\mathfrak{a}_{m\ell})}{(m\ell)^n}$$

for all $\ell \in \mathbf{N}$. So taking the limit as ℓ gets large, Lemma 3.8 below ensures that the volume of $\{\mathfrak{a}_\bullet\}$ is bounded above by the rational number $e(\mathfrak{a}_m)/m^n$, where $e(\mathfrak{a}_m)$ denotes the multiplicity of the ideal $\mathfrak{a}_m$.

Example 3.5. (i) The volume of the trivial graded family $\{\mathfrak{a}^m\}$ of powers of a fixed ideal $\mathfrak{a}$ is equal to the multiplicity of the ideal. Likewise, the volume of $\{\overline{\mathfrak{a}^m}\}$ of integral closures of a fixed power of an ideal $\mathfrak{a}$ is also the multiplicity of $\mathfrak{a}$.

(ii) Let ν be the valuation on $k(x, y)$ given by "order of vanishing at the origin" in Example 1.6(i). As we have seen, the valuation ideals of ν on $k[x, y]$ are given by $\mathfrak{a}_m = (x, y)^m$. In this case, the volume is the multiplicity of the maximal ideal (x, y) in $k[x, y]$, which is one.

(iii) Let ν be the monomial valuation of Example 1.6(iii). Then each $\mathfrak{a}_m \subset k[x, y]$ is generated by the monomials $x^a y^b$, where $a + \pi b \geq m$. So the length of $k[x, y]/\mathfrak{a}_m$ is equal to the number of integer points in the first quadrant of the Cartesian plane inside the triangle bounded by the a-axis, the b-axis, and the line $a + \pi b = m$. The area of this triangle is roughly $\frac{1}{2\pi}m^2$. Taking the limit, the volume of ν on Spec $k[x, y]$ is $\frac{1}{\pi}$. An evident modification of this shows that any positive real number can occur as the volume of a graded family of ideals in $k[x, y]$.

(iv) Let ν be the arc valuation of Example 1.6(iv). Then the quotients R/q_m are spanned by the residues of $1, x, x^2, \ldots, x^{m-1}$, so the length of R/q_m is m and the volume of ν is zero. However, the 1-volume is 1; cf. Remarks 3.7 and 3.17.

Remark 3.6. The volume of the family of ideals associated to a divisorial valuation was considered by Cutkosky and Srinivas in [3]. In the two-dimensional case, they show that this invariant is always a rational number: this essentially reflects the existence of Zariski decompositions. However Kuronya [15] gives an

example of a four dimensional divisorial valuation with irrational volume. His construction makes use of Cutkosky's curves in $\mathbf{P}^3$ having irrational Castelnuovo-Mumford regularity [2].

Remark 3.7. It is also possible to define the *p*-volume of a graded system $\mathfrak{a}_\bullet$ for any $p \leq n = \dim R$ as

$$\limsup_{m \in \Phi} \frac{\text{length}\,(R/\mathfrak{a}_m)}{m^p/p!}.$$

In the current paper, we will not pursue this further. See, however, Remark 3.17.

LEMMA 3.8. *Let* $\{\mathfrak{a}_t\}_{t \in \Phi}$ *be a graded system of* $\mathfrak{m}$-*primary ideals in a Noetherian local ring* $(R, \mathfrak{m})$ *of dimension n satisfying* (24). *Then for any fixed positive* $m \in \Phi$,

$$\limsup_{t \in \Phi} \frac{\text{length}\,(R/\mathfrak{a}_t)}{t^n} = \limsup_{\ell \in \mathbf{N}} \frac{\text{length}\,(R/\mathfrak{a}_{m\ell})}{(m\ell)^n}.$$

In particular, the volume of the mth Veronese graded subsystem $\{\mathfrak{a}_{m\ell}\}_\ell$ *is given by*

$$\text{vol}(\{\mathfrak{a}_{m\ell}\}_{\ell \in \mathbf{N}}) = m^n \text{vol}(\{\mathfrak{a}_t\}_{t \in \Phi}).$$

Remark 3.9. It follows from Lemma 3.8 that volume can be defined as

$$\limsup_{\ell \in \mathbf{N}} \frac{\text{length}\,(R/\mathfrak{a}_{\ell m})}{(\ell m)^n/n!},$$

where *m* is any fixed nonzero real number. In particular, with the convention that $1 \in \Phi$, the volume is

$$\limsup_{m \in \mathbf{N}} \frac{\text{length}\,(R/\mathfrak{a}_m)}{m^n/n!}.$$

Proof of Lemma 3.8. For each $t \in \Phi$, we have

$$m \cdot \left[\frac{t}{m}\right] \leq t < m \cdot \left[\frac{t}{m}\right] + m,$$

where $[\frac{t}{m}]$ denotes, as usual, the greatest integer less than or equal to the real number $\frac{t}{m}$. Setting $\ell = [\frac{t}{m}]$, we have thanks to (24):

$$\mathfrak{a}_{(\ell+1)m} \subset \mathfrak{a}_t \subseteq \mathfrak{a}_{\ell m},$$

so that

$$\frac{\text{length}\,(R/\mathfrak{a}_{(\ell+1)m})}{t^n} \geq \frac{\text{length}\,(R/\mathfrak{a}_t)}{t^n} \geq \frac{\text{length}\,(R/\mathfrak{a}_{\ell m})}{t^n}.$$

Since $\lim_{t\to\infty} \frac{t}{[\frac{t}{m}]m} = 1$, one has

$$\limsup_{\ell\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_{\ell m})}{(\ell m)^n} = \limsup_{t\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_{\ell m})}{t^n},$$

and likewise with ℓ replaced by $\ell + 1$. Thus

$$\limsup_{\ell\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_{(\ell+1)m})}{((\ell+1)m)^n} \geq \limsup_{t\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_t)}{t^n} \geq \limsup_{\ell\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_{\ell m})}{(\ell m)^n}.$$

Since the expression on the left here is equal to the expression on the right, the lemma is proved.   □

The next proposition shows that from the point of view of multiplier ideals, graded families with zero volume are trivial.

PROPOSITION 3.10. *Let $\mathfrak{a}_\bullet$ be a graded family of $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$ essentially of finite type over a field of characteristic zero. Assume that $\mathfrak{a}_\bullet$ satisfies (24). If $\{\mathfrak{a}_\bullet\}$ has volume zero, then each of its asymptotic multiplier ideals $\mathfrak{j}_m(\mathfrak{a}_\bullet)$ is the unit ideal.*

Proof. Fix any $m \in \Phi$. Then for all $\ell \in \mathbf{N}$, we have

$$\mathfrak{a}_{m\ell} \subseteq \mathfrak{j}_m^\ell,$$

whence $e(\mathfrak{j}_m) \geq \text{vol}(\mathfrak{a}_\bullet)$ thanks to the previous Lemma. The assertion follows. □

The following proposition allows us to deduce Corollary C from the Introduction.

PROPOSITION 3.11. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n, essentially of finite type over a field of characteristic zero. Let $\{\mathfrak{a}_m\}_{m\in\Phi}$ be a graded family of $\mathfrak{m}$-primary ideals of R satisfing (24), and let $\{\mathfrak{j}_m\}_m$ be the associated sequence of asymptotic multiplier ideals. Assume that there is a fixed nonzero element $\delta \in R$ such that*

(25) $$\delta \cdot \mathfrak{j}_m \subseteq \mathfrak{a}_m \quad \text{for all} \quad m \in \Phi.$$

Then

$$\text{vol}(\mathfrak{a}_\bullet) = \limsup_{m\to\infty} \frac{e(\mathfrak{a}_m)}{m^n} = \limsup_{m\to\infty} \frac{e(\mathfrak{j}_m)}{m^n},$$

where $e(\mathfrak{a})$ denotes the multiplicity of the ideal $\mathfrak{a}$ in the local ring R.

(As in Lemma 3.8, the limits can be taken over all $m \in \Phi$ or just over all positive integer multiples of a fixed element in Φ. For graded systems indexed by the natural numbers $\mathbf{N}$, one does not need to assume the filtration condition (24).)

Proof. Given any index $m \in \Phi$, set $\mathfrak{d}_m = (\mathfrak{a}_m : \mathfrak{j}_m)$. This is an $\mathfrak{m}$-primary ideal, and since all the $\mathfrak{d}_m$ contain the fixed element δ one verifies that

$$\limsup_{m \to \infty} \frac{e(\mathfrak{d}_m)}{m^n} = 0.$$

Indeed, since $\mathfrak{a}_1^m \subset \mathfrak{a}_m \subset \mathfrak{d}_m$, we see that $(\mathfrak{a}_1^m + \delta) \subset \mathfrak{d}_m$ for all m. Thus $e(\mathfrak{d}_m) \le e(\overline{\mathfrak{a}_1}^m) = m^{n-1} e(\overline{\mathfrak{a}_1})$, where $\overline{\mathfrak{a}_1}$ is the image of the ideal $\mathfrak{a}_1$ in the $(n-1)$-dimensional ring $R/(\delta)$.

Now fix a large index $m \in \Phi$. Then for all ℓ we have

$$(\mathfrak{d}_m \mathfrak{j}_m)^\ell \subseteq \mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{m\ell} \subseteq \mathfrak{j}_{m\ell} \subseteq \mathfrak{j}_m^\ell,$$

and so

$$\frac{\text{length}\,(R/\mathfrak{j}_m^\ell)}{\ell^n} \le \frac{\text{length}\,(R/\mathfrak{a}_{m\ell})}{\ell^n} \le \frac{\text{length}\,(R/\mathfrak{a}_m^\ell)}{\ell^n} \le \frac{\text{length}\,(R/(\mathfrak{d}_m \mathfrak{j}_m)^\ell)}{\ell^n}$$

for all ℓ. Taking the limit as ℓ goes to infinity, we find that

$$(26) \qquad e(\mathfrak{j}_m) \le \text{vol}(\{\mathfrak{a}_{m\ell}\}_\ell) \le e(\mathfrak{a}_m) \le e(\mathfrak{d}_m \mathfrak{j}_m),$$

where $\{\mathfrak{a}_{m\ell}\}_\ell$ is the mth Veronese subgraded sequence of $\{\mathfrak{a}_\bullet\}$.

But now note that dividing by m^n, the expressions on the left and right here (namely $\frac{e(\mathfrak{j}_m)}{m^n}$ and $\frac{e(\mathfrak{d}_m \mathfrak{j}_m)}{m^n}$) have the same limit superior as m gets large. In fact, Teissier's Minkowski Inequality [24, p. 39] implies that

$$e(\mathfrak{d}_m \mathfrak{j}_m)^{1/n} \le e(\mathfrak{d}_m)^{1/n} + e(\mathfrak{j}_m)^{1/n}$$

for all m, whence

$$\frac{e(\mathfrak{d}_m \mathfrak{j}_m)^{1/n}}{m} \le \frac{e(\mathfrak{d}_m)^{1/n}}{m} + \frac{e(\mathfrak{j}_m)^{1/n}}{m}$$

for all m. But $\lim_{m \to \infty} \frac{e(\mathfrak{d}_m)}{m^n} = 0$, and so

$$\limsup_{m \to \infty} \frac{e(\mathfrak{d}_m \mathfrak{j}_m)}{m^n} \le \limsup_{m \to \infty} \frac{e(\mathfrak{j}_m)}{m^n}.$$

On the other hand, since $e(\mathfrak{j}_m) \leq e(\delta_m \mathfrak{j}_m)$, the reverse inequality always holds. Finally, using Lemma 3.8, we conclude from (26) that

$$\limsup_{m \to \infty} \frac{e(\mathfrak{j}_m)}{m^n} = \mathrm{vol}(\mathfrak{a}_\bullet) = \limsup_{m \to \infty} \frac{e(\mathfrak{a}_m)}{m^n},$$

as claimed.   □

Remark 3.12. The proof shows also that the "volume" of $\{\mathfrak{j}_m\}$ is equal to the volume of the graded system $\{\mathfrak{a}_m\}$ (even though $\{\mathfrak{j}_m\}$ itself is not a graded system).

Proof of Corollary C. To prove Corollary C from the Introduction, let ν be a rank one Abhyankar valuation centered at a smooth point on a complex variety, and let R be the local ring of that point. By Theorem 2.3, there exists a nonzero element δ such that $\delta \mathfrak{j}_m \subset \mathfrak{a}_m$ for all $m \in \Phi$. Thus Corollary C is an immediate consequence of Proposition 3.11.   □

Remark 3.13. Another example of a graded family satisfying (25) is given by the base loci of the linear series of a big divisor. Specifically, fix a big divisor L on a smooth projective variety X, and let $\mathfrak{b}'_m \subset \mathcal{O}_X$ be the base ideal of the linear system $|mL|$. The components of the base locus stabilize as $m \to \infty$, so choose one component and localize along it to get a graded family of ideals $\mathfrak{b}_m$ in the local ring along the generic point of the component. One can show that there exists D such that $\mathcal{O}_X(-D) \cdot \mathfrak{j}_m \subset \mathfrak{b}_m$ for all m (see [16], Chapter 10). So the conclusion of Proposition 3.11 holds for $\mathfrak{b}_\bullet$.

Remark 3.14. The conclusion of Proposition 3.11 holds under the weaker assumption that there is a family of nonzero $\mathfrak{m}$-primary ideals $\mathfrak{d}_m$, with $\limsup \frac{e(\mathfrak{d}_m)}{m^n} = 0$, such that $\mathfrak{d}_m \cdot \mathfrak{j}_m \subseteq \mathfrak{a}_m$. As far as we know, it is possible that *every* graded system $\mathfrak{a}_m$ has this property, namely that the sequence $\mathfrak{a}_m$ is "tightly bound" to the sequence $\mathfrak{j}_m$ in the sense that $\limsup \frac{e(a_m:j_m)}{m^n}$ tends to zero as m goes to infinity. In particular, Proposition 3.11 may hold for a completely arbitrary graded system of $\mathfrak{m}$-primary ideals, so in particular, for an arbitrary valuation. Since we posed this question in an earlier version of this manuscript, Mircea Mustata has shed some light on the question of whether Proposition 3.11 holds more generally. Specifically, he shows that the volume of an arbitrary graded system of ideals is equal to the limit of the normalized multiplicities $\frac{e(\mathfrak{a}_m)}{m^n}$ in general—that is, that the first equality in the conclusion of Proposition 3.11 holds without the assumption that there exists a δ such that $\delta \mathfrak{j}_m \subset \mathfrak{a}_m$ for all m; see [19]. In particular, Mustata shows that Corollary C holds for any rank one valuation, Abhyankar or not. However, the relationship with the sequence of asymptotic multiplier ideals (as well as the second equality in Proposition 3.11) remains an open question in the general case.

Theorem 2.3 implies that the volume of an Abhyankar valuation on any model is positive, and we saw in Example 3.5 (iv) that the volume of a non-Abhyankar valuation can be zero. However, it is not the case that a valuation has positive volume if and only if it is Abhyankar, as the example below shows.

Example 3.15. In [26, pp. 102–104], there is a construction of a valuation on $K = k(x, y)$ with value group an arbitrary additive subgroup of the rational numbers; see also [25, §10, Example 12]. Using this, we can construct a non-Abhyankar valuation of arbitrary volume (even normalizing so that $\nu(\mathfrak{m}) = 1$).

Let $\nu(y) = 1$ and set $\nu(x) = \beta_0 > 1$, some rational number. Let c_0 be the smallest positive integer such that $c_0\beta_0 \in \mathbf{Z}$. As in [26], there exists a valuation so that the polynomial

$$q_1 = x^{c_0} - y^{\beta_0 c_0}$$

has value β_1 equal to any rational number greater than (or equal to) the "expected value" of $\beta_0 c_0$. Let us choose this value so that $\beta_1 = \frac{d_1}{c_1} > \beta_0 c_0$, where d_i and c_i are relatively prime positive integers with c_1 relatively prime to c_0.

This process can be repeated, so that we can construct a valuation having the values on x, y, and q_1 as already specified, and having arbitrary rational value $\beta_2 \geq \beta_1 c_1$ on the polynomial

$$q_2 = q_1^{c_1} + y^{\beta_1 c_1}.$$

Again, we make this choice of β_2 so that the smallest positive integer c_2 such that $c_2\beta_2 \in \mathbf{Z}$ is relatively prime to each of the preceding c_i.

In this way, we inductively construct a sequence of polynomials q_i, rational numbers β_i, and positive integers c_i with the following properties:

$$q_{i+1} = q_i^{c_i} + y^{\beta_i c_i},$$

and

$$\beta_{i+1} > \beta_i c_i,$$

where c_i is the smallest positive integer such that $c_i\beta_i \in \mathbf{Z}$, and c_i is relatively prime to the product $c_0 c_1 \cdots c_{i-1}$. As shown (even more generally) in [26], this uniquely defines a valuation ν on $k(x, y)$, such that $\nu(q_i) = \beta_i$. Indeed, using the Euclidean algorithm, and setting $q_{-1} = y$ and $q_0 = x$, every polynomial has a unique expression as a sum of "monomials" in the q_i:

$$q_{-1}^{a_{-1}} q_0^{a_0} q_1^{a_1} \cdots q_t^{a_t}$$

where a_{-1} is arbitrary but the remaining exponents a_i satisfy $a_i < c_i$. Because each of these "monomials" has a distinct value, the valuation ideals on $k[x, y]$

have the form

$$\mathfrak{a}_m = \left(\left\{ q_{-1}^{a_{-1}} q_0^{a_0} q_1^{a_1} \cdots q_t^{a_t} \ \middle| \ \sum_{j=-1}^{t} \beta_i a_i \geq m; a_j \leq c_j - 1 \text{ for } j \geq 0 \right\} \right).$$

In particular, the quotients $k[x, y]/\mathfrak{a}_m$ have vector space basis consisting of "monomials"

$$q_{-1}^{a_{-1}} q_0^{a_0} q_1^{a_1} \cdots q_t^{a_t} \quad \text{where} \quad \sum_{j=-1}^{t} \beta_i a_i < m \quad \text{and} \quad a_j \leq c_j - 1 \text{ for } j \geq 0.$$

Although the number of products t here can be arbitrary, note that for each fixed m, we only need t up to the greatest integer such that $\beta_t < m$.

So computing the volume amounts to counting the number of monomials in this basis. A computation shows that the volume of ν is the limit of the following decreasing sequence of rational numbers

$$\alpha_i = \frac{1}{\beta_0} \left(\frac{c_0 \beta_0}{\beta_1} \right) \cdots \left(\frac{c_i \beta_i}{\beta_{i+1}} \right).$$

(Calculation hint: this is the limit of the subsequence $\{2! \frac{\lambda(k[x,y]/\mathfrak{a}_m)}{m^2}\}$ indexed by $m = c_t \beta_t$, which bounds the volume below. On the other hand, we can approximate ν by a sequence of valuations ν_i which take the values β_j on q_j for $j \leq i$, and the "expected values" on the remaining q_i (which is to say that the corresponding sequences of β_j and c_j satisfy $\beta_j = c_{j-1}\beta_{j-1}$ for $j > i$). Then just observe that the volume of each ν_i bounds the volume of ν above and compute that the volume of ν_i is α_i.)

By choosing the values of β_i and c_i appropriately, one can make this limit be any nonnegative real number (note that we have normalized so that $\beta_{-1} = \nu(y) = 1$). For a completely explicit example, take c_i to be the standard enumeration of the prime numbers ($c_0 = 2, c_1 = 3, \ldots$), and set $\beta_{i+1} = c_i \beta_i + \frac{1}{c_{i+1}}$. Then the volume turns out to be the reciprocal of the infinite sum

$$1 + \frac{1}{c_0} + \frac{1}{c_0 c_1} + \frac{1}{c_0 c_1 c_3} + \cdots,$$

which approaches a real number between $\frac{1}{2}$ and 1.

Discussion 3.16. (The associated graded algebra of a valuation) Fix a rank one valuation ν on a function field K/k centered a local domain $(R, \mathfrak{m})$ and let Φ be the corresponding value semigroup $\nu(R)$. The *associated graded algebra* of

the valuation on R is

$$\mathrm{gr}_\nu R = \bigoplus_{m \in \Phi} \mathfrak{a}_m / \mathfrak{a}_{>m},$$

where $\mathfrak{a}_{>m}$ denotes the valuation ideal $\{f \mid \nu(f) > m\}$. It is easy to check that $\mathrm{gr}_\nu R$ is a domain, but it is not finitely generated over $R/\mathfrak{m}$ (its degree zero piece) in general. The transcendence degree of $\mathrm{gr}_\nu R$ over $R/\mathfrak{m}$ is equal to the rational rank of ν plus the transcendence degree of $R_\nu/\mathfrak{m}_\nu$ over $R/\mathfrak{m}$, where $\mathfrak{m}_\nu$ denotes the maximal ideal of the valuation ring R_ν. If $\mathrm{gr}_\nu R$ is finitely generated, therefore, its Krull dimension is equal to this sum. In this case, the associated graded ring has dimension equal to the dimension of R if and only if the valuation is Abhyankar. (Here we are using a slightly different, but equivalent (for function fields), form of Abhyankar's inequality (16) which says that $\mathrm{rat.rank}\,\nu + \mathrm{trans.deg}_{R/\mathfrak{m}}(R_\nu/\mathfrak{m}_\nu) \leq R$; this inequality differs from (16) by addition of the same number, namely the transcendence degree of $R/\mathfrak{m}$ over k, to both sides.)

When the associated graded ring of ν is finitely generated and $\mathbf{N}$-graded, the volume of ν has a simple interpretation in terms of $\mathrm{gr}_\nu R$, namely it is equal to the Hilbert multiplicity of ν. Recall that if $A = \oplus A_m$ is a finitely generated $\mathbf{N}$-graded domain over a field A_0 containing nonzero elements of every degree, then there exists a positive rational number e such that

$$(27) \qquad\qquad \dim A_m = e \frac{m^{n-1}}{(n-1)!} + O(m^{n-2})$$

where n is the Krull dimension of A and $\dim$ denotes the dimension over the field A_0. This number e is called the *Hilbert multiplicity* of A. (To see this, note that for some r, the Veronese subalgebra $A^{(r)}$ is generated in degree one (or r), so A decomposes as a direct sum of $A^{(r)}$-modules $A_{(0)} \oplus A_{(1)} \oplus \cdots \oplus A_{(r-1)}$, where $A_{(i)} = \oplus_{j \in \mathbf{N}} A_{jr+i}$. Thus each of the modules $A_{(i)}$ has some multiplicity e_i over the ring $A^{(r)}$. In general, these multiplicities can be different, but if A has elements of every degree, one shows that the e_i are all equal. Indeed, to see that $e_i = e_{i'}$, just take a nonzero element $x \in A_{i-i'}$ and note that the cokernel of the injective map $A_{(i')} \xrightarrow{x} A_{(i)}$ has dimension strictly less than the common dimension of $A_{(i)}$ and $A_{(i')}$. Thus $A_{(i)}$ and $A_{(i')}$ necessarily have the same multiplicity, e. Thus the Hilbert polynomials of all the $A_{(i)}$ have the same leading terms, leading to the formula for the dimension above (with e suitably normalized).)

Thus

$$\mathrm{length}\,(R/\mathfrak{a}_m) = \sum_{i=0}^{m-1} \mathrm{length}\,(\mathfrak{a}_m/\mathfrak{a}_{m+1}) = \frac{e}{(n-1)!} \left[\sum_{i=0}^{m-1} i^{(n-1)} + O(m^{n-2}) \right],$$

and since $\sum_{i=0}^{m-1} [i^{(n-1)} + O(m^{n-2})] = \frac{m^n}{n} + O(m^{n-1})$, we see that

$$\lim_{m\to\infty} \frac{\text{length}\,(R/\mathfrak{a}_m)}{m^n/n!} = e.$$

Remark 3.17. When $\text{gr}_\nu R$ is finitely generated, the preceding discussion indicates that it is natural to consider the *p*-volume of ν on R, where $p = \text{rat.rank}\,\nu + \text{trans.deg}_{R/\mathfrak{m}}(R_\nu/\mathfrak{m}_\nu)$; see Example 3.5(iv). However, Example 3.15 indicates that the *p*-volume need not be finite in general, even when the graded ring has transcendence degree *p*.

4. Generalizations and further applications. Let *D* be an arbitrary effective divisor on a smooth variety *X* and let $\pi\colon X \longrightarrow Y$ be a proper birational map to a smooth variety *Y* that collapses *D* to a point. The collection

(28) $\{\pi_*\mathcal{O}_X(-mD)\}_{m\in\mathbf{N}}$

forms a graded family of ideals in $\mathcal{O}_Y$. Although this is not the graded family of a valuation, it can be handled by the methods developed here for families of valuation ideals because it is an intersection of graded families of valuation ideals.

Definition 4.1. Let $\{\{a_m^\lambda\}_{m\in\Phi}\}_{\lambda\in\Lambda}$ be an arbitrary collection of graded families, all indexed by the same semigroup $\Phi \subset \mathbf{R}$. The intersection graded family is defined by

$$\bigcap_{\lambda\in\Lambda} \mathfrak{a}_\bullet^\lambda := \left\{ \bigcap_{\lambda\in\Lambda} \mathfrak{a}_m^\lambda \right\}_{m\in\Phi}.$$

Note that if each $\mathfrak{a}_\bullet^\lambda$ satisfies (24), then so too does their intersection.

The asymptotic multiplier ideals of an intersection family satisfy

(29) $\mathfrak{j}_m\left(\bigcap_{\lambda\in\Lambda} \mathfrak{a}_\bullet^\lambda\right) \subset \bigcap_{\lambda\in\Lambda} \mathfrak{j}_m(\{\mathfrak{a}_\bullet^\lambda\})$

Indeed, for each λ, we have $\{\bigcap_{\lambda\in\Lambda} \mathfrak{a}_m^\lambda\} \subset \mathfrak{a}_m^\lambda$ for all *m*, so the corresponding inclusion holds for the asymptotic multiplier ideals.

Let $\mathcal{S}$ be any collection of rank one valuations centered on a domain *R*. For each $m \in \mathbf{R}_{\geq 0}$, set

(30) $\mathfrak{a}_m = \{f \in R \mid \nu(f) \geq m \text{ for all } \nu \in \mathcal{S}\}.$

The collection $\{\mathfrak{a}_m\}_{m\in\mathbf{R}_{\geq 0}}$ forms a graded family of ideals in R indexed by the nonnegative real numbers (or by considering only distinct such ideals, some subsemigroup of $\mathbf{R}$.) This graded family is the intersection, over all $\nu \in \mathcal{S}$, of the graded families $\{q_m(\nu)\}_{m\in\mathbf{R}_{\geq 0}}$ of ν-valuation ideals.

As a variant, one can also assign multiplicities to the valuations in $\mathcal{S}$. Say for each ν in $\mathcal{S}$, we assign some positive real number e_ν. Then for each $m \in \mathbf{R}$, set

$$(31) \qquad \mathfrak{a}_m = \left\{ f \in R \mid \frac{\nu(f)}{e_\nu} \geq m \text{ for all } \nu \in \mathcal{S} \right\}.$$

The collection $\{\mathfrak{a}_m\}_{m\in\mathbf{R}}$ forms a graded family of ideals in R, indexed by (some subsemigroup of) the real numbers. For example, the graded family (28) above is of this form: If $D = \sum e_i D_i$ where the D_i are prime divisors, then the set $\mathcal{S}$ is the set of valuations ν_i given by order of vanishing along D_i and the multiplicities e_{ν_i} are the coefficients e_i. These graded families are also intersections: the graded family (31) is the intersection, over all ν in $\mathcal{S}$, of the graded families $\{q_{e_\nu m}(\nu)\}_m$ of the e_νth Veronese subfamily of the graded family of valuation ideals of ν. Alternatively, it can be interpreted as an intersection of graded families of valuation ideals: it is the intersection of the graded families of valuation ideals in the set $\mathcal{S}'$, where the set $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing each valuation ν in $\mathcal{S}$ by the valuation $\frac{1}{e_\nu}\nu$. So this is really no more general than the intersection (30) discussed in the previous paragraph. Once again, condition (24) is satisfied by these families.

One can also define the product of two graded families $\{\mathfrak{a}_\bullet\}_{m\in\Phi}$ and $\{\mathfrak{b}_\bullet\}_{m\in\Phi}$ by

$$\{\mathfrak{a}_m\mathfrak{b}_m\}_{m\in\Phi}.$$

Note that the asymptotic multiplier ideals of the product graded family satisfy

$$(32) \qquad \mathfrak{j}_m(\mathfrak{a}_\bullet\mathfrak{b}_\bullet) \subseteq \mathfrak{j}_m(\mathfrak{a}_\bullet)\mathfrak{j}_m(\mathfrak{b}_\bullet),$$

since, for all large p, we have the inclusion of "usual" multiplier ideals $\mathcal{J}(\frac{1}{p} \cdot (\mathfrak{a}_{mp}\mathfrak{b}_{mp})) \subset \mathcal{J}(\frac{1}{p} \cdot \mathfrak{a}_{mp})\mathcal{J}(\frac{1}{p} \cdot \mathfrak{b}_{mp})$ by subadditivity [6]. By induction, the product of any finite number of graded families indexed by the same semi-group is defined, and the same multiplicativity of the asymptotic multiplier ideals holds.

Our Main Theorem 2.3 and its ensuing corollaries can be extended to graded families arising as finite intersections or products of Abhyankar valuations. Explicitly:

COROLLARY 4.2. *Let $\mathcal{S}$ be a finite collection of rank one Abhyankar valuations of a function field K/k of characteristic zero, all centered on some local ring R of a model of K/k. Let $\{\mathfrak{a}_\bullet\}$ denote either the corresponding intersection or the*

corresponding product graded family of ideals. Then there exists a nonzero element $\delta \in R$ such that

$$\delta \mathfrak{j}_m \subset \mathfrak{a}_m$$

for all $m \in \mathbf{R}$, where $\{\mathfrak{j}_m\}$ is the associated sequence of asymptotic multiplier ideals. Furthermore, the conclusions of Theorem A and Corollary C hold for this graded family of ideals.

Proof. This is immediate from Theorem 2.3. For each $\nu \in \mathcal{S}$, let δ_ν be the nonzero element guaranteed by Theorem 2.3 and let δ be their product. The desired inclusion follows from (29) or from (32), respectively, for the intersection and the product case. □

Proof of Corollary B. For the graded system (28) of a divisor D, Corollary 4.2 immediately implies Corollary B from the Introduction. It also guarantees the existence of a nonzero δ such that

$$\delta^\ell(\pi_*\mathcal{O}_X(-m\ell D)) \subseteq (\pi_*\mathcal{O}_X(-mD))^\ell$$

$\ell, m \in \mathbf{N}$ with $m \gg 0$. □

Finally, we point out the following Minkowski Inequality for intersections and products of graded families.

COROLLARY 4.3. (Minkowski Inequality) *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two graded families of $\mathfrak{m}$-primary ideals in a regular local ring essentially of finite type over a field of characteristic zero. Assume that the hypothesis of Proposition 3.11 holds for each of these systems. Then*

$$\mathrm{vol}\left(\mathfrak{a}_\bullet \bigcap \mathfrak{b}_\bullet\right)^{1/n} \le \mathrm{vol}(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} \le \mathrm{vol}(\mathfrak{a}_\bullet)^{1/n} + \mathrm{vol}(\mathfrak{b}_\bullet)^{1/n}.$$

Proof. Since $\mathfrak{a}_m \mathfrak{b}_m \subseteq \mathfrak{a}_m \bigcap \mathfrak{b}_m$ for all m, it is evident that $\mathrm{vol}(\mathfrak{a}_\bullet \bigcap \mathfrak{b}_\bullet) \le \mathrm{vol}(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)$. So it is enough to prove the inequality for product families.

It follows from the subadditivity relation (32) that if the hypothesis of Proposition 3.11 holds for each of $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ then it holds also for the product system. This being said, the result follows from Proposition 3.11 and Teissier's Minkowski inequality for multiplicities. Indeed,

$$\mathrm{vol}(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} = \left[\limsup_m \frac{e(\mathfrak{a}_m \mathfrak{b}_m)}{m^n}\right]^{1/n}$$

$$\le \limsup_m \left\{\left[\frac{e(\mathfrak{a}_m)}{m^n}\right]^{1/n} + \left[\frac{e(\mathfrak{b}_m)}{m^n}\right]^{1/n}\right\}$$

$$\leq \limsup_m \left\{ \left[\frac{e(\mathfrak{a}_m)}{m^n} \right]^{1/n} \right\} + \limsup_m \left\{ \left[\frac{e(\mathfrak{b}_m)}{m^n} \right]^{1/n} \right\}$$

$$= \operatorname{vol}(\mathfrak{a}_\bullet)^{1/n} + \operatorname{vol}(\mathfrak{b}_\bullet)^{1/n},$$

as required. $\square$

Remark 4.4. Note that the hypothesis of Proposition 3.11 is satisfied in particular for graded families that are finite products or intersections of graded families of Abhyankar valuations. This is the content of Corollary 4.2. More generally, the product or intersection graded family of any finite collection of graded families that satisfy the hypothesis also satisfy the hypothesis. Indeed, we verified this in the proof of Corollary 4.3 for products using (32) and the same argument, using (29) instead of (32), works for intersections.

Example 4.5. As a special case of the Minkowski inequality, fix an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a local ring R of a smooth complex variety and let $\nu_1, \ldots, \nu_r$ be the associated Rees valuations of $\mathfrak{a}$. (The definition of Rees valuations is recalled in the Introduction.) Set $e_i = \min_{f \in I} \nu_i(f)$. Then the associated graded family

$$\mathfrak{a}_m = \left\{ f \in R \mid \frac{\nu_i(f)}{e_i} \geq m \text{ for } i = 1, 2, \ldots, r \right\}$$

is nothing more than the graded family of integral closures of powers of $\mathfrak{a}$ of Example 1.3(ii). In particular, its volume is the multiplicity of $\mathfrak{a}$. This family is a finite intersection of graded families of Abhyankar valuations, so Corollary 4.3 can be applied. This produces the bound

$$e(\mathfrak{a})^{1/n} \leq e_1 \operatorname{vol}(\nu_1)^{1/n} + \cdots + e_r \operatorname{vol}(\nu_r)^{1/n}.$$

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REFERENCES

[1] S. S. Abhyankar, On the valuations centered in a local domain, *Amer. J. Math.* **78** (1956), 321–348.

[2] D. Cutkosky, Irrational asymptotic behavior of Castelnuovo-Mumford Regularity, *J. Reine Angew. Math.* **522** (2000), 93–103.

[3] D. Cutkosky and V. Srinivas, On a problem of Zariski on dimensions of linear systems, *Ann. of Math.* **137** (1993), 531–559.

[4] J.-P. Demailly, *Analytic Methods and Multiplier Ideals Sheaves, School on Vanishing Theorems and Effective Results in Algebraic Geometry, 2000, ICTP Lect. Notes*, vol. 6, Abdus Salam International Centre for Theoretical Physics, Trieste, 2001.

[5] ―――――, Méthodes L^2 et résultats effectifs en géométrie algébrique, Séminaire Bourbaki, exposé n° 852, novembre 1998, 38pp.

[6] J.-P. Demailly, L. Ein and R. Lazarsfeld, A subadditivity property of multiplier ideals, *Michigan Math. J.* (special volume in honor of W. Fulton) **48** (2000), 137–156.

[7] L. Ein, Multiplier ideals, vanishing theorems and applications, *Algebraic Geometry – Santa Cruz 1995, Proc. Sympos. Pure Math.*, vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 203–219.

[8] L. Ein, R. Lazarsfeld, and K. E. Smith, Uniform bounds and symbolic powers on smooth varieties, *Invent. Math.* **144** (2001), 241–252.

[9] R. Hartshorne, *Algebraic Geometry, Grad. Texts in Math.*, vol. 52, Springer-Verlag, New York, 1993.

[10] J. Howald, Multiplier ideals of monomial ideals, *Trans. Amer. Math. Soc.* **353** (2001), 2665–2671.

[11] R. Hübl and I. Swanson, Discrete valuations centered on local domains, *J. Pure Appl. Algebra* **161** (2001), 145–166.

[12] S. Izumi, A measure of integrity for local analytic algebras, *Publ. Res. Inst. Math. Sci.* **21** (1985), 719–735.

[13] Y. Kawamata, Deformations of canonical singularities, *J. Amer. Math. Soc.* **12** (1999), 519–527.

[14] H. Knaf and F.-V. Kuhlmann, Abhyankar places admit local uniformization in any characteristic, preprint, available at the Valuation Theory homepage http://math.usask.ca/fvk/Valth.html.

[15] A. Kuronya, A divisorial valuation with irrational volume, preprint, math.AG/0206228.

[16] R. Lazarsfeld, *Positivity in Algebraic Geometry*, book in preparation, http://www.math.lsa.umich.edu/~rlaz

[17] J. Lipman, Adjoints of ideals in regular local rings, *Math. Res. Lett.* **1** (1994), 739–755.

[18] J. Milne, *Étale Cohomology, Princeton Math. Ser.*, vol. 33, Princeton University Press, Princeton, NJ, 1980.

[19] M. Mustata, On multiplicities of graded sequences of ideals, preprint, math.AG/0203235.

[20] A. Nadel, Multiplier ideal sheaves and Khler-Einstein metrics of positive scalar curvature, *Ann. of Math.* **132** (1990), 549–596.

[21] D. Rees, Izumi's Theorem, *Commutative Algebra* (M. Hochster, C. Huneke, and P. Sally, eds.), Springer-Verlag, New York, 1989, pp. 407–416.

[22] Y.-T. Siu, Invariance of plurigenera, *Invent. Math.* **134** (1998), 661–673.

[23] M. Spivakovsky, Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990), 107–156.

[24] B. Teissier, Sur une inégalité á Minkowski pour les multiplicités, Appendix to: D. Eisenbud and H. I. Levine, The degree of a C^∞ map germ, *Ann. of Math.* **106** (1977), 38–44.

[25] M. Vaquié, Valuations, *Resolution of Singularities (Obergurgl, 1997), Progr. Math.*, vol. 181, Birkhäuser, Basel, 2000, pp. 539–590.

[26] O. Zariski and P. Samuel, *Commutative Algebra II, Grad. Texts in Math.*, vol. 29, Springer-Verlag, New York, 1960.