UNIQUE ERGODICITY ON COMPACT HOMOGENEOUS SPACES

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Abstract. Extending results of a number of authors, we prove that if U is the unipotent radical of a solvable epimorphic subgroup of an algebraic group G, then the action of U on G/Γ is uniquely ergodic for every cocompact lattice Γ in G. This gives examples of uniquely ergodic and minimal two-dimensional flows on homogeneous spaces of arbitrarily high dimension. Our main tools are Ratner classification of ergodic invariant measures for the action of a unipotent subgroup on a homogeneous space, and a simple lemma (the 'Cone Lemma') about representations of epimorphic subgroups.

In his 1972 paper [F] Hillel Furstenberg proved that the horocycle flow on the unit tangent bundle to any compact Riemann surface is uniquely ergodic (i.e., admits a unique finite U-invariant Borel measure). This flow may be realized as the action of the subgroup U of upper-triangular matrices in $L = \mathrm{SL}(2, \mathbb{R})$ on the quotient of L by a cocompact lattice Λ, acting by

$$u(\ell\Lambda) = (u\ell)\Lambda.$$

Such an action is called a subgroup action on a homogeneous space. It follows from unique ergodicity that the action is minimal (i.e., any U-orbit is dense).

Various generalizations of Furstenberg's result were subsequently obtained by a number of authors (see [V, Bo, EP, D1]). In these generalizations what is proved is unique ergodicity of the action of U on L/Λ, where L is an algebraic group, U an algebraic subgroup which is unipotent (all its elements have all eigenvalues equal to 1 in a representation of L as a matrix group; in other words all its elements are unipotent) and Λ is a cocompact lattice. Typically in these results the dimension of the acting group U may be quite large. In this note we generalize these results. For example, for any simple noncompact L, we obtain a 2-dimensional unipotent U which acts uniquely ergodically on L/Λ for any cocompact lattice Λ.

The two essential ingredients we shall use are Marina Ratner's theorems regarding actions of a unipotent subgroup on a homogeneous space, and a simple result concerning epimorphic subgroups. Recall that Ratner's Measure Classification Theorem (see [R1], Theorem 1) asserts that if L, Λ are as above, and U is a subgroup of L generated by unipotents elements, then for any U-invariant U-ergodic measure μ there are a connected subgroup H and $x \in L/\Lambda$, such that H contains U and μ is an H-invariant measure supported on the closed orbit Hx. Recall also that Ratner's Orbit-Closure Theorem (see [R2], Theorem A) asserts that if L, Λ are as above, and U is a subgroup of L generated by unipotent elements, then for any $x \in L/\Lambda$ there exists a connected subgroup H containing U such that $\overline{Ux} = Hx$ and Hx supports an H-invariant probability measure. Ratner's results are in fact stronger, see [R3] for a survey.

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Let $F < G$ be an inclusion of real algebraic groups. Recall that F is *epimorphic* in G if for any rational representation $\rho : G \to \mathrm{GL}(V)$, and any $v \in V$,

$$\rho(F)v = v \Rightarrow \rho(G)v = v.$$

In [Mo], Shahar Mozes proved that this representation-theoretic property has implications for invariant measures on homogeneous spaces. His theorem is the following:

Theorem 0.1. *Let $F < G < L$ be an inclusion of real algebraic groups, where G is generated by unipotent elements, and F is epimorphic in G. Let Λ be a discrete subgroup of L. Then any F-invariant probability measure on L/Λ is G-invariant. In particular, if $G = L$ then the action of F on L/Λ is uniquely ergodic.*

Note that in this result Λ is not assumed to be cocompact. Our goal in this paper is to see to what extent the conclusion of Theorem 0.1 can be strengthened under the additional assumption that Λ is cocompact. We should mention that by a result of Gregory A. Margulis (see [Ma]) the cocompactness condition is satisfied whenever L/Λ admits a uniquely ergodic action of a unipotent subgroup.

By a theorem of Frederic Bien and Armand Borel (see [BB], Theorem 2), for any algebraic group G generated by unipotent elements and any epimorphic subgroup F, there is a subgroup F' of F which is also epimorphic in G and is of the form $F' = T \cdot U$, where U is unipotent and T is an $\mathbb{R}$-split algebric torus normalizing U. We now describe the setup for our results.

Basic Assumptions: Let $U < G < L$ be an inclusion of real algebraic groups, with U unipotent and G generated by unipotents. Assume that there is an $\mathbb{R}$-split torus T such that T normalizes U and $T \cdot U$ is epimorphic in G. Let Λ be a subgroup of L such that L/Λ is compact and carries a finite L-invariant measure.

Note that any simple noncompact G contains a 2-dimensional subgroup U satisfying the Basic Assumptions (see [BB], §5 (b)).

These are our results:

Theorem 0.2. *Let U, G, L, Λ be as in the Basic Assumptions. Suppose also that $L = G$. Then the action of U on G/Λ is uniquely ergodic and minimal.*

Theorem 0.3. *Let U, G, L, Λ be as in the Basic Assumptions. Suppose also that G is normal in L. Then any U-invariant closed set, as well as any U-invariant measure, is also G-invariant.*

Theorem 0.4. *Let U, G, L, Λ be as in the Basic Assumptions. Suppose also that the action of G on L/Λ is either minimal or uniquely ergodic. Then the action of U is both minimal and uniquely ergodic.*

Theorems 0.2,0.3 and 0.4 all follow from the following:

Lemma 0.5. *Let U, G, L, Λ be as in the Basic Assumptions. Assume that Λ is discrete. Let H be a connected subgroup containing U as in the conclusion of either Ratner's Measure Classification Theorem or Ratner's Orbit-Closure Theorem. Then for some $\ell \in L$, $G \subset \ell H \ell^{-1}$.*

Lemma 0.5 follows in turn from the following simple Lemma, which is a variant of what we have called the 'cone lemma' (see [W], Lemma 1):

Lemma 0.6. *Let G be an algebraic group with no nontrivial rational characters, and let $T \cdot U$ be epimorphic in G, where U is unipotent and T is an $\mathbb{R}$-split torus normalizing U. Then for any representation (of real algebraic groups) $\rho : G \to \mathrm{GL}(V)$, there is $t \in T$ such that for any $\rho(U)$-invariant vector v, $\rho(t^n)v$ converges to a $\rho(G)$-invariant vector as $n \to \infty$.*

As was pointed to the author by Nimish Shah, the ideas used in proving Lemma 0.5 also imply the following result for the non-compact case:

Theorem 0.7. *Let $U < L$ be an inclusion of real algebraic groups, where U is unipotent and L is generated by the unipotent elements contained in it. Let T be an $\mathbb{R}$-split torus in L, normalizing U, such that TU is epimorphic in L. Then there is an open subsemigroup $T^- \subset T$ (a 'cone') such that for any lattice Λ in L and any $x \in L/\Lambda$, if T^-x is a non-escaping trajectory then $\overline{Ux} = L/\Lambda$.*

Theorem 0.7 improves a result proved by Dani in 1985 (see [D2], Theorem 1.6).

The paper is organized as follows. First we prove Lemma 0.5. Then we deduce Theorems 0.2, 0.3, 0.4. We then show by an example that the Basic Assumptions alone do not guarantee that any U-invariant measure on L/Λ is G-invariant. This explains why the additional hypotheses in our theorems are needed. We then prove Theorem 0.7. Finally we sketch, for the reader's convenience, the proof of Lemma 0.6.

Proof of Lemma 0.5: Let U, G, L, Λ, H be as in the statement of the Lemma. Thus $\overline{U\ell_0\Lambda} = H\ell_0\Lambda$ for some $\ell_0 \in L$. Let T be as in the Basic Assumptions. Let $d = \dim H$ and let ρ be the representation of L on its d-dimensional Lie subalgebras. That is,

$$\rho = \bigwedge_1^d \mathrm{Ad} : L \to \mathrm{GL}(V) \text{ where } V = \bigwedge_1^d \mathrm{Lie}(L).$$

Let v_H be a nonzero element in the one-dimensional subspace of V corresponding to $\mathrm{Lie}(H)$.

We first claim that $\rho(L)v_H$ is closed in V. Indeed, it is proved in [DMa], Theorem 3.4 that $\rho(\ell_0\Lambda\ell_0^{-1})v_H$ is discrete. Hence if $\rho(\ell_n)v_H \to w$ then passing to a subsequence we can write, by the cocompactness of Λ,

$$\ell_n = \tilde{\ell}_n\lambda_n, \text{ where } \lambda_n \in \ell_0\Lambda\ell_0^{-1} \text{ and } \tilde{\ell}_n \to \tilde{\ell}.$$

Thus

$$\rho(\lambda_n)v_H \to \rho(\tilde{\ell}^{-1})w$$

and, by discreteness,

$$\rho(\tilde{\ell}^{-1})w \in \rho(\ell_0\Lambda\ell_0^{-1})v_H,$$

so $w \in \rho(L)v_H$.

Since $U \subset H$, $\rho(U)$ stabilizes the line through v_H and since U has no rational characters, v_H is $\rho(U)$-invariant. Therefore we may apply Lemma 0.6 to ρ and to v_H. We obtain that for some $t \in T$, $\rho(t^n)v_H \to v_0$ where v_0 is $\rho(G)$-invariant. Since $\rho(L)v_H$ is closed there exists $\ell \in L$ such that $v_0 = \rho(\ell)v_H$, that is v_0 corresponds to the Lie algebra of $H' = \ell H\ell^{-1}$, and H' is normalized by G. Since T normalizes U, for any n we have $U \subset t^nHt^{-n}$. This implies that also $U \subset H'$.

We now claim that $G \subset H'$. Indeed (replacing H' if necessary by the subgroup of H' generated by the unipotent elements in H'), we can assume that H' is generated by unipotents. Therefore H' is algebraic (this follows from [Sh], Lemma 2.9). Let

$$G_0 = G \cap H' \text{ and let } \pi : G \to G/G_0$$

be the natural quotient map. Since H' is algebraic, $\pi(T \cdot U)$ is epimorphic in G/G_0, and since $U \subset H'$, $\pi(T \cdot U) = \pi(T)$ is an algebraic torus. Since proper toral subgroups are never epimorphic (see [BHM], Theorem 2) this implies that $G/G_0 = \pi(T)$. But G/G_0 is generated by unipotents since G is, whereas $\pi(T)$ consists of semisimple elements. Thus G/G_0 is trivial, whence $G \subset H'$.

□

Deduction of Theorems 0.2, 0.3, 0.4:

First let us show we may assume that Λ is discrete. For this we use an argument of Dave Witte (see [Wi]). Let L_0 be the Zariski closure of Λ. Then Λ is a lattice in L_0. The space L/L_0 is the support of an L-invariant finite measure, namely the projection of the L-invariant measure on L/Λ. From Dani's version of the Borel density theorem (see [Wi], Theorem 4.2) it follows that L_0 contains an algebraic subgroup L_1 which is normal and cocompact in L. Since G is generated by unipotents, the projection of G onto L/L_1 is trivial, and therefore $G < L_1$. In particular $G < L_0$. Thus replacing L by L_0 we may assume that Λ is Zariski dense in L. Let Λ^0 be the connected component of the identity in Λ. Then Λ normalizes Λ^0 and by Zariski density, so does L. We can now quotient out by Λ^0; this means that there is an isomorphism, given by

$$\ell\Lambda \mapsto \ell\Lambda^0(\Lambda/\Lambda^0),$$

between the action of G on L/Λ and the action of $G/(G \cap \Lambda^0)$ on $(L/\Lambda^0)/(\Lambda/\Lambda^0)$. Since Λ/Λ^0 is discrete, this proves our claim.

Suppose G is normal in L. Given any orbit-closure $\overline{Ux}$ (respectively, a U-invariant U-ergodic measure μ), we get from Ratner's Orbit-Closure Theorem (resp., Ratner's Measure Classification Theorem) a subgroup H such that $\overline{Ux} = Hx$ (respectively, $U \subset H$ and μ is H-invariant). From Lemma 0.5 we get that $G \subset H' = \ell H \ell^{-1}$, and since G is normal, this implies that $G \subset H$. This proves Theorem 0.3. Theorem 0.2 is a special case.

Now let us prove Theorem 0.4. It follows from Ratner's theorems that for groups generated by unipotents, minimality and unique ergodicity are equivalent properties. So it will suffice to prove that if the action of G is minimal, so is the action of U.

Let $x \in L/\Lambda$ and $\overline{Ux} = Hx$. Let $H' = \ell H \ell^{-1}$ be as in Lemma 0.5. Then

$$\overline{Ux} = \overline{Hx} = \overline{\ell^{-1}H'\ell x} = \ell^{-1}\overline{H'\ell x} \supset \ell^{-1}\overline{G\ell x} = L/\Lambda.$$

Therefore the action of U is also minimal.

□

An Example: Let $L = \mathrm{SL}(3, \mathbb{R})$,

$$G_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \bigcap L,$$

and

$$U = \left\{ \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

It is possible to find a cocompact lattice Λ in L such that $\Lambda \cap G_1$ is a lattice in G_1 (such an example was communicated to me by Mary Rees). Now let

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $gUg^{-1} = U$ and $G_2 = gG_1g^{-1} \neq G_1$. It is easy to see that both U, G_1, L, Λ and U, G_2, L, Λ satisfy the Basic Assumptions, and (e.g. using the main result of [F]) that $\overline{U\Lambda} = G_1\Lambda$. In particular, $\overline{U\Lambda} \neq G_2\Lambda$. Letting μ be the G-invariant measure on $\overline{G_2\Lambda}$ gives a counterexample to the following statement, which is analogous to Theorem 0.1: Under the Basic Assumptions, for any $x \in L/\Lambda$, any U-invariant measure on L/Λ is G-invariant. Thus some additional assumption on G, as in Theorems 0.2, 0.3, 0.4, is required.

Proof of Theorem 0.7: The proof of Lemma 0.6 in fact implies a stronger statement: for any representation $\rho : G \to \mathrm{GL}(V)$ there is an open subsemigroup $T^- \subset T$ such that for any $\rho(U)$-invariant vector $v \in V$ and any sequence $\{t_n\}$ with $t_n \in T^-$ and $t_n \to \infty$ (in T), the sequence $\rho(t_n)v$ tends to a $\rho(G)$-invariant vector. See [W] for more details. So now let us take for T^- the semigroup corresponding to the natural action of L on

$$\bigoplus_{k=1}^{n} \bigwedge^{k} \mathrm{Lie}(L). \tag{1}$$

Let $x = \ell_0\Lambda$, and suppose that T^-x is non-escaping, that is, there exists a compact $K \subset L$ and a sequence $t_n \in T^-$ such that $t_n \to \infty$ and $t_n\ell_0 \in K\Lambda$. Thus $t_n \in K\ell_0\Lambda\ell_0^{-1}$. Let $\overline{Ux} = Hx$ and consider again the representation $\rho : L \to \mathrm{GL}(V)$ and the vector $v_H \in V$ as in the proof of Lemma 0.5. Notice that this representation is contained as a summand in (1).

Arguing as in the proof of the Lemma we see that $\rho(K\ell_0\Lambda\ell_0^{-1})v_H$ is closed. On the other hand Lemma 0.6 (in fact the stronger version alluded to above) shows that $\rho(t_n)v_H$ tends to a $\rho(L)$-invariant vector as $n \to \infty$. This implies as in the proof of Lemma 0.5 that a conjugate H' of H contains U and is normalized by L. Again as in the proof of Lemma 0.5 we get that $L \subset H'$. ☐

Proof of Lemma 0.6: Let W be the subspace of V consisting of $\rho(U)$-invariant vectors. Then $v \in W$ and W is $\rho(T)$-invariant since T normalizes U. Since T is an $\mathbb{R}$-split torus, we can write

$$W = \bigoplus_{\chi \in \Psi} W_\chi,$$

where Ψ is a subset of the set of rational characters on T and

$$\forall w \in W_\chi,\ t \in T, \quad \rho(t)w = \chi(t)w.$$

Denote the trivial character by 0 and for $\chi \in \Psi$ let $d\chi$ denote the derivative, which is just a linear functional on $\mathrm{Lie}(T)$. Write

$$v = w_0 + \sum_{\chi \in \Psi - \{0\}} w_\chi.$$

Since $T \cdot U$ is epimorphic in G, w_0 is $\rho(G)$-invariant. We will prove that for some $t \in T$, $\rho(t^n)v \to w_0$ by showing that there exists t such that

$$d\chi(t) < 0 \text{ for all } \chi \in \Psi - \{0\}.$$

For this it suffices to show that 0 is not contained in the convex hull (over $\mathbb{R}$) of $\{d\chi : \chi \in \Psi - \{0\}\}$. If we had constants $a_\chi \in \mathbb{R}_+$ such that

$$\sum_{\chi \in \Psi - \{0\}} a_\chi d\chi = 0, \quad \sum a_\chi = 1,$$

then using the fact that the derivatives of the rational characters of T form a discrete $\mathbb{Z}$-module in $\mathrm{Lie}(T)^*$ it follows (see [W] for more details) that there are $b_\chi \in \mathbb{Z}_+$, not all zero, such that

$$\sum_{\chi \in \Psi - \{0\}} b_\chi d\chi = 0. \tag{2}$$

Consider then the action of G on the vector

$$w = \bigotimes_{\chi \in \Psi - \{0\}} w_\chi^{\otimes b_\chi},$$

in the space $\bigotimes_1^d V$, where $d = \sum b_\chi$. Then it follows from (2) that w is invariant under the action of $\rho_0(T \cdot U)$ and hence of $\rho_0(G)$, where $\rho_0 = \bigotimes_1^d \rho$. Thus the line through each w_χ is invariant under $\rho(G)$. However, since G has no rational characters, this implies that each w_χ is $\rho(G)$-invariant, which implies that $\Psi = \{0\}$. □

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