

On Fibers and Local Connectivity of Compact Sets in \mathbb{C}

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Abstract

A frequent problem in holomorphic dynamics is to prove local connectivity of Julia sets and of many points of the Mandelbrot set; local connectivity has many interesting implications. The intention of this paper is to present a new point of view for this problem: we introduce *fibers* of these sets (Definition 2.3), and the goal becomes to show that fibers are “trivial”, i.e. they consist of single points. The idea is to show “shrinking of puzzle pieces” without using specific puzzles. This implies local connectivity at these points, but triviality of fibers is a somewhat stronger property than local connectivity. Local connectivity proofs in holomorphic dynamics often actually yield that fibers are trivial, and this extra knowledge is sometimes useful.

Since we believe that fibers may be useful in further situations, we discuss their properties for arbitrary compact connected and full sets in the complex plane. This allows to use them for connected filled-in Julia sets of polynomials, and we deduce for example that infinitely renormalizable polynomials of the form $z^d + c$ have the property that the impression of any dynamic ray at a rational angle is a single point. An appendix reviews known topological properties of compact, connected and full sets in the plane.

The definition of fibers grew out of a new brief proof that the Mandelbrot set is locally connected at every Misiurewicz point and at every point on the boundary of a hyperbolic component. This proof works also for “Multibrot sets”, which are the higher degree cousins of the Mandelbrot set. These sets are discussed in a self-contained sequel [S2]. Finally, we relate triviality of fibers to tuning and renormalization in [S3].

1 Introduction

A great deal of work in holomorphic dynamics has been done in recent years trying to prove local connectivity of Julia sets and of many points of the Mandelbrot set, notably by Yoccoz, Lyubich, Levin, van Strien, Petersen and others. One reason for this work is that the topology of Julia sets and the Mandelbrot set is completely described once local connectivity is known. Another reason is that local connectivity of the Mandelbrot set implies that hyperbolicity is dense in the space of quadratic polynomials, and that the dynamics can completely be classified by its combinatorics plus multipliers of attracting orbits.

In this paper, we introduce *fibers* of Mandelbrot and Julia sets and shift the focus from local connectivity to a closely related but somewhat stronger concept which we call triviality of fibers. It can be observed that people often prove that fibers are trivial when they only speak about local connectivity. However, triviality of fibers has quite a few useful properties: it allows to draw some conclusions which do not follow from local connectivity, and it makes several proofs more transparent. On the other hand, the concept of trivial fibers is not too restrictive: every compact connected and full subset of the complex plane which is locally connected has only trivial fibers for an appropriate choice of external rays used in the construction of fibers.

A fundamental construction in holomorphic dynamics is called the *puzzle*, introduced by Branner, Hubbard and Yoccoz. A typical proof of local connectivity consists in establishing *shrinking of puzzle pieces* around certain points. This is exactly the model for fibers: the fiber of a point is the collection of all points which will always be in the same puzzle piece, no matter how the puzzle was constructed. Our arguments will thus never use specific puzzles.

This paper is the first in a series: we introduce fibers and discuss their properties for arbitrary compact connected and full subsets of the complex plane. In particular, we explain the relation between triviality of fibers, local connectivity and landing properties of external rays (Section 2). It turns out that it is possible to construct certain bad subsets of \mathbb{C} for which fibers behave rather badly. However, we will give a criterion in Lemma 2.7 which will ensure that fibers are well-behaved, and this criterion will usually be satisfied in holomorphic dynamics.

In Section 3, we apply fibers to connected filled-in Julia sets of polynomials and show that they are generally quite well-behaved. As a new result, we show that many Julia sets have the property that all periodic external rays have impressions consisting only of their landing points. These Julia sets include infinitely renormalizable Julia sets of polynomials with a single critical point. We will need Thurston's No Wandering Triangles Theorem, which we cite here with his proof and permission.

The paper concludes with an appendix about compact connected full (and sometimes locally connected) subsets of the complex plane. Several well known results which are needed elsewhere in the paper are collected there, often with proofs included for easier reference.

In [S2], we will use fibers to give a new proof that the Mandelbrot set (and more generally Multibrot sets) have trivial fibers at Misiurewicz points and at all boundary points of hyperbolic components, including roots of primitive components. An imme-

diate corollary will be local connectivity at these points. Finally in [S3], we will discuss how triviality of fibers is related to renormalization and tuning: in parameter space, it is preserved under tuning, and any Julia set of the form $z \mapsto z^d + c$ has all its fibers trivial if and only if any of its renormalizations has this property; again, the same follows for local connectivity of these sets.

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2 Fibers and Local Connectivity

Our goal in this section is to introduce fibers of compact connected and full subsets of \mathbb{C} . Fibers will be the topological building blocks. We will discuss triviality of fibers and local connectivity as two closely related concepts which are the focus of interest of a lot of work, related for example to the Mandelbrot set.

Throughout this section, let K be a connected, compact and full subset of \mathbb{C} (“full” means that the complement has no bounded components). External rays of K are defined as inverse images of radial rays under the Riemann map sending the exterior of K to the exterior of the unit disk, normalized so as to fix ∞ with positive real derivative (in the special case that K has conformal radius one, this means that the Riemann map is tangent to the identity at ∞). When dealing with dynamic and parameter planes, we will speak of “dynamic rays” and “parameter rays” instead of external rays.

Definition 2.1 (Limit Set and Impression of External Ray)

We denote the external ray of K at angle ϑ by $R_K(\vartheta)$. Its limit set is $L_K(\vartheta) := \overline{R_K(\vartheta)} \cap K$. The impression of the ray is the set

$$I_K(\vartheta) := \bigcap_{\varepsilon > 0} \overline{\bigcup_{|\varphi - \vartheta| < \varepsilon} L_K(\varphi)} .$$

We say that the external ray at angle ϑ lands if its limit set is a single point.

Equivalently, limit set and impression can be defined as the sets of all possible limits

$$L_K(\vartheta) = \lim_{r \searrow 1} \Phi_K^{-1}(re^{2\pi i \vartheta}) \quad \text{and} \quad I_K(\vartheta) = \lim_{r \searrow 1, \varphi \rightarrow \vartheta} \Phi_K^{-1}(re^{2\pi i \varphi}) ,$$

where $\Phi_K: \overline{\mathbb{C}} - K \rightarrow \overline{\mathbb{C}} - \overline{\mathbb{D}}$ is the normalized Riemann map. The impression obviously contains the limit set, and both are compact, connected and non-empty. It may well happen that an external ray lands while its impression is a continuum. As usual,

we measure external angles in full turns so that they live in $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Let $Q \subset \mathbb{S}^1$ be any *countable* subset of angles such that all the external rays at angles in Q land. (One could allow larger sets Q , for example the set of all angles such that the corresponding rays land; by Fatou’s Theorem, this set has full measure in \mathbb{S}^1 . However, in all the applications we have in mind, the set Q will be countable anyway, and the countability hypothesis makes a few arguments more convenient; see also the remarks after Lemma 2.7.) In most cases, Q will be the set of rational angles, in particular when discussing Multibrot sets and *monic* polynomials (however, when there are Siegel disks, we need to enlarge Q). We will often loosely speak of an “external ray in Q ” when we mean an external ray such that its external angle is in Q , thus identifying rays with their angles.

The landing properties of external rays are studied by Carathéodory Theory, which investigates into which pieces the boundary of K can be cut by external rays landing there (see for example Milnor [M1, Sections 15 and 16]; recently, Petersen [Pt] has refined the study of this theory). We are going to do a related study here, but we will look at the set K from inside as well as from outside.

Definition 2.2 (Separation Line)

A separation line will be two external rays with angles in Q which land at a common point on ∂K , or two such rays which land at different points, together with a simple curve in the interior of K connecting the two landing points. Two points $z, z' \in K$ can be separated if there is a separation line γ avoiding z and z' such that these two points are in different connected components of $\mathbb{C} - \gamma$.

The separation line should also contain the landing points of the two rays. The curve in the interior of K must land at the same points as the two rays (where landing is understood in the same sense as for rays). Therefore, any separation line will cut the complex plane into two open parts. We will use these lines to define fibers of K and to construct connected neighborhoods of a point when proving local connectivity at this point.

When an interior component of K is equipped with an arbitrary base point, one might require the separation line within this component to be the union of two “internal rays”: since the interior component is simply connected, there is a Riemann map from the component to \mathbb{D} sending the base point to 0, and this map is unique up to rotation. Internal rays are then inverse images of radial lines, and by Lindelöf’s Theorem A.5 in the appendix, any point which is accessible by a curve is in fact the landing point of a ray. As far as the boundary of K is concerned, there is nothing lost in restricting to internal rays. We will not need to make this restriction.

Definition 2.3 (Fibers and Triviality)

For any point $z \in K$, consider the set of points in K which cannot be separated from z . In this set, the connected component containing z will be called the fiber of z . We say that it is trivial if it consists of the point z alone.

REMARK. Mandelbrot, Multibrot, and Julia sets are often studied with the help of a partition called a “Branner-Hubbard-Yoccoz jigsaw puzzle”, and a lot of work is devoted

to showing that “puzzle pieces shrink to points” (expressed by Douady as “points are points”). The idea behind fibers is to capture the essential properties of these puzzles without using any details about the exact construction of the puzzle. Douady’s joke “points are points” can then be replaced by the more precise (but dull) “fibers are points”.

For the Mandelbrot and Multibrot sets, fibers and their triviality are related to combinatorial classes and combinatorial rigidity. These differ exactly at hyperbolic components: entire hyperbolic components form combinatorial classes together with part of their boundaries, while we want to distinguish all their points; compare [S2]. For parameter spaces, another way of saying that the fiber of a point is trivial is that the space is “fiber rigid” at this point. In order to avoid overusing the word “rigidity”, we have decided not to use it for general sets K or for Julia sets and rather speak of “triviality of fibers”.

REMARK. For the given definition of fibers, it is possible to construct compact connected full sets $K \subset \mathbb{C}$ for which fibers behave badly. However, for the applications we have in mind, fibers usually have quite nice properties because we can choose Q so that the landing points of the selected rays have trivial fibers: see Lemma 2.7 and the remarks thereafter. We have not been successful in finding a satisfactory definition of fibers which has similar pleasant properties from the start for arbitrary sets K without becoming too complicated for the sets we are interested in. Most of the problems are related to interior components of K . If there is no interior, which is the case for many interesting Julia sets, the situation generally becomes quite a bit easier.

In the following paragraph, we will describe some “bad” possibilities of fibers for appropriately constructed sets K , in order to show what we need to have in mind in our proofs.

For certain points z , it may happen that the set of points in K which cannot be separated from z is disconnected. This occurs whenever there is a component U of the interior of K which has exactly two boundary points p, q each of which is accessible from within U and the landing point of a ray in Q (see Figure 2). Then every point in U can be separated from any other point in \bar{U} except p and q . With our definition, the fiber of every point in U is trivial, while the fibers of p and q contain all of \bar{U} . This same example also shows that the relation “ z_1 is in the fiber of z_2 ” need not be symmetric or transitive. Fibers of different points may also intersect without being equal: as an example, take the filled-in Julia set of $z^2 - 1$ (the “Basilica”) and let Q be the set of external angles of the form $a/(3 \cdot 2^k)$ for integers $k \geq 0$ and $a > 0$; these are exactly the angles of external rays landing together with another ray. Not allowing separation lines through the interior of K (or changing the topology so that curves in the interior of K cannot land at landing points of rays in Q), then the fiber of any interior point in K is the closure of its connected component of the interior of K , and two such components may have intersecting closures. — Fibers may thus have some rather unpleasant properties. One could try to remedy this by defining new fibers to be the smallest equivalence classes which are topologically closed and which contain entire fibers in the sense above. However, it would then be possible that some point could be separated from every other point without its fiber being trivial; in fact, the fiber could be all of K . As mentioned above, in our applications we will usually be able to

choose our rays in Q so that the fibers of their landing points are trivial, and all these problems disappear by Lemma 2.7.

We begin by collecting a couple of useful properties of fibers which are true in general.

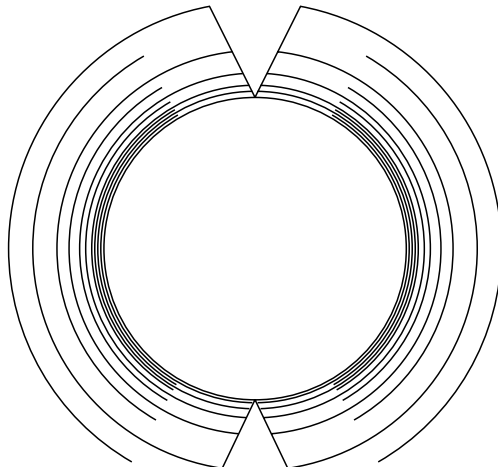


Figure 1: A compact connected and full set K such that an interior component has exactly two boundary points which are simultaneously accessible from inside and outside. Any separation line through the interior must pass both of these boundary points, so these two boundary points cannot be separated from any interior point.

Lemma 2.4 (Properties of Fibers)

Fibers have the following properties:

1. *Every fiber is compact, connected, and full.*
2. *Any connected component of the interior of K is either contained in a single fiber, or the fiber of each of its points is trivial; the latter happens if and only if at least two rays in Q land on the boundary of this connected component such that their landing points are accessible from the inside of the component.*
3. *Let z, z' be two points in K . If z' is not in the fiber of z , then z and z' can be separated using a separation line avoiding z and z' , except in the following situation: z is in the interior of K , the interior component of K containing z has exactly two boundary points which are landing points of rays in Q and can be connected by a curve in the interior of K , and both of these boundary points have non-trivial fibers.*
4. *Finally, if a fiber consists of more than a single point, then its boundary is contained in the boundary of K .*

PROOF. Any separation line avoiding z obviously separates an open subset of K from z , so the fiber of z is an intersection of closed sets and thus closed. It is connected by definition.

Let U be a connected component of the interior of K . If at least two rays with angles in Q land on the boundary of U such that there is a curve in U landing at the landing points of the rays, then every point in U can easily be separated from any other point in U , and the fiber of every point in U is trivial. Conversely, if not all of U is in the same fiber, there must be a curve in U connecting two landing points of rays in Q . But then the fiber of every point in U is trivial. (However, fibers of boundary points of U might not be trivial.)

Let z be an arbitrary point in K and assume first that the set of separation lines is countable. Let K_n be the closure of the connected component of K containing z in the complement of the first n separation lines avoiding z . It is compact and connected. It is also full because the complements of $\mathbb{C} - K_n$ are open and connected, and their unions are then also open and connected. The fiber of z is the nested intersection of all the K_n and thus full for the same reason.

Any point $z' \in K - K_n$ can then be separated from z by a separation line avoiding both z and z' , unless two separation lines meet more than once and together separate a point from z which is not separated by any single separation line. But this can happen only if z is in the interior of K and has a trivial fiber. Let U be the connected component of the interior of K containing z . If at least three boundary points of U are landing points of rays in Q and of curves from within U , then z can easily be separated from any point in $K - \{z\}$ by a separation line avoiding both points. If this does not happen, but the fiber of z is trivial, there must be exactly two such boundary points. But then the fibers of these boundary points must both contain all of U .

Since we have assumed the set Q of rays to be countable, the choice of the two external rays used for a separation line is also countable. Any two separation lines using the same two external rays must either coincide, or they must traverse the same interior component of K . Therefore, a countable collection of separation lines is always sufficient, and the proof above works in general. All fibers are thus full.

Now suppose that a point z is a boundary point of a fiber Y and an interior point of K . Then the connected component of the interior of K containing z must contain a non self-intersecting curve connecting the landing points of two rays in Q , and $Y = \{z\}$.

□

Lemma 2.5 (Impression is in Single Fiber)

For an external ray which lands (in particular for rays in Q), the impression is contained in the fiber of its landing point. For a ray which is not in Q (even if it does not land), the impression is contained in the fiber of any point in the impression.

PROOF. For a ray in Q , let z be its landing point; for a ray not in Q , let z be any point in the impression. Then $z \in \partial K$. We want to show that any point $z' \in K$ which is not in the fiber of z cannot be in the impression of the ray. But this is obvious because z and z' are separated by a separation line (Lemma 2.4), and no impression can extend over this separation. □

REMARK. It is not quite true that the impression is contained in the fiber of any point from the impression: if a ray in Q is part of a separating ray pair, then the impression

may extend over both sides of the separation line, while fibers of points from different sides cannot contain each other. However, the fiber of the landing point will still contain the entire impression.

Lemma 2.6 (Boundary Points are in Impression)

Every boundary point of K is in the impression of at least one external ray. If the fiber of a boundary point is trivial, then at least one external ray lands there.

PROOF. Let $z \in \partial K$ and let (z_n) be a sequence of points in $\mathbb{C} - K$ tending to z . The external angles of (z_n) must then have at least one limit $\vartheta \in \mathbb{S}^1$, so that z is in the impression of the ray at angle ϑ . If the fiber of z is trivial, then z can be separated from any $z' \in K$ and the impression of the ray is $\{z\}$, which implies in particular that the ray lands at z . \square

The next lemma shows that fibers behave particularly nicely if the rays in Q land at points with trivial fibers.

Lemma 2.7 (When Fibers Behave Nicely)

If the landing points of all the rays in Q have trivial fibers, then the fibers of any two points are either equal or disjoint, and the set K splits into fibers as equivalence classes of points with coinciding fibers. In that case, there is an obvious map from external angles to fibers of K via impressions of external rays. This map is surjective onto the set of fibers meeting ∂K

PROOF. The relation “ z_1 is in the fiber of z_2 ” is always reflexive. When the landing points of rays in Q have trivial fibers, then this relation is also symmetric by Lemma 2.4. In order to show transitivity, assume that two points z_1 and z_2 are both in the fiber of z_0 . If they are not in the fibers of each other, then the two points can be separated by a separation line avoiding z_1 and z_2 (Lemma 2.4). If such a separation line can avoid z_0 , then these two points cannot both be in the fiber of z_0 . The only separation between z_1 and z_2 therefore runs through the point z_0 , so z_0 cannot be in the interior of K and rays in Q land at z_0 . By assumption, the fiber of z_0 consists of z_0 alone. Any two points with intersecting fibers thus have indeed equal fibers. The map from external angles to fibers exists by Lemma 2.5. It is surjective by Lemma 2.6. \square

REMARK. The situation described in this lemma is what we want fibers to be: we want to speak of “fibers of K ” rather than having to specify which point of K any fiber is seen from. This is one reason not to make Q unnecessarily large, or it would be harder to establish this “nice” property. We will show in [S2] that the Mandelbrot and Multibrot sets have this property: this amounts to showing that they have trivial fibers at the boundary of hyperbolic components (including the roots of primitive components) and at Misiurewicz points. Also, most Julia sets have “nice” fibers (Section 3).

If the set K has trivial fibers at all the landing points of rays in Q , then it is not hard to show that the quotient of K by identifying points with coinciding fibers is a compact connected locally connected Hausdorff space (for the proof of local connectivity, see the proof of Proposition 2.9 below). In fact, the topological pair $(\overline{\mathbb{C}}, K)$ modulo this

equivalence relation is homeomorphic to the topological pair (\mathbb{S}^2, K') for a compact connected locally connected set K' : this is due to Moore's Theorem assuring exactly that (see Daverman [Da]).

Definition 2.8 (Local Connectivity)

A compact connected set $K \subset \mathbb{C}$ is called locally connected at a point $z \in K$ if every neighborhood of z contains a subneighborhood intersecting K in a connected set. If this subneighborhood can always be chosen open, then K is said to be openly locally connected at z . We say that K is locally connected if it is locally connected at each of its points.

REMARK. At a point z , open local connectivity is a strictly stronger condition than simply local connectivity. However, the entire set K is locally connected if and only if it is openly locally connected: see Milnor [M1, Section 16]. We will discuss important properties of locally connected sets in \mathbb{C} in the appendix.

The following proposition will be the motor for many proofs of local connectivity.

Proposition 2.9 (Trivial Fibers Yield Local Connectivity)

If a point of a compact connected full set $K \subset \mathbb{C}$ has a trivial fiber, then K is openly locally connected at this point. Moreover, if the external ray at angle ϑ lands at a point z with trivial fiber, then for any sequence of external angles converging to ϑ , the corresponding impressions converge to $\{z\}$. In particular, if all the fibers of K are trivial, then K is locally connected, all external rays land, and the landing points depend continuously on the angle.

PROOF. Consider a point $z \in K$ with trivial fiber. If z is in the interior of K , then K is trivially openly locally connected at z . Otherwise, let U be an open neighborhood of z . By Lemma 2.4, any point z' in $K - U$ can be separated from z such that the separation avoids z and z' . The region cut off from z is open; what is left is a neighborhood of z having connected intersection with K . By compactness of $K - U$, a finite number of such cuts suffices to remove every point outside U , leaving another neighborhood of z intersecting K in a connected set. Removing the finitely many cut boundaries, an open neighborhood remains, and K is openly locally connected at z . Similarly, if z is the landing point of the ϑ -ray, then external rays with angles sufficiently close to ϑ will have their entire impressions in \overline{U} (although the rays need not land). \square

REMARK. The last statements of the proposition are always equivalent by Carathéodory's Theorem A.2. This is another illustration of how closely fibers are related to Carathéodory theory.

The converse to Proposition 2.9 is not true: local connectivity at a certain point does not imply that the fiber of this point is trivial. A counterexample is given in Figure 2. However, if the set Q in the definition of fibers is sufficiently big, then local connectivity and triviality of fibers are equivalent for the entire set. We give a general proof here, to be used for Julia sets in Section 3; for the Multibrot sets, there will be a direct proof in [S2]. (The following more local version of this result seems plausible: whenever a point $z \in \partial K$ has a neighborhood in K such that K is locally connected in this entire neighborhood, then the fiber of z is trivial for an appropriate choice of Q .)

Proposition 2.10 (Local Connectivity Makes Fibers Trivial)

Let $K \subset \mathbb{C}$ be a compact connected full set which is locally connected. Suppose that Q is a dense subset of \mathbb{R}/\mathbb{Z} . Then all fibers of K are trivial provided that the following three conditions are satisfied:

1. whenever three external rays land at a common point, all their angles are in Q ;
2. if there exists an open interval I of angles such that all the rays with angles in I land at different points, and each of their landing points is also the landing point of some other ray, then there exist angles $\vartheta \in I \cap Q$ and $\vartheta' \in Q$ such that the corresponding rays land together.
3. if a point on the boundary of an interior component of K disconnects K , then all the external rays of this point are in Q .

REMARK. In our applications, the second condition is usually void because the landing points of the rays in I would define an embedded arc in K which contains no branches or decorations on at least one side. The only quadratic polynomial where this condition applies is $z^2 - 2$ for which the Julia set is an interval (for z^2 , where the Julia set is a circle, no two rays land together). In addition, the requirement in this condition is usually satisfied anyway: in general, there is a dense subset of external angles contained in Q such that the corresponding rays land together with another ray in Q .

In the third condition, the disconnecting boundary point is the landing point of at least two external rays by Lemma A.8.

PROOF. Since K is locally connected, it is pathwise connected by Lemma A.1. Consider a connected component of the interior of K and let Y be its closure. Then there is a dense subset of ∂Y (with respect to the topology of \mathbb{C}) consisting of points which are landing points of rays in Q : if U is an open set intersecting ∂Y , then it either contains a boundary point of Y disconnecting K (which is the landing point of a ray in Q by the third hypothesis), or density of Q supplies a landing point of a ray in Q within $\partial Y \cap U$. Since local connectivity of K is equivalent to local connectivity of ∂K (Carathéodory's Theorem A.2), every boundary point of Y is accessible from the inside of Y . By Lemma 2.4, the fiber of every point in the interior of Y is trivial, and every boundary point of Y can be separated from any other point in Y as well. Hence no fiber of any point contains more than a single point on the closure of any connected component of the interior.

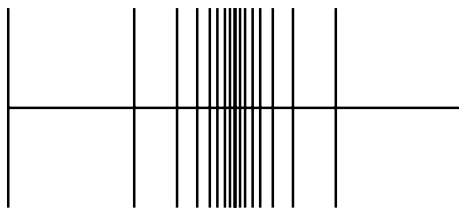


Figure 2: A compact connected full set which is not locally connected. It is locally connected at the center, but the fiber of the center contains a vertical line segment, no matter which rays are used to construct the fibers.

Suppose that there is a fiber which is not trivial and denote it Y . It has no interior, so we have $Y \subset \partial K$ by Lemma 2.4. Let $z_1 \neq z_2$ be two points in Y and let $\gamma \subset \partial K$ be a simple closed curve connecting them; such a curve exists by local connectivity of ∂K . We have $\gamma \subset Y$ because otherwise $\gamma \cup Y$ would enclose an open subset of \mathbb{C} and thus an interior component of K , and the fiber Y would meet more than a single point on the closure of this interior component. Any point z on the interior of γ is the landing point of at least two external rays, one from either side of γ : this is because the curve γ cuts every sufficiently small disk D around z in at least two parts, and both parts must intersect the exterior of K . No interior point of γ can be the landing point of three or more external rays because otherwise we could separate γ and thus Y . Let α be an external angle of z ; then rays at angles near α must land near z , and if they did not land on γ , then γ would have a branch point near z . Therefore, rays with angles sufficiently close to α land at interior points of γ , and by the second assumption, some of them must be in Q and landing together with another ray in Q . This ray must come in from the other side of γ , and we can separate Y again.

It follows that every fiber of a locally connected set K is trivial, provided that Q is sufficiently large so as to satisfy the stated conditions. \square

REMARK. For any compact connected and full $K \subset \mathbb{C}$ which is locally connected, there is always a countable collection Q of external angles for which all the fiber become trivial: The second condition requires only countably many rays. The first and third conditions specify countably many points: the number of branch points is countable by Lemma 2.11 below; similarly, the number of interior components is obviously countable, and each has at most countably many disconnecting boundary points by Corollary A.11. The problem is that some of these points might be the landing points of uncountably many external rays. Even in that case, the number of connected components any such point disconnects K into is countable by Lemma A.12, and countably many rays at every branch point suffice to separate any two of the connected components of the complement. The proposition remains true with these weakened hypotheses. However, in our applications only finitely many rays land at any single point and the given form of the proposition suffices.

Lemma 2.11 (Branch Points Countable)

For any compact connected and full subset K of \mathbb{C} and any $\varepsilon > 0$, the number of points which are the landing points of at least three external rays with mutual distance at least ε is finite and bounded above independently of K . In particular, the number of points which are the landing points of at least three rays is countable.

REMARK. The “distance between external angles” will be the distance between their external angles in \mathbb{R}/\mathbb{Z} , so that the maximal distance is $1/2$.

PROOF. We will follow a suggestion of Saeed Zakeri. Parametrize the boundary of \mathbb{D} by external angles in \mathbb{R}/\mathbb{Z} . When three external rays land at a common point, mark this by a Euclidean triangle in \mathbb{D} with vertices at the boundary points of \mathbb{D} corresponding to the external angles of the rays. Triples of rays landing at distinct points will then give rise to disjoint triangles. If all the angles of the triangle have mutual distance at

least $\varepsilon > 0$, then the Euclidean area of the triangle will be bounded below. Since the total area of the disk is finite, the number of such triangles is finite. The second claim follows. \square

3 Fibers of Filled-in Julia Sets

In this section, we will apply the general concept of fibers from Section 2 to connected filled-in Julia sets, where the set Q of external angles will always contain the rational numbers \mathbb{Q}/\mathbb{Z} and sometimes countably many further angles. We will always assume the set Q to be forward and backward invariant under multiplication by the degree, so that the set of corresponding dynamic rays is invariant under the dynamics. Several of the results in this section will be valid for arbitrary connected Julia sets of polynomials (which we will then always assume to be monic), while others are proved only for unicritical polynomials.

First we show that every bounded Fatou component has zero or infinitely many boundary points which are accessible from inside and outside, which makes the relation “is in the fiber of” symmetric. We will then discuss branch points of unicritical Julia sets: the analog to the Branch Theorem for the Multibrot sets [S2, Theorem 2.2] is Thurston’s No Wandering Triangles Theorem 3.3. We know that local connectivity and triviality of all fibers are equivalent for some choice Q of external angles. The No Wandering Triangles Theorem will allow to specify the set Q .

For the Multibrot sets, local connectivity implies that every connected component of the interior is hyperbolic; similarly, a corollary to Thurston’s theorem is that locally connected Julia sets of unicritical polynomials do not have wandering domains, i.e., all their Fatou components are eventually periodic. This result holds for arbitrary rational maps by Sullivan’s Theorem, and we will assume it throughout.

Finally, we will establish the “nice” situation of Lemma 2.7 for certain Julia sets: landing points of rational rays have trivial fibers, so the filled-in Julia sets split into equivalence classes of points with coinciding fibers. The analogous statement for the Multibrot sets is discussed in [S2].

Lemma 3.1 (Accessibility of Interior Components of Julia Sets)

Consider an arbitrary polynomial with connected filled-in Julia set. Every bounded Fatou component corresponding to an attracting or rationally indifferent periodic orbit has infinitely many boundary points which are landing points of dynamic rays at rational angles and which are also accessible from within the component.

Whenever any bounded Fatou component eventually lands on a periodic orbit of Siegel disks and has a single boundary point which is accessible from inside and which is also the landing point of an external dynamic ray, then every Fatou component on the same grand orbit has countably many such boundary points. This always happens when the Julia set is locally connected. However, the corresponding external angles are in no case rational.

PROOF. Denote the filled-in Julia set by K and consider a bounded periodic Fatou component. If this Fatou component belongs to an attracting or rationally indifferent

orbit, there is at least one boundary point which is fixed under the first return map of the Fatou component. This point must be repelling or rationally indifferent. It is thus the landing point of at least one rational dynamic ray, and it is accessible from within its Fatou component. Every Fatou component which eventually maps onto U then inherits countably many points on its boundary which are all accessible from inside and which are landing points of rational dynamic rays.

The only further type of Fatou components of polynomials are Siegel disks and their preimages. If a boundary point z of a Siegel disk is accessible both from inside and outside, it is the landing point of an external and of an internal ray (by Lindelöf's Theorem A.5; we define internal rays with respect to the periodic point at the center as the base point). Since the dynamics on internal rays is an irrational rotation, the point z cannot be periodic: otherwise, it would be the landing point of two (and even countably many) internal rays, and the region between them would have to be contained entirely within the Siegel disk because the filled-in Julia set is full and the boundary of the Siegel disk is contained in the boundary of the filled-in Julia set. But then an open interval of internal angles would have to land at the same point, which is a contradiction to the theorem of the Riesz brothers [M1, Theorem A.3]. Any boundary point accessible from inside and outside thus gives rise to countably many such points, and their external angles are all irrational. Again, every Fatou component which eventually maps onto this Siegel disk inherits countably many boundary points with the specified property.

If the filled-in Julia set is locally connected, then there are many such boundary points: any boundary point of the Siegel disk which is accessible from inside will do the job, and these are dense (in fact, by Lemma A.3, the boundary of the Siegel disk itself is locally connected, and each of its boundary points is accessible from inside and outside). \square

The following corollary shows that the relation “is in the fiber of” is symmetric for arbitrary connected Julia sets.

Corollary 3.2 (Fibers are Symmetric)

Consider an arbitrary polynomial with connected filled-in Julia set K . Define fibers of K using an arbitrary choice of the set Q of external angles which is forward and backward invariant (subject to the usual two conditions that Q be countable and that all rays with angles in Q actually land). Let $z, z' \in K$ be two points such that z' is not in the fiber of z . Then there is a separation line separating z and z' which avoids these two points, and z is not in the fiber of z' .

PROOF. By Lemma 2.4, the claim can fail only if z is in the interior of K and the connected component of the interior of K which contains z has exactly two boundary points which are accessible from inside and outside. But every Fatou component will eventually map onto a periodic Fatou component corresponding to an attracting or rationally indifferent periodic point or onto a Siegel disk. For those Fatou components, the number of boundary points which are accessible from inside and outside is either zero or infinite. \square

The principal goal in this section is to specify a choice Q of external angles for which the fibers of a locally connected unicritical Julia set are trivial. We have to check

three conditions in Proposition 2.10: the first one is easy to satisfy and the second one is usually void. For the third condition, we need a theorem due to Thurston [T, Theorem II.5.2] which is still unpublished. It is the dynamic analog to the Branch Theorem [S2, Theorem 2.2] for the Multibrot sets, stating that branch points have rational external angles. Thurston states his theorem only for quadratic polynomials, but his proof works for all unicritical polynomials. With his permission, we give his proof here. It is slightly modified using an idea of Saeed Zakeri.

Theorem 3.3 (No Wandering Triangles)

If three dynamic rays of a unicritical polynomial with connected Julia set land at a common point, then the landing point is either periodic or preperiodic, or it eventually maps through a critical point.

REMARK. If the landing point is on a repelling or rationally indifferent orbit, then the rays are all periodic or all preperiodic and have thus rational angles. The only other conceivable case is that the landing point is a Cremer point and all the rays landing there are irrational. As far as I know, it is not known whether that can possibly happen.

Thurston proves his theorem in an abstract setting using “laminations”, related to the pinched disk model of the Julia set. That way, he does not have to worry whether certain dynamic rays land at all. We will use the theorem only for Julia sets which are locally connected, so all dynamic rays land and there is no Cremer point.

PROOF. External angles are parametrized by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$; identify this set with $\partial\mathbb{D}$. Assume that three dynamic rays at angles $\vartheta_1, \vartheta_2, \vartheta_3$ land at a common point. If the theorem is false, then the forward orbit never repeats and never maps through the critical point. We will suppose that in the following. The three angles are necessarily irrational and will remain distinct under forward iteration. For every $k \geq 0$, the dynamic rays at angles $d^k\vartheta_1, d^k\vartheta_2, d^k\vartheta_3$ also land at a common point.

On $\partial\mathbb{D}$, connect the three points ϑ_i pairwise by Euclidean straight lines, yielding a Euclidean triangle in \mathbb{D} which represents the landing point of these three rays: every side of the triangle stands for a ray pair. Since ray pairs landing at different points do not cross and all the landing points are different, we obtain an infinite sequence of disjoint image triangles connecting the angles $d^k\vartheta_1, d^k\vartheta_2, d^k\vartheta_3$: a wandering triangle.

Because of the d -fold rotation symmetry of the Julia sets, every triangle has $d - 1$ rotated counterparts, and adding these in still leaves the triangles non-intersecting: each of these extra triangles corresponds to the landing point of three rays which maps in one step onto the orbit of the initial triangle. This is where we are using the assumption that the polynomials are unicritical.

We will measure the lengths of a triangle side (i.e., of a ray pair) as usual as the unsigned distance along \mathbb{S}^1 between the corresponding angles. The maximum distance between any two points is therefore $1/2$, realized for points straight across. In fact, because of the rotation symmetry and since triangle sides never cross, no side can have length $1/d$ or more (except for rays landing at the critical point, which is the center of symmetry; in that case, we discard the initial triangle and consider only the remaining orbit). If a side of a triangle has length $s < 1/d$, then after multiplication by d , the image side will have length $\min\{ds, 1 - ds\}$ (measuring the short way around the circle),

so that sides with lengths less than $1/(d+1)$ will be mapped to longer sides, while those with lengths greater than $1/(d+1)$ will shrink in length. Short sides of length ε are images of sides of length ε/d or of length $1/d - \varepsilon/d$, so they are images of very short or of very long sides.

By Lemma 2.11, there can be only finitely many points which are landing points of three dynamic rays with mutual distance at least ε , for any $\varepsilon > 0$. Therefore, if there is a wandering triangle, then the lengths of the respective shortest sides must converge to zero. It follows that there can be no upper bound less than $1/d$ for the lengths of sides because a new shortest side can be the image only of a very long side. Therefore, there exists a sequence k_1, k_2, \dots of iteration steps such that the longest side of the k_1 -th image of the wandering triangle has length $l_1 > 1/(d+1)$ and the image after k_{i+1} steps has a longest side of length $l_{i+1} > l_i$. Denote the respective triangles by T_i and denote the lengths of its other two sides by l'_i and l''_i such that $l_i \geq l'_i \geq l''_i$. We want this sequence to be maximal in the following sense: the first image of T_i with a side of length exceeding l_i is already T_{i+1} .

The side of T_i with length l_i and its $d-1$ symmetric rotates cut the disk into $d+1$ pieces, of which one contains the origin and is rotation symmetric. Denote this piece by C_i . Since $l_{i+1} > l_i > 1/(d+1)$, the side with length l_{i+1} must be contained in C_i , together with the triangle T_{i+1} it belongs to. Therefore, we also have $l'_{i+1} > l_i$: two sides of a new triangle will be longer than the longest side of an old triangle. It follows that $l_{i+1} > l'_{i+1} > l''_{i+1}$ with strict inequality; this holds for every i .

We claim that the two long sides of any triangle T_i will, after $k_{i+1} - k_i$ iterations, map onto the two long sides of T_{i+1} . Indeed, the shortest side of T_{i+1} has length less than $1/d - l_i$ because this is the length of the intervals in which C_i meets \mathbb{S}^1 . However, the image of the longest side of T_i has length $1 - dl_i = d(1/d - l_i)$, so it is already too long for the shortest side of T_{i+1} ; the image of the middle side of T_i is even longer. If the two long sides of T_i want to become shorter, they must first be longer. The first time that this happens they are on the triangle T_{i+1} , proving the claim.

Perhaps not unexpectedly, we obtain a contradiction by looking at the orbit of the shortest sides, which must always map to the shortest sides. No matter how short it started, it will eventually have length at least $1/(d+1)$ and might then get shorter. But in order to map to the shortest side of a triangle T_{i+1} , it must have been very short in T_i or longer than l_i . The second option is clearly impossible, and the first can happen only a finite number of times. To acquire a new shortest length, it must have been very long before, and that happens only at the T_i . Here is the contradiction. \square

REMARK. This theorem has recently been generalized by Kiwi [Ki] to arbitrary polynomials with connected Julia sets: he has a “No Wandering Polygon” Theorem, but the number of sides of his polygons depends on the degree.

Lemma 3.4 (Dynamics of Fibers)

For any polynomial with connected Julia set and any choice of the set Q which is forward and backward invariant, the dynamics maps the fiber of any point as a possibly branched cover onto the fiber of the image point.

PROOF. Let K be the filled-in Julia set of a polynomial p having degree d . We know

from Corollary 3.2 that whenever one point is not in the fiber of another, then these two points can be separated by a separation line avoiding both points.

Choose a point $z \in K$, let $z' := p(z)$, let Y' be the fiber of z' and let Y_0 be the connected component of $p^{-1}(Y')$ containing z . Since $p: \mathbb{C} \rightarrow \mathbb{C}$ is a branched covering, its restriction to Y_0 is also a branched covering. Denote the fiber of z by Y . We will first show that $Y \subset Y_0$.

Choose an arbitrary point $z'_1 \in K - Y'$. Then there is a separation line separating z' and z . If this separation line does not contain a critical value, then its pull-back under p will be d separation lines, and every inverse image of z'_1 can be separated from z by one of them. If the separation line does contain a critical value, then it is still possible to separate every inverse image of z'_1 from z by a separation line made up of parts of the inverse images of the given separation line. Therefore, $Y \subset Y_0$.

If already $Y = Y_0$, then we are done. If not, let z_1 be a point in $Y_0 - Y$ and consider a separation line γ between z and z_1 . If it is a ray pair which does not land at a critical point of p , then the image of γ is another separation line. Since γ runs through Y_0 and separates it, its image will run through Y and separate it. This is impossible. If γ is a separation line running through an interior component U of K and the images of its two dynamic rays are different, then the image of γ is again a separation line, possibly after modifying it within $p(U)$ so that the new separation traverses $p(U)$ in a simple curve. All the fibers of points in U and $p(U)$ will then be trivial. Since γ disconnects Y_0 , which is connected, the landing point of at least one of the two dynamic rays in γ will have a neighborhood in Y_0 which is disconnected by γ . The new separation line will then separate Y' at the image point, which is again impossible.

Therefore, if $Y \neq Y_0$, then any separation line γ which separates Y_0 has the property that its two dynamic rays have the same image rays, or it is a ray pair landing at a critical point. If γ runs through an interior component U of K , then there are countably many further dynamic rays landing at U which are accessible from inside, and it is easy to manufacture a new separation line which still separates Y_0 but which will not collapse when mapped forward, so the argument above applies: an impossibility again. The last case is that γ is a ray pair landing at a critical point. Removing from Y_0 the part which is separated from z , it is easy to check that p induces a covering from the rest onto Y : the only place where we have to check this is at the landing point of the ray pair, and there is no problem. Since there are only finitely many critical points, and these have only finitely many rays landing, there are only finitely many such ray pairs. After finitely many cuts in Y_0 , we obtain the fiber Y , and $p: Y \rightarrow Y'$ is a branched covering. If there are branch points at all, these are critical points of p . \square

We will now show that, at least for many Julia sets, the landing points of rational rays have trivial fibers. The corresponding statement for Multibrot sets can be found in [S2].

Theorem 3.5 (Repelling Periodic Points Have Trivial Fibers)

Consider a polynomial with connected filled-in Julia set and define its fibers for $Q = \mathbb{Q}/\mathbb{Z}$, together with the grand orbits of all the rays landing at those critical values which are on the boundary of periodic Siegel disks (if any). Let z be a repelling periodic or preperiodic point and suppose that all the points on its forward orbit can be separated

from all the critical values and from all the points on closures of periodic bounded Fatou components. Then the fiber of z is trivial.

PROOF. By Lemma 3.4, the fiber of any point is trivial whenever it every maps to a point with trivial fiber. Therefore, we may assume that z is periodic. By switching to an iterate, we may assume z to be a fixed point. Denote the corresponding (iterated) polynomial by p and let K be its filled-in Julia set.

Every point on the closure of a periodic bounded Fatou component can be separated from z by a separation line. Since every such line separates from z an open subset of the closure of this periodic Fatou component, a finite number of separation lines suffices to separate the entire closure of this Fatou component (in fact, a single line will do the job). The total number of periodic Fatou components is finite, so there is a finite number of separation lines separating z from all the bounded periodic Fatou components and from all the critical values. Denote this collection of ray pairs by S_0 and let U_0 be the neighborhood of z which is not separated from z by separation lines in S_0 .

Consider all the separation lines in S_0 which are not ray pairs. They will then traverse bounded Fatou components, so all but finitely many of their images under forward iteration will intersect bounded periodic Fatou components. Therefore, only finitely many of these forward images can intersect and cut U_0 , and none of them can meet z . A similar argument applies to those ray pairs in S_0 which have irrational external angles, so they necessarily land on the boundary of periodic or preperiodic Siegel disks. Let U_1 be the connected component of z in U_0 minus these finitely many separation lines.

Now we look at separation lines bounding U_1 which are ray pairs at rational angles. Their landing points are periodic or preperiodic. Then all these separating ray pairs have finite forward orbits. Consider all the finitely many ray pairs on these forward orbits, except those landing at z . They might possibly disconnect U_1 . Let U_2 be the connected component of z in U_1 minus these finitely many ray pairs. Consider an arbitrary equipotential of K and let U be the subset of U_2 within this equipotential. Then p restricted to U is a conformal isomorphism onto its image, and p cannot send boundary points of U into the interior of U .

Since U is full and contains no critical point, the branch of p^{-1} fixing z can be extended throughout U . All the ray pairs and separation lines bounding U are mapped into U or to its boundary: if they are mapped outside of U , then a separation line on the boundary of U is inside $p^{-1}(U)$, and mapping $p^{-1}(U)$ forward under p sends a bounding ray pair into U , which we had excluded above. Since the equipotential bounding U is mapped to a lower equipotential under p^{-1} , the branch of p^{-1} fixing z maps U into itself.

Therefore, the restriction of p^{-1} to U is a holomorphic self-map of U with an attracting fixed point at z . Each of the finitely many separation lines bounding U is either mapped eventually into U , or it is periodic. The latter case is impossible because the separation line would necessarily have to be a ray pair at rational angles, all parabolic periodic points are separated from z by assumption, and repelling periodic points would have to attract nearby points under iteration of $p^{-1}(U)$, while the interior of U has to converge to z by Schwarz' Lemma.

Therefore, all of \overline{U} converges to z under iteration of $p^{-1}(U)$. For every $\varepsilon > 0$, there is an n such that $p^{o(-n)}(\overline{U})$ is contained in the ε -neighborhood of z . But that means that no point $z' \in K$ with $|z' - z| > \varepsilon$ can be in the fiber of z . Since ε was arbitrary, the fiber of z is trivial. \square

REMARK. It is important to require that z can be separated from closures of periodic Siegel disks. The separation from other periodic bounded Fatou components (attracting or parabolic) is for convenience and does not seem essential. Similarly, a related proof will probably transfer the proof from repelling to parabolic periodic points. For unicritical polynomials, the presence of attracting or parabolic orbits makes all the fibers of the Julia set trivial anyway.

From now on, we will restrict to filled-in Julia sets of unicritical polynomials. For these, we can now specify a set Q for which triviality of all fibers is equivalent to local connectivity of the Julia set. We already know from Proposition 2.9 that triviality of fibers implies local connectivity, so we only state the converse.

Proposition 3.6 (Locally Connected Julia Sets have Trivial Fibers)

If the filled-in Julia set of a unicritical polynomial is locally connected, then all its fibers are trivial for the choice $Q = \mathbb{Q}/\mathbb{Z}$ unless there is a Siegel disk; in that case, all fibers are trivial when $Q = \mathbb{Q}/\mathbb{Z}$ together with the grand orbits of the angles of all the rays landing at the critical value.

REMARK. A locally connected Julia set of a polynomial can never have a Cremer point; see Milnor [M1, Corollary 18.6]. In the case of a Siegel disk, all the rays we really need are the rays in \mathbb{Q}/\mathbb{Z} and those landing at the critical point and on its backwards orbit; the extra rays are just taken in to have invariance of the rays in Q under the dynamics. We will see below that a single ray lands at the critical value and at every point of its forward orbit. The separation lines through periodic Siegel disks which we can obtain from such rays can be replaced by lines through precritical points.

PROOF. The No Wandering Triangles Theorem implies that three or more rays landing at a common point either have rational angles, or the landing point eventually maps through the critical point.

First we discuss the case that the filled-in Julia set has no interior. Being locally connected, it is a *dendrite*: any pair of points can be connected by a unique arc within the Julia set (Lemma A.1). Separation lines are just ray pairs at rational angles.

The critical point cuts the Julia set into two parts, to be labelled 0 and 1, and this partition defines a symbolic itinerary for any point which is not a pre-critical point. The subset of the Julia set with identical first k entries in the itinerary is connected, and no two points have identical itineraries forever (otherwise, an entire interval of external angles would have to have the same itinerary). Therefore, precritical points are dense on any subarc of the Julia set.

Within the dendrite Julia set, the critical orbit spans an invariant subtree (a post-critically infinite Hubbard tree), and the critical value is an endpoint of this tree. It follows that the critical point cannot be a branch point of the Hubbard tree, so all its branch points are periodic or preperiodic.

The critical value is a limit point of periodic points in the tree: if z_n is a precritical point on the tree such that the interval between z_n and c contains no point which maps before z_n onto the critical point, then there is a homeomorphic forward image of the interval $[z_n, c]$ which maps z_n onto c , producing a periodic point on this interval. By density of precritical points, periodic points are dense on every subarc of the Julia set. It follows that any two given points in the Julia set can be separated by a periodic point, which is necessarily repelling, and the rays landing at this periodic point separate the two given points. Therefore, all fibers are trivial, even when Q only contains periodic angles.

We now consider the case that the filled-in Julia set has interior. We will prove the result by checking the conditions in Proposition 2.10.

If the bounded Fatou components correspond to an attracting or parabolic orbit, then the critical orbit is in the Fatou set and $Q = \mathbb{Q}/\mathbb{Z}$; otherwise, we have a Siegel disk and the critical point is in the Julia set. In that case, Q contains countably many further rays. In both cases, all the external angles of branch points are in Q by the No Wandering Triangles Theorem, and the first condition of Proposition 2.10 is always satisfied. Moreover, the number of rays landing at any given point is well known to be finite.

If there is an open interval of external angles of length less than $1/d$ such that all the corresponding dynamic rays land at different points, then multiplication by d yields another longer interval with the same property. Restricting to a subinterval of length $1/d^n$ for an appropriate integer n and iterating this argument, it follows that all dynamic rays land at different points. The second condition is thus always void.

We are assuming that there is a periodic cycle of bounded Fatou components. Let U be one such component and let z_1 be a boundary point of U which disconnects the filled-in Julia set. Then at least two dynamic rays land at z_1 by Lemma A.8, but the total number of rays at z_1 is always finite. Let ϑ_1 and ϑ'_1 be the angles of two rays landing at z_1 so that they separate as much as possible from U . Denote the period of U by n . Iterating the n -th iterate of the polynomial, we obtain a sequence z_2, z_3, \dots of boundary points of U and two sequences of dynamic rays at angles $\vartheta_2, \vartheta_3, \dots$ and $\vartheta'_2, \vartheta'_3, \dots$.

Each ray pair $(\vartheta_k, \vartheta'_k)$ cuts away an open interval of external angles from U , so that the projection (as defined after Lemma A.10) of external rays within such an interval yields the point z_k . If all the points z_k are different and all $\vartheta_k \neq \vartheta'_k$, then they will cut away infinitely many intervals which must all be disjoint. Therefore, their lengths must shrink to zero. However, when such intervals are short, then their lengths are multiplied by the degree d of the polynomial in every step and by d^n under the first return map of U , so there will always be intervals with lengths bounded below. This is a contradiction.

Therefore, either the point z_1 is periodic or preperiodic and then its external angles are rational, or it is in the backwards orbit of the critical point. In both cases, its external angles are in Q , satisfying the third condition of Proposition 2.10 and finishing the proof also in the case when there are bounded Fatou components. \square

REMARK. The second proof given also applies when there are no bounded Fatou

components, but it gives a weaker result because it specifies a larger choice of Q .

We now state some observations which came out of the proof. They are all known.

Corollary 3.7 (Disconnecting Boundary Points of Fatou Components)

Let z be a boundary point of a bounded Fatou component of a unicritical polynomial and assume that it disconnects the Julia set. If the Fatou component corresponds to an attracting or parabolic periodic orbit (in which case the Julia set is known to be locally connected), then z is a periodic or preperiodic point. If the Fatou component corresponds to a Siegel disk, and we assume the Julia set to be locally connected, then z will eventually map to the critical point. In particular, the critical point of a unicritical polynomial with a locally connected Julia set featuring a Siegel disk is on the boundary of one of the periodic components of the Siegel disk, and the critical value is the landing point of a unique dynamic ray. □

Lemma 3.8 (Critical Point in Periodic Fiber)

Consider a unicritical polynomial and set $Q = \mathbb{R}/\mathbb{Z}$. If the fiber containing the critical point is periodic of some period n , then the polynomial is n -renormalizable and the critical fiber contains an indifferent or superattracting periodic point of period n .

PROOF. The statement is void or trivial if the Julia set is locally connected, so we can in particular exclude hyperbolic or parabolic Julia sets. For other parameters on the closure of the main hyperbolic component of a Multibrot set, no two rational dynamic rays land together, and the entire Julia set is a single fiber. The claim holds trivially for $n = 1$. Otherwise, there is a unique repelling fixed point which is the landing point of at least two dynamic rays. Denote this fixed point by α . The rays landing at α separate the critical point from the critical value. If the critical fiber is periodic, its period must be at least two.

Let $Q' \subset Q$ be the union of the rays at α together with their entire backwards orbits. These are the rays usually used in the construction of the Yoccoz puzzle. The critical fiber corresponding to these rays will still be periodic of some period n' dividing n , again with $n' \geq 2$. It is quite easy to see and well known that the polynomial is now n' -renormalizable (see e.g. Milnor [M2, Lemma 2]). After n' -renormalization, we have a new unicritical polynomial with equal degree, and the critical fiber is still periodic of period n/n' . If we are now on the closure of the main hyperbolic component of the Multibrot set, the entire Julia set is a single fiber, the critical fiber has period 1 and contains a non-repelling fixed point, and if it is attracting, then all fibers are trivial. It follows that $n = n'$. For the original polynomial, the critical fiber must contain an indifferent or superattracting periodic point of period dividing n .

If the renormalized polynomial is not on the closure of the main hyperbolic component, then the period of the critical fiber is again at least 2, and we can repeat the argument. Since the period of the critical fiber is reduced in every step, we must land after finitely many steps on the closure of the main hyperbolic component. □

Corollary 3.9 (Impressions of Rational Dynamic Rays)

For any unicritical polynomial in \mathbf{M}_d without indifferent periodic points, the impression of any dynamic ray at a rational angle is always a single point. Fibers of any two points (for $Q = \mathbb{Q}/\mathbb{Z}$) are either disjoint or equal and have the “nice” property of Lemma 2.7.

PROOF. If the critical fiber is periodic, then there is either a superattracting or an indifferent orbit by Lemma 3.8. The indifferent case is excluded. If there is a superattracting orbit, or if the critical fiber is not periodic, then every repelling periodic point can be separated from the critical value. By Theorem 3.5, the fibers of repelling periodic points are trivial, and they contain the entire impressions of all the rays landing there by Lemma 2.5.

This is obvious since rational dynamic rays always land, and the impression of any ray is contained in the fiber of its landing point by Lemma 2.5. This establishes the “nice” situation of Lemma 2.7, and the Julia set splits into equivalence classes of points having intersecting and thus identical fibers. \square

REMARK. For non-infinitely renormalizable quadratic polynomials, this is a special case of a theorem of Yoccoz [H, Theorem II]. Very recently, J. Kiwi [Ki] has independently proved this theorem for arbitrary polynomials with connected Julia sets and with all periodic points repelling.

A Compact Connected Full Sets in the Plane

In this appendix, we will discuss compact connected full (and sometimes locally connected) subsets in \mathbb{C} and describe certain properties which we will need in the main text. Local connectivity has been defined in Definition 2.8.

Of principal importance is that local connectivity implies pathwise connectivity, i.e., any two points can be connected by a continuous image of an interval. In fact, we can connect them by a homeomorphic image of an interval, a property known as arcwise connectivity.

Lemma A.1 (Local Connectivity Implies Arcwise Connectivity)

Every compact connected and locally connected subset of \mathbb{C} is arcwise connected and locally arcwise connected. \square

For a proof, see Douady and Hubbard [DH1, Exposé II], or Milnor [M1, Section 16].

Another important result is Carathéodory’s Theorem, which is also described in [DH1] and [M1].

Theorem A.2 (Carathéodory’s Theorem)

Let K be a compact connected and full subset of \mathbb{C} . Then K is locally connected if and only if ∂K is locally connected, or if and only if all the external rays of K land with the landing points depending continuously on the external angles. In that case, every boundary point is the landing point of at least one external angle. \square

If K is locally connected, then the map from external angles to the corresponding landing points is known as the *Carathéodory loop* of K , and it is surjective onto ∂K .

Lemma A.3 (Interior Component Locally Connected)

Consider a compact, connected and full subset of \mathbb{C} which is locally connected. Then any connected component of the interior has locally connected boundary.

PROOF. Denote the original locally connected set by K , let U_0 be an interior component of K and let K_0 be the closure of U_0 . Let $z \in \partial K_0$ and let U be an open neighborhood of K_0 . By local connectivity of K , there is a neighborhood $V \subset U$ of z such that $V \cap K$ is connected. We claim that $V \cap K_0$ is connected.

Suppose not. Then let K_1 and K_2 be two connected components of $V \cap K_0$. Since V is open, both K_1 and K_2 contain interior points of K_0 , and there is a curve γ in the interior of K_0 connecting K_1 and K_2 . Obviously, this curve cannot be contained entirely within V . Since it connects two points in $V \cap K$, the set $V \cap K$ and γ together disconnect $\mathbb{C} - (V \cap K)$. Let W be open subset of $\mathbb{C} - (V \cap K)$ which is disconnected from ∞ by the curve γ . Since $\partial W \subset K$ and K is full, we have $W \subset K$. And since the interior component U_0 intersects W , it follows $W \subset U_0$. The entire boundary of W , except the part on γ , is contained in V by construction, and it is also contained in K_0 . Therefore, K_1 and K_2 are connected within $V \cap K_0$, contrary to our assumption. \square

REMARK. It seems plausible that a subset K_0 of a compact connected full and locally connected set $K \subset \mathbb{C}$ is always locally connected whenever it contains any interior component which it meets.

We can now apply Carathéodory's Theorem to any connected component of the interior of K : its closure is locally connected by Lemma A.3 above, so its boundary is locally connected by Carathéodory's Theorem, and we then have "internal rays" with respect to any base point in the interior: since this interior component must be simply connected, it has a Riemann map to \mathbb{D} sending the base point to the origin, and the inverse of radial lines under this Riemann map will be internal rays. Carathéodory's Theorem then says that all internal rays land, and the landing points depend continuously on the angle. Since no two internal rays can land at the same point (because the closure of the interior component must be full), the landing points of the rays induce a homeomorphism between \mathbb{S}^1 and the boundary of the interior component.

In the remainder of this section, we will consider a fixed compact connected and full set $K \subset \mathbb{C}$. For the moment, we do not require it to be locally connected, but we will later add this hypothesis.

Lemma A.4 (Rays Landing at Common Point)

If two external rays land at a common point, then they separate K .

PROOF. The angles of the rays cut \mathbb{S}^1 into two parts. If the rays do not separate K , then all the external rays with angles in one of the two parts of \mathbb{S}^1 must land at z . But the set of external angles corresponding to the same landing point always has measure zero by the Theorem of Riesz [M1, Theorem 15.3 or A.3]. \square

The following result will be important for us at several places. Its proof can be found, for example, in Ahlfors [Ah, Theorem 3.5].

Theorem A.5 (Lindelöf's Theorem)

If there is a curve $\gamma : [0, 1) \rightarrow \mathbb{C} - K$ which converges to a point $z \in \partial K$, then there is a unique external ray of K which lands at z and which is homotopic to γ in $\mathbb{C} - K$. \square

Definition A.6 (Access to Boundary Point)

Let $z \in \partial K$. An access of z is a choice, for every Euclidean disk D_r of radius r around z , of a connected component V_r of $D_r - K$, such that $V_r \subset V_s$ whenever $r < s$.

REMARK. It is not true that, for $r < s$, we must have $V_r = V_s \cap D_r$: there might be a connected component of $K - \{z\}$ which separates $V_s \cap D_r$, and V_r is one of its parts. The following lemma justifies the term ‘‘access’’.

Lemma A.7 (Ray in Access)

For every access of z , there is a curve in $\mathbb{C} - K$ landing at z running entirely through the domains V_r in the definition of the access, and visiting all of them. Two such curves can never separate K . Exactly one of these curves is an external ray.

PROOF. For positive integers k and $r_k = 1/k$, let $V_k := V_{r_k}$, and let z_k be arbitrary points in V_k . Since the V_k are open and nested, there are curves γ_k in V_k connecting z_k to z_{k+1} . Together, they form a curve γ starting at z_1 , remaining in V_1 and necessarily converging to z , i.e., landing at z . This curve obviously satisfies the given conditions. If two such curves, without their landing point z , were to separate K , then K would have to be disconnected, a contradiction. If the curves, together with their landing point, surround some part of K , then there is a radius $r > 0$ such that both remaining parts of K contain points at distance greater than r from z . But then these curves cannot stay forever in the same region V_r , so they correspond to different accesses. Finally, exactly one of these curves is an external ray by Lindelöf’s Theorem A.5. \square

Lemma A.8 (Landing of Rays and Disconnecting Points)

The number of external rays landing at any point $z \in \partial K$ equals the number of connected components of $K - \{z\}$. Between any pair of external rays, there is a connected component, and conversely. \square

PROOF. By Carathéodory’s Theorem A.2, every boundary point of K is the landing point of at least one external ray. Any pair of rays landing at z separates K by Lemma A.4, so we only have to show the converse. Let K_1 and K_2 be two components of $K - \{z\}$ and let $r > 0$ be such that both components contain points at distance r from z . For positive integers k , let D_k be the Euclidean disk of radius $1/k$ around z . For $k_0 > 1/r$, $D_{k_0} - K$ is disconnected because the sets K_i separate it. For any connected component of $D_{k_0} - K$ containing z on its closure, there is an access to z (and possibly many), and there must be two accesses to z separating K_1 and K_2 .

It follows that any finite collection of rays landing at z produces equally many connected components of $K - \{z\}$ between them, and any finite collection of connected components gives rise to equally many rays separating them. The numbers of rays and connected components are thus either both finite and equal, or they are both infinite. \square

REMARK. Even if K is locally connected, the number of external rays landing at any single point is not necessarily finite or even countable; see Lemma A.12. However, in

our applications, this number will always be finite (although not necessarily bounded over the branch points of K): for connected Julia sets of unicritical polynomials, three or more rays landing at the same point are always preperiodic or periodic by Thurston's No Wandering Triangles Theorem 3.3 (except for the rays on the inverse orbit of the critical point, but if they are not eventually periodic, then their number can be at most twice the degree); for the Mandelbrot set, three rays can land together only at Misiurewicz points [S2, 3.16], and the number of rays landing there is always finite.

From now on, we assume the set K to be locally connected, in addition to the requirements that it be compact, connected and full. In the following, we will collect several properties of such sets.

Definition A.9 (Branch Point)

A branch point of a compact connected locally connected full set K is a point which is the landing point of at least three external rays; equivalently, it is a point which disconnects K into at least three parts.

Lemma A.10 (Projection onto Interior Components)

Let K_0 be the closure of a connected component of the interior of K (a compact connected locally connected full subset of \mathbb{C} , let z be a point in the interior of K_0 and let z' be a point in $K - K_0$. Then there is a unique point in ∂K_0 through which every curve in K connecting z' to z must run. This point disconnects K so that z and z' are in different connected components.

PROOF. Let γ_1 and γ_2 be two curves connecting z' to z . Such a curve meets ∂K_0 in a compact set, so starting from z' , there will be a first point when the curve reaches ∂K_0 . Replacing the rest of the curve by a curve within the interior of K_0 landing at the same point (which is possible by Carathéodory's Theorem A.2 since ∂K_0 is locally connected by Lemma A.3), we may assume that the curve meets ∂K_0 once. If the curves γ_i meet ∂K_0 in different points, then the curves, together with K_0 , enclose a subset of \mathbb{C} containing boundary points of K_0 in its interior. Since $\partial K_0 \subset \partial K$, this contradicts the assumption that K is full.

Denoting the unique boundary point thus constructed by \tilde{z} , we now claim that $K - \tilde{z}$ is disconnected. If it is not, we prove that it is still arcwise connected: $K - \tilde{z}$ is locally arcwise connected, so the set of points in the path component of z is open. Any limit point different from \tilde{z} in the path component is also within the path component because the limit point has a path connected neighborhood in K . Therefore, we can connect z to z' by a path within $K - \{\tilde{z}\}$, contradicting uniqueness of \tilde{z} . \square

REMARK. This way, we obtain a canonical *projection* (which is a retraction) of K onto K_0 : this projection is the identity on K_0 , and outside of K_0 it maps to ∂K_0 by the construction above. It is not hard to see that this projection is continuous. It is locally constant on $K - K_0$. This projection has been introduced by Douady and Hubbard in [DH1, Exposé II.5]. From Lemma A.8, it follows that the projection image of any point $z' \in K - K_0$ is the landing point of at least two external rays.

Corollary A.11 (Projection Images Countable)

Let K_0 be the closure of a connected component of the interior of K (compact connected locally connected and full). The projection of $K - K_0$ onto K_0 takes images in at most countably many points.

PROOF. Every image point is the landing point of at least two external rays, so it separates an open set of external angles from K_0 . Different projection points obviously separate different sets of external angles. The total sum of external angles thus separated is finite, so the number of projection points must be countable. \square

Lemma A.12 (Countably Many Branches)

The number of connected components of $K - \{z\}$ is always countable when K is compact connected locally connected and full. However, the external angles of z may form a Cantor set.

PROOF. For any connected component of $K - \{z\}$, pick a point within and let α be an external angle of this point. Then, by continuity of landing points, external rays at angles sufficiently close to α land in the same connected component of $K - \{z\}$, so every connected component takes up an open set of external angles in \mathbb{S}^1 .

It follows that the external angles of z form the complement of a dense open subset of \mathbb{C} , and if the connected components of $K - \{z\}$ are arranged so that between any two of them there is always another one, then the open subsets corresponding to any connected component always have disjoint closures, and their complement is a Cantor set. \square

REMARK. As mentioned above, in all the cases of interest to us the number of rays landing at a single point will be finite.

Lemma A.13 (Countably Many Branch Points)

The number of branch points of and compact connected locally connected and full subset of \mathbb{C} is countable.

PROOF. This is a special case of Lemma 2.11, the proof of which was self-contained. \square

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