http://dx.doi.org/10.1090/conm/207/02719

Contemporary Mathematics
Volume **207**, 1997

Nilpotent Cones and Sheaves on K3 Surfaces

Ron Donagi, Lawrence Ein and Robert Lazarsfeld

In Memoriam: Professor Wei-Liang Chow

Introduction

The purpose of this note is to point out a relation between two constructions that have attracted considerable attention among algebraic geometers. On the one hand, Hitchin [**H**] has introduced a beautiful completely integrable dynamical system living on the cotangent bundle $\mathbb{T}^*\mathcal{U}_C$ of the moduli space $\mathcal{U}_C$ of stable vector bundles of given rank and degree on an algebraic curve C. On the other hand, Mukai [**M**] has studied the space $\mathcal{M}$ parametrizing stable sheaves of given numerical type on a $K3$ surface S, proving in particular that $\mathcal{M}$ is a symplectic variety. We will argue here that it is profitable to view (certain cases of) the Mukai contsruction as a deformation of the Hitchin system. The idea is that if C is a hyperplane section of a $K3$-surface S, then S can be deformed to the cone K over the canonical embedding of C. Our deformation of $\mathcal{M}$ to the Hitchin system (given in §1) is based on exactly this deformation of $C \subset S$ to its normal cone.

We apply this deformation in §2 to study the "nilpotent cone" Nilp($\mathcal{M}$) in the Mukai system, which parametrizes certain sheaves on a non-reduced curve on S. Drawing on Laumon's analysis [**L**] of the corresponding locus in $\mathbb{T}^*\mathcal{U}_C$, we show that Nilp($\mathcal{M}$) is a Lagrangian subvariety of $\mathcal{M}$ provided at least that certain numerical restrictions are satisfied. Finally, in §3, we sketch an elementary result describing in detail the nilpotent cones of the two systems in the rank two case. In Hitchin's setting, this consists of a union of vector bundles over various symmetric products of the base curve C. In the Mukai space, the corresponding locus consists of *affine* bundles having the same underlying linear structure as in the Hitchin case.

Consideration of the Mukai and Hitchin systems as moduli spaces of sheaves on a fixed surface suggests some general questions as to when such a moduli space has a symplectic, Poisson, or completely integrable structure. One beautiful answer has recently been discovered by Markman [**Mkm**]: the moduli space of "Lagrangian"

1991 *Mathematics Subject Classification.* Primary14F05, 14J28.

Partially supported by the NSA Grant MDA904-92-H3047 and NSF Grant DMS 95-03249
Partially supported by NSF Grant DMS-93-02513
Partially supported by NSF Grant DMS 94-00815

sheaves on a symplectic variety is a completely integrable system, fibered by the support map. We discuss this and some open questions in the final section.

The reader will note that we do not prove any particularly substantive theorems in this note. Nonetheless, we hope that the connections and analogies that we observe may be of some interest.

We work throughout over the complex numbers $\mathbf{C}$. If E is a vector bundle on a variety X, viewed say as a locally free sheaf, we denote by $\mathbb{E} \longrightarrow X$ the total space of E, considered as an algebraic variety mapping to X.

§1. The systems of Hitchin and Mukai, and the Deformation

We start by briefly recalling Hitchin's construction. Let C be a fixed smooth projective curve of genus $g \geq 2$, let K_C be the canonical bundle of C, and denote by $\mathcal{U} = \mathcal{U}_C(n, e)$ the moduli space parametrizing stable rank n vector bundles on C, of degree e. Then $\mathcal{U}$ is a smooth variety, of dimension $\tilde{g} =_{\text{def}} n^2(g-1)+1$. In a well-known manner, one can identify the cotangent bundle $\mathbb{T}^*\mathcal{U}$ with the set of all pairs consisting of a stable rank n vector bundle (up to isomorphism) plus a K_C-valued endomorphism of that bundle:

$$\mathbb{T}^*\mathcal{U} = \{(E, \phi) \mid \phi : E \longrightarrow E \otimes K_C\}.$$

Let $B = \Gamma(C, K_C) \oplus \Gamma(C, K_C^2) \oplus \cdots \oplus \Gamma(C, K_C^n)$. Then one has a natural morphism of varieties:

$$H: \ \mathbb{T}^*\mathcal{U} \longrightarrow B = \mathbf{C}^{\tilde{g}}, \tag{1.1}$$

given by $h(E, \phi) = P_\phi$, where

$$P_\phi = (-\text{tr } \phi, \text{tr } \Lambda^2\phi, \ldots, (-1)^n \det \phi) \in B$$

is the "characteristic polynomial" of ϕ. Hitchin proves that H is an algebraically completely integrable Hamiltonian system with respect to the natural symplectic structure on $\mathbb{T}^*\mathcal{U}$. It follows that if $P \in B$ is a general point, then $H^{-1}(P)$ can be realized as a Zariski-open subset of an abelian variety J_P. These abelian varieties can also be explained via the spectral construction. In fact, an element $P \in B$, viewed as a characteristic polynomial, determines a *spectral curve*

$$D_P \subset \mathbb{K}_C$$

mapping n-to-one onto C. Fibre-wise over $x \in C$, D_P parametrizes the eigenvalues of $\phi(x)$. Then for general $P \in W$, $J_P = Jac(D_P)$. We refer to [**BNR**] for a discussion of spectral curves, and for an algebro-geometric approach to Hitchin's results.

Turning to Mukai's work, let S be a smooth projective polarized $K3$ surface. Mukai's beautiful observation [**M**] is that the moduli spaces parametrizing simple sheaves on S are smooth, and carry natural symplectic structures. We will be concerned with the following special case. Fix a smooth curve $C \subset S$ of genus $g \geq 2$, and suppose for simplicity that C is a very ample divisor on S. Given integers $k, n \in \mathbf{Z}$, with $n \geq 1$, denote by

$$\overline{\mathcal{M}} = \mathcal{M}^k_{|nC|}(S)$$

the moduli space parametrizing pairs consisting of a divisor

$$D \in \overline{B} =_{\text{def}} |nC|,$$

plus a semi-stable coherent sheaf $\mathcal{E}$ supported on D having the same Hilbert polynomial as a line bundle of degree k on a smooth member $D' \in |nC|$. We refer to [**S**], §1, for the precise definition of semi-stability, and for the proof that $\overline{\mathcal{M}}$ exists.* We will say that such a sheaf $\mathcal{E}$ has *numerical rank one and degree* k. For example, if E is a semi-stable vector bundle of rank n and degree $k + (n^2 - n)(1 - g)$ on C, then we may view E as a coherent sheaf on the non-reduced curve $R = nC \subset \overline{S}$, and the pair (E, R) determines a element of $\overline{\mathcal{M}}$. Since a semi-stable sheaf is simple, it follows from [**M**] that the open subset $\mathcal{M} \subset \overline{\mathcal{M}}$ consisting of stable sheaves is a symplectic manifold. As in [**LeP**], §2.3, taking schematic supports via Fitting ideals defines a morphism

$$\overline{M} : \overline{\mathcal{M}} \longrightarrow \overline{B} = \mathbf{P}^{\tilde{g}}, \tag{1.2}$$

where as above $\tilde{g} = \dim |nC| = n^2(g - 1) + 1$.

Just as in the Hitchin system, one has

Lemma 1.3. *For a smooth curve $D \in |nC|$, the fibre $\overline{M}^{-1}([D]) = Pic^k(D)$ is a Lagrangian submanifold of $\mathcal{M}$.*

PROOF. One computes that D has genus $\tilde{g}$, and hence

$$\dim \mathcal{M} = 2\tilde{g} = 2 \dim \overline{M}^{-1}(D)$$

along $\overline{M}^{-1}(D)$. So it is enough to show that the symplectic form vanishes on the fibres. Let L be a line bundle of degree k on D. Then the corresponding point $[L] \in \overline{\mathcal{M}}$ lies in the stable locus $\mathcal{M} \subset \overline{\mathcal{M}}$. The tangent space $T_{[L]}\mathcal{M}$ to $\mathcal{M}$ at $[L]$ is identified with $\mathrm{Ext}^1_S(L, L)$, whereas the tangent space $T_{[L]}Pic(D)$ to the fibre is given by the subspace $\mathrm{Ext}^1_D(L, L) = \mathrm{Ext}^1_D(\mathcal{O}_D, \mathcal{O}_D)$. It follows from Mukai's construction [**M**] that the restriction of the symplectic form to $T_{[L]}Pic(D)$ factors through the cup-product map

$$\mathrm{Ext}^1_D(L, L) \times \mathrm{Ext}^1_D(L, L) \longrightarrow \mathrm{Ext}^2_D(L, L).$$

But $\mathrm{Ext}^2_D(L, L) = 0$, and the lemma follows. □

Remark 1.4 In both the Hitchin and Mukai systems, the Jacobians $J(D)$ of smooth curves $D \in |nC|$ are realized as Lagrangian submanifolds of a symplectic manifold. It is shown in [**DM1**] and [**DM2**] that such a situation is determined by a family of cubic tensors $c_D \in Sym^3 H^0(D, K_D)^*$. In each setting, the cubic at D is given by the extension class

$$e_D \in \mathrm{Ext}^1_D(K_D, T_D) = H^0(D, K_D^3)^*$$

of the normal bundle sequence

$$0 \longrightarrow T_D \longrightarrow T_F|D \longrightarrow N_{D/F} \longrightarrow 0,$$

where $F = \overline{\mathbb{K}}_C$ or $F = S$ as the case may be.

We now wish to show that one can degenerate the Mukai system (1.2) to the Hitchin system. Note to begin with that Hitchin's system admits a natural compactification, as follows. Let $\overline{\mathbb{K}}_C$ denote the one-point compactification of $\mathbb{K}_C$, i.e.

*Note that one the requirements for semi-stability is that $\mathcal{E}$ have "pure dimension one", i.e. that it be free of embedded components.

the projective completion $\mathbf{P}(K_C \oplus \mathcal{O}_C)$ with the section at infinity blown down to a point. (So if C is non-hyperelliptic, $\overline{\mathbb{K}}_C$ is the cone over the canonical embedding of C.) View $C \subset \mathbb{K}_C \subset \overline{\mathbb{K}}_C$ as embedded by the zero-section. Then the spectral curves $D_P \subset \overline{\mathbb{K}}_C$ lie in the linear series $\overline{B} =_{\text{def}} |nC| = \mathbf{P}^{\tilde{g}}$, and one may identify the affine space B as the subset of $\overline{B}$ formed by those curves not passing through the vertex of $\overline{\mathbb{K}}_C$. Given an integer k, we consider pairs consisting of a curve $D \in |nC|$ plus a semi-stable coherent sheaf $\mathcal{E}$ supported on D having the same Hilbert polynomial as a line bundle of degree k on a smooth member $D' \in |nC|$. We will say as above that such a sheaf $\mathcal{E}$ has numerical rank one and degree k. According to Simpson [**S**], there is a projective moduli space

$$\overline{\mathcal{H}} = \overline{\mathcal{H}}^k_{|nC|}(\overline{\mathbb{K}}_C)$$

parametrizing isomorphism classes of all such, where naturally we view $\overline{\mathbb{K}}_C$ as polarized by the ample divisor C. Taking supports defines as before a morphism

$$\overline{H} : \overline{\mathcal{H}} \longrightarrow \overline{B}. \tag{1.5}$$

Via the spectral construction $\mathbb{T}^*\mathcal{U}_C$ is realized as an open subset of $\overline{H}^{-1}(B)$, with $\overline{H}$ restricting to the Hitchin map. We denote by $\mathcal{H} \subset \overline{\mathcal{H}}$ the open subset parametrizing stable sheaves.

Now consider a $K3$ surface $S \subset \mathbf{P}^g$ containing a smooth curve $C \subset S$ of genus g as a hyperplane section. The point is simply to exploit the elementary fact that one can degenerate S to the cone $\overline{\mathbb{K}}_C$ over C. Specifically, let $X_0 \subset \mathbf{P}^{g+1}$ be the cone over S, and denote by $X \longrightarrow X_0$ the blowing-up of X_0 along C. The pencil of hyperplanes in $\mathbf{P}^{g+1}$ passing through C gives rise to a mapping

$$f : X \longrightarrow \mathbf{P}^1.$$

There is a distinguished point $O \in \mathbf{P}^1$ such that

$$f^{-1}(O) = \overline{\mathbb{K}}_C;$$

for all other points $O \neq t \in \mathbf{P}^1$, $f^{-1}(t) \cong S$. The map f is thus a deformation of $C \subset S$ to the cone over C. It was used for example in [**P**], and is closely related to the well-known deformation (cf. [**F**]) of $C \subset S$ to its normal cone. (The latter is essentially the blow-up of X at the vertex of $\overline{\mathbb{K}}_C \subset X$.) Note that C embeds naturally as an ample divisor in each of the fibres X_t of f, and that X carries a polarization whose restriction to each fibre is $\mathcal{O}_{X_t}(C)$.

Now recall that Simpson [**S**] constructs moduli spaces of sheaves in a relative setting. Let

$$\overline{\mathcal{W}} = \overline{\mathcal{W}}^k_{|nC|} \longrightarrow \mathbf{P}^1$$

be the moduli space, projective over $\mathbf{P}^1$, parametrizing semistable sheaves of numerical rank one and degree k contained in the fibres of f. Equivalently, $\overline{\mathcal{W}}$ may be described as the moduli space parametrizing semi-stable sheaves of the given numerical type on X which are supported in a fibre of f. Denote by

$$\overline{\mathcal{B}} \longrightarrow \mathbf{P}^1$$

the $\mathbf{P}^{\tilde{g}}$-bundle whose fibre over $t \in \mathbf{P}^1$ is the linear series $|\mathcal{O}_{X_t}(nC)|$. Then Le Potier's construction in [**LeP**] globalizes to define a support map

$$\overline{W} : \overline{\mathcal{W}} \longrightarrow \overline{\mathcal{B}} \tag{1.6}$$

of schemes over $\mathbf{P}^1$. Given $t \in \mathbf{P}^1$, let

$$\overline{\mathcal{W}}_t \longrightarrow \overline{\mathcal{B}}_t$$

denote the the fibres of (1.6) over t. Then for $O \neq t \in \mathbf{P}^1$ one has $\overline{\mathcal{W}}_t \cong \overline{\mathcal{M}}$, whereas $\overline{\mathcal{W}}_O \cong \overline{\mathcal{H}}$. Thus (1.6) defines the required degeneration of the Mukai to the Hitchin system.

§2. The Nilpotent Cone in Mukai Space

In the "classical" Hitchin system $H : \mathbb{T}^*\mathcal{U}_C \longrightarrow B$, the *nilpotent cone* consists of all pairs (E, ϕ) such that the endomorphism $\phi : E \longrightarrow E \otimes K_C$ is nilpotent. A basic theorem of Laumon [**L**] states that this cone is a Lagrangian subvariety of $\mathbb{T}^*\mathcal{U}_C$. In this section, we use the deformation (1.6) to deduce the corresponding statement for the Mukai system under certain numerical restrictions.

The first point is to define the nilpotent cones in the present setting. To this end, note that if $\phi : E \longrightarrow E \otimes K_C$ is a nilpotent endomorphism of a semi-stable bundle E on C, then the corresponding spectral curve is the non-reduced scheme $R = nC \subset \mathbb{K}_C$, i.e. R is the $(n-1)^{\text{st}}$ infinitesimal neighborhood of the zero section $C \subset \mathbb{K}_C$. Therefore we define the nilpotent variety $\mathrm{Nilp}(\overline{\mathcal{H}}) \subset \overline{\mathcal{H}} = \overline{\mathcal{H}}^k_{|nC|}(\overline{\mathbb{K}}_C)$ to be the scheme

$$\mathrm{Nilp}(\overline{\mathcal{H}}) = \overline{H}^{-1}([R])$$

parametrizing all semi-stable sheaves of numerical rank one and degree k supported on $R = nC \subset \overline{\mathbb{K}}_C$. As noted above, a semi-stable vector bundle on C of degree $e = k + (n^2 - n)(1-g)$ determines a point in $\mathrm{Nilp}(\overline{\mathcal{H}})$, i.e. there is natural inclusion $\mathcal{U}_C = \mathcal{U}_C(n, e) \subset \mathrm{Nilp}(\overline{\mathcal{H}})$; in the classical Hitchin setting, this is just embedding of the zero-section in $\mathbb{T}^*\mathcal{U}_C$. We write $\mathrm{Nilp}(\mathcal{H})$ for the corresponding locus of stable sheaves. Similarly, suppose the curve C lies as a hyperplane section of a K3 surface S. We define the nilpotent subvariety of Mukai space to be the set

$$\mathrm{Nilp}(\overline{\mathcal{M}}) = \overline{M}^{-1}([R])$$

consisting of all semi-stable sheaves of the appropriate numerical type supported on the subscheme $R = nC \subset S$, where as above $\overline{\mathcal{M}} = \overline{\mathcal{M}}^k_{|nC|}(S)$. The analogous locus of stable sheaves is denoted by $\mathrm{Nilp}(\mathcal{M}) \subset \mathcal{M}$. Again one has an inclusion $\mathcal{U}_C \subset \mathrm{Nilp}(\overline{\mathcal{M}})$. Observe also that the nilpotent subvariety $\mathrm{Nilp}(\overline{\mathcal{M}})$ degenerates to a subvariety of $\mathrm{Nilp}(\overline{\mathcal{H}})$ under the deformation (1.6).

Our aim here is to prove the following:

Theorem 2.1. *Assume that k and n are coprime. Then* $\mathrm{Nilp}(\mathcal{M})$ *is a Lagrangian subvariety of $\mathcal{M}$.*

Recall from [**L**], Appendix A, that this means that $\mathrm{Nilp}(\mathcal{M})$ contains a Zariski-open dense set which is everywhere of half the dimension of $\mathcal{M}$, and on which the symplectic form vanishes.

We start with

Lemma 2.2. *Under the hypotheses of (2.1), every component of* $\mathrm{Nilp}(\overline{\mathcal{M}})$ *has dimension $\tilde{g}$.*

PROOF. Let $\overline{\mathcal{H}}_0$ [resp. $\mathcal{H}_0$] denote the open subset of $\overline{\mathcal{H}}$ [resp. $\mathcal{H}$] parametrizing semi-stable [resp. stable] sheaves supported on $\mathbb{K}_C \subset \overline{\mathbb{K}}_C$. Note that sheaves on $\overline{\mathbb{K}}_C - \{\text{vertex}\}$ are exactly those corresponding via the spectral construction to Higgs pairs $(E, \phi : E \longrightarrow E \otimes K_C)$. We claim to begin with that under the numerical hypotheses of the Theorem,

$$(*) \qquad \overline{\mathcal{H}}_0 = \mathcal{H}_0.$$

In fact, suppose that $\mathcal{E}$ is sheaf of pure dimension one on $\overline{\mathbb{K}}_C - \{\text{vertex}\}$. Then stability (or semi-stability) of $\mathcal{E}$ is in the first instance defined à la Gieseker by means of the Hilbert polynomial $p_{\mathcal{E}}(t) = \chi(\mathcal{E}(tC))$ determined by the polarization $C \subset \overline{\mathbb{K}}_C$. However C is linearly equivalent on $\mathbb{K}_C$ to a divisor $(2g-2)F$ where F is the pull-back of a divisor of degree one on C under the bundle projection $\mathbb{K}_C \longrightarrow C$, and it is equivalent to compute stability with respect to the Hilbert polynomial

$$q_{\mathcal{E}}(t) = \chi(\mathcal{E}(tF)).$$

Now

$$q_{\mathcal{E}}(t) = nt + \left(k + n^2(1-g)\right),$$

and hence if k and n are relatively prime, then so are the two coefficients $r = n$ and $a = k + n^2(1-g)$ of $q_{\mathcal{E}}(t)$. But from the definition of q-stability it follows immediately that if $\mathcal{E}$ is semi-stable, then it is automatically stable. [In effect, we are proving that (semi-)stability of $\mathcal{E}$ is equivalent to (semi-)stability of the corresponding Higgs pair (E, ϕ).]

We next wish to invoke Laumon's theorem that the Hitchin nilpotent cone is Lagrangian. The main theorem of [L] states that the nilpotent cone is Lagrangian in the moduli stack $T^*\mathrm{Fib}$, which parametrizes isomorphism classes of arbitrary Higgs pairs (E, ϕ). In particular, each component of this nilpotent cone has dimension (as a stack) $\tilde{g} - 1$, which is the dimension of Fib. (The difference of 1 is due to the presence of scalar automorphisms on every vector bundle.) But as we have just seen, points of $\overline{\mathcal{H}}_0$ are given by *stable* pairs (E, ϕ), and since stable pairs are simple by [**S**],§1, we conclude that each component of $\mathrm{Nilp}(\overline{\mathcal{H}})$ is Lagrangian of dimension $\tilde{g}$ in $\overline{\mathcal{H}}_0$.

Now under the deformation (1.6), $\mathrm{Nilp}(\overline{\mathcal{M}})$ specializes to $\mathrm{Nilp}(\overline{\mathcal{H}})$. Hence it follows from the semi-continuity of fibre dimensions that every component of $\mathrm{Nilp}(\overline{\mathcal{M}})$ has dimension $\leq \tilde{g}$. On the other hand, Lemma 1.3 shows that $\mathrm{Nilp}(\overline{\mathcal{M}})$ is itself a specialization of the Lagrangian subvarieties $\overline{\mathcal{M}}^{-1}([D])$ for general $D \in |nC|$, and this implies the reverse inequality. □

Remark 2.3. It seems plausible that Lemma 2.2 remains valid without the numerical hypotheses of the Theorem. However we do not know how to rule out the possibility that there are whole components of $\mathrm{Nilp}(\overline{\mathcal{H}})$ contained in the singular locus of $\overline{\mathcal{H}}$, in which case Laumon's theorem does not seem to apply. Note in any event that it is only in Lemma 2.2 that we use that k and n are coprime.

Keeping the hypotheses of the Theorem, it remains to show that the symplectic form on $\mathcal{M}$ vanishes on any component F of $\mathrm{Nilp}(\mathcal{M})$. Denote by $\overline{F}$ the closure of F in $\mathrm{Nilp}(\overline{\mathcal{M}})$, so that $\overline{F}$ is an irreducible component of $\mathrm{Nilp}(\overline{\mathcal{M}})$. Now $\mathcal{M}$ has dimension $2\tilde{g}$ at every point of F, so it follows from Lemma 2.2 that $\overline{F}$ is contained in an irreducible component W of $\overline{\mathcal{M}}$ that maps onto $\overline{B} = |nC|$. Choose a smooth curve $T \subset \overline{B}$ passing through $[R] = [nC] \in \overline{B}$, and meeting the open subset of $\overline{B}$ parametrizing smooth members of $|nC|$. Let $W_T = W \cap \overline{M}^{-1}(T) \subset \overline{\mathcal{M}}$, and denote by $m : W_T \longrightarrow T$ the projection. Then every component of W_T has dimension $\geq \tilde{g}+1$, whereas $\overline{F}$, which has dimension $\tilde{g}$, is an irreducible component of $\mathrm{Nilp}(\overline{\mathcal{M}}) \supseteq m^{-1}([R])$. It follows that there is an irreducible component Z of W_T containing $\overline{F}$ and mapping onto T. Write $g : Z \longrightarrow T$ for the projection. Thus $\overline{F}$ is a component of $g^{-1}(0)$, whereas for t in a punctured neighborhood of $[R] \in T$, it follows from (1.3) that the fibre $Z_t = g^{-1}(t)$ is a Lagrangian submanifold of $\mathcal{M}$. Therefore the assertion follows from the following lemma, which states in effect that a limit of Lagrangian submanifolds is a Lagrangian subvariety.

Lemma 2.4. *Let (M, ω) be a symplectic variety, let T be (the germ of) a smooth curve with a marked point $0 \in T$. Suppose that $Z \subset M \times T$ is an irreducible variety such that for $0 \neq t \in T$ the fibre $Z_t \subset M$ is a smooth subvariety on which the symplectic form ω vanishes. Then ω also vanishes at the general point of any component of the special fibre $Z_0 \subset M$.*

PROOF. Let $p : Z \longrightarrow T$ denote projection onto the second factor. By Mumford's semi-stable reduction theorem [**K**], after a base change $(T', 0') \longrightarrow (T, 0)$ and blowings up over the central fibre, we can construct a new family $q : Y \longrightarrow T'$ such that the central fibre $q^{-1}(0')$ is a reduced normal crossing divisor, and all other fibres are smooth. Then the sheaf $\Omega^2_{Y/T'}$ of relative two-forms along q is locally free outside the subset $G \subset Y$ of codimension ≥ 2 where q fails to be smooth. Let $f : Y \longrightarrow M$ denote the composition $Y \longrightarrow Z \longrightarrow M$. Then $\eta =_{\mathrm{def}} f^*(\omega)$ vanishes on the general fibre of q, and hence vanishes off G. In particular, η vanishes at the general point of each component D_i of $q^{-1}(0')$. But each component W of $Z_0 \subset Z$ is dominated by some D_i, so ω must vanish at the general point of W. □

This completes the proof of Theorem 2.1.

§3. Comparison of Nilpotent Cones in Rank Two

In this section, we state without proof a detailed description of the nilpotent cones in the case $n = 2$. For simplicity we limit attention to stable sheaves, and the classical Hitchin setting, although we do not need the numerical hypotheses of Theorem 2.1.

We start with the Hitchin system. Consider the first infinitesimal neighborhood $R = 2C \subset \mathbb{K}_C$. Via the spectral construction one has an identification

$$\mathrm{Nilp}(\mathcal{H}) = H^{-1}(0) = \left\{ (E, \phi) \,\middle|\, \begin{array}{l} \text{E a stable rank 2 bundle on } C, \\ \phi : E \longrightarrow E \otimes K_C \text{ with } \phi^2 = 0 \end{array} \right\}.$$

Laumon [**L**] has enumerated the components of the nilpotent cone in general, but in the present case the description is elementary to obtain by hand. In fact, suppose $0 \neq \phi$ is nilpotent. Then ϕ has generic rank one. Let $D \subset C$ be the effective divisor on which ϕ vanishes, and denote by A the line bundle $\mathrm{im}\phi \subset E \otimes K_C$. Then E sits in the exact sequence

$$0 \longrightarrow A(D) \otimes K_C^{-1} \xrightarrow{\alpha} E \longrightarrow A \longrightarrow 0,$$

and ϕ factors as the composition

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & A \\ & & \big\downarrow{\cdot D} \\ & & A(D) \xrightarrow{\alpha\otimes 1} E \otimes K_C. \end{array}$$

Such nilpotent endomorphisms may be parametrized as follows. Set $e = \deg E$, and fix an integer

$$0 \le d < 2g-2, \quad \text{with} \quad d \equiv e \pmod 2.$$

Put $a = (e + 2g - 2 - d)/2$, and define:

$$\mathbb{N}_d = \left\{ (A, D, \epsilon) \mid A \in Pic^a(C),\ D \in Sym^d(C),\ \epsilon \in \operatorname{Ext}^1(A, A(D) \otimes K_C^{-1}) \right\}.$$

Note that there is a natural map

$$\pi : \mathbb{N}_d \longrightarrow Pic^a(C) \times Sym^d(C)$$

which realizes $\mathbb{N}_d$ as the total space of a vector bundle whose fibre over the point (A, D) is the vector space $H^1(\mathcal{O}(D) \otimes K_C^{-1})$. In the special case at hand, we may state one of Laumon's results as follows:

PROPOSITION 3.1. *The irreducible components of the nilpotent cone* Nilp$\mathcal{H}$ *consist of the moduli space $\mathcal{U}_C(2,e)$ of stable bundles, together with the Zariski closures in $\mathcal{H}$ of suitable Zariski-open subsets $N_d^+ \subset \mathbb{N}_d$.* □

(The open subsets N_d^+ arise because not every rank two bundle determined by an element in $\mathbb{N}_d$ is stable.)

Turning to the Mukai setting, recall that the nilpotent subvariety Nilp$(\mathcal{M})$ para- metrizes stable sheaves of numerical rank one and degree k on the infinitesimal neighborhood $R = 2C \subset S$ of C in the K3 surface S. At this point the question no longer has anything to do with K3 surfaces, and it clarifies matters to generalize slightly.

Let C be a smooth curve of genus g, and let R be any "ribbon" on C with normal bundle K_C, i.e. the scheme arising from a double structure on C, with normal bundle K_C. Thus $\mathcal{O}_R$ sits in an exact sequence:

$$0 \longrightarrow K_C^{-1} \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Such double structures are classified by $H^1(C, K_C^{-2})$, and given $r \in H^1(C, K_C^{-2})$ we write R_r for the corresponding scheme. For example, R_0 is just the first infinitesimal neighborhood of the zero-section $C \subset \mathbb{K}_C$. Fixing some polarization on R, we define $\mathcal{M}(r) = \mathcal{M}^k(R_r)$ to be the moduli space of stable (in particular, Cohen-Macaulay) sheaves of numerical rank one and degree k on R_r.

The sheaves in question have one of two types. First, a rank two stable vector bundle on C of degree $e = k + 2 - 2g$, considered as an $\mathcal{O}_R$ module, has numerical rank one and degree k. The second type of sheaves consists of stable $\mathcal{O}_R$-modules $\mathcal{E}$ whose restrictions $\mathcal{E} \otimes \mathcal{O}_C$ to $C \subset R$ have rank 1. Fix such a sheaf $\mathcal{E}$. One shows that canonically associated to $\mathcal{E}$ there is an effective divisor $D \subset C$ on C supported on the set where $\mathcal{E}$ fails to be locally isomorphic to $\mathcal{O}_R$: we write $D = \operatorname{Sing}(\mathcal{E})$. Put

$$A = \mathcal{E} \otimes \mathcal{O}_C \ / \ \text{torsion}.$$

One has an exact sequence $0 \longrightarrow A(D) \otimes K_C^{-1} \longrightarrow \mathcal{E} \longrightarrow A \longrightarrow 0$ of $\mathcal{O}_R$-modules. Set

$$\mathbb{N}_d(r) = \{\mathcal{E} \in \mathcal{M}(r) \mid \deg \operatorname{Sing}(E) = d\}.$$

Then there is a morphism

$$\pi : \mathbb{N}_d(r) \longrightarrow Pic^a(C) \times Sym^d(C) \quad \text{via} \quad \mathcal{E} \mapsto (A, D),$$

where as above $D = \operatorname{Sing}(\mathcal{E})$, and $a = (k - d)/2$.

We may now state the result:

Theorem 3.2. *(i). The irreducible components of $\mathcal{M}^k(r)$ are the Zariski closures of*

$$\mathbb{N}_\infty(r) =_{\text{def}} \mathcal{U}_C(2, e) \qquad (\textit{for } e = k + 2 - 2g)$$

and the sets

$$\mathbb{N}_d(r) \qquad (\textit{for } 0 \leq d < 2g - 2 \textit{ and } d \equiv k \pmod 2).$$

(ii). The map $\pi : \mathbb{N}_d(r) \longrightarrow Pic^a(C) \times Sym^d(C)$ is an affine bundle, with underlying vector bundle $\mathbb{N}_d = \mathbb{N}_d(0)$. In general it is not itself a vector bundle.

For example consider the case $d = 0$. Then $\mathbb{N}_0(r) = Pic^k(R_r)$, and the morphism $\pi : Pic^k(R_r) \longrightarrow Pic^a(C)$ comes from the exact sequence

$$0 \longrightarrow H^1(C, K_C^*) \longrightarrow Pic^k(R_r) \longrightarrow Pic^a(C) \longrightarrow 0 \tag{3.3}$$

of algebraic groups. The last statement of the Theorem is illustrated by the fact that $Pic^k(R_r)$ has the structure of an affine bundle, but not in general a vector bundle, over $Pic^a(C)$. Note that the Theorem can be seen simply as a statement about the geometry of double curves, but it seems to us that it is only in the context of the Hitchin-Laumon nilpotent cone that it becomes natural. The proof of the Theorem is not terribly difficult but it is fairly long, and we will not give details here. We remark that a similar analysis holds for the moduli space of stable sheaves on a "ribbon" R with any normal bundle N, at least when N has sufficiently large degree.

There is some interesting additional geometry connected with this situation. Let $\mathcal{E}$ be a sheaf on R, corresponding to a point in $\mathbb{N}_d(r)$, and set $D = \operatorname{Sing} \mathcal{E}$. One can show that there exists a line bundle $\mathcal{L}$ on R such that

$$\mathcal{E} = \operatorname{elm}_D(\mathcal{L}) =_{\text{def}} \ker\{\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_D\}.$$

This globalizes to a morphism

$$f : Pic^k(R) \times Sym^d(C) \longrightarrow \mathbb{N}_d(r)$$

given by $f(\mathcal{L}, D) = \operatorname{elm}_D(\mathcal{L})$. In fact, f is a map of affine bundles over $Pic^a(C) \times Sym^d(C)$, and for some questions it reduces the analysis of $\mathbb{N}_d(r)$ to the case $d = 0$.

Recall next that if X is any variety, and if E is a vector bundle on X, then the set of all isomorphism classes of affine bundles on X with underlying linear structure E is classified by the cohomology group $H^1(X, E)$. It is an amusing exercise to compute the characteristic classes of the various affine bundles appearing in the Theorem. When $d = 0$, for example, the bundle (3.3) is classified by an element in

$$H^1(Pic(C), H^1(K_C^{-1}) \otimes_{\mathbf{C}} \mathcal{O}_{Pic(C)}) = H^1(C, K_C^{-1}) \otimes H^1(C, \mathcal{O}_C).$$

As one might expect, the element in question is just the image of the "ribbon class" r under the natural map $H^1(K_C^{-2}) \longrightarrow H^1(\mathcal{O}_C) \otimes H^1(K_C^{-1})$ dual to the multiplication $H^0(K_C) \otimes H^0(K_C^2) \longrightarrow H^0(K_C^3)$.

Finally, note that it follows immediately from (3.2.ii) that every component of $\mathcal{M}(r)$ has dimension $g+d+\operatorname{rank}\mathbb{N}_d = 4g-3 = \tilde{g}$. Returning to the Mukai nilpotent cone $\operatorname{Nilp}(\mathcal{M})$ determined by a $K3$-"ribbon" $R = 2C \subset S$, this implies that Lemma 2.2 remains valid when $n = 2$ without the hypothesis that k be odd. Hence in the rank two case, we do not need any restrictions on k in Theorem 2.1.

§4. Concluding Remarks and Open Questions

It would be quite interesting to generalize the analysis of §3 to higher rank. Let R be a "ribbon" (or "tape") of order n and normal bundle N, i.e. a multiplicity n structure on a smooth curve C which looks locally like the $(n-1)^{\text{st}}$ infinitesimal neighborhood of C on a surface. Thus we suppose that R is filtered by subschemes

$$C = C_1 \subset C_2 \subset \cdots \subset C_{n-1} \subset C_n = R,$$

where the C_i sit in exact sequences

$$0 \longrightarrow N^{-(i-1)} \longrightarrow \mathcal{O}_{C_i} \longrightarrow \mathcal{O}_{C_{i-1}} \longrightarrow 0.$$

For example, one has the "split" ribbon R_0, i.e. the $(n-1)^{\text{st}}$ neighborhood of the zero-section in the normal bundle $\mathbb{N}$. Consider a Cohen-Macaulay $\mathcal{O}_R$-module $\mathcal{E}$ of numerical rank one. Define

$$\nu_i = \operatorname{rank}_{\mathcal{O}_C} \ker\{\mathcal{E} \otimes \mathcal{O}_{C_i} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{C_{i-1}}\}$$

(and $\nu_1 = \operatorname{rank}(\mathcal{E} \otimes \mathcal{O}_C)$). One has $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, and we call the vector $\nu = (\nu_1, \ldots, \nu_n)$ the *type* of $\mathcal{E}$. For example, when $n = 2$ there are two possible types: $(2,0)$, corresponding to a rank two vector bundle on C, and $(1,1)$, corresponding to a sheaf $\mathcal{E}$ whose restriction to C has generic rank one. As another example, on the split ribbon R_0, $\mathcal{E}$ is given by a vector bundle E of rank n on C together with a nilpotent endomorphism $\phi : E \longrightarrow E \otimes N$, and then $\nu_i = \operatorname{rank}(\ker \phi^i / \ker \phi^{i-1})$. One can associate to a sheaf $\mathcal{E}$ of type ν vector bundles E_i on C of rank ν_i (generalizing the line bundle A in §3) and sky-scraper sheaves Δ_i on C (generalizing the divisor $D = \operatorname{Sing}(E)$ in §3). The type – and, within a given type, the degrees of the E_i and Δ_i – give discrete invariants of the components of the space of stable sheaves on R. For type $(1,1,\ldots,1)$ one can work out the picture in some detail, much as in §3. However for general types, the analysis is less clear. One can hope that the sort of construction with "elementary transformations" indicated at the end of the previous section can at least reduce the study of the general case to the "defectless" situation where all $\Delta_i = 0$.

Much more generally, one can attempt to replace the $K3$ surface by a higher dimensional symplectic variety S. It is not clear at the moment exactly when one should expect to have an analogue of Mukai's theorem, i.e. a natural symplectic structure on appropriate moduli spaces of sheaves on S. The case of vector bundles on S was proved by Kobayashi [**Ko**]. There are however examples of such moduli spaces (parametrizing line bundles on a divisor in S) which have odd dimension, and so cannot admit any algebraic symplectic structure, cf. [**DM2**]. A beautiful new idea of Markman [**Mkm**] is that Mukai's results do extend to the moduli of Lagrangian sheaves, i.e. those sheaves on a symplectic variety S whose support in S is itself Lagrangian. Further, a version of this space comes with a support map to a Lagrangian-Hilbert base B, and this map makes it into an algebraically completely integrable system. More precisely, there is a closed two-form on the non-singular locus of the moduli space of Lagrangian sheaves, which can be constructed in terms of a field of cubics on the base B as in Remark 1.4, cf. [**DM1**]. Under some mild conditions, this form is non-degenerate at all points which correspond to line bundles over a non-singular Lagrangian support. There are several variations which replace the symplectic structures (on S and on the resulting moduli spaces) with Poisson or quasisymplectic structures, respectively. This whole picture can be considered as a non-linear analogue, and sometimes as a deformation, of Simpson's moduli space of Higgs bundles on a variety [**S**], or of the more general moduli spaces of vector bundles with arbitrarily twisted endomorphisms studied e.g. in [**DM2**], in the same sense as Mukai's space appears here as a deformation of Hitchin's.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE (M/C 249), UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE, COLLEGE OF LIBERAL ARTS AND SCIENCE, CHICAGO, IL 60607-7045

DEPARTMENT OF MATHEMATICS, U.C.L.A., LOS ANGELES, CA 90095