Infinitesimal Thurston Rigidity and the Fatou-Shishikura Inequality

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1. Introduction

Any rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $D > 1$ has infinitely many periodic points, by the Fundamental Theorem of Algebra. The finiteness of the set of *nonrepelling* points is a cornerstone of complex analytic dynamics. Recall that the cycle

$$\langle x \rangle = \{x, ..., f^{\kappa-1}(x)\} \quad \text{is} \quad \begin{cases} \textit{superattracting} & \text{if } \rho = 0 \\ \textit{attracting} & \text{if } 0 < |\rho| < 1 \\ \textit{indifferent} & \text{if } |\rho| = 1 \\ \textit{repelling} & \text{if } |\rho| > 1, \end{cases}$$

where $\rho = (f^\kappa)'(x)$ is the corresponding *eigenvalue.* An indifferent cycle is *rationally indifferent* if ρ is a root of unity, and *irrationally indifferent* otherwise. The assumption $D > 1$ guarantees that every rationally indifferent cycle is *parabolic*, the first return to x being expressible as

$$\zeta \mapsto \rho(\zeta + \zeta^{N+1} + \alpha\zeta^{2N+1}) + O(\zeta^{2N+2}) \tag{1}$$

in a suitable local coordinate: if ρ is a primitive n-root of unity then $N = \nu n$ for some positive integer ν (see [1] or [12]). The *Fatou-Shishikura Inequality* asserts that there are at most $2D - 2$ nonrepelling cycles, each parabolic cycle of the form (1) counting as $\nu \geq 1$.

Here we present a new and independent proof of a refined Fatou-Shishikura Inequality. The refinement concerns a more generous convention for counting parabolic cycles: in terms of the normal form (1) we associate to each cycle $\langle x \rangle$ the quantity

$$\gamma_{\langle x \rangle} = \begin{cases} 0 & \text{if } \langle x \rangle \text{ is repelling or superattracting} \\ 1 & \text{if } \langle x \rangle \text{ is attracting or irrationally indifferent} \\ \nu & \text{if } \langle x \rangle \text{ is } \textit{parabolic-repelling} \quad (\Re\beta > 0) \\ \nu + 1 & \text{if } \langle x \rangle \text{ is } \textit{parabolic-attracting} \quad (\Re\beta < 0) \\ & \qquad \text{or } \textit{parabolic-indifferent} \ (\Re\beta = 0), \end{cases}$$

where $\beta = \frac{N+1}{2} - \alpha$ (in view of (5) below, this invariant behaves iteratively like $\frac{1}{\log \rho}$). Our count of nonrepelling cycles of f is $\gamma(f) = \sum_{\langle x \rangle \subset \mathbb{P}^1} \gamma_{\langle x \rangle}$ which *a priori* might be infinite. We denote by $\delta(f)$ the number of infinite tails of critical orbits; this quantity is certainly no greater than $2D - 2$ (the number of critical points), but if there are any critical orbit relations then it will be smaller. Our refinement of the Fatou-Shishikura Inequality is the following:

Theorem 1. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $D > 1$. Then $\gamma(f) \leq \delta(f)$.*

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This formulation of the Fatou-Shishikura Inequality has the advantage that the degree no longer explicitly appears, so that the appropriate extension to transcendental maps is an assertion with content; the expanded account [5] of our argument contains a uniform treatment for all *finite type* analytic maps. We recover the usual formulation by observing that there are at most $2D - 2 - \delta(f)$ superattracting cycles:

Corollary 1. *The number of superattracting, attracting, or indifferent cycles is at most $2D - 2$, for any rational map of degree $D > 1$.*

This result has a long history. Fatou [6] and Julia [9] both proved that any attracting cycle must attract a critical point; in fact, a parabolic cycle must have a critical point in each of the ν cycles of petals (see [1] or [12] for details). The relation between critical points and irrationally indifferent cycles is rather more subtle. It is fairly easy to show that any Cremer cycle lies in the postcritical accumulation, and the same is true for Siegel disk boundaries; however, the same critical point might well have an orbit which accumulates on several such features, so this consideration does not even show that the set of indifferent cycles is finite (Kiwi [10] has recently circumvented this difficulty in the polynomial case). Fatou instead found a perturbative proof that the number of nonrepelling cycles is at most $4D - 4$: loosely speaking, half the indifferent cycles become attracting after a random perturbation (see [12]). Douady and Hubbard [2] proved the sharp bound for polynomials ($D-1$ nonrepelling cycles in $\mathbb{C}$) using the theory of polynomial-like mappings, and finally Shishikura [14] proved the sharp bound $2D - 2$ in complete generality. Shishikura employs quasiconformal surgery to construct perturbations where all the nonrepelling cycles become attracting. His discussion of the irrationally indifferent case is especially delicate; in particular, the treatment of Cremer cycles requires a careful comparison of asymptotics. Shishikura shows further, using another beautiful surgery, that the same bound applies to the total count of nonrepelling cycles augmented by twice the number of Herman ring cycles.

Our proof of the Fatou-Shishikura Inequality is nonperturbative, and rather more algebraic; in particular, we completely sidestep the classical finiteness theorem for attracting cycles. The underlying mechanism is a suitable extension of *Infinitesimal Thurston Rigidity*: the injectivity of the linear operator $\nabla_f = I - f_*$ on spaces of meromorphic quadratic differentials. More precisely, the infinitesimal content of Thurston's Uniqueness Theorem (see [3] and also [11] for Global Rigidity and Thurston's much harder Existence Theorem) is the assertion $f_*q \neq q$ for nonzero quadratic differentials q having at worst simple poles. The novel feature of our extension is an allowance for multiple poles: for $q \in \ker \nabla_f$ these are necessarily situated along cycles, so that the associated infinities dynamically cancel, leaving finite residues whose signs reflect the cycles' dynamical character (compare the proof [8] of the Jenkins General Coefficient Theorem). We address the issue of Herman rings in [5], and a planned sequel will discuss pertubative implications: we will extend the considerations of [4] to prove smoothness and transversality for dynamically defined loci in parameter spaces.

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2. Contraction, Injectivity and Finiteness

Let $\mathcal{M}(\mathbb{P}^1)$ be the $\mathbb{C}$-linear space of all meromorphic quadratic differentials q on the Riemann sphere. We denote by $\mathcal{Q}(\mathbb{P}^1)$ the subspace consisting of all $q \in \mathcal{M}(\mathbb{P}^1)$ with at worst simple poles. Recall that $q \in \mathcal{Q}(\mathbb{P}^1)$ if and only if $||q|| < \infty$, where $||q|| = \int_{\mathbb{P}^1} |q|$ is the total mass of the associated area form $|q|$; see [7] and also [8] for the standard details. The quotient $\mathcal{M}(\mathbb{P}^1)/\mathcal{Q}(\mathbb{P}^1)$ is canonically isomorphic to $\mathcal{D}(\mathbb{P}^1) = \bigoplus_{x\in\mathbb{P}^1} \mathcal{D}_x(\mathbb{P}^1)$, where $\mathcal{D}_x(\mathbb{P}^1)$ is the space of all *algebraic divergences* at x: polar parts, of order at most -2, of germs of meromorphic quadratic differentials. The algebraic divergence of q at x is the corresponding class $[q]_x \in \mathcal{D}_x(\mathbb{P}^1)$, and the total algebraic divergence is $[q] = \sum_{x\in\mathbb{P}^1}[q]_x$. We write $\mathcal{M}(\mathbb{P}^1, A)$ for the subspace consisting of all $q \in \mathcal{M}(\mathbb{P}^1)$ whose poles lie in a given set $A \subseteq \mathbb{P}^1$, and we denote by $\mathcal{Q}(\mathbb{P}^1, A)$ the corresponding subspace of $\mathcal{Q}(\mathbb{P}^1)$. For well-known cohomological reasons (see [13]), $\dim \mathcal{Q}(\mathbb{P}^1, A) = \#A - 3$ so long as $\#A \geq 3$, and then $\mathcal{M}(\mathbb{P}^1, A)/\mathcal{Q}(\mathbb{P}^1, A)$ is canonically isomorphic to $\mathcal{D}(\mathbb{P}^1, A) = \bigoplus_{x\in A} \mathcal{D}_x(\mathbb{P}^1)$.

Recall that we may *pullback* any quadratic differential q on $\mathbb{P}^1$ by any analytic map $\phi : U \to \mathbb{P}^1$ to obtain a quadratic differential ϕ^*q on U; the pullback of the associated area form is $\phi^*|q| = |\phi^*q|$. If q is meromorphic at $\phi(x)$ then ϕ^*q is meromorphic at x: indeed,

$$\mathrm{ord}_x\phi^*q = \deg_x \phi \cdot (\mathrm{ord}_{\phi(x)}q + 2) - 2 \tag{2}$$

so a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ induces a pullback operator $f^* : \mathcal{M}(\mathbb{P}^1) \to \mathcal{M}(\mathbb{P}^1)$ which restricts to an endomorphism of $\mathcal{Q}(\mathbb{P}^1)$. Now consider the corresponding *pushforward* operator $f_* : \mathcal{M}(\mathbb{P}^1) \to \mathcal{M}(\mathbb{P}^1)$: by definition,

$$f_*q = \sum_{\phi} \phi^*q$$

where ϕ ranges over the inverse branches of f, so that

$$f_*f^*q = Dq \tag{3}$$

for a rational map of degree D. As $||f_*q|| = \int_{\mathbb{P}^1} |f_*q| \leq \int_{\mathbb{P}^1} f_*|q| = \int_{\mathbb{P}^1} |q| = ||q||$ by the Triangle Inequality, the operator f_* restricts to an endomorphism of $\mathcal{Q}(\mathbb{P}^1)$; indeed, it follows directly from (2) that

$$\mathrm{ord}_x f_*q \geq \max_{w\in f^{-1}(x)} \left(\frac{\mathrm{ord}_w q + 2}{\deg_w f} - 2\right) \tag{4}$$

for each $x \in \mathbb{P}^1$, so $\mathrm{ord}_x f_*q \geq -1$ if $\mathrm{ord}_w q \geq -1$ for every $w \in f^{-1}(x)$. Furthermore, $\mathrm{ord}_x f_*q \geq 0$ if $\mathrm{ord}_w q \geq 0$ for every such w, except possibly when some w is a critical point; we write $S(f)$ for the set of *critical values*, so that $P(f) = \bigcup_{k=0}^{\infty} f^k(S(f))$ is the *postcritical set.* It follows that $f_*\mathcal{M}(\mathbb{P}^1, A) \subseteq \mathcal{M}(\mathbb{P}^1, f(A) \cup S(f))$, and in particular

$$f_*\mathcal{Q}(\mathbb{P}^1, A) \subseteq \mathcal{Q}(\mathbb{P}^1, f(A) \cup S(f)), \qquad \text{for any} A \subseteq \mathbb{P}^1;$$

similarly, $f_*\mathcal{D}(\mathbb{P}^1, A) \subseteq \mathcal{D}(\mathbb{P}^1, f(A))$ where $f_*\colon \mathcal{D}(\mathbb{P}^1) \to \mathcal{D}(\mathbb{P}^1)$ is the induced endomorphism.

Consider the $\mathbb{C}$-linear endomorphism $\nabla_f = I - f_*$ of $\mathcal{M}(\mathbb{P}^1)$. This operator restricts to an endomorphism of $\mathcal{Q}(\mathbb{P}^1)$, so there is also an induced endomorphism ∇_f of $\mathcal{D}(\mathbb{P}^1)$. It follows from (4) that the space $\mathcal{D}(f) = \ker \nabla_f|_{\mathcal{D}(\mathbb{P}^1)}$ of *invariant divergences* is computed cycle by cycle: to be precise, $\mathcal{D}(f) = \bigoplus_{\langle x\rangle\subset\mathbb{P}^1} \mathcal{D}_{\langle x\rangle}(f)$ where $\mathcal{D}_{\langle x\rangle}(f) = \ker \nabla_f|_{\mathcal{D}(\mathbb{P}^1,\langle x\rangle)}$. Moreover, it suffices to compute these spaces for fixed points: indeed, if x is a point of period κ then the projection $\mathcal{D}(\mathbb{P}^1, \langle x \rangle) \to \mathcal{D}_x(\mathbb{P}^1)$ restricts to an isomorphism $\mathcal{D}_{\langle x \rangle}(f) \to \mathcal{D}_x(f^\kappa)$. We carry out this computation in Section 3: we describe $\mathcal{D}_{\langle x \rangle}(f)$ in terms of the formal invariants ρ, ν and α, then calculate the *dynamical residue*

$$\mathcal{D}_{\langle x \rangle}(f) \ni [q]_{\langle x \rangle} \longmapsto \mathrm{Res}_{\langle x \rangle}(f : q) \in \mathbb{R}$$

which measures the local creation or destruction of mass by f_*. The relevant conclusions are summarized in:

Proposition 1. *The set* $\mathcal{D}^\flat_{\langle x \rangle}(f)$ *of all* $[q]_{\langle x \rangle} \in \mathcal{D}_{\langle x \rangle}(f)$ *with* $\mathrm{Res}_{\langle x \rangle}(f : q) \le 0$ *is a* $\mathbb{C}$*-linear subspace of dimension* $\gamma_{\langle x \rangle}$.

We write $\mathcal{D}^\flat(f)$ for the subspace $\bigoplus_{\langle x \rangle \subset \mathbb{P}^1} \mathcal{D}^\flat_{\langle x \rangle}(f)$ of $\mathcal{D}(f)$; for $A \subseteq \mathbb{P}^1$ we denote by $\mathcal{D}^\flat(f, A)$ the corresponding subspace of $\mathcal{D}(f, A) = \bigoplus_{\langle x \rangle \subseteq A} \mathcal{D}_{\langle x \rangle}(f)$. Consider the subspace

$$\mathcal{Q}^\flat(f) = \{q \in \mathcal{M}(\mathbb{P}^1) : [q] \in \mathcal{D}^\flat(f)\}$$

of the $\mathbb{C}$-linear space $\mathcal{Q}(f) = \{q \in \mathcal{M}(\mathbb{P}^1) : [q] \in \mathcal{D}(f)\} = {\nabla_f}^{-1}\mathcal{Q}(\mathbb{P}^1) \supseteq \ker \nabla_f$. Our main result is the following:

Proposition 2. *Let* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *be a rational map of degree* $D > 1$*, and assume that* f *is not a Lattès example. Then* $\nabla_f : \mathcal{Q}^\flat(f) \to \mathcal{Q}(\mathbb{P}^1)$ *is injective.*

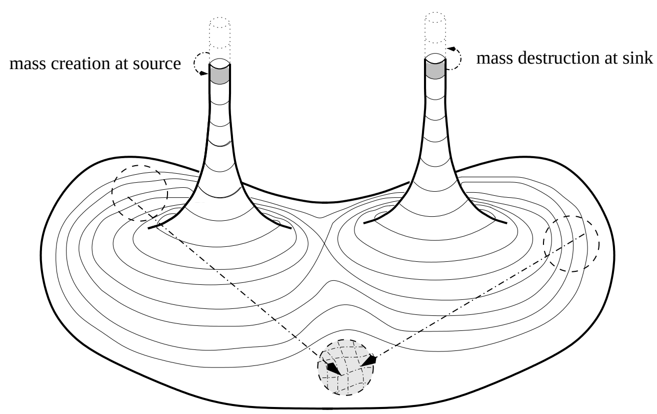

Figure 1: Net contraction

We prove this algebraic assertion by combining two measure-theoretic observations. The first generalizes the *Contraction Principle* behind Thurston Rigidity: that cancellation due to phase incoherence results in a well-defined decrease in mass. For nonintegrable quadratic differentials this loss may be offset by the creation of new mass at (parabolic-)repelling cycles or compounded by the further destruction of mass at (parabolic-)attracting cycles, and our second consideration is a *Balance Principle* which accordingly constains $\ker \nabla_f$. Figure 1 is a caricature of this argument, which we present in Section 4; it is not much harder to show that no eigenvalue of $f_* : \mathcal{M}(\mathbb{P}^1) \to \mathcal{M}(\mathbb{P}^1)$ lies on the unit circle [5].

Proof of Theorem 1: We may assume without loss of generality that f is not a Lattès example, as such a map has only repelling periodic points. Let A be any finite set of the form $B \cup C$, where $B \subseteq P(f)$ is an initial segment including all critical orbit relations and C consists of nonrepelling cycles, and set $A^+ = A \cup f(A)$. If $C \neq \varnothing$ then $\#A \geq 3$, so that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Q}(\mathbb{P}^1, A) & \longrightarrow & \mathcal{M}(\mathbb{P}^1, A) & \longrightarrow & \mathcal{D}(\mathbb{P}^1, A) & \longrightarrow & 0 \\ & & \downarrow \nabla_f & & \downarrow \nabla_f & & \downarrow \nabla_f & & \\ 0 & \longrightarrow & \mathcal{Q}(\mathbb{P}^1, A^+) & \longrightarrow & \mathcal{M}(\mathbb{P}^1, A^+) & \longrightarrow & \mathcal{D}(\mathbb{P}^1, A^+) & \longrightarrow & 0 \end{array}$$

is a commutative diagram of linear maps. As both rows are exact we may apply the Serpent Lemma (an elementary diagram chase from homological algebra - see [13]) to obtain an exact sequence

$$\ker \nabla_f|_{\mathcal{M}(\mathbb{P}^1,A)} \longrightarrow \mathcal{D}(f, A) \stackrel{\chi}{\longrightarrow} \mathcal{Q}(\mathbb{P}^1, A^+)/\nabla_f \mathcal{Q}(\mathbb{P}^1, A).$$

In view of Proposition 2, the restrictions $\chi|_{\mathcal{D}^\flat(f,A)}$ and $\nabla_f|_{\mathcal{Q}(\mathbb{P}^1,A)}$ are injective; thus,

$$\begin{aligned} \dim \mathcal{D}^\flat(f, A) &\leq \dim \mathcal{Q}(\mathbb{P}^1, A^+)/\nabla_f \mathcal{Q}(\mathbb{P}^1, A) = \dim \mathcal{Q}(\mathbb{P}^1, A^+) - \dim \nabla_f \mathcal{Q}(\mathbb{P}^1, A) \\ &= \dim \mathcal{Q}(\mathbb{P}^1, A^+) - \dim \mathcal{Q}(\mathbb{P}^1, A) = \#(A^+ - A) = \delta(f) \end{aligned}$$

so $\gamma(f) = \sup_A \sum_{\langle x \rangle \subseteq A} \gamma_{\langle x \rangle} = \sup_A \dim \mathcal{D}^\flat(f, A) \leq \delta(f)$ by Proposition 1. □

3. Local Considerations

Here we prove Proposition 1: that $\dim \mathcal{D}^\flat_{\langle x \rangle}(f) = \gamma_{\langle x \rangle}$ for each cycle $\langle x \rangle$ of a rational map f. Note that $\mathcal{D}_{\langle x \rangle}(f)$ depends only the local behavior of f along $\langle x \rangle$: indeed, it follows from (4) that if $q \in \mathcal{M}(\mathbb{P}^1)$ with $[f_* q]_{\langle x \rangle} = \lambda [q]_{\langle x \rangle} \neq 0$ then $[q]_w = 0$ for every $w \notin \langle x \rangle$ in the backward orbit of $\langle x \rangle$. It similarly follows that $\mathrm{ord}_x q = -2$ if $\langle x \rangle$ is superattracting, in which case $\lambda = \frac{1}{\deg_x f^\kappa}$ so that $[f_* q]_{\langle x \rangle} \neq [q]_{\langle x \rangle}$; on the other hand, if $\rho \neq 0$ then $[f_* q]_{\langle x \rangle} = \lambda [q]_{\langle x \rangle}$ precisely when $[f^* q]_{\langle x \rangle} = \frac{1}{\lambda} [q]_{\langle x \rangle}$.

Lemma 1. *The space $\mathcal{D}_{\langle x \rangle}(f)$ of invariant divergences is computed in terms of the formal invariants of $\langle x \rangle$:*

- *If $\langle x \rangle$ is superattracting then $\mathcal{D}_{\langle x \rangle}(f) = 0$.*
- *If $\langle x \rangle$ is attracting, repelling or irrationally indifferent then $\mathcal{D}_{\langle x \rangle}(f)$ is the 1-dimensional space generated by $\frac{d\zeta^2}{\zeta^2}$, for any local coordinate ζ vanishing at x.*
- *If $\langle x \rangle$ is parabolic then $\mathcal{D}_{\langle x \rangle}(f)$ is the direct sum of the ν-dimensional subspace $\mathcal{D}^\circ_{\langle x \rangle}(f)$ generated by*

  $$\frac{d\zeta^2}{\zeta^2}, \ldots, \frac{d\zeta^2}{\zeta^{\ell n+2}}, \ldots, \frac{d\zeta^2}{\zeta^{N-n+2}}$$

  and the 1-dimensional subspace generated by

  $$\frac{d\zeta^2}{(\zeta^{N+1} - \beta \zeta^{2N+1})^2} = \frac{d\zeta^2}{\zeta^{2N+2}} + 2\beta \frac{d\zeta^2}{\zeta^{N+2}} + 3\beta^2 \frac{d\zeta^2}{\zeta^2} + O\left(\frac{d\zeta^2}{\zeta}\right),$$

  for any local coordinate ζ as in (1).

Proof: In view of the discussion above, it suffices to determine when $[f^*q]_{\langle x\rangle} = [q]_{\langle x\rangle}$. Assume without loss of generality that x is a fixed point with $\rho \neq 0$, and let ζ be any local coordinate vanishing at x; as $f^*\frac{d\zeta^2}{\zeta^{j+2}} = \frac{(\rho+O(\zeta))^2}{(\rho\zeta+O(\zeta^2))^{j+2}}\,d\zeta^2 = \rho^{-j}\frac{d\zeta^2}{\zeta^{j+2}} + O\left(\frac{d\zeta^2}{\zeta^{j+1}}\right)$ for any integer j, it follows that $\mathcal{D}_x(f) = \mathbb{C}[\frac{d\zeta^2}{\zeta^2}]_x$ unless ρ is a root of unity. Suppose now that x is parabolic, and let ζ be a local coordinate as in (1); then

$$\begin{cases} f^*\zeta^k &= \left(\rho\left(\zeta+\zeta^{N+1}+\left(\frac{N+1}{2}-\beta\right)\zeta^{2N+1}+O(\zeta^{2N+2})\right)\right)^k \\ &= \rho^k\zeta^k\left(1+k\zeta^N+k\left(\frac{N+k}{2}-\beta\right)\zeta^{2N}+O(\zeta^{2N+1})\right), \\ f^*d\zeta^2 &= \left(\rho\left(1+(N+1)\zeta^N+(2N+1)\left(\frac{N+1}{2}-\beta\right)\zeta^{2N}+O(\zeta^{2N+1})\right)d\zeta\right)^2 \\ &= \rho^2(1+(2N+2)\zeta^N+(3N^2+5N+2-(4N+2)\beta)\zeta^{2N}+O(\zeta^{2N+1}))d\zeta^2, \end{cases}$$

so

$$f^*\frac{d\zeta^2}{\zeta^{j+2}} = \bar{\rho}^j\left(1+(2N-j)\zeta^N+(3N^2-\tfrac{5}{2}jN+\tfrac{1}{2}j^2+(j-4N)\beta)\zeta^{2N}+O(\zeta^{2N+1})\right)\frac{d\zeta^2}{\zeta^{j+2}}.$$

In particular,

$$\begin{aligned} \mathrm{ord}_x\left(f^*\frac{d\zeta^2}{\zeta^{j+2}}-\frac{d\zeta^2}{\zeta^{j+2}}\right) &\geq N-(j+2) \text{ if } n|j, \text{ with equality for } j\neq 2N, \text{ and} \\ \mathrm{ord}_x\left(f^*\frac{d\zeta^2}{\zeta^{j+2}}-\frac{d\zeta^2}{\zeta^{j+2}}\right) &= -(j+2) \text{ otherwise;} \end{aligned}$$

as

$$\begin{aligned} f^*\frac{d\zeta^2}{\zeta^{2N+2}}+2\beta f^*\frac{d\zeta^2}{\zeta^{N+2}} &= (1-2N\beta\zeta^{2N})\frac{d\zeta^2}{\zeta^{2N+2}}+2\beta(1+N\zeta^N)\frac{d\zeta^2}{\zeta^{N+2}}+O\left(\frac{d\zeta^2}{\zeta}\right) \\ &= \frac{d\zeta^2}{\zeta^{2N+2}}+2\beta\frac{d\zeta^2}{\zeta^{N+2}}+O\left(\frac{d\zeta^2}{\zeta}\right), \end{aligned}$$

it follows that $\mathcal{D}_x(f) = \mathbb{C}\left[\frac{d\zeta^2}{(\zeta^{N+1}-\beta\zeta^{2N+1})^2}\right]_x \oplus \mathcal{D}^\circ_x(f)$. □

These bases are actually canonical. Indeed, $[\frac{d\zeta^2}{\zeta^2}]_x$ is independent of the choice of local coordinate ζ, and $[\frac{d\zeta^2}{\zeta^{\ell n+2}}]_x$ is invariant under coordinate changes $\zeta \mapsto \rho^j\zeta + O(\zeta^{\ell n+2})$ for $j \in \mathbb{Z}$, while $[q_f]_x = \left[\frac{d\zeta^2}{(\zeta^{N+1}-\beta\zeta^{2N+1})^2}\right]_x$ is invariant under the coordinate changes respecting the normal form (1); more precisely, consideration of the change of variable $Z^N = \tau\zeta^N$ shows that $[q_f]_x \in \mathcal{D}_{\langle x\rangle}(F)$ for any germ

$$F(\zeta) = e^{2\pi ik/N}\left(\zeta+\tau\zeta^{N+1}+\left(\frac{N+1}{2}\tau^2-\beta\right)\zeta^{2N+1}+O(\zeta^{2N+2})\right)$$

with $(k,\tau) \in \mathbb{Z}\times\mathbb{C}$. As

$$f^m(\zeta) = \rho^m\left(\zeta+m\zeta^{N+1}+\left(\frac{N+1}{2}m^2-\beta\right)\zeta^{2N+1}+O(\zeta^{2N+2})\right) \tag{5}$$

it similarly follows that $[q_f]_x = \frac{1}{m^2}[q_{f^m}]_x$ for any integer m. In fact, there is always a unique normalized *formal* quadratic differential q_f with $f^*q_f = q_f$. If x is formally linearizable then $q_f = \frac{d\zeta^2}{\zeta^2}$ for any choice of formal linearizing coordinate ζ. Otherwise, x is parabolic and $q_f = \frac{1}{n^2}q_{f^n} = \frac{1}{n^2}\eta^2_{f^n}$ where η_{f^n} is the formal linear differential dual to the unique formal vector field v_{f^n} whose formal exponential is f^n:

necessarily, $v_{f^n} = n\left(\zeta^{N+1} - \beta\zeta^{2N+1} + O(\zeta^{2N+2})\right)\frac{\partial}{\partial\zeta}$ in any local coordinate as in (1). We pursue this more intrinsic approach in [5].

Recall that if ξ is a smooth vector field and ϖ is a smooth 2-form, then the flux across an oriented smooth curve Γ is $\int_\Gamma \iota_\xi\varpi$, where ι_ξ is the interior product with respect to ξ. In particular, if ϖ has an isolated singularity at x then there is a flux across ∂U for any sufficiently small smoothly bounded neighborhood; in view of Cartan's Formula $\mathcal{L}_\xi\vartheta = \iota_\xi d\vartheta + d\iota_\xi\vartheta$, it follows by Stokes Theorem that there is a well-defined asymptotic flux

$$\int_{\partial U}\iota_\xi\varpi - \int_{\partial U}\mathcal{L}_\xi\varpi = \lim_{U\searrow x}\int_{\partial U}\iota_\xi\varpi$$

from any singularity where the Lie derivative $\mathcal{L}_\xi\varpi$ remains integrable. Similarly, if $[q]_{\langle x\rangle} \in \mathcal{D}_{\langle x\rangle}(f)$ then the local action of f_* creates or destroys a well-defined quantity of mass. Indeed, the net q-mass exiting $U \supset \langle x\rangle$ is $\int_{f(U)-U}|q| - \int_{U-f(U)}|q|$; moreover, if $\tilde{U} \supset \langle x\rangle$ is contained in $U\cap f(U)$ then $\int_{f(U)-U}|q| - \int_{U-f(U)}|q| = \int_{f(U)-\tilde{U}}|q| - \int_{U-\tilde{U}}|q|$ and $\int_{f(U)-f(\tilde{U})}|q| - \int_{U-\tilde{U}}|q| = \int_{U-\tilde{U}}(f^*|q|-|q|)$, hence

$$\left(\int_{f(U)-U}|q| - \int_{U-f(U)}|q|\right) - \left(\int_{f(\tilde{U})-\tilde{U}}|q| - \int_{\tilde{U}-f(\tilde{U})}|q|\right) = \int_{U-\tilde{U}}(f^*|q|-|q|),$$

so that the quantity $\int_{f(U)-U}|q| - \int_{U-f(U)}|q| - \int_U(f^*|q|-|q|)$ is independent of U. This quantity makes sense because $|\,f^*|q|-|q|\,| \le |f^*q-q|$ is locally integrable. We set

$$\begin{aligned}\operatorname{Res}_{\langle x\rangle}(f:q) &= \frac{1}{2\pi}\left(\int_{f(U)-U}|q| - \int_{U-f(U)}|q| - \int_U(f^*|q|-|q|)\right)\\ &= \frac{1}{2\pi}\lim_{U\searrow\langle x\rangle}\left(\int_{f(U)-U}|q| - \int_{U-f(U)}|q|\right);\end{aligned}$$

as $|\,|q|-|\hat{q}|\,| \le |q-\hat{q}|$ is locally integrable for quadratic differentials q and $\hat{q}$ with the same algebraic divergence, the invariant $\operatorname{Res}_{\langle x\rangle}(f:q)$ depends only on $[q]_{\langle x\rangle}$. Note that $\operatorname{Res}_x(f^m:q) = m\operatorname{Res}_x(f:q)$ for a fixed point x and any integer m, because

$$\int_{f^m(U)-U}|q| - \int_{U-f^m(U)}|q| = \sum_{j=0}^{m-1}\left(\int_{f^{j+1}(U)-f^j(U)}|q| - \int_{f^j(U)-f^{j+1}(U)}|q|\right);$$

similarly, $\operatorname{Res}_{\langle x\rangle}(f:q) = \operatorname{Res}_x(f^\kappa:q)$ for any κ-cycle $\langle x\rangle$, as we may take U to be the disjoint union $\bigcup_{j=0}^{\kappa-1} f(U_x)$ for an appropriately small neighborhood $U_x \ni x$.

Lemma 2. *The dynamical residue* $\operatorname{Res}_{\langle x\rangle}(f:q)$ *is computed in terms of the formal invariants of* $\langle x\rangle$*:*

- *If* $\langle x\rangle$ *is attracting, indifferent or repelling then* $\operatorname{Res}_{\langle x\rangle}(f:q) = |c|\log|\rho|$ *for the invariant divergence* $[q]_{\langle x\rangle} = c[\frac{d\zeta^2}{\zeta^2}]_{\langle x\rangle}$.

- *If* $\langle x\rangle$ *is parabolic then* $\operatorname{Res}_{\langle x\rangle}(f:q) = |c|\Re\beta$ *for any invariant divergence* $[q]_{\langle x\rangle}$ *in the hyperplane* $c[q_f]_{\langle x\rangle} + \mathcal{D}^\circ_{\langle x\rangle}(f)$.

Proof: Assume without loss of generality that x is a fixed point, and let ζ be a local coordinate vanishing at x. The choice of $U \ni x$ is immaterial so we may further assume that ∂U is smooth; it follows by Stokes Theorem that

$$\begin{aligned}\mathrm{Res}_x(f:q) &= \frac{1}{2\pi}\left(\int_{\partial f(U)}\vartheta \;-\; \int_{\partial f(U)}\vartheta \;-\; \int_U (f^*|q|-|q|)\right)\\ &= \frac{1}{2\pi}\left(\int_{\partial U}(f^*\vartheta-\vartheta) \;-\; \int_U (f^*|q|-|q|)\right) = \frac{1}{2\pi}\lim_{U\searrow x}\int_{\partial U}(f^*\vartheta-\vartheta)\end{aligned}$$

for any 1-form ϑ with $|q| = d\vartheta$. Let $t \mapsto f_t$ be a smooth path of holomorphic germs connecting the identity f_0 to $f = f_1$, and consider the holomorphic vector fields v_t with $\frac{d}{dt}f_t = v_t \circ f_t$. Then

$$\begin{aligned}f^*\vartheta-\vartheta &= \int_0^1\left(\frac{d}{dt}f_t^*\vartheta\right)dt = \int_0^1 f_t^*(\mathcal{L}_{\xi_t}\vartheta)\,dt = \int_0^1 \mathcal{L}_{f_t^*\xi_t}(f_t^*\vartheta)\,dt\\ &= \int_0^1 \iota_{f_t^*\xi_t}(f_t^*|q|)\,dt \;+\; d\left(\int_0^1 \iota_{f_t^*\xi_t}(f_t^*\vartheta)\,dt\right)\end{aligned}$$

by Cartan's Formula, where $\xi_t = 2\Re v_t$ corresponds to v_t under the standard identification of real and complex tangent spaces; as $\iota_{f_t^*\xi_t}(f_t^*|q|) = \Re\iota_{2f_t^*v_t}(\Re f_t^*|q|)$ and $\Re f_t^*|q| = f_t^*|q|$, it follows that

$$\mathrm{Res}_x(f:q) = \frac{1}{\pi}\lim_{U\searrow x}\Re\int_{\partial U}\int_0^1 \iota_{f_t^*v_t}(f_t^*|q|)\,dt.$$

In particular, if $f_t^*v_t = v + O\left(\zeta^k\frac{\partial}{\partial\zeta}\right)$ for some $k \geq -\mathrm{ord}_x q$ and if $f_t^*q = q + O\left(\frac{d\zeta^2}{\zeta}\right)$, then

$$\int_0^1 \iota_{f_t^*v_t}(f_t^*|q|)\,dt = \int_0^1\left(\iota_v|q| + O(d\bar\zeta)\right)dt = \iota_v|q| + O(d\bar\zeta),$$

so that

$$\mathrm{Res}_x(f:q) = \frac{1}{\pi}\lim_{U\searrow x}\Re\int_{\partial U}\left(\iota_v|q| + O(d\bar\zeta)\right) = \Re\lim_{U\searrow x}\frac{1}{\pi}\int_{\partial U}\iota_v|q| = \Re\lim_{U\searrow x}\frac{1}{\pi}\int_{\partial U}\overline{\iota_v|q|}.$$

Suppose first that $q = c\frac{d\zeta^2}{\zeta^2}$, and take $f_t(\zeta) = e^{t\log\rho}\zeta + t\left(f(\zeta)-\rho\zeta\right)$ for some choice of $\log\rho$. Then $f_t(\zeta) = e^{t\log\rho}\zeta + O(\zeta^2)$, so

$$f_t^*v_t = \frac{df_t}{dt}\Big/\frac{df_t}{d\zeta} = \left(\frac{(\log\rho)e^{t\log\rho}\zeta + O(\zeta^2)}{e^{t\log\rho}+O(\zeta)}\right)\frac{\partial}{\partial\zeta} = v + O\left(\zeta^2\frac{\partial}{\partial\zeta}\right)$$

where $v = (\log\rho)\zeta\frac{\partial}{\partial\zeta}$, and $f_t^*q = q + O\left(\frac{d\zeta^2}{\zeta}\right)$ because $\mathrm{ord}_x q \geq -2$; as $|d\zeta^2| = \frac{i}{2}d\zeta\wedge d\bar\zeta$ and therefore $\iota_v|q| = (\log\rho)\zeta\left(\frac{i|c|}{2|\zeta|^2}\right)\iota_{\frac{\partial}{\partial\zeta}}\left(d\zeta\wedge d\bar\zeta\right) = \frac{i}{2}|c|(\log\rho)\frac{d\bar\zeta}{\bar\zeta}$, it follows that

$$\mathrm{Res}_x(f:q) = \Re\frac{1}{2\pi i}\oint|c|\overline{(\log\rho)}\frac{d\zeta}{\zeta} = |c|\log|\rho|.$$

Assume now that x is parabolic; let ζ be a local coordinate as in (1), and take

$$f_t(\zeta) = \zeta + t\left(f^n(\zeta) - \zeta + \tfrac{N+1}{2}(t-1)n^2\zeta^{2N+1}\right)$$

so that $f_1 = f^n$. Then $f_t(\zeta) = \zeta + tn\zeta^{N+1} + tn\left(\frac{N+1}{2}tn - \beta\right)\zeta^{2N+1} + O(\zeta^{2N+2})$ by (5), so

$$\begin{aligned} f_t^* v_t &= \frac{df_t}{dt}\Big/\frac{df_t}{d\zeta} = \left(\frac{n\zeta^{N+1} + n\left((N+1)tn - \beta\right)\zeta^{2N+1} + O(\zeta^{2N+2})}{1 + tn(N+1)\zeta^N + O(\zeta^{2N})}\right)\frac{\partial}{\partial\zeta} \\ &= v + O\left(\zeta^{2N+2}\frac{\partial}{\partial\zeta}\right) \end{aligned}$$

where $v = n(\zeta^{N+1} - \beta\zeta^{2N+1})\frac{\partial}{\partial\zeta}$. By the remarks following Lemma 1, if $q = c\frac{d\zeta^2}{(\zeta^{N+1}-\beta\zeta^{2N+1})^2} + O\left(\frac{d\zeta^2}{\zeta^{N+1}}\right)$ then $f_t^* q = q + O\left(\frac{d\zeta^2}{\zeta}\right)$; as

$$\begin{aligned} \iota_v|q| &= n\left(\zeta^{N+1} - \beta\zeta^{2N+1}\right)\left(\frac{i}{2}|c|\frac{1 + O(\zeta^{N+1})}{|\zeta^{N+1} - \beta\zeta^{2N+1}|^2}\right)\iota_{\frac{\partial}{\partial\zeta}}\left(d\zeta \wedge d\bar\zeta\right) \\ &= \frac{i}{2}n|c|\left(\frac{1}{\overline{\zeta^{N+1} - \beta\zeta^{2N+1}}} + O(1)\right)d\bar\zeta = \frac{i}{2}n|c|\left(\frac{d\bar\zeta}{\bar\zeta^{N+1}} + \bar\beta\frac{d\bar\zeta}{\bar\zeta}\right) + O(d\bar\zeta), \end{aligned}$$

it follows that $\mathrm{Res}_x(f : q) = \frac{1}{n}\mathrm{Res}_x(f^n : q) = |c|\Re\frac{1}{2\pi i}\oint\left(\frac{d\zeta}{\zeta^{N+1}} + \beta\frac{d\zeta}{\zeta}\right) = |c|\Re\beta$. $\square$

Recall that $\mathcal{D}^\flat_{\langle x\rangle}(f) = \{[q]_{\langle x\rangle} \in \mathcal{D}_{\langle x\rangle}(f) : \mathrm{Res}_{\langle x\rangle}(f : q) \le 0\}$ by definition. The above discussion shows that

$$\mathcal{D}^\flat_{\langle x\rangle}(f) = \begin{cases} 0 & \text{if } \langle x\rangle \text{ is repelling or superattracting} \\ \mathbb{C}[\frac{d\zeta^2}{\zeta^2}]_{\langle x\rangle} & \text{if } \langle x\rangle \text{ is attracting or irrationally indifferent} \\ \mathcal{D}^\circ_{\langle x\rangle}(f) & \text{if } \langle x\rangle \text{ is parabolic-repelling} \\ \mathcal{D}_{\langle x\rangle}(f) & \text{if } \langle x\rangle \text{ is parabolic-attracting or parabolic-indifferent,} \end{cases}$$

and Proposition 1 now follows from the definition of $\gamma_{\langle x\rangle}$.

4. Global Considerations

We now prove Proposition 2: the injectivity of $\nabla_f : \mathcal{Q}^\flat(f) \to \mathcal{Q}(\mathbb{P}^1)$. Recall that $||f_*q|| \le ||q||$ for any quadratic differential q, but note that this inequality is vacuous when $||q|| = \infty$; nevertheless, we may still identify the mass decrease due to cancellation as

$$\mathrm{Dec}(f : q) = \int_{\mathbb{P}^1}(f_*|q| - |f_*q|)$$

which is always nonnegative and might be infinite. Clearly, $\mathrm{Dec}(f : q) = ||q|| - ||f_*q||$ for any integrable q, so the following extension of Thurston's Contraction Principle already shows injectivity on $\mathcal{Q}(\mathbb{P}^1)$:

Lemma 3. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $D > 1$; assume that f is not a Lattès example, and let $q \ne 0$ be a meromorphic quadratic differential in* $\ker\nabla_f$. *Then* $\mathrm{Dec}(f : q) > 0$.

Proof: Fix an open disk $U \subset \mathbb{P}^1 - S(f)$ and an inverse branch $\phi : U \to \mathbb{P}^1$. If $\mathrm{Dec}(f : q) = 0$ then $|\phi^* q + \psi^* q| = |\phi^* q| + |\psi^* q|$ for any inverse branch $\psi : U \to \mathbb{P}^1$, so the meromorphic function $\lambda_\psi = \frac{\psi^* q}{\phi^* q}$ is almost everywhere real and positive. As a real-valued meromorphic function is necessarily constant, $f_* q = \lambda \phi^* q$ on U, so $f^* f_* q = \lambda f^* \phi^* q = \lambda q$ on $\phi(U)$ and therefore globally, where $\lambda = \sum_\psi \lambda_\psi \in \mathbb{R}^+$; it follows from (3) that $D f_* q = f_* f^* f_* q = \lambda f_* q$, so in fact $\lambda = D$.

As $\nabla_f q = 0$, it further follows that $f^* q = f^* f_* q = Dq$, and we claim that this is impossible unless f is a Lattès example. Indeed, if $f^* q = \lambda q$ for any $\lambda \neq 0$ then $\mathrm{ord}_x f^* q = \mathrm{ord}_x q$ for each point x on the Riemann sphere. In view of (2), the finite set

$$\Phi = \{x \in \mathbb{P}^1 : \mathrm{ord}_x q \leq -2 \text{ or } \mathrm{ord}_x q \geq 1\}$$

is backward invariant, hence $\#\Phi \leq 2$ with any $x \in \Phi$ superattracting of period 1 or 2 (see [1] or [12]); but then $\mathrm{ord}_x q = -2$ which is only possible for $\lambda = D^2$, so $\Phi = \emptyset$. In particular, q is nowhere vanishing with only simple poles; in fact, there are precisely four such poles, as $\sum_{x \in \mathbb{P}^1} \mathrm{ord}_x q = -4$ for any quadratic differential on $\mathbb{P}^1$. It now follows from (2) that the set of poles is forward invariant, that every preimage of a pole is either a pole or a simple critical point, that every critical value is a pole, and that no critical point is a pole: this is precisely the description of the Lattès examples given in [3]. □

To complete the proof of Proposition 2, we show $\mathrm{Dec}(f : q) = 0$ for any quadratic differential q in $\mathcal{Q}^\flat(f) \cap \ker \nabla_f$. Consider the total dynamical residue

$$\mathrm{Res}(f : q) = \sum_{\langle x \rangle \subset \mathbb{P}^1} \mathrm{Res}_{\langle x \rangle}(f : q);$$

note that this quantity is defined whenever $q \in \mathcal{Q}(f)$, and that $\mathrm{Res}(f : q) \leq 0$ for any $q \in \mathcal{Q}^\flat(f)$ As $\mathrm{Dec}(f : q) \geq 0$ for any $q \in \mathcal{M}(\mathbb{P}^1)$, the desired conclusion follows from the Balance Principle: that

$$\mathrm{Dec}(f : q) = 2\pi \mathrm{Res}(f : q)$$

for every $q \in \ker \nabla_f$. This identity is an immediate consequence of the following:

Lemma 4. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map, and let q be a meromorphic quadratic differential in $\mathcal{Q}(f)$. Then $||\nabla_f q|| \geq |\mathrm{Dec}(f : q) - 2\pi\, \mathrm{Res}(f : q)|$.*

Proof: Recall that $E = \{x \in \mathbb{P}^1 : \mathrm{ord}_x q \leq -2\}$ consists of finitely many cycles, none superattracting. Let $U_x \ni x$ be pairwise disjoint open disks in $\mathbb{P}^1 - S(f)$, and set $U = \bigcup_{x \in E} U_x$; we may arrange that $U \cap f^{-1}(U) = \bigcup_{x \in E} \big(U_x \cap \phi(U_{f(x)})\big)$ where $\phi : U \to \mathbb{P}^1$ with $\phi(E) = E$ consists of the distinguished inverse branches of f.

We claim that $f_*|q| - |f_* q|$ is integrable, so that $\mathrm{Dec}(f : q) < \infty$: indeed, both $f_*|q|$ and $|f_* q|$ are integrable on $\mathbb{P}^1 - U$, and the restriction of $f_*|q| - |f_* q|$ to U is

$$(\phi^*|q| - |f_* q|) + \sum_{\psi \neq \phi} \psi^* |q|$$

where $|\,\phi^*|q| - |f_*q|\,| \leq \left|\sum_{\psi\neq\phi}\psi^*q\right| \leq \sum_{\psi\neq\phi}\psi^*|q|$. Moreover, $|\,|q| - |f_*q|\,| \leq |q - f_*q|$ so $|q| - |f_*q|$ is integrable, and it follows that $|q| - f_*|q|$ is also integrable: in fact,

$$\begin{aligned}\int_{\mathbb{P}^1-U}(|q| - f_*|q|) &= \int_{\mathbb{P}^1-U}|q| - \int_{\mathbb{P}^1-f^{-1}(U)}|q| \\ &= \int_{f^{-1}(U)-\phi(U)}|q| + \int_{\phi(U)-U}|q| - \int_{U-\phi(U)}|q|\end{aligned}$$

and

$$\begin{aligned}\int_U(|q| - f_*|q|) &= \int_U(|q| - \phi^*|q|) - \int_U(f_*|q| - \phi^*|q|) \\ &= \int_U(|q| - \phi^*|q|) - \int_{f^{-1}(U)-\phi(U)}|q|\end{aligned}$$

so

$$\begin{aligned}\int_{\mathbb{P}^1}(|q| - f_*|q|) &= \int_{\phi(U)-U}|q| - \int_{U-\phi(U)}|q| - \int_U(\phi^*|q| - |q|) \\ &= 2\pi\sum_{\langle x\rangle\subseteq E}\mathrm{Res}_{\langle x\rangle}(\phi : q) = -2\pi\,\mathrm{Res}(f : q).\end{aligned}$$

Consequently,

$$||\nabla_f q|| \geq \left|\int_{\mathbb{P}^1}(f_*|q| - |f_*q|) + \int_{\mathbb{P}^1}(|q| - f_*|q|)\right| = |\mathrm{Dec}(f : q) - 2\pi\,\mathrm{Res}(f : q)|\,.$$

□

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