ZERO-ESTIMATES, INTERSECTION THEORY, AND A THEOREM OF DEMAILLY

LAWRENCE EIN[1], ROBERT LAZARSFELD[2], MICHAEL NAKAMAYE[3]

Introduction

Let X be a smooth complex projective variety of dimension n, and let A be an ample line bundle on X. Fujita has conjectured that the adjoint bundle $\mathcal{O}_X(K_X+mA)$ is basepoint-free if $m \geq n+1$ and very ample if $m \geq n+2$. These conjectures and related questions have attracted considerable attention in recent years because they would extend to varieties of arbitrary dimension the most elementary facts about linear series on curves.

In the seminal paper [De1], Demailly drew on deep analytic tools to make the first serious attack on Fujita's conjectures. More generally, starting with a nef line bundle L, Demailly gives numerical and geometric positivity conditions on L to guarantee that the adjoint linear series $|K_X+L|$ be free or very ample or separate any given number of points and jets. He deduces for example that if A is ample, then $\mathcal{O}_X(2K_X+12n^nA)$ is very ample. While the numbers are rather far from Fujita's predictions, this was the first effective criterion for very ampleness, and it represented a real breakthrough.

Starting with [EL1], it soon became clear that one could also obtain effective (and eventually stronger) results using algebraic methods. Specifically, in [EL1], [Kol1], [Siu1], [Siu2] [De2] and most recently in [AS] (as explained and extended by Kollár [Kol2]), the cohomological ideas pioneered by Kawamata, Reid, Shokurov and others in connection with the minimal model program are applied to obtain effective results on adjoint and pluricanonical linear series. It seems fair to say that by now it is relatively quick to obtain many of the statements occuring in [De1].

But if recent work has largely superceeded many of the specific results of [De1], the geometric ideas underlying that paper remain exceedingly interesting, and they are likely to have other applications in the future. However the apparently analytic nature of Demailly's techniques have limited their defusion in the algebro-geometric community. Our purpose here is to explain in algebro-geometric language the geometric underpinnings of [De1], and to develop algebro-geometric analogues of some of the analytic parts of Demailly's argument.

One of the fascinating aspects of this story is that Demailly's work is very close

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to some ideas that come up in diophantine approximation and transcendence theory. Specifically, one can see the crucial self-intersection formula of [De1] as proving a very refined and novel sort of *zero estimate*. This viewpoint has been very helpful in our thinking about these matters, and we attempt to emphasize the connection here. We also present a conjecture that would put the formula in question into the context of contemporary intersection theory. Some related ideas are applied in a deformation theoretic setting in the paper [EKL].

There is very little here for which we claim any real novelty. We offer these notes in the spirit of *pro bono* work, in the hope that an explanation of the geometric ideas of [De1] may be of some interest to algebraic geometers who – like us – are less than fluent in the analytic language in which that paper is written. We hope that they may contribute to the fruitful interchange that has developed between the algebraic and analytic viewpoints on these matters. The application to adjoint series that we give is slightly weaker than Demailly's because we were unable to prove in general a certain inequality on the self-intersection of the moving part of a linear series. We circumvent this problem with a hyperplane section argument, but this leads to a slight degrading of the numbers. Since the bulk of these notes were written, we have learned that Fujita has proven the general case of the inequality in question in his very interesting note [Fuj] (apparently also motivated by understanding [De1] algebro-geometrically). This means one could give an algebraic proof of the the full statements of [De1]. But since it is in any event the methods rather than the specific results that are of interest here, we have decided not to incorporate Fujita's theorem in our presentation. Finally, we refer the reader to [Laz], §7, for an account of Demailly's approach in the particularly elementary and transparent two dimensional case, and to Demailly's notes [De4] for an overview of the analytic techniques.

We work throughout over the complex numbers $\mathbf{C}$. If X is a projective variety, and L is a line bundle on X, we write indifferently $\int_X c_1(L)^n$, $c_1(L)^n$ or simply L^n for the top Chern number of L. However if $V \subset X$ is a subvariety of dimension k, we generally denote the degree $\int_V c_1(L|V)^k$ by the short-hand $L^k \cdot V$. We trust that this variable notation will not cause undue confusion.

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§1. Informal Overview of Strategy.

This section is devoted to an informal overview, in algebro-geometric language, of the rough strategy of [De1]. Our hope is that an impression of the overall picture and main issues will render subsequent sections easier to absorb. We ignore here entirely the problem of trying to weld together the various constants that appear in different parts of the discussion. However we do state and prove some Lemmas in the generality required for later use.

To set the stage, we start by describing the cohomological result on which the whole approach depends. Let X be a smooth projective variety of dimension n, let $S \subset X$ be a finite set, and let L be an ample line bundle on X. Fix an integer $s \geq 0$.

Theorem 1.1. *Suppose there exists a divisor $E \in |kL|$ satisfying the following properties:*

(1.1.1). *For every $x \in S, \mathrm{mult}_x(E) \geq k(n+s)$;*

(1.1.2). *There is an open neighborhood $U \subset X$ of S such that*

$$\mathrm{mult}_y(E) < k \ \textit{ for all } y \in U - S.$$

Then

$$H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_S^{s+1}) = 0.$$

In particular, $|K_X + L|$ generates s-jets at every point of S, i.e. the evaluation map

$$H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow \bigoplus_{x \in S} H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is surjective.

The Theorem is proved by applying vanishing for $\mathbf{Q}$-divisors on a blow-up of X, or analytically via multiplier ideals and Nadel's Vanishing theorem. We refer to [EV2], (7.5),(7.7) or [De2], (5.6) for details, or to [Laz], (6.4), for the particularly simple and transparent case of surfaces. We will say that a divisor $E \in |kL|$ has an *almost isolated singularity of index* $\geq (n+s)$ along S if it satisfies the hypotheses (1.1.1) and (1.1.2) of the Theorem.

Imagine now that we want to apply Theorem 1.1 to show that $|K_X + L|$ generates s-jets at a given point x. Taking $S = \{x\}$, condition (1.1.1) is very easy to arrange. In fact, by Riemann-Roch:

$$h^0(X, \mathcal{O}_X(kL)) = L^n \cdot \frac{k^n}{n!} + o(k^n).$$

On the other hand, it is

$$\binom{k(n+s)+n}{n} = (n+s)^n \cdot \frac{k^n}{n!} + o(k^n)$$

conditions to impose multiplicity $> k(n+s)$ at a given point. Therefore if

$$\int_X c_1(L)^n > (n+s)^n, \tag{1.2}$$

then for $k \gg 0$ there exists a divisor $E \in |kL|$ for which (1.1.1) holds.[4] Note that (1.2) will certainly be satisfied if for instance $L = (n+s+1)A$ for some ample bundle A.

The essential difficulty arises in trying to guarantee (1.1.2): how can one control the singularities of a divisor E obtained by imposing high multiplicity at a given point? A very similar problem comes up in many situations in diophantine approximation and transcendence theory. Typically one constructs a polynomial (or section of a line bundle) σ vanishing to high order at a some point, and the hard part is to show that σ does not vanish identically, or to too high order at some other point.[5] One line of attack on these questions goes under the general heading of *zero-estimates*. We will try to give a rough idea of the strategy in the present situation.

As a matter of notation, given a divisor $E \in |kL|$, define the *index* of E at a point $x \in X$ to be the "normalized multiplicity":

$$\mathrm{ind}_x(E) = \frac{\mathrm{mult}_x(E)}{k}.$$

As we will work asymptotically with the linear series $|kL|$ for $k \gg 0$, this is the convenient measure of singularity. For $\sigma \in \mathbf{Q}^+$, put

$$Z_\sigma(E) = \{x \in E \mid \mathrm{ind}_x(E) \geq \sigma\}.$$

Thus $Z_\sigma(E)$ is a Zariski-closed subset of X. It carries a natural scheme structure, locally defined by the vanishing of all partial derivatives of orders $< k\sigma$ of a local equation for E.

To explain the idea of the zero estimates, suppose that $E \in |kL|$ is a divisor, and $x \in X$ is a point such that

$$\mathrm{ind}_x(E) > \alpha$$

for some rational $\alpha > 1$, and assume that x is not an almost isolated singularity of E. Then for elementary dimensional reasons, there must exist an irreducible subvariety

[4] We remark that it is already in the construction of E – or more precisely, a current T which plays an analogous role – that Demailly's approach begins to become analytic. His idea is to invoke Yau's solution of the Calabi conjecture to produce a sequence of metrics on L with a given amount of mass concentrated closer and closer to the finite set S. Then he passes to a limit to produce T.

[5] Schmidt's notes [Sch] contain a very nice overview of how these different steps come into the proof of Roth's theorem.

$V \subset X$ of positive dimension passing through x such that the index of E "jumps" by at least $\frac{1}{n}$ along V. More precisely, V is an irreducible component of both $Z_\sigma(E)$ and $Z_{\sigma+(1/n)}(E)$ for suitable σ, so that E has index $\geq \sigma + \frac{1}{n}$ along V, but there exists an open set $U \subset X$ meeting V such that E has index $< \sigma$ on $U - V$. This is a typical "gap" argument which we shall formalize and prove in Lemma 1.5 below.

At this point there are two possibilities. If E has an almost isolated singularity of index $> \alpha$ at x, then we are in a position to apply Theorem 1.1. Alternatively, we can find a positive dimensional "multiplicity jumping" subvariety $V \ni x$ as above. In classical situations – e.g. when $X = \mathbf{P}^n$ – one obtains in the latter case an absolute upper bound on the degree of V:

Proposition 1.3. *Let $X = \mathbf{P}^n$ and $L = \mathcal{O}_{P^n}(1)$, and fix $\epsilon > 0$. Then there is a constant $C = C(n, \epsilon)$ depending only on n and ϵ with the following property. Suppose that*

$$E \in |kL| \ (k \gg 0)$$

is a hypersurface of degree k, and $V \subset E$ is an irreducible component of both $Z_\sigma(E)$ and $Z_{\sigma+\epsilon}(E)$ for some $\sigma > 0$. Then

$$\deg(V) \leq C.$$

One may view Proposition 1.3 as the "one-factor" case of the Faltings Product Theorem [Falt], Theorem 3.1, and we will give the elementary proof shortly.

Now return to an arbitrary smooth projective variety X, and imagine – which unfortunately is not true – that one had an analogous statement for the jumping locus V associated to a divisor $E \in |kL|$ on X. Consider as before a divisor E with index $> \alpha$ a given point $x \in X$. Then if E does not have an almost isolated singularity at x, the gap argument and the imaginary extension of Proposition 1.3 would yield a positive dimensional subvariety $V \subset X$ of bounded degree with respect to L. On the other hand, since we are interested in very positive line bundles L — e.g. $L = mA$ for A ample and large m — we are free to assume that there are no subvarieties of small degree with respect to L. So we would have a mechanism to guarantee the presence of an almost isolated singularity, and we would be done!

In a word, the basic strategy will be to develop an appropriate extension of Proposition 1.3. In order to understand the issues involved, let us recall the

Proof of Proposition 1.3. Let $c = \operatorname{codim}(V, \mathbf{P}^n)$, and denote by

$$F \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k))$$

the equation defining E. Then $Z_\sigma(E)$ is cut out by homogeneous polynomials of degrees $\leq k$, to wit all the partial derivatives $\{D(F)\}$ of F of order less than σk in the homogeneous coordinates on $\mathbf{P}^n$. Therefore V is generically cut out by the $\{D(F)\}$. On the other hand, being also a component of $Z_{\sigma+\epsilon}(E)$, V is contained in the common zeroes of all the partial derivatives $\{D'(F)\}$ for D' of order $< (\sigma+\epsilon)k$. In particular, each of the $[\sigma k]^{\text{th}}$ partials $D(F)$ must vanish to order $\geq [\epsilon k]$ along V. In other words, V is an irreducible component of the common zeroes of a collection of hypersurfaces of degrees $\leq k$, each of which vanishes to order $\geq k\alpha$ along V, where $\alpha = \frac{[k\epsilon]}{k} \sim \epsilon$. Lemma 1.4 below then shows that

$$(k\alpha)^c \cdot \deg(V) \leq k^c.$$

The Proposition follows. $\square$

Lemma 1.4. [Compare [Falt], Thm.3.1 and [Fult],12.4] *Let X be a smooth projective variety of dimension n, and let M be an ample line bundle on X. Assume given a collection*

$$s_1, \ldots, s_N \in \Gamma(X, M)$$

of sections of M whose common zeroes contain a codimension c subvariety $V \subset X$ as an irreducible component. Suppose moreover that each s_i vanishes to order $\geq a$ along V. Then for any ample line bundle L,

$$\deg_L(V) \leq \frac{(M^c \cdot L^{n-c})}{a^c}.$$

Proof. After possibly replacing each s_i by a linear combination of the sections in question, we can assume that V is an irreducible component of the zero locus $Z = \{s_1 = s_2 = \ldots = s_c = 0\}$. Let $E_i \subset X$ be the divisor of s_i. According to the general theory of [Fult], the intersection product $E_1 \cdot E_2 \cdot \ldots \cdot E_c = (M^c) \in A^c(X)$ can be decomposed into a sum of classes $\gamma_j \in A^c(X)$ coming from various "distinguished subvarieties" of $E_1 \cap \cdots \cap E_c$ determined by intersecting the regular embedding

$$E_1 \times \cdots \times E_c \subset X \times \cdots \times X$$

with the small diagonal $X \subset X \times \cdots \times X$. Since the E_i intersect properly along V, V itself appears as one of these components, and since $\mathrm{mult}_V(E_i) \geq a$, the cycle $[V]$ occurs with coefficient $\geq a^c$ in the decomposition (cf. [Fult], (12.4.8)). Hence

$$(L^{n-c} \cdot M^c) = a^c \cdot \deg_L(V) + \sum{}' \deg_L(\gamma_j),$$

where the sum on the right is taken over all the distinguished components other than V. Therefore it is enough to show that $\deg_L(\gamma_j) \geq 0$. But this follows from [FL] (cf. [Fult], (12.2)) since the restriction to X of the normal bundle $N_{E_1 \times \cdots \times E_c / X \times \cdots \times X}$ is ample, and hence nef. $\square$

Remark. The Lemma and proof remain valid assuming only that M is nef. Faltings [Falt] (Proposition 2.3) gives an elmentary proof in the case $X = \mathbf{P}^n$ which avoids most of the intersection theory quoted here.

Looking over the proof of Proposition 1.3, one sees that the first essential point about working on $\mathbf{P}^n$ is that one is able to differentiate homogeneous polynomials. In fact, a similar argument works on an arbitrary smooth projective variety X provided that one controls the positivity of the tangent bundle TX of X. The next section is devoted to this rather direct extension of Proposition 1.3 to general X. However the resulting statement (Theorem 2.1) contains a factor of $\int_X c_1(L)^n$ on the right hand side, so one does not directly get an absolute upper bound on the degree of the multiplicity-jumping locus V. The most interesting part of the approach — and this is where Demailly's argument becomes essentially analytic — starts with the observation that the number of degrees of freedom in choosing a divisor E satisfying (1.1.1) grows with $\int_X c_1(L)^n$. So for the application to adjoint linear series, one will want to apply (1.3) or (2.1) to the general divisor E in a rather large linear series. Motivated by some hints in [De1], (11.1), we develop in §3 some machinery for obtaining in effect a bound on the degree of the jumping locus V that takes into account the dimension of the linear series in which E moves. We give the application of this "zero-estimate with moving parts" (Theorem 3.9) in the spirit of [De1] to adjoint series in §4.

We conclude this section with a formulation and proof of the "gap" lemma that was used above:

Lemma 1.5. *Let X be a smooth irreducible variety of dimension n, and let $E \in |kL|$ be a divisor on X with $\operatorname{ind}_x(E) > \alpha$ for some $x \in X$. Choose rational numbers*

$$0 = \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq \beta_{n+1} = \alpha,$$

and set

$$Z_0 = X, \quad Z_j = Z_{\beta_j}(E) = \left\{ x \in E \;\middle|\; \operatorname{ind}_x(E) \geq \beta_j \right\} \quad (1 \leq j \leq n+1),$$

so that $Z_1 = E$. Then for at least one index $1 \leq i \leq n$, there exists a subvariety $V \subset X$ passing through x which is an irreducible component of both Z_i and Z_{i+1}.

Thus $\mathrm{ind}_y(E) \geq \beta_{i+1}$ for every $y \in V$, while there exists an open set $U \subset X$ meeting V such that $\mathrm{ind}_w(E) < \beta_i$ for all $w \in U - V$. In other words, the index of E "jumps" by at least $(\beta_{i+1} - \beta_i)$ along V. For example, assuming $\alpha > 1$ and taking

$$\beta_{n+1} = \alpha,\ \beta_n = \left(\frac{n-1}{n}\right),\ \ldots,\ \beta_2 = \left(\frac{1}{n}\right),\ \beta_1 = 0,$$

we deduce the fact stated above that if x is not an almost isolated singularity of E of index $> \alpha$ then for some σ, $Z_\sigma(E)$ and $Z_{\sigma+(1/n)}(E)$ share an irreducible component of positive dimension.

Proof of Lemma 1.5. The sets Z_j lie in a chain

$$Z_{n+1} \ \subseteq\ Z_n \ \subseteq \ldots \subseteq Z_1 \ \subseteq Z_0 \ =\ X.$$

Starting with Z_{n+1} and working up in dimension, we can choose irreducible components V_j of Z_j passing through x such that $V_{j+1} \subseteq V_j$. So we arrive at a chain of irreducible varieties:

$$V_{n+1} \ \subseteq\ V_n \ \subseteq \ldots \subseteq V_1 \ \subseteq V_0 \ =\ X. \tag{1.5.1}$$

But since X is irreducible of dimension n, at least two consecutive links in the chain must coincide, say $V_i = V_{i+1}$, and we take $V = V_i$. $\square$

For later use, we record finally a variant:

Lemma 1.6. *In the situation of Lemma 1.5, there exists an index c such that Z_c and Z_{c+1} share an irreducible component V having codimension c in X.*

Sketch of Proof. Consider the chain (1.5.1) constructed in the proof of (1.5). It is enough to prove the existence of an index c such that $V_c = V_{c+1}$ has codimension c, and this is a purely combinatorial fact. In brief, let k be the largest index such that $V_k = V_{k+1}$. Then V_k has dimension $\geq n - k$, and if equality holds we can choose $c = k$. If $\dim V_k > n - k$, then in the chain

$$V_k \ =\ V_{k+1} \ \subseteq\ \ldots \ \subseteq V_1 \ \subseteq V_0 = X$$

there are again two consecutive members that concide, and we can apply a suitable inductive statement. We leave details to the reader. $\square$

§2. Derivatives and the Self-Intersection Inequality.

By way of warm-up for §3, we prove in this section a simple special case of Demailly's self-intersection inequality [De1], (10.7), (10.8), [De3]. The result in question gives a first generalization of Proposition 1.3 to arbitrary smooth projective varieties, and — while elementary — it is already interesting from the standpoint of enumerative geometry.[6] Our impression is that even this clasically oriented part of Demailly's theorem isn't well known in the algebro-geometric community, so we felt it worthwhile to write out a proof here (which differs only in language from [De1]). The argument follows closely the model of Proposition 1.3, the main new point being that now one has to keep track of the positivity of the tangent bundle in order to make sense of differentiating sections of a line bundle. It is natural to expect that one should also be able to give a proof of Theorem 2.1 based on the general machinery of contemporary intersection theory. This leads us to formulate a conjecture of a local nature concerning normal cones associated to hypersurfaces. We present this conjecture at the end of the section, and we hope it may be of some interest to intersection theorists.

We start with some formalism concerning vector bundles and $\mathbf{Q}$-divisors. Let E be a vector bundle on a smooth projective variety X, and let A be a $\mathbf{Q}$-divisor on X, or more generally an element of $\mathrm{Pic}(X) \otimes \mathbf{Q}$. As in [Myk], it will be convenient to deal with formal expressions of the form $E(A)$, which we think of as the "$\mathbf{Q}$-vector bundle" resulting from twisting E by A. We will want to discuss in particular the positivity of such objects, and this reduces in the usual way to statements about positivity of $\mathbf{Q}$-divisors (whose meaning is clear). Specifically, we say that $E(A)$ is *ample* if the $\mathbf{Q}$-divisor class

$$c_1(\mathcal{O}_{\mathbf{P}(E)})(1) + \pi^* A \in NS(\mathbf{P}(E)) \otimes \mathbf{Q}$$

is ample on the projectivization $\pi : \mathbf{P}(E) \longrightarrow X$. Most of the elementary formal properties of ample vector bundles hold in this context. For example, if $E(A)$ is ample, then so too is $Sym^{\ell}(E)(\ell A)$. Or again, given an exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ of vector bundles, if $E'(A)$ and $E''(A)$ are ample, then $E(A)$ is ample. These (and similar asertions) are most easily proved by taking a finite flat covering $f : Y \longrightarrow X$ such that f^*A is (numerically equivalent to) a genuine line bundle, and reducing to the analogous statements for ample vector bundles on Y. Nefness of $E(A)$ is defined similarly, and in fact for nefness there is no difficulty in allowing $A \in \mathrm{Pic}(X) \otimes \mathbf{R}$.

Our goal now is to prove the following version of Demailly's self-intersection inequality. We keep the notation of §1 concerning the index of a divisor and the corresponding multiplicity loci.

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[6] We stress however that the statement here ignores the most interesting and subtle part of Demailly's inequality, to wit the presence of a contribution from the absolutely continuous part T_{abc} of the current with which he works. The purpose of §3 will be to develop an algebro-geometric analogue of this more delicate result.

Theorem 2.1. ([De1], Cor. 10.8) *Let X be a smooth projective variety of dimension n, let L be an ample line bundle on X, and assume that $\delta \geq 0$ is a real number such that the twisted tangent bundle $TX(\delta L)$ is nef. Given rational numbers $\epsilon, \sigma > 0$, suppose that*

$$E \in |kL|$$

is a divisor containing a subvariety V, of codimension c in X, such that V is an irreducible component of both $Z_\sigma(E)$ and $Z_{\sigma+\epsilon}(E)$. Then

$$\deg_L(V) \leq \frac{(1+\sigma\delta)^c}{\epsilon^c} \int_X c_1(L)^n.$$

Observe that when $X = \mathbf{P}^n$ one can take $\delta = 0$, and the statement reduces to Proposition 1.3. As in the proof of that Proposition, the essential point is to study the equations defining various multiplicity loci of E. So we start with some remarks about differentiating sections of a line bundle.

Let B be any line bundle on the smooth projective variety X. Recall that there are so-called bundles of principal parts P^ℓ_B associated to B: the fibre of P^ℓ_B at a given point $x \in X$ is the space $B \otimes \mathcal{O}_X / \mathcal{I}_x^{\ell+1}$ of ℓ-jets at x of sections of B. These jet bundles are connected by exact sequences:

$$0 \longrightarrow \mathrm{Sym}^\ell(\Omega^1_X) \otimes B \longrightarrow P^\ell_B \longrightarrow P^{\ell-1}_B \longrightarrow 0 \tag{2.2}$$

and a basic fact is that any section $\phi \in \Gamma(X, B)$ lifts to a section $j_\ell(\phi) \in \Gamma(X, P^\ell_B)$. The vector bundle

$$\mathcal{D}^\ell_B = Hom(P^\ell_B, B) = \check{P}^\ell_B \otimes B$$

is then the sheaf of differential operators of order $\leq \ell$ on B. They sit in exact sequences

$$0 \longrightarrow \mathcal{D}^{\ell-1}_B \longrightarrow \mathcal{D}^\ell_B \longrightarrow Sym^\ell(TX) \longrightarrow 0 \tag{2.3}$$

dual to (2.2). A section $\phi \in \Gamma(X, B)$ determines a homomorphism of vector bundles

$$d_\ell(\phi) : \mathcal{D}^\ell_B \longrightarrow B$$

defined for example via the transpose of $j_\ell(\phi)$. Locally, representing ϕ by a function f, $d_\ell(\phi)$ is just the map which takes a differential operator D to the function $D(f)$. It follows that $d_\ell(\phi)$ is zero precisely at the locus where ϕ vanishes to order $> \ell$. More precisely, let E be the divisor of ϕ, and consider the multiplicity locus

$$\Sigma_\ell = \Sigma_\ell(E) = \left\{ x \in X \;\middle|\; \mathrm{mult}_x(E) > \ell \right\},$$

with its natural scheme structure. Then the image of $d_\ell(\phi)$ is just the ideal sheaf of this scheme, i.e. one has a surjective sheaf homomorphism:

$$\mathcal{D}^\ell_B \longrightarrow B \otimes \mathcal{I}_{\Sigma_\ell} \tag{2.4.}$$

Now suppose that $A \in \mathrm{Pic}(X) \otimes \mathbf{Q}$ is an ample $\mathbf{Q}$-line bundle such that the $\mathbf{Q}$-vector bundle $TX(A)$ is ample.

Lemma 2.5. *If ℓ is any sufficiently large and divisible integer such that ℓA is a genuine line bundle (i.e. integral), then*

$$\mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A)$$

is globally generated for all line bundles B.

Proof. Let H be a fixed very ample divisor on X. We shall show that if ℓ is any sufficiently large and divisble integer $\mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A)$ is a 0-regular sheaf with respect to $\mathcal{O}_X(H)$. It is well known that such a 0-regular sheaf is generated by its global sections. First we will show that there is an integer ℓ_0 such that for all integers $\ell \geq \ell_0$, the sheaves $\mathrm{Sym}^\ell TX \otimes \mathcal{O}_X(kA)$ are 0-regular with respect to H, for all k such that $k \geq \ell$ and kA is a genuine line bundle. This means we will need to show that

$$H^i(Sym^\ell TX \otimes \mathcal{O}_X(kA) \otimes \mathcal{O}_X(-iH)) = 0, \text{for all } i > 0.$$

Let $\pi : \mathbf{P}(TX) \longrightarrow X$ be the projectivization of TX. Then we observe that

(2.6)

$$H^i(\mathrm{Sym}^\ell TX \otimes \mathcal{O}_X(kA - iH)) = H^i(K_{\mathbf{P}(TX)} \otimes \mathcal{O}_{\mathbf{P}(TX)}(\ell) \otimes \pi^* \mathcal{O}_X(kA - iH) \otimes K^{-1}_{\mathbf{P}(TX)}).$$

Since $\mathcal{O}_{\mathbf{P}(TX)}(1) \otimes \pi^* A$ is ample, we see that for sufficiently large ℓ_0 the cohomology groups in (2.6) vanish if $\ell \geq \ell_0$, $k \geq \ell$ and kA is a geniune line bundle by Kodaira's theorem. By Serre vanishing theorem, we can find an integer ℓ_1 such that for all $k \geq \ell_1$ the sheaves $\mathrm{Sym}^j TX \otimes \mathcal{O}_X(kA)$ are 0-regular when kA is a genuine line bundle, and $j = 0, 1, ..., \ell_0$. By (2.3), we see that $\mathcal{D}_B^\ell \otimes \mathcal{O}_X(kA)$ is 0-regular if $\ell \geq Max(\ell_0, \ell_1)$ and $k \geq \ell$. $\square$

Suppose now given as above a divisor $E \in |B|$. Then for sufficiently large and divisible ℓ one has a surjective map of sheaves

$$\mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A) \longrightarrow \mathcal{O}_X(B + \ell A) \otimes \mathcal{I}_{\Sigma_\ell}.$$

Therefore $\mathcal{I}_{\Sigma_\ell}(B + \ell A)$ is generated by its global sections. We will refer to sections in the image of

$$H^0(X, \mathcal{D}_B^\ell \otimes \mathcal{O}_X(\ell A)) \longrightarrow H^0(X, \mathcal{O}_X(B + \ell A))$$

as *differential sections* of order ℓ since they arise by a process of differentiation. For instance suppose $X = \mathbf{P}^n$, $L = \mathcal{O}_{\mathbf{P}^n}(1)$ and $B = \mathcal{O}_{\mathbf{P}^n}(k)$. Then $\mathcal{D}_B^\ell = \oplus \mathcal{O}_{\mathbf{P}^n}(\ell)$. We can take $\delta = -1$ and then the differential sections are exactly the partial derivatives of order ℓ of the homogeneous polynomial defining E.

Now we give the

Proof of Theorem 2.1. We can increase δ slightly and assume that $\delta \in \mathbf{Q}$, and that $TX(\delta L)$ is ample, for then the general statement of the Theorem follows by passing to the limit in δ. Similarly, we may decrease ϵ slightly, so we may assume that $\mathrm{mult}_V(E) > k(\sigma+\epsilon)$. Note also that it suffices to prove the theorem for $aE \in |akL|$ for any positive $a \in \mathbf{Z}$. Set

$$\ell =_{\mathrm{def}} \sigma k \quad , \quad m =_{\mathrm{def}} \epsilon k.$$

Replacing k by ak where a is suffiently divisble, we may assume that ℓ, m and $\ell\delta$ are all positive integers. Futhermore, by (2.5) we may assume that $\mathcal{D}^{\ell}_{kL} \otimes \mathcal{O}_X(\ell\delta L)$ is generated by its sections. The hypothesis then is that V is a common irreducible component of the two multiplicity loci $\Sigma_{\ell-1} = \Sigma_{\ell-1}(E)$ and $\Sigma_{m+\ell} = \Sigma_{m+\ell}(E)$.

We claim first that

$$(*) \qquad \mathcal{I}_{\Sigma_l} \subseteq \mathcal{I}_V^{(m)},$$

where $\mathcal{I}_V^{(m)}$ denotes the symbolic power sheaf of all functions vanishing to order $\geq m$ on V. This follows from the hypothesis on V just as in (1.3). In fact, let f be a local equation for E on some open set U. Then $\mathcal{I}_{\Sigma_l}$ is locally generated by all the functions $\{D(f)\}$ for $D \in \mathcal{D}^{\ell}(U)$ a differential operator of order $\leq \ell$. On the other hand, since V is an irreducible component of $\Sigma_{\ell+m}$, $D'(F) \in \mathcal{I}_V(U)$ for every $D' \in \mathcal{D}^{\ell+m}(U)$. Hence all the $D(f)$ $(D \in \mathcal{D}^{\ell})$ vanish to order $\geq m$ on V, as asserted.

We now apply the discussion following the statement of Theorem 2.1 with $B = kL$ and $A = \delta L$. The sheaf I_{Σ_ℓ} is generated by finitely many differential sections of order ℓ:

$$s_1, \ \ldots \ , s_N \in \Gamma(X, \mathcal{O}_X(kL + \ell\delta L)).$$

In particular, the s_i cut out Σ_ℓ as a set, and hence contain V as an irreducible component of their common zeroes. On the other hand, by (*), the s_i vanish to order $\geq m$ on V. Then Lemma 1.4 implies that

$$\begin{aligned} \deg_L(V) &\leq \frac{((k+\ell\delta)L)^c}{(m)^c} \cdot L^{n-c} \\ &= \frac{(k+\ell\delta)^c}{m^c} \int_X c_1(L)^n \\ &= \frac{(1+\sigma\delta)^c}{\epsilon^c} \int_X c_1(L)^n. \end{aligned}$$

The theorem follows. $\square$

Remark 2.7. F. Angelini has verified that a similar argument proves the following more precise statement of [De1], Cor. 10.8. Given a divisor $E \in |kL|$ and a subvariety $V \subset X$, define the *jumping values* $b_1, \ldots, b_{n+1}$ of E with respect to V to be the integers

$$b_p = b_p(E) = \min\left\{\mu > 0 \;\middle|\; \mathrm{codim}_x(\Sigma_\mu(E), X) \geq p \ \forall x \in V\right\}.$$

Thus

$$0 = b_1 \leq b_2 \leq \ldots \leq b_{n+1} = \max_{x \in V} \ \mathrm{mult}_x(E).$$

For each $1 \leq p \leq n$ let $\{T_{p,\alpha}\}$ be the irreducible components of $\Sigma_{b_p}(E)$ meeting V, and denote by $\nu_{p,\alpha}$ the generic multiplicity of E along $T_{p,\alpha}$. Then

$$\text{(2.7.1)} \quad \sum_{\alpha=1}^{\infty} (\nu_{p,\alpha} - b_1) \cdot \ldots \cdot (\nu_{p,\alpha} - b_p) \cdot \deg_L(T_{p,\alpha}) \leq (k + b_1\delta) \cdot \ldots \cdot (k + b_p\delta) \cdot \int_X c_1(L)^n.$$

The idea, roughly speaking, is to argue as above but using differential sections of orders $b_1, \ldots, b_p$.

As a particularly interesting example, due also to Angelini, let $(X, \mathcal{O}_X(\Theta))$ be a principally polarized abelian variety of dimenion n, with $\Theta \subset X$ the theta divisor on X. Kollár has shown that $\mathrm{codim}\Sigma_p(\Theta) \geq p+1$, and so in particular Θ has no $(n+1)$-fold points. Suppose that Θ has an n-fold point at x, and consider the jumping numbers with respect to $\{x\}$. By Kollár's theorem,

$$b_i \leq i - 1 \quad (1 \leq i \leq n).$$

Applying (2.7.1) with $p = n$ we find:

$$(n - b_1) \cdot \ldots \cdot (n - b_n) \leq \int_X c_1(L)^n = n!.$$

Therefore $b_i = i-1$ for all i. Let $\{T_\alpha\}$ be the irreducible components of Σ_{n-2}. Applying (2.7.1) with $p = n-1$ then gives $\sum \deg T_\alpha \leq n$. If $\cup T_\alpha$ generates X, then it follows from the Matsusaka-Ran criterion that X is a Jacobian, in which case it is easy to see that in fact $(X, \mathcal{O}_X(\Theta))$ must split as a product of elliptic curves. This proves for example that if X is simple, then Θ can have no n-fold points. By a similar sort of argument, Smith and Varley [SV] have recently established that the presence of an n-fold point implies in general that $(X, \mathcal{O}_X(\Theta))$ is a product of elliptic curves, the additional point being to show that Σ_{n-2} generates X.

We conclude this section with a conjecture suggesting how Theorem 2.1 should fit into the framework of contemporary intersection theory, as in [Fult]. Let X be any smooth irreducible variety of dimension n, and let $E \subset X$ be any effective divisor on X. Suppose that $V \subset X$ is an irreducible subvariety of codimension c in X which is simultaneously an irreducible component of the two multiplicity loci $\Sigma_\ell(E)$ and $\Sigma_{\ell+m}(E)$. Our conjecture asserts roughly speaking that $[V]$ always occurs with multiplicity at least m^c as a distinguished component of the Fulton-MacPherson c-fold self-intersection class of E in X.

More precisely, consider the c-fold products

$$E^c = E \times \cdots \times E \subset X \times \cdots \times X = X^c.$$

Let $C = NC(E, E^c)$ denote the (affine) normal cone to the diagonal embedding $E \subset E^c$, so that C has pure dimension $c(n-1)$. Similarly, let $T = N(X, X^c) \cong \oplus^{c-1} TX$ be the normal bundle to the diagonal $X \subset X^c$, with projection map $p : T \longrightarrow X$. Then $C \subset T$, and the Fulton-MacPherson class $E \cdot \ldots \cdot E$ (c times) is obtained roughly speaking by intersecting $[C]$ with the zero-section in T.

Conjecture 2.8. *The inverse image $p^{-1}(V)$ is an irreducible component of C, and moreover $[p^{-1}(V)]$ appears with coefficient $\geq m^c$ in the fundamental class $[C]$.*

One may view this as a local analogue of Theorem 2.1. The global statement should then follow by using the results of [FL] to control the other contributions to $E \cdot \ldots \cdot E$ in terms of the nefness of the vector bundle T. One expects a similar statement in the setting of (2.7), with the multiplicity of $[p^{-1}V]$ being estimated in the spirit of (2.7.1).

§3. Graded Linear Series.

This section contains the main technical result of the paper, the self-intersection theorem with moving parts (Theorem 3.9). This is an algebro-geometric analogue of the corresponding inequality of Demailly. Although the techniques employed in [De1] are largely analytic, the basic idea here is suggested by Demailly in the remark following [De1] Corollary 10.8. The reader is also encouraged to compare this section with the proof of Dyson's Lemma by Esnault and Viehweg [EV1]. Although Esnault and Viehweg ignore the moving part in the proof of their main theorem, they indicate how to improve Dyson's Lemma by accounting for the moving part in [EV1] §10. The theorem of Esnault and Viehweg is revisited from this point of view in [Nak].

As suggested by the overview in §1, the main theorem will involve an "asymptotic" analysis of a linear system defined by imposing an index condition on a complete linear series $|kL|$ for $k \gg 0$. We start by introducing some formalism designed to facilitate the discussion of such linear series.

Let X be a smooth projective variety and let L be an ample line bundle on X. Consider the graded algebra

$$R = \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_X(kL)).$$

Suppose that

$$A \subseteq R$$

is a graded subalgebra; we will call A a *graded linear series*. For technical reasons, we will often assume that A_k is nonzero for all sufficiently large k.

Example 3.1. (a). Let B be a big divisor such that $L(-B)$ is effective. Then $\bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_X(kB))$ is a graded linear series.

(b). Let V be a subvariety of X and α a positive real number. Let

$$A_k = \left\{ s \in H^0(\mathcal{O}_X(kL)) \mid \operatorname{mult}_x(s) \geq k\alpha \text{ for all } x \in V \right\}$$

and define

$$A^{\alpha}(V, X) = \bigoplus_{k=0}^{\infty} A_k.$$

One checks that $A^{\alpha}(V, X)$ is a graded linear series and it is the primary example which will appear in the sequel.

(c). Suppose $Y \subset X$ is a smooth subvariety and let

$$R_Y = \bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_Y(kL)).$$

Given a graded linear series $A \subset R$ on X, let A_Y denote the restriction of A to Y. Then it is easy to verify that $A_Y \subset R_Y$ is a graded linear series.

Definition 3.2. Let A be a graded linear series and let

$$a(k) = \dim(A_k) \text{ for } k \geq 0.$$

Define the degree of A by

$$\rho(A) = \limsup_{k \to \infty} \frac{a(k) n!}{k^n}.$$

The invariant $\rho(A)$ measures the moving part of the graded linear series A or equivalently the degree of freedom in choosing a divisor $E_k \in |A_k|$ for $k \gg 0$.

Example 3.3. (a). $\rho(R) = c_1(L)^n$

(b). Let x be a point of X. Then

$$\rho(A^\alpha(x,X)) \geq c_1(L)^n - \alpha^n,$$

as one verifies by counting the number of linear conditions imposed by requiring multplicity $\geq k\alpha$ at x.

The next two lemmas demonstrate that the invariant ρ behaves well under such operations as taking a Veronese subalgebra or restricting the graded linear series to a hyperplane.

Lemma 3.4. *Let $A \subset R$ be a graded linear series such that $A_k \neq 0$ for all $k \gg 0$. Fix a positive integer d and consider the Veronese subalgebra: $V_d = \bigoplus_{k=0}^\infty A_{kd} \subset \bigoplus_{k=0}^\infty H^0(\mathcal{O}_X(kdL))$. Then*

$$\rho(V_d) = d^n \rho(A). \tag{3.4.1}$$

Proof. It is clear from Definition 3.2 that $\rho(V_d) \leq d^n\rho(A)$. Let $V_d(k)$ denote the k^{th} graded piece of V_d and let

$$v_d(k) = \dim\ V_d(k).$$

Write $\rho = \rho(A)$. We will prove Lemma 3.4 by showing that for any $\epsilon > 0$ there exists a sequence of integers k_i such that $k_i \to \infty$ and

$$v_d(k_i) \geq \frac{(k_i d)^n}{n!}(\rho(1-\epsilon)) \text{ for all } i.$$

Fix a large positive integer c such that $a(m) > 0$ for all $m \geq c$. It follows from the definition of ρ that there exists an infinite increasing sequence of integers $\{m_i\}$ such that

$$\frac{a(m_i)n!}{m_i^n} \geq \rho\left(1 - \frac{\epsilon}{2}\right) \text{ for all } i.$$

Replacing $\{m_i\}$ by a subsequence if necesary, we may also assume that all the m_i's are sufficiently large so that

$$\left(\frac{m_i}{m_i + c + d}\right)^n \geq 1 - \frac{\epsilon}{2}.$$

Next observe that there is a unique integer b_i such that $m_i + c < b_i \leq m_i + c + d$ and b_i is divisible by d. We will write b_i as $k_i d$. Since $k_i d - m_i > c$ it follows that there is a nonzero section $\sigma \in A_{k_i d - m_i}$. Multiplication by σ gives an injection $A_{m_i} \hookrightarrow A_{k_i d} = V_d(k_i)$

and consequently $v_d(k_i) \geq a(m_i)$. Thus

$$\begin{aligned} \frac{v_d(k_i)n!}{(k_i d)^n} &\geq \frac{a(m_i)n!}{m_i^n} \frac{m_i^n}{(k_i d)^n} \\ &\geq \rho \left(1 - \frac{\epsilon}{2}\right) \left(1 - \frac{\epsilon}{2}\right) \\ &\geq \rho(1-\epsilon). \end{aligned}$$

This completes the proof of Lemma 3.4. $\square$

Next we investigate the behaviour of ρ under restriction to hyperplanes. For this we focus only on graded linear series of the form $A^\alpha(V, X)$.

Lemma 3.5. *Let $A = A^\alpha(V, X)$ be the graded linear series defined in 3.1(b). Suppose that $|m_0 L|$ is free and choose a general divisor $H \in |m_0 L|$. Set $W = V \cap H$. Then*

$$\rho(A^\alpha(W, H)) \geq m_0 \rho(A).$$

Proof. As above, put

$$R_H = \bigoplus_k H^0(\mathcal{O}_H(kL)).$$

Let $B = B_H \subset R_H$ be the restriction of A to H (cf. Example 3.1 (c)). There is a natural surjective map $A \longrightarrow B$ and we denote the kernel by C. Thus for all $k \geq 0$ there is an exact sequence

$$0 \longrightarrow C_k \longrightarrow A_k \longrightarrow B_k \longrightarrow 0.$$

Let $a(k) = \dim A_k$, $b(k) = \dim B_k$, and $c(k) = \dim C_k$. Observe that $B \subset A^\alpha(W, H)$. In order to prove Lemma 3.5, it will be sufficent to show that

$$\rho(B) = \limsup_{k\to\infty} \frac{b(k)(n-1)!}{k^{n-1}} \geq m_0 \rho(A). \tag{3.5.1}$$

Let s_0 be the section of $m_0 L$ defining H. It follows from the definition of C_k that every $t \in C_k$ can be expressed as $t = s_0 s'$ with $s' \in H^0(X, \mathcal{O}_X((k - m_0)L))$. Furthermore, since H is general, we can assume that it does not contain any irreducible component of V and hence

$$\mathrm{mult}_x(s') \geq k\alpha \quad \text{for all} \quad x \in V.$$

Let
$$C'_k = \{s \in H^0(X, \mathcal{O}_X((k-m_0)L)) \mid \mathrm{mult}_x(s) \geq k\alpha \text{ for all } \ x \in V\}.$$
The above remarks show that there is an injection $C_k \hookrightarrow C'_k$. Since $C'_k \subset A_{k-m_0}$ it follows that $c(k) \leq a(k-m_0)$ and hence $b(k) \geq a(k) - a(k-m_0)$ for all $k \geq 0$. In particular, when k is divisible by m_0, this yields
$$b(tm_0) \geq a(tm_0) - a((t-1)m_0).$$
Summing the above inequalities, we obtain
$$\sum_{t=0}^{N} b(tm_0) \geq a(Nm_0).$$
By Lemma 3.4
$$\limsup_{N\to\infty} \left(\frac{a(Nm_0)n!}{(Nm_0)^n}\right) = \rho(A).$$
Hence Lemma 3.5 follows from the following simple lemma on partial sums:

Lemma 3.6. *Let $g(k)$ be a sequence of real numbers. Let $f(N) = \sum_{j=0}^{N} g(j)$ and suppose that $\limsup_k (f(k)/k^n) = \rho$. Then $\limsup_k \left(g(k)/k^{n-1}\right) \geq n\rho$.*

Proof. Suppose Lemma 3.6 is false. Then there exists a real number $\epsilon > 0$ and a positive integer $N_0 > 0$ such that $g(j)/j^{n-1} \leq n\rho - \epsilon$ for all $j \geq N_0$. Then for $k \gg 0$
$$f(k) = \sum_{j=0}^{k} g(j) \leq (n\rho - \epsilon)\frac{k^n}{n} + o(k^{n-1})$$
which contradicts our assumption on $f(k)$. □

Applying Lemma 3.5 inductively gives

Corollary 3.7. *Let $V \subset X$ and α be as in Lemma 3.5. Suppose that $|m_0L|$ is free and that Λ is a complete intersection of $n-c$ general divisors in $|m_0L|$. Let $S = \Lambda \cap V$. Then*
$$\rho(A^\alpha(S,\Lambda)) \geq m_0^{n-c}\rho(A^\alpha(V,X)). \quad \square$$

Let
$$A = \bigoplus_{k=0}^{\infty} A_k \subseteq \bigoplus_{k=0}^{\infty} H^0(X, kL)$$

be a graded linear series and choose for each k a general divisor $E_k \in |A_k|$. For any rational number $\sigma > 0$ define

$$Z_\sigma(k) = \{x \in X \mid \mathrm{mult}_x(E_k) \geq k\sigma\}.$$

The following lemma shows that these loci stabilize for $k \gg 0$.

Lemma 3.8. *There is a positive integer k_0 such that*

$$Z_\sigma(k_1) = Z_\sigma(k_2) \textit{ for all } k_1, k_2 \geq k_0.$$

Proof of Lemma 3.8. As σ is fixed in Lemma 3.8, we will write $Z(k)$ for $Z_\sigma(k)$ in order to alleviate notation. Suppose $a \geq 2$ is a positive integer. We claim that there exists a positive integer $k(a)$ such that

$$Z(c) \subseteq Z(a) \quad \text{whenever} \quad c \geq k(a). \tag{3.8.1}$$

To prove the claim, suppose that $x \notin Z(a)$ so that there exists $\eta > 0$ satisfying

$$\mathrm{mult}_x(E_a) \leq a\sigma - \eta.$$

Note that since the index is a discrete invariant, η is bounded below independly of x; in fact if $m\sigma \in \mathbf{Z}$ then $\eta \geq 1/m$. Suppose b is a positive integer relatively prime to a. Then any integer $c \geq ab$ can be expressed as

$$c = \alpha a + \beta b, \quad \alpha, \beta \in \mathbf{Z} \text{ and } 0 \leq \beta \leq a.$$

Consider the divisor

$$F_c = \alpha E_a + \beta E_b \in |A_c|.$$

Then

$$\begin{aligned} \mathrm{mult}_x(F_c) &= \alpha \cdot \mathrm{mult}_x(E_a) + \beta \cdot \mathrm{mult}_x(E_b) \\ &\leq \alpha a\sigma - \alpha\eta + \beta \cdot \mathrm{mult}_x(E_b) \\ &\leq c\left(\sigma - \frac{\eta\left(1 - \frac{\beta b}{c}\right)}{a} + \frac{\beta \cdot \mathrm{mult}_x(E_b)}{c}\right). \end{aligned}$$

Since η, β, and b are bounded independent of c, it follows that $\mathrm{mult}_x(F_c) < c\sigma$ for $c \gg 0$. On the other hand, if $E_c \in |A_c|$ is a *general* divisor, then $\mathrm{mult}_x(E_c) \leq \mathrm{mult}_x(F_c)$. Hence $x \notin Z(c)$ for all sufficiently large c as claimed.

If $Z(c) = Z(a)$ for all $c \gg 0$ then we are finished. If not, then by (3.8.1) there exists $a' > 0$ such that $Z(a') \subsetneq Z(a)$. The argument of (3.8.1) can then be repeated with a' in place of a. This process cannot go on indefinitely and this establishes Lemma 3.8. $\square$

Using Lemma 3.8 we can define a subvariety $Z_\sigma(A) \subset X$ by

$$Z_\sigma(A) = Z_\sigma(k) \text{ for } k \gg 0.$$

The main result of this section is the "one factor" product theorem with moving parts:

Theorem 3.9. *Let X be a smooth projective variety of dimension n. Let L be an ample line bundle on X and assume that δ is a non-negative real number such that $TX(\delta L)$ is nef. Let*

$$A \subset \bigoplus H^0(\mathcal{O}_X(kL))$$

be a graded linear series. Suppose that there are rational numbers $\sigma, \epsilon > 0$ and an irreducible subvariety $V \subset X$ of codimension $c < n$ which is an irreducible component of both $Z_\sigma(A)$ and $Z_{\sigma+\epsilon}(A)$. Then

$$\deg_L(V) \leq \left(\frac{1+\sigma\delta}{\epsilon}\right)^c (c_1(L)^n - \rho(A)).$$

Proof of Theorem 3.9. As in the proof of Theorem 2.1, we may assume that $TX(\delta L)$ is ample and δ is rational by replacing δ with a slightly larger rational number. Fix a sufficiently large and divisible integer k_0 such that $k_0\sigma$, $k_0\epsilon$, $k_0\delta\sigma$, and $k_0\epsilon/(1+\sigma\delta)$ are all integers. By Lemma 3.8, we may assume that for all $k \geq k_0$,

$$Z_\sigma(A) = Z_\sigma(A_k),$$
$$Z_{\sigma+\epsilon}(A) = Z_{\sigma+\epsilon}(A_k).$$

By Lemma 2.2, we may also assume that the sheaves $\mathcal{D}_{kL}^{k\sigma} \otimes \mathcal{O}_X(k\delta\sigma L)$ are generated by global sections for all k divisble by k_0. Let $s_k \in A_k$ be a general section and E_k its divisor of zeroes. If $x \notin Z_\sigma(A)$ then $\mathrm{mult}_x(E_k) < k\sigma$. Thus there exists a differential operator

$$D \in H^0\left(\mathcal{D}_{kL}^{k\sigma} \otimes \mathcal{O}_X(k\delta\sigma L)\right)$$

such that the differential section $D(s_k) \in H^0\left(\mathcal{O}_X(k(1+\sigma\delta)L)\right)$ does not vanish at x. Furthermore, since the order of D is $\leq k\sigma$,

$$\mathrm{mult}_V(D(s_k)) \geq k(\sigma+\epsilon) - k\sigma = k\epsilon.$$

Since V is an irreducible component of $Z_\sigma(A)$, V is also an irreducible component of the base locus of the linear system

$$\left|\mathcal{I}_V^{(k\epsilon)} \otimes \mathcal{O}_X(k(1+\delta\sigma)L)\right| \text{ for all } k \text{ divisble by } k_0.$$

Letting $\alpha = \epsilon/(1+\delta\sigma)$, this shows that V is an irreducible component of $|A^\alpha(V,X)_m|$ for all m divisible by $k_0(1+\sigma\delta)$. Lemma 3.10 below then implies that

$$\text{(3.9.1)} \qquad \deg_L(V) \leq \frac{c_1(L)^n - \rho}{\alpha^c} = \left(\frac{1+\sigma\delta}{\epsilon}\right)^c (c_1(L)^n - \rho(A^\alpha(V,X)))\,.$$

But since $\alpha \leq \sigma + \epsilon$ and since $V \subset Z_{\sigma+\epsilon}(A)$ it follows that

$$A_k \subset A^\alpha(V,X)_k \text{ for all } k \text{ divisible by } k_0(1+\sigma\delta).$$

Hence, using Lemma 3.4, $\deg(A^\alpha(V,X)) \geq \rho(A) = \rho$; combining this with 3.9.1 completes the proof of Theorem 3.9. $\square$

Lemma 3.10. *Let $V \subset X$ be an irreducible subvariety of codimension $c < n$ and let α be a positive real number. Let $\rho = \rho(A^\alpha(V,X))$. Suppose that there is a positive integer k_0 such that V is an irreducible component of the base locus of the linear system $|A^\alpha(V,X)_k|$ for all $k \gg 0$ divisible by k_0. Then*

$$\deg_L(V) \leq \left(\frac{1}{\alpha}\right)^c (c_1(L)^n - \rho)$$

Proof.

Fix a positive integer m_0 such that $|m_0L|$ is free of base points. Choose general divisors $D_1 \dots , D_{n-c} \in |m_0L|$ and let

$$\Lambda = \bigcap_{i=1}^{n-c} D_i.$$

Also define

$$S = \Lambda \cap V.$$

Since the divisors D_i are general, S is a finite set of $\deg_{m_0L}(V)$ distinct points. The hypotheses of Lemma 3.10 imply that for all k sufficiently large and divisible there is a divisor $E_k \in H^0(\Lambda, \mathcal{O}(k\Lambda))$ with an almost isolated singularity at every point of S.

The key observation in the proof of Lemma 3.10 is that Theorem 1.1 gives surjectivity statements onto skyscraper sheaves and this provides a means of bounding the size of S, or equivalently, of bounding $\deg_L(V)$; the moving part ρ enters by considering the kernel of these surjective maps.

We will show that

$$\operatorname{card}(S) \leq \frac{m_0^{n-c}}{\alpha^c}(c_1(L)^n - \rho). \tag{3.10.1}$$

It is clear, in light of fact that $\operatorname{card}(S) = \deg_{m_0 L}(V)$, that (3.10.1) implies Lemma 3.10. Choose two positive numbers α_1, α_2 slightly less than α, such that $\alpha_1 < \alpha_2 < \alpha$. Replacing k_0 by a suitably high multiple, we may assume that V is an irreduible component of the base locus of $|A^\alpha(V,X)_d|$ for all d divisible by k_0. Next fix a sufficiently divisible positive integer k such that $k\alpha_1$, $k\alpha_2$, $k\alpha$, $k\alpha_2/k_0\alpha$ and k/k_0 are all integers. In order to apply Theorem 1.1, we also require k to be sufficiently large so that

$$k\left(1 - \frac{\alpha_2}{\alpha}\right) L - K_\Lambda \text{ is ample on } \Lambda \tag{3.10.2}$$

and

$$k(\alpha_2 - \alpha_1) > c. \tag{3.10.3}$$

Choose a general divisor $D \in |A^\alpha(V,X)_{k\alpha_2/\alpha}|$ and let

$$D_\Lambda = D \cap \Lambda.$$

Since V is an irreducible component of the base locus of $|A^\alpha(V,X)_{k\alpha_2/\alpha}|$ and since D is general, it follows that D_Λ has isolated singularities along S. Moreover,

$$\operatorname{mult}_S(D_\Lambda) \geq k\alpha\frac{\alpha_2}{\alpha} = k\alpha_2 > k\alpha_1 + c,$$

where the last inequality follows by (3.10.3). Applying Theorem 1.1 to D_Λ gives

$$H^1\left(\mathcal{I}_S^{k\alpha_1} \otimes \mathcal{O}\left(K_\Lambda + \frac{k\alpha_2}{\alpha}L\right)\right) = 0. \tag{3.10.4}$$

By (3.10.2), $kL - (K_\Lambda + k\alpha_2/\alpha L)$ is ample. This, together with (3.10.4) implies that

$$H^1\left(\mathcal{I}_S^{k\alpha_1} \otimes \mathcal{O}_\Lambda(kL)\right) = 0.$$

Hence the following sequence is exact:
(3.10.5)

$$0 \longrightarrow H^0\left(\mathcal{I}_S^{k\alpha_1} \otimes \mathcal{O}_\Lambda(kL)\right) \longrightarrow H^0(\mathcal{O}_\Lambda(kL)) \longrightarrow H^0\left(\mathcal{O}_\Lambda(kL)/\mathcal{I}_S^{k\alpha_1}\right) \longrightarrow 0.$$

To prove (3.10.1) it remains to compute the dimension of the three terms in (3.10.5). Since $\alpha_1 \leq \alpha$, $\dim(A^{\alpha_1}(S,\Lambda)_k) \geq \dim(A^{\alpha}(S,\Lambda)_k))$. Lemma 3.4 and Lemma 3.7 imply that

$$\limsup_{k_0|k} \dim(A^{\alpha_1}(S,\Lambda)_k)\frac{c!}{k^c} \geq m_0^{n-c}\rho.$$

By the Riemann-Roch theorem,

$$h^0(\mathcal{O}_\Lambda(kL)) = m_0^{n-c}(c_1(L)^n)\frac{k^c}{c!} + O(k^{c-1}).$$

Also

$$h^0\left(\mathcal{O}_\Lambda(kL)/\mathcal{I}_S^{k\alpha_1}\right) = \mathrm{card}(S)\frac{(k\alpha_1)^c}{c!} + O(k^{c-1}).$$

We deduce from (3.10.5), that

$$\mathrm{card}(S) \leq \frac{(m_0)^{n-c}}{(\alpha_1)^c}(c_1(L)^n - \rho).$$

As α_1 approaches α we obtain the inequlaity (3.10.1) and this concludes the proof of Lemma 3.10. □

§4. Application to Adjoint Series.

In this section, following [De1], we address the problem of producing sections of $|K_X + L|$ for a suitably positive line bundle L. The logic of the argument is simple: as outlined in §1, we choose a divisor $E_k \in |kL|$ with very high multiplicity at a point $x \in X$. If E_k has an almost isolated singularity of index $n + s$ at x then Theorem 1.1 guarantees that $|K_X + L|$ separates s-jets at x. If the singularity of E_k at x is not isolated, then E_k has large index along some positive dimensional subvariety V containing x; the techniques developed in §3 will allow us to choose V in such a fashion that there is an absolute upper bound for $\deg_L(V)$. Thus if L is sufficiently positive to rule out the latter case, $|K_X + L|$ must separate s-jets at x.

Theorem 4.1. *Let X be a smooth projective variety of dimension n and fix a point $x \in X$. Let L be an ample line bundle on X satisfying $c_1(L)^n > (n+s)^n$. Suppose that δ is a non-negative real number such that $TX(\delta L)$ is nef. Then either:*

(a) *the linear system $|K_X + L|$ separates s-jets at x, or*

(b) *there is a subvariety V of codimension c containing x such that*

$$\deg_L V \leq (n - 1 + (c-1)\delta)^c(n+s)^n.$$

Proof. Fix a rational number α satisfying

$$(n+s)^n < \alpha^n < c_1(L)^n.$$

Let $A = A^\alpha(x, X)$. Note that $|A^\alpha(x, X)_k| \neq 0$ for all $k \gg 0$ and by Example 3.3 (b), $\rho(A) \geq c_1(L)^n - \alpha^n$. For each $k > 0$ choose a general divisor $E_k \in |A^\alpha(x, X)|$. Let

$$\beta_j = \frac{j-1}{n-1} \;\text{ for } 1 \leq j \leq n$$

and let $\beta_{n+1} = \alpha$. Set $Z_j = Z_{\beta_j}(A)$. By Lemma 1.6, there exists an irreducible subvariety V of codimension c containing x which is a common irreducible component of Z_c and Z_{c+1}. If $c = n$ then for all $k \gg 0$, E_k has an isolated singularity of index $\geq n + s$ at x and consequently Theorem 1.1 implies that $|K_X + L|$ separates s-jets at x. So suppose that $c < n$. Theorem 3.9 applies in this situation with $\sigma = \frac{c-1}{n-1}$, $\epsilon = 1/(n-1)$, and $\rho = c_1(L)^n - \alpha^n$, giving

$$\deg_L V \leq (n - 1 + (c-1)\delta)^c \alpha^n.$$

Letting α approach $(n + s)$ gives Theorem 4.1. $\square$

Theorem 4.1 is similar to the main theorem of [De1] although the upper bound for $\deg_L V$ is worse and there are no β_i's in the statement because (to simplify the exposition) we chose explicit values in the proof. The weaker upper bound on $\deg_L V$ stems from the fact that we passed to hyperplane sections in the proof of Theorem 3.9. This could be avoided by using an interesting theorem of Fujita [Fuj], which would allow one to recover Demailly's statements *in toto*. However the basic idea is not much different.

A negative feature of Theorem 4.1 is the dependence on the positivity of TX; Demailly removes this dependence through an iteration argument which we present here. The argument allows one to prove a very ampleness criterion for line bundles of the form $2K_X + B$ where B is a suitably positive ample line bundle.

Proposition 4.2. *Let X be a smooth projective variety dimension n and let B be an ample line bundle on X. Assume that $A = K_X \otimes B$ is also ample and that for all subvarieties V of codimension c:*

$$\deg_A(V) > (n - 1 + (c-1)(2n+1))^c (n+1)^n, \quad 1 \leq c \leq n-1. \tag{4.2.1}$$

Then $K_X + A = 2K_X + B$ is very ample.

Proof. We will use Theorem 4.1 to show the following inductive statement:

(4.2.2) If $2K_X + 2^k B$ is very ample then $2K_X + 2^{k-1}B$ is also very ample for $k \geq 1$.

Since $2K_X+2^m B$ is very ample for sufficiently large m, (4.2.2) and descending induction on m show that $2K_X + B$ is very ample establishing Proposition 4.2. It remains to show (4.2.2). It is well known that for any very ample line bundle L, $TX(K_X + nL)$ is generated by its global sections. In particular, since $2K_X + 2^k B$ is very ample by assumption, this implies that $TX\left(K_X + n(2K_X + 2^k B)\right)$ is *nef*. Hence we can apply Theorem 4.1 with $L = K_X + 2^{k-1}B$ and $\delta = (2n+1)$. Since $k-1 \geq 0$, (4.2.1) rules out case (b) of Theorem 4.1 and hence $\left|2K_X + 2^{k-1}B\right|$ separates 1-jets at every point $x \in X$. Similarly, one can show that $\left|2K_X + 2^{k-1}B\right|$ separates any two distinct points of X. Thus $2K_X + 2^{k-1}B$ is very ample and (4.4.2) is established.

Recall that $K_X + (n+1)L$ is *nef* for any ample line bundle L on X by a theorem of Mori. Now the hypothesis (4.2.1) of Proposition 4.2 can of course be guaranteed by choosing $B = mL$ for an ample line bundle L and a suitably positive integer m:

Corollary 4.3. *Let X be a smooth projective variety of dimension n and let L be an ample line bundle on X. Then $2K_X + mL$ is very ample for*

$$m > n + 1 + ((2n+2)(n-1))^{n-1}(n+1)^n.$$

Finally we state without proof a variant of Theorem 4.1 which depends on the positivity of K_X rather than the positivity of TX.

Theorem 4.4. *Let X be a smooth projective variety of dimension n and let L be an ample line bundle on X satisfying $c_1(L)^n > (n+s)^n$. Let b be a non-negative number such that $-K_X + bL$ is nef. Suppose that m_0 is a positive integer such that $|m_0 L|$ is free. Then for any point $x \in X$ either*

(a). $|K_X + L|$ *separates s-jets at x or*

(b). *There exists a codimension c subvariety V containing x and satisfying*

$$\deg_L V \leq \left(b + m_0(n-c) + \frac{n!}{c!}\right)^c (n+s)^n.$$

Remark 4.5. (a). We can make theorem 4.4 effective by using Kollár's effective base point free theorem. According to this result, $|m_0 L|$ is base point free for

$$m_0 = 2(n+2)!(n + \ulcorner b \urcorner).$$

(In some preliminary notes on (4.4) we overlooked the necessity to invoke Kollár's theorem at this point, and accordingly claimed a somewhat better result. We apologize for any confusion that may have resulted from this error.)

(b). As Demailly pointed out, one can also deduce Theorem 4.4 from Theorem 4.1 and Corollary 4.3. Under the same assumption as in Theorem 4.4, we can apply 4.3 to deduce that $TX(\delta L)$ is *nef*, if $\delta > (2n+1)b + n(n+1+((n-1)(2n+2))^{n-1}(n+1)^n)$. Now we can apply Theorem 4.1 to deduce a result similar to Theorem 4.4.

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Lawrence EIN
Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60680
e-mail: U22425@uic.edu

Robert LAZARSFELD
Department of Mathematics
University of California, Los Angeles
Los Angeles, CA 90024
e-mail: rkl@math.ucla.edu

Michael NAKAMAYE
Department of Mathematics
Harvard University
Cambridge, MA 02138
e-mail: nakamaye@math.harvard.edu