REGULAR POLYNOMIAL ENDOMORPHISMS OF $\mathbf{C}^k$

ERIC BEDFORD* AND MATTIAS JONSSON**

§0 Introduction

Let $f = (f_1, \ldots, f_k) : \mathbf{C}^k \to \mathbf{C}^k$ be a mapping such that each f_j is a polynomial of degree d. We consider the behavior of f as a dynamical system. That is, we consider the behavior of the iterates $f^n = f \circ \cdots \circ f$ as $n \to \infty$. The points of most interest are those whose forward orbits show recurrent behavior. Thus we are led to focus on the set K of points whose forward orbits are bounded. The point of this paper is to present an approach to studying K by working instead at the "points at infinity" and then making a descent to K via "external rays."

To illustrate this, let us consider the case $k = 1$. The recurrent dynamics of a polynomial mapping $p : \mathbf{C} \to \mathbf{C}$ is carried by the set K, and the chaotic dynamics take place on the Julia set $J := \partial K$. A polynomial mapping may be characterized as a holomorphic mapping which extends continuously to the Riemann sphere. The point at infinity is completely invariant, and it is possible to find a holomorphic function φ defined in a neighborhood of ∞, called the Böttcher coordinate, such that φ conjugates p to the model mapping $\sigma(\zeta) = \zeta^d$ in a neighborhood of infinity. If the set K is connected, then φ has an analytic continuation to a conformal equivalence $\varphi : \mathbf{C} - K \to \mathbf{C} - \bar{\mathbf{D}}$ with the complement of the closed unit disk. Thus the dynamics of the restriction $p|(\mathbf{C} - K)$ is conjugate to σ on all of $\mathbf{C} - \bar{\mathbf{D}}$. If the inverse φ^{-1} can be extended continuously to $\partial\mathbf{D}$, then $p|J$ is represented as a quotient of $\sigma|\partial\mathbf{D}$.

Another point of view is to consider the point at infinity as the pole for the Green function for $\mathbf{C} - K$. This serves as the starting point for the use of potential-theoretic methods in the study of polynomial mappings, as was introduced by Brolin [Bro] and further developed by Sibony (see [CG]) and Tortrat [T]. A natural way to descend from infinity to the set J is to follow the gradient lines of the Green function, which are also known as "external rays." External rays were introduced by Douady and Hubbard and have been developed into a powerful tool for studying the relationship between the mappings $p|(\mathbf{C} - K)$ and $p|J$.

In our approach, we view $\mathbf{C}^k$ as an affine coordinate chart in $\mathbf{P}^k$. Thus $\Pi := \mathbf{P}^k - \mathbf{C}^k$, the hyperplane at infinity, is isomorphic to $\mathbf{C}^{k-1}$. We study polynomial mappings f of degree $d \geq 2$ which are regular, which means that f extends continuously (and thus holomorphically) to $\mathbf{P}^k$. It follows that Π is completely invariant, and we let f_Π denote the induced dynamical system at infinity. Further, Π is (super)-attracting in the normal direction, so the basin A of points which are attracted to Π in forward time is an open set containing Π. This gives a completely invariant partition $\mathbf{P}^k = K \cup A$.

---

* Supported in part by the NSF.

** Supported in part by the Swedish Natural Science Council.

Stony Brook IMS Preprint #1998/3
April 1998

We will make use of methods of the dynamics of holomorphic mappings of $\mathbf{P}^k$ and $\Pi \simeq \mathbf{P}^{k-1}$. In particular, we will make use of the approach introduced in [HP] and developed more generally and systematically in [FS1–3]. Namely, there is an invariant current T on $\mathbf{P}^k$, and the exterior powers $T^l := T \wedge \cdots \wedge T$, $1 \le l \le k$, are well defined, positive, closed currents of bidegree (l,l). The supports $J_l := \operatorname{supp}(T^l)$ serve as a family of intermediate Julia sets. In the maximal degree we have measures $\mu := T^k$ on $\mathbf{P}^k$ and $\mu_\Pi := T_\Pi^{k-1}$ (corresponding to f_Π) on Π. We will denote their supports by J and J_Π, respectively. The importance of μ is shown by the fact that it is ergodic (see Fornæss-Sibony [FS2–3]); it is balanced and has maximal entropy ($= k \log d$); and μ describes the distribution of periodic points (see Briend [Bri]). The corresponding properties hold for μ_Π and f_Π.

Closely related to the current T is the function G, defined in (1.1), which measures the superexponential rate at which an orbit approaches Π. The function G coincides with the plurisubharmonic Green function in $\mathbf{C}^k$ for the set K, and G has a logarithmic singularity along Π. In our passage from the case $k = 1$ to $k > 1$, we replace the point at infinity by the set J_Π. We replace the one-dimensional set $\mathbf{C} - K$ by the set $A \cap \operatorname{supp}(T^{k-1})$. The usual critical locus of f is the set $\mathcal{C}$ where the Jacobian determinant of f vanishes. For the purpose of dynamical study, we consider the critical measure $\mu_c := [\mathcal{C}] \wedge T^{k-1}$. This is motivated by another critical measure defined in [BS2]. A major theme of this paper is to explore the interplay between the dynamical measures μ and μ_Π, the critical measure μ_c, and the current T^{k-1}.

As a first example we consider the Lyapunov exponents, which measure the rate of expansion of the mapping f with respect to a measure. We concern ourselves here with $\Lambda(f) = \Lambda(f, \mu)$, which is defined as the sum of all the Lyapunov exponents of f with respect to μ, and thus $\Lambda(f)$ measures the infinitesimal exponential rate of Lebesgue volume expansion. We show (Theorem 3.2) that the Lyapunov exponents of (f, μ) and (f_Π, μ_Π) are related by the formula

$$\Lambda(f) = \Lambda(f_\Pi) + \log d + \int G \, \mu_c.$$

This generalizes formulas of Przytycki [Pr] and Jonsson [J1] and is analogous to a formula in [BS5]. We are grateful to B. Berndtsson for showing us how to greatly simplify our previous proof.

The connection between J_Π and J is mediated by the stable manifolds

$$W^s(a) = \{x \in \mathbf{P}^k : \lim_{n \to +\infty} \operatorname{dist}(f^n a, f^n x) = 0\}$$

and

$$W^s_{\text{loc}}(a) = \{x \in \mathbf{P}^k : \operatorname{dist}(f^n a, f^n x) < \delta, n \ge 0\}$$

for $a \in J_\Pi$. By Pesin Theory there is a $\delta = \delta(a) > 0$ for μ_Π a.e. a such that $W^s_{\text{loc}}(a)$ is a complex disk. In using the potential-theoretic approach, we work with a local stable manifold as a current of integration $[W^s_{\text{loc}}(a)]$. (In general, a component of the global stable manifold is not locally closed and does not have locally bounded area, so $W^s(a)$

does not define a current of integration.) A crucial property of T^{k-1} is that it has, in a measure-theoretic sense, a laminar structure on A, made up of currents of integration over pieces of stable manifolds. The strongest example of laminarity is Theorem 5.1: if f_Π is uniformly expanding on J_Π, then T^{k-1} has the uniform laminar structure (5.1) in a neighborhood of Π. (This result was also obtained by Peng [Pe].) For a general map, we establish two sorts of laminar structure (see Theorems 6.4 and 6.10).

The set $W^s(a)$ is a manifold (with singularities), which contains the Pesin disk $W^s_{\rm loc}(a)$, as well as $W^s_{\rm loc}(b)$ whenever $b \in J_\Pi$ and $f^n_\Pi a = f^n_\Pi b$ for some $n \geq 1$. The restriction $G|W^s(a)$ is harmonic for μ_Π-a.e a and has no critical points on $W^s_{\rm loc}(a)$. We define the set $\mathcal{E}_a$ as the set of gradient lines of $G|W^s(a)$ emanating from a (the point at infinity). Thus $\mathcal{E}_a$ is parametrized by the angle a gradient line makes inside $W^s(a)$ at a. Let $\mathcal{E}$ be the union of all $\mathcal{E}_a$ over μ_Π-a.e. a. The elements of $\mathcal{E}$ are called external rays. It follows that the measure $\nu := \mu_\Pi \otimes \frac{d\theta}{2\pi}$ is well defined on $\mathcal{E}$. Using the (non-uniform) laminar structure in the general case, we are able to show that for ν a.e. ray $\gamma \in \mathcal{E}$, there is a well defined endpoint $e(\gamma) \in J$. The basic connection between ν and μ is given by transport along the external rays (Theorem 7.3): $e_*\nu = \mu$.

In case the critical measure μ_c puts no mass near Π, and $k = 2$, the family of Pesin disks in fact extends to a Riemann surface lamination near Π (Theorem 8.8). To obtain results with better control over the geometry, we assume that f_Π is uniformly expanding on J_Π; thus by the Stable Manifold Theorem, $\mathcal{W}^s_{\rm loc}(J_\Pi) = \{W^s_{\rm loc}(a) : a \in J_\Pi\}$ is a Riemann surface lamination for a uniform $\delta > 0$. We use this lamination to obtain a local conjugacy with a model mapping. Let $C(J_\Pi)$ denote the set of complex lines through the origin in $\mathbf{C}^k$ corresponding to points of J_Π. Our canonical model is given by the restriction of f_h, the homogeneous part of f, to $C(J_\Pi)$. If f_Π is uniformly expanding on J_Π, then the restriction $f_h|C(J_\Pi)$ is conjugate to the restriction $f|\mathcal{W}^s_{\rm loc}(J_\Pi)$ in a neighborhood of J_Π (Theorem 4.3). This conjugacy, which was also found by G. Peng [Pe], can be viewed as a local Böttcher coordinate for f. A global version is given in Theorem 7.4.

The general approach of starting with f_Π to study f is dual in spirit to that of Hubbard and Papadopol [HP], who study a superattracting fixed point by working on the fiber $\mathbf{P}^{k-1}$ over the blowup. From this point of view, Theorem 4.3 is analogous to Theorem 9.3 of [HP]. A different approach was given by S. Heinemann [H], who works directly on K without appeal to Π or f_Π.

The final part of this paper addresses the question of when the landing map $e : \mathcal{E} \to J$ is continuous. When possible, we would like to replace statements about almost every ray with statements about every ray. In the case $k = 1$, the continuity of e is equivalent to local connectedness of J. In dimension $k = 2$, Π is the Riemann sphere, and f_Π is a rational map. If f_Π is hyperbolic, then $\mathcal{E}$ is homeomorphic to $J_\Pi \times S^1$. In order to obtain the continuity of e, we assume that f is Axiom A. Thus the nonwandering set is the closure of the set of periodic points and can be written as the union $S_0 \cup S_1 \cup S_2$, where S_j is a (uniformly) hyperbolic set with unstable index j. Our main result about the landing map is Theorem 10.2: *If $k = 2$, if f is Axiom A, if $f^{-1}S_2 = S_2$, and if $\mu_c(A) = 0$, then $e : \mathcal{E} \to J$ is a continuous surjection.*

The contents of this paper are as follows: Sections 1 and 2 recall some basic notation and results, mostly on currents and potential theory. §3 is devoted to the proof of the formula for the sum of the Lyapunov exponents of μ. In §4 we discuss the Stable Manifold Theorem in the context of stable disks through points of J_Π. We show that $f|W^s(J_\Pi)$ is conjugate to the canonical model in a neighborhood of J_Π. In §5 we show that T^{k-1} is laminar in a neighborhood of Π when f_Π is uniformly expanding on J_Π. This serves as a prototype for our results in §6, where we discuss laminarity in the general case. The difficulties of §6 arise from two sources. First is the fact that in general (without uniform expansion on J_Π) we need to use Pesin Theory to obtain our stable manifolds, and the geometry of the Pesin disks is not controlled. The second source of difficulty, which is present already in the uniformly expanding case, comes from working globally in A, since in general the global stable manifolds do not define currents of integration. In §7, we define external rays and show that ν almost every external ray has a well-defined landing point, and the endpoint map pushes the measure ν forward to μ. We also give a global version (Theorem 7.4) of the local conjugacy result of §4. §8 discusses the structure of the support of T^{k-1} inside A for general maps f. First we show that for μ_Π a.e. a, the global stable manifold $W^s(a)$ is dense in $A \cap \operatorname{supp}(T^{k-1})$. Then we show (Theorem 8.8) that if $\mu_c(A) = 0$, then the family of local Pesin disks in A is contained in a Riemann surface lamination by disks which are proper in A. The property Axiom A is discussed for endomorphisms of $\mathbf{C}^2$ in §9, and §10 is devoted to the proof of Theorem 10.2, which gives the continuous landing of the external rays. Appendix A analyzes the behavior of the homogeneous (canonical) model. Appendix B serves as a reference for some of the hyperbolicity results that are used in §8 and 9.

List of notation

f	regular polynomial endomorphism of $\mathbf{C}^k$ of degree d.
f_h	homogeneous part of f of degree d.
Π	hyperplane at infinity.
f_Π	restriction of f to Π.
$\mathcal{C}$	critical set of f.
$\mathcal{C}_\Pi$	critical set of f_Π.
$\mathbf{C}^k_*$	$\mathbf{C}^k - \{0\}$.
π	projection of $\mathbf{C}^k_*$ on Π or $\mathbf{C}^{k+1}_*$ on $\mathbf{P}^k$.
A	basin of Π.
K	complement of A.
G	Green function for f.
G_h	homogeneous Green function for f_h.
$\tilde{f}$	homogeneous map on $\mathbf{C}^{k+1}_*$.
$\tilde{G}$	homogeneous Green function for $\tilde{f}$.
ρ_G	Robin function for G.
T	invariant current for f.
T_h	homogeneous invariant current for f_h.

T_Π	invariant current for f_Π.
μ	T^k.
μ_Π	T_Π^{k-1}.
J	support of μ.
J_Π	support of μ_Π.
L_a	line in $\mathbf{P}^k$ through 0 associated with $a \in \Pi$.
$\Lambda(f)$	sum of Lyapunov exponents of f with respect to μ.
$\Lambda(f_\Pi)$	sum of Lyapunov exponents of f_Π with respect to μ_Π.
μ_c	critical measure: $\mu_c = [\mathcal{C}] \wedge T^{k-1}$.
$W^s_{\mathrm{loc}}(a)$	local stable manifold at a.
A_0	subset of A where $G > R_0$.
A_n	$f^{-n}(A_0)$.
$W^s(a)$	global stable manifold of a.
$W^s(a, f_\Pi)$	global stable manifold of a with respect to f_Π.
$W^s(J_\Pi)$	stable set of J_Π for f.
$W^s(J_\Pi, f_h)$	stable set of J_Π for f_h.
$W^s_0(a)$	local stable disk for f at $a \in J_\Pi$.
$W^s_0(a, f_h)$	local stable disk for f_h at a (subset of complex line).
$\mathcal{W}^s(J_\Pi)$	stable lamination for f.
$C(J_\Pi)$	complex homogeneous cone over J_Π.
Ψ	conjugation of $f_h\vert W^s(J_\Pi, f_h)$ to $f\vert W^s(J_\Pi)$.
$W^s_{-m}(a)$	$W^s_{\mathrm{loc}}(a) \cap A_{-m}$.
$Z_{a,n}$	component of $W^s(a) \cap A_n$ containing a.
$N_n(a)$	number of points in $Z_{a,n} \cap \Pi$.
$\mathcal{C}_n$	$\bigcup_{0 \le j \le n-1} f^{-j}(\mathcal{C})$ (critical set of f^n).
$\mathcal{C}_\infty$	$\bigcup_{n \ge 0} \mathcal{C}_n$.
$S_{a,n}$	union of incomplete gradient lines in $Z_{a,n}$.
$W_{a,n}$	component of $Z_{a,n} - S_{a,n}$ containing a.
W_a	union of $W_{a,n}$ over $n \ge 0$.
φ_a	uniformizing map on W_a.
H_a	range of φ_a (hedgehog domain).
ψ_a	inverse of φ_a.
$\mathcal{E}_a$	set of external rays in W_a.
$\mathcal{E}$	set of external rays.
σ	map on $\mathcal{E}$.
ν	invariant measure on $\mathcal{E}$.

e_r	endpoint map on $\mathcal{E}$ to level $G = r$.
e	endpoint map on $\mathcal{E}$.
G_r	$\max(G, r)$.
S	union of incomplete gradient lines.
S_h	union of rays in $W^s(J_\Pi, f_h)$ corresponding to S.
$\hat{q}$	history of a point q, $\hat{q} = (q_i)_{i\leq 0}$.
$\hat{f}$	shift map:$\hat{f}((q_i)) = (q_{i+1})$.
S_i	union of basic sets of unstable index i in K.
$W^u_{\mathrm{loc}}(\hat{q})$	local unstable manifold at $\hat{q}$.
$W^u(\hat{q})$	global unstable manifold at $\hat{q}$.
$W^u(J)$	backwards attracting basin for J.
$W^u(S_1)$	unstable set of S_1.

§1 Regular polynomial endomorphisms and their Green functions

In the following two sections we summarize several basic results that we will use. Additional details may be found in [HP], [FS1–3], and [U]. We recommend the unified treatment in [FS3]. Throughout this paper, we will let f be a *regular polynomial endomorphism* of $\mathbf{C}^k$ of degree $d \geq 2$. This means that the components of f are polynomials of degree d, and the homogeneous part f_h of degree d satisfies $f_h^{-1}(0) = \{0\}$. Alternatively, f is regular if and only if $\liminf |f(z)|/|z|^d > 0$ as $|z| \to \infty$. Such mappings are also called strict polynomials by Heinemann [H].

Let $z = (z_1, \ldots, z_k)$ denote (inhomogeneous) coordinates on $\mathbf{C}^k$, and let $[z : t] = [z_1 : \ldots : z_k : t]$ denote homogeneous coordinates on $\mathbf{P}^k$. We fix the embedding of $\mathbf{C}^k$ into $\mathbf{P}^k$ given by $z \mapsto [z : 1]$. Thus $\Pi = \{t = 0\}$ corresponds to the hyperplane at infinity, and Π may be identified with with $\mathbf{P}^{k-1}$ using homogeneous coordinates $[z] = [z_1 : \ldots : z_k]$. We equip $\mathbf{P}^k$ with the Fubini-Study metric and measure distances and volumes in that metric unless otherwise stated.

A regular polynomial endomorphism f extends to a holomorphic endomorphism of $\mathbf{P}^k$, still denoted by f, which may be defined by the formula $f[z : t] = [t^d f(z/t) : t^d]$. In fact, a holomorphic endomorphism of $\mathbf{P}^k$ has a completely invariant hyperplane exactly when it is conjugate to a regular polynomial endomorphism of $\mathbf{C}^k$. We let f_Π denote the restriction of f to Π. Under the identification $\Pi \cong \mathbf{P}^{k-1}$, f_Π is given by $[z] \mapsto [f_h(z)]$. Let $\mathcal{C}$, $\mathcal{C}_{\mathbf{P}^k}$ and $\mathcal{C}_\Pi$ denote the critical sets of f as a map of $\mathbf{C}^k$, $\mathbf{P}^k$ and Π, respectively. Then we have $\mathcal{C}_{\mathbf{P}^k} = \mathcal{C} \cup \Pi$ and $\mathcal{C}_\Pi = \overline{\mathcal{C}} \cap \Pi$.

The model for our study of regular polynomial automorphisms is the case when $f = f_h$ is a homogeneous mapping of $\mathbf{C}^k$. In this case we write $\mathbf{C}^k_* = \mathbf{C}^k - \{0\}$, and we let $\pi : \mathbf{C}^k_* \to \Pi$ be the projection given by $\pi(z) = [z]$. It is evident that π gives a semiconjugacy from f_h to f_Π: $\pi \circ f_h = f_\Pi \circ \pi$. In fact, f_h is essentially a skew product over f_Π, as is shown in Appendix A.

In the general case, we let K be the compact set of points in $\mathbf{C}^k$ with bounded forward orbits and define $A := \mathbf{P}^k - K$. Thus A is the basin of attraction of Π. The function

$$G(z) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z)| \tag{1.1}$$

gives the (super-exponential) rate at which the orbit of $z \in \mathbf{C}^k$ approaches Π. This is continuous and plurisubharmonic (psh) on $\mathbf{C}^k$ and coincides with the pluri-complex Green function of K. We will therefore also call G the Green function of f. The homogeneous Green function for the homogeneous part f_h of f of maximal degree d is defined in an analogous way, namely as

$$G_h(z) = \lim_{n\to\infty} d^{-n} \log |f_h^n(z)|.$$

The functions G and G_h are continuous on $\mathbf{C}^k$ and $\mathbf{C}^k_*$, respectively. We use log instead of $\log^+$ so that G_h is logarithmically homogeneous.

We may also define a homogeneous map $\tilde{f}$ on $\mathbf{C}^{k+1}$ by $\tilde{f}(z,t) = (t^d f(z/t), t^d)$. The pair $(\tilde{f}, f)$ has properties analogous to those of (f_h, f_Π). The projection $\pi : \mathbf{C}^{k+1}_* \to \mathbf{P}^k$ given by $\pi(z,t) = [z:t]$ semiconjugates $\tilde{f}$ to f: $f \circ \pi = \pi \circ \tilde{f}$. We define the homogeneous Green function $\tilde{G}$ for $\tilde{f}$ by

$$\tilde{G}(z,t) := \lim_{n\to\infty} d^{-n} \log |\tilde{f}^n(z,t)| \tag{1.2}$$

for $(z,t) \in \mathbf{C}^{k+1}_*$. The connection between $\tilde{G}$, G and G_h is $\tilde{G}(z,1) = G(z)$ and $\tilde{G}(z,0) = G_h(z)$. This leads us to the following asymptotic formulas for G and G_h near Π. Here ρ_G denotes the Robin function of G (cf. [BT2]).

Lemma 1.1. *The asymptotics of G and G_h at Π are given by*

$$\begin{aligned} G_h(z) &= \log|z| + \rho_G[z] \\ G(z) &= \log|z| + \rho_G[z] + o(1), \end{aligned}$$

where ρ_G is continuous on Π. Here $[z] = \pi(z)$ is the projection of z on Π defined above.

Proof. Since G_h is homogeneous we have

$$G_h(z) = \log|z| + G_h(z/|z|).$$

Here the second term is continuous in z and depends only on the projection $[z]$ of z on Π. Hence there exists a continuous function ρ_G on Π such that $G_h(z/|z|) = \rho_G[z]$. This proves the first formula. To prove the second we write

$$\begin{aligned} G(z) &= \tilde{G}(z,1) \\ &= \log|z| + \tilde{G}(z/|z|, 0) + (\tilde{G}(z/|z|, 1/|z|) - \tilde{G}(z/|z|, 0)) \\ &= \log|z| + \rho_G[z] + o(1), \end{aligned}$$

where the last line follows from the continuity of $\tilde{G}$ on $\mathbf{C}^{k+1}_*$. □

The next result, implicitly contained in [FS2], is crucial.

Lemma 1.2. *If $M \subset \mathbf{C}^k$ is a complex manifold, and if the iterates of f, restricted to M, are a normal family, then $G|M$ is pluriharmonic on M.*

Proof. We may assume that $M \subset A$. Passing to a subsequence of the iterates $f^n = (f^n_{(1)}, \ldots, f^n_{(k)})$, we may assume that on M we have $|f^n_{(j)}/f^n_{(1)}|$ bounded and $f^n_{(1)} \to \infty$ as $j \to \infty$. Thus $\log|f^n| = \log|f^n_{(1)}| + \frac{1}{2}\log\sum|f^n_{(j)}/f^n_{(1)}|^2$, so $\log|f^n|$ is written as a pluriharmonic function plus something bounded. Dividing by d^n and letting $n \to \infty$, we have that $G|M$ is pluriharmonic. □

§2 Invariant Currents

Here we assemble some basic facts about currents and give the definitions of the invariant currents that are defined in terms of the Green functions. Let Ω be a complex Hermitian manifold. If M is a positive current of bidimension (p,p) on Ω, then M is representable by integration. This means that there is a total variation measure $\|M\|$ and a measurable family of (p,p)-vectors $\vec{m}$ of unit length with respect to the Hermitian metric, such that we have the polar decomposition $M = \vec{m}\|M\|$. In terms of a test form ϕ, this means that $\langle M, \phi\rangle = \int \langle \vec{m}(x), \phi(x)\rangle_x \, \|M\|(x)$, where $\langle\cdot,\cdot\rangle_x$ denotes the pointwise pairing between vectors and covectors. This representation allows us to treat positive currents as measures.

Let Z is a closed subset of Ω, and let M be a positive current of bidimension on $\Omega - Z$. If the total variation measure $\|M\|$ has locally bounded mass near Z, then we may make the *trivial extension* of M to Ω by extending the domain of definition of $\|M\|$ to Ω, and setting $\|M\|(Z) = 0$.

For a general current M, we may define $M \llcorner \beta$, the contraction with a smooth form β, as the current which acts on a test form ϕ according to $\langle M\llcorner\beta, \phi\rangle = \langle M, \beta\wedge\phi\rangle$. For a positive current M and a Borel set S, we may also define the restriction $M\llcorner S$ by restricting $\|M\|$ to S. This is the same as the contraction of M by the function which is 1 on S and 0 elsewhere.

Let Ω and Ω' be complex manifolds and $g : \Omega' \to \Omega$ a holomorphic mapping. We define the pullback g^*S of a positive closed current S on Ω in two cases. The first is when g is a submersion (eg. $g = \pi$): g^*S is then defined by integrating over the fibers of g. The second case is when g is a finite branched cover (eg. g is a regular polynomial endomorphism). If S puts no mass on the critical image $g\mathcal{C}$, then S coincides with the trivial extension to Ω of $S\llcorner(\Omega - g\mathcal{C})$. We define g^*S to be the trivial extension to Ω' of $(g|_{(\Omega'-g^{-1}g\mathcal{C})})^*(S\llcorner(\Omega - g\mathcal{C}))$.

The two sorts of currents for which we will take pullbacks are as follows. First, if $[M]$ is a current of integration over a complex manifold, then our definition gives $g^*[M] = [g^{-1}M]$ both in the case where g is a submersion and when $M \cap g\mathcal{C}$ does not contain an open subset of M. Second, if locally $S \le (dd^c u)^j$ for a bounded psh function u, then S puts no mass on any complex analytic set and thus no mass on $g\mathcal{C}$. Similarly, $(dd^c(u\circ g))^j$ puts no mass on $g^{-1}g\mathcal{C}$, and $g^*(dd^c u)^j = (dd^c(u\circ g))^j$. Thus g^*S is well defined.

Before we define currents on $\mathbf{P}^k$, we recall the structure of $\mathbf{P}^k$ as a complex manifold. If U_j is the open subset of $\mathbf{P}^k$ where $z_j \ne 0$, then

$$[z_1 : \ldots : z_k : t] \to (z_1/z_j, \ldots, 1, \ldots, z_k/z_j, t/z_j) = (\xi_1, \ldots, 1, \ldots, \xi_{k+1})$$

is a section of the bundle $\pi : \mathbf{C}^{k+1}_* \to \mathbf{P}^k$ over U_j and $\xi = (\xi_1, \ldots, \xi_{k+1})$ (with the jth coordinate missing) defines a biholomorphism between U_j and $\mathbf{C}^k$.

We define invariant currents on $\mathbf{C}^k$ by $T_{\mathbf{C}^k} := \frac{1}{2\pi} dd^c G$ and $T_{h,\mathbf{C}^k} := \frac{1}{2\pi} dd^c G_h$. To define the invariant currents on $\mathbf{P}^k$, we define g_j on the coordinate chart U_j in terms of the ξ-coordinates by

$$G(\xi) = \log\frac{1}{|\xi_{k+1}|} + g_j(\xi).$$

By Lemma 1.1, g_j has a continuous extension from $U_j - \Pi$ to U_j. A property of the operator dd^c is that $dd^c g_j$ can put no mass on a pluripolar set if g_j is bounded and psh.

Since $dd^cG = dd^cg_j$ on $U_j - \Pi$, it follows that this formula defines a positive, closed current on U_j, which coincides with the trivial extension of $T_{\mathbf{C}^k}$ from $\mathbf{C}^k \cap U_j$ to U_j, and these definitions on U_j fit together to give $T_{\mathbf{P}^k}$. A similar formula serves to define $T_{h,\mathbf{P}^k}$ as a positive, closed current on the affine coordinate chart U_j, and this coincides with the trivial extension of $T_{h,\mathbf{C}^k}$. Since $T_{\mathbf{P}^k}$ and $T_{h,\mathbf{P}^k}$ are the trivial extensions of $T_{\mathbf{C}^k}$ and $T_{h,\mathbf{C}^k}$, respectively, we will just denote these currents as T and T_h.

We recall that if $S = dd^cu$ is a positive closed current of bidegree $(1,1)$ on a complex manifold with continuous potential u, and M is a complex submanifold, then the slice $S|_M$ is well-defined and is equal to the current on M defined by $S|_M = dd^c|_M(u|_M)$. We have seen that $T = T_{\mathbf{P}^k}$ has a local, continuous psh potential everywhere on $\mathbf{P}^k$. Thus we may define the current $T_\Pi := T|_\Pi$ as the slice current. By Lemma 1.1, we also see that T_Π is also given as the slice of the homogeneous current $T_\Pi = T_h|_\Pi$.

The positive closed currents of bidegree $(1,1)$ on $\mathbf{P}^k$ are characterized in terms of logarithmically homogeneous psh functions on $\mathbf{C}^{k+1}$. In terms of this characterization, we have $\pi^*T_{\mathbf{P}^k} = \frac{1}{2\pi}dd^c\tilde{G}$, with $\tilde{G}$ from (1.2), and T_Π is the unique positive, closed current T_Π on Π such that $\pi^*T_\Pi = T_h$.

Since the current T has a locally defined continuous potential, we may define the exterior powers T^l for $1 \le l \le k$. Using the fact that for a bounded, psh function g_j on U_j, $(dd^cg_j)^l$ puts no mass on a pluripolar set (and thus no mass on Π) for $1 \le l \le k$, we see that the trivial extension of $T^l_{\mathbf{C}^k}$ to $\mathbf{P}^k$ is given by $T^l_{\mathbf{P}^k}$. Thus we may denote the exterior powers of our currents simply as T^l without ambiguity. The currents T^l, $1 \le l \le k$, are positive and closed on $\mathbf{P}^k$ and satisfy $f^*T^l = d^lT^l$.

The same arguments and properties, e.g. $f_h^*T_h^l = d^lT_h^l$, apply to T_h, with the small complication that the potential $G_h = \log|z| + O(1)$ has a logarithmic singularity at the origin. Let us observe, however, that by a familiar calculation $(dd^c\log|z|)^l$ is equal to $(2\pi)^k$ times the point mass at the origin if $l = k$; and is a positive, closed current which is absolutely continuous with respect to Lebesgue measure if $l \le k-1$. Now the Comparison Theorem (see [BT1]) may be applied in the standard way to G_h and $\log|z|$, to conclude that $(dd^cG_h)^l$ is $(2\pi)^k$ times the point mass at the origin if $l = k$, and puts no mass on the origin if $l < k$.

Most important for us will be the currents T^{k-1}, T_h^{k-1}, of bidimension (1,1), and $\mu := T^k$ and $\mu_\Pi := T_\Pi^{k-1}$, of bidimension (0,0). Note that μ and μ_Π are represented by probability measures on $\mathbf{C}^k$ and Π, respectively. We will denote their supports by $J := \mathrm{supp}(\mu)$ and $J_\Pi := \mathrm{supp}(\mu_\Pi)$.

Remark. In the notation of [HP] the latter two sets would be called J_k and $J_{\Pi,k-1}$, respectively. We use J and J_Π for brevity, as we will not be using the other intermediate Julia sets.

Currents that appear in complex dynamics often have a laminar structure. Let Ω be a complex manifold with a Hermitian metric. Let (A,ν) be a measure space, and let $a \mapsto M_a$ denote a measurable family of positive currents on Ω with the property that for every relatively compact domain $\Omega_0 \subset \Omega$ we have

$$\int_{a\in A} \nu(a)\, \|M_a\|(\Omega_0) < \infty. \tag{2.1}$$

It follows from (2.1) that we may define a positive current $S = \int_{a\in A} \nu(a)\, M_a$, where the action on a test form ϕ is given by

$$\langle S, \phi\rangle := \int_{a\in A} \nu(a)\, \langle M_a, \phi\rangle.$$

We refer to S as *laminar* if for almost every a and b, either $M_a = M_b$, or the supports of M_a and M_b are disjoint. If in addition $a \to M_a$ is continuous, then the we say that S is *uniformly laminar*. We note that if the currents M_a are closed in Ω, then so is S.

A consequence of positivity is that the total variation measure $\|M_a\|$ is equivalent to the measure $M_a \llcorner \beta^p$, where β is any strictly positive (1,1)-form, and (p,p) is the bidimension of M_a. Equivalent here means that the two measures are bounded above and below by each other on compact subsets of Ω, with the constant depending only on β. Since $S \llcorner \beta^p = \int \nu(a) M_a \llcorner \beta^p$, we conclude that the total variation measure of S is equivalent to the integral of the total variation measures of the currents M_a:

$$\|S\| \sim \int \nu(a)\, \|M_a\|. \tag{2.2}$$

The case that will appear in the sequel is where M_a is the current of integration over a 1-dimensional complex variety in Ω. An example is the following result.

Proposition 2.1. *The following holds on* $\mathbf{P}^k$*:*

$$T_h^{k-1} = \int [L_a]\, \mu_\Pi(a), \tag{2.3}$$

where $L_a = \overline{\pi^{-1}(a)}$ *is the complex line in* $\mathbf{P}^k$ *passing through* $a \in \Pi$ *and the origin in* $\mathbf{C}^k$*.*

Proof. We know that T_h^l puts no mass on the origin for $l < k$, and no mass on Π in any case. Since taking the wedge product commutes with taking pullback, the identity $T_h = \pi^* T_\Pi$ gives us $T_h^l = \pi^*(T_\Pi^l)$ on $\mathbf{C}_*^k$, and hence on $\mathbf{P}^k$ if $l < k$. Hence, by the definition of π^* as integration over the fibers of π, we have

$$T_h^{k-1} = \int \left[\pi^{-1}(a)\right]\, \mu_\Pi(a).$$

Since $\{a\}$ and $\{0\}$ are sets of measure zero with respect to L_a, it follows that $[L_a]$ and $\left[\pi^{-1}(a)\right]$ define the same current on $\mathbf{P}^k$. Therefore, the equation above yields (2.3). $\square$

Next we show that the laminar structure of a current is preserved under wedge products. Let S and X be positive closed currents on a complex Hermitian manifold Ω and suppose that X is of bidegree $(1,1)$. For any ball Ω_0, there is a psh function ψ on Ω_0 such that we may write $X = dd^c\psi$ on Ω_0. In order to define $X \wedge S$, it suffices to define the action on any test form ϕ on Ω_0 as

$$\langle X \wedge S, \phi\rangle := \langle \psi\, S, dd^c\phi\rangle. \tag{2.4}$$

If we have
$$\int_{\Omega_0} |\psi| \, \|S\| < \infty,$$
then (2.4) defines a positive, closed current. Note that, in this case, if ψ_m is a sequence of smooth, psh functions decreasing to ψ, then $dd^c\psi_m \wedge S$ converges in the sense of currents to $X \wedge S = dd^c\psi \wedge S$.

Let us recall the estimate of Alexander and Taylor [AT]. If u is a bounded, psh function on Ω, then for $1 \leq j \leq k$, the current $(dd^c u)^j$ has the property that
$$\int_{\Omega_0} |\psi| \, \|(dd^c u)^j\| < \infty$$
for any relatively compact domain $\Omega_0 \subset \Omega$ and any psh function ψ on Ω. Thus, if S is a positive, closed current such that locally there exists a bounded, psh function u such that
$$S \leq (dd^c u)^j, \tag{2.5}$$
then the integral in formula (2.4) converges and defines $X \wedge S$ as a positive, closed current on Ω.

Lemma 2.2. *Let M_a, $a \in A$, be a measurable family of positive, closed currents on Ω, and let ν be a measure on A such that (2.1) holds. Suppose, too, that the current $S = \int_{a\in A} \nu(a) \, M_a$ satisfies (2.5). If X is a positive, closed current of bidegree (1,1) with local potential ψ, then for ν almost every a, $X \wedge M_a$ is a well-defined positive closed current on Ω. Further, we have*
$$X \wedge S = \int_{a\in A} \nu(a) \, (X \wedge M_a),$$
where $X \wedge S$ is defined according to (2.4).

Proof. Let $\Omega' \subset \Omega$ be a domain where there is a psh function ψ with $dd^c\psi = X$. By (2.2) and (2.5), we have
$$\int \nu(a) \, (\int_{\Omega_0} |\psi| \, \|M_a\|) \sim \int_{\Omega_0} |\psi| \, \|S\| < \infty \tag{2.6}$$
for every relatively compact $\Omega_0 \subset \Omega'$. It follows that for ν almost every a we have $\int_{\Omega_0} |\psi| \, \|M_a\| < \infty$ for all relatively compact Ω_0, and thus $X \wedge M_a$ is well defined.

Let ψ_m denote a sequence of smooth, psh functions decreasing to ψ. If ϕ is a test form on Ω_0, then the smooth current $X_m := dd^c\psi_m$ satisfies
$$\langle X_m \wedge S, \phi\rangle = \langle \psi_m S, dd^c\phi\rangle = \int_{a\in A} \nu(a) \langle \psi_m M_a, dd^c\phi\rangle.$$
The left hand side converges to $X \wedge S$ as $m \to \infty$. For fixed a, the integrand on the right hand side converges to $\langle \psi M_a, dd^c\phi\rangle = \langle X \wedge M_a, \phi\rangle$ as $m \to \infty$. Further, we have
$$|\langle \psi_m M_a, dd^c\phi\rangle| \leq C_\phi \int |\psi| \|M_a\|,$$
where C_ϕ does not depend on m or a, so the lemma follows from (2.6) and the Dominated Convergence Theorem. □

We observe that taking pullbacks respects the laminarity of a positive current S.

Lemma 2.3. *Let Ω, Ω' be complex manifolds and let $g : \Omega' \to \Omega$ be a branched covering. Let M_a, $a \in A$, be a measurable family of positive currents on Ω, and let ν be a measure on A such that (2.1) holds. Then*

$$g^* S = \int_{a \in A} \nu(a)\, g^* M_a.$$

Proof. This follows immediately from the definition of g^*S and g^*M_a. □

The object dual to the pullback is the pushforward. If S is a positive, closed current on Ω', then $g_* S$ is the current on Ω defined by $\langle g_* S, \varphi \rangle = \langle S, g^* \varphi \rangle$ for all test forms φ. If g is a finite branched covering, then the pushforward may be thought of, locally, as the inverse of the pullback so that $g_* S = \sum (g_j^{-1})^* S$, where the sum is taken over all the branches g_j^{-1} of g^{-1}. Applying this to the d^k local inverses of a regular polynomial endomorphism, we get that

$$f_* T^l = d^{k-l} T, \quad \text{and} \quad f_*(G T^{k-1}) = G T^{k-1}. \tag{2.7}$$

§3 Lyapunov exponents.

In this section we prove a formula for the sum of the Lyapunov exponents of a regular polynomial endomorphism f of $\mathbf{C}^k$. The proof below, which depends significantly on ideas of Berndtsson [Be], simplifies the argument in a previous version of this paper [J2]. A special case of (3.1) below was proved in [J1].

Let us recall the notion of Lyapunov exponents. For more details we refer to [Y]. The sum of the Lyapunov exponents of f with respect to μ is the number $\Lambda(f) = \Lambda(f, \mu)$ given by

$$\lim_{n \to \infty} \frac{1}{n} \log |\det Df^n(x)| = \Lambda(f),$$

for μ-a.e. $x \in \mathbf{P}^k$. That this is well-defined is part of the statement of Oseledec's Theorem. Hence $\Lambda(f)$ measures asymptotic infinitesimal Lebesgue volume growth of f^n at μ-a.e. point. The individual Lyapunov exponents measure the asymptotic growth of the derivative of f^n in different directions; we will not give the precise definition since we do not need it.

Our formula for $\Lambda(f)$ will involve the integral of the Green function against a critical measure so we begin by defining the latter measure as

$$\mu_c := [\mathcal{C}] \wedge T^{k-1} = \frac{1}{2\pi} dd^c H \wedge T^{k-1},$$

where $H = \log |\det Df|$.

Then μ_c is a well-defined positive measure because T has continuous local potentials, and the mass of μ_c is finite (= the degree of the critical locus).

Lemma 3.1. *Let $f = f_h$ be any homogeneous regular polynomial endomorphism of $\mathbf{C}^k$, and let $|\det(Df)|$ and $|\det(Df_\Pi)|$ be the Jacobians of f and f_Π in the Euclidean metric on $\mathbf{C}^k$ and the Fubini-Study metric on Π, respectively. Then*

$$|\det(Df)(z)| = d \cdot \left(\frac{|f(z)|}{|z|}\right)^k |\det(Df_\Pi)[z]|.$$

Proof. Pick any $z_0 \in \mathbf{C}^k_*$. After pre- and post-composing with dilations and unitary maps, we may assume that $f(z_0) = z_0 = (0, \ldots, 0, 1)$. Since z_0 and $[z_0]$ are now fixed points, the choices of metrics are irrelevant when computing the Jacobians. We use local coordinates (ξ, s) on $\mathbf{P}^k$ and ξ on Π, where $\xi_i = z_i/z_k$ for $1 \le i \le k-1$ and $s = t/z_k$. In these coordinates, the homogeneity of f allows us to write

$$\begin{aligned} f(\xi, s) &= (f_1(\xi,1)/f_k(\xi,1), \ldots, f_{k-1}(\xi,1)/f_k(\xi,1), s^d/f_k(\xi,1)),\\ f_\Pi(\xi) &= (f_1(\xi,1)/f_k(\xi,1), \ldots, f_{k-1}(\xi,1)/f_k(\xi,1)). \end{aligned}$$

Since the first $k-1$ coordinates in $f(\xi, s)$ do not depend on s, we see that

$$\det Df(\xi, s)|_{(\xi,s)=(0,1)} = d \cdot \det Df_\Pi(\xi)|_{\xi=0}.$$

We introduce the factors $|f(z)|^k$ and $|z|^k$ because of the pre- and post-compositions with dilations. This completes the proof. □

Theorem 3.2. *If f is any regular polynomial endomorphism of $\mathbf{C}^k$, then*

$$\Lambda(f) = \log d + \Lambda(f_\Pi) + \int G\, \mu_c. \tag{3.1}$$

Proof. From the Ergodic Theorem and the definition of μ we have

$$\begin{aligned} \Lambda(f) &= \int_{\mathbf{C}^k} H\, \mu \\ &= \frac{1}{2\pi} \int_{\mathbf{C}^k} H dd^c G \wedge T^{k-1}. \end{aligned} \tag{3.2}$$

Extend G to a function $\hat{G}$ on $\mathbf{P}^k$ by declaring $\hat{G} = \infty$ on Π. Then $\hat{G}$ is locally integrable, and $dd^c\hat{G} = 2\pi(T - [\Pi])$. Note that the current of integration over Π appears because $\hat{G}$ has a $+\infty$ logarithmic singularity along Π. We have $H = k(d-1)\log|z| + O(1)$ near $\Pi - \mathcal{C}_\Pi$, so we define $\hat{H}$ on $\mathbf{C}^k$ by $\hat{H}(z) = H(z) + k(\log^+|z| - \log^+|f(z)|)$. It follows that $\hat{H}$ extends to a locally integrable function on $\mathbf{P}^k$ which is continuous outside $\mathcal{C} \cup \mathcal{C}_\Pi$. Note that if $x \in \Pi$, then $\hat{H}(x)$ depends only on the homogeneous part of f of degree d. Thus $\hat{H} = \log d + \log|\det Df_\Pi|$ on Π by Lemma 3.1.

By the invariance of $\mu = \frac{1}{2\pi} dd^c G \wedge T^{k-1}$, we have

$$\int_{\mathbf{C}^k} (\log^+|z| - \log^+|f(z)|)\, dd^c G \wedge T^{k-1} = 0.$$

Thus the last integral in (3.2) equals

$$\frac{1}{2\pi}\int_{\mathbf{C}^k} \hat{H} dd^c G \wedge T^{k-1}.$$

Using the formula $dd^c\hat{G} = 2\pi(T - [\Pi])$ and the fact that $dd^c G \wedge T^{k-1}$ is supported on $\mathbf{C}^k$ we see that this equals

$$\frac{1}{2\pi}\int_{\mathbf{P}^k} \hat{H} dd^c \hat{G} \wedge T^{k-1} + \int_{\Pi} \hat{H}\,\mu_\Pi. \tag{3.3}$$

By Stokes' Theorem the first term in (3.3) is

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbf{P}^k} \hat{G}\, dd^c \hat{H} \wedge T^{k-1} &= \frac{1}{2\pi}\int_{\mathbf{C}^k} G\, dd^c \hat{H} \wedge T^{k-1} \\
&= \frac{1}{2\pi}\int_{\mathbf{C}^k} G\, dd^c (H + k(\log^+|z| - \log^+|f(z)|)) \wedge T^{k-1} \\
&= \frac{1}{2\pi}\int_{\mathbf{C}^k} G\, dd^c H \wedge T^{k-1} \\
&= \int G\,\mu_c.
\end{aligned}$$

The first equality holds because $dd^c\hat{H}\wedge T^{k-1}$ puts no mass on Π, and the third equality follows because $f_*(GT^{k-1}) = GT^{k-1}$ by (2.7), so

$$\begin{aligned}
\int_{\mathbf{C}^k} (dd^c \log^+|f(z)|)\, GT^{k-1} &= \int_{\mathbf{C}^k} (f^* dd^c \log^+|z|)\, GT^{k-1} \\
= \int_{\mathbf{C}^k} (dd^c \log^+|z|)\, f_*(GT^{k-1}) &= \int_{\mathbf{C}^k} (dd^c \log^+|z|)\, GT^{k-1}.
\end{aligned}$$

Further, by the remark above the second term in (3.3) is equal to

$$\int_\Pi (\log d + \log|\det Df_\Pi|)\,\mu_\Pi = \log d + \Lambda(f_\Pi).$$

This completes the proof. □

Corollary 3.3. $\Lambda(f) \geq \frac{k+1}{2}\log d$.

Proof. It is a result of Briend [Bri] that the Lyapunov exponents of f_Π with respect to μ_Π are bounded below by $\frac{1}{2}\log d$ (see §6). In particular, $\Lambda(f_\Pi) \geq \frac{k-1}{2}\log d$, so Theorem 3.2 gives $\Lambda(f) \geq \log d + \frac{k-1}{2}\log d = \frac{k+1}{2}\log d$. □

§4 Stable manifolds and a local model near Π.

In §5–§7 we will show how the current $T^{k-1}\llcorner A$ has a laminar structure, and how this allows us to describe μ in terms of external rays. The laminar structure is easier to handle —and visualize—in the case when f_Π is uniformly expanding on J_Π. The expansion enables us to invoke the (uniform) stable manifold theorem, and thus provides us with a continuous family of local stable manifolds. These manifolds define currents of integration, which are responsible for the laminar structure of T^{k-1} in a neighborhood of Π.

In this section we prove a version of the stable manifold theorem. We also show that f, restricted to the union of the local stable manifolds, is conjugate to the homogeneous map f_h. The homogeneous map f_h may be seen as the canonical model for f near Π and the conjugacy as a generalization of the Böttcher coordinate at infinity for one-dimensional polynomials.

Let us therefore assume that f_Π is uniformly expanding on J_Π. This means that there exist constants $c > 0$ and $\lambda > 1$ such that

$$|Df_x^n v| \geq c\lambda^n |v| \quad x \in J_\Pi,\ v \in T_x\Pi,\ n \geq 1. \tag{4.1}$$

If f is expanding on J_Π and $a \in J_\Pi$, then the tangent space $T_a\mathbf{P}^k$ splits into a direct sum $E^u(a) \oplus E^s(a)$, where $E^u(a) = T_a\Pi$ and $E^s(a)$ is the eigenspace of Df_a associated with the zero eigenvalue. We clearly have $Df_a(E^{u/s}(a)) \subset E^{u/s}(f_\Pi a)$, and $E^{u/s}(a)$ depends continuously on a. Therefore, with the definition given in Appendix B, f_Π is hyperbolic on J_Π.

The stable manifold theorem (see [Ru, p. 96] or [PS, Theorem 5.2]) asserts that there is a *local stable manifold* $W^s_{\mathrm{loc}}(a)$ at each point of a in J_Π. This is defined by

$$W^s_{\mathrm{loc}}(a) := \{x \in \mathbf{P}^k : d(f^j x, f^j a) < \delta \text{ for all } j \geq 0\} \tag{4.2}$$

for small $\delta > 0$, and is an embedded real 2-dimensional disk. In fact, since f is holomorphic, $W^s_{\mathrm{loc}}(a)$ is a complex disk, i.e. the image of an injective holomorphic immersion of $\mathbf{D}$. Moreover, the local stable manifolds depend continuously on a in the C^1 topology.

It will be convenient to work with neighborhoods of Π defined in terms of the Green function, so let $A_0 := \{G > R_0\}$ and $A_n = f^{-n}A_0 = \{G > d^{-n}R_0\}$, where $R_0 > 0$ and $n \in \mathbf{Z}$. Thus $A_n \subset A_{n+1}$, $\bigcap_n A_n = \Pi$ and $\bigcup_n A_n = A$.

Since $W^s_{\mathrm{loc}}(a) \cap \Pi = \{a\}$, it follows from Lemma 1.2 that $G|W^s_{\mathrm{loc}}(a)$ is harmonic on the complement of a and equal to $+\infty$ at a. If we choose R_0 greater than the maximum of G on $\partial W^s_{\mathrm{loc}}(a)$, it follows from the maximum principle that

$$W^s_0(a) := W^s_{\mathrm{loc}}(a) \cap A_0$$

is a properly embedded disk in A_0 for all $a \in J_\Pi$. We call $W^s_0(a)$ the *local stable disk* at a.

We also define global stable manifolds by

$$W^s(a) = \{x \in \mathbf{P}^k : d(f^j x, f^j a) \to 0 \text{ as } j \to \infty\}.$$

In contrast to the diffeomorphism case, the global stable manifolds may have singular points. Notice also that $W^s(a)$ contains all the local stable manifolds $W^s_{\mathrm{loc}}(b)$ for $b \in J_\Pi$ with $f^n_\Pi b = f^n_\Pi a$, $n \geq 0$. We will in fact prove that $W^s(a)$ is dense in the support of $T^{k-1} \llcorner A$ (see Corollary 8.5). The global stable manifolds may have infinitely many components or be connected (see the example following Proposition 4.2).

In Theorem 4.3 we will show that f is conjugate to f_h. In the proof of this theorem we will use a holomorphic homotopy between f and f_h, defined by $f_\tau = f_h + \tau(f - f_h)$ for $\tau \in \mathbf{C}$. Note that $\tau = 0$ and $\tau = 1$ correspond to f_h and f, respectively, and that the restriction of f_τ to Π is f_Π for all τ. Hence there are local stable manifolds for f_τ for all τ. We will need to control the dependence of these manifolds on τ. To get this control, we prove a version of the Stable Manifold Theorem adapted to our situation.

It will be natural to consider the *stable set of* J_Π, i.e.

$$W^s(J_\Pi) = W^s(J_\Pi, f) = \{x \in \mathbf{P}^k : d(f^n x, J_\Pi) \to 0 \text{ as } j \to \infty\}.$$

Note that the expansion of f_Π on J_Π and the superattracting nature of Π implies that $W^s(J_\Pi)$ is closed in A and that $W^s(J_\Pi) \cap \Pi = J_\Pi$.

Let G_τ be the Green function for f_τ, $A_{0,\tau} = \{G_\tau > R_0\}$, etc.

Theorem 4.1. *If δ is small enough and R_0 is large enough, then for all τ with $|\tau| < 2$ the following holds*

(1) $W^s_{\mathrm{loc}}(a; f_\tau)$ is proper in $A_{0,\tau}$ for $a \in J_\Pi$ and $W^s_{0,\tau}(a) := W^s_{\mathrm{loc}}(a; f_\tau) \cap A_{0,\tau}$ is a properly embedded disk in $A_{0,\tau}$.

(2) $W^s_{0,\tau}(a)$ is the connected component of $W^s(a, f_\tau) \cap A_{0,\tau}$ containing a. In particular, $W^s_{0,\tau}(a)$ does not depend on the choice of δ.

(3) $W^s_{0,\tau}(a)$ depends continuously on a and holomorphically on τ.

(4) G_τ is harmonic and has no critical points on $W^s_{0,\tau}(a)$.

Proof. To avoid cumbersome notation we will write f instead of f_τ. However, it is important that the constructions below hold uniformly in τ (for $|\tau| < 2$).

Our first task is to define good coordinate charts. Pick $a = [a_1 : \ldots : a_k] \in J_\Pi$. After a unitary change of coordinates we may assume that $a = [0 : \ldots : 0 : 1]$. Let $\zeta = (\zeta_1, \ldots, \zeta_{k-1})$, where $\zeta_j = z_j/z_k$ and let $t = 1/z_k$. We denote the ball $|\zeta| < \epsilon_1$ by $U_a = U_a(\epsilon_1)$, the disk $|t| < \epsilon_2$ by $V_a = V_a(\epsilon_2)$ and the box $U_a \times V_a$ by $B_a = B_a(\epsilon) = B_a(\epsilon_1, \epsilon_2)$ for $\epsilon_1, \epsilon_2 > 0$. Note that Π corresponds to $\{t = 0\}$ and the line L_a to $\{\zeta = 0\}$. Also, the Euclidean metric on B_a and the Fubini-Study metric on $\mathbf{P}^k$ differ by at most a multiplicative constant $C \geq 1$ and C is close to one if ϵ_1 and ϵ_2 are small.

We may find an iterate f^N, such that (4.1) holds with $n = N$, $c = 1$ and $\lambda = 3C$. Thus, if ϵ_1 and ϵ_2 small enough, then we have

(i) If $a, b \in J_\Pi$ and $a \neq b$, then there is an $n \geq 0$ such that $d(f^n_\Pi a, f^n_\Pi b) > 3C\epsilon_1$.

(ii) If a,$b \in J_\Pi$, $a \neq b$ and $f^N_\Pi a = f^N_\Pi b$, then $\overline{B_a} \cap \overline{B_b} = \emptyset$.

(iii) If $a \in J_\Pi$, then f^N has no critical points in $\overline{B_a} - \Pi$.

We define a vertical disk in B_a to be a disk of the form $\{\zeta = \text{const}\}$ and a vertical-like disk to be the graph of a holomorphic map $U_a \to V_a$. Similarly we define horizontal and horizontal-like disks (these have codimension 1).

By choosing $1 \gg \epsilon_1 \gg \epsilon_2 > 0$, we get that for all $a \in J_\Pi$ and for all $f = f_\tau$ with $|\tau| < 2$:

(iv) $f^N(B_a) \cap B_{f^N_\Pi a} \subset U_{f^N_\Pi a} \times \frac{1}{2} V_{f^N_\Pi a}$.

(v) $f^{-N}(B_{f^N_\Pi a}) \cap B_a \subset \frac{1}{2} U_a \times V_a$.

(vi) If Σ is a horizontal disk in B_a, then $f^N(\Sigma) \cap B_{f^N_\Pi a}$ is a horizontal-like disk in $B_{f^N_\Pi a}$ and the restriction of f^N to $\Sigma \cap f^{-N}(B_{f^N_\Pi a})$ is a biholomorphism.

Conditions (iv)–(vi) are illustrated in Figure 1 (with N=1).

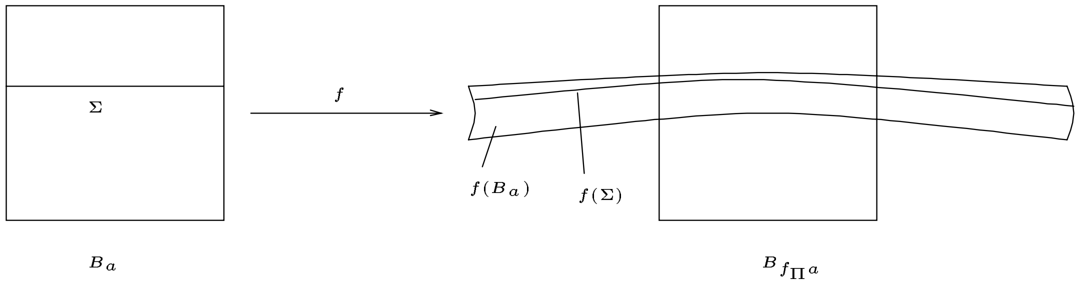

Figure 1.

To produce stable manifolds we have to iterate backwards. We claim that

(vii) If Σ' is a vertical-like disk in $B_{f_\Pi^N a}$, then $f^{-N}(\Sigma') \cap B_a$ is a vertical-like disk in B_a.

To see this, note that $f^{-N}(\Sigma') \cap B_a$ is an analytic variety in B_a. Let Σ be a horizontal disk in B_a. We claim that $f^N(\Sigma)$ intersects Σ' in exactly one point. Indeed, by (v) we may write $f^N(\Sigma) \cap B_{f_\Pi^N a} = \{t = g(\zeta)\}$ and $\Sigma' = \{\zeta = h(t)\}$, where g and h are holomorphic. Hence the intersection between these two sets is the unique fixed point of the holomorphic map $g \circ h : U_{f_\Pi^N a} \to \frac{1}{2} U_{f_\Pi^N a}$. By (v) it follows that Σ intersects $f^{-N}(\Sigma') \cap B_a$ in exactly one point. This proves that the latter set is a vertical-like disk.

Now define $B_a^n = B_a \cap f^{-N}(B_{f_\Pi^N a}) \cap \ldots \cap f^{-nN}(B_{f_\Pi^{nN} a})$ for $n \geq 0$ and $B_a^\infty = \cap_{n\geq 0} B_a^n$ Using the Kobayashi metric on U_a, it follows from (v) and (vi) that there is a constant $\kappa > 0$, independent of a and n, such that the diameter of $\Sigma \cap B_a^n$ is less that $\kappa 2^{-n}$ for every horizontal disk Σ in B_a. We claim that B_a^∞ is a vertical-like disk in B_a. Indeed, the estimate above implies that $\Sigma \cap B_a^\infty$ consists of at most one point for every Σ. On the other hand, a repeated application of (vii) shows that the set $\gamma_n(a)$, defined inductively by $\gamma_0(a) = \{0\} \times V_a$ and $\gamma_n(a) = f^{-N}(\gamma_{n-1}(f_\Pi^N a)) \cap B_a$, is a vertical-like disk in B_a, contained in B_a^n. Hence any limit of a subsequence $\gamma_{n_j}(a)$ is a vertical-like disk in B_a. By the remark above, this disk must be exactly B_a^∞. We also see that $\gamma_n(a)$, and hence B_a^∞, depends holomorphically on τ.

Clearly $f(B_a^\infty) \subset B_{f_\Pi a}^\infty$ for all $a \in J_\Pi$. We claim that the sets B_a^∞ are pairwise disjoint. Suppose $a \neq b$. By (i) there exists an $n \geq 0$ such that $f_\Pi^n b \notin 3B_{f_\Pi^n a}$. Hence $B_{f_\Pi^n a}^\infty \cap B_{f_\Pi^n b}^\infty = \emptyset$, so B_a^∞ and B_b^∞ are disjoint.

We next show that the disks B_a^∞ depend continuously on a. Note that if $\epsilon_2' < \epsilon_2$ and $\epsilon' = (\epsilon_1, \epsilon_2')$, then $B_a^\infty(\epsilon')$ is the restriction to $V_a(\epsilon_2')$ of the vertical-like disk defining $B_a^\infty(\epsilon)$. Pick any $\epsilon_2' < \epsilon_2$ and M be larger than the Lipschitz constant for all f_τ on $\mathbf{P}^k$. Assume that b is close to a and choose n maximal so that $M^{nN} C^3 d(a,b) < (\epsilon_1^2 + \epsilon_2^2)^{1/2} - (\epsilon_1^2 + \epsilon_2'^2)^{1/2}$. Then a simple calculation shows that B_b^∞ is contained in B_a^n, and the latter set intersects every horizontal disk in a set of diameter at most $\kappa 2^{-n}$. Hence B_a^∞ depends continuously on a.

It follows from the superattractive nature of Π that if ϵ_2 is small enough, then $d(fx, fa) < d(x, a)$ whenever $a \in J_\Pi$ and $x \in B_a^\infty$. Hence, if $\delta > 0$ is small enough, then $W_{\text{loc}}^s(a) = \{x \in B_a^\infty : d(x,a) < \delta\}$. Thus $W_{\text{loc}}^s(a)$ is a complex disk, compactly contained in B_a, depending continuously on a and holomorphically on τ. Thus $W_0^s(a)$ depends continuously on a and holomorphically on τ.

For $a \in J_\Pi$ in a neighborhood of $a_0 \in J_\Pi$, let ζ denote a local holomorphic coordinate for $W^s_{\mathrm{loc}}(a)$ such that $\zeta = 0$ corresponds to a. By Lemma 1.1 and Lemma 1.2, the restriction $G|W^s_{\mathrm{loc}}(a)$ has the form $\log|\zeta| + g_a(\zeta)$, where g_a is bounded and harmonic in a neighborhood of $\zeta = 0$. It follows that for ζ sufficiently small, $G|W^s_{\mathrm{loc}}(a)$ has no critical points. Thus for R_0 sufficiently large, $G|W^s_0(a)$ has no critical points.

We claim that if $a \in J_\Pi$, then

$$f^{-1}W^s_0(a) \cap A_0 = \bigcup_{f_\Pi b = a} W^s_0(b). \tag{4.3}$$

Indeed, $X := f^{-1}W^s_0(a) \cap A_0$ is a subvariety of A_0. Any component of X must meet Π, and this must happen at a point in $f_\Pi^{-1}a$. But then a neighborhood of Π in X is contained in the union of B^∞_b, where $b \in f_\Pi^{-1}a$. Thus (4.3) holds. Repeating the same argument, we see that

$$f^{-j}W^s_0(a) \cap A_0 = \bigcup_{f^j_\Pi b = a} W^s_0(b). \tag{4.4}$$

It remains to be seen that $W^s_0(a)$ is the connected component of $W^s(a) \cap A_0$ containing a. We may assume that R_0 is so large that $W^s(J_\Pi) \cap A_0$ is contained in the union of the boxes B_a. In particular $W^s(a) \cap A_0$ is contained in the union these boxes for all $a \in J_\Pi$.

Thus, by the definition of B^∞_a, we see that if $x \in W^s(a) \cap A_0$, then $f^{nN}(x) \in B^\infty_{f^{nN}_\Pi a}$ for large n. Thus (4.4) implies that $x \in \bigcup_{f^{nN}_\Pi b = a} W^s_0(b)$. Hence we have shown that

$$W^s(a) \cap A_0 = \bigcup_{b \in W^s(a, f_\Pi)} W^s_0(b), \tag{4.5}$$

where $W^s(a, f_\Pi) = \cup_{j \geq 0} f_\Pi^{-j} f^n_\Pi a$. Since the disks $W^s_0(a)$ are disjoint, it follows that $W^s_0(a)$ is the connected component of $W^s(a) \cap A_0$ containing a. □

Proposition 4.2. *For R_0 large enough we have*

$$W^s(J_\Pi) \cap A_0 = \bigcup_{a \in J_\Pi} W^s_0(a).$$

We let $\mathcal{W}^s(J_\Pi)$ denote the partition of $W^s(J_\Pi)$ by global stable manifolds. Proposition 4.2 implies that $\mathcal{W}^s(J_\Pi) \cap A_0$ is a Riemann surface lamination (see [C] for the definition). Now the iterates of f are local biholomorphisms outside the set $\mathcal{C}_\infty := \bigcup_{n \geq 0} f^{-n}(\mathcal{C})$. The expansion of f_Π on J_Π implies that $\mathcal{C}_\infty \cap W^s(J_\Pi)$ is closed and nowhere dense in $W^s(J_\Pi)$. Thus $\mathcal{W}^s(J_\Pi) - \mathcal{C}_\infty$ is also a lamination. In Figure 2 we see how the local stable disks in A_0 can join at higher levels and create a lamination whose leaves are not simply connected.

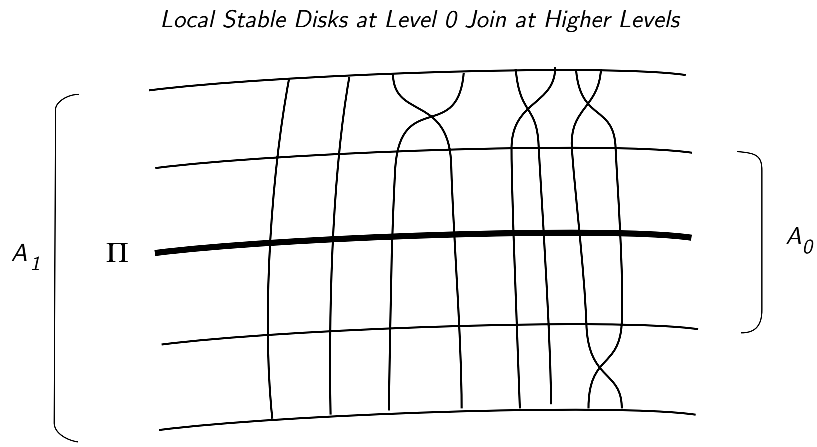

Figure 2.

Proof of Proposition 4.2. This can be proved by observing that the natural extension $\widehat{J_\Pi}$ has local product structure (see Proposition B.6), but we will give a direct proof. The inclusion "$\supset$" is trivial. After replacing f by an iterate we may assume that (4.1) holds with $n = c = 1$ and $\lambda = 3$. Let $M \geq 1$ be larger than the Lipschitz constant for f on $\mathbf{P}^k$. Let $\eta > 0$ be so small that if $a \in J_\Pi$, then all branches of f_Π^{-1} are single-valued on the ball $B(f_\Pi a, 4M\eta)$ in Π and the branch mapping $f_\Pi a$ to a maps $B(f_\Pi a, 4M\eta)$ into the ball $B(a, 2M\eta)$. Now let $x \in W^s(J_\Pi) \cap A_0$. Let n be so large that $d(f^{n+j}x, J_\Pi) < \eta$ for $j \geq 0$ and pick points $a_j \in J_\Pi$ such that $d(f^{n+j}x, a_j) < \eta$ for $j \geq 0$. Then $(a_j)_{j\geq 0}$ is an $2M\eta$-pseudoorbit in J_Π, i.e. $d(f_\Pi a_j, a_{j+1}) < 2M\eta$. Let g_j be the branch of f_Π^{-1} on $B(f_\Pi a_j, 4M\eta)$ mapping $f_\Pi a_j$ to a_j. Then $g_j(a_{j+1}) \in B(f_\Pi a_{j-1}, 4M\eta)$ so the point $b^{(j)} := g_0 \circ \ldots \circ g_j(a_{j+1})$ is well-defined. Moreover $d(f_\Pi^i(b^{(j)}), a_i) < 2M\eta$ for $0 \leq i \leq j$. Letting $j \to \infty$ and using the compactness of J_Π we find a point $b \in J_\Pi$ such that $d(f_\Pi^i b, a_i) < 3M\eta$ for all $i \geq 0$. Hence $d(f^{n+i}x, f_\Pi^i b) < 4M\eta$ for all $i \geq 0$. Assume that $4CM\eta < \epsilon$, with C and ϵ from the proof of Theorem 4.1. It follows that $f^n x \in W_0^s(b)$, so by (4.4) we have $x \in W_0^s(c)$ for some $c \in f_\Pi^{-n} b$. This completes the proof. $\square$

Remark. Proposition 4.2 holds for f_τ for $|\tau| < 2$ (with a uniform R_0).

The following family of examples shows that the behavior of the lamination $\mathcal{W}^s(J_\Pi)$ can be simple when the leaves are simply connected, and complicated otherwise.

Example. Let $f(z, w) = (z^2 + c, w^2)$. We use the affine coordinate $\zeta = w/z$ on Π, so that $f_\Pi(\zeta) = \zeta^2$, and $J_\Pi = \{|\zeta| = 1\}$. Let K_c denote the (1-dimensional) filled Julia set of $p_c(z) = z^2 + c$, and let $G_c(z) = \log|z| + o(1)$ denote the Green function for K_c. It follows that $G(z, w) = \max(G_c(z), \log^+|w|)$, and $W^s(J_\Pi) = \{G_c(z) = \log^+|w| > 0\}$. We may choose a harmonic conjugate function $G_c^*(z)$ such that $\phi_c(z) := \exp(G_c(z) + iG_c^*(z)) \approx z$ is single-valued and analytic for z large. For $\zeta \in J_\Pi$, the local stable manifold $W_0^s(\zeta)$ is given (for z large) as the graph $w = \zeta\phi_c(z)$. If c belongs to the Mandelbrot set, then K_c is connected, and ϕ_c extends analytically to $\mathbf{C} - K_c$. The stable manifolds are then countable unions of closed disks in A, each of which is a graph of the form $\{w = \zeta\phi_c(z) : z \in \mathbf{C} - K_c\}$, $\zeta \in J_\Pi$.

In case K_c is not connected, we let $\Phi : \mathbf{C} \times \mathbf{C}^* \to \mathbf{C} \times \mathbf{C}^*$ be the biholomorphic mapping given by $(u, v) = \Phi(z, w) = (z/w, 1/w)$. Thus $h := \Phi \circ f \circ \Phi^{-1}$ is given by $h(u, v) = (u^2 + cv^2, v^2)$, which is a homogeneous mapping. Let $G_h(u, v)$ denote the logarithmically homogeneous Green function associated to h. Thus

$$G_h(u, v) = G_h(uw, vw) - \log |w| = G_h(z, 1) - \log |w| = G_c(z) - \log |w|.$$

The set $\{G_h < 0\}$ is the basin of attraction of $(0, 0)$ for the mapping h, and the boundary of the basin is given by $\{G_h = 0\}$.

The set $\{G_c(z) = G_c(u/v) > 0, G_h(u, v) = 0\} \subset \{G_h = 0\}$ is the image of $W^s(J_\Pi) = \{G_c(z) > 0, G_c(z) - \log |w| = 0\}$ under the mapping Φ. Thus Φ transfers the Riemann surface lamination of the one set to the lamination of the other. The set $\{G_c(z) > 0, G_h(u, v) = 0\}$ is exactly that part of the boundary of the basin where G_h is pluriharmonic, and thus it part of the boundary that lies inside the Fatou set. The leaves of the Riemann surface lamination of this set have been studied by Hubbard and Papadopol [HP] and Barrett [Ba] and are shown to be dense and to have infinite topological type as well as other complicated behaviors.

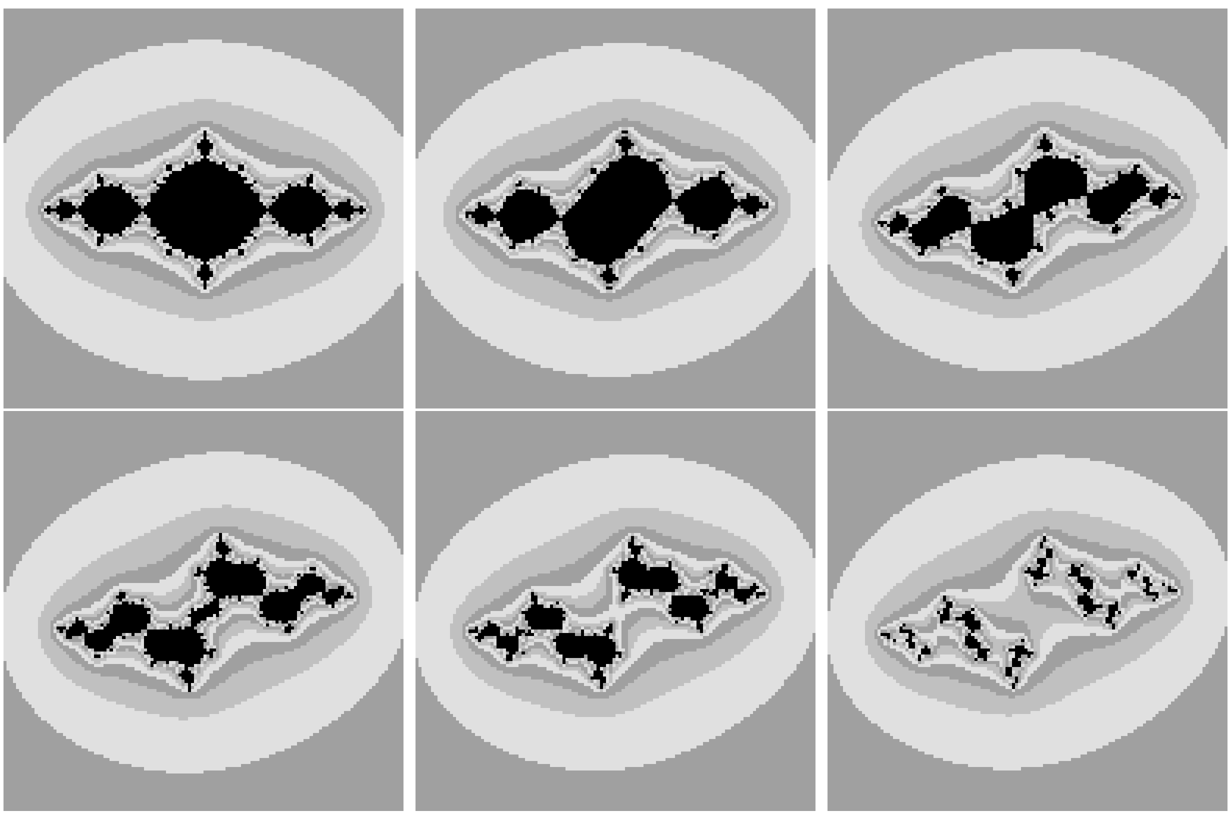

Figure 3.

Figure 3 shows slices of $W^s(J_\Pi)$ by complex lines $\{z = c\}$ for the map $f(z, w) = (z^2 - 0.1, w^2 - z^2 + 0.2z - 0.5i)$. In the coordinate $\zeta = w/z$, we have $f_\Pi(\zeta) = \zeta^2 - 1$. The first picture is the Julia set of the map $\zeta \to \zeta^2 - 1$. By Proposition 4.2, the slices above converge (suitably scaled) to this picture as $c \to \infty$. The remaining five pictures show the slices by the lines $\{z = 2\}$, $\{z = 1.3\}$, $\{z = 1.2\}$, $\{z = 1.15\}$, $\{z = 1.1\}$.

We will now show that f, restricted to $W^s(J_\Pi) \cap A_0$, is conjugate to the canonical model f_h. Note that if $f = f_h$ is homogeneous, then $W^s(J_\Pi) = C(J_\Pi) \cap A$, where $C(J_\Pi)$ is the complex cone of lines L_a, for $a \in J_\Pi$. Let $A_{0,h} := \{G_h > R_0\}$.

Theorem 4.3. *Suppose that f_Π is uniformly expanding on J_Π. If R_0 is large enough, then there is a homeomorphism $\Psi : W^s(J_\Pi, f_h) \cap A_{0,h} \to W^s(J_\Pi, f) \cap A_0$ conjugating f_h and f. Further, $G \circ \Psi = G_h$ and the restriction of Ψ to the local stable disk $W^s_0(a, f_h)$ is a biholomorphism onto $W^s_0(a, f)$ for all $a \in J_\Pi$.*

Remark. Using the Green functions G_h and G, it is easy to construct a biholomorphism of $W^s_0(a, f_h)$ onto $W^s_0(a, f)$ taking G_h to G. Such a biholomorphism is unique up to a rotation of the disk $W^s_0(a, f_h)$. The difficulty in constructing Ψ is to chose these rotations in a continuous way. Note that the disk $W^s_0(a, f)$ will not, in general, be tangent to $W^s_0(a, f_h)$ at a. On the other hand, if $f = f_h$, then we may use $\Psi = \mathrm{id}$. Our approach will be to use a holomorphically varying homotopy between f and f_h.

Proof. The idea is to define the conjugacy Ψ as $\lim_{n\to\infty} f^{-n} \circ f_h^n$, the difficulty being to define f^{-n}. We will use the notation from the proof of Theorem 4.1.

Fix $a \in J_\Pi$, $n \geq 1$ and τ with $|\tau| < 2$. Write $f_\tau = f_h + \tau(f - f_h)$. Let Δ_n be the disk defined by

$$\Delta_n = f_h^{nN} W^s_0(a, f_h) = W^s_0(f_\Pi^{nN} a, f_h) \cap A_{-nN,h}.$$

Then $f_\tau^{-nN}(\Delta_n)$ is a variety in $f_\tau^{-nN} A_{-nN,0}$, all of whose components must meet Π at some point in $f_\Pi^{-nN} f_\Pi^{nN} a$. Hence, we may write

$$f_\tau^{-nN}(\Delta_n) = \bigcup_{b \in f_\Pi^{-nN} f_\Pi^{nN} a} \beta_{b,n},$$

where $\beta_{b,n}$ is contained in a vertical-like disk in B_b. Thus f_τ^{nN} maps $\beta_{b,n}$ onto Δ_n as a branched covering of degree d, branched only at a.

Hence there are d^n locally defined branches of $f_\tau^{-nN} \circ f_h^{nN}$ mapping $W^s_0(a, f_h) - \{a\}$ into $\beta_{a,n} \subset B_a$. These branches depend holomorphically on τ. Let $\psi_{a,\tau,n}$ be the branch obtained by analytic continuation of $\psi_{a,0,n} = \mathrm{id}$. Then $\psi_{a,\tau,n}$ is well-defined on $W^s_0(a, f_h)$, depends continuously on a and holomorphically on τ.

The mappings $\psi_{a,\tau,n}$ map $W^s_0(a, f_h) \times \{|\tau| < 2\}$ into B_a, hence they form a normal family. We claim that in fact $\psi_{a,\tau,n}$ converges as $n \to \infty$. To see this, we first note that $d^{-n} G_h \circ f_\tau^n \to G_\tau$ uniformly on compact subsets of $(A_{0,h} - \Pi) \times \{|\tau| < 2\}$ by Lemma 1.1. Hence any limit point $\psi_{a,\tau}$ of $\psi_{a,\tau,n}$ must have the following properties:

(i) $\psi_{a,\tau}$ depends holomorphically on τ.
(ii) $G_\tau \circ \psi_{a,\tau} = G_h$.
(iii) $\psi_{a,\tau}$ maps $W^s_0(a, f_h)$ biholomorphically onto $W^s_0(a, f_\tau)$.
(iv) $\psi_{a,0} = \mathrm{id}$.

Now suppose that $\psi'_{a,\tau}$ and $\psi''_{a,\tau}$ are two such limits. By (iii) the mapping $\nu_{a,\tau} := (\psi''_{a,\tau})^{-1} \circ \psi'_{a,\tau}$ is a biholomorphism of the disk $W^s_0(a, f_h)$. By (i) $\nu_{a,\tau}$ depends holomorphically on τ and by (ii) $\nu_{a,\tau}$ is a rotation for all τ. Hence $\nu_{a,\tau}$ is a constant (not depending on τ) times the identity, so by (iv) $\nu_{a,\tau} = \mathrm{id}$, i.e. $\psi'_{a,\tau} = \psi''_{a,\tau}$ for all τ.

Thus $\psi_{a,\tau,n}$ converges to a map $\psi_{a,\tau}$ having the properties (i)–(iv). Define Ψ_τ by $\Psi_\tau = \psi_{a,\tau}$ on $W^s_0(a, f_h)$. Then Ψ_τ is a bijection of $W^s(J_\Pi, f_h) \cap A_{0,h}$ onto $W^s(J_\Pi, f_\tau) \cap A_{0,\tau}$.

We claim that Ψ_τ conjugates f_h to f_τ. This amounts to showing that, for all $a \in J_\Pi$, $\psi_{f_\Pi a,\tau} \circ f_h = f_\tau \circ \psi_{a,\tau}$ on $W^s_0(a, f_h)$. Now these two mappings are both branched coverings of $W^s_0(a, f_h)$ onto $W^s_0(f_\Pi a, f_\tau) \cap A_{-1,\tau}$ of degree d, branched only at a. Moreover, we have

$$G_\tau \circ \psi_{f_\Pi a,\tau} \circ f_h = d \cdot G_h = G_\tau \circ f_\tau \circ \psi_{a,\tau}.$$

Hence there exists a complex number $\nu_{a,\tau}$ of unit modulus such that

$$f_\tau \circ \psi_{a,\tau} \circ \nu_{a,\tau} = \psi_{f_\Pi a,\tau} \circ f_h.$$

Since $\nu_{a,\tau}$ depends holomorpically on τ and $\nu_{a,0} = 1$ we see that $\nu_{a,\tau} = 1$ for all τ.

We complete the proof by showing that Ψ_τ is continuous for all τ. Fix $a \in J_\Pi$ and pick parametrizations $\chi_b : \hat{\mathbf{C}} - \bar{\mathbf{D}}_{R_0} \to W^s_0(b, f_h)$ for $b \in J_\Pi$ close to a, such that χ_b depends continuously on b and $G_h \circ \chi_b = \log|\cdot|$. It suffices to prove that $\psi_{b,\tau} \circ \chi_b$ converges to $\psi_{a,\tau} \circ \chi_a$ as $b \to a$. Again this follows, using the Green functions and the holomorphic dependence on τ. □

Remark. The above result is similar to Theorem 9.3 in [HP].

§5 Uniform laminar structure of T^{k-1} near Π.

The next two sections will be devoted to the laminarity of the current T^{k-1} on A. In §6, we cover some rather general situations. It is worthwhile, however, to start with the case where f_Π is uniformly expanding on J_Π, and we obtain (5.1), which is our strongest laminarity property.

Theorem 5.1. *If f_Π is expanding on J_Π and R_0 is large enough, then*

$$T^{k-1} \llcorner A_0 = \int [W^s_0(a)]\ \mu_\Pi(a). \tag{5.1}$$

Proof. It follows from Proposition 2.1 that

$$\begin{aligned}\frac{1}{d^{j(k-1)}}\left(f^j\right)^*\left(T_h^{k-1} \llcorner A_0\right) \llcorner A_0 \\ = \frac{1}{d^{j(k-1)}}\left(\left(f^j\right)^* \int [L_a \cap A_0]\, \mu_\Pi(a)\right) \llcorner A_0,\end{aligned}$$

for all $j \geq 0$. We will show: (1) the left hand side tends to $T^{k-1} \llcorner A_0$ as $j \to \infty$, and (2) the right hand side tends to $\int [W^s_0(a)]\, \mu_\Pi(a)$ as $j \to \infty$.

To prove (1) it suffices to show that $d^{-j} G_h \circ f^j \to G$ uniformly on $A_0 - \Pi$. But from Lemma 1.1 we know that $G - G_h$ is bounded on A_0. Thus

$$\begin{aligned}\frac{1}{d^j} G_h \circ f^j - G &= \frac{1}{d^j}\,(G_h - G) \circ f^j \\ &= O\left(\frac{1}{d^j}\right).\end{aligned}$$

To show (2), we use Lemma 2.3 and calculate

$$
\begin{aligned}
\frac{1}{d^{(k-1)j}}&\left(\left(f^j\right)^* \int [L_a \cap A_0]\ \mu_\Pi(a)\right) \llcorner A_0 \\
&= \int \frac{1}{d^{(k-1)j}} \left[f^{-j}\left(L_a \cap A_0\right) \cap A_0\right]\ \mu_\Pi(a) \\
&= \int \frac{1}{d^{(k-1)j}} \left[f^{-j}\left(L_{f_\Pi^j a}\right) \cap A_0\right]\ \mu_\Pi(a),
\end{aligned}
$$

where we have used the invariance of μ_Π. From the proof of Theorem 4.1 we know that $f^{-j}L_{f_\Pi^j a} \cap A_0$ is a union of $d^{(k-1)j}$ disjoint complex disks $\gamma_j(b)$, over $b \in f_\Pi^{-j} f_\Pi^j a$ (at least if j is a multiple of N, with N from the same proof). Hence we get

$$
\begin{aligned}
\int \frac{1}{d^{(k-1)j}} \left[f^{-j}\left(L_{f_\Pi^j a}\right) \cap A_0\right]\ \mu_\Pi(a) &= \int \frac{1}{d^{(k-1)j}} \sum_{b \in f_\Pi^{-j} f_\Pi^j a} [\gamma_j(b)]\ \mu_\Pi(a) \\
&= \int [\gamma_j(a)]\ \mu_\Pi(a),
\end{aligned}
$$

since $f_\Pi^* \mu_\Pi = d^{k-1}\mu_\Pi$. Moreover, from the same proof it follows that $\gamma_j(a)$ converges to the local stable disk $W_0^s(a)$ in C^1-topology, uniformly in a. Hence the last line above converges to $\int [W_0^s(a)]\, \mu_\Pi(a)$ as $j \to \infty$, completing the proof. □

Theorem 5.1 allows us to describe the support of $T^{k-1} \llcorner A$ in dynamical terms.

Corollary 5.2. *If f_Π is expanding on J_Π, then* $\operatorname{supp}(T^{k-1}) \cap A = W^s(J_\Pi)$.

Proof. It follows from Theorem 5.1, from the continuity of $a \to W_0^s(a)$ and from Proposition 4.2 that the support of $T^{k-1} \llcorner A_0$ is equal to $W^s(J_\Pi) \cap A_0$. This proves the corollary, because the sets $\operatorname{supp}(T^{k-1}) \cap A$ and $W^s(J_\Pi)$ are both completely invariant and any compact subset of either of them is mapped by some iterate of f into A_0. □

Another consequence of Theorem 5.1 is that T^{k-1} has a uniform laminar structure on $A_n = f^{-n}A_0$ for every $n \geq 0$, hence on every relatively compact subset of A.

Corollary 5.3. *For every $n \geq 0$ we have*

$$
T^{k-1} \llcorner A_n = \int \frac{1}{d^{(k-1)n}} \left[f^{-n} W_0^s(f_\Pi^n a)\right]\ \mu_\Pi(a). \tag{5.2}
$$

Proof. This is an easy consequence of Theorem 5.1. Indeed,

$$
\begin{aligned}
T^{k-1} \llcorner A_n &= \frac{1}{d^{(k-1)n}} (f^n)^* \left(T^{k-1} \llcorner A_0\right) \\
&= \frac{1}{d^{(k-1)n}} (f^n)^* \left(\int [W_0^s(a)]\ \mu_\Pi(a)\right) \\
&= \frac{1}{d^{(k-1)n}} \int \left[f^{-n}\left(W_0^s(a)\right)\right]\ \mu_\Pi(a) \\
&= \int \frac{1}{d^{(k-1)n}} \left[f^{-n}\left(W_0^s\left(f^n(a)\right)\right)\right]\ \mu_\Pi(a).
\end{aligned}
$$

□

§6 Laminar Structure of T^{k-1} on A.

The main goal of this section is to show that the current $T^{k-1} \llcorner A$ has a laminar structure. We have seen in the previous section that if f_Π is uniformly expanding on J_Π, then $T^{k-1} \llcorner A$ has a uniformly laminar structure in a neighborhood of Π with respect to the Riemann surface lamination given by the local stable disks $W_0^s(a)$. Also, T^{k-1} is uniformly laminar on A_n for each $n \geq 0$, hence on each compact subset of A.

Here we show that there is a nonuniform laminar structure in general. When the expansion of f_Π is not uniform, we still have stable manifolds by Pesin theory. Without uniformity, however, we are not able to bound the topological type of the stable manifolds in a neighborhood of Π. Despite this, there is a formulation of the laminarity of $T^{k-1} \llcorner A_n$ (Theorem 6.4) in terms of currents of integration over subvarieties of A_n. This allows us to express the restriction of the critical measure μ_c to A_n as an intersection product with the critical locus (Corollary 6.5). A more global laminar formulation for $T^{k-1} \llcorner A$ (Theorem 6.10) is obtained by subdividing the manifolds in the lamination into disks, which is done by cutting along the gradient lines of G.

Our starting point is the result by Briend [Bri] that the Lyapunov exponents of f_Π with respect to μ_Π are strictly positive. More precisely, for μ_Π-almost every a and all $v \in T_a\Pi$, $v \neq 0$, we have

$$\liminf_{j\to\infty} \frac{1}{j} \log |Df_\Pi^j(a)v| \geq \frac{1}{2}|v| \log d.$$

Thus, if we set $E_a^u = T_a\Pi$, then f is (nonuniformly) expanding on the subspace E_a^u. When we consider the mapping f at a point a of $\Pi - \mathcal{C}_\Pi$, there is a unique one-dimensional subspace E_a^s of $T_a\mathbf{P}^k$ such that the restriction of Df to E_a^s is zero. We also have $Df(T_a\Pi) \subset T_{f_\Pi a}\Pi$. Note that $\mu_\Pi(\mathcal{C}_\Pi) = 0$, since $\mathcal{C}_\Pi$ is pluripolar in Π and μ_Π has continuous local potentials. Hence, for μ_Π-a.e. a the tangent space of $\mathbf{P}^k$ at a is the direct sum of two subspaces on which Df^j is asymptotically expanding and contracting, respectively. In other words, μ_Π is a hyperbolic measure for f.

By Pesin theory there exists a local stable manifold through almost every point of J_Π. In general, we write

$$W^s_{\mathrm{loc}}(a) = \{x \in \mathbf{P}^k : d(f^j x, f^j a) < \delta \ \forall j \geq 0\}$$

for small $\delta > 0$. Since μ_Π is a hyperbolic measure, it is a consequence of Pesin Theory that for μ_Π-almost every $a \in J_\Pi$ there exists a $\delta = \delta(a) > 0$ such that $W^s_{\mathrm{loc}}(a)$ is an embedded real 2-dimensional disk in $\mathbf{P}^k$, tangent to E_a^s at a. Since f is holomorphic, $W^s_{\mathrm{loc}}(a)$ is in fact a complex disk in $\mathbf{P}^k$. We may choose $m = m(a) \geq 0$ such that $W^s_{\mathrm{loc}}(a)$ is proper in the neighborhood A_{-m} of Π.

The precise statement from Pesin Theory that we will need is the following, which is an adaptation of Corollary 5.3 of [PS].

For every $\eta > 0$ there exists a compact subset $F = F_\eta$ of J_Π with $\mu_\Pi(F) \geq 1 - \eta$ and an integer $m = m(\eta) \geq 0$ such that the following holds:

(a) F has no isolated points and does not intersect the set $\bigcup_{j\in\mathbf{Z}} f^j(\mathcal{C}_\Pi)$.

(b) *For each $a \in F$, the local stable manifold $W^s_{\text{loc}}(a)$ is proper in A_{-m} and $W^s_{-m}(a) := W^s_{\text{loc}}(a) \cap A_{-m}$ is a properly embedded disk in A_{-m}. The Green function G is harmonic on $W^s_{-m}(a) - \{a\}$ and has no critical point there.*

(c) *The map $a \to W^s_{-m}(a)$ is continuous in the C^1 topology and the set of disks $\{W^s_{-m}(a) : a \in F\}$ defines a Riemann surface lamination in A_{-m}.*

(d) *For each $a \in F$, we let L_a denote the complex line in $\mathbf{P}^k$ defined by a, and we let $D_j(a)$ denote the component of $f^{-j}(L_{f^j_\Pi a}) \cap A_{-m}$ containing a. Then $D_j(a)$ is a complex disk in A_{-m} for $j \geq 0$ and $D_j(a)$ converges in the C^1 topology to $W^s_{-m}(a)$ as $j \to \infty$. This convergence is uniform in a for $a \in F$.*

Property (a) is a consequence of the fact that μ_Π does not give mass to pluripolar sets. For (b), we know from Lemma 1.2 that G is harmonic on the local stable manifolds. Since the local stable manifolds vary continuously over F, we see from Lemma 1.1 that G has no critical points on $W^s_{\text{loc}}(a)$ if δ is small enough. Thus we may choose $m \geq 0$ so that $W^s_{\text{loc}}(a) \cap A_{-m}$ is a properly embedded complex disk in A_{-m}. Properties (c) and (d) follow from the construction of the local stable manifolds as in [PS].

We will call a set F satisfying (a)–(d) a *Pesin box* and the associated disks $W^s_{-m}(a)$ *Pesin disks*.

In general we must let $m(\eta) \to \infty$ as $\eta \to 0$ to insure that $W^s_{\text{loc}}(a)$ is a proper disk in A_{-m} for all $a \in F$. Figure 4 illustrates this phenomenon.

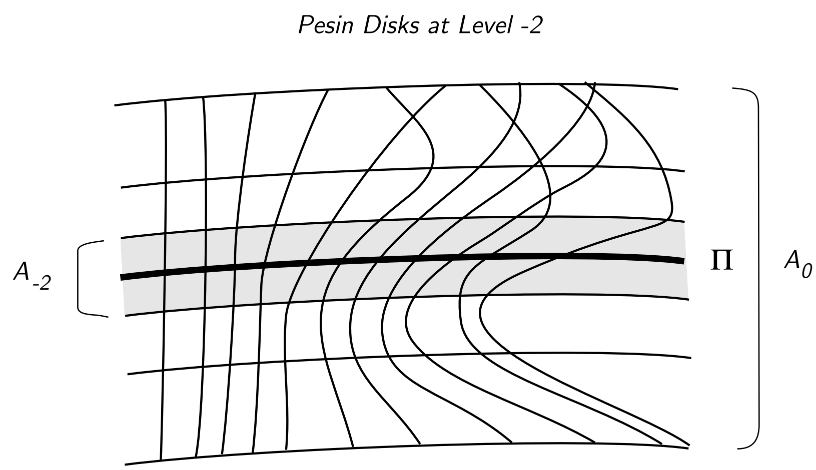

Figure 4.

In §5 we saw that the (uniform) hyperbolicity of f on J_Π implies that T^{k-1} has a uniform laminar structure in a neighborhood of Π. The following lemma is a corresponding result in the nonuniformly hyperbolic case. By restricting to a small neighborhood A_{-m} of Π we get many Pesin disks and these account for a large part of T^{k-1} in this neighborhood.

Lemma 6.1. *Given $\eta > 0$ there exists $m \geq 0$ and a Pesin box $F = F_\eta$ having properties (a)–(d) above such that*

$$T^{k-1} \llcorner A_{-m} = \int_{a \in F} [W^s_{-m}(a)]\, \mu_\Pi(a) + S \llcorner A_{-m}, \tag{6.1}$$

where S is a positive closed current on $\mathbf{P}^k$ with $\|S\| \le \eta$.

Proof. We let F satisfy properties (a)–(d) above. From Proposition 2.1 we have

$$\begin{aligned} T_h^{k-1} \llcorner A_{-m} &= \int_{J_\Pi} [L_a \cap A_{-m}]\, \mu_\Pi(a) \\ &= \int_{J_\Pi} [L_{f^j a} \cap A_{-m}]\, \mu_\Pi(a) = \int_F + \int_{^cF}, \end{aligned}$$

where we have used the invariance of μ_Π. Now we apply f^{*j} to this equation, divide by $d^{(k-1)j}$, restrict to A_{-m} and let $j \to \infty$. As in Theorem 5.1 we see that the left hand side then converges to $T^{k-1} \llcorner A_{-m}$ and, using (b), $\frac{1}{d^{(k-1)j}} f^{*j}(\int_F) \llcorner A_{-m}$ converges to $\int_F [W^s_{-m}(a)]\, \mu_\Pi(a)$.

Finally, we consider $\frac{1}{d^{(k-1)j}} f^{*j}(\int_{^cF}) \llcorner A_{-m}$. The mass norm of the current of integration over a curve of degree r in $\mathbf{P}^k$ is r (up to a constant only depending on the volume of $\mathbf{P}^k$). In particular, $\|L_a\| = 1$. It follows that the currents $\frac{1}{d^{(k-1)j}} f^{*j}(\int_{^cF} [L_{f^j a}])$ have mass norms that are bounded by η. Passing to a subsequence, we obtain a limit S which has the desired properties in (6.1). □

Corollary 6.2. *Given the assumptions in Lemma 6.1 and $n \ge 0$, we have*

$$T^{k-1} \llcorner A_n = \int_F \frac{1}{d^{(k-1)(n+m)}} [f^{-(n+m)} W^s_{-m}(a)]\, \mu_\Pi(a) + S \llcorner A_n, \tag{6.2}$$

where S is a positive, closed current on $\mathbf{P}^k$ with mass norm bounded by η.

Proof. We pull (6.1) back by f^{n+m} and divide by $d^{(k-1)(n+m)}$. Note that the proof of Lemma 6.1 shows that the current S in (6.1) satisfies $\|(f^{n+m})^* S\| \le d^{(k-1)(n+m)}$. □

Corollary 6.2 shows that we can approximate T^{k-1} on A_n by laminar currents over closed varieties in A_n. We want to pass to the limit and obtain an exact formula, just as in (5.2).

Let us fix a Pesin box $E_0 \subset J_\Pi$ satisfying (a)–(d) above with some $\eta > 0$. We may choose R_0 so that $m = m(\eta) = 0$. By Poincaré recurrence, the set

$$E = \{a \in J_\Pi : f_\Pi^n a \in E_0 \text{ for infinitely many } n \ge 0\} \tag{6.3}$$

has full measure. It is clearly invariant under f_Π. Recall the notion of the global stable manifold $W^s(a)$ of a point $a \in J_\Pi$. In general, $W^s(a)$ will be a very complicated object, but we do have the following.

Lemma 6.3. *If $a \in E$, and $n \ge 0$, then $W^s(a) \cap A_n$ is a disjoint countable union of connected varieties in A_n, each of which intersects Π transversely at finitely many points.*

Proof. Let $n \le n_1 < n_2 < \dots$ be return times for a to E_0, i.e. $f_\Pi^{n_j} a \in E_0$. Let $Z_a^{(j)} = f^{-n_j} W_0^s(f_\Pi^{n_j} a) \cap A_n$ for $j \ge 1$. Then $Z_a^{(j)}$ is a variety in A_n which intersects Π transversely at finitely many points because of property (a). Further, $f^{n_{j+1}-n_j}$ maps $W_0^s(f_\Pi^{n_j} a)$ into

$W_0^s(f_\Pi^{n_{j+1}}a)$, so $Z_a^{(j)} \subset Z_a^{(j+1)}$ and by definition of $W^s(a)$ we have $W^s(a)\cap A_n = \bigcup_{j\geq 1} Z_a^{(j)}$. We complete the proof by showing that $Z_a^{(j)}$ is open in $Z_a^{(j+1)}$. Take $x \in Z_a^{(j)}$ and let $y \in Z_a^{(j+1)}$ be close to x. We have to show that $f^{n_j}y \in W_0^s(f_\Pi^{n_j}a)$. If y is close to x, then the orbit $(f^l f^{n_j} y)_{l\geq 0}$ stays close to $(f^l f^{n_j} a)_{l\geq 0}$ for $0 \leq l \leq n_{j+1} - n_j$. But $f^{n_{j+1}}y \in W_0^s(f_\Pi^{n_{j+1}}a)$, so the same is true for $l \geq n_{j+1} - n_j$. Hence $f^{n_j}y \in W_0^s(f_\Pi^{n_j}a)$ and we are done. □

Theorem 6.4. *Given any regular polynomial endomorphism of $\mathbf{C}^k$ there is a set $E \subset J_\Pi$ with $\mu_\Pi(E) = 1$ such that the following holds. If $a \in E$, $n \geq 0$ and $Z_{a,n}$ is the connected component of $W^s(a) \cap A_n$ containing a, then $Z_{a,n}$ is a one-dimensional subvariety of A_n intersecting Π in $N_n(a) < \infty$ points, and*

$$T^{k-1} \llcorner A_n = \int_E \frac{1}{N_n(a)}[Z_{a,n}]\, \mu_\Pi(a). \tag{6.4}$$

Further, the restriction of G to $Z_{a,n}$ is harmonic outside the singular locus of $Z_{a,n}$ for all $a \in E$.

Proof. The first part follows from Lemma 6.3 and the harmonicity of G from Lemma 1.2. It remains to show (6.4). Given $\eta > 0$, let $F = F_\eta$ be a Pesin box satisfying (a)–(d) above with $m = m(\eta) \geq 0$. If $a \in f_\Pi^{-(n+m)}F_\eta \cap E$, then $f^{-(n+m)}(W_{-m}^s(f_\Pi^{n+m}a))$ is a subvariety of A_n, contained in $W^s(a)$. By Lemma 6.3 we may therefore write

$$f^{-(n+m)}(W_{-m}^s(f_\Pi^{n+m}a)) = \bigcup_{b\in f_\Pi^{-(n+m)} f_\Pi^{n+m} a} Z_{b,n},$$

and, by definition of $N_n(b)$,

$$\sum_{b\in f_\Pi^{-(n+m)} f_\Pi^{n+m} a} \frac{1}{d^{(k-1)(n+m)}}[f^{-(n+m)}(W_{-m}^s(f_\Pi^{n+m}b))] = \sum_{b\in f_\Pi^{-(n+m)} f_\Pi^{n+m} a} \frac{1}{N_n(b)}[Z_{b,n}].$$

This, Corollary 6.2, and the invariance of μ_Π yield

$$T^{k-1} \llcorner A_n = \int_{f_\Pi^{-(n+m)} F\cap E} \frac{1}{N_n(a)}[Z_{a,n}]\, \mu_\Pi(a) + S \llcorner A_n.$$

Theorem 6.4 follows by letting $\eta \to 0$. □

Corollary 6.5. *With the assumptions and notation of Theorem 6.4, we have*

$$\mu_c \llcorner A_n = \int_E \frac{1}{N_n(a)}[\mathcal{C} \cap Z_{a,n}]\, \mu_\Pi(a). \tag{6.5}$$

Further, $\mu_c \llcorner A_n = 0$ holds if and only if $\mathcal{C} \cap Z_{a,n} = \emptyset$ for almost every a and $\mu_c \llcorner A = 0$ if and only if $\mathcal{C} \cap W^s(a) = \emptyset$ for almost every a.

Proof. The formula (6.5) follows from Lemma 2.2 and Corollary 6.4. It follows directly from (6.5) that $\mu_c \llcorner A_n = 0$ if and only if $\mathcal{C} \cap Z_{a,n} = \emptyset$ for almost every a. By Lemma 6.3 this happens if and only if $\mathcal{C} \cap W^s(a) \cap A_n = \emptyset$. Letting $n \to \infty$ we get $\mu_c \llcorner A = 0$ if and only if $\mathcal{C} \cap W^s(a) = \emptyset$ for almost every a. □

Formula (6.4) exhibits $T^{k-1} \llcorner A_n$ as a laminar current using currents of integration over (closed) subvarieties $Z_{a,n}$ in A_n. These varieties $Z_{a,n}$ are subsets of the global stable manifolds $W^s(a)$. We would like to have a laminar structure for T^{k-1} in the larger set A. We could try to do this by attempting to extend the varieties $Z_{a,n}$ analytically to subvarieties of A. However, in §4 we gave an example of a map f (with f_Π uniformly expanding on J_Π) where the global stable manifolds $W^s(a)$ are connected and have locally infinite area in A. The analytic continuations of $Z_{a,n}$ would be $W^s(a)$ in this case, hence would not define currents of integration.

Nevertheless, we will show that $T^{k-1} \llcorner A$ does have a laminar structure. We will accomplish this by dividing the global stable manifolds $W^s(a)$ into disks W_a. These disks are not in general closed in A. The construction of W_a consists of cutting $W^s(a)$ along gradient lines of G. These gradient lines will be used in §7 to exhibit μ as a quotient of the product of μ_Π and Lebesgue measure on the circle.

Let E_0 and E be the sets defined above. Recall that $G|_{W_0^s(a)}$ has no critical points for $a \in E_0$. Let $\mathcal{C}_n = \mathcal{C} \cup f^{-1}\mathcal{C} \cdots \cup f^{-(n-1)}\mathcal{C}$ be the critical set of f^n. Let us fix $a \in E$ and $n \geq 0$ and recall the definition of $Z_{a,n}$ above. Suppose that $Z_{a,n} \subset \mathcal{C}_n$. Since the restriction of G to Z_a is harmonic outside the singular locus, $G|_{Z_a}$ cannot be bounded above, so $Z_{a,n} \cap \Pi$ is nonempty. It follows that $Z_{a,n} \cap \mathcal{C}_n \cap \Pi \neq \emptyset$, which contradicts property (a). Thus $\mathcal{C}_n \cap Z_{a,n}$ is a discrete set. In fact it is finite, because $\mathcal{C}_n \cap Z_{a,n+1}$ is also discrete. Now f^n is a local biholomorphism on $\mathbf{C}^k - \mathcal{C}_n$, so f^{-n} serves to transfer certain properties from the Pesin disks of E_0. Specifically, let n be such that $f_\Pi^n a \in E_0$. Such an n will be called a *return time*. If n is a return time, then $Z_{a,n} - \mathcal{C}_n$ is a manifold, and the restriction of G to this manifold is a harmonic function without critical points.

We wish to define gradient lines of G. There is a unique tangent line to the level sets of $G|_{Z_{a,n}}$ at points off of $\mathcal{C}_n$. By the conformal structure, we may define the gradient vector of G to be the tangent to $Z_{a,n} - \mathcal{C}_n$ which is orthogonal to the level line, and which points in the direction of increasing G. A *gradient line* is then an integral curve of the gradient vector field. Equivalently, a gradient line is, locally, a level set for a harmonic conjugate to G. We note that these so-called gradient lines are an artifact of the conformal structure. We could have as well defined τ-gradient lines, which make an angle of τ with respect to the gradient inside the tangent space to $Z_{a,n} - \mathcal{C}_n$.

We say that a gradient line γ is *complete* if it is a complete orbit of the gradient vector field, and if $\sup_\gamma G = +\infty$. We let $S_{a,n}$ denote the set $Z_{a,n} \cap \mathcal{C}_n$, together with points of $Z_{a,n} - \mathcal{C}_n$ which are not contained in complete gradient lines.

Lemma 6.6. *The set $S_{a,n} - \mathcal{C}_n$ consists of a finite number of (incomplete) gradient lines.*

Proof. If γ is an incomplete gradient line, it follows that the closure of γ must contain a singular point of the gradient field. Let us fix a point $p \in Z_{a,n} \cap \mathcal{C}_n$. It suffices to show that there are only finitely many gradient lines whose closures contain p. For this, we let $h : \Delta \to Z_{a,n}$ be a holomorphic mapping with $h(0) = p$, and which is a homeomorphism with its image. It follows that $G \circ h(\zeta) = \Re(a_l \zeta^l) + \cdots$. By further composing with a conformal map fixing the origin, we may achieve that $G \circ h(\zeta) = \Re(\zeta^l)$. Since the gradient lines are conformally invariant, it follows that they are mapped into the gradient lines of the function $\Re(\zeta^l)$, which are a finite family of straight lines through the origin. $\square$

It follows that $S_{a,n}$ is a closed subset of $Z_{a,n}$, and thus $Z_{a,n} - S_{a,n}$ is a manifold and an open subset of the subvariety $Z_{a,n} \subset A_n$. An alternative definition of $Z_{a,n} - S_{a,n}$ is that it is the largest open subset of $Z_{a,n} - \mathcal{C}_n$ which is invariant under the (positive) gradient flow. For every point $x \in Z_{a,n} - S_{a,n}$, the gradient line starting at x approaches a unique point $a' \in \Pi$.

Let $W_{a',n}$ denote the connected component of $Z_{a,n} - S_{a,n}$ containing a'. Then the gradient lines of all points of $W_{a',n}$ must approach a'. Thus $W_{a',n} \cap \Pi = \{a'\}$. From the paragraph above, we see that $W_{a,n}$ consists of all of the complete gradient lines in $Z_{a,n} - S_{a,n}$ that emanate from a. It is evident, then, that the gradient lines serve as a sort of exponential map from the tangent space of $W_{a,n}$ at a to $W_{a,n}$.

If m is the number given in conditions (a)–(d) from Pesin Theory given above, then $Z_{a,-m}$ contains no critical points, and so for $a \in E$,

$$W^s_{-m}(a) = W_a \cap A_{-m} = Z_{a,-m}.$$

Lemma 6.7. *If n is a return time for a, then $Z_{a,n} - S_{a,n}$ consists of $N_n(a)$ components, each of which is of the form $W_{a',n}$ for some $a' \in f_\Pi^{-n} f_\Pi^n a$. The manifolds $W_{a',n}$ are simply connected. If $n' > n$ is another return time, then $W_{a,n} \subset W_{a,n'}$, and*

$$W_{a,n'} - W_{a,n} \subset A_{n'} - A_n.$$

Proof. The first statement is a consequence from the discussion above. For the simple connectivity of $W_{a,n}$, we let σ denote a simple, closed curve in $W_{a,n}$. Moving σ along the gradient flow of G is a homotopy, which takes σ to a neighborhood of a, where it is contractible. The last statements follow because $W_{a,n}$ is the union of complete gradient lines emanating from a. □

We define $W_a = \bigcup_n W_{a,n}$, where the union is taken over any sequence of return times $n \to \infty$.

Corollary 6.8. *W_a is a simply connected Riemann surface, and the topology of W_a as a manifold coincides with the topology of W_a as a topological subspace of A. If n is a return time for a, then $W_{a,n} = W_a \cap A_n$.*

Lemma 6.9. *For any $n \geq 0$ and any $a \in E$ we have*

$$[Z_{a,n}] = \sum_{b \in Z_{a,n} \cap \Pi} [W_b \cap A_n]. \tag{6.6}$$

Proof. First suppose n is a return time for a. Then we know from Lemma 6.7 that

$$Z_{a,n} = \bigcup_{b \in Z_{a,n} \cap \Pi} W_{b,n} \cup S_{a,n}.$$

Thus (6.6) follows from the fact that $S_{a,n}$ has zero area (Hausdorff 2-dimensional measure) in $\mathbf{P}^k$. This in turn is a consequence of the fact that each incomplete gradient line is the countable union of real analytic arcs of finite length (and thus zero area).

If n is not a return time for a, then let $n' > n$ be a return time. Thus

$$Z_{a,n'} = \bigcup_{b \in Z_{a,n'} \cap \Pi} W_{b,n'} \cup S_{a,n'},$$

and we get (6.6) by intersecting this formula with $Z_{a,n}$ and using the fact that $S_{a,n'}$ has zero area. □

From (6.4), (6.6), and the definition of $N_n(a)$ we get

$$T^{k-1} \llcorner A_n = \int_E [W_a \cap A_n] \, \mu_\Pi(a). \tag{6.7}$$

If we let $n \to \infty$ in (6.7), then we have a sequence which is eventually stationary in the sense that it is constant on the open set A_j if $n \geq j$. This yields the following.

Theorem 6.10. *If E is the set defined by (6.3), then for $a \in E$, there exists a complex disk W_a in A such that*

$$T^{k-1} \llcorner A = \int_E [W_a] \, \mu_\Pi(a).$$

In particular, the disk W_a has finite area for μ_Π almost every a.

§7 External Rays, Global Model

In this Section, we do two things. First we define the set $\mathcal{E}$ of external rays and show that $\mathcal{E}$ carries a natural measure ν. Further, there is an endpoint mapping $e : \mathcal{E} \to J$ defined almost everywhere, and this mapping satisfies $e_*\nu = \mu$ (Theorem 7.3). Second, we prove Theorem 7.4, which is a more global version of Theorem 4.3: the conjugacy is given between $f| \bigcup_{a \in J_\Pi} W_a$ and the the restriction of f_h to a union of hedgehogs.

Let a be a point of E, as defined in (6.3). The set W_a, defined in §6, is a simply connected Riemann surface, and $G|_{W_a}$ is harmonic. Let G_a^* be a harmonic conjugate for $G|_{W_a}$, well-defined modulo $2\pi i$ and is unique up to an additive constant. Since $G|_{W_a}$ has a logarithmic pole at a, the function

$$\varphi_a := e^{G + iG_a^*} : W_a \to \mathbf{C}$$

is analytic on W_a, and φ_a is locally injective on W_a near a. Note that $\log |\varphi_a| = G_a$, and this condition determines φ_a uniquely up to multiplication with a constant of unit modulus. Since G_a^* is constant on the gradient lines of G, these gradient lines are taken to radial lines. By the construction of W_a, there are at most finitely many gradient lines γ with the property that $\inf_\gamma G \geq d^{-n} R_0 > 0$ for any $n \geq 0$. Thus the range $H_a := \varphi_a(W_a)$ is a hedgehog domain of the form

$$H_a = \hat{\mathbf{C}} - \left(\bar{\mathbf{D}} \cup \bigcup_{j=1}^{N} (e^{i\theta_j}, e^{r_j + i\theta_j}] \right)$$

for some $0 \leq N \leq \infty$, and $r_j > 0$ is a sequence of points with $r \to 0$. The case $N = 0$ is interpreted as $H_a = \hat{\mathbf{C}} - \bar{\mathbf{D}}$. Since W_a is invariant under the gradient flow, it follows that φ_a is injective. We let

$$\psi_a := \varphi_a^{-1} : H_a \to W_a \tag{7.1}$$

denote the inverse. Thus the gradient lines in W_a correspond to the images under ψ_a of the rays in H_a.

Let $\mathcal{E}_a$ denote the set of all gradient lines in W_a for $a \in E$, and let $\mathcal{E}$ be the union of all $\mathcal{E}_a$. For $a \in E$, the gradient lines are naturally parametrized by a choice of argument θ. The function ψ_a represents one assignment of argument. Although θ is not uniquely defined, the induced measure $\frac{d\theta}{2\pi}$ is on $\mathcal{E}_a$. Thus the measure $\nu = \mu_\Pi \otimes \frac{d\theta}{2\pi}$ is well defined on $\mathcal{E}$. Note that f maps gradient lines to gradient lines. Thus f induces a measurable mapping $\sigma : \mathcal{E} \to \mathcal{E}$.

We would like to assign an endpoint to ν almost every ray. For $\gamma \in \mathcal{E}$ and $r > 0$ we set $e_r(\gamma) = \gamma \cap \{G = r\}$. For each $a \in E$, e_r is defined for all but possibly finitely many rays lying in W_a and we will write $e_{a,r}$ for the restriction of e_r to the rays in W_a. The mapping ψ_a represents e_r in the sense that if γ_θ is the ray in W_a corresponding to argument θ, then

$$e_r(\gamma_\theta) = e_{a,r}(\gamma_\theta) = \psi_a(e^{r+i\theta}).$$

Lemma 7.1. *If $a \in E$, and if W_a has finite area as a subset of $\mathbf{P}^k$, then*

$$\lim_{r\to 0^+} e_r(\gamma_\theta) = \lim_{r\to 0^+} \psi_a(e^{r+i\theta})$$

exists for almost every θ.

Proof. We work in affine coordinates in $\mathbf{C}^k \subset \mathbf{P}^k$. Let $\tilde{a} \in \mathbf{C}^k$ denote a point with $|\tilde{a}| = 1$ such that $\pi(\tilde{a}) = a \in \Pi$. Thus we may write $\psi_a(\zeta) = \zeta^{-1}\tilde{a} + h_a(\zeta)$, where h_a is analytic on H_a. Away from the hyperplane at infinity, the Euclidean metric on $\mathbf{C}^k$ is equivalent to the Fubini-Study metric on $\mathbf{P}^k$. The condition that W_a has finite area in $\mathbf{P}^k$ is equivalent to $\int_{H_a} |\nabla h_a|^2 < \infty$. It follows that

$$\int_0^1 |\nabla h_a(re^{i\theta})|^2 r dr = \int_0^1 \left| \frac{\partial h_a(re^{i\theta})}{\partial r} \right|^2 r dr < \infty$$

for almost every θ. Thus radial limits exist for these values of θ. □

It follows that there is a measurable mapping $e : \mathcal{E} \to \partial K$ such that

$$e(\gamma) = \lim_{r\to 0^+} e_r(\gamma)$$

for ν a.e. γ. Our next step will be to show that the mapping e pushes ν forward to μ. We will write $G_r := \max(G, r)$.

Lemma 7.2. *For $a \in E$,*

$$(e_{a,r})_* d\theta = dd^c_{W_a} G_r|_{W_a}.$$

Proof. Let us first note that the statment of the Lemma is conformally invariant. Under a conformal transformation, the Green function transforms by composition; the gradient lines and level sets are preserved, and so the map $e_{a,r}$ transforms by composition. Similarly, the operator dd^c on the right hand side is invariant, so the right hand side transforms correctly. Now we transform under the map ψ_a. Since ψ_a is nonsingular at infinity, the measure $d\theta$ is preserved. The image of W_a is H_a, and G is taken to $\log|\zeta|$. The mapping $e_{a,r}$ then takes the angle θ to the point $re^{i\theta}$ and the Lemma is reduced to an elementary calculation involving $\log|z|$. □

Theorem 7.3. *The endpoint mapping e satisfies $e_*\nu = \mu$. Further, $f \circ e = e \circ \sigma$, where $\sigma : \mathcal{E} \to \mathcal{E}$ is the map induced by f.*

Proof. Integrating the formula of Lemma 7.2 with respect to μ_Π, and dividing by 2π, we obtain

$$(e_r)_*\nu = \int \frac{1}{2\pi} dd^c_{W_a} G_r|_{W_a}\, \mu_\Pi(a). \tag{7.2}$$

Let us choose n sufficiently large that $\{G \geq r\} \subset A_n$. By Theorem 6.4 we have

$$T^{k-1} \llcorner A_n = \int_E \frac{1}{N_n(a)} [Z_{a,n}]\, \mu_\Pi(a),$$

and by Lemma 2.2 with $X = \frac{1}{2\pi} dd^c G_r$, we have

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = \int_E \frac{1}{2\pi N_n(a)} dd^c G_r \wedge [Z_{a,n}]\, \mu_\Pi(a). \tag{7.3}$$

Here we have dropped $\llcorner A_n$ because the support of $dd^c G_r$ is contained in A_n. By Lemma 6.9, we have

$$[Z_{a,n}] = \sum_{b \in Z_{a,n} \cap \Pi} [W_b \cap A_n],$$

where the sum has $N_n(a)$ terms. Since G_r is continuous, the wedge product of $dd^c G_r$ with the current of integration over $Z = Z_{a,n}$ is equal to the slice measure $dd^c_Z G_r|_Z$. Since G_r is bounded, $dd^c G_r|_Z$ can put no mass on a polar set. In particular there can be no mass on the intersection of a gradient line of G with $\{G = r\}$, which is an isolated point. Thus we have

$$\begin{aligned} dd^c G_r \wedge [Z_{a,n}] &= \sum_{b \in Z_{a,n} \cap \Pi} dd^c G_r \wedge [W_b \cap A_n] \\ &= \sum_{b \in Z_{a,n} \cap \Pi} dd^c G_r \wedge [W_b]. \end{aligned}$$

Again the $\llcorner A_n$ is dropped, since $dd^c G_r$ is supported in A_n. Thus (7.3) becomes

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = \int_E \frac{1}{2\pi} dd^c G_r \wedge [W_a]\, \mu_\Pi(a).$$

By the continuity of G_r, we have $dd^c G_r \wedge [W_a] = dd^c_{W_a} G_r|_{W_a}$, so that by (7.2) we have

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = (e_r)_*\nu.$$

Finally, as we let $r \to 0$, the left hand side converges to $\mu = \frac{1}{2\pi} dd^c G \wedge T^{k-1}$, and the right hand side converges to $e_*\nu$, which shows that $e_*\nu = \mu$.

It is evident from the definition of σ that $f \circ e_{dr} = e_r \circ \sigma$. Thus $f \circ e = e \circ \sigma$. $\square$

Example. Let $f(z_1, z_2) = (z_1^2 + c_1, z_2^2 + c_2)$. Since f is a product, $J = J_{c_1} \times J_{c_2}$, where J_c is the one-dimensional Julia set of $p_c(z) = z^2 + c$. In the coordinate $\zeta = w/z$, $f_\Pi(\zeta) = \zeta^2$, so $J_\Pi = \{|\zeta| = 1\}$ and f_Π is uniformly expanding on J_Π. Using the local canonical model (Theorem 4.3) and Proposition A.2, we see that $\mathcal{E}$ is homeomorphic to $J_\Pi \times S^1 = S^1 \times S^1$. In the example at hand, the homeomorphism is given as follows. For $\zeta = e^{2\pi i\phi} \in J_\Pi$ and $e^{2\pi i\theta} \in S^1$, we associate the external ray $\gamma = \gamma(\phi, \theta) \in \mathcal{E}$ with the properties: the projection of γ to the z_1 axis is the external ray $\gamma(\theta)$ for p_{c_1}, which is asymptotic to $re^{2\pi i\theta}$ at infinity, and the projection of γ to the z_2 axis is the external ray $\gamma(\phi + \theta)$ for p_{c_2}. If the endpoint map e is defined, then we may write it as $e(\phi, \theta) = (e_{c_1}(\phi), e_{c_2}(\phi + \theta))$. Any identifications in the quotient by e (i.e. pairs (ϕ', θ'), (ϕ'', θ'') with $e(\phi', \theta') = e(\phi'', \theta'')$) are thus determined by the identifications in the quotients by e_{c_1} and e_{c_2}.

We conclude this section by proving a global version of the conjugacy given by Theorem 4.3. First note that if f_Π is uniformly expanding on J_Π, then we can assume that $E_0 = J_\Pi$ and that $W^s(J_\Pi) \cap A_0 \cap \mathcal{C}_\infty = \emptyset$, where $\mathcal{C}_\infty = \bigcup_{j \geq 0} f^{-j}\mathcal{C}$. Thus $W^s(J_\Pi) \cap \mathcal{C}_\infty$ is closed and nowhere dense in $W^s(J_\Pi)$, and $W^s(J_\Pi) - \mathcal{C}_\infty$ is a Riemann surface lamination on which the gradient flow induced by G is defined. For $x \in W^s(J_\Pi) - \mathcal{C}_\infty$, let $\gamma(x)$ be the gradient line containing x. Define a numbers $r(x) \in (0, \infty]$, $s(x) \in [0, \infty)$ for $x \in W^s(J_\Pi)$ be declaring $r(x) = s(x) = G(x)$ if $x \in \mathcal{C}_\infty$ and $r(x) = \sup\{G(y) : y \in \gamma(x)\}$, $s(x) = \inf\{G(y) : y \in \gamma(x)\}$ for $x \in W^s(J_\Pi) - \mathcal{C}_\infty$. Let $S := \{x \in W^s(J_\Pi) : r(x) < \infty\}$. Then S is the union of incomplete gradient lines.

Theorem 7.4. *If f_Π is uniformly expanding on J_Π, then the following hold:*

(1) The set S of points contained in incomplete gradient lines is closed and nowhere dense in $W^s(J_\Pi)$. Further, $W^s(J_\Pi) - S$ is a Riemann surface lamination in $A - S$. The leaves of this lamination are exactly the disks W_a and these are properly embedded in $A - S$.

(2) There is a closed and nowhere dense subset S_h of $W^s(J_\Pi, f_h)$ such that Ψ, defined in Theorem 4.3 extends to a homeomorphism

$$\Psi : W^s(J_\Pi, f_h) - S_h \to W^s(J_\Pi, f) - S,$$

conjugating f_h to f. The set S_h is a union of real rays through the origin in $\mathbf{C}^k$ and if D_a is the disk in $W^s(J_\Pi, f_h)$ passing through $a \in J_\Pi$, then Ψ maps $D_a - S_h$ biholomorphically onto W_a.

(3) We may identify the set $\mathcal{E}$ of external rays with the boundary of $W^s(J_\Pi, f_h)$, i.e. the union of the circles ∂D_a, $a \in J_\Pi$. With this identification, the maps e_r are defined by $e_r(\gamma) = \Psi(e^r \cdot \gamma)$ and $e(\gamma) = \lim_{r \to 0} \Psi(e^r\gamma)$.

(4) We have $\mu_c(A) = 0$ if and only if $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$. In this case, $S = S_h = \emptyset$ and hence there is a conjugacy $\Psi : W^s(J_\Pi, f_h) \to W^s(J_\Pi, f)$ conjugating f_h to f.

Proof. (1) We first prove that S is closed. Note that $S \cap A_0 = \emptyset$. Take $x \in W^s(J_\Pi) - S$. By definition there is a gradient line $\gamma = \gamma(x)$ containing x and ending at a point $a \in J_\Pi$. We have to show that the same is true for all points in a neighborhood of x in $W^s(J_\Pi)$. We may assume that $x \notin A_0$, because otherwise there is nothing to prove. Pick $t > 0$ and $n \geq 0$ with $s(x) < t < G(x)$ and $R_0 < d^n t < R_0 d$. Thus $f^n\gamma$ is a complete gradient line in A_{-1}. In fact, all gradient lines are complete in A_0, so there is a complete gradient line γ' in A_0 containing $f^n\gamma$. We have $\gamma \cap C_{-n} = \emptyset$. Thus there exists a branch g of f^{-n}, defined in a neighborhood of $\gamma' \cap \{G > d^n t\}$, such that $g \circ f^n = \mathrm{id}$ on $\gamma \cap \{G > t\}$. The map Ψ is defined on $W^s(J_\Pi, f_h)$ and $\gamma'' := \Psi^{-1}(\gamma')$ is a gradient line (i.e. a real line segment) in $A_{0,h}$. Let U be a simply connected neighborhood of $\gamma'' \cap \{d^n rt < G_h < \infty\}$ in $W^s(J_\Pi, f_h) \cap \{d^n t < G_h < \infty\}$ consisting of gradient lines for G_h. Then $\Psi(U)$ is a simply connected neighborhood of $\gamma' \cap \{d^n t < G < \infty\}$ consisting of gradient lines for G. We may assume that g is defined in $\Psi(U)$. Thus $g \circ \Psi(U)$ is an open set in $W^s(J_\Pi) \cap \{t < G < \infty\}$ consisting of gradient lines in $\{t < G < \infty\}$. Since $x \in g \circ \Psi(U)$ we have proved that S is closed in $W^s(J_\Pi)$.

Since S is closed, it is also nowhere dense if it contains no relative interior of $W^s(J_\Pi)$. But if S were to contain a relative interior point of $W^s(J_\Pi)$, then it contains relative interior of a global stable manifold $W^s(a)$. However, this is not possible, since $S \cap W^s(a)$ consists of at most a countable number of curves.

(2) Let D_a be the disk in $W^s(J_\Pi, f_h)$ associated with $a \in J_\Pi$. The discussion in the beginning of §7 shows that there is a uniquely defined closed subset $S_{a,h}$ of D_a such that $G_h > R_0$ on $S_{a,h}$ and such that $\Psi|_{D_a \cap \{G_h > R_0\}}$ extends to a biholomorphism of $D_a - S_{a,h}$ onto W_a. In fact, if we identify D_a with $\hat{\mathbf{C}} - \bar{\mathbf{D}}$, then $D_a - S_{a,h}$ equals the set H_a and $\Psi|_{D_a - S_{a,h}} = \psi_a$ defined in (7.1). Let $S_h = \bigcup_{a \in J_\Pi} S_{a,h}$. Clearly $S_{a,h} \cap A_{0,h} = \emptyset$. By analytic continuation in D_a we get $f \circ \Psi = \Psi \circ f_h$ on $W^s(J_\Pi, f_h) - S_h$. We show that S_h is closed in $W^s(J_\Pi, f_h)$. Take $x \in W^s(J_\Pi) - S_h$. Thus $x \in D_a - S_{a,h}$ for some $a \in J_\Pi$. Pick $0 < s < t < 1$ such that the point $sx \in D_a - S_{a,h}$ and let γ be the gradient line containing x. Choose $n \geq 0$ such that $R_0 < d^n sx < dR_0$ and let γ_1 be the gradient line in $A_{0,h}$ containing $f_h^n(x)$. Then $\gamma' = \Psi(\gamma)$ is a gradient line for G containing $\Psi(sx)$ and ending at a and $\gamma_1' = \Psi(\gamma_1)$ is a gradient line in A_0 containing $f^n(\Psi(x))$. There is a neighborhood U of $\Psi(tx)$ consisting of gradient lines γ'' for G such that $r(\gamma'') = \infty$ and $s(\gamma'') < t$. Let h be a branch of f_h^{-n} defined near γ_1 such that $g(f_h^n(x)) = x$. If U is small enough, then $g \circ \Psi^{-1}(U)$ is a neighborhood of x in $W^s(J_\Pi, f_h)$ disjoint from S_h. Thus S_h is closed in $W^s(J_\Pi, f_h)$. Since S_h intersects each disk of $W^s(J_\Pi, f_h)$ in a nowhere dense set, it is nowhere dense.

To finish the proof, we note that (3) is a consequence of (1) and (2), and (4) follows from Corollary 6.5, from Propostion 4.2, and from (2). □

Remark. Both the external rays and the global model are not "canonical" in the sense that they were not defined by dynamical behaviors. In the definitions of $\mathcal{E}$ and the hedgehog sets H_a, it is equally natural to replace the "gradient" lines by the family of curves in $W^s(a)$ which cross the level sets of $G|W^s(a)$ at a constant angle τ. The flexibility of considering values of τ different from $\frac{\pi}{2}$ has proved useful in the case $k = 1$ (see [Le]).

§8 Properties of the Support of T^{k-1}

In this Section we show that under rather general conditions the (local) stable disks given by Pesin theory actually determine the set $A \cap \mathrm{supp}(T^{k-1})$. First we show that $W^s(a)$ is dense in the support of $T^{k-1} \llcorner A$ for μ_Π-a.e. a. As in the case of polynomial automorphisms of $\mathbf{C}^2$ ([BLS, Prop 2.9]) we do this by proving a convergence result for currents. See also [FS4, Cor 5.13]. The other result of this Section is that when the critical measure μ_c vanishes on A_n, the closure of the Pesin family of disks (and thus support of $T^{k-1} \llcorner A_n$) has a uniformly laminar structure.

First we prove some convergence results for measures on J_Π. The main tool is Lemma 8.1 below, due to Fornæss and Sibony. Given $a \in J_\Pi$ and $j \geq 0$, define the measures $\nu_{a,j}$ and $\nu'_{a,j}$ by

$$\nu'_{a,j} := \frac{1}{d^{(k-1)j}} (f_\Pi^j)^* \delta_a = \frac{1}{d^{(k-1)j}} \sum_{b \in f_\Pi^{-j} a} \delta_b$$

and

$$\nu_{a,j} := \nu'_{f_\Pi^j a, j} = \frac{1}{d^{(k-1)j}} \sum_{b \in f_\Pi^{-j} f_\Pi^j a} \delta_b.$$

Lemma 8.1 ([FS3, Lemma 8.3]). *There is a constant $C > 0$ such that if ϕ is a C^2 test function on Π and $s > 0$, and*

$$E'(\phi, s, j) := \{a \in J_\Pi : |\langle \nu'_{a,j}, \phi \rangle - \langle \mu_\Pi, \phi \rangle| > s\},$$

then

$$\mu_\Pi(E'(\phi, s, j)) \leq \frac{C|\phi|_{C^2}}{d^j s}.$$

Lemma 8.2. *As $j \to \infty$, we have $\nu_{a,j} \to \mu_\Pi$ for μ_Π-a.e. a. If f_Π is expanding on J_Π, then $\nu_{a,j} \to \mu_\Pi$ for every $a \in J_\Pi$.*

Proof. Fix $\phi \in C^2$. It is sufficient to prove that $\langle \nu_{a,j}, \phi \rangle \to \langle \mu_\Pi, \phi \rangle$ for almost every a. Let

$$E_j := \{a \in \Pi : |\langle \nu_{a,j}, \phi \rangle - \langle \mu_\Pi, \phi \rangle| > d^{-j/2}\}.$$

By applying Lemma 8.1 with $s = d^{-j/2}$ and using the invariance of μ_Π we get $\mu_\Pi(E_j) \leq Cd^{-j/2}$ (the industrious reader may check that Lemma 8.3 in [FS3] remains valid if s depends on j). It follows that the set of a such that $a \notin E_j$ for sufficiently large j has full measure. Clearly $\langle \nu_{a,j}, \phi \rangle \to \langle \mu_\Pi, \phi \rangle$ for these a.

If f_Π is expanding on J_Π, then there is an $\epsilon > 0$ such that all branches of f_Π^{-j} are single-valued on balls in Π of radius ϵ centered at points in J_Π. Further, the diameters of the preimages under f^j of these balls tend to zero uniformly as $j \to \infty$. It follows that if $d(a, a') < \epsilon$, then $\langle \nu'_{a,j}, \phi \rangle - \langle \nu'_{a',j}, \phi \rangle \to 0$.

Now let $a \in J_\Pi$ be given. If j is large enough, then there is a point $b_j \notin E_j$ close to $f_\Pi^j a$. Thus

$$\langle \nu_{a,j}, \phi \rangle = \langle \nu'_{f_\Pi^j a, j}, \phi \rangle \to \langle \mu_\Pi, \phi \rangle \text{ as } j \to \infty.$$

□

Remark. The statements of Lemma 8.2 also hold with the measures $\nu_{a,j}$ replaced by $\nu'_{a,j}$.

Corollary 8.3. *$W^s(a, f_\Pi)$ is dense in J_Π for μ_Π-a.e. $a \in J_\Pi$. If f_Π is expanding on J_Π, then this holds for every $a \in J_\Pi$.*

Now we consider convergence of currents.

Proposition 8.4. *For almost every $a \in J_\Pi$ we have*

$$\liminf_{j\to\infty} \frac{1}{d^{(k-1)j}} f^{j*}[W_{f_\Pi^j a}] \geq T^{k-1} \llcorner A. \tag{8.1}$$

If f_Π is expanding on J_Π, then

$$\lim_{j\to\infty} \frac{1}{d^{(k-1)j}} f^{j*}[W_{f_\Pi^j a}] = T^{k-1} \llcorner A \tag{8.2}$$

for every $a \in J_\Pi$.

Proof. Fix $n \geq 0$. It suffices to show (8.1) and (8.2) on A_n.

Fix $\eta > 0$ and let $F = F_\eta \subset E$ be a Pesin box with $\mu_\Pi(F) \geq 1 - \eta$ satisfying (a)–(d) above. Since the Pesin disks $W^s_{-m}(b) = W_b \cap A_{-m}$ depend continuously on b on F, it follows from Lemma 8.2 that

$$\frac{1}{d^{(k-1)j}} \sum_{b \in f_\Pi^{-j} f_\Pi^j a \cap F} [W_b \cap A_{-m}] \to \int_F [W_b \cap A_{-m}]\, \mu_\Pi(b)$$

as $j \to \infty$ for almost every a. Hence, if we define

$$X_j(a) = \frac{1}{d^{(k-1)j}} f^{j*}[W_{f_\Pi^j(a)}],$$

then

$$\liminf_{j\to\infty} X_j(a) \llcorner A_{-m} \geq \int_F [W_b \cap A_{-m}]\, \mu_\Pi(b)$$

for almost every a. If we pull this back by f^{n+m}, then we get

$$\liminf_{j\to\infty} X_j(a) \llcorner A_n \geq \int h[W_b \cap A_n]\, \mu_\Pi(b), \tag{8.3}$$

where $h = h_\eta = d^{-(k-1)(m+n)} f_*^{n+m} \chi_F$. In particular we have $0 \leq h \leq 1$ and $\int h = \mu_\Pi(F)$. By letting $\eta \to 0$ we get $h_\eta \to 1$ so by dominated convergence we find that the right hand side of (8.3) converges to $\int [W_b \cap A_n]\, \mu_\Pi(b) = T^{k-1} \llcorner A_n$ for almost every a.

If f_Π is expanding on J_Π, then we may take $F = J_\Pi$ and use the second part of Lemma 8.2. Thus $X_j(a) \llcorner A_n \to T^{k-1} \llcorner A_n$ for every $a \in J_\Pi$. □

Corollary 8.5. *For almost every $a \in J_\Pi$ we have $\overline{W^s(a)} = \operatorname{supp}(T^{k-1} \llcorner A)$. If f_Π is uniformly expanding on J_Π, then this holds for every $a \in J_\Pi$.*

Proof. It is clear that $W_a \subset W^s(a)$ for all $a \in E$. We claim that $W_a \subset \operatorname{supp}(T^{k-1} \llcorner A)$. By the construction of W_a it suffices to show that $W_0^s(a)$ is contained in the support of $T^{k-1} \llcorner A$ for $a \in E_0$. But this follows from the continuity of $W_0^s(a)$ on E_0, from the fact that E_0 has no isolated points and from (6.4).

Thus $\overline{W^s(a)} \subset \operatorname{supp}(T^{k-1} \llcorner A)$. The reverse inclusion is a consequence of Proposition 8.4. □

If f_Π is uniformly expanding on J_Π, then by increasing R_0, we have $\mu_c(A_0) = 0$. We now consider the property

$$\mu_c(A_n) = [\mathcal{C}] \wedge (T^{k-1} \llcorner A_n) = 0$$

for some $n \in \mathbf{Z}$, without assuming uniform expansion on J_Π. By Corollary 6.5, this implies that $\mathcal{C} \cap Z_{a,n} = \emptyset$ for μ_Π almost every $a \in E$. Using the invariance of μ_Π and the fact that $f Z_{a,n} \subset Z_{f_\Pi a,n}$ we also get that $Z_{a,n} \cap \mathcal{C}_\infty = \emptyset$ for almost every $a \in E$, where $\mathcal{C}_\infty = \bigcup_{j \geq 0} f^{-j}\mathcal{C}$.

For such a, the construction of W_a involves the removal of no gradient lines in A_n. Thus $W_a \cap A_n = Z_{a,n}$, and $W_a \cap A_n$ is a properly embedded disk in A_n. Further, for these a, the mapping ψ_a, defined by (7.1) maps the disk $\Delta_n = \{|\zeta| > \exp(d^{-n}R_0)\}$ biholomorphically onto $W_a \cap A_n$. Let us summarize this.

Proposition 8.6.

(1) If $\mu_c(A_n) = 0$, then for almost every a, $W_{a,n} := W_a \cap A_n$ is a properly embedded disk in A_n, and the restriction of ψ_a to Δ_n is a biholomorphism onto $W_{a,n}$.

(2) If $\mu_c(A) = 0$, then for almost every a W_a is a properly embedded disk in A, and ψ_a maps $\Delta = \{|\zeta| > 1\}$ biholomorphically onto W_a. Further, $\psi_a : \Delta \to \mathbf{P}^k - J$ is proper.

Proof. Everything except the last statement follows from the discussion above. By Theorem 7.3, it follows that for μ_Π-almost every a the boundary values of the disk $\psi_a : \Delta \to W_a$ lie inside J for almost every θ. Thus $\psi_a : \Delta \to \mathbf{P}^k - J$ is proper by the theorem of Alexander (see [A] and [Ro]). □

Suppose $\mu_c(A_n) = 0$. We want to show that the family of disks $W_{a,n}$ extends to a Riemann surface lamination in A_n. Let Γ_n denote all the uniformizing mappings $\psi_a : \Delta_n \to W_{a,n}$.

Lemma 8.7. *Either Γ_n is a normal family, or there is a nonconstant holomorphic mapping $h : \mathbf{C} \to \Pi$ such that $h(\mathbf{C}) \subset \Pi - E$.*

Proof. If Γ_n is not a normal family, there is a sequence $\{\psi_j\} \subset \Gamma_n$ without a convergent subsequence. By the renormalization technique of Brody [La, pp. 68–71] there is a sequence $r_j \to \infty$ and a sequence of Möbius transformations $\rho_j : \{|\zeta| < r_j\} \to \Delta_n$ such that $\psi_j \circ \rho_j$ converges to a nonconstant mapping $h : \mathbf{C} \to \mathbf{P}^k$. Further, $G \circ \psi_j \circ \rho_j$ is a sequence of positive, superharmonic functions which converge normally on $\mathbf{C}$ to $G \circ h$. Since every positive, superharmonic function on $\mathbf{C}$ is constant (or $\equiv +\infty$), it follows that either $h(\mathbf{C}) \subset \{G = c\}$ or $h(\mathbf{C}) \subset \Pi$. The set $\{G = c\}$ is a compact subset of $\mathbf{C}^k$, so we cannot have $h(\mathbf{C}) \subset \{G = c\}$. Thus we have $h(\mathbf{C}) \subset \Pi$. Finally, the disks $\{W_a\}$ are pairwise disjoint, so an application of the Hurwitz Theorem shows that either $h(\mathbf{C}) \subset W_a$ or $h(\mathbf{C}) \cap W_a = \emptyset$. The first case is not possible since h is nonconstant, so we have $h(\mathbf{C}) \subset \Pi - \{a\}$. Thus $h(\mathbf{C}) \subset \Pi - E$. □

Remark. If $k = 2$, there can be no nonconstant mapping of $\mathbf{C}$ into $\Pi - E$, so Γ_n is necessarily a normal family. For $k > 2$ we let $X_1, X_2, \ldots$ denote the distinct irreducible components in $\bigcup_{j \in \mathbf{Z}} f_\Pi^j \mathcal{C}_\Pi$. Since $\mu_c(A_n) = 0$ we conclude, using the Hurwitz Theorem, that for any X_j we have either $h(\mathbf{C}) \subset X_j$ or $h(\mathbf{C}) \subset \Pi - X_j$. Thus Γ_n is a normal family if for any finite family $X_{j_1}, \ldots, X_{j_p}$, there is no nonconstant mapping of $\mathbf{C}$ into $X_{j_1} \cup \ldots \cup X_{j_p} - \bigcup' X_i$, where $\bigcup'$ denotes the union of the remaining components. Since normality of Γ_n is the essential ingredient of the following theorem, it would appear to apply in many cases with $k > 2$.

Theorem 8.8. *If $\mu_c(A_n) = 0$ and $k = 2$, then for each $a \in J_\Pi$ there is a complex disk $W_{a,n}$ which is properly embedded in A_n, such that $a \mapsto W_{a,n}$ is continuous, and such that $W_{a,n} \cap W_{b,n} = \emptyset$ if $a \neq b$. Thus the family $\{W_{a,n} : a \in J_\Pi\}$ is a Riemann surface lamination in A_n. Further, we have*

$$T \llcorner A_n = \int [W_{a,n}]\, \mu_\Pi(a) \tag{8.4}$$

and for each a we have $W_{a,n} \cap \mathcal{C} = \emptyset$ or $W_{a,n} \subset \mathcal{C} \cup \{a\}$. If $\mu_c(A) = 0$, then all of the above conclusions hold on A.

Proof. By Lemma 8.7 and the remark above, Γ_n is a normal family. Let $\tilde\Gamma_n$ denote the mappings $\psi : \Delta_n \to A_n$ which are normal limits of Γ_n. Note that $G \circ \psi(\zeta) = \log|\zeta|$ for any $\psi \in \tilde\Gamma_n$. Since the disks W_a are pairwise disjoint, it follows from the Hurwitz Theorem that if $\psi', \psi'' \in \tilde\Gamma_n$, then either $\psi'(\Delta_n) = \psi''(\Delta_n)$ or $\psi'(\Delta_n) \cap \psi''(\Delta_n) = \emptyset$. Thus for each $a \in J_\Pi$ there is a unique image $W_{a,n} := \psi(\Delta_n)$, $\psi \in \tilde\Gamma_n$, which contains a. The continuity of $a \mapsto W_{a,n}$ follows from the normality of Γ_n. We have $Z_{a,n} = W_{a,n}$ and $N_n(a) = 1$ for almost every a and hence (8.4) follows from (6.4). Applying Lemma 2.2 to (8.4) we obtain

$$\mu_c \llcorner A_n = \int [W_{a,n} \cap \mathcal{C}]\, \mu_\Pi(a) = 0. \tag{8.5}$$

By continuity of $a \to W_{a,n}$, the property that $W_{a,n} \cap \mathcal{C} \neq \emptyset$ but $W_{a,n} - (\mathcal{C} \cup \{a\}) \neq \emptyset$ is open in J_Π. Thus this property never holds by (8.5). □

Example. Let $f(z,w) = (z^2, \frac{1}{4}z^2 + w^2)$. In the coordinate $\zeta = w/z$ on Π, we have $f_\Pi(\zeta) = \zeta^2 + \frac{1}{4}$. Let $K_\Pi \subset \Pi$ denote the filled Julia set for f_Π. The point $\zeta = \frac{1}{2}$ is a parabolic fixed point, and all points of $\{\frac{1}{2}\} \cup \mathrm{int}(K_\Pi) \subset \Pi$ approach $\{\zeta = \frac{1}{2}\}$ in forward time. Thus the stable set $W^s(\frac{1}{2})$ for f contains a neighborhood of $\frac{1}{2}$ inside the cone of complex lines $C(\{\frac{1}{2}\} \cup \mathrm{int}(K_\Pi))$, which contains an open set in $\mathbf{P}^k$.

Since $f = f_h$ is homogeneous, each (local) Pesin disk $W^s_{-m}(a)$ is the complement of a closed disk (centered at the origin) inside the complex line L_a. Thus the family of Pesin disks has an extension to the lamination inside the complex cone of lines $C(J_\Pi)$. In the example at hand, the critical locus is $\mathcal{C} = \{z = 0\} \cup \{w = 0\}$, and T_h is supported on $C(J_\Pi)$. Since $\{\zeta = 0, \infty\}$ is disjoint from J_Π, and A is a neighborhood of Π disjoint from $0 \in \mathbf{C}^2$, it follows that $\mu_c \llcorner A = [\mathcal{C}] \wedge T_h \llcorner A = 0$. Thus this example also satisfies the hypotheses of Theorem 8.8.

§9 Axiom A in $\mathbf{C}^2$.

In the next section we will impose certain hyperbolicity assumptions (see Definition 10.1) on the dynamics on f in order to prove that all of the external rays land (and land continuously) on J. Most of these assumptions are related to Axiom A, which was introduced by Smale as a property of a smooth dynamical system which enables the understanding of its global dynamics. In this Section, we discuss Axiom A in the setting of polynomial endomorphisms of $\mathbf{C}^2$, chiefly to clarify our assumptions in Definition 10.1.

The literature on hyperbolic dynamics is vast, but most expositions consider only diffeomorphisms. A regular polynomial endomorphism of $\mathbf{C}^2$ of degree $d \geq 2$ is not invertible, and the hyperbolic theory is slightly different. There seems to be no general, detailed treatment of exactly the results we need, so we will give further definitions and results in Appendix B. More details can be found in [J2]. We also refer to [FS4], where the authors study hyperbolic endomorphisms of $\mathbf{P}^2$.

Suppose that f is a regular polynomial endomorphism of $\mathbf{C}^2$; as usual we regard f as a holomorphic map of $\mathbf{P}^2$. Since f is not injective, we will often have to work with histories of points instead of the points themselves. Precisely, a *history* of a point $x \in \mathbf{P}^2$ is a sequence $(x_i)_{i\leq 0}$ of points in $\mathbf{P}^2$ such that $x_0 = x$ and $fx_i = x_{i+1}$ for all $i < 0$. We will use the notation $\hat{x}$ for a history (x_i).

Let L be a compact subset of $\mathbf{P}^2$ with $fL = L$. We refer to Appendix B for a definition of what it means for f to be (uniformly) hyperbolic on L. Let us only recall that the definition involves the compact set $\hat{L}$ of histories in L. The pair $(\hat{L}, \hat{f})$, where $\hat{f}$ is the left shift on $\hat{L}$, is often called the natural extension of $f|_L$. There is a natural projection $\pi : \hat{L} \to L$ such that $\pi(\hat{x}) = x_0$. We say that L has unstable index i if the stable bundle E^s has constant dimension $2 - i$ on L. If L has unstable index 2, then f is said to be (uniformly) expanding on L (see Appendix B for an alternative definition). If f is hyperbolic on L, then to every point in $x \in L$ and every history $\hat{x} \in \hat{L}$ there is an associated local stable and unstable manifold respectively, defined by

$$W^s_{\mathrm{loc}}(x) = \{y \in \mathbf{P}^2 : d(f^i y, f^i x) < \delta \ \forall i \geq 0\}$$
$$W^u_{\mathrm{loc}}(\hat{x}) = \{y \in \mathbf{P}^2 : \exists \hat{y} \in \widehat{\mathbf{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) < \delta \ \forall i \leq 0\},$$

for small $\delta > 0$. Then $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(\hat{x})$ are complex disks of $\mathbf{P}^2$. If f is uniformly expanding on L, then the local stable manifolds are empty and the local unstable manifold at $\hat{x}$ is a neighborhood of x_0 in $\mathbf{P}^2$.

We also define global stable and unstable manifolds by declaring

$$W^s(x) = \{y \in \mathbf{P}^2 : d(f^i y, f^i x) \to 0 \text{ as } i \to \infty\}$$
$$W^u(\hat{x}) = \{y \in \mathbf{P}^2 : \exists \hat{y} \in \widehat{\mathbf{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) \to 0 \text{ as } i \to -\infty\}.$$

Note that if $n \geq 0$, $y \in L$ and $f^n y = f^n x$, then $W^s(x)$ contains $W^s_{\mathrm{loc}}(y)$. Hence the global stable manifolds are in general large and quite complicated objects (compare with Corollary 8.5). Both the stable and unstable manifolds may have singularities; this is in contrast to the case of polynomial automorphisms of $\mathbf{C}^2$, where they are immersed copies of $\mathbf{C}$ [BS1],

We now turn to Axiom A regular polynomial endomorphisms of $\mathbf{C}^2$. A point $x \in \mathbf{P}^2$ is wandering if for every neighborhood V of x there exists an $n \geq 1$ such that $f^n(V) \cap V \neq \emptyset$. The *non-wandering set* Ω of f is the set of all non-wandering points; it is a compact set. A regular polynomial endomorphism f of $\mathbf{C}^2$ is *Axiom A* if the periodic points of f are dense in Ω and f is hyperbolic on Ω. If f is Axiom A, then Smale's spectral decomposition theorem (Theorem B.9) asserts that Ω can be written in a unique way as a finite union of disjoint compact invariant sets Ω_j, called basic sets, such that $f|_{\Omega_j}$ is transitive, i.e. has a dense orbit. Thus each basic set has a well-defined unstable index.

Let us investigate what the possible basic sets are for a Axiom A regular polynomial endomorphism f of $\mathbf{C}^2$. To do this, we first observe that the four sets Π, $\mathbf{C}^2 - K$, $\mathrm{int}(K)$ and ∂K are all completely invariant and see what basic sets each one of them may contain.

To begin with, it is clear that $\Omega(f) \cap \Pi = \Omega(f_\Pi)$. Now f_Π is a rational map and from one-dimensional dynamics we know that f_Π is Axiom A if and only if f_Π is uniformly expanding on J_Π (see [M]). Hence, if f is Axiom A, then the basic sets in Π are J_Π, which is of unstable index 1, and a finite union of attracting periodic points, all of whose unstable index is zero.

All the points in the open set $\mathbf{C}^2 - K$ are attracted to Π so $(\mathbf{C}^2 - K) \cap \Omega$ is empty. It is clear that $\{f^n\}$ is normal on the interior of K, so if f is Axiom A, then the only basic sets in $\mathrm{int}(K)$ are attracting periodic points, all of whose unstable index are zero.

The boundary of K contains the most complicated dynamics. Clearly, no basic sets in ∂K can have unstable index 0. Let S_2 and S_1 be the union of the basic sets in ∂K of index 2 and 1, respectively. We note that S_1 can be empty, as in the example $f(z,w) = (z^2 + c, w^2 + c)$, with c outside the Mandelbrot set. On the other hand, J is a basic set of unstable index 2 (see [FS2, Theorem 7.4]), so $J \subset S_2$. The question arises whether this inclusion is ever strict or, equivalently, whether f can have repelling periodic points outside J. Hubbard and Papadopol [HP] have in fact given an example of a regular polynomial endomorphism of $\mathbf{C}^2$ with a repelling periodic point outside J but is seems difficult to check whether their map can be made Axiom A. In any case we have the following.

Lemma 9.1. *Let f be an Axiom A regular polynomial endomorphism of $\mathbf{C}^2$. Then $f^{-1}S_2 = S_2$ if and only if $S_2 = J$, i.e. if all repelling periodic points are contained in J.*

Remark. A proof is given in [FS4]. We give it here for the convenience of the reader.

Proof. The "only if" part is trivial since $f^{-1}(J) = J$, so suppose that f is Axiom A and $f^{-1}(S_2) = S_2$ but $S_2 \neq J$. Let N be an open neighborhood of J such that $f^{-1}(N) \subset N$ and $\cap_{n\geq 0} f^{-n}(N) = J$. Then $N - J$ contains only wandering points, so $S_2 - J$ is at a positive distance from J and is therefore a completely invariant compact set. Let N' be an open neighborhood of $S_2 - J$ disjoint from J with $f^{-1}(N') \subset N'$. Then N' has positive capacity and if $x \in N'$ then $(f^n)^*\delta_x/d^{2n}$ cannot converge to μ as $n \to \infty$. This contradicts Lemma 8.3 in [FS3]. □

Let f be an Axiom A regular polynomial endomorphism of $\mathbf{C}^2$ with $f^{-1}S_2 = S_2$. It follows from Corollary B.10 and the above discussion that any history of a point $\mathbf{C}^2$ which is not an attracting periodic point must converge to either J or S_1. We define the unstable set of J to be the set of points in $\mathbf{C}^2$ all of whose histories converge to J, i.e.

$$W^u(J) = \{x \in \mathbf{C}^2 : (\hat{x} \in \widehat{\mathbf{C}^2}, \pi(\hat{x}) = x) \Rightarrow x_i \to J \text{ as } i \to -\infty\}.$$

We note that this definition differs from the one in [FS4], where $W^u(J)$ is defined as the set of points having *at least* one history converging to J. On the other hand we define the unstable set of S_1 as

$$W^u(S_1) = \{x \in \mathbf{C}^2 : \exists \hat{x} \in \widehat{\mathbf{C}^2}, \pi(\hat{x}) = x, x_i \to S_1 \text{ as } i \to -\infty\}.$$

Let N be a neighborhood of J in $\mathbf{C}^2$ as in the proof of Lemma 9.1. Clearly $N \subset W^u(J)$ and every point in $\mathbf{C}^2$ which is not an attracting periodic point is contained in precisely one of the sets $W^u(J)$ and $W^u(S_1)$.

Lemma 9.2. *If $x \in W^u(J)$, then there exists an $n \geq 0$ such that $f^{-n}(x) \subset N$. In particular, $W^u(J)$ is open in $\mathbf{C}^2$ and $W^u(S_1)$ is closed in $\mathbf{C}^2$ except possibly at some of the attracting periodic points.*

Proof. Let Z be the set of points y in $\mathbf{C}^2$ such that for all $n \geq 0$, there is a point in $f^{-n}(y)$ outside N. It is clear that if $y \in Z$, then y has at least one preimage in Z, so every point $y \in Z$ has a whole history inside Z. Such a history cannot converge to J so it follows that $Z \cap W^u(J) = \emptyset$, which completes the proof. □

For the proof of the main result in §10 (Theorem 10.2), we will work with slightly weaker hyperbolicity hypotheses.

Definition 9.3. *A regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (†) if the following four properties hold:*
(†1) f_Π is uniformly expanding on J_Π.
(†2) f is uniformly expanding on J.
(†3) The nonwandering set of f in ∂K consists of J and a hyperbolic set S_1 of unstable index 1.
(†4) $W^u(S_1) = \bigcup_{\hat{x} \in \hat{S}_1} W^u(\hat{x})$.

Proposition 9.4. *Let f be an Axiom A regular polynomial endomorphism of $\mathbf{C}^2$ with $f^{-1}S_2 = S_2$. Then f satisfies condition (†).*

Proof. From the above discussion we know that f satisfies conditions (†1), (†2) and (†3), and (†4) follows from Corollary B.10. □

§10 Continuous landing of rays in $\mathbf{C}^2$.

So far we have been able to understand the dynamics in the set $W^s(J_\Pi)$, or at least on the support of $T^{k-1} \llcorner A$, for rather general f. In this Section, we approach the dynamics of f on J by proving Theorem 10.2, which shows that $e : \mathcal{E} \to J$ is a continuous surjection (under suitable assumptions). Our approach is restricted to the case $k = 2$ because we work with unstable manifolds $W^u(\hat{q})$ as Riemann surfaces; if $k > 2$, the unstable manifolds can have dimension > 1. If $k = 2$, then Π is the Riemann sphere, and f_Π is a rational mapping. Up to §8, the only hyperbolicity assumption that we have been concerned with has been uniform expansion on J_Π, which for $k = 2$ means that f_Π is a hyperbolic rational mapping. The reason for this is that we have dealt with the dynamics on A and not directly on K. For the continuity of $e : \mathcal{E} \to J$ we need to consider the dynamics on K (or, rather, ∂K). Notice that hyperbolicity of f_Π does not exclude complicated dynamics on K.

To motivate (‡5) in Definition 10.1 below, let us revisit the example presented after Theorem 7.3, namely $f(z_1, z_2) = (z_1^2 + c_1, z_2^2 + c_2)$. We have $J = J_{c_1} \times J_{c_2}$, where J_c is the one-dimensional Julia set of $p_c(z) = z^2 + c$. Further, $\mathcal{E} \simeq S^1 \times S^1$, so $\mathcal{E}$ is connected and locally connected. Thus, if e maps $\mathcal{E}$ continuously onto J, then J also is connected and locally connected. This, in turn, is equivalent to J_{c_j} being connected and locally connected for $j = 1, 2$. Now J_c is connected if and only if the critical point 0 of p_c is not in the basin of attraction of infinity, i.e. the parameter value c is in the Mandelbrot set. Using this one can see that J is connected if and only if $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$. The question of whether J_c is locally connected is more delicate, but a sufficient condition is that J_c is connected and p_c is uniformly expanding on J_c. Thus J is locally connected if J is connected and f is uniformly expanding on J.

Definition 10.1. *We say that a regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (‡) if f satisfies condition (†) in Definition 9.3, and $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$, i.e. if the following five properties hold:*
(‡1) f_Π is uniformly expanding on J_Π.
(‡2) f is uniformly expanding on J.
(‡3) The nonwandering set of f in ∂K consists of J and a (possibly empty) hyperbolic set S_1 of unstable index 1.
(‡4) $W^u(S_1) = \bigcup_{\hat{x} \in \hat{S}_1} W^u(\hat{x})$.
(‡5) $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$ (or, equivalently, $\mu_c(A) = 0$).

Let us comment on these conditions. It follows from Proposition 9.4 that if f is Axiom A, $f^{-1}(S_2) = S_2$, and satisfies (‡5), then f satisfies (‡). Using this, one can show that perturbations of the map $f(z, w) = (z^d, w^d)$ satisfy (‡).

Conditions (‡1) and (‡5) guarantee that $e_r : \mathcal{E} \to \{G = r\}$ is well defined and continuous for $r > 0$ (in general it is defined almost everywhere on $\mathcal{E}$). We have $e = \lim e_r$ as $r \to 0$, and $e_r(\mathcal{E}) = f^{-1} e_{dr}(\mathcal{E})$. Thus, once we know that $e_r(\mathcal{E})$ is in a small neighborhood of J for all sufficiently small r, then condition (‡2) helps us to show that e_r converges uniformly as $r \to 0$. However, we only know that e_r accumulates on ∂K, which in general is a larger set than J. In fact, the main difficult in proving continuity for e is to show that $e_r(\mathcal{E})$ accumulates only at J. In order to do this, we use properties (‡3) and (‡4).

Let us make some remarks about the connection between the endpoint map e and the conjugacy $\Psi : W^s(J_\Pi, f_h) \to W^s(J_\Pi, f)$ between f_h to f given in Theorem 7.4. Let $\Delta_a := L_a \cap A_h$ be the disk in $W^s(J_\Pi, f_h)$ corresponding to a. We may identify the set $\mathcal{E}_a$ of external rays in W_a with $\partial\Delta_a$ and $\mathcal{E}$ with the boundary of $W^s(J_\Pi, f_h)$, i.e. the union of $\mathcal{E}_a$ over $a \in J_\Pi$. This defines the topology on $\mathcal{E}$. With these identifications we have $e_r(\gamma) = \Psi(e^r\gamma)$ for $r > 0$. It follows that e_r is well defined and continuous for all $r > 0$. Thus Ψ extends continuously to $\mathcal{E}$ if and only if e is continuous, and in this case e coincides with the restriction of Ψ to $\mathcal{E}$. The selfmap f_h on $W^s(J_\Pi, f_h)$ induces the selfmap σ of $\mathcal{E}$.

Our main goal is to prove the following result.

Theorem 10.2. *If the regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (‡), then e maps $\mathcal{E}$ Hölder continuously onto J and $f \circ e = e \circ \sigma$. In particular, the conclusions hold if f is Axiom A, if $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$, and if the expanding part S_2 of the nonwandering set of f satisfies $f^{-1}(S_2) = S_2$.*

As mentioned above, the main difficulty in proving Theorem 10.2 is to show that the external rays accumulate only at J. In particular, there must be no heteroclinic intersection between S_1 and J_Π, i.e. no complete orbit $(x_i)_{i \in \mathbf{Z}}$ such that $x_i \to S_1$ as $i \to -\infty$ and $x_i \to J_\Pi$ as $i \to \infty$.

Lemma 10.3. *If f satisfies condition (‡), then $W^s(J_\Pi) \cap W^u(S_1) = \emptyset$.*

We postpone the proof of Lemma 10.3 for the moment and head towards the proof of Theorem 10.2.

We may identify the disk Δ_a in $W^s(J_\Pi, f_h)$ with $\Delta = \{|\zeta| > 1\}$ in such a way that $G_h(\zeta) = \log|\zeta|$. The restriction of Ψ to Δ_a then induces a conformal equivalence $\psi_a : \Delta \to W_a$ such that $G \circ \psi_a(\zeta) = \log|\zeta|$. This last condition defines ψ_a uniquely up to rotation; the precise choice of rotation will not be important in what follows. Given a choice of ψ_a, the conjugacy between f_h and f translates into

$$f \circ \psi_a(\zeta) = \psi_{f_\Pi a}(\nu_a \zeta^d), \tag{10.1}$$

where $|\nu_a| = 1$.

We first show that the maps ψ_a are uniformly Hölder continuous.

Lemma 10.4. *There exist constants $\alpha > 0$ and $C > 0$ such that*

$$d(\psi_a(\zeta), \psi_a(\zeta')) \le C d(\zeta, \zeta')^\alpha, \tag{10.2}$$

for all $a \in J_\Pi$ and $\zeta, \zeta' \in \Delta$.

Proof. The expansion of f on J implies that there exists a neighborhood N of J with $f^{-1}(N) \subset N$, $\lambda > 1$ and a metric equivalent to the Euclidean metric such that $|Df(x)v| \ge \lambda|v|$ for all $x \in N$ and all $v \in T_x\mathbf{C}^2$ with respect to this metric. By Lemma 9.2 and Lemma 10.3 we know that the set $W^s(J_\Pi) \cap \{1 \le G \le d\}$ is a compact subset of the open set $W^u(J)$ so by pulling back by f we see that there exists an $R > 1$ such that $\psi_a\{1 < |\zeta| \le R\}) \subset N$ for all a. Let $\alpha > 0$ be so small that $d^\alpha < \lambda$ and assume that R is so small that $R^{d-1} d^\alpha < \lambda$. It is sufficient to prove (10.2) for $1 < |\zeta|, |\zeta'| \le R$.

By differentiating (10.1) and using the estimates above we get that, for $1 < |\zeta| < R^{d^{-1}}$,

$$|D\psi_a(\zeta)| \le \lambda^{-1} \left| D\psi_{f_\Pi a}\left(\nu_a \zeta^d\right)\right| d|\zeta|^{d-1}. \tag{10.3}$$

Define

$$m(r) = \sup_{a \in J_\Pi} \sup_{|\zeta| = r} |D\psi_a(\zeta)|,$$

for $1 < r \le R$. Then there exists a constant $C' < \infty$ such that

$$m(r) \le C'(r-1)^{\alpha - 1}, \tag{10.4}$$

for $R^{d^{-1}} \le r \le R$. Using the estimate (10.3) we prove inductively that (10.4) holds for $1 < r \le R$. Thus (10.2) follows by integrating (10.4). □

Proof of Theorem 10.2. We know that e_r is continuous for each $r > 0$. Lemma 10.4 shows that e_r converges uniformly on $\mathcal{E}_a$ for each a. Thus e_r converges uniformly to e, so e is continuous and it follows from Theorem 7.3 that $e(\mathcal{E}) = J$. We have $f \circ e_r = e_{dr} \circ \sigma$, so be letting $r \to 0$ we get $f \circ e = e \circ \sigma$. It remains to be seen that e is Hölder continuous.

Let N and λ be as in the proof of Lemma 10.4 and let $\epsilon > 0$ be small. We may assume that $d(fx, fy) \geq \lambda d(x, y)$ for $x, y \in N$, $d(x, y) \leq \epsilon$, and that $d(\sigma\gamma, \sigma\gamma') \geq \lambda d(\gamma, \gamma')$ for $\gamma, \gamma' \in \mathcal{E}$, $d(\gamma, \gamma') \leq \epsilon$. There is a number M so that $d(\sigma\gamma, \sigma\gamma') \leq M d(\gamma, \gamma')$ for all $\gamma, \gamma' \in \mathcal{E}$. Pick $\alpha > 0$ such that $M^\alpha < \lambda$. Let $C > 0$ be so large that $d(e_r(\gamma), e_r(\gamma)) \leq C d(\gamma, \gamma')^\alpha$ if $d(\gamma, \gamma') \geq \epsilon$ and $r > 0$.

Since $e(\mathcal{E}) = J$, there exists an $R > 0$ such that $e_r(\mathcal{E}) \subset N$ for $0 < r \leq R$. Now suppose $r \leq R d^{-j}$ for some $j \geq 0$ and that $d(\gamma, \gamma') \geq \epsilon/\lambda^j$. Then

$$\begin{aligned} d(e_r(\gamma), e_r(\gamma')) &\leq \lambda^{-j} d(e_{d^j r}\sigma^j\gamma, e_{d^j r}\sigma^j\gamma') \\ &\leq \lambda^{-j} C d(\sigma^j\gamma, \sigma^j\gamma')^\alpha \\ &\leq \lambda^{-j} M^{\alpha j} C d(\gamma, \gamma')^\alpha \\ &\leq C d(\gamma, \gamma')^\alpha \end{aligned}$$

The theorem thus follows by letting $r \to 0$. □

We now turn to the proof of Lemma 10.3 and proceed in a number of steps. An intersection between W_a and $W^s(J_\Pi)$ is a *heteroclinic connection*. There are two types, as shown in Figure 5. That is, the intersection between W_a and $W^u(S_1)$ can be either a discrete set or a relatively open set. The next Lemma says that the discrete intersection does not occur.

Two Types of Heteroclinic Connections

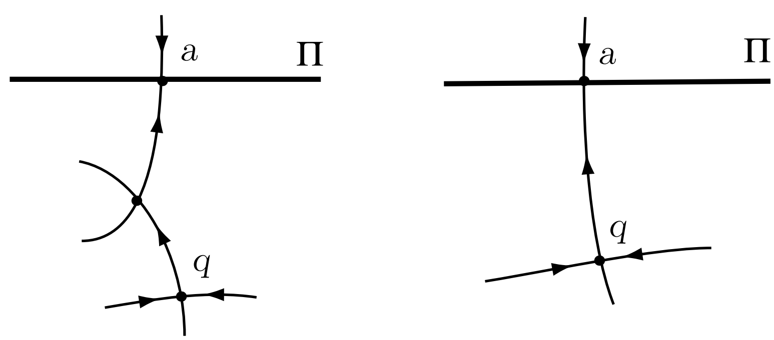

Figure 5.

Lemma 10.5. *Let W_a be the stable disk of a point $a \in J_\Pi$. Then either $W_a \cap W^u(S_1) = \emptyset$ or there exists a point $\hat{x} \in \hat{S}_1$ such that $W_a^* \subset W^u(\hat{x})$, where $W_a^* = W_a - \{a\}$.*

The key observation in proving the dichotomy is the following.

Lemma 10.6. *If U is a simply connected open subset of a punctured stable disk W_a^*, then all branches of $f^{-i}|_U$ for all $i > 0$ are well-defined and holomorphic on U and they form a normal family there.*

Proof. That the branches are well-defined follows from condition (‡5). If V is relatively compact in U then all branches of f^{-i} on V map V into a fixed compact subset of $\mathbf{C}^2$. Thus they form a normal family on U. □

Proof of Lemma 10.5. Suppose that $y \in W_a \cap W^u(S_1)$. Then by condition (‡4) there exists a point $\hat{x} \in \hat{S}_1$ such that $y \in W^u(\hat{x})$, i.e. y has a history $\hat{y}$ such that $d(y_i, x_i) \to 0$ as $i \to -\infty$. Let U be any simply connected open subset of W_a^* containing y and let g_i be the unique sequence of branches of $f^{-i}|_U$ such that $g_i(y) = y_i$. Then $\{g_i\}$ is equicontinuous by Lemma 10.6, so there is a small neighborhood V of y in U such that the maximal distance from $g_i(V)$ to x_i is uniformly small as $i \to \infty$. Hence $V \subset W^u(\hat{x})$ and, by normality of $\{g_i\}$, $U \subset W^u(\hat{x})$. Since U was arbitrary it follows that $W_a^* \subset W^u(\hat{x})$. □

Corollary 10.7. *Let J'_Π be the set $a \in J_\Pi$ such that $W_a^* \subset W^u(S_1)$. Then J'_Π is closed, $f_\Pi(J'_\Pi) = J'_\Pi$ and $J'_\Pi \neq J_\Pi$.*

Proof. If $a \notin J'_\Pi$, then $W_a^* \cap W^u(S_1) = \emptyset$ by Lemma 10.5. Hence $W_a \cap \{G = 1\}$ is a compact subset of the open set $W^u(J)$ so by continuity there is an open neighborhood X of a in J_Π such that $W_b \cap \{G = 1\} \subset W^u(J)$ for all $b \in X$. By Lemma 10.5 it follows that $X \cap J'_\Pi = \emptyset$ and we conclude that $J_\Pi - J'_\Pi$ is open. That $f_\Pi(J'_\Pi) = J'_\Pi$ follows from the fact that $f(W^u(S_1)) = W^u(S_1)$.

Finally suppose $J'_\Pi = J_\Pi$. Then $W^s(J_\Pi) \subset J_\Pi \cup W^u(S_1)$, so $W^s(J_\Pi)$ does not intersect $W^u(J)$. This contradicts Theorem 7.3, because $W^u(J)$ contains a neighborhood of $J = \operatorname{supp}(\mu)$. □

We say that a stable disk W_b *lands* on J if ψ_b extends continuously to S^1 and $\psi_b(S^1) \subset J$. This does not depend on the specific choice of parametrization ψ_b.

Lemma 10.8. *There exists a dense set of $b \in J_\Pi$ such that W_b lands on J.*

Proof. Since periodic points are dense in J_Π and $J_\Pi - J'_\Pi$ is open and nonempty, we can find a periodic point $b' \in J_\Pi - J'_\Pi$, say of period n. Furthermore, f is expanding on J, so there exists a neighborhood N of J and $\lambda > 1$ with $f^{-1}(N) \subset N$ and

$$|Df^n(y)v| \geq \lambda^n |v|, \tag{10.5}$$

for all $y \in N$ and all tangent vectors v (we may have to increase n). Now the annulus $\psi_{b'}\{2^{1/d^n} \leq |\zeta| \leq 2\}$ in $W_{b'}$ is a compact subset of $W^u(J)$, so the inverse images under sufficiently high iterates of f of points in this annulus will be in N. In particular, since b' is periodic, it follows that there exists an $R > 1$ such that $\psi_{b'}\{1 < |\zeta| \leq R\} \subset N$. Then, using the estimate (10.5) above, we may prove that $\psi_{b'}$ extends to a Hölder continuous map of $\bar{\Delta}$, mapping S^1 into J. The proof is very similar to the proof of Lemma 10.4 and is therefore omitted.

We conclude that $W_{b'}$ lands on J and so does W_b for all preimages b of b' under iterates of f. Such preimages are dense in J_Π. □

Figure 6 illustrates the effect of a heteroclinic connection. Here W_a^* is in the unstable set of S_1 whereas W_b lands on J. The stable disks in the middle are of the form W_{b_n}, where b_n are preimages of b converging to a. Note that the disks W_{b_n} are very "bent" for large n. If the W_{b_n}'s were graphs over W_a, then this would contradict the maximum principle. We cannot show directly that W_{b_n} are graphs over W_a, but we will nevertheless prove, using the maximum principle, that the picture is impossible.

Heteroclinic Connection between J_Π and S_1

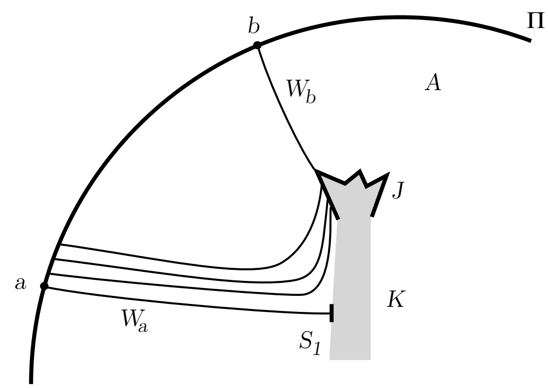

Figure 6.

It follows from Lemma 10.5 that for each $a \in J'_\Pi$ there exists a (not necessarily unique) history $\widehat{p_a}$ in S_1 such that $W^*_a \subset W^u(\widehat{p_a})$. In general, an unstable manifold $W^u(\hat{q})$ of a history $\hat{q}$ in S_1 is a complicated object, but, as we will see, the information that $W^*_a \subset W^u(\widehat{p_a})$ implies that $W^u(\widehat{p_a})$ is in fact an algebraic subvariety of $\mathbf{C}^2$. Recall that the image of a holomorphic map of a compact Riemann surface into $\mathbf{P}^2$ is an algebraic variety. The authors thank Jeffrey Diller for useful conversations on the proof of the following result.

Lemma 10.9. *If $J'_\Pi \neq \emptyset$, then there exists an $a \in J'_\Pi$ such that $W^u(\widehat{p_a})$ is an algebraic subvariety of $\mathbf{C}^2$.*

Proof. Take any point $a \in J'_\Pi$, a complete orbit $(a_i)_{i\in\mathbf{Z}}$ with $a_0 = a$ and a complete orbit $(p_i)_{i\in\mathbf{Z}}$ be a complete orbit in S_1 such that $W^*_{a_i} \subset W^u((p_{i+j})_{j\leq 0})$ for all i. We write $\widehat{p_i}$ for the history $(p_{i+j})_{j\leq 0}$. If $\delta > 0$ is small enough, then the local unstable manifolds $W^u_{\mathrm{loc}}(\widehat{p_i})$ are complex disks for all i and there exist biholomorphisms $\eta_i : \mathbf{D}_{\delta_i} \to W^u_{\mathrm{loc}}(\widehat{p_i})$ with $|D\eta_i(0)| = 1$ and complex numbers $\lambda_i \neq 0$ such that

$$\eta_i(\lambda_{i-1}\zeta) = f(\eta_{i-1}(\zeta)), \tag{10.6}$$

for all i and all $|\zeta| < \delta_{i-1}$. Since f is hyperbolic on S_1, the numbers δ_i are uniformly bounded from below and $\lambda_{i-n}\cdots\lambda_{i-1} \to \infty$ as $n \to \infty$ for all i, so (10.6) allows us to extend η_i to maps of $\mathbf{C}$ into $W^u(\widehat{p_i})$ by defining

$$\eta_i(\lambda_{i-n}\cdots\lambda_{i-1}\zeta) = f^n(\eta_{i-n}(\zeta)),$$

for $n \geq 0$.

The maps η_i are surjective by the definition of $W^u(\widehat{p_i})$ but they need not be injective. However, the global unstable manifolds $W^u(\widehat{p_i})$ have a natural structure as abstract Riemann surfaces given by the maps η_i. More precisely, for each i we define a Riemann surface X_i as the quotient $\mathbf{C}/\sim$, where $z \sim w$ if there are open sets $U \ni z$ and $V \ni w$ such that $\eta_i(U) = \eta_i(V)$. Then the map η_i factors as $\eta_i = \eta'_i \circ \pi_i$, where $\pi_i : \mathbf{C} \to X_i$ is surjective, $\eta_i' : X_i \to \mathbf{C}^2$ is locally injective and the set of points $(z,w) \in X_i \times X_i$ with $z \neq w$ and $\eta_i'(z) = \eta_i'(w)$ is discrete. We will be sloppy and make no distinction between the unstable manifold $W^u(\widehat{p}_i)$ and the Riemann surface X_i. Hence we will sometimes view $W^u(\widehat{p}_i)$ as a subset of $\mathbf{C}^2$ and sometimes as an abstract Riemann surface. The precise meaning should be clear from the context.

Now the Riemann surface $W^u(\widehat{p}_i)$ cannot be hyperbolic, because η_i maps $\mathbf{C}$ into it so $W^u(\widehat{p}_i)$ is biholomorphic to $\mathbf{C}^*$, $\mathbf{C}$, $\mathbf{P}^1$ or a torus. The last two cases cannot occur, because then $W^u(\widehat{p}_i)$ would be an algebraic subvariety of $\mathbf{P}^2$, which is impossible (because $W^u(\widehat{p}_i) \cap \Pi = \emptyset$). Hence $W^u(\widehat{p}_i)$ is biholomorphic to $\mathbf{C}^*$ or $\mathbf{C}$ for all i.

Write W_i instead of W_{a_i} and note that $W^u(\widehat{p}_i)$ has an open subset biholomorphic to W_i^*. Let Σ_i be the Riemann surface obtained from $W^u(\widehat{p}_i)$ by filling in the hole at a_i. Then Σ_i is biholomorphic to $\mathbf{C}$ or $\mathbf{P}^1$ for all i. If Σ_i is biholomorphic to $\mathbf{P}^1$ for some i, then Σ_i is an algebraic subvariety of $\mathbf{P}^2$ (in fact a line) and we are done, so assume that Σ_i is biholomorphic to $\mathbf{C}$ for all i.

Suppose that $(\Sigma_i - W_i) \cap W^s(J_\Pi) \neq \emptyset$ for some i. Then $(\Sigma_i - W_i) \cap W_b \neq \emptyset$ for some $b \in J_\Pi$, $b \neq a_i$. By the dichotomy given in Lemma 10.5 we then have that $W_b^* \subset (\Sigma_i - W_i)$ so by filling in the hole at b we see that the closure of Σ_i in $\mathbf{P}^2$ is an algebraic subvariety of $\mathbf{P}^2$, which implies that $W^u(\widehat{p}_i)$ is algebraic in this case too.

Let us now suppose that Σ_i is biholomorphic to $\mathbf{C}$ and that $(\Sigma_i - W_i) \cap W^s(J_\Pi) = \emptyset$ for all i. Pick biholomorphisms $\chi_i : \mathbf{C} \to \Sigma_i$ such that $\chi_i(0) = a_i$. Note that f induces holomorphic maps of Σ_i onto Σ_{i+1}. Hence we may define entire maps h_i by $\chi_i \circ h_i = f \circ \chi_{i-1}$ for all i. The restriction of f to W_{i-1} is a branched covering of W_i of degree d, branched only at a_i. This implies that $h_i(\zeta) = \zeta^d \exp(u_i(\zeta))$ where u_i is entire. Moreover, the condition $(\Sigma_i - W_i) \cap W^s(J_\Pi) = \emptyset$ implies that the inverse image of W_i in Σ_{i-1} is exactly W_{i-1}. Therefore $\limsup |h_i(\zeta)| > 0$ as $|\zeta| \to \infty$ and this is only possible if u_i is constant. Hence we may write $h_i(\zeta) = c_i \zeta^d$ for some constants $c_i \neq 0$.

Define $g_i = G \circ \chi_i$. Then, for each i, $g_i \geq 0$ is continuous and subharmonic on $\mathbf{C}^*$. The equation $G \circ f = d\,G$ translates into $g_i \circ h_i = d\,g_{i-1}$, i.e. $g_i(c_i\zeta^d) = d\,g_{i-1}(\zeta)$. Iterating this we see that $g_i(\zeta)$ depends only on $|\zeta|$. Thus, by the maximum principle, for each i there exists an R_i such that either $g_i = 0$ on $|\zeta| > R_i$ or $g_i > 0$ for $|\zeta| > R_i$.

If $g_i = 0$ for $|\zeta| > R_i$, then χ_i maps $|\zeta| > R_i$ into the bounded set K and therefore extends to a holomorphic map of $\mathbf{P}^1$ into $\mathbf{P}^2$. Hence $W^u(\widehat{p}_i)$ is algebraic.

If $g_i > 0$ for $|\zeta| > R_i$, then χ_i maps $|\zeta| > R_i$ into $\mathbf{C}^2 - K$, and by our previous assumption, the image does not intersect $W^s(J_\Pi) = \operatorname{supp}(T \llcorner A)$, so g_i is harmonic on $|\zeta| > R_i$. Hence there exist constants $A_i > 0$ and B_i such that $g_i(\zeta) = A_i \log|\zeta| + B_i$ for $|\zeta| > R_i$. Since $G(x) = \log|x| + O(1)$ as $x \to \Pi$, this implies that $|\chi_i(\zeta)| \leq C|\zeta|^{A_i}$ as $\zeta \to \infty$, so again χ_i extends to a holomorphic map of $\mathbf{P}^1$ into $\mathbf{P}^2$. Hence $W^u(\widehat{p}_i)$ is algebraic, which completes the proof of Lemma 10.9. □

We are now in position to prove Lemma 10.3.

Proof of Lemma 10.3. Suppose that $W^s(J_\Pi) \cap W^u(S_1) \neq \emptyset$. Then $J_\Pi' \neq \emptyset$ so Lemma 10.9 shows that there exist $a \in J_\Pi$, a history $\widehat{p}$ in S_1 and an irreducible polynomial $P(z,w)$ such that $W_a^* \subset W^u(\widehat{p}) = \{P = 0\}$. Clearly $W^u(\widehat{p}) \cap J = \emptyset$ so there exists an $\epsilon > 0$ such that $|P| \geq 2\epsilon$ on J.

By Lemma 10.8 there is a dense set of b's such that W_b lands on J. If we choose b close enough to a, then by continuity W_b will intersect the open set $|P| < \epsilon$. Let U be a component of $\{\zeta \in \Delta^* : |P(\psi_b(\zeta))| < \epsilon\}$. Then U is relatively compact in Δ^*. Further, P is a nonzero holomorphic function on U, so $-\log|P|$ is harmonic on U. But $|P| < \epsilon$ on U and $|P| = \epsilon$ on ∂U, contradicting the maximum principle for $-\log|P|$ on U. This completes the proof of Lemma 10.3. □

Corollary 10.10. *If f satisfies condition (‡) and J_Π is connected, then J is connected. If J_Π is also locally connected, then so is J.* □

Proof. If J_Π is connected (and locally connected) then $\mathcal{E}$ is connected (and locally connected). □

Remark. In order to prove that e maps $\mathcal{E}$ continuously onto J, we could do without the assumption that f_Π is expanding on J_Π. It would be sufficient to assume that $\mu_c(A) = 0$, and that (‡2)–(‡4) hold. Indeed, by Theorem 8.8 we still have a disk lamination $\{W_a : a \in J_\Pi\}$ and e_r is defined and continuous on $\mathcal{E}$ for all $r > 0$. The only place where the uniform expansion of f_Π on J_Π is used, is to get Hölder continuity of e. However, we always get continuity of e.

Questions. Is J a *finite* quotient of $\mathcal{E}$, i.e. is $\#e^{-1}(x)$ uniformly bounded on J? What are the possible identifications on $\mathcal{E}$ introduced by e?

§Appendix A. The homogeneous model

The model for our study of regular polynomial automorphisms is the case when $f = f_h$ is a homogeneous mapping of $\mathbf{C}^k$. Here we show that f_h is essentially a skew product over f_Π. If g is a homogeneous polynomial of degree N, we let $V = \mathbf{C}^k \cap \{g = 0\}$ and $V_\Pi = \Pi \cap \{g = 0\}$. We let Σ_N denote the multiplicative group of the Nth roots of unity in $\mathbf{C}$. We let $\check{\mathbf{C}}_* = \mathbf{C}_*/\Sigma_N$ denote the quotient. With this complex structure, the mapping $\mathbf{C}_* \to \mathbf{C}_*$ given by $\lambda \mapsto \lambda^N$ descends to an isomorphism $\mathbf{C}_* \to \check{\mathbf{C}}_*$. Similarly, we define the (finite) quotient $\check{\mathbf{C}}^k_* := \mathbf{C}^k_*/\Sigma_N$. Thus we have a holomorphic mapping

$$s : \Pi - V_\Pi \to \check{\mathbf{C}}^k_*$$

given by $s(z) = g^{-1/N}(z)z$. It follows that the mapping

$$\psi : \mathbf{C}_* \times (\Pi - V_\Pi) \to \check{\mathbf{C}}^k_* - V$$

given by $\psi(\lambda, [z]) = \lambda^{1/N}s(z)$ is biholomorphic. The homogeneous mapping descends to a finite quotient mapping

$$\check{f}_h : \check{\mathbf{C}}^k_* - (V \cup f_h^{-1}V) \to \check{\mathbf{C}}^k_*.$$

Since s is a section of the bundle $\pi : \check{\mathbf{C}}^k_* \to \Pi$, it follows that $s \circ f_\Pi(z)$ and $\check{f}_h \circ s[z]$ define the same line in $\check{\mathbf{C}}^k_*$. Thus $\chi(z) := f_h(s(z))/s(f_\Pi(z))$ is an N valued holomorphic function on $\Pi - V_\Pi$. A short calculation shows that

$$\psi^{-1} \circ \check{f}_h \circ \psi(\lambda, z) = (\chi^N(z)\lambda^d, f_\Pi(z)) \qquad (A.1)$$

where χ^N is a (single-valued) analytic function on $\Pi - V_\Pi$.

We note that for $a \in \Pi$, $W^s_{\mathrm{loc}}(a, f_h)$ is contained in the line L_a. Let $C(J_\Pi)$ denote the union of J_Π and the complex homogeneous cone in $\mathbf{C}^k_*$ over J_Π. Our canonical model in §4 is given by the restriction of f_h to $C(J_\Pi)$. Let $\check{C}(J_\Pi)$ denote the quotient of $C(J_\Pi)$ in $\check{\mathbf{C}}^k_* \cup \Pi$.

Proposition A.1. *Suppose that $\mathcal{C}_\Pi \cap J_\Pi = \emptyset$. Then there is a continuous function η on J_Π with unit modulus such that the restriction of $\check{f}_h$ to $\check{C}(J_\Pi)$ is conjugate to the self-mapping of $\mathbf{C}_* \times J_\Pi$ given by*

$$(\lambda, z) \mapsto (\eta(z)\lambda^d, f_\Pi(z)).$$

In addition, if η has a continuous logarithm on J_Π, then $\check{f}_h|_{\check{C}(J_\Pi)}$ is conjugate to $(\lambda, [z]) \mapsto (\lambda^d, f_\Pi[z])$.

Proof. If we let g denote the Jacobian determinant of f_h on $\mathbf{C}^k$, then g is a homogeneous polynomial of degree $N = k(d-1)$. In particular, $V = \{g = 0\}$ is the critical locus of f_h, and V_Π is disjoint from J_Π. Thus we may represent $\check{f}_h$ as in $(A.1)$.

Let us define $\phi(\lambda, z) = (\alpha(z)\lambda, z)$. Then $\phi^{-1} \circ \psi^{-1} \circ \check{f}_h \circ \psi \circ \phi$ is given by

$$(\lambda, z) \mapsto (\alpha^d(z)\alpha^{-1}(f_\Pi z)\chi^N(z)\lambda^d, f_\Pi z).$$

Thus we wish to find $\alpha : J_\Pi \to \mathbf{R}$ such that

$$\alpha^d(z)\alpha^{-1}(f_\Pi z)\,|\chi(z)| = 1. \tag{$A.2$}$$

Taking logarithms, we have

$$\log \alpha(z) = -\frac{1}{d}\log|\chi^N(z)| + \frac{1}{d}\log|\alpha(f_\Pi z)|.$$

Applying f^j_Π, dividing by d^j, and summing over j, we have that

$$\log \alpha(z) = -\sum_{j=0}^{\infty} \frac{1}{d^{j+1}} \log|\chi^N(f^j_\Pi z)| \tag{$A.3$}$$

is continuous on J_Π, and α solves (A.2). Setting $\eta = \chi^N/|\chi^N|$, we have the desired form of $\check{f}_h$.

Finally, if η has a continuous logarithm, we may solve (A.2) without taking absolute value. □

Examples. We consider two mappings: $f_1(z_1, z_2) = (z_1^2, z_2^2)$ and $f_2(z_1, z_2) = (z_2^2, z_1^2)$. We consider the function $g(z) = z_1$, the coordinate $\zeta = z_2/z_1$ on $\Pi - \{[0:1]\}$, and the mapping $\psi : \mathbf{C}_* \times (\Pi - \{[0:1]\} \to \mathbf{C}^2_*$ given by $\psi(\lambda, \zeta) = \lambda(1, \zeta) = (\lambda, \lambda\zeta)$. In both cases, $J_\Pi = \{|\zeta| = 1\}$. The associated mappings on J_Π are $f_{1,\Pi}(\zeta) = \zeta^2$ and $f_{2,\Pi}(\zeta) = \zeta^{-2}$. The normal forms given in the Proposition are

$$\psi^{-1} f_1 \psi(\lambda, \zeta) = (\lambda^2, \zeta^2), \qquad \psi^{-1} f_2 \psi(\lambda, \zeta) = (\zeta^2\lambda^2, \zeta^{-2}).$$

There is no continuous function $\alpha : J_\Pi \to \mathbf{C}_*$ solving $(A.3)$ for f_2. For in this case we would have $\alpha^2(\zeta)\alpha^{-1}(\zeta^{-2})\zeta^2 = 1$. But if A is the (integer) winding number of $\alpha\{|\zeta| = 1\}$ about 0 in $\mathbf{C}_*$, then the winding numbers of $\alpha^2(\zeta)$ and $\alpha^{-1}(\zeta^{-2})$ are each $2A$, so we must have $2A + 2A + 2 = 0$, which is a contradiction.

In dimension $k = 2$ we can often assume that $N = 1$.

Proposition A.2. *Suppose that $k = 2$ and that $J_\Pi \neq \Pi$. Then there is a continuous function η on J_Π with unit modulus such that the restriction of f_h to $C(J_\Pi)$ is conjugate to the self-mapping of $\mathbf{C}_* \times J_\Pi$ given by*

$$(\lambda, z) \mapsto (\eta(z)\lambda^d, f_\Pi(z)).$$

In addition, if η has a continuous logarithm on J_Π, then $f_h|_{C(J_\Pi)}$ is conjugate to $(\lambda, [z]) \mapsto (\lambda^d, f_\Pi[z])$.

Proof. The proof is the same as for Proposition A.1, except that we use $g(z, w) = a_2 z - a_1 w$, where $a = [a_1 : a_2] \in \Pi - J_\Pi$. $\square$

§Appendix B. Hyperbolicity for endomorphisms.

In this appendix we present some basic results on hyperbolicity for smooth endomorphisms. More details can be found in [J2]. Our main references are [Ru] and [PS]; see also [FS4]. No proofs are given in this appendix; they can be found in the above references.

Let f be a C^∞ endomorphism of a finite-dimensional Riemannian manifold M. Let L be a compact subset of M with $f(L) = L$ and define

$$\hat{L} = \{(x_i)_{i \le 0} : x_i \in L, f x_i = x_{i+1}\}.$$

Then $\hat{L}$ is a closed subset of $L^{\mathbf{N}}$, hence compact. We will use the notation $\hat{x}$ for a point $(x_i)_{i\le 0}$ in $\hat{L}$. The restriction $f|_L$ lifts to a homeomorphism $\hat{f}$ of $\hat{L}$ given by $\hat{f}((x_i)) = (x_{i+1})$. There is a natural projection π from $\hat{L}$ to L sending $\hat{x}$ to x_0 and the pullback under π of the restriction to L of the tangent bundle of M is a bundle on $\hat{L}$ which we call the tangent bundle $T_{\hat{L}}$. Explicitly, a point in $T_{\hat{L}}$ is of the form $(\hat{x}, v)$ where $\hat{x} \in \hat{L}$ and v is a tangent vector in $T_{x_0}M$. The derivative Df lifts to a map $D\hat{f}$ of $T_{\hat{L}}$ in a natural way.

Now f is *hyperbolic* on L if there exists a continuous splitting $T_{\hat{L}} = E^u \oplus E^s$ which is invariant under $D\hat{f}$ and such that $D\hat{f}$ is expanding on E^u and contracting on E^s. More precisely, $D\hat{f}(E^{u/s}) \subset E^{u/s}$ and there are constants $c > 0$ and $\lambda > 1$ such that for all $n \ge 1$

$$|D\hat{f}^n v| \ge c\lambda^n |v| \quad v \in E^u$$
$$|D\hat{f}^n v| \le c^{-1}\lambda^{-n} |v| \quad v \in E^s.$$

Remark. Such a map is called prehyperbolic in [Ru].

Remark. It is possible to make a smooth change of metric in a neighborhood of L and obtain $c = 1$ in the equation above.

Note that whereas the fiber of the unstable bundle E^u at a point $\hat{x} \in \hat{L}$ depends on the whole history $\hat{x}$ of x_0, the fiber of E^s at $\hat{x}$ depends only on the point x_0. Hence the dimension of the fiber of E^u at a point $\hat{x}$ depends only on x_0, so the dimensions of the fibers of the bundles E^u and E^s are locally constant.

As a special case of the above we say that f is expanding on L if the bundle E^s is trivial. This means that there exist constants $c > 0$ and $\lambda > 1$ such that $|D\hat{f}^n(x)v| \ge c\lambda^n|v|$ for all $x \in L$, $v \in T_x M$ and all $n \ge 1$.

A basic result in hyperbolic dynamics is the stable manifold theorem. For each point p in L and each history $\hat{q}$ in $\hat{L}$, we define local stable and unstable manifolds by

$$\begin{aligned} W^s_{\text{loc}}(p) &= \{y \in M : d(f^i y, f^i p) < \delta \ \forall i \geq 0\} \\ W^u_{\text{loc}}(\hat{q}) &= \{y \in M : \exists \hat{y}, \pi(\hat{y}) = y, d(y_i, q_i) < \delta \ \forall i \leq 0\}, \end{aligned}$$

for small $\delta > 0$.

The following theorem asserts that the local (un)stable manifolds are indeed nice objects. [Ru, §15] contains an outline of a proof, whereas [PS, Theorem 5.2] proves a more general theorem).

Theorem B.1 (Stable Manifold Theorem). *If δ is small enough, then*
(i) For all $p \in L$ and all $\hat{q} \in \hat{L}$, $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(\hat{q})$ are embedded C^∞ balls of M tangent to $E^s(p)$ and $E^u(\hat{q})$ at p and q_0, respectively.
(ii) $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(\hat{q})$ depend continuously on p and $\hat{q}$, respectively.
(iii) If $x \in W^s_{\text{loc}}(p)$, then $d(f^n x, f^n p) \to 0$ exponentially fast as $n \to \infty$. Similarly, every point x in $W^u_{\text{loc}}(\hat{q})$ has a unique history $\hat{x}$ such that $x_j \in W^u_{\text{loc}}(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ and $d(x_j, q_j) \to 0$ exponentially fast as $j \to -\infty$.

If δ is small enough, then by continuity $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(\hat{q})$ are almost flat, i.e. C^1 close to the tangents at p and q_0, respectively for all $p \in L$ and all $q \in \hat{L}$. Therefore $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(q)$ intersect in at most one point.

Definition B.2. *We say that L has local product structure if δ can be chosen so that $W^s_{\text{loc}}(p) \cap W^s_{\text{loc}}(\hat{q}) \subset L$ for all p and $\hat{q}$.*

If L has local product structure, $p \in L$, $\hat{q} \in \hat{L}$ and if p,q_0 are sufficiently close, then $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(\hat{q})$ intersect in exactly one point $x \in L$ and x has a history $\hat{x}$ such that $x_j \in W^u_{\text{loc}}(\hat{f}^j(\hat{q}))$ for all $j \leq 0$. It is not a priori clear that $\hat{x} \in \hat{L}$, i.e. that $x_j \in L$ for all $j \leq 0$. We therefore make another definition.

Definition B.3. *We say that $\hat{L}$ has local product structure if δ can be chosen so that if the intersection $W^s_{\text{loc}}(p) \cap W^s_{\text{loc}}(\hat{q})$ is nonempty, then it consists of a unique point $x \in L$ and the unique history $\hat{x}$ of x with $x_j \in W^u_{\text{loc}}(\hat{f}^j \hat{q})$ for all $j \leq 0$ is contained in $\hat{L}$.*

Definition B.4. *Let $\eta > 0$. An η-pseudoorbit in M is a sequence $(x_i)_{[t_1,t_2]}$, where $-\infty \leq t_1 < t_2 \leq \infty$, such that $d(fx_i, x_{i+1}) < \delta$ for $t_1 \leq i < t_2$. An η-pseudoorbit $(x_i)_{[t_1,t_2]}$ is ϵ-shadowed by an orbit $(y_i)_{[t_1,t_2]}$ if $d(y_i, x_i) < \epsilon$ for all $i \in [t_1, t_2]$.*

For proofs of the remaining results in this appendix see [J2].

Theorem B.5 (Shadowing Lemma). *Suppose that L is hyperbolic and that $\hat{L}$ has local product structure. Then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in L can be ϵ-shadowed by an orbit in L.*

Using shadowing we control the orbits of f staying near L in positive or negative time.

Proposition B.6 (Fundamental Neighborhood). *Let L be a hyperbolic set for a map f. Assume that $\hat{L}$ has local product structure. Then L has a neighborhood U in M such that*

(i) If $x \in U$ and $f^j x \in U$ for all $j \geq 0$, then $x \in W^s_{\mathrm{loc}}(p)$ for some $p \in L$.
(ii) If $x \in U$ and x has a history $\hat{x}$ with $x_i \in U$ for all $i \leq 0$, then $x \in W^u_{\mathrm{loc}}(\hat{q})$ for some $\hat{q} \in \hat{L}$. More precisely $d(x_i, q_i) < \delta$ for all $i \leq 0$.
(iii) If $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in U then $x_i \in L$ for all i.

Next we consider Axiom A endomorphisms. A point $x \in M$ is *wandering* if it has a neighborhood V such that $f^n(V) \cap V = \emptyset$ for all $n \geq 1$; otherwise it is called *non-wandering*. The *non-wandering set* Ω of f is the set of all non-wandering points; it is a closed set.

Definition B.7. *A map f is said to be Axiom A if its non-wandering set satisfies*

(i) Ω is compact.
(ii) Periodic points are dense in Ω.
(iii) f is hyperbolic on Ω.

Remark. If Ω satisfies (ii), then $f(\Omega) = \Omega$, so (iii) makes sense. Also, if f is Axiom A, then periodic points (under $\hat{f}$) are dense in $\hat{\Omega}$.

The next proposition shows that the preceding results apply to open Axiom A endomorphisms.

Proposition B.8. *If f is Axiom A and open, then $\hat{\Omega}$ has local product structure.*

Theorem B.9 (Spectral decomposition). *If f is Axiom A, then Ω can be written in a unique way as a disjoint union $\Omega = \cup_{i=1}^{l} \Omega_i$, where each Ω_i is compact, satisfies $f(\Omega_i) = \Omega_i$ and f is transitive on Ω_i. The sets Ω_i are called the basic sets of f. Morover, each Ω_i can be further decomposed into a finite disjoint union $\Omega_i = \cup_{1 \leq j \leq n_i} \Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $f(\Omega_{i,j}) = \Omega_{i,j+1}$ ($\Omega_{i,n_i+1} = \Omega_{i,1}$) and f^{n_i} is topologically mixing on each $\Omega_{i,j}$.*

Our final result in this appendix describes forward and backward orbits for an Axiom A endomorphism.

Corollary B.10. *Assume that f is Axiom A and M is compact.*

(i) If $x \in M$, then there is a unique basic set Ω_i such that $f^j x \to \Omega_i$ as $j \to \infty$. Moreover, there is a (not necessarily unique) $p \in \Omega_i$ such that $d(f^j x, f^j p) \to 0$ as $j \to \infty$.
(ii) If $\hat{x} \in \hat{M}$, then there is a unique basic set Ω_i such that $x_j \to \Omega_i$ as $j \to -\infty$. Moreover, there is a (not necessarily unique) $\hat{q} \in \widehat{\Omega_i}$ such that $d(x_j, q_j) \to 0$ as $j \to -\infty$.

References

[A] H. Alexander, Gromov's method and Bennequin's problem. Invent. Math. 125, 135–148 (1996).

[AT] H. Alexander, B. A. Taylor, Comparison of two capacities in $\mathbf{C}^n$, Math. Z. 186, 407–417 (1984).

[Ba] D. Barrett, Holomorphic extension from boundaries with concentrated Levi form, Indiana U. Math. J., 44 (1995), 1075–1087.

[Be] B. Berndtsson, Personal communication.

[BLS] E. Bedford, M. Lyubich, J. Smillie, Polynomial diffeomorphims of $\mathbf{C}^2$ IV: The measure of maximal entropy and laminar currents. Invent. Math. 112, 77–125 (1993).

[BS1] E. Bedford, J. Smillie, Polynomial diffeomorphims of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity. Invent. Math. 103, 69–99 (1991).

[BS2] E. Bedford, J. Smillie, Polynomial diffeomorphims of $\mathbf{C}^2$ V: Critical points and Lyapunov exponents. To appear in J. Geom. Anal.

[BT1] E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions. Acta Math. 149, 1–39 (1982).

[BT2] E. Bedford, B. A. Taylor, Plurisubharmonic functions with logarithmic singularities. Ann. Inst. Fourier (Grenoble) 38, 133-171 (1988).

[Bri] J.-Y. Briend. Exposants de Liapounoff des endomorphismes holomorphes de $\mathbf{CP}^k$. Thesis, Université Paul Sabatier–Toulouse III, 1997.

[Bro] H. Brolin, Invariant sets under iteration of rational functions. Ark. Mat. 6, 103–144 (1965).

[C] A. Candel, Uniformization of surface laminations. Ann. Scient. Éc. Norm. Sup., 4 série t. 26 (1993), 489–516.

[CG] L. Carleson, T. W. Gamelin, Complex dynamics. Springer-Verlag (1993).

[FS1] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension I. Astérisque 222, 201–231 (1994).

[FS2] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension II. In Annals of Mathematics Studies 137, 135–182. Princeton University Press, 1995.

[FS3] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension, In P. M. Gauthier, G. Sabidussi, editors, Complex potential theory, pages 131–186. Kluwer Academic Publishers, 1994.

[FS4] J. E. Fornæss, N. Sibony, Hyperbolic maps on $\mathbf{P}^2$. Math. Ann., to appear.

[H] S. Heinemann, Julia sets for holomorphic endomorphisms of $\mathbf{C}^n$. Ergodic Theory Dynam. Systems 16, 1275–1296 (1996).

[HP] J. H. Hubbard, P. Papadopol, Superattractive fixed points in $\mathbf{C}^n$. Indiana Univ. Math. J. 43, 321–365 (1994).

[J1] M. Jonsson, Sums of Lyapunov exponents for some polynomial maps of $\mathbf{C}^2$. To appear in Ergodic Theory Dynam. Systems.

[J2] M. Jonsson, Thesis, Royal Institute of Technology, 1997.

[La] S. Lang, Introduction to complex hyperbolic spaces. Springer-Verlag, 1987.

[Le] G. Levin, Disconnected Julia set and rotation sets, Ann. scient. Éc. Norm. Sup., t. 29, (1996) 1–22.

[M] J. Milnor, Dynamics in one complex variable: introductory lectures. Preprint SUNY Stony Brook (1990).

[Pe] G. Peng. On the dynamics of non-degenerate polynomial endomorphisms of $\mathbf{C}^2$. PhD thesis, CUNY, 1997.

[Pr] F. Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. Invent. Math. 80, 161–179 (1985).

[PS] C. Pugh, M. Shub, Ergodic attractors. Trans. Amer. Math. Soc. 312, 1–54 (1989).

[Ro] J.-P. Rosay, A remark on the paper by H. Alexander on Bennequin's problem. Invent. Math. 126, 625–627 (1996).

[Ru] D. Ruelle, Elements of differentiable dynamics and bifurcation theory. Academic Press, 1989.

[T] P. Tortrat, Aspects potentialistes de l'itération des polynômes. Séminaire de Théorie du Potentiel. Lecture Notes in Math 1235, 195–209 (1987).

[U] T. Ueda, Fatou sets in complex dynamics on projective spaces. J. Math. Soc. Japan 46, 545–555 (1994).

[Y] L.-S. Young, Ergodic theory of differentiable dynamical systems. In Branner, B., Hjorth, P., editors, Real and complex dynamical systems, pages 293–336, Kluwer Academic Publishers (1995).

Eric Bedford
Indiana University
Bloomington, IN 47405
bedford@indiana.edu

Mattias Jonsson
Royal Institute of Technology
100 44 Stockholm, Sweden
mjo@math.kth.se