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Positivity and complexity of ideal sheaves

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Introduction

The problem of bounding the "complexity" of a polynomial ideal in terms of the degrees of its generators has attracted a great deal of interest in recent years. Results in this direction go back at least as far as the classical work [17] of Hermann on the ideal membership problem, and the effective Nullstellensatz of Brownawell [4] and Kollár [21] marks a major recent advance. With the development of computational algebraic geometry the question has taken on increasing importance, and it came into particularly clear focus through the influential paper [3] of Bayer and Mumford. More recently, the theorem of [8] and [20] concerning regularity of powers raises the question of bounding the complexity of powers of an ideal, and suggests that asymptotically the picture should become very clean.

The aim of the present paper is to examine some of the results and questions of [3], [30] and [7] from a geometric perspective, in the spirit of [12]. Our thesis is that much of this material is clarified, and parts rendered transparent, when viewed through the lenses of vanishing theorems and intersection theory. Specifically, motivated by the work [25], [26] of Paoletti we introduce an invariant $s(\mathcal{J})$ that measures in effect how much one has to twist an ideal $\mathcal{J}$ in order to make

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it positive. Degree bounds on generators of $\mathcal{J}$ yield bounds on this s-invariant, but in general $s(\mathcal{J})$ can be small even when the degrees of generators are large. We prove that the s-invariant $s(\mathcal{J})$ computes the asymptotic regularity of large powers of an ideal sheaf, and bounds the asymptotic behaviour of several other natural measures of complexity considered in [3] and [30]. We also show that this invariant behaves very well with respect to natural geometric and algebraic operations. This leads for example to a considerably simplified analogue of the construction from [7] of varieties with irrational asymptotic regularity.

Turning to a more detailed overview of the contents of this paper, we start by fixing the setting in which we shall work. Denote by X an irreducible nonsingular projective variety of dimension n defined over an algebraically closed field K of arbitrary characteristic, and let H be a fixed ample divisor (class) on X. The most natural and important example is of course $X = \mathbf{P}^n$ and H a hyperplane, and in fact little essential will be lost to the reader who focuses on this classical case. However since we are working geometrically it seems natural to consider general varieties, and in the end it is no harder to do so. Given an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, consider the blowing-up $\nu : \mathrm{Bl}_{\mathcal{J}}(X) \longrightarrow X$ of $\mathcal{J}$, with exceptional divisor F. We define

$$s_H(\mathcal{J}) \;=\; \inf\{s \in \mathbf{R} \mid \nu^*(sH) - F \text{ is an ample } \mathbf{R}\text{-divisor on } \mathrm{Bl}_{\mathcal{J}}(X)\,\}.$$

One has $s_H(\mathcal{J}) \leq d_H(\mathcal{J})$, where $d_H(\mathcal{J})$ denotes the least integer $d > 0$ such that $\mathcal{J}(dH)$ is globally generated, but in general the inequality is strict.[1] This *s-invariant* is closely related the Seshadri constants introduced by Demailly, and has been studied by Paoletti when $\mathcal{J}$ is the ideal sheaf of a smooth subvariety of X. (See Remark 1.3 below). In a general way, our goal is to bound the “complexity" of $\mathcal{J}$ (or at least its powers) in terms of this invariant.

We do not use the term “complexity" here in any technical sense. Rather, guided by [3], we consider various natural invariants that each give a picture of how complicated one might consider $\mathcal{J}$ to be:

(0). The degrees of the irreducible components of the zero-locus $\mathrm{Zeroes}(\mathcal{J}) \subseteq X$ of $\mathcal{J}$;

(1). The degrees of all the “associated subvarieties" of $\mathrm{Zeroes}(\mathcal{J})$, including those corresponding to the embedded primes in a primary decomposition of $\mathcal{J}$;

(2). For H very ample, the *Castelnuovo-Mumford regularity* $\mathrm{reg}(\mathcal{J})$ of $\mathcal{J}$, which measures roughly speaking the cohomological complexity of $\mathcal{J}$;

(3). The *index of nilpotency* $\mathrm{nilp}(\mathcal{J})$ of $\mathcal{J}$, i.e. the least integer $t > 0$ such that

$$\left(\sqrt{\mathcal{J}}\right)^t \subseteq \mathcal{J}.$$

[1] In fact, it is possible for $s_H(\mathcal{J})$ to be irrational.

In settings (0) and (1), one can ask also for degree bounds after having attached multiplicities to the components in question: allowing embedded components, this leads to what Bayer and Mumford call the *arithmetic degree* of $\mathcal{J}$. The index of nilpotency is closely related to the effective Nullstellensatz of Kollár and [12], and various relations among these invariants have been established ([3], [30], [24]). In the classical situation, where $\mathcal{J}$ is replaced by a homogeneous ideal I generated by forms of degree d, it is elementary to obtain a Bezout-type bound for (0), while the main theorem of [21] gives the analogous statement for (3). However, Bayer and Mumford observe that there cannot exist singly exponential bounds in d for the regularity or arithmetic degree.

In the direction of (0) and (1), one has:

Proposition A. *Let $s = s_H(\mathcal{J})$. Then*

$$\sum s^{\dim Z} \cdot \deg_H Z \ \leq \ s^n \cdot \deg_H X, \tag{0.1}$$

where the sum is taken over all irreducible components of $\mathrm{Zeroes}(\mathcal{J})$. *If $\mathcal{J}$ is integrally closed, then the same inequality holds including in the sum also the embedded associated subvarieties of* $\mathrm{Zeroes}(\mathcal{J})$.

The first assertion follows already from the positivity theorems for intersection classes established in [16], although the elementary direct approach of [12] also applies. The stronger statement for integrally closed ideals, while elementary, seems to have been overlooked. We also give examples (Example 2.8) to show that if $\mathcal{J}$ is not integrally closed, then one cannot bound the number of embedded components in terms of $s_H(\mathcal{J})$.

Assume now that H is very ample. In this setting the Castelnuovo-Mumford regularity $\mathrm{reg}_H(\mathcal{J})$ of $\mathcal{J}$ (with respect to $\mathcal{O}_X(H)$) can be defined just as in the classical case $X = \mathbf{P}^n$.

Theorem B. *One has the equalities*

$$\lim_{p\to\infty} \frac{\mathrm{reg}_H(\mathcal{J}^p)}{p} \ = \ \lim_{p\to\infty} \frac{d_H(\mathcal{J}^p)}{p} \ = \ s_H(\mathcal{J}),$$

where given an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, $d_H(\mathcal{I})$ denotes as above the least integer d such that $\mathcal{I}(dH)$ is globally generated.

Thus the s-invariant $s_H(\mathcal{J})$ governs exactly the asymoptotic regularity of powers of $\mathcal{J}$. As indicated above, this result was suggested by the theorems of [8] and [20], which prove the analogue of the first equality for homogeneous ideals.

Continuing to assume that H is very ample, one can define as in [3] the codimension k contribution $\mathrm{adeg}^k_H(\mathcal{J})$ to arithmetic degree of $\mathcal{J}$, which measures (taking into account suitable multiplicities) the degrees of the codimension k irreducible and embedded components of the scheme defined by $\mathcal{J}$. As in [3] there are upper bounds — at least asymptotically — for this degree in terms of the regularity, and we deduce

Corollary C. *Denote by* $\operatorname{adeg}_H^k(\mathcal{J}^p)$ *the codimension* k *contribution to the arithmetic degree of* $\mathcal{J}^p$. *Then*

$$\limsup_{p\to\infty} \frac{\operatorname{adeg}_H^k(\mathcal{J}^p)}{p^k} \leq \frac{s_H(\mathcal{J})^k}{k!} \cdot \deg_H(X).$$

In general this statement is the best possible: for instance equality holds for complete intersections of hypersurfaces of the same degree in projective space. As in the case of Theorem B, the simple asymptotic statement contrasts with the examples presented in [3] showing that there cannot be a singly exponential bound for $\operatorname{adeg}^k(\mathcal{J})$ in terms of $d_H(\mathcal{J})$ (let alone in terms of $s_H(\mathcal{J})$).

Turning finally to the index of nilpotency, one can canonically attach to $\mathcal{J}$ an integer $r(\mathcal{J})$ arising as the maximum of the multiplicities of the irreducible components of the exceptional divisor of the normalized blow-up of $\mathcal{J}$. These multiplicities appear in a Bezout-type bound strengthening Proposition A, which in particular gives rise to the inequality $r(\mathcal{J}) \leq s_+^n \cdot \deg_H X$, where $s_+ = \max\{1, s_H(\mathcal{J})\}$. The results of [12] (and also [18]) show that $\left(\sqrt{\mathcal{J}}\right)^{n\cdot r(\mathcal{J})} \subseteq \mathcal{J}$ and more generally that

$$(*) \qquad \left(\sqrt{\mathcal{J}}\right)^{r(\mathcal{J})\cdot(n+p-1)} \subseteq \mathcal{J}^p$$

for all $p \geq 1$. Note that (*) leads to the asymptotic statement $\limsup_{p\to\infty} \frac{(\operatorname{nilp}(\mathcal{J}^p))}{p} \leq s_+^n \cdot \deg_H X$.

Motivated by the influence of the s-invariant in these questions, we study its behavior under natural geometric and algebraic operations. In this direction we prove for example:

Proposition D. *Let* $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{O}_X$ *be ideal sheaves on* X. *Then one has:*

$$s_H\big(\mathcal{J}_1 \cdot \mathcal{J}_2\big) \leq s_H(\mathcal{J}_1) + s_H(\mathcal{J}_2)$$
$$s_H\big(\mathcal{J}_1 + \mathcal{J}_2\big) \leq \max\big\{s_H(\mathcal{J}_1)\,,\ s_H(\mathcal{J}_2)\big\}.$$

Moreover, if $\overline{\mathcal{J}}$ *denotes the integral closure of an ideal* $\mathcal{J}$, *then* $s_H(\overline{\mathcal{J}}) = s_H(\mathcal{J})$.

In view of Theorem , this result shows that the asymptotic regularity of large powers of an ideal satisfies much better formal algebraic properties than are known or expected to hold for the regularity of an ideal itself.

Our exposition is organized as follows. In §1 we define and study the s-invariant measuring the positivity of an ideal sheaf. Degree and nilpotency bounds — which for the most part involve only minor modifications to results from [12] — are given in §2. Finally, in §3 we consider asymptotic bounds on the regularity and arithmetic degree of large powers of an ideal.

Finally, a word about assumptions. Many (but not all) of our results do not require the ambient variety X to be non-singular. However in order to avoid changing hypotheses throughout the body of the exposition, we make a blanket assumption of smoothness (which in any event is the most natural situation geometrically). We indicate in Remarks when this hypothesis can be relaxed.

1. The s-invariant of an ideal sheaf

In the present section we define and study the s-invariant of an ideal sheaf with respect to an ample divisor.

We start by fixing some notation that will remain in force throughout the paper. Let X be a non-singular irreducible projective variety defined over an algebraically closed field K of arbitrary characteristic, and consider a fixed coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. Denote by

$$\nu : W = \mathrm{Bl}_{\mathcal{J}}(X) \longrightarrow X$$

the blowing up of X along $\mathcal{J}$. Then $\mathcal{J}$ becomes locally principal on W, i.e there is an effective Cartier divisor F on W (namely the excptional divisor of ν) such that

$$\mathcal{J} \cdot \mathcal{O}_W = \mathcal{O}_W(-F).$$

Fix now an ample divisor (class) H on X. If $s \gg 0$, then $\nu^*(sH) - F$ is ample on W thanks to the fact that $-F$ is ample for ν. In order to measure the positivity $\mathcal{J}$ with respect to H, we ask how small one can take s to be while keeping the class in question non-negative:

Definition 1.1. *The s-invariant of $\mathcal{J}$ with respect to H is defined to be the positive real number*

$$s_H(\mathcal{J}) \ = \ \min \left\{ s \in \mathbf{R} \mid \nu^*(sH) - F \text{ is nef } \right\}.$$

Here $\nu^*(sH) - F$ is considered as an $\mathbf{R}$-divisor (class) on W,[2] and to say that it is nef means by definition that

$$s \cdot \left(\nu^* H \cdot C'\right) \geq \left(F \cdot C'\right)$$

for every effective curve $C' \subset W$.

Remark 1.2. Suppose that $f : Y \longrightarrow X$ is a surjective morphism of projective varieties with the property that $\mathcal{J} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ for some effective Cartier divisor E on Y. Then f factors through ν. Recalling that nefness can be tested after pull-back by a surjective morphism, it follows that

$$s_H(\mathcal{J}) \ = \ \min \left\{ s \in \mathbf{R} \mid f^*(sH) - E \text{ is nef } \right\}. \square$$

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[2] By an $\mathbf{R}$-divisor on a variety V we understand an element of $\mathrm{Div}(V) \otimes \mathbf{R}$, $\mathrm{Div}(V)$ denoting the group of Cartier divisors on V. An $\mathbf{R}$-divisor class is a numerical equivalence class of $\mathbf{R}$-divisors.

Remark 1.3. (Seshadri constants). Following [9] and [25] it would be natural to define the *Seshadri constant* $\epsilon_H(\mathcal{J})$ of $\mathcal{J}$ with respect to H to be the reciprocal

$$\epsilon_H(\mathcal{J}) \;=\; \frac{1}{s_H(\mathcal{J})}.$$

However Definition 1.1 is more convenient for our purposes, and we use a different name in order to avoid the possibility of confusion. When $\mathcal{J}$ is the ideal sheaf of a point $x \in X$ and $L = \mathcal{O}_X(H)$, the invariant $\epsilon(L, x) = \epsilon_H(\mathcal{J})$ was introduced by Demailly as a measure of the local positivity of L at x. The behavior of these Seshadri constants in this case is very interesting and they have been the focus of considerable study (cf. [11], [10], [22], [1], [2]). When $\mathcal{J}$ is the ideal of a smooth subvariety the Seshadri constants $\epsilon_H(\mathcal{J})$ were studied in the interesting papers ([25], [26]), of Paoletti, who considers especially smooth curves in threefolds. Several of the results in the present note are simple generalizations of statements appearing in and suggested by Paoletti's work, particularly [25], §3. In the past, however, it was unclear how to use geometric methods to study these invariants for arbitrary ideals. One of our main technical observations is that so long as one is content with asymptotic statements for powers, one doesn't need restrictions on the geometry of $\mathcal{J}$. This also motivates our study in the present section of the algebraic properties of the s-invariant. □

We start by comparing this invariant to the twists needed to generate $\mathcal{J}$. As customary, set

$$(1.3) \qquad d(\mathcal{J}) \;=\; d_H(\mathcal{J}) \;=\; \min\Big\{\, d \in \mathbf{Z} \mid \mathcal{J}(dH) \text{isgloballygenerated} \,\Big\}.$$

The following is due to Paoletti:

Lemma 1.4. *One has the inequality*

$$s_H(\mathcal{J}) \;\leq\; d_H(\mathcal{J}).$$

More generally,

$$s_H(\mathcal{J}) \;\leq\; \frac{d_H(\mathcal{J}^p)}{p}$$

for every integer $p \geq 1$.

Proof. In fact, supppose that $\mathcal{J}(dH)$ is globally generated. Then

$$\nu^{-1}\mathcal{J}(dH) \;=\; \mathcal{O}_V\big(\nu^*(dH) - E\big)$$

is likewise globally generated and hence nef. Therefore $s_H(\mathcal{J}) \leq d$. The second assertion is proven similarly.

Remark 1.5. We will see later (Theorem 3.2) that if H is very ample, then in fact $s_H(\mathcal{J}) = \lim \frac{d_H(\mathcal{J}^p)}{p}$. □

Example 1.6. (Schemes cut out by quadrics). Take $X = \mathbf{P}^n$ and H a hyperplane, and suppose that $\mathcal{J}$ is generated by quadrics, i.e. that $\mathcal{J}(2)$ is spanned by its global sections. Assume in addition that the zero-locus $Z = \text{Zeroes}(\mathcal{J})$ is not a linear space. Then $s_H(\mathcal{J}) = 2$. Indeed, the previous Lemma shows that $s_H(\mathcal{J}) \leq 2$, and by taking C' in (1.1) to be the proper transform of a general secant line to Z, one sees that $s_H(\mathcal{J}) \geq 2$.

Example 1.7. (Irrational **s***-invariants).* A construction used on several occasions by the first author (cf. [6]) leads to examples where $s_H(\mathcal{J})$ is irrational. This of course also gives examples where $s_H(\mathcal{J}) < d_H(\mathcal{J})$.[3] Take A to be an abelian surface with Picard number $\rho(A) \geq 3$ (for example A might be the product of two copies of an elliptic curve). Denote by $\text{Nef}(A) \subset NS(A)_{\mathbf{R}}$ the cone of numerically effective real divisor classes. Then, as on any abelian surface,

$$\text{Nef}(A) \ = \ \{\alpha \in NS(A)_{\mathbf{R}} \mid (\alpha^2) \geq 0\ ,\ (\alpha \cdot h) \geq 0\ \},$$

h being any ample class. But the Hodge Index theorem shows that the intersection form has type $(+, -, \dots, -)$ on $NS(A)_{\mathbf{R}}$, and therefore $\text{Nef}(A)$ is a circular cone. At least on suitable A, can then find an effective curve $C \subset A$, plus an ample divisor class H such that the ray passing through $-[C]$ in the direction of $[H]$ meets the boundary of $\text{Nef}(A)$ at an irrational point, i.e.

$$\inf \left\{ s > 0 \mid sH - C \in \text{Nef}(A) \right\} \notin \mathbf{Q}.$$

Taking $\mathcal{J} = \mathcal{O}_A(-C)$, this means that $s_H(\mathcal{J})$ is irrational. Note that one can replace H by aH and C by $C + bH$ $(a, b \in \mathbf{N})$, and so arrive at examples with C and H arbitrarily positive. □

Remark 1.8. (Algebraic **s***-invariants).* As in the case of "punctual" Seshadri constants [28], it follows from a theorem of Campana and Peternell [5] that s-invariants are always algebraic numbers. In fact, Campana and Peternell show that if η is a nef **R**-divisor class on a variety Y which is not ample, then there exists an irreducible subvariety $Z \subseteq Y$ such that $\int_Z \eta^{\dim Z} = 0$. Applying this to $\eta = \nu^*(sH) - F$ on $W = \text{Bl}_{\mathcal{J}}(X)$ yields an integer polynomial satisfied by $s = s_H(\mathcal{J})$. □

Remark 1.9. (Paoletti's geometric interpretation of the **s***-invariant).* Suppose that $Y \subset X$ is a smooth subvariety with normal bundle $N = N_{Y/X}$, and set $\mathcal{J} = \mathcal{I}_{Y/X}$. Then Paoletti [25], p. 487, shows that the s-invariant $t_H(\mathcal{J})$ has a simple geometric interpretation, as follows. Consider first a non-constant mapping f :

[3] See also Example 2.8.

$C \longrightarrow X$ from a smooth curve to X. If $f(C) \not\subseteq Y$, then $f^{-1}\mathcal{J} \subset \mathcal{O}_C$ is an ideal of finite colength in $\mathcal{O}_C$, and we define

$$s'_H(\mathcal{J}) \;=\; \sup_{\substack{f:C\longrightarrow X\\ f(C)\not\subseteq Y}} \left\{ \frac{\operatorname{colength}(f^{-1}\mathcal{J})}{(C \cdot_f H)} \right\},$$

where $(C \cdot_f H)$ denotes the degree of the divisor f^*H on C. Next, put

$$s''_H(\mathcal{J}) \;=\; \inf\big\{\, s > 0 \mid N^*(sH) \text{ isnef } \big\},$$

the nefness of a bundle twisted by an **Q** or **R** divisor being defined in the evident manner (cf. [23], Chapter 2). Then

$$s_H(\mathcal{J}) \;=\; \max\{\, s'_H(\mathcal{J})\,,\; s''_H(\mathcal{J})\,\}.$$

In fact, given $f : C \longrightarrow X$ as above, let $f' : C \longrightarrow W$ be the proper transform of f. Then $\operatorname{colength}(f^{-1}\mathcal{J}) = \big(C \cdot_{f'} F\big)$, and consequently $s'_H(\mathcal{J})$ is the least real number $s' > 0$ such that $\nu_0^*(s'H) - F$ has non-negative degree on every curve $C' \subset W$ not lying in the exceptional divisor $F \subset W$. Similarly, $s''_H(\mathcal{J})$ controls the nefness of $\mathcal{O}_F\big(\nu_0^*(s''H) - F\big)$.

For constructing examples, it is useful to understand something about how the s-invariant behaves in “chains" of subvarieties. With X as before, consider then a sequence of non-singular irreducible subvarieties.

$$Z \subseteq Y \subseteq X,$$

and fix an ample divisor H on X. There are three naturally defined ideal sheaves in this setting, and we can consider the correponding s-invariants

$$s_H(\mathcal{I}_{Z/X}) \;,\;\; s_H(\mathcal{I}_{Y/X}) \;\text{ and }\; s_H(\mathcal{I}_{Z/Y});$$

in the third case we view H as an ample divisor on Y, and compute on the blow-up of Y. One evidently has the inequality $s_H(\mathcal{I}_{Z/Y}) \leq s_H(\mathcal{I}_{Z/X})$, and in favorable situations the two invariants in question coincide:

Proposition 1.10. *In the situation just described, assume that* $s_H(\mathcal{I}_{Y/X}) < s_H(\mathcal{I}_{Z/Y})$. *Then*

$$s_H(\mathcal{I}_{Z/X}) = s_H(\mathcal{I}_{Z/Y}).$$

Proof. We keep the notation introduced in Remark 1.9. It is evident that

$$s'_H(\mathcal{I}_{Z/X}) \;\leq\; \max\{\, s'_H(\mathcal{I}_{Z/Y})\,,\; s'_H(\mathcal{I}_{Y/X})\,\},$$

and it follows from the conormal bundle sequence $0 \to N^*_{Y/X}|Z \to N^*_{Z/X} \to N^*_{Z/Y} \to 0$ that

$$s''_H(\mathcal{I}_{Z/X}) \;\leq\; \max\{\, s''_H(\mathcal{I}_{Z/Y})\,,\; s''_H(\mathcal{I}_{Y/X})\,\}.$$

The assertion is then a consequence of Remark 1.9.

Example 1.11. (Irrational **s***-invariants on projective space).* One can combine Example 1.7 with the previous Proposition to arrive at a quick example of a curve $C \subset \mathbf{P}^r$, with ideal sheaf $\mathcal{J} = \mathcal{I}_{C/\mathbf{P}^r}$, such that $s_H(\mathcal{J})$ is irrational, H being the hyperplane class. Specifically, take a very ample divisor H on an abelian surface A, plus a curve $C \subset X$, such that $s_H(\mathcal{O}_A(-C))$ is an irrational number > 2, and such that A is cut out by quadrics under the embedding $A \subset \mathbf{P} = \mathbf{P}^r$ defined by H. [Starting with any C and H giving irrational invariant, first replace H by $4H$ to ensure that $\mathcal{I}_{A/\mathbf{P}}$ is generated by quadrics, and then replace C by $C + mH$ for $m \gg 0$ to make $s_H(\mathcal{O}_A(-C)) > 2$.] Then $s_H(\mathcal{I}_{A/\mathbf{P}}) = 2$, so by applying the previous example to the chain $C \subset A \subset \mathbf{P}^r$, we find that $s_H(\mathcal{J}) = s_H(\mathcal{O}_A(-C))$. Examples of curves in $\mathbf{P}^3$ having irrational s-invariant were given by the first author in [7][4], but they involved more computation. (The present approach has the additional advantage that it actually works over any algebraically closed ground field.) □

We conclude this section with a result giving some algebraic properties of the s-invariant:

Proposition 1.12. *Let X be an irreducible projective variety, and H an ample divisor on X.*

(i). Given ideal sheaves $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{O}_X$, one has the inequalities:

$$\begin{aligned} s_H\big(\mathcal{J}_1 \cdot \mathcal{J}_2\big) &\leq s_H(\mathcal{J}_1) + s_H(\mathcal{J}_2) \\ s_H\big(\mathcal{J}_1 + \mathcal{J}_2\big) &\leq \max\big\{ s_H(\mathcal{J}_1)\,,\ s_H(\mathcal{J}_2) \big\}. \end{aligned}$$

(ii). If $\overline{\mathcal{J}} \subseteq \mathcal{O}_X$ denotes the integral closure of $\mathcal{J}$, then

$$s_H(\overline{\mathcal{J}}) = s_H(\mathcal{J}).$$

For basic facts about the integral closure of an ideal, see [29].

Proof. We will apply Remark 1.2. Thus for (i), let $f : Y \longrightarrow X$ be a surjective mapping from an irreducible variety Y which dominates the blowings-up of X along $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_1 + \mathcal{J}_2$. Thus Y carries effective Cartier divisors E_1, E_2 and E_{12} characterized by

$$\mathcal{J}_1 \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_1)\ ,\ \ \mathcal{J}_1 \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_1)\,,\ \big(\mathcal{J}_1 + \mathcal{J}_2\big) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_{12}).$$

Note that then

$$\big(\mathcal{J}_1\mathcal{J}_2\big) \cdot \mathcal{O}_Y = \mathcal{O}_Y\big(- (E_1 + E_2)\,\big). \tag{*}$$

[4] The cited paper deals with curves having irrational asymptotic Castelnuovo-Mumford regularity, but Theorem 3.2 shows that this is the same as irrational s-invariant.

Write $s_1 = s_H(\mathcal{J}_1)$ and $s_2 = s_H(\mathcal{J}_2)$. Then $f^*(s_1H) - E_1$ and $f^*(s_2H) - E_2$ are nef on Y, and consequently so is their sum $f^*\big((s_1 + s_2)H\big) - (E_1 + E_2)$. The first inequality in (i) then follows from (*). For the second, set $s = \max\{s_1, s_2\}$ and note that one has a surjective map

$$\mathcal{O}_Y(-E_1) \oplus \mathcal{O}_Y(-E_2) \longrightarrow \mathcal{O}_Y(-E_{12})$$

of vector bundles on Y. By definition of s, the bundle on the left becomes nef when twisted by the **R**-divisor $f^*(sH)$. Since quotients of nef bundles are nef, this implies that $f^*(sH) - E_{12}$ is nef, and the required inequality follows.[5]

For (ii), we use the fact (cf. [29], p.330) that $\mathrm{Bl}_{\mathcal{J}}(X)$ and $\mathrm{Bl}_{\overline{\mathcal{J}}}(X)$ have the same normalization V, which sits in a commutative diagram:

$$\begin{tikzcd}
& V \arrow[dl] \arrow[dr] \arrow[dd, "\mu"] & \\
\mathrm{Bl}_{\overline{\mathcal{J}}}(X) \arrow[dr, "\overline{\nu}"'] & & \mathrm{Bl}_{\mathcal{J}}(X) \arrow[dl, "\nu"] \\
& X &
\end{tikzcd}$$

Moreover, the exceptional divisors F and $\overline{F}$ of ν and ν pull back to the same divisor E on V. Invoking again Remark 1.2 one has

$$s_H(\mathcal{J}) \;=\; \inf\{s > 0 \mid \mu^*(sH) - E \text{ isnef}\} \;=\; s_H(\overline{\mathcal{J}}),$$

as required.

Remark 1.13. Definition 1.1 and (1.4) remain valid on singular varieties. In (1.12) it would be enough to assume that X is normal. □

2. Degree and nilpotency bounds

In the present section, we show how the s-invariant governs bounds on the degrees of zeroes of an ideal and its index of nilpotency. For the most part this involves only small modifications to computations appearing for instance in [12], so we shall be brief.

We start by fixing some additional notation. Let X be a non-singular irreducible quasi-projective variety of dimension n — which for the moment we do

[5] We are implicitly using here the fact that nefness makes sense for twists of bundles by **Q**- or **R**-divisors, and that the usual formal properties are satisfied. These facts are worked out in Chapter 2 of the forthcoming book [23], but the reader can easily verify the required assertion directly by considering the evident **R**-divisors on the projectivization $\mathbf{P}\big(\mathcal{O}_Y(-E_1) \oplus \mathcal{O}_Y(-E_2)\big) \longrightarrow X$.

not assume to be projective — and suppose that $\mathcal{J} \subset \mathcal{O}_X$ is a coherent sheaf of ideals on X. As before we denote by $\nu : W = \mathrm{Bl}_{\mathcal{J}}(X) \longrightarrow X$ the blowing up of $\mathcal{J}$, with exceptional divisor F. Consider now the normalization $p : V \longrightarrow W$ of W, with $\mu : V \longrightarrow X$ the natural composition:

$$\begin{tikzcd} V \arrow[r, "p"'] \arrow[rr, bend left, "\mu"] & W \arrow[r, "\nu"'] & X. \end{tikzcd}$$

We denote by $E = p^*F$ the pull-back to V of the exceptional divisor F on W. Thus E is an effective Cartier divisor on V, and

$$\mathcal{J} \cdot \mathcal{O}_V \;=\; \mathcal{O}_V(-E).$$

Note that $\mathcal{O}_V(-E)$ is ample relative to μ, and in particular is ample on every fibre of μ.

Now E determines a Weil divisor on V, say

$$[E] \;=\; \sum_{i=1}^{t} r_i \cdot [E_i],$$

where the E_i are the irreducible components of the support of E, and $r_i > 0$. Set

$$Z_i \;=\; \mu(E_i) \;\subseteq\; X,$$

so that Z_i is a reduced and irreducible subvariety of X. Following [15], the Z_i are called the *distinguished subvarieties* of $\mathcal{J}$. (Note that several of the E_i may have the same image in X, in which case there will be repetitions among the Z_i. However this doesn't cause any problems.) Denoting by

$$Z \;=\; \mathrm{Zeroes}\big(\sqrt{\mathcal{J}}\big)$$

the reduced zero-locus of $\mathcal{J}$, one has then the decomposition

$$Z \;=\; \cup Z_i$$

of Z as a union of distinguished subvarieties. Thus each irreducible component of Z is distinguished, but there can be “embedded" distinguished subvarieties as well. We refer to the positive integer r_i as the coefficient attached to Z_i, and we define

$$r(\mathcal{J}) \;=_{\mathrm{def}}\; \max\{r_i\}. \tag{2.1}$$

The following result, implicit in [12] and independently observed by Hickle [18], shows that the invariant $r(\mathcal{J})$ controls the index of nilpotency of $\mathcal{J}$:

Theorem 2.1. *One has*

$$\left(\sqrt{\mathcal{J}}\right)^{n\cdot r(\mathcal{J})} \subseteq \mathcal{J}.$$

More generally, $\left(\sqrt{\mathcal{J}}\right)^{(n+1-p)\cdot r(\mathcal{J})} \subseteq \mathcal{J}^p$ *for every integer* $p \geq 1$.

Sketch of Proof. One checks right away as in [12], (2.1) and (2.4), that

$$\left(\sqrt{\mathcal{J}}\right)^{\ell\cdot r(\mathcal{J})} \subseteq \mu_* \mathcal{O}_V(-\ell E) = \overline{\mathcal{J}^\ell}$$

for every $\ell \geq 0$. The stated inclusions then follow from the Briançon–Skoda theorem (cf. [19]). □

In order to give Theorem 2.1 some real content, one needs an upper bound on $r(\mathcal{J})$. It would be interesting to know whether one can give useful statements in a purely local setting. However globally they follow (Corollary 2.3) from the fact that one has Bezout-type inequalities for the degrees of the distinguished subvarieties in terms of the s-invariant of $\mathcal{J}$.

Assume henceforth that X is projective, and fix an ample divior class H on X.

Proposition 2.2. *Let* $s = s_H(\mathcal{J})$ *be the* s*-invariant of* $\mathcal{J}$ *with respect to* H. *Then*

$$\sum_{i=1}^{t} r_i \cdot s^{\dim Z_i} \cdot \deg_H Z_i \;\leq\; s^n \cdot \deg_H X,$$

where for any subvariety $V \subseteq X$, $\deg_H V = \left(H^{\dim V} \cdot V\right)$ *denotes the degree of* V *with respect to* H.

Corollary 2.3. *In the situation of the Proposition one has*

$$\sum s^{\dim Z_i} \deg_H Z_i \;\leq\; s^n \; \deg_H X,$$

and the integer $r(\mathcal{J}) = \max\{r_i\}$ *satisfies*

$$r(\mathcal{J}) \leq s_+^n \cdot \deg_H X,$$

where $s_+ = \max\{1, s_H(\mathcal{J})\}$. □

The Proposition can be deduced from general positivity results due to Fulton and the third author [16]. However following [12] we indicate a direct proof using classical intersection theory.

Sketch of proof of Proposition 2.2. Consider the classes

$$h = [\mu^* H] \;,\; m = [\mu^*(sH) - E] \in \mathrm{NS}(V)_{\mathbf{R}}$$

in the vector space of numerical equivalence classes on X with real coefficients. Thus m is a nef class — so in particular $\int_V m^n \geq 0$ and $\int_{E_i} (s \cdot h)^j \cdot m^{n-1-j} \geq 0$ for all i and j — and $[E] = s \cdot h - m$. Arguing as in the proof of Proposition 3.1 in [12], one then finds that

$$\begin{aligned} s^n \cdot \deg_H(X) &= \int_V (s \cdot h)^n \\ &\geq \int_V \big((s \cdot h)^n - m^n\big) \\ &= \int_V \Big((s \cdot h) - m\Big)\Big(\sum_{j=0}^{n-1} (s \cdot h)^j \cdot m^{n-1-j}\Big) \\ &= \int_{[E]} \Big(\sum_{j=0}^{n-1} (s \cdot h)^j \cdot m^{n-1-j}\Big) \\ &\geq \sum_{i=1}^{t} r_i \cdot \int_{E_i} (s \cdot h)^{\dim(Z_i)} \cdot m^{n-1-\dim(Z_i)}, \\ &\geq \sum_{i=1}^{t} r_i \cdot s^{\dim Z_i} \cdot \deg_H Z_i, \end{aligned}$$

as required. □

Remark 2.4. Suppose that $Z_1, \ldots, Z_p$ are the (distinct) irreducible components of Z. Then arguing as in [15] (4.3.4) and (12.2.9) one finds

$$\sum_{i=1}^{p} e_{Z_i}(\mathcal{J}) \cdot s^{\dim Z_i} \cdot \deg_H Z_i \;\leq\; s^n \cdot \deg_H X,$$

where $e_{Z_i}(\mathcal{J})$ is the Samuel multiplicity of $\mathcal{J}$ along Z_i. □

One does not expect Bezout-type inequalities such as 2.3 to capture the embedded components of $\mathcal{J}$ in the sense of primary decomposition (see [3], [13], [21] and Example 2.8 below). Somewhat unexpectedly, however, the situation is different when $\mathcal{J}$ is integrally closed:

Corollary 2.5. *Assume that $\mathcal{J}$ is integrally closed, and let $Y_1, \ldots, Y_q \subseteq X$ be the irreducible subvarieties defined by all the associated primes of $\mathcal{J}$ (minimal or embedded). Then*

$$\sum_{j=1}^{q} s^{\dim Y_j} \deg_H Y_j \;\leq\; s^n \;\deg X.$$

Proof. It is enough to show that every associated subvariety is distinguished. To this end, let $\mathfrak{q}_i = \mu_* \mathcal{O}_Y\big(-r_i E_i\big)$ be the sheaf of all functions on X whose pull-backs to V vanish to order $\geq r_i$ along the Weil divisor E_i. Then $\mathfrak{q}_i \subseteq \mathcal{O}_X$ is a primary ideal, and one has

$$\begin{aligned} \overline{\mathcal{J}} &= \mu_* \mathcal{O}_Y(-E) \\ &= \bigcap_{i=1}^{t} \mu_* \mathcal{O}_Y\big(-r_i E_i\big). \end{aligned}$$

Since $\mathcal{J} = \overline{\mathcal{J}}$ this means that we have the (possibly redundant) primary decomposition $\mathcal{J} = \cap \mathfrak{q}_i$. In particular every associated prime of $\mathcal{J}$ must occur as the radical of one of the $\mathfrak{q}_i$, i.e. as one of the distinguished subvarieties.

Remark 2.6. The argument just given to show that each associated subvariety of $\overline{\mathcal{J}}$ is distinguished appears a number of times in the literature (e.g. [18]). However it seems to have been overlooked that this leads to degree bounds on associated subvarieties for integrally closed ideals. □

Remark 2.7. The same argument shows more generally that for any ideal $\mathcal{J}$, the bound $\sum s^{\dim Y} \deg_H Y \leq s^n \deg_H X$ holds if one sums over all subvarieties Y defined by an associated prime ideal of the integral closure $\overline{\mathcal{J}^p}$ of some power of $\mathcal{J}$. □

Example 2.8. (Pathological ideals with fixed **s***-invariant).* We construct here a family of ideals having fixed s-invariant but arbitrarily many embedded points. The same examples will show that the regularity bounds presented in the next section only hold asymptotically. For simplicity we work over the complex numbers $\mathbf{C}$, but in fact one could deal with an arbitrary algebraically closed ground-field.

In order to highlight the underlying geometric picture, we start with a local discussion. Working in affine three-space $X = \mathbf{A}^3$ with coordinates x, y, t, consider the ideal $\mathfrak{a} = \mathfrak{a}_p = (x^2, p(t) \cdot xy, y^2) \in \mathbf{C}[x, y, t]$, where $p(t) \in \mathbf{C}[t]$ is a polynomial in t. Then the zeroes of $p(t)$ along the line ℓ defined by $\{x = y = 0\}$ are embedded points of $\mathfrak{a}$. On the other hand, let $f : Y = \mathrm{Bl}_{(x,y)}(X) \longrightarrow X$ is the blowing up of X along ℓ, with exceptional divisor E. Then one checks that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-2E)$, so in other words on Y the ideal $\mathfrak{a}$ cannot be distinguished from the square $(x, y)^2$ of the ideal of ℓ.[6] The idea is that in the global setting, the s-invariant will be computed on the blow-up of the line (Example 1.2), and so cannot detect the embedded points.

This local construction is easily globalized. Take $X = \mathbf{P}^3$ with homogeneous coordinates X, Y, Z, W, fix a homogeneous polynomial $P_d = P_d(Z, W) \in$

[6] Geometrically, the important point is that for every complex number $a \in \mathbf{C}$, the homogeneous polynomials x^2, $p(a)xy$, $y^2 \in \Gamma\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2)\big)$ span a base-point linear series. More algebraically, observe that already $(x^2, y^2) \cdot \mathcal{O}_Y = \mathcal{O}_X(-2E)$.

$\mathbf{C}[Z, W]$ of degree d, and let $\mathcal{J} = \mathcal{J}_P \subseteq \mathcal{O}_{\mathbf{P}^3}$ be the ideal sheaf spanned by the homogeneous polynomials X^2, $P_d \cdot XY$ and Y^2. Denoting by $L \subseteq \mathbf{P}^3$ the line $\{X = Y = 0\}$, one sees as above that the zeroes of P_d along L are embedded points of $\mathcal{J}$, so for general P there will be d such. As before let $Y = \mathrm{Bl}_L(\mathbf{P}^3) \longrightarrow \mathbf{P}^3$ be the blowing-up of L, with exceptional divisor E. Then $\mathcal{J}_P \cdot \mathcal{O}_Y = \mathcal{O}_Y(-2E)$, so it follows from Example 1.2 that $s_H(\mathcal{J}_P) = 2$ for every P (H being the hyperplane divisor). In particular, the number of embedded points cannot be bounded in terms of the s-invariant. □

Remark 2.9. Proposition 2.2 does not require X non-singular. The smoothnes of X is however used when the Briançon-Skoda theorem is invoked in 2.1. However Huneke [19] has established analogues of this result which would lead to statements on arbitrary X.

3. Asymptotic regularity and degree bounds

In the present section we bound the "complexity" of large powers of an ideal sheaf in terms of its s-invariant.

As above, let X be a non-singular irreducible projective variety of dimension n. We assume in this section that H is a *very* ample divisor on X. In this case the Castelnuovo-Mumford regularity of a coherent sheaf $\mathcal{F}$ on X is defined just as in the classical setting of projective space:

Definition 3.1. *A coherent sheaf $\mathcal{F}$ is m-regular (with respect to H) if*

$$H^i\big(X, \mathcal{F}\big((m-i)H\big)\big) = 0 \ \text{ for } i > 0.$$

The regularity $\mathrm{reg}_H(\mathcal{F})$ *of $\mathcal{F}$ is the least integer m for which $\mathcal{F}$ is m-regular.*[7] □

Just as in the classical case, if $\mathcal{F}$ is m-regular for some integer m, then $\mathcal{F}(mH)$ is globally generated, and $\mathcal{F}$ is also $(m+1)$-regular. We view the regularity of a sheaf as a measure of its cohomological complexity. When $X = \mathbf{P}^n$, this regularity has a well-known interpretation as bounding the degrees of the generators of the modules of syzygies of the module corresponding to $\mathcal{F}$ (see [3]).

Fix now an ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ with s-invariant $s_H(\mathcal{J})$. As above we denote by $d_H(\mathcal{J}^p)$ the least integer $d \geq 0$ such that $\mathcal{J}^p(dH)$ is globally generated.

Theorem 3.2. *The quantities $\frac{\mathrm{reg}_H(\mathcal{J}^p)}{p}$ and $\frac{d_H(\mathcal{J}^p)}{p}$ tend to limits as $p \to \infty$, and one has:*

$$\lim_{p\to\infty} \frac{\mathrm{reg}_H(\mathcal{J}^p)}{p} = \lim_{p\to\infty} \frac{d_H(\mathcal{J}^p)}{p} = s_H(\mathcal{J}).$$

[7] If $\mathcal{F}$ is m-regular for every $m \in \mathbf{Z}$ — which will occur if and only if $\mathcal{F}$ is supported on a finite set — we put $\mathrm{reg}_H(\mathcal{F}) = -\infty$.

Proof. Set $d_p = d_H(\mathcal{J}^p)$ and $r_p = \mathrm{reg}_H(\mathcal{J}^p)$. Note to begin with that $d_{\ell+m} \leq d_\ell + d_m$ for all $\ell, m \geq 0$, from which it follows that the limit $\lim_{p\to\infty} \frac{d_p}{p}$ exists. Call this limit $\bar{d}$. We will prove the theorem by establishing (from right to left) the inequalities

$$\limsup \frac{r_p}{p} \ \leq\ s_H(\mathcal{J}) \ \leq\ \bar{d} \ \leq\ \liminf \frac{r_p}{p}. \tag{3.1}$$

Starting with the right-most inequality in (3.1), recall that if $\mathcal{J}^p$ is m-regular with respect to H then $\mathcal{J}^p(mH)$ is globally generated. Therefore $d_p \leq r_p$ for every $p > 0$, and in particular $\lim \frac{d_p}{p} \leq \liminf \frac{r_p}{p}$.

We next show that $s_H(\mathcal{J}) \leq \lim \frac{d_p}{p} = \bar{d}$. To this end, fix any $\epsilon > 0$. Then we can choose large positive integers $p_0, q_0 > 0$ such that

$$\frac{d_{p_0}}{p_0} \ \leq\ \frac{q_0}{p_0} \ \leq\ \bar{d} + \epsilon,$$

so that in particular $\mathcal{J}^{p_0}(q_0H)$ is globally generated. Writing as before $\nu : W = \mathrm{Bl}_{\mathcal{J}}(X) \to X$ for the blow-up of $\mathcal{J}$, with exceptional divisor F, it follows that $\nu^*(q_0H) - p_0F$ is globally generated and hence nef. Therefore $s_H(\mathcal{J}) \leq \frac{q_0}{p_0} \leq \bar{d} + \epsilon$, as required.

It remains to prove that $\limsup \frac{r_p}{p} \leq s_H(\mathcal{J})$. To this end we use a theorem of Fujita [14] to the effect that Serre Vanishing remains valid even after twisting by arbitrary nef divisors. Specifically, consider an irreducible projective variety V, and fix an ample divisor A plus a coherent sheaf $\mathcal{F}$ on V. Fujita shows that there is an integer $m_0 = m_0(A, \mathcal{F})$ such that for any nef divisor B:

$$H^i\big(V, \mathcal{F}(mA + B)\big) = 0 \ \text{ forall } i > 0 \text{ and } m \geq m_0. \tag{*}$$

(The important point here is that m_0 is independent of B.)

We propose to apply (*) on the blowing-up $W = \mathrm{Bl}_{\mathcal{J}}(X)$ of $\mathcal{J}$. Given $\epsilon > 0$, choose large integers q_0, p_0 such that

$$s_H(\mathcal{J}) \ <\ \frac{q_0}{p_0} \ <\ s_H(\mathcal{J}) + \frac{\epsilon}{2}.$$

Then $\nu^*(q_0H) - p_0F$ is ample, so there exists an integer m_0 such that if $m \geq m_0$ then for any nef divisor P on W, the bundles associated to the divisors $\nu^*(mq_0H) - mp_0F + P$ have vanishing higher cohomology. Now fix any integer $p \geq m_0p_0$, and write

$$p = mp_0 + p_1 \ \text{ with } \ 0 \leq p_1 < p_0 \text{ and } m \geq m_0.$$

Then $\nu^*(q_0H) - p_1F$ is nef (in fact ample), and consequently we have the vanishing of the higher cohomology of the line bundle

$$\mathcal{O}_W\big(\, \nu^*\,((m+1)q_0H\,) - pF\,\big).$$

It now follows from Lemma 3.3 below — and this is the crucial point — that

$$H^i\Big(X\,,\ \mathcal{J}^p\big((m+1)q_0H\big)\Big) = 0 \ \text{ for } i > 0$$

provided that p is sufficiently large. Therefore $\mathcal{J}^p$ is $\big((m+1)q_0+n\big)$-regular for $p \gg 0$, and consequently

$$\frac{r_p}{p} \ \le \ \frac{(m+1)q_0+n}{p} \ \le \ \frac{q_0}{p_0} + \frac{q_0+n}{mp_0}.$$

By taking p (and hence also m) to be large enough, we can arrange that the second term on the right is $\le \frac{\epsilon}{2}$, so that $\frac{r_p}{p} \le s_H(\mathcal{J}) + \epsilon$ for $p \gg 0$. Therefore $\limsup \frac{r_p}{p} \le s_H(\mathcal{J})$, and we are done. □

The following Lemma played an essential role in the proof just completed. It shows that one can realize large powers of an ideal $\mathcal{J} \subseteq \mathcal{O}_X$ geometrically from the natural divisor on the blow-up.[8] This fact is surely not new, but we include a proof for the convenience of the reader.

Lemma 3.3. *Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal sheaf on X, and*

$$\nu : W = \mathrm{Bl}_{\mathcal{J}}(X) \longrightarrow X$$

the blowing-up of $\mathcal{J}$, with exceptional divisor F. There exists an integer $p_0 > 0$ with the property that if $p \ge p_0$, then

$$(*) \qquad \nu_*\mathcal{O}_W\big(-pF\big) = \mathcal{J}^p,$$

and for any divisor D on X:

$$H^i\big(X, \mathcal{J}^p(D)\big) = H^i\big(W, \mathcal{O}_W(\nu^*D - pF)\big)$$

for all $i \ge 0$.

Proof. Since $\mathcal{O}_W(-F)$ is ample for ν, it follows from Grothendieck-Serre vanishing that

$$R^j\nu_*\mathcal{O}_W(-pF) = 0 \quad \text{for } j > 0 \text{ and } p \gg 0.$$

The isomorphism on global cohomology groups is then a consequence of (*) thanks to the Leray spectral sequence.

As for (*), the assertion is local on X, so we may assume that X is affine. Choosing generators $g_1, \dots, g_r \in \mathcal{J}$ gives rise to a surjection $\mathcal{O}_X^r \longrightarrow \mathcal{J}$, which in turn determines an embedding

$$W = \mathrm{Bl}_{\mathcal{J}}(X) \ \subseteq \ \mathbf{P}\big(\mathcal{O}_X^r\big) = \mathbf{P}_X^{r-1}$$

[8] If $\mathcal{J}$ defined a smooth subvariety of X, then the corresponding statement would be true for all powers.

in such a way that $\mathcal{O}_{\mathbf{P}_X^{r-1}}(1) \mid W = \mathcal{O}_W(-F)$. Write $\pi : \mathbf{P}_X^{r-1} = X \times \mathbf{P}^{r-1} \longrightarrow X$ for the projection. Serre vanishing for π, applied to the ideal sheaf $\mathcal{I}_{W/\mathbf{P}_X^{r-1}}$, shows that if $p \gg 0$ then the natural homomorphism

$$(**) \qquad \pi_* \mathcal{O}_{\mathbf{P}_X^{r-1}}(p) \longrightarrow \pi_* \mathcal{O}_W(-pF)$$

is surjective. On the other hand, recalling that $\pi_* \mathcal{O}_{\mathbf{P}_X^{r-1}}(k) = S^k\big(\mathcal{O}_X^r\big)$ for every $k \geq 0$, one sees that the image of (**) is exactly $\mathcal{J}^p$. It follows that $\nu_* \mathcal{O}_W\big(-pF\big) = \mathcal{J}^p$ for $p \gg 0$, as asserted. □

Remark 3.4. The use of Serre Vanishing in the proof of Theorem 3.2 was suggested by Demailly's proof of Theorem 6.4 in [9]. Proposition 3.3 of [25] uses a similar argument to prove a result for zero-loci of vector bundles that is rather close in spirit to 3.2. □

Finally, we turn to asymptotic bounds on the arithmetic degree of $\mathcal{J}^p$. In a general way we follow the approach of Bayer and Mumford, suitably geometrized. We start by recalling the definition of the arithmetic degree from the viewpoint of [21].

Assume then that X is a non-singular irreducible projective variety of dimension n carrying a fixed ample divisor class H, and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a canonical filtration

$$0 \subseteq \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \ldots \subseteq \mathcal{F}^1 \subseteq \mathcal{F}^0 = \mathcal{F}$$

where $\mathcal{F}^k \subseteq \mathcal{F}$ is the subsheaf consisting of sections whose support has codimension $\geq k$ in X. As in [15], Example 18.3.11, one can in a natural way associate to the quotient $\mathcal{F}^k/\mathcal{F}^{k+1}$ a codimension k cycle $[\mathcal{F}^k/\mathcal{F}^{k+1}] \in Z^k(X)$, and then the codimension k contribution to the arithmetic degree of $\mathcal{F}$ is defined to be

$$\mathrm{adeg}_H^k\big(\mathcal{F}\big) \;=\; \deg_H\big([\mathcal{F}^k/\mathcal{F}^{k+1}]\big),$$

where as indicated the degree of $[\mathcal{F}^k/\mathcal{F}^{k+1}]$ is computed with respect to the fixed polarization H. For an ideal $\mathcal{J} \subseteq \mathcal{O}_X$ one sets $\mathrm{adeg}_H^k\big(\mathcal{J}\big) = \mathrm{adeg}_H^k\big(\mathcal{O}_X/\mathcal{J}\big)$. Thus $\mathrm{adeg}^k(\mathcal{J})$ measures the degrees of the codimension k components of $\mathcal{J}$ (both mimimal and embedded), counted with suitable multiplicities.

A variant of the following Lemma was implicitly used by Bayer and Mumford in a similar context, and re-examined in [24] .

Lemma 3.5. *Still assuming that H is very ample, let $D \subseteq X$ be a general divisor linearly equivalent to H, and let $\mathcal{F}_D = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ denote the restiction of $\mathcal{F}$ to D. If $k \leq n-1$ then*

$$\mathrm{adeg}_H^k\big(\mathcal{F}_D\big) \;=\; \mathrm{adeg}_H^k\big(\mathcal{F}\big),$$

where the degree on the left is computed with respect to the ample line bundle $\mathcal{O}_D(H)$ on D.

Indication of Proof. The essential point is to show that if $\mathcal{M}$ is an equidimensional $\mathcal{O}_X$-module without embedded components, then the restriction $\mathcal{M}_D$ of $\mathcal{M}$ to D is also equidimensional without embedded components (see [24] for an argument in a similar setting). Once one knows this, one can deduce the lemma from the fact [15], Examples (18.3.6) and (18.3.11), that $\deg_H\big([\mathcal{F}^k/\mathcal{F}^{k+1}]\big)$ governs the leading term of the Hilbert polynomial of the sheaf in question. We leave details to the reader. □

In the spirit of [3], Proposition 3.6, we show that — at least asymptotically — the arithmetic degrees of large powers of an ideal are bounded in terms of their regularity:

Theorem 3.6. *Suppose as above that X is a smooth irreducible projective variety, and assume that H is a very ample divisor on X. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal sheaf on X, and set $\overline{\mathrm{reg}}_H(\mathcal{J}) = \lim \frac{\mathrm{reg}_H(\mathcal{J}^p)}{p}$. Then for every $0 \le k \le n$:*

$$\limsup_{p\to\infty} \frac{\mathrm{adeg}_H^k\big(\mathcal{J}^p\big)}{p^k} \le \frac{\big(\overline{reg}_H(\mathcal{J})\big)^k}{k!} \cdot \deg_H(X).$$

Corollary 3.7. *In the situation of the Proposition,*

$$\limsup_{p\to\infty} \frac{\mathrm{adeg}_H^k\big(\mathcal{J}^p\big)}{p^k} \le \frac{s_H(\mathcal{J})^k}{k!} \cdot \deg_H(X).\square$$

Proof of Theorem 3.6. Passing to a suitable field extension (which leaves both the regularity and the arithmetic degree unchanged), we can assume without loss of generality that the ground field is uncountable. Let $D \in |H|$ be a general divisor linearly equivalent to H, and consider the restriction $\mathcal{J}_D = \mathcal{J} \cdot \mathcal{O}_D$ of $\mathcal{J}$ to D. According to a theorem of Ratliff [27] there are only finitely many prime ideals which appear as associated primes for any of the ideals $\mathcal{J}^p$ for $p \ge 1$. So we may assume that $\mathcal{O}_X(-D)$ does not contain any of these primes, so that the sequence

$$0 \longrightarrow \mathcal{J}^p(-D) \xrightarrow{\cdot D} \mathcal{J}^p \longrightarrow \mathcal{J}_D^p \longrightarrow 0$$

is exact for every p. This sequence shows that $\mathrm{reg}_H(\mathcal{J}_D^p) \le \mathrm{reg}_H(\mathcal{J}^p)$ for every p, where by abuse of notation we are writing H for the class of the restriction $\mathcal{O}_D(H)$ to D. Consequently $\overline{\mathrm{reg}}_H(\mathcal{J}_D) \le \overline{\mathrm{reg}}_H(\mathcal{J})$. Similarly, Lemma 3.5 shows that $\mathrm{adeg}_H^k(\mathcal{J}^p) \le \mathrm{adeg}_H^k(\mathcal{J}_D^p)$ for fixed p provided that $k \le n-1$. As we are now working over an uncountable ground field, we can assume by taking D to be very general that this holds simultaneously for all $p \ge 1$. Since of course also $\deg_H D = \deg_H X$, if $k \le n-1$ it therefore suffices to prove the Proposition for D. So by induction on $n = \dim X$ we can assume that $k = n$.

Supposing then that $k = n$, we need to bound as a function of $p \gg 0$ the length of the (finitely supported) subsheaf

$$\mathcal{Q}_p \subseteq \mathcal{O}_X/\mathcal{J}^p$$

of sections having zero-dimensional support. Equivalently, we need to bound for $p \gg 0$ the dimension $h^0\big(X, \mathcal{Q}_p\big)$. To this end, observe first of all that for every integer $q \in \mathbf{N}$ there is an inclusion

(3.2)
$$H^0\big(X, \mathcal{Q}_p\big) \cong H^0\big(X, \mathcal{Q}_p \otimes \mathcal{O}_X(qH)\big) \subseteq H^0\big(X, \big(\mathcal{O}_X/\mathcal{J}^p\big) \otimes \mathcal{O}_X(qH)\big).$$

The plan is to estimate the dimension of the group on the right for a suitable integer q. Fix $\epsilon > 0$ plus large integers $p, q \gg 0$ such that

$$\overline{\mathrm{reg}}_H(\mathcal{J}) + \epsilon > \tfrac{q}{p} > \tfrac{r_p}{p}, \tag{3.3}$$

where $r_p = \mathrm{reg}_H(\mathcal{J}^p)$. Then $H^1\big(X, \mathcal{J}^p \otimes \mathcal{O}_X(qH)\big) = 0$, and so the exact sequence

$$0 \longrightarrow \mathcal{J}^p \otimes \mathcal{O}_X(qH) \longrightarrow \mathcal{O}_X(qH) \longrightarrow \big(\mathcal{O}_X/\mathcal{J}^p\big) \otimes \mathcal{O}_X(qH) \longrightarrow 0$$

together with (3.2) shows that

$$\begin{aligned} \mathrm{adeg}^n_H\big(\mathcal{J}^p\big) &= h^0\big(X, \mathcal{Q}_p\big) \\ &\leq h^0\big(X, (\mathcal{O}_X/\mathcal{J}^p) \otimes \mathcal{O}_X(qH)\big) \\ &\leq h^0\big(X, \mathcal{O}_X(qH)\big). \end{aligned} \tag{3.4}$$

But Riemann-Roch implies that as a function of q,

$$h^0\big(X, \mathcal{O}_X(qH)\big) = \frac{q^n}{n!} \cdot \deg_H(X) + O(q^{n-1}).$$

It follows from (3.3) that by taking p (and hence q) sufficiently large, and ϵ sufficiently small, we can arrange that

$$\frac{1}{p^n} \cdot h^0\big(X, \mathcal{O}_X(qH)\big) \leq \frac{\big(\overline{\mathrm{reg}}_H(\mathcal{J})\big)^n}{n!} \cdot \deg_H(X) + C\epsilon$$

where C is a constant. The result then follows from (3.4). □

Remark 3.8. (Non-asymptotic pathology). In Example 2.8, we constructed for each $d \geq 1$ ideals $\mathcal{J} = \mathcal{J}_d$ on $\mathbf{P}^3$ with fixed s-invariant $s_H(\mathcal{J}_d) = 2$, but having d embedded points. This shows that one cannot bound the arithmetic degree of an ideal in terms of its s-invariant. One easily checks that the regularity of the ideals $\mathcal{J}_d$ also goes to infinity with d. So by the same token, the regularity of a given ideal cannot be bounded in terms of its s-invariant. This pathology contrasts with results of Bayer and Mumford [3] showing that there are (multiply exponential) bounds for the regularity and arithmetic degree of a homogeneous ideal in terms of the degrees of its generators. The overall picture that seems to emerge is that the singly exponential Bezout-type bounds appearing in [3] are explained geometrically, i.e. in terms of the s-invariant, whereas the multiply-exponential bounds on regularity and arithmetic degree are more algebraic in nature. □

Remark 3.9. Theorems 3.2 and 3.6 do not require that X be smooth. (In Lemma 3.5 one can use the hypothesis that H is very ample to reduce to the case when $X = \mathbf{P}^n$.) □

References

1. T. Bauer, Seshadri constants and periods of polarized abelian varieties (with an appendix by the author and Tomasz Szemberg) Math. Ann. **312** (1998), 607–623
2. T. Bauer, Seshadri constants on algebraic surfaces, Math. Ann. **313** (1999), 547–583
3. D. Bayer and D. Mumford, What can be computed in algebraic geometry? In: Computational Algebraic Geometry and Commutative Algebra (Eisenbud and Robbiano, eds.) Cambridge Univ. Press, 1993, pp. 1–48
4. D. Brownawell, Bounds for the degrees in the Nullstellensatz, Ann. Math **126** (1987), 57–591
5. F. Campana and T. Peternell, Algebraicity of the ample cone of projective varieties, J. Reine Angew. Math. **407** (1990), 160–166
6. S.D. Cutkosky, Zariski decomposition of divisors on algebraic varieties, Duke Math. J. **53** (1986), 149–156
7. S. D. Cutkosky, Irrational asymptotic behaviour of Castelnuovo regularity, to appear in Journal für die reine und angewandte Mathematik
8. S.D. Cutkosky, J. Herzog and Trung N.V, Asymptotic behaviour of the Castelnuovo-Mumford regularity, Compositio Math. **118** (1999), 243–261
9. J.-P. Demailly, Singular Hermitian metrics on positive line bundles, in Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math.**1507**, 1992, 87–104
10. L. Ein, O. Küchle and R. Lazarsfeld, Local positivity of ample line bundles, J. Differential Geometry **42** (1995), 193–219
11. L. Ein and R. Lazarsfeld, Seshadri constants on smooth surfaces, in Journées de Géométrie Algébrique d'Orsay, Astérique **218** (1993), 177–186
12. L. Ein and R. Lazarsfeld, A geometric effective Nullstellensatz, Invent. Math. **137** (1999), 427–448
13. H. Flenner, On imbedded components of improper intersections in projective space Arch. Math. (Basel) **48** (1987), 533–537
14. T. Fujita, Vanishing theorems for semipositive line bundles, in Algebraic Geometry, Proceedings, Tokyo/Kyoto 1982, Lect. Notes in Math. **1016** (1983), 519–528
15. W. Fulton, Intersection Theory (Second Edition), Springer-Verlag, Berlin (1998)
16. W. Fulton and R. Lazarsfeld, Positivity and excess intersection, in Enumerative and Classical Geometry, Nice, 1981, Progress in Math. **24** (1982), 97–105
17. G. Hermann, Die Frage der endlich vielen Schritte in der Theorie der Polynomialideale, Math. Ann. **95** (1926), 736–788
18. M. Hickle, Solution d'une conjecture de C. Berenstein–A.Yger et invariants de contact à l'infini (to appear)

19. C. Huneke, Tight closure and its applications, CBMS Regional Conference Series in Mathematics **88** (1996)
20. V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity, Proc. Amer. Math. Soc. **128** (2000), 407–411
21. J. Kollár, Sharp effective Nullstellensatz, J. Amer. Math. Soc. **1** (1988), 963–975
22. R. Lazarsfeld, Lengths of periods and Seshadri constants of abelian varieties, Math. Res. Lett. **3** (1996), 439–447
23. R. Lazarsfeld, Positivity in Algebraic Geometry, book in preparation
24. C. Miyazaki and W. Vogel, Arithmetic and geometric degrees of graded modules, in Commutative algebra, algebraic geometry, and computational methods (Hanoi, 1996), Springer, Singapore, 1999, pp. 97–112
25. R. Paoletti, Seshadri constants, gonality of space curves, and restriction of stable bundles, J. Differential Geom. **40** (1994), 475–504
26. R. Paoletti, Seshadri positive curves in a smooth projective 3-fold. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **6** (1995), 259–274
27. L. Ratliff, On prime divisors of I^n, n large, Michigan Math. J. **23** (1976), 337–352
28. A. Steffans, Remarks on Seshadri constants, Math. Z. **227** (1998), 505–510
29. B. Teissier, Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney, in Algebraic Geometry (La Rábida, 1981). Lecture Notes in Math.**961** (1982), 314–491
30. W. Vasconcelos, Computational Methods in Commutative Algebra and Algebraic Geometry, Springer, 1998