BIACCESSIBLILITY IN QUADRATIC JULIA SETS II: THE SIEGEL AND CREMER CASES

SAEED ZAKERI

Abstract. Let f be a quadratic polynomial which has an irrationally indifferent fixed point α. Let z be a biaccessible point in the Julia set of f. Then:

- In the Siegel case, the orbit of z must eventually hit the critical point of f.
- In the Cremer case, the orbit of z must eventually hit the fixed point α.

Siegel polynomials with biaccessible critical point certainly exist, but in the Cremer case it is possible that biaccessible points can never exist.

As a corollary, we conclude that the set of biaccessible points in the Julia set of a Siegel or Cremer quadratic polynomial has Brolin measure zero.

§1. Introduction. Let f be a polynomial map of the complex plane $\mathbb{C}$. A fixed point $z = f(z)$ is called *indifferent* if the *multiplier* $\lambda = f'(z)$ has the form $e^{2\pi i\theta}$, where the *rotation number* θ belongs to $\mathbb{R}/\mathbb{Z}$. We call z *irrationally indifferent* if θ is irrational so that λ is on the unit circle but not a root of unity.

Let z be an irrationally indifferent fixed point of f. When f is holomorphically linearizable about z, we call z a *Siegel* fixed point. On the other hand, when z is nonlinearizable, it is called a *Cremer* fixed point.

In this paper we only consider quadratic polynomials. Such a polynomial, which we can put in the normal form

$$f : z \mapsto z^2 + c, \tag{1}$$

has two fixed points $(1 \pm \sqrt{1-4c})/2$ which are distinct if and only if $c \neq 1/4$. If $c \notin [1/4, \infty)$, so that the two fixed points have distinct real parts, then by convention the fixed point which is further to the left is called α and the other fixed point $1-\alpha$ is called β. The corresponding multipliers are $\lambda = 2\alpha$ and $2-\lambda = 2\beta$, with $|\lambda| < |2-\lambda|$. Evidently only the α-fixed point can be indifferent. The critical value parameter c is then given by

$$c = \lambda(2-\lambda)/4.$$

Therefore, the set of all quadratic polynomials which have an indifferent fixed point is a cardioid in the c-plane parametrized by λ on the unit circle. The set of quadratic polynomials with an irrationally indifferent fixed point is then a dense subset of this cardioid.

We call a quadratic polynomial f in (1) *Siegel* or *Cremer* if the α-fixed point is irrationally indifferent and has the corresponding property.

By the theorem of Brjuno-Yoccoz [**Yo**], f is a Siegel polynomial if and only if $\theta = \frac{1}{2\pi i}\log f'(\alpha)$ satisfies the *Brjuno condition*:

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty, \tag{2}$$

where the q_n appear as the denominators of the rational approximations coming from the continued fraction expansion of θ.

Recall that the *filled Julia set* of f is

$$K(f) = \{z \in \mathbb{C} : \text{The orbit } \{f^{\circ n}(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the *Julia set* of f is the topological boundary of the filled Julia set:

$$J(f) = \partial K(f).$$

Both sets are nonempty, compact, connected and the filled Julia set is full, i.e., the complement $\mathbb{C} \setminus K(f)$ is connected. Every connected component of the interior of $K(f)$ is a topological disk called a *bounded Fatou component* of f. In the Siegel case, the component S of the interior of $K(f)$ containing the fixed point α is called the *Siegel disk* of f on which the action of f is holomorphically conjugate to the rigid rotation $z \mapsto e^{2\pi i\theta} z$.

Since $f(z) = f(-z)$ by (1), the Julia set $J(f)$ is invariant under the $180°$ rotation $\tau : z \mapsto -z$. If U is an open Jordan domain in the plane such that $\overline{U} \cap \tau(\overline{U}) = \emptyset$, it follows that f is univalent in some Jordan domain V containing the closure $\overline{U}$.

According to Sullivan, every bounded Fatou component must eventually map to the immediate basin of attraction of an attracting periodic point, or to an attracting petal for a parabolic periodic point, or to a periodic Siegel disk for f (see for example [**Mi1**]). On the other hand, by [**Do1**] a polynomial of degree $d \geq 2$ can have at most $d-1$ nonrepelling periodic orbits. It follows that in the Siegel case, every bounded Fatou component eventually maps to the Siegel disk S centered at α. In the Cremer case, however, we simply conclude that $K(f)$ has no interior, so that $K(f) = J(f)$.

§2. Accessibility. Given a quadratic polynomial f as in (1) with connected Julia set, there exists a unique conformal isomorphism

$$\varphi : \overline{\mathbb{C}} \setminus K(f) \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}},$$

called the *Böttcher map*, with $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$, which conjugates f to the squaring map:

$$\varphi(f(z)) = (\varphi(z))^2. \tag{3}$$

The φ-preimages of the radial lines and circles centered at the origin are called the *external rays* and *equipotentials* of $K(f)$, respectively. The external ray R_t, by definition, is

$$\varphi^{-1}\{re^{2\pi it} : r > 1\},$$

where $t \in \mathbb{R}/\mathbb{Z}$ is called the *angle* of the ray. From (3) it follows that

$$f(R_t) = R_{2t \ (mod\ 1)}.$$

More generally, let us consider an arbitrary compact, connected, full set $K \subset \mathbb{C}$, and let $\varphi_K : \overline{\mathbb{C}} \setminus K \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the unique conformal isomorphism with $\varphi_K(\infty) = \infty$ and $\varphi_K'(\infty) > 0$ given by the Riemann Mapping Theorem. We can define the external rays R_t for K in a similar way. We say that R_t *lands* at $p \in \partial K$ if $\lim_{r \to 1} \varphi_K^{-1}(re^{2\pi it}) = p$. A point $p \in \partial K$ is called *accessible* if there exists a simple arc in $\mathbb{C} \setminus K$ which starts at infinity and terminates at p. According to a theorem of Lindelöf (see for example [**Ru**], p. 259), p is accessible exactly when there exists an external ray landing at p. We call p *biaccessible* if it is accessible through at least two distinct external rays. By a theorem of F. and M. Riesz [**Mi1**], $K \setminus \{p\}$ is disconnected whenever p is biaccessible. It is interesting that the converse is also true. More precisely, if there are at least $n > 1$ connected components of $K \setminus \{p\}$, then at least n distinct external rays land at p (see for example [**Mc**], p. 85).

In the case $K = K(f)$ is the filled Julia set of a quadratic polynomial f in (1), note that $\tau(R_t) = R_{t+1/2}$. Hence if R_t lands at some $p \in J(f)$, then $R_{t+1/2}$ lands at $\tau(p) = -p$.

§3. Arithmetical conditions. As it is suggested by the theorem of Brjuno-Yoccoz, the behavior of the orbits near the indifferent fixed point is intimately connected to the arithmetical properties of the rotation number θ. There are certain classes of irrational numbers which are of special interest in holomorphic dynamics and in this paper we will be working with some of them. Let

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be the continued fraction expansion of θ, where all the a_i are positive integers, and

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

be the n-th rational approximation of θ. We say that

- θ is of *constant type* (we write $\theta \in \mathcal{CT}$) if $\sup_n a_n < +\infty$.
- θ is *Diophantine* (we write $\theta \in \mathcal{D}$) if there exist positive constants C and ν such that for every rational number $0 \leq p/q < 1$, we have $|\theta - p/q| > C/q^\nu$. This condition is equivalent to $\sup_n(\log q_{n+1}/\log q_n) < +\infty$.
- θ is of *Yoccoz type* (we write $\theta \in \mathcal{H}$) if every analytic circle diffeomorphism with rotation number θ is analytically

  linearizable. (An explicit arithmetical description of $\mathcal{H}$ is given by Yoccoz although it is not easy to explain; see [**Yo**].)

  A closely related condition, which we denote by $\mathcal{H}'$, is defined as follows: $\theta \in \mathcal{H}'$ if and only if every analytic circle diffeomorphism with rotation number θ, with no periodic orbit in some neighborhood of the circle, is analytically linearizable [**PM1**].
- θ is of *Brjuno type* (we write $\theta \in \mathcal{B}$) if θ satisfies the condition (2).

We have the proper inclusions $\mathcal{H} \subset \mathcal{H}'$ and $\mathcal{CT} \subset \mathcal{D} \subset \mathcal{H} \subset \mathcal{B}$. It is not hard to show that $\mathcal{D}$, hence $\mathcal{H}, \mathcal{H}'$ and $\mathcal{B}$, are sets of full measure in $\mathbb{R}/\mathbb{Z}$.

§4. Basic results. Very little is known about the topology of the Julia set of f in the Siegel or Cremer case or the dynamics of f on its Julia set. The following theorem summarizes the basic results in the Cremer case:

Theorem 1. *Let f in (1) be a Cremer quadratic polynomial, so that $\theta \notin \mathcal{B}$. Then*

(a) *The Julia set $J(f)$ cannot be locally-connected* [**Su**].
(b) *Every neighborhood of the Cremer fixed point α contains infinitely many repelling periodic orbits of f* [**Yo**].
(c) *The critical point 0 is recurrent, i.e., it belongs to the closure of its orbit $\{f^{\circ n}(0)\}_{n>0}$* [**Ma**].
(d) *The critical point 0 is not accessible from $\mathbb{C} \setminus J(f)$* [**Ki**].

See also [**Sø**] for the so-called "Douady's nonlanding Theorem" which partially explains why the Julia set of a generic Cremer quadratic polynomial fails to be locally-connected.

In the Siegel case, we know a little bit more, but still the situation is far from being fully understood.

Theorem 2. *Let f in (1) be a Siegel quadratic polynomial, so that $\theta \in \mathcal{B}$. Let S denote the Siegel disk of f. Then*

(a) *If $\theta \in \mathcal{CT}$, then the boundary ∂S is a quasicircle which contains the critical point 0* [**Do2**]. *The Julia set $J(f)$ is locally-connected and has measure zero* [**Pe**].
(b) *If $\theta \in \mathcal{H}$, then $0 \in \partial S$* [**He1**].
(c) *For some rotation numbers $\theta \in \mathcal{B} \backslash \mathcal{H}$, the entire orbit of 0 is disjoint from ∂S* [**He2**]. *In this case, $J(f)$ cannot be locally-connected* [**Do2**].
(d) *For every $\theta \in \mathcal{B}$, the critical point 0 is recurrent.*

Part (b) was proved by Herman for $\theta \in \mathcal{D}$, but his proof works equally well for $\theta \in \mathcal{H}$. We will include a very short proof for the latter case in section **§5**. The proof of part (d) goes as follows: If $0 \in \partial S$, then by classical Fatou-Julia theory, every point in ∂S is in the closure of the orbit of 0 [**Mi1**], and recurrence follows. If $0 \notin \partial S$ and 0 is not recurrent, then by [**Ma**] the invariant set ∂S is expanding, i.e., there is a constant $\lambda > 1$ and a positive integer k such that $|(f^{\circ k})'(z)| > \lambda$ for all $z \in \partial S$. It follows that the same inequality holds over some neighborhood U of ∂S with a slightly smaller $\lambda_1 > 1$. We may as well assume that $U \cap S$ is invariant. Take a small disk $V \Subset U \cap S$. Since $f^{\circ k}|_{U \cap S}$ is holomorphically conjugate to the rigid rotation $z \mapsto e^{2\pi i k\theta} z$, there exists a sequence $n_j \to \infty$ such that $f^{\circ k^{n_j}}$ converges uniformly to the identity map on V as $j \to \infty$. But this is impossible since for all $z \in V$, $|(f^{\circ k^{n_j}})'(z)| > \lambda_1^{n_j} \to \infty$.

Comparing the two theorems, we notice that the Cremer case and the Siegel case with $0 \notin \partial S$ share many properties. This is a general philosophy which is partially explained by the theory of "hedgehogs" introduced recently by Perez-Marco [**PM1**] (see section **§5** below).

Inspired by this similarity, one expects the following to be true:

Conjecture. *Let f be a Siegel quadratic polynomial and $0 \notin \partial S$. Then*

(i) *Every neighborhood of ∂S contains infinitely many repelling periodic orbits of f.*
(ii) *The critical point 0 is not accessible from $\mathbb{C} \backslash K(f)$.*

By an argument similar to [**Ki**], one can show that (i) implies (ii) (see also Proposition 3).

§5. Hedgehogs. Let f be a Siegel or Cremer quadratic polynomial as in (1). Let U be a simply connected domain with compact closure which contains the closure of the Siegel disk S in the linearizable case, or the indifferent fixed point α in the nonlinearizable case. Suppose that f is univalent in a neighborhood of the closure $\overline{U}$. Then there exists a set $H = H_U$ with the following properties:

(i) $\alpha \in H \subset \overline{U}$,
(ii) H is compact, connected and full,
(iii) $\partial H \cap \partial U$ is nonempty,
(iv) $\partial H \subset J(f)$,

(v) $f(H) = H$.

Note that H has nonempty interior if and only if α is linearizable. In this case our assumption that f is univalent on U implies that the critical point is off the boundary of the Siegel disk. Clearly $H \supset \overline{S}$.

Such an H is called a *hedgehog* for the restriction $f|_U : U \to \mathbb{C}$. See Fig. 1(a) for the Cremer case and (b) for the Siegel case. (We would like to emphasize that the topology of a hedgehog is infinitely more complicated than anything we can possibly sketch!) The existence of such totally invariant sets is proved by Perez-Marco [**PM1**].

Note that in the Siegel case, one can get totally invariant sets H with the above properties (i)-(v) even if ∂U intersects the closure $\overline{S}$. But in this case the existence of H is not hard to show because we can simply take H as $\overline{S}$ or a compact invariant piece with analytic boundary inside the Siegel disk (see Fig. 1(c) and (d)).

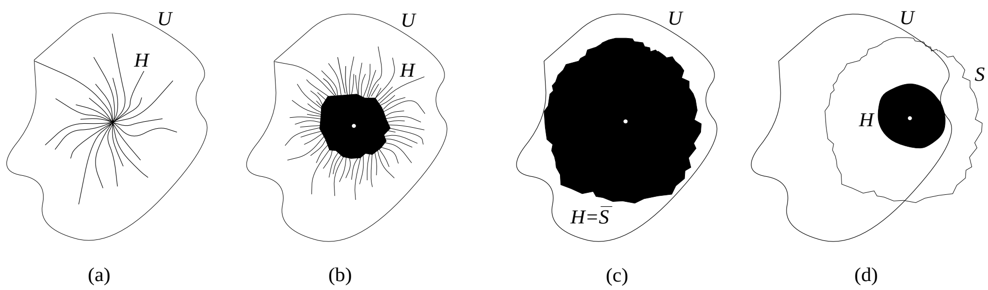

FIGURE 1

Hedgehogs turn out to be useful because of the following nice construction: Uniformize the complement $\mathbb{C} \setminus H$ by the Riemann map $\psi : \mathbb{C} \setminus H \to \mathbb{C} \setminus \overline{\mathbb{D}}$ and consider the induced map $g = \psi \circ f \circ \psi^{-1}$ which is defined (by (v) above) and holomorphic in an open annulus $\{z \in \mathbb{C} : 1 < |z| < r\}$. Use the Schwarz Reflection Principle to extend g to the annulus $\{z \in \mathbb{C} : r^{-1} < |z| < r\}$. The restriction of g to the unit circle $\mathbb{T}$ will then be a real-analytic diffeomorphism whose rotation number is exactly $\frac{1}{2\pi i} \log f'(\alpha) = \theta \in \mathbb{R}/\mathbb{Z}$ (see [**PM1**]). This allows us to transfer results from the more developed theory of circle diffeomorphisms to the less explored theory of indifferent fixed points of holomorphic maps.

Using the above construction, it is not hard to prove the following fact (see [**PM2**]):

Proposition 1. *Let p be a point in a hedgehog H which is biaccessible from outside of H. Then $p \in \partial S$ in the Siegel case and $p = \alpha$ in the Cremer case.*

In fact, let us assume that we are in the Siegel case and $p \notin \partial S$. Then one can find a simple arc γ in $\mathbb{C} \setminus H$ which starts and terminates at p and does not encircle the indifferent fixed point α. Let D be the bounded connected component of $\mathbb{C} \setminus (H \cup \gamma)$. Evidently $\overline{D}$ is disjoint from $\overline{S}$. The topological disk $D' = \psi(D)$ is bounded by the simple arc $\psi(\gamma)$ and an interval I on the unit circle. (The fact that $\psi(\gamma)$ actually lands from both sides on the unit circle follows from general theory of conformal mappings; see for example [**Po**], page 29.) Since g has irrational rotation number on the unit circle $\mathbb{T}$, for some integer N we have $\bigcup_{i=0}^{N} g^{\circ i}(I) = \mathbb{T}$. By choosing γ close enough to H, we can assume that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are all defined on D' and $\bigcup_{i=0}^{N} g^{\circ i}(D')$ contains an entire outer neighborhood of $\mathbb{T}$. It follows that $\bigcup_{i=0}^{N} f^{\circ i}(D)$ covers an entire deleted neighborhood of H. Therefore, some iterate $f^{\circ i}(\overline{D})$ intersects ∂S. Since $f^{\circ i}$ is univalent on $\overline{D} \cup \overline{S}$, it follows that $\overline{D} \cap \partial S \neq \emptyset$, which contradicts our assumption. The proof in the Cremer case is similar.

The construction of the circle maps associated with hedgehogs as described above gives short proofs for some interesting facts. As the first example, we prove that there are no periodic points on ∂S when the critical point 0 is off this boundary. This fact will be used in the proof of Theorem 3 (see [**PM1**] for a general proof in the case of indifferent germs; the fact that we are working with polynomials makes the proof even shorter).

First we need the following lemma:

Lemma 1. *Let f be a Siegel quadratic polynomial as in (1) whose critical point 0 is off the boundary ∂S of the Siegel disk. Then the closure $\overline{S}$ is full and f acts injectively on it.*

It is reasonable to speculate that the closure of any bounded Fatou component for a quadratic polynomial is full. This is known to be true except when the polynomial has a periodic Siegel disk S with the critical point on its boundary ∂S. In this case, we do not know if ∂S can separate the plane into more than two connected components (a so-called "Lakes of Wada" example in plane topology [**H-Y**]).

Proof. Since $f(z) = f(-z)$ for all z, if f is not injective on $\overline{S}$, there must be a pair of symmetric points p and $-p = \tau(p)$ in ∂S. Since $J(f)$ has a 180° rotational symmetry, $f^{-1}(S) = S \cup \tau(S)$. So p and $-p$ also belong to $\partial(\tau(S))$. Consider the connected component V of $\mathbb{C} \smallsetminus (\overline{S} \cup \overline{\tau(S)})$ which contains the critical point 0. Since V is open and $\partial V \subset J(f)$, it follows from the Maximum Principle that V has to be a bounded Fatou component of f. This contradicts the fact that $0 \in J(f)$.

Let us now assume that $\overline{S}$ is not full and let U be a bounded component of $\mathbb{C} \smallsetminus \overline{S}$. Since $\partial U \subset \partial S \subset J(f)$, it follows again from the Maximum Principle that U has to be a bounded Fatou component of f, hence it eventually maps to S, i.e., $f^{\circ n}(U) = S$ for some $n \geq 1$. Therefore $f^{\circ n-1}(U) = \tau(S)$. But the boundary of $f^{\circ n-1}(U)$ is a subset of ∂S, which implies that the common boundary $\partial S \cap \partial(\tau(S))$ is nonempty. This contradicts the fact that $f|_{\partial S}$ is injective. □

Proposition 2. *Let f be a Siegel quadratic polynomial whose critical point 0 is off the boundary ∂S. Then there are no periodic points on ∂S.*

Proof. By the above lemma $\overline{S}$ is full and f acts injectively on it, so we can find a Jordan domain U containing $\overline{S}$ such that $f|_{\overline{U}}$ is univalent. Consider a hedgehog $H = H_U$ for the restriction $f|_U$. Clearly $H \supset \overline{S}$. Suppose that there is a periodic point on ∂S which is necessarily repelling. Then there exists a rational external ray R landing at this point, hence $f^{\circ n}(R) = R$ for some $n \geq 1$ (see for example [**Mi1**]). Consider the induced map $g = \psi \circ f \circ \psi^{-1}$ as described above, and look at the arc $\gamma = \psi(R)$. It is a standard fact that γ has to land at some point $p \in \mathbb{T}$ [**Po**] and $g^{\circ n}(p) = p$. But this contradicts the fact that the rotation number of g is irrational. □

In the second application, we prove Theorem 2(b): We want to show that $\theta \in \mathcal{H}$ implies $0 \in \partial S$. If not, by Lemma 1 $\overline{S}$ is full and $f|_{\overline{S}}$ is univalent. Consider a Jordan domain U, a hedgehog H_U and the induced circle map g as in the above proof. Since the rotation number of g belongs to $\mathcal{H}$, g is analytically linearizable. The linearization can be extended holomorphically to an annulus neighborhood of the unit circle $\mathbb{T}$. Pulling this neighborhood back, we find a larger domain containing S on which f is linearizable, which contradicts the definition of a Siegel disk.

As a final application, we prove the following:

Proposition 3. *Let f be a Siegel quadratic polynomial whose critical point 0 is off the boundary ∂S. If $\theta \in \mathcal{H}'$, then every neighborhood of ∂S contains infinitely many repelling periodic orbits. Hence, the critical point 0 is not accessible from $\mathbb{C} \backslash K(f)$.*

Proof. Consider the hedgehog construction as in the proof of Proposition 2 or the above proof for Theorem 2(b). If there are no periodic orbits in some neighborhood of ∂S, it follows that g has no periodic orbit in some neighborhood of $\mathbb{T}$ either. Since the rotation number of g is $\theta \in \mathcal{H}'$, g has to be linearizable. Now we can get a contradiction as in the above proof for Theorem 2(b). So every neighborhood of ∂S must contain infinitely many periodic orbits. The fact that this implies nonaccessibility of 0 follows easily by an argument similar to [**Ki**]. $\square$

§6. Wakes. To see the behavior of rays near infinity, it will be convenient to add a circle at infinity $\mathbb{T}_\infty \simeq \mathbb{R}/\mathbb{Z}$ to the complex plane to obtain a closed disk © topologized in the natural way. We denote the point $\lim_{r\to\infty} re^{2\pi it}$ on $\mathbb{T}_\infty$ simply by $\infty \cdot e^{2\pi it}$. The action of f in (1) on the complex plane extends continuously to © by

$$f(\infty \cdot e^{2\pi it}) = \infty \cdot e^{4\pi it}, \tag{4}$$

which is just the doubling map on $\mathbb{T}_\infty$. Note that the symmetry $f(z) = f(-z)$ also extends to © if we define $-\infty \cdot e^{2\pi it} = \infty \cdot e^{2\pi i(t+1/2)}$.

Definition. Let f be a quadratic polynomial as in (1) with connected Julia set. Let $z \neq \alpha$ be a biaccessible point in $J(f)$ with two distinct rays R and R' landing on it. We call (R, R') a *ray pair.* By the Jordan Curve Theorem, $R \cup R' \cup \{z\}$ cuts the plane into two open topological disks. By the *wake* W of the ray pair (R, R') we mean the connected component of $\mathbb{C} \backslash (R \cup R' \cup \{z\})$ which does not contain the fixed point α. The other component is called the *co-wake* and it is denoted by $\check{W}$. Point z is called the *root* of W. The *angle* $a(W)$ of the wake is just the (normalized) measure of $\overline{W} \cap \mathbb{T}_\infty$. Clearly $a(W) + a(\check{W}) = 1$ (see Fig. 2(a)).

Since distinct external rays are disjoint, it follows that any two wakes with distinct roots are either disjoint or nested.

In the following lemma we collect basic properties of wakes (compare with [**G-M**] or [**Mi2**]):

Lemma 2. *Let $z \in J(f)$ be a biaccessible point, $z \neq \alpha$, and let W be a wake with root z.*

(a) *If $z \neq 0$, then $a(W) > 1/2$ if and only if W contains the critical point 0.*

(b) *If $a(W) = 1/2$, then z must be the critical point 0. Conversely, if there is any ray R landing at 0, then $R' = \tau(R)$ also lands at 0 and the two rays span a wake W with $a(W) = 1/2$.*

(c) *Let $a(W) < 1/2$ and $f(z) \neq \alpha$. Then $f(W)$ is a wake or a co-wake with root $f(z)$, depending on whether $-\alpha \notin W$ or $-\alpha \in W$. In either case, $f : W \to f(W)$ is a conformal isomorphism and $a(f(W)) = 2a(W)$.*

Proof. Let W be the wake of a ray pair (R, R').

(a) Let $0 \in W$ and $a(W) < 1/2$. Consider the symmetric region $\tau(W)$ whose angle is equal to $a(W)$. W and $\tau(W)$ intersect since both contain 0 (see Fig. 2(b)). On the other hand, $\overline{W} \cap \overline{\tau(W)} \cap \mathbb{T}_\infty = \emptyset$ because $a(W) < 1/2$. Since $\overline{W}$ and $\overline{\tau(W)}$ are both homeomorphic to closed disks, it follows that the ray pairs (R, R') and $(\tau(R), \tau(R'))$ must intersect, which is a contradiction. Therefore $a(W) > 1/2$ if $0 \in W$.

On the other hand, let $a(W) > 1/2$. Then the angle of the co-wake $\check{W}$ has to be less than $1/2$, so by the above argument $0 \notin \check{W}$, or $0 \in W$. This proves (a).

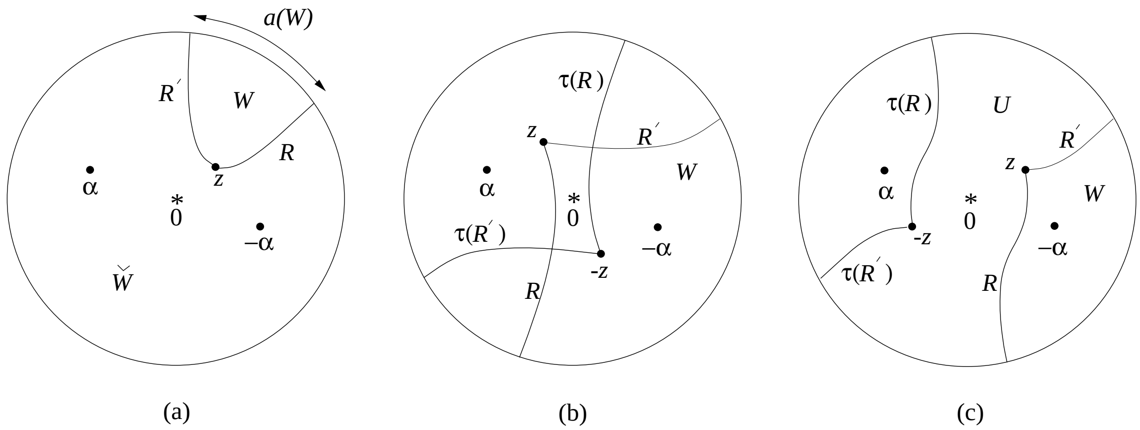

FIGURE 2

(b) If $a(W) = 1/2$, then $R' = \tau(R)$. Hence $z = \tau(z)$ by continuity, which means $z = 0$. The converse is trivial.

(c) If $a(W) < 1/2$, then the ray pairs (R, R') and $(\tau(R), \tau(R'))$ cut the plane into simply connected domains W, $\tau(W)$ and an open set U which is either a simply connected domain or the disjoint union of two simply connected domains depending on whether $z \neq 0$ or $z = 0$. By (a), $0 \notin W \cup \tau(W)$. Consider the ray pair $(f(R), f(R'))$ landing at $f(z)$, and let W' be the corresponding wake. The pull-back of W' by f either consists of the disjoint union $W \sqcup \tau(W)$ or the open set U (see Fig. 2(c)). In the first case, f maps W to W' isomorphically and $-\alpha \notin W$. In the second case, however, we must have $-\alpha \in W$, $\alpha \in \tau(W)$, and both W and $\tau(W)$ map isomorphically to the co-wake $\check{W}'$. The fact that $a(f(W)) = 2a(W)$ simply follows from (4). □

§7. The main theorem. Now we are in a position to state and prove the main theorem of this paper:

Theorem 3. *Let f be a quadratic polynomial as in (1) which has an irrationally indifferent fixed point α. Let z be a biaccessible point in the Julia set of f. Then:*

- *In the Siegel case, the orbit of z must eventually hit the critical point* 0.
- *In the Cremer case, the orbit of z must eventually hit the fixed point α.*

(Compare [**S-Z**] where this same result for the Cremer case is proved by a somewhat different argument.)

In the Siegel case, if the critical point 0 is accessible, then exactly two rays land there (see the proof of Lemma 2(b)). This happens, for example, when $\theta \in \mathcal{CT}$, since in this case by Theorem 2(a) the Julia set is locally-connected. On the other hand, for some rotation numbers $\theta \in \mathcal{B} \cap \mathcal{H}'$, the critical point is not accessible so that there are no biaccessible points in the Julia set (see Corollary 1).

In the Cremer case, if the fixed point α is accessible, then infinitely many rays land there. In fact, if R_t lands at α, then t is irrational and every $R_{2^n t}$ lands at α also. However, there is no known example where one can decide whether α is accessible or not.

Theorem 3 can be viewed from a more general perspective: Let f be any quadratic polynomial with connected Julia set $J(f)$. There is a unique measure μ of maximal entropy $\log 2$, called the *Brolin measure*, which is supported on $J(f)$. In fact, μ coincides with the harmonic measure induced by the radial limits of the inverse of the Böttcher map (see for example [**E-L**]). It is a standard fact that μ is f-invariant and ergodic. For $z \in J(f)$, let $v(z)$ denote the number of external rays which land at z. (In Milnor's terminology [**Mi2**], this is called the *valence* of z.) For $0 \leq n \leq \infty$ define $J_n = \{z \in J(f) : v(z) = n\}$. It follows from elementary plane topology that the union $\bigcup_{n \geq 3} J_n$ is at most countable (see [**Po**], page 36). On the other hand, the fact that almost every external ray (with respect to the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$) lands shows that $\mu(J_0) = 0$. Putting these two facts together, we conclude that $J(f) = J_1 \cup J_2$ up to a set of μ-measure zero. Note that $v(f(z)) = v(z)$ unless z is the critical point. Therefore, if we neglect the grand orbit of the critical point which has μ-measure zero, it follows that both J_1 and J_2 must be f-invariant subsets of the Julia set. Ergodicity of μ then shows that up to a set of μ-measure zero, either $J(f) = J_1$ or $J(f) = J_2$. For example, in the first part of this paper [**Za**], it is shown that for any quadratic polynomial f with locally-connected Julia set, $J(f) = J_1$ must be the case unless f is the *Chebyshev* map $z \mapsto z^2 - 2$. For this map the Julia set is the closed interval $[-2, 2]$ and every point is the landing point of exactly two rays except for the endpoints ± 2 where unique rays land, so that $J(f) = J_2$. Theorem 3 in particular proves that if f has an irrationally indifferent fixed point, then $J = J_1$ up to a set of Brolin measure zero, thus covering some non locally-connected cases. It is conjectured that $J = J_1$ is true for *every* non-Chebyshev quadratic polynomial.

The proof of Theorem 3 is based on the following lemma:

Lemma 3. *Let f be a Siegel or Cremer quadratic polynomial as in (1). Assume that there exists a biaccessible point in $J(f)$ whose orbit never hits the critical point 0 or the fixed point α. Then there exists a ray pair which separates α from 0.*

Proof. Let $z \in J(f)$ be such a biaccessible point and (R, R') be a ray pair which lands at z. Consider the associated wake W_0 with root z. Since $z \neq 0$, we have $a(W_0) \neq 1/2$ by Lemma 2(b). If $a(W_0) > 1/2$, then $0 \in W_0$ by Lemma 2(a) and (R, R') separates α from 0. Let us consider the case where $a(W_0) < 1/2$. If $-\alpha \in W_0$, then (R, R') must separate $-\alpha$ from 0 because by Lemma 2(a), $0 \notin W_0$. It follows that the symmetric ray pair $(\tau(R), \tau(R'))$ separates α from 0. If, however, $-\alpha \notin W_0$, then by Lemma 2(c), $W_1 = f(W_0)$ is a wake with root $z_1 = f(z)$ with angle $a(W_1) = 2a(W_0)$.

Now we can replace W_0 by W_1 in the above argument. If either $a(W_1) > 1/2$ or $a(W_1) < 1/2$ and $-\alpha \in W_1$, we can find a ray pair separating α from 0. Otherwise, we consider the new wake $W_2 = f(W_1)$ with angle $a(W_2) = 2^2 a(W_0)$. Since each passage $W_i \mapsto W_{i+1}$ implies doubling the angles, this process must stop at some stage, and this proves the lemma. □

Proof of Theorem 3. It will be more convenient to consider the Cremer case first. Suppose that the orbit of z never hits α. Since the critical point is not accessible by Theorem 1(d), Lemma 3 gives us a ray pair (R, R') landing at some point $p \in J(f)$ which separates α from 0. Let W be the corresponding wake with root p and consider the co-wake $\check{W}$. The restriction of f to the closure of $\check{W}$ is univalent since otherwise this closure would intersect the closure of the symmetric domain $\tau(\check{W})$, which is impossible since $a(\check{W}) < 1/2$. To work with a Jordan domain in the plane we cut off $\check{W}$ along an equipotential curve and call the resulting domain U (see Fig. 3(a)).

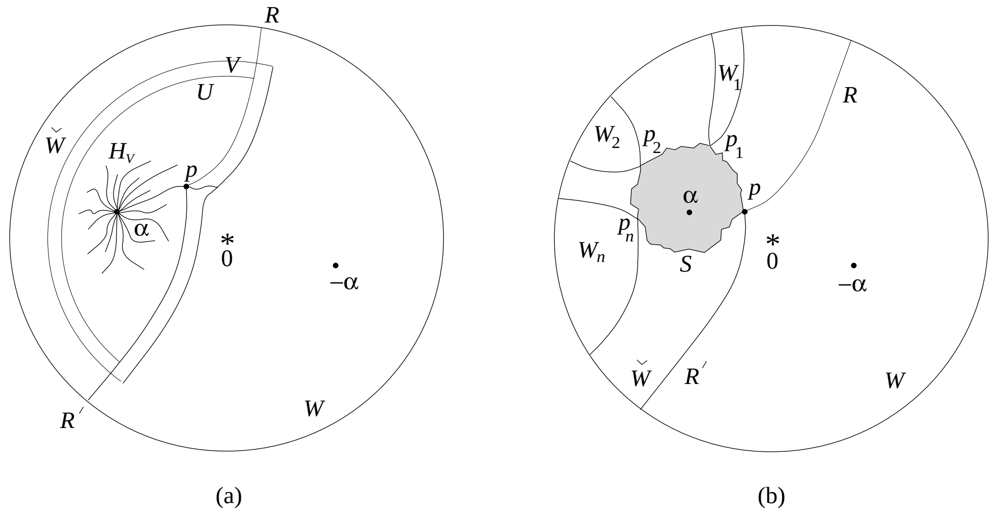

FIGURE 3

Now we consider a hedgehog H_U for the restriction $f|_U : U \to \mathbb{C}$ as given in section **§5**. Note that p is the only point of the Julia set on the boundary of U and that $H_U \subset J(f)$ has to intersect this boundary. Therefore, we simply have $p \in H_U$. Let us consider a slightly larger Jordan domain $V \supset \overline{U}$ with compact closure such that $f|_{\overline{V}}$ is still univalent. The hedgehog H_V for the restriction $f|_V : V \to \mathbb{C}$ has to contain p also and reach the boundary of V. Since p is biaccessible from outside of the Julia set, it follows that $H_V \setminus \{p\}$ is disconnected. Therefore, p is biaccessible from outside of H_V. This contradicts Proposition 1, and finishes the proof of the theorem in the Cremer case.

Let us now assume that we are in the Siegel case. If the orbit of z eventually hits the critical point 0, there is nothing to prove. Otherwise, since this orbit trivially cannot hit the fixed point $\alpha \in S$, we are again in the situation of Lemma 3. Therefore, there exists a ray pair (R, R') landing at a point $p \in J(f)$ which separates α from 0. In particular the critical point 0 is off the boundary ∂S of the Siegel disk. Then the same argument as in the Cremer case with an application of Proposition 1 shows that p must belong to ∂S.

As before, let W be the wake of the ray pair (R, R'), with root p. Then by construction W contains the critical point 0 while the co-wake $\check{W}$ contains the Siegel disk S and has its boundary touching $\overline{S}$ only at p. The point p is not periodic by Proposition 2. Hence the successive images $p_n = f^{\circ n}(p) \in \partial S$ are all contained in $\check{W}$ for $n \geq 1$. Therefore each wake W_n corresponding to the ray pair $(f^{\circ n}(R), f^{\circ n}(R'))$, with root point p_n, is also contained in $\check{W}$ (see Fig. 3(b)). In particular, none of these wakes contains the critical point. Hence $a(W_{n+1}) = 2a(W_n) < 1/2$ for all n by Lemma 2(c), which is clearly impossible. The contradiction shows that the orbit of z must eventually hit the critical point. $\square$

By Proposition 3, we have the following corollary:

Corollary 1. *Let f be a Siegel quadratic polynomial with $0 \notin \partial S$ and $\theta \in \mathcal{H}'$. Then there are no biaccessible points in $J(f)$ at all.*

By Lemma 2(b), every wake with angle 1/2 must have its root at the critical point 0. The converse is not true for arbitrary quadratic polynomials. For example, the real Feigenbaum map $z \mapsto z^2 - 1.401155\cdots$ has four distinct external rays landing on its critical point (compare with [**J-H**]). However, in the case of a Siegel quadratic polynomial, the critical point 0 is the landing point of *at most* one ray pair $(R_t, \tau(R_t))$ (In the Cremer case, there are no such ray pairs by Theorem 1(d)). This is nontrivial and follows from the statement that every Siegel or Cremer quadratic on the boundary of the main cardioid of the Mandelbrot set is the landing point of a unique parameter ray [**G-M**]. In fact, one can explicitly compute the angle of the candidate ray pair $(R_t, \tau(R_t))$ which may land at 0 by

$$t = \sum_{0<p/q<\theta} 2^{-(q+1)}.$$

It is interesting that the uniqueness of such t also follows from Theorem 3:

Corollary 2. *Let f be a Siegel quadratic polynomial as in (1). Then, no point in the Julia set $J(f)$ is the landing point of more than two rays. In particular, at most one ray pair lands at the critical point* 0.

Proof. By Theorem 3 it suffices to prove the corollary for the critical point. Suppose that there is a ray pair (R, R') which lands at 0 such that $R' \neq \tau(R)$. It follows that $(f(R), f(R'))$ is a ray pair which lands at the critical value c. By Theorem 3, the orbit of c must eventually hit the critical point 0. But this means that 0 is periodic, which is impossible. $\square$

Acknowledgements. This paper is an expanded version of a joint note with Dierk Schleicher which was written during my visit to the University of Münich in July 1997, and covered the Cremer case with a rather different proof [**S-Z**]. I would like to thank him for sharing his ideas which led to the joint work, which have been useful in the present paper as well. I am also grateful to Jack Milnor for his support and interest in this work. In particular, he suggested the idea of working with wakes, which simplified and unified part of the proof for both the Siegel and Cremer cases.

References

[**Do1**] A. Douady, *Systemes dynamiques holomorphes*, Asterisque, **105-106** (1983) 36-63.

[**Do2**] A. Douady, *Disques de Siegel at aneaux de Herman*, Seminar Bourbaki, Asterisque, **152-153** (1987) 151-172.

[**E-L**] A. Eremenko and M. Lyubich, *The dynamics of analytic transformations*, Leningrad Math. J., **1** (1990) 563-634.

[**G-M**] L. Goldberg and J. Milnor, *Fixed points of polynomial maps II*, Ann. Sci. Ec. Norm. Sup., **26** (1993) 51-98.

[**He1**] M. Herman, *Are there critical points on the boundaries of singular domains?*, Comm. Math. Phys., **99** (1985) 593-612.

[**He2**] M. Herman, *Conjugaison quasisymetrique des diffeomorphismes des cercle a des rotations et applications aux disques singuliers de Siegel*, Manuscript.

[**H-Y**] J. Hocking and G. Young, *Topology*, Addison Wesley, 1961.

[**J-H**] Y. Jiang and J. Hu, *The Julia set of the Feigenbaum quadratic polynomial is locally-connected*, Preprint, 1993.

[**Ki**] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, SUNY at Stony Brook IMS preprint 1996/2.

[**Ma**] R. Mañe, *On a lemma of Fatou*, Bol. Soc. Bras. Mat., **24** (1993) 1-12.

[**Mc**] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math Studies, vol. 135, 1994.

[**Mi1**] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, SUNY at Stony Brook IMS preprint 1990/5.

[**Mi2**] J. Milnor, *Periodic orbits, external rays, and the Mandelbrot set: An expository account*, see http://www.math.sunysb.edu/~jack.

[**Pe**] C. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Acta Math., **177** (1996) 163-224.

[**PM1**] R. Perez-Marco, *Fixed points and circle maps*, Prepublications Math. Orsay, 1994.

[**PM2**] R. Perez-Marco, *Topology of Julia sets and hedgehogs*, Prepublications Math. Orsay, 1994.

[**Po**] C. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, 1992.

[**Ru**] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.

[**S-Z**] D. Schleicher and S. Zakeri, *On biaccessible points in the Julia set of a Cremer quadratic polynomial*, SUNY at Stony Brook IMS preprint 1998/1.

[**Sø**] D. Sørensen, *Accumulation theorems of quadratic polynomials*, Ergod. Th. and Dynam. Sys., **16** (1996) 555-590.

[**Su**] D. Sullivan, *Conformal Dynamical Systems*, Springer Lecture Notes in Mathematics, **1007** (1983) 725-752.

[**Yo**] J.C. Yoccoz, *Petits Diviseurs en Dimension 1*, Asterisque **231**, 1995.

[**Za**] S. Zakeri, *Biaccessibility in quadratic Julia sets I: The locally-connected case*, SUNY at Stony Brook IMS preprint 1998/1.

Department of Mathematics, SUNY at Stony Brook, NY 11794

E-mail address: zakeri@math.sunysb.edu