Bounded Hyperbolic Components of Quadratic Rational Maps

Adam Lawrence Epstein

September 1, 1997

Abstract

Let $\mathcal{H}$ be a hyperbolic component of quadratic rational maps possessing two distinct attracting cycles. We show that $\mathcal{H}$ has compact closure in moduli space if and only if neither attractor is a fixed point.

Contents

1 Introduction

From the perspective of dynamics, the simplest rational maps are *hyperbolic*: every critical point tends under iteration to some attracting periodic cycle. Such maps constitute an open and conjecturally dense set in parameter space [8], whose components are referred to as *hyperbolic components*. Maps in the same component are quasiconformally conjugate near the Julia set, and thus have essentially identical dynamics if critical orbit relations are ignored.

Stony Brook IMS Preprint #1997/9
September, 1997

The family $P_c(z) = z^2 + c$ of quadratic polynomials contains one unbounded component, namely $\mathbb{C} - M$ where

$$M = \{c : \ J(P_c) \text{ is connected}\}$$

is the much-studied Mandelbrot set, and infinitely many bounded components; the latter are simply connected regions with smooth real-algebraic boundary, and are naturally parameterized by the eigenvalue $\rho \in \mathbb{D}$ of the unique attracting cycle. Matters become more involved when there are at least two free critical points. The two-parameter families of normalized quadratic rational maps and normalized cubic polynomials are often considered in parallel, as their hyperbolic components admit similar descriptions: there is a single component of maps with totally disconnected Julia set and all other components are topological 4-cells [11, 17]. One essential difference is that cubic polynomials with connected Julia set form a compact set in parameter space; in particular, every hyperbolic component of maps with two distinct attractors is precompact. By contrast, while many unbounded hyperbolic components of quadratic rational maps have been identified [6, 15], bounded components have yet to be exhibited.

Hyperbolic components may also be discussed in the context of Kleinian groups and their quotient 3-manifolds. For finitely generated hyperbolic groups with connected limit set - those whose quotient has incompressible boundary - the corresponding hyperbolic component is precompact if and only the limit set is a Sierpinski carpet: the complement of a countable dense union of Jordan domains with disjoint closures whose diameters tend to zero. Guided by Sullivan's dictionary between these subjects, McMullen conjectured that hyperbolic rational maps with Sierpinski carpet Julia set lie in bounded hyperbolic components [9]. Pilgrim has suggested more precisely that a hyperbolic component is bounded when the Julia set is *almost* a Sierpinski carpet: for example, if every Fatou component is a Jordan domain and no two Fatou components have closures which intersect in more than one periodic point. Here we establish precompactness for hyperbolic components of quadratic rational maps with two attracting cycles, provided that neither attractor is a fixed point. While it is known in this case that every Fatou component is a Jordan domain [16], our largely algebraic arguments do not exploit the topology of the Julia set.

We begin in Section 2 with a review of the theory of the holomorphic index. The index formula

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 1$$

relating the eigenvalues of the three fixed points is fundamental to Milnor's description [12] of the moduli space of quadratic rational maps. We survey this work in Section 3 and show in particular that a sequence of maps is bounded in moduli space if and only if there is an upper bound on the eigenvalues of the fixed points. Moduli space is readily parametrized through the choice of a normal form. For certain purposes it is convenient to work with the family

$$f_{\alpha,\beta}(z) = z\frac{(1-\alpha)z + \alpha(1-\beta)}{\beta(1-\alpha)z + (1-\beta)}$$

of maps fixing 0,∞, and 1 with eigenvalues α, β, and $\gamma = \frac{2-(\alpha+\beta)}{1-\alpha\beta}$; in other settings it is more useful to work with the family

$$F_{\gamma,\delta}(z) = \frac{\gamma z}{z^2 + \delta z + 1}$$

of maps with critical points ± 1 and a fixed point at 0 with eigenvalue γ.

In Section 4 we study the limiting dynamics of unbounded sequences in moduli space. Milnor showed that such sequences accumulate at a restricted set of points on a natural infinity locus [12], provided that there are cycles with the same period $n > 1$ and uniformly bounded eigenvalues. We sharpen this and related observations in order to show that suitably normalized iterates take limits in the family

$$G_T(z) = z + T + \frac{1}{z}$$

as anticipated by considerations in the thesis of Stimson [20]. Cycles with bounded eigenvalue tend in the limit to cycles of G_T or to points in the backward orbit of the parabolic fixed point at ∞; in the latter case this backward orbit contains a critical point. In particular, if the maps in the sequence lie in a hyperbolic component where there are two nonfixed attractors then G_T must have either two nonrepelling cycles, one nonrepelling cycle and one preperiodic critical point, or two preperiodic critical points, in addition to the parabolic fixed point at ∞. As discussed in Section 5, this violation of the Fatou-Shishikura bound on the number of nonrepelling cycles yields the desired contradiction.

Section 6 gives an intersection-theoretic reinterpretation based on Milnor's observation that $\mathrm{Per}_n(\rho)$, the locus of conjugacy classes of maps with an n-cycle of eigenvalue ρ, is an algebraic curve whose degree depends only on n. The explicit formulas in [12] yield a short independent proof of boundedness in the special case of maps with one attracting cycle of period 2 and another of period 3. These considerations suggest a combinatorial expression for the intersection cycle at infinity of a pair of such curves.

Acknowledgments.

The author wishes to thank Jack Milnor and Kevin Pilgrim for sharing their insight. Further thanks are due to Misha Yampolsky for assistance with the graphics which were graciously provided by Jack Milnor. This research has been generously supported by the Institute for Mathematical Sciences at Stony Brook.

2 Local Invariants

Let g be analytic on $U \subseteq \mathbb{C}$ and $\zeta \in U$ with $g(\zeta) = \zeta$. Assuming that g is not the identity, the *topological multiplicity* is defined as the positive integer

$$\mathrm{mult}_g(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{1 - g'(z)}{z - g(z)}\, dz$$

where Γ is any sufficiently small positively oriented rectifiable Jordan curve enclosing ζ; the *holomorphic index* is similarly defined as the complex number

$$\mathrm{ind}_g(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - g(z)}\, dz.$$

One easily checks that these quantities are invariant under holomorphic change of coordinates and can thereby be sensibly defined for $\zeta = \infty$; moreover, $\mathrm{mult}_g(\zeta) = 1$ if and only if the *eigenvalue* $\rho = g'(\zeta)$ differs from 1, and then

$$\mathrm{ind}_g(\zeta) = \frac{1}{1-\rho}. \tag{1}$$

Furthermore, if $|\rho| \neq 1$ or $\rho = 1$ then $\mathrm{mult}_{g^n}(\zeta) = \mathrm{mult}_g(\zeta)$ for every $n \geq 1$.

It follows from Cauchy's Integral Formula that

$$\begin{aligned} \sum_{\zeta = g(\zeta) \in V} \mathrm{mult}_g(\zeta) &= \frac{1}{2\pi i} \int_{\partial V} \frac{1 - g'(z)}{z - g(z)}\, dz \\ \sum_{\zeta = g(\zeta) \in V} \mathrm{ind}_g(\zeta) &= \frac{1}{2\pi i} \int_{\partial V} \frac{1}{z - g(z)}\, dz \end{aligned}$$

for open V with $\overline{V} \subseteq U \subseteq \mathbb{C}$ and with rectifiable boundary containing no fixed points. These sums evidently depend continuously on g. For rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree d, one sees from the Residue Theorem that

$$\sum_{\zeta = g(\zeta) \in \hat{\mathbf{C}}} \mathrm{mult}_g(\zeta) = d + 1; \tag{2}$$

the *Holomorphic Index Formula*

$$\sum_{\zeta = g(\zeta) \in \hat{\mathbf{C}}} \mathrm{ind}_g(\zeta) = 1 \tag{3}$$

follows similarly. We denote $\mathrm{Fix}(g)$ the unordered $(d+1)$-tuple of fixed points listed with multiplicity. In general, we denote such collections of possibly identical points as $[x_1, \ldots, x_n]$. We similarly write $\mathrm{Crit}(g)$ for the unordered $(2d-2)$-tuple of critical points; note that there are at least two distinct critical points when $d \geq 2$.

A fixed point ζ of an analytic map g is said to be *attracting, indifferent*, or *repelling* according as the eigenvalue ρ is less than, equal to, or greater than 1. If $\rho = e^{2\pi i p/q}$ where $(p, q) = 1$ and g^q is not the identity, then ζ is *parabolic*. A simple calculation then shows that $\mathrm{mult}_{g^q}(\zeta) = \ell q + 1$ for some positive integer ℓ; we refer to ℓ as the *degeneracy*, and say that ζ is a *degenerate parabolic* fixed point when $\ell \geq 2$. In view of (1), if $\mathrm{mult}_g(\zeta) = 1$ then ζ is attracting, indifferent, or repelling according as the real part of $\mathrm{ind}_g(\zeta)$ is greater than, equal to, or less than $\frac{1}{2}$. Following [1] we say that a parabolic fixed point ζ with eigenvalue $e^{2\pi ip/q}$ is

$$\begin{array}{lll} \textit{parabolic}-\textit{attracting} & \text{when} & \Re\,\mathrm{ind}_{g^q}(\zeta) > \frac{\ell q+1}{2}, \\ \textit{parabolic}-\textit{indifferent} & \text{when} & \Re\,\mathrm{ind}_{g^q}(\zeta) = \frac{\ell q+1}{2}, \\ \textit{parabolic}-\textit{repelling} & \text{when} & \Re\,\mathrm{ind}_{g^q}(\zeta) < \frac{\ell q+1}{2}. \end{array}$$

More generally, we say that ζ is *periodic* under g when $g^n(\zeta)=\zeta$ for some $n\geq 1$, the least such n being referred to as the *period*. The multiplicity, index, and eigenvalue of the cycle $\langle\zeta\rangle=\{\zeta,\dots,g^{n-1}(\zeta)\}$ are the corresponding invariants of ζ as a fixed point of g^n. It follows from the definition of multiplicity that a generic perturbation of g splits an n-cycle with eigenvalue $\rho=e^{2\pi ip/q}$ and degeneracy ℓ into an n-cycle with eigenvalue close to ρ and an ℓ-tuple of nq-cycles with eigenvalues close to 1. Continuity of the local index sum implies:

Lemma 1 *Let g be analytic on U with a parabolic n-cycle $\langle\zeta\rangle$ of eigenvalue $e^{2\pi ip/q}$. Further let g_k be analytic with $g_k\to g$ locally uniformly on U, and with n-cycles $\langle\zeta_k^{[0]}\rangle$ and nq-cycles $\langle\zeta_k^{[1]}\rangle,\dots,\langle\zeta_k^{[\ell]}\rangle$ converging to $\langle\zeta\rangle$. If all $\langle\zeta_k^{[j]}\rangle$ are attracting for k sufficiently large then $\langle\zeta\rangle$ is parabolic-attracting or parabolic-indifferent.*

Assume now that g is rational of degree d. The *basin* of an attracting cycle $\langle\zeta\rangle$ is the open set consisting of all points $z\in\widehat{\mathbb{C}}$ with $g^n(z)\to\langle\zeta\rangle$. We refer to the connected component containing $\xi\in\langle\zeta\rangle$ as the *immediate basin* of ξ. The basin of a parabolic cycle is similarly defined as the open set of all $z\in\widehat{\mathbb{C}}$ with $\langle\zeta\rangle\not\ni g^n(z)\to\langle\zeta\rangle$, the ℓq components adjoining ξ forming the immediate basin of ξ. In both cases, the immediate basin of $\langle\zeta\rangle$ is taken to be the union of the immediate basins of the points in the cycle. Fatou established the fundamental fact that each cycle of components of the immediate basin of an attracting or parabolic cycle always contains at least one critical value with infinite forward orbit [10]. In particular, counting degeneracy there are at most $2d-2$ attracting and parabolic cycles. Shishikura extended this bound to the total count of nonrepelling cycles [18], and the author proved a refined inequality where the contribution of each parabolic-attracting and parabolic-indifferent cycle is augmented by one [1]; consideration of the return maps on Ecalle cylinders shows in fact that there are at least $\ell+1$ critical values with infinite forward orbit in the immediate basin of a parabolic-attracting or parabolic-indifferent cycle of degeneracy ℓ.

Consider the family

$$G_T(z)=z+T+\frac{1}{z}$$

of quadratic rational maps with critical points ±1 and a degenerate fixed point at ∞ with eigenvalue 1 and holomorphic index $1-\frac{1}{T^2}$; by convention, $G_\infty\equiv\infty$. The Fatou-Shishikura Inequality has the following consequences in this special case:

Lemma 2 *Let $G=G_T$ where $T\in\mathbb{C}$.*

- *If $T=0$ then ∞ is a degenerate parabolic fixed point. Neither critical point is preperiodic and all other cycles are repelling.*

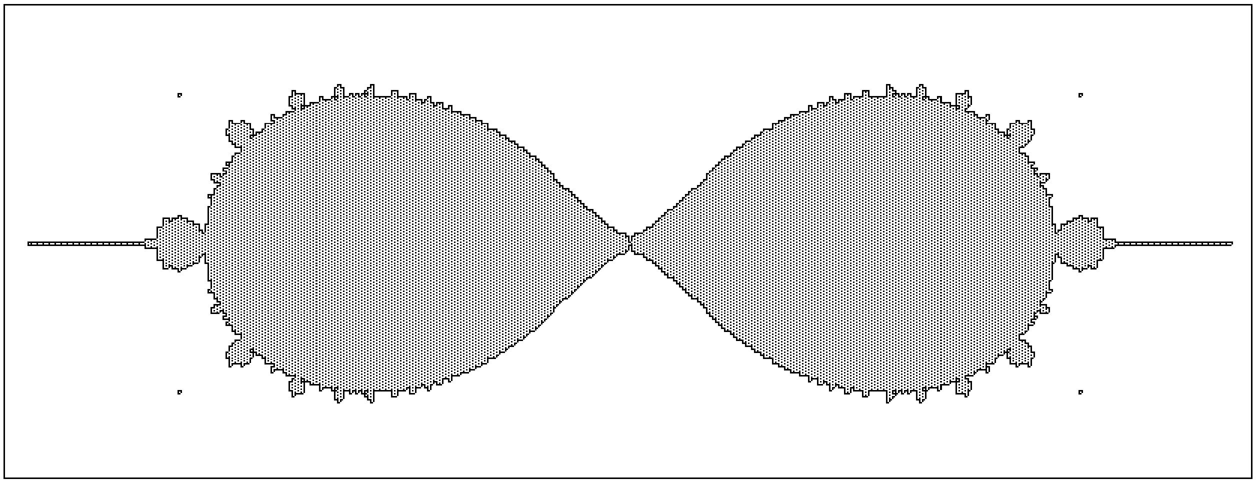

Figure 1: Bifurcation locus for the family $G_\kappa(z) = z + \kappa + \frac{1}{z}$.

- *If* $T \neq 0$ *and* $\langle \zeta \rangle$ *is attracting or indifferent, then neither critical point is preperiodic and all other cycles are repelling; if parabolic, then* $\langle \zeta \rangle$ *is nondegenerate parabolic-repelling.*
- *If* $T \neq 0$ *and either critical point is preperiodic, then the other critical point has infinite forward orbit and all other cycles are repelling.*

3 Normal Forms

We naturally identify the space of all quadratic rational maps

$$\mathcal{RAT}_2 = \left\{ g(z) = \frac{A_2 z^2 + A_1 z + A_0}{B_2 z^2 + B_1 z + B_0} : \ \deg g = 2 \right\}$$

with the open subvariety of projective space $\mathbb{P}^5$ where the resultant

$$\det \begin{pmatrix} A_2 & A_1 & A_0 & 0 \\ 0 & A_2 & A_1 & A_0 \\ B_2 & B_1 & B_0 & 0 \\ 0 & B_2 & B_1 & B_0 \end{pmatrix}$$

is nonvanishing. Various technical purposes require that we work in the spaces

$$\begin{aligned} \mathcal{RAT}_2^{\times} &= \left\{ (g;\ \chi^+, \chi^-) \in \mathcal{RAT}_2 \times \widehat{\mathbb{C}}^2 :\ \mathrm{Crit}(g) = [\chi^+, \chi^-] \right\} \\ \mathcal{RAT}_2^{\circ} &= \left\{ (g;\ a, b, c) \in \mathcal{RAT}_2 \times \widehat{\mathbb{C}}^3 :\ \mathrm{Fix}(g) = [a, b, c] \right\} \\ \mathcal{RAT}_2^{\otimes} &= \left\{ (g;\ \chi^+, \chi^-;\ a, b, c) \in \mathcal{RAT}_2 \times \widehat{\mathbb{C}}^5 :\ \begin{array}{c} \mathrm{Crit}(g) = [\chi^+, \chi^-] \\ \mathrm{Fix}(g) = [a, b, c] \end{array} \right\} \end{aligned}$$

where the critical points, fixed points, or both have been marked. The quotients under the conjugation action of the Möbius group are the *moduli spaces*

$$\begin{aligned}\mathbf{Rat}_2 &= \mathcal{RAT}_2/\mathrm{PSL}_2\mathbb{C}\\ \mathbf{Rat}_2^{\times} &= \mathcal{RAT}_2^{\times}/\mathrm{PSL}_2\mathbb{C}\\ \mathbf{Rat}_2^{\circ} &= \mathcal{RAT}_2^{\circ}/\mathrm{PSL}_2\mathbb{C}\\ \mathbf{Rat}_2^{\otimes} &= \mathcal{RAT}_2^{\otimes}/\mathrm{PSL}_2\mathbb{C}\end{aligned}$$

all varieties of complex dimension 2.

Writing α, β, γ for the eigenvalues of the fixed points a, b, c we see from (3) that

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 1$$

so long as $\alpha, \beta, \gamma \neq 1$, and

$$\alpha\beta\gamma - (\alpha + \beta + \gamma) + 2 = 0 \tag{4}$$

always; in particular,

$$\gamma = \frac{2 - (\alpha + \beta)}{1 - \alpha\beta}.$$

Let $[(g;\ \chi^+, \chi^-\ ; a, b, c)]]$ be a class in $\mathbf{Rat}_2^{\otimes}$. Provided that $\chi^+ \neq c \neq \chi^-$, there is a unique representative of the form $(F;\ +1, -1;\ a, b, 0)$ where

$$F(z) = F_{\gamma,\delta}(z) = \frac{\gamma z}{z^2 + \delta z + 1} \tag{5}$$

for some $\gamma, \delta \in \mathbb{C}$ with $\gamma \neq 0$; moreover, every class in $\mathbf{Rat}_2^{\times}$ has a representative of this form. As

$$F'_{\gamma,\delta}(z) = \frac{\gamma(1 - z^2)}{z^2 + \delta z + 1}$$

it follows that

$$\begin{aligned}\alpha = \frac{1 - a^2}{\gamma} &= \frac{\delta a + 2}{\gamma} - 1\\ \beta = \frac{1 - b^2}{\gamma} &= \frac{\delta b + 2}{\gamma} - 1\end{aligned}$$

with

$$\{a, b\} = \left\{ \frac{-\delta \pm \sqrt{\delta^2 - 4(1 - \gamma)}}{2} \right\}.$$

Alternatively, provided that $a \neq b \neq c \neq a$ there is a unique representative of the form

$$f_{\alpha,\beta}(z) = z\frac{(1 - \alpha)z + \alpha(1 - \beta)}{\beta(1 - \alpha)z + (1 - \beta)} \tag{6}$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha, \beta, \alpha\beta \neq 1$. Writing

$$f_{\alpha,\beta}(z) = z\frac{(1-\alpha)(z-1)+\epsilon}{\beta(1-\alpha)(z-1)+\epsilon} \tag{7}$$

$$= \frac{z}{\beta}\left[\frac{z-\nu}{z-\mu}\right] = \frac{z}{\beta}\left[1+\frac{\mu\epsilon}{z-\mu}\right], \tag{8}$$

where

$$\epsilon = 1-\alpha\beta = \frac{(1-\alpha)(1-\beta)}{(\gamma-1)} \tag{9}$$

and

$$\mu = \frac{\beta-1}{\beta-\alpha\beta} = 1-\frac{\epsilon}{\beta(1-\alpha)} \tag{10}$$

$$\nu = \frac{\alpha\beta-\alpha}{1-\alpha} = 1-\frac{\epsilon}{1-\alpha}, \tag{11}$$

we see that $f_{\alpha,\beta}(\mu) = \infty$ and $f_{\alpha,\beta}(\nu) = 0$. Calculating the derivative

$$f'_{\alpha,\beta}(z) = \frac{\beta(1-\alpha)^2(z-1)^2+(1+\beta)(1-\alpha)(z-1)\epsilon+(2-\alpha-\beta)\epsilon}{[\beta(1-\alpha)(z-1)+\epsilon]^2} \tag{12}$$

$$= \frac{1}{\beta}\left[1+\frac{\mu\nu-\mu^2}{(z-\mu)^2}\right] = \frac{1}{\beta}\left[1-\frac{\mu^2\epsilon}{(z-\mu)^2}\right], \tag{13}$$

we find that

$$\mu = \frac{\chi^+ + \chi^-}{2}$$

$$\epsilon = \left(\frac{\chi^+ - \chi^-}{\chi^+ + \chi^-}\right)^2$$

whence

$$\chi^{\pm} = \mu(1 \pm \sqrt{\epsilon})$$

for the appropriate choice of $\sqrt{\epsilon}$.

Assuming both restrictions on the marked points, there is a unique Möbius transformation

$$\phi(z) = \frac{bz-ab}{az-ab} = \frac{(\mu^2\epsilon-\mu^2+\mu)z+\mu\sqrt{\epsilon}}{(1-\mu)z+\mu\sqrt{\epsilon}} \tag{14}$$

sending $+1, -1, a, b, 0$ to $\chi^+, \chi^-, 0, \infty, 1$. Clearly,

$$F_{\gamma,\delta} = \phi^{-1} \circ f_{\alpha,\beta} \circ \phi$$

where

$$(\alpha,\beta) = \left(\frac{4\chi^+\chi^- - 2\chi^+\chi^-(\chi^+ + \chi^-)}{(\chi^+ + \chi^-)^2 - 2\chi^+\chi^-(\chi^+ + \chi^-)}, \frac{2(\chi^+ + \chi^-) - 4\chi^+\chi^-}{2(\chi^+ + \chi^-) - (\chi^+ + \chi^-)^2}\right)$$

and

$$(\gamma, \delta) = (1 - ab\,,\ -a - b).$$

Recall that the elementary symmetric functions

$$\begin{aligned} X(\alpha,\beta,\gamma) &= \alpha+\beta+\gamma \\ Y(\alpha,\beta,\gamma) &= \alpha\beta+\alpha\gamma+\beta\gamma \\ Z(\alpha,\beta,\gamma) &= \alpha\beta\gamma \end{aligned}$$

together determine $[\alpha, \beta, \gamma]$. It follows from (4) that

$$\mathbf{Rat}_2^\circ \ni [f;\ a, b, c] \rightsquigarrow (X(\alpha,\beta,\gamma), Y(\alpha,\beta,\gamma), Z(\alpha,\beta,\gamma)) \in \mathbb{C}^3$$

descends to a map $\mathbf{Rat}_2 \to \mathbb{C}^3$ with image in the hyperplane

$$\{(X, Y, Z) \in \mathbb{C}^3 :\ Z = X - 2\},$$

and we obtain

$$j : \mathbf{Rat}_2 \to \mathbb{C}^2$$

on composing with the projection $\mathbb{C}^3 \ni (X, Y, Z) \rightsquigarrow (X, Y) \in \mathbb{C}^2$. Consideration of the normal forms (5) and (6) shows that an unordered triple $[\alpha, \beta, \gamma]$ satisfying (4) determines a unique class in $\mathbf{Rat}_2$, and thus j is an isomorphism. As $\{(X(\alpha_k, \beta_k, \gamma_k), Y(\alpha_k, \beta_k, \gamma_k)\}$ and $\{\alpha_k, \beta_k, \gamma_k\}$ are simultaneously bounded or unbounded, we recover Milnor's observation [12]:

Lemma 3 *Let g_k be quadratic rational maps with eigenvalues $\alpha_k, \beta_k, \gamma_k$ at the fixed points a, b, c. Then $[g_k]$ is bounded in $\mathbf{Rat}_2$ if and only if $\{\alpha_k, \beta_k, \gamma_k\}$ is bounded in $\mathbb{C}$.*

4 Limit Dynamics

Our first goal is the following:

Proposition 1 *Let g_k be quadratic rational maps with eigenvalues $\alpha_k, \beta_k, \gamma_k$ at the fixed points a, b, c, where α_k and β_k converge in $\widehat{\mathbb{C}}$ and $\gamma_k \to \infty$. Assume that there are cycles $\langle z_k \rangle$ with the same period $n > 1$ and uniformly bounded eigenvalues. Then*

$$\alpha_k = \omega + O(\sqrt{\epsilon_k}) \qquad \textit{and} \qquad \beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$$

as $k \to \infty$, where $\omega \neq 1$ is a q-th root of unity for some $q \leq n$ and

$$\epsilon_k = 1 - \alpha_k \beta_k = O\left(\frac{1}{\gamma_k}\right).$$

The proof requires several preliminary lemmas and the elimination of various special cases. Let $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ satisfying (4), and suppose that $\gamma_k \to \infty$. Inspection of (9), (10), and (11) shows that

- $\alpha_k \to \infty$ if and only if $\beta_k \to 0$, and vice-versa;
- $\alpha_k \to 1$ if and only if $\beta_k \to 1$, and vice-versa;
- $\epsilon_k = o(\alpha_k - 1)$ if β_k is bounded, and $\epsilon_k = o(\beta_k - 1)$ if α_k is bounded;
- $\epsilon_k = O(\gamma_k^{-1})$ if both α_k and β_k are bounded;
- $\mu_k \to 1$ if $\alpha_k \neq 1$ is bounded, and $\nu_k \to 1$ if $\beta_k \neq 1$ is bounded;
- Both μ_k and ν_k are $1 + O(\epsilon_k)$ if α_k, hence also β_k, is bounded away from $\{0, 1, \infty\}$.

Recall from (14) that the choice of $\sqrt{\epsilon_k}$, corresponding to a marking of the critical points, specifies a Möbius transformation ϕ_k which conjugates f_{α_k,β_k} to some F_{γ_k,δ_k}. It follows from these observations that

$$\phi_k(z) = 1 + z\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \tag{15}$$

on compact sets in $\mathbb{C}$, provided that α_k and β_k are bounded away from $\{0, 1, \infty\}$.

Let $f_k = f_{\alpha_k,\beta_k}$ where $\gamma_k \to \infty$ and $\alpha_k \to \alpha_\infty \in \mathbb{C}^*$. By (7) and (8),

$$\frac{f_k(z_k)}{z_k} = 1 + o(1) \quad \text{if } z_k - 1 = o(\epsilon_k)$$

and

$$f_k(z) \to \alpha_\infty z \quad \text{locally uniformly on } \widehat{\mathbb{C}} - \{0, 1, \infty\}$$

Assuming further that $\alpha_\infty \neq 1$, we have

$$\frac{f_k(z_k)}{z_k} = \alpha_k \left[1 + \frac{\epsilon_k}{z_k - 1} + o\left(\frac{\epsilon_k}{z_k - 1}\right) + O(\epsilon_k)\right]$$

when $\epsilon_k = o(z_k - 1)$, and thus

$$\frac{f_k(z_k)}{z_k} = \begin{cases} \alpha_k + o(1) & \text{if } \epsilon_k = o(z_k - 1) \\ \alpha_k(1 + \frac{1}{\tau}\sqrt{\epsilon_k}) + o(\sqrt{\epsilon_k}) & \text{if } z_k = 1 + \tau\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \text{ for } \tau \in \mathbb{C}^* \\ \alpha_k + o(\sqrt{\epsilon_k}) & \text{if } \sqrt{\epsilon_k} = o(z_k - 1) \\ \alpha_k + O(\epsilon_k) & \text{if } z_k \text{ is bounded away from } 1; \end{cases} \tag{16}$$

moreover,

$$\frac{f_k(z_k) - 1}{z_k - 1} = \frac{(z_k - \beta_k)(1 - \alpha_k) + \epsilon_k}{\beta_k(1 - \alpha_k)(z_k - 1) + \epsilon_k} \to \infty \tag{17}$$

whenever $z_k \to 1$.

Observe that if $z_k = 1 + O(\sqrt{\epsilon_k})$ and $f_k^n(z_k) = 1 + O(\sqrt{\epsilon_k})$ then

$$1 + O(\sqrt{\epsilon_k}) = \frac{f_k^n(z_k)}{z_k} = \prod_{j=0}^{n-1} \frac{f_k^{j+1}(z_k)}{f_k^j(z_k)} = \alpha_k^n + O(\sqrt{\epsilon_k}),$$

unless $f_k^j(z_{k_\ell}) = 1 + o(\sqrt{\epsilon_k})$ for some $0 \leq j < n$ and $k_\ell \to \infty$. Applying (15) we deduce:

Lemma 4 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$ and $\alpha_k \to \alpha_\infty \notin \{0, 1, \infty\}$, hence $\beta_k \to \beta_\infty = \alpha_\infty^{-1}$. Suppose that $z_k \in \widehat{\mathbb{C}}$ with $F_k^j(z_k) \to \zeta^{(j)} \in \widehat{\mathbb{C}}$ for $0 \leq j \leq n$, where $\zeta^{(0)} \notin \{0, \infty\}$ and $\zeta^{(n)} \neq \infty$. If $\zeta^{(j)} \neq 0$ for $0 < j < n$ then*

$$\alpha_k = \alpha_\infty + O(\sqrt{\epsilon_k}) \qquad \textit{and} \qquad \beta_k = \beta_\infty + O(\sqrt{\epsilon_k})$$

and $\alpha_\infty^n = 1 = \beta_\infty^n$; moreover, if $\zeta^{(j)} = \infty$ for $0 < j < n$ then α_∞, hence also β_∞, is a primitive n-th root of unity.

Assume now that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a root of unity, and suppose that $z_k = 1 + o(\sqrt{\epsilon_k})$ but $\epsilon_k = o(z_k - 1)$. Then

$$f_k^n(z_k) = \omega^n \left[1 + \frac{\epsilon_k}{z_k - 1} + o\left(\frac{\epsilon_k}{z_k - 1}\right)\right]$$

for $n \geq 1$ by (16) and induction, and thus (15) implies:

Lemma 5 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a root of unity, and let $z_k \in \widehat{\mathbb{C}}$ with $z_k \to 0$. If $F_k^n(z_k)$ is bounded for some $n > 1$ then $z_k = O(\sqrt{\epsilon_k})$.*

Suppose now that $f_k^n(z_k) = z_k$ where $z_k \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$ and $n > 1$. As

$$1 = \frac{f_k^n(z_k) - 1}{z_k - 1} = \prod_{j=0}^{n-1} \frac{f_k^{j+1}(z_k) - 1}{f_k^j(z_k) - 1},$$

it follows from (17) that ζ_k is bounded away from 1 for some $\zeta_k \in \langle z_k \rangle$. Similarly,

$$\alpha_k^n = \begin{cases} 1 + o(1) & \text{if } \min_{\zeta \in \langle z_k \rangle} \left|\frac{\zeta - 1}{\epsilon_k}\right| \to \infty \\ 1 + O(\sqrt{\epsilon_k}) & \text{if } \min_{\zeta \in \langle z_k \rangle} \left|\frac{\zeta - 1}{\sqrt{\epsilon_k}}\right| \text{ is bounded away from } 0 \\ 1 + o(\sqrt{\epsilon_k}) & \text{if } \min_{\zeta \in \langle z_k \rangle} \left|\frac{\zeta - 1}{\sqrt{\epsilon_k}}\right| \to \infty \\ 1 + O(\epsilon_k) & \text{if } \langle z_k \rangle \text{ is bounded away from } 1. \end{cases}$$

by (16). Combining these observations with Lemma 4, we obtain:

Lemma 6 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a root of unity, and let $\langle z_k \rangle$ be cycles of period $n > 1$. If $\langle z_k \rangle \to \mathcal{Z} \subset \widehat{\mathbb{C}}$ then $\infty \in \mathcal{Z}$. Moreover:*

- *If $\mathcal{Z} \neq \{0, \infty\}$ then $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$.*
- *If $\mathcal{Z} = \{\infty\}$ then $\alpha_k = \omega + o(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + o(\sqrt{\epsilon_k})$.*

Assume now that α_k, hence also β_k, is bounded away from $\{0, 1, \infty\}$. It follows from (12) that

$$f_k'(z_k) = \frac{(2 - \alpha_k - \beta_k)\epsilon_k + o(\epsilon_k)}{o(\epsilon_k)} \to \infty$$

if $z_k - 1 = o(\sqrt{\epsilon_k})$. On the other hand, if $\sqrt{\epsilon_k} = O(z_k - 1)$ then (13) implies

$$f_k'(z_k) = \frac{1}{\beta_k}\left[1 - \frac{\epsilon_k}{(z_k - 1)^2} + o\left(\frac{\epsilon_k}{(z_k - 1)^2}\right)\right]$$

whence

$$f_k'(z_k) = \begin{cases} \alpha_k \left(1 - \frac{1}{\tau^2}\right) + o(1) & \text{if } z_k = 1 + \tau\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \text{ for } \tau \in \mathbb{C}^*, \\ \alpha_k + o(1) & \text{if } \sqrt{\epsilon_k} = o(z_k - 1). \end{cases}$$

In particular:

Lemma 7 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive q-th root of unity, and let $\langle z_k \rangle \to \mathcal{Z}$ be cycles of period $n > 1$ and eigenvalues ρ_k. If $0 \not\in \mathcal{Z}$ then $q|n$ and ρ_k is bounded; moreover, if $\mathcal{Z} = \{\infty\}$ then $\rho_k \to 1$.*

We similarly deduce:

Lemma 8 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that α_k, hence also β_k, is bounded away from $\{0, 1, \infty\}$, and let $\langle z_k \rangle \to \mathcal{Z}$ be cycles of period $n > 1$ and eigenvalues ρ_k. If $0 \in \mathcal{Z}$ and ρ_k is bounded then $+1 \in \mathcal{Z}$ or $-1 \in \mathcal{Z}$.*

Proof of Proposition 1: Assume that $\alpha_k \to \alpha_\infty$. If $\alpha_\infty \notin \{0, 1, \infty\}$ then also $\beta_\infty \notin \{0, 1, \infty\}$. In particular, we may represent each class

$$[(g_k;\ a_k, b_k, c_k)]] \in \mathbf{Rat}_2^\circ$$

by a map $F_k = F_{\gamma_k,\delta_k}$. Recall that we are given n-cycles $\langle z_k \rangle$ with uniformly bounded eigenvalues ρ_k. Passing to a subsequence if necessary, we may assume that $\langle z_k \rangle \to \mathcal{Z} \subseteq \hat{\mathbf{C}}$. In view of Lemma 8, either $\mathcal{Z} = \{\infty\}$ or $\mathcal{Z} \cap \mathbb{C}^* \neq \emptyset$, and the conclusion follows by Lemma 6.

Suppose next that α_∞, hence also β_∞, is in $\{0, \infty\}$. Permuting the fixed points if necessary, we may assume on passage to a subsequence that $\alpha_k \to \infty$ and $\alpha_k = O(\gamma_k)$. Following Milnor [12] we work with the representatives $(\hat{f}_k;\ 0, \infty, c_k)$ where

$$\hat{f}_k(z) = z\frac{z + \alpha_k}{\beta_k z + 1}$$

and

$$c_k = \frac{1 - \alpha_k}{1 - \beta_k} = -\alpha_k + o(\alpha_k).$$

Calculating the derivative

$$\hat{f}_k'(z) = \frac{\beta_k z^2 + 2z + \alpha_k}{(\beta_k z + 1)^2}$$

we see that $\hat{f}_k'(z) = \alpha_k + O(1)$ on the disc $|z| < 4$. In particular, $\hat{f}_k$ is univalent on $|z| < 4$ with image containing the disc $|z| < 3|c_k|$, and both critical values lie outside the latter region. Consequently, there are univalent inverse branches A_k and C_k, fixing 0 and c_k, defined on the disc $|z| < 3|c_k|$. As $D_k = \{z : |2z - c_k| < 2|c_k|\}$ lies in the image of the disc $|z| < 4$, it follows that $A_k'(D_k) = O(\alpha_k^{-1})$ on D_k and $A_k(D_k) \subset D_k$. On the other hand,

$$C_k'(z) = O(\gamma_k^{-1}) = O(\alpha_k^{-1})$$

for $|z| < \frac{5}{2}|c_k|$ by the compactness of normalized univalent functions; consequently, $|C_k(z) - c_k| = O(c_k\gamma_k^{-1}) = O(1)$ for $|z - c_k| < \frac{3}{2}|c_k|$, and in particular $C_k(D_k) \subset D_k$. We deduce that $J(\hat{f}_k) \subset \hat{f}_k^{-1}(D_k)$ is a Cantor set containing all periodic points other than the fixed point at ∞. Thus, $\langle z_k \rangle \subset J(\hat{f}_k)$ and $\rho_k^{-1} = O(\alpha_k^{-n})$ whence $\rho_k \to \infty$.

It remains to treat the case $\alpha_\infty = 1 = \beta_\infty$. Now it is advantageous to choose representatives $(\hat{g}_k;\ \infty, b_k, c_k)$ where

$$\hat{g}_k(z) = \frac{(\alpha_k\gamma_k - 1)z^2 + (\alpha_k^2\gamma_k - \alpha_k^2)z + \alpha_k^2}{(\alpha_k^2\gamma_k - \alpha_k)z}$$

and

$$(b_k, c_k) = \left(\frac{\alpha_k}{\alpha_k - 1}, \frac{\alpha_k}{1 - \alpha_k\gamma_k}\right) \to (\infty, 0).$$

Notice that $\hat{g}_k(z) \to z + 1$ locally uniformly on $\widehat{\mathbb{C}} - \{0\}$, and thus $\hat{g}_k^n(z) \to z + n$ locally uniformly on $\widehat{\mathbb{C}} - \{-(n-1), \ldots, 0\}$. As the translation $z \rightsquigarrow z + 1$ has a fixed point of multiplicity 2 at ∞ and no other periodic points, b_k and ∞ are the only fixed points of $\hat{g}_k^n$ outside the circle $|z| = n$. We may therefore assume without loss of generality that

$$\hat{g}_k^j(z_k) \to \zeta^{(j)} \in \{-(n-1), \ldots, 0\}$$

with $\zeta^{(j+1)} = \zeta^{(j)}+1$ whenever $\zeta^{(j)} \neq 0$. It follows that some $\zeta^{(j)} = 0$, whence $\hat{g}_k^j(z_k) = O(\gamma_k^{-1})$ as $\hat{g}_k^{j+1}(z_k)$ is bounded away from 1. Calculating the derivative

$$\hat{g}_k'(z) = \frac{1}{\alpha_k} - \frac{\alpha_k}{(\alpha_k\gamma_k - 1)z^2}$$

we see that $\hat{g}_k'(\hat{g}_k^j(z_k)) \to \infty$ when $\zeta^{(j)} = 0$, while $\hat{g}_k'(\hat{g}_k^j(z_k)) \to 1$ otherwise, and we conclude that $\rho_k \to \infty$. □

Assume that $\alpha_k = \omega[1 + \tau\sqrt{\epsilon_k}] + o(\sqrt{\epsilon_k})$, hence $\beta_k = \bar{\omega}[1 - \tau\sqrt{\epsilon_k}] + o(\sqrt{\epsilon_k})$, where $\omega \neq 1$ is a primitive q-th root of unity and $\tau \in \mathbb{C}$. For $z \in \mathbb{C}^*$ and $0 \leq j < q$, it follows from (16) that

$$f_k^q\left(\bar{\omega}^j[1 + z\sqrt{\epsilon_k}]\right) = \bar{\omega}^j\left[1 + \left(z + q\tau + \frac{1}{z + j\tau}\right)\sqrt{\epsilon_k}\right]) + o(\sqrt{\epsilon_k})$$

whence

$$\psi_{k,(j)}^{-1} \circ f_k^q \circ \psi_{k,(j)}(z) \to z + q\tau + \frac{1}{z + j\tau} = G_{q\tau}(z + j\tau) - j\tau \tag{18}$$

locally uniformly on $\mathbb{C}^*$, where

$$\psi_{k,(j)}(z) = \bar{\omega}^j(1 + z\sqrt{\epsilon_k}).$$

Similarly, if $\alpha_k \to \omega$ but $\frac{\alpha_k - \omega}{\sqrt{\epsilon_k}} \to \infty$ then

$$\frac{f_k^q\left(\bar{\omega}^j(1 + z\sqrt{\epsilon_k})\right)}{\alpha_k^q} = \begin{cases} 1 + \left(z + \frac{1}{z}\right)\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) & j = 0 \\ \bar{\omega}^j(1 + z\sqrt{\epsilon_k}) + o(\sqrt{\epsilon_k}) & j \neq 0 \end{cases}$$

and thus

$$\psi_{k,(j)}^{-1} \circ f_k^q \circ \psi_{k,(j)}(z) \to \infty$$

locally uniformly on $\mathbb{C}^*$. Applying (15) to the case $j = 0$, we deduce:

Proposition 2 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive q-th root of unity, and assume further that $\frac{\alpha_k - \omega}{\sqrt{\epsilon_k}} \to \omega\tau$, hence $\frac{\beta_k - \bar{\omega}}{\sqrt{\epsilon_k}} \to -\bar{\omega}\tau$, for some $\tau \in \widehat{\mathbb{C}}$. Then*

$$F_k^q \to G_{q\tau}$$

locally uniformly on $\mathbb{C}^*$.

Recalling Lemmas 6 and 8, we observe:

Proposition 3 *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive q-th root of unity. Assume further that $F_k^q \to G_T$ for some $T \in \mathbb{C}$, and let $\langle z_k \rangle \to \mathcal{Z}$ be cycles of period $n > 1$ and eigenvalues $\rho_k \to \rho_\infty \in \mathbb{C}$. Then $G_T(\mathcal{Z}) \subseteq \mathcal{Z}$. Furthermore:*

- *If* $\mathcal{Z} = \{\infty\}$ *then* $T = 0$.
- *If* $0 \in \mathcal{Z}$ *then* $G_T^m(\chi) = 0$, *whence* $G_T^{m+1}(\chi) = \infty = G_T^{m+2}(\chi)$, *for some* $\chi \in \{+1, -1\}$ *and* $1 \leq m < \frac{n}{q}$.
- *Otherwise,* $\mathcal{Z} = \langle\zeta\rangle \cup \{\infty\}$ *where* $\langle\zeta\rangle \subset \mathbb{C}$ *is a cycle of period* $m = \frac{n}{q}$ *and eigenvalue* ρ_∞, *or possibly a parabolic cycle of lower period if* $\rho_\infty = 1$.

Conversely, given an m-cycle $\langle\zeta\rangle$ of G_T there exist mq-cycles of F_k converging to $\langle\zeta\rangle \cup \{\infty\}$. In particular, for $T \neq 0$ there is a unique finite fixed point $\zeta = -\frac{1}{T}$ with eigenvalue $1 - T^2$, hence $\langle z_k\rangle \to \{\zeta, \infty\}$ for some q-cycles $\langle z_k\rangle$. As $\text{mult}_{G_T}(\zeta) = 1$, it follows from Lemma 6 that $\langle\hat{z}_k\rangle \to \{0, \infty\}$ for every convergent sequence of q-cycles $\langle\hat{z}_k\rangle \neq \langle z_k\rangle$. In view of Lemma 8, the eigenvalues of $\langle\hat{z}_k\rangle$ tend to ∞, as do those of all ℓ-cycles where $\ell \not\in \{1, q\}$ divides q, and thus

$$\frac{1}{1-\alpha_k^q} + \frac{1}{1-\beta_k^q} \to 1 - \frac{1}{T^2} \tag{19}$$

by (3). On the other hand, for $T = 0$ there is only the fixed point at ∞, so every convergent sequence of q-cycles of F_k tends to $\{\infty\}$ or $\{0, \infty\}$. The validity of (19) in this case is a particular consequence of the following:

Proposition 4 *Let* $F_k = F_{\gamma_k, \delta_k}$ *where* $\gamma_k \to \infty$. *Assume that* $\alpha_k \to \omega$, *hence* $\beta_k \to \bar{\omega}$, *where* $\omega \neq 1$ *is a primitive* q*-th root of unity, and let* $\langle z_k\rangle$ *be cycles of period* $n > 1$. *If* $\langle z_k\rangle \to \{\infty\}$ *then* $n = q$, *and every convergent sequence of* q*-cycles* $\langle\hat{z}_k\rangle \neq \langle z_k\rangle$ *tends to* $\{0, \infty\}$.

Proof: In view of Lemma 7, we may assume without loss of generality that $n = mq$ for some positive integer m. By (15), it is enough to show that for r and k sufficiently large at most one mq-cycle of f_k lies completely inside

$$V_k^r = \widehat{\mathbb{C}} - \bigcup_{j=1}^{q-1} \overline{D}_{k,(j)}^r$$

where $D_{k,(j)}^r = \{z \in \mathbb{C} : \ |z - \bar{\omega}^j| < r\sqrt{|\epsilon_k|}\}$. It follows from (16) that $f_k^{-mq}(\infty) \cap \overline{V}_k^r = \{\infty\}$ for large r and k, and thus all of the $2^{mq}-1$ finite poles of f_k^{mq} lie in $\bigcup_{j=0}^{q-1} D_{k,(j)}^r$. Consequently,

$$\sum_{z=f_k^{mq}(z)\in V_k^r} \text{mult}_{f_k^{mq}}(z) \ = \ 2^{mq} + 1 \ - \sum_{z=f_k^{mq}(z)\in\hat{\mathbf{C}}-\overline{V}_k^r} \text{mult}_{f_k^{mq}}(z)$$

provided that f_k^{mq} has no fixed points on ∂V_k^r, whence

$$\sum_{z=f_k^{mq}(z)\in V_k^r} \text{mult}_{f_k^{mq}}(z) = 2 \ - \ \sum_{j=0}^{q-1} \frac{1}{2\pi i} \int_{\partial D_{k,(j)}^r} \frac{1 - (f_k^{mq})'(z)}{z - f_k^{mq}(z)}\, dz$$

by the Argument Principle.

Observe that $G_0(z) = z + \frac{1}{z}$ has a fixed point of multiplicity 3 at ∞, and thus G_0^m has $2^{mq} - 2$ finite fixed points and $2^{mq} - 1$ finite poles. It follows as above that

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{1 - (G_0^m)'(z)}{z - G_0^m(z)}\, dz \;=\; -1$$

so long as $r > \max\{|z| :\ z \in \mathbb{C} \text{ and } z - G_0^m(z) \in \{0, \infty\}\}$. In view of (18),

$$\begin{aligned}\frac{1}{2\pi i} \int_{\partial D^r_{k,(j)}} \frac{1 - (f_k^{mq})'(z)}{z - f_k^{mq}(z)}\, dz \;&=\; \frac{1}{2\pi i} \int_{|z|=r} \frac{1 - \left(\psi_{k,(j)}^{-1} \circ f_k^{mq} \circ \psi_{k,(j)}\right)'(z)}{z - \left(\psi_{k,(j)}^{-1} \circ f_k^{mq} \circ \psi_{k,(j)}\right)(z)}\, dz \\ &=\; \frac{1}{2\pi i} \int_{|z|=r} \frac{1 - (G_0^m)'(z)}{z - G_0^m(z)}\, dz\end{aligned}$$

when k is sufficiently large, and thus

$$\sum_{z = f_k^{mq}(z) \in V_k^r} \text{mult}_{f_k^{mq}}(z) = q + 2.$$

We deduce that $f_k^{mq}(z) = z \in V_k^r$ implies $f_k^q(z) = z$ for large r and k depending only on m. If $\alpha_k^q = 1$ then $\text{mult}_{f_k^{mq}}(0) = q + 1$ and $\text{mult}_{f_k^{mq}}(\infty) = 1$, while if $\beta_k^q = 1$ then $\text{mult}_{f_k^{mq}}(0) = 1$ and $\text{mult}_{f_k^{mq}}(\infty) = q + 1$; in these cases f_k^{mq} has no other fixed points in V_k^r. Otherwise,

$$\text{mult}_{f_k^{mq}}(0) = 1 = \text{mult}_{f_k^{mq}}(\infty)$$

and it follows from (16) that the remaining q fixed points of f_k^q in V_k^r constitute a q-cycle of f_k. $\Box$

In view of Fatou's Theorem, the second assertion in Proposition 3 is sharpened by:

Proposition 5 *Let $F_k = F_{\gamma_k, \delta_k}$ where $\gamma_k \to \infty$, and let z_k be attracting points of period $n > 1$ with immediate basins B_k. If $z_k \to 0$ then $B_k \to 0$.*

Proof: In view of Proposition **1** we may assume without loss of generality that $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ where $\omega \neq 1$ is a root of unity. If k is large then $z_k \in \mathbb{D}$, so for $j \geq 0$ there are unique components $W_k^j \ni z_k$ of $F_k^{-nj}(\mathbb{D})$. We claim first that $W_k^1 \to 0$; otherwise, as W_k^1 is connected there exist $k_\ell \to \infty$ and $w_{k_\ell} \in W_{k_\ell}^1$ with $\sqrt{\epsilon_{k_\ell}} = o(w_{k_\ell})$, contradicting Lemma 5. In particular, $W_k^1 \subset \mathbb{D}$ and thus $W_k^{j+1} \subset W_k^j$ for $j \geq 0$ and sufficiently large k. We denote W_k^∞ the component of z_k in the interior of $\bigcap_{j=0}^\infty W_k^j$ and contend that $W_k^\infty = B_k$. By definition, if $\zeta \in B_k$ there exists open $U \ni \zeta$ such that $F_k^{nj}(U) \subset \mathbb{D}$ when j is large, while if $\zeta \in \partial W_k^\infty$ there exist $\zeta_j \to \zeta$ with $F_k^{nj}(\zeta_j) \in \partial\mathbb{D}$. Thus, $B_k \cap \partial W_k^\infty = \emptyset$ and as $z_k \in B_k$ it follows that $B_k \subseteq W_k^\infty$; conversely, $W_k^\infty \subseteq B_k$ as F_k^{nj} is bounded, hence normal, on W_k^∞. $\Box$

5 Precompactness

Recall that a rational map is *hyperbolic* if and only if the orbit of every critical point tends to some attracting cycle. As discussed in [11, 17], there are four configurations for quadratics:

B Both critical points lie in the immediate basin of the same attracting cycle, but in different components.

C Both critical points lie in the basin of the same attracting cycle, but only one lies in the immediate basin.

D The critical points lie in the immediate basins of distinct attracting cycles.

E Both critical points lie in the same component of the immediate basin of an attracting fixed point.

There is in fact a unique hyperbolic component of type E consisting of maps with totally disconnected Julia set. This component is unbounded; see [11] for details. Our main result is that components of type D are bounded, so long as neither attractor is a fixed point:

Theorem 1 *Let g_k be quadratic rational maps, each having distinct nonrepelling cycles of periods $n^{\pm} > 1$. Then $[g_k]$ is bounded in $\mathbf{Rat}_2$.*

Proof: It follows from the proof of the Fatou-Shishikura Inequality that we lose no generality in assuming that these cycles are attracting [18]. Suppose to the contrary that $[g_k]$ is unbounded in $\mathbf{Rat}_2$. By Lemma 3 we may, passing to a subsequence if necessary, choose representatives $F_k = F_{\gamma_k,\delta_k} \in [g_k]$ where $\gamma_k \to \infty$. Let $\langle z_k^{\pm} \rangle$ be the corresponding $n^{\pm}$-cycles of F_k. Without loss of generality, $\langle z_k^{\pm} \rangle \to \mathcal{Z}^{\pm} \subset \widehat{\mathbb{C}}$; it follows from Propositions 1 and 2 that we may also assume that $\alpha_k \to e^{2\pi i p/q}$ and $F_k^q \to G_T$ for some $q \geq 2$ and $T \in \mathbb{C}$. In view of Fatou's Theorem we may label the critical points so that ± 1 lies in the immediate basin of $\langle z_k^{\pm} \rangle$.

By Proposition 3, if $\mathcal{Z}^{\pm} = \{\infty\}$ then $T = 0$ and thus also $\mathcal{Z}^{\mp} = \{\infty\}$ by Lemma 2, in contradiction to Proposition 4. Thus, $\mathcal{Z}^{\pm} = \langle \zeta^{\pm} \rangle \cup \{\infty\}$ for some nonrepelling cycle $\langle \zeta^{\pm} \rangle \subset \mathbb{C}$ of G_T, or else $G_T^{m_{\pm}}(\pm 1) = 0$ for some $m_{\pm} \geq 1$ as a consequence of Proposition 5. Applying Lemma 2, we deduce that $\mathcal{Z}^{+} - \{\infty\} = \langle \zeta \rangle = \mathcal{Z}^{-} - \{\infty\}$ for some cycle $\langle \zeta \rangle$. It follows from Lemma 1 that $\langle \zeta \rangle$ is parabolic-attracting or parabolic-indifferent, once again contradicting Lemma 2. □

The same considerations apply when there is one nonrepelling cycle along with a preperiodic critical point:

Theorem 2 *Let g_k be quadratic rational maps with nonrepelling n-cycles $\langle z_k \rangle$ where $n > 1$. Assume further that $g_k^{\ell}(\chi_k) \in \langle \hat{z}_k \rangle$ for some $\ell > 0$, critical points χ_k, and $\hat{n}$-cycles $\langle \hat{z}_k \rangle$. Then $[g_k]$ is bounded in $\mathbf{Rat}_2$.*

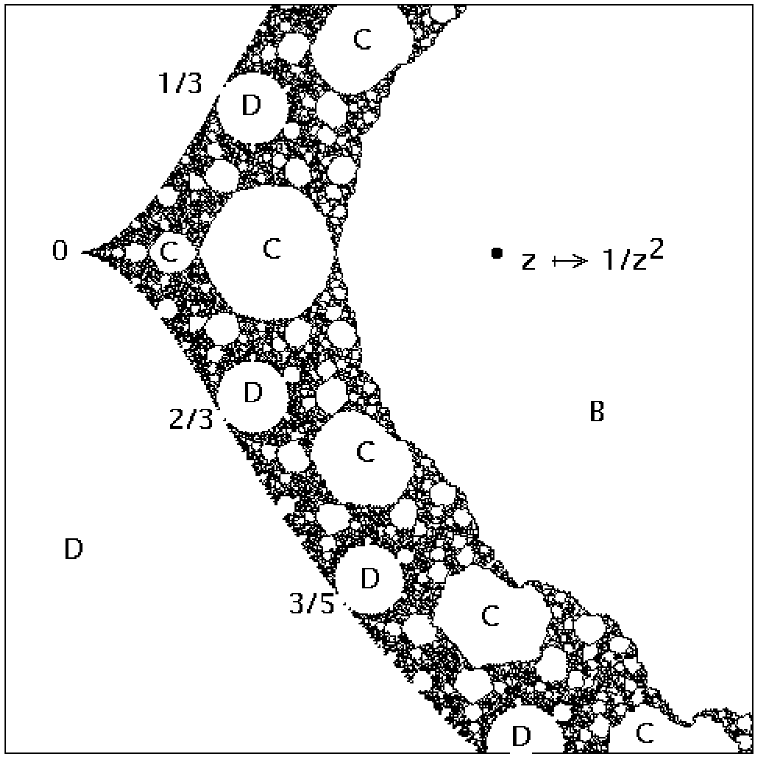

Figure 2: Bifurcation locus in $\mathrm{Per}_2(0)$.

The exceptional type D components are known to be unbounded; see Lemma 10 below. Many, though not all type D maps arise as *matings* of pairs of hyperbolic quadratic polynomials. In this construction, the filled-in Julia sets are glued back-to-back along complex-conjugate prime ends; see [2, 21] for further details. It is tempting to speculate that our arguments could be refined to establish precompactness for large portions of the mating locus, but our results in this direction are rather limited at present

Examination of Figure 2 suggests that the type C components are all bounded. This would follow immediately from our arguments if it could be shown in this case that $F_k^{-1}(\mathcal{B}_k) - \mathcal{B}_k \to 0$ where $\mathcal{B}_k$ is the immediate basin of the unique attracting cycle. There are evidently many unbounded type B components. Makienko [6] has obtained a degree-independent sufficient condition for unboundedness, loosely speaking the existence of a family of closed Poincaré geodesics on the basin quotient with lifts linking to separate the Julia set; see also [15]. On the other hand, there are type B maps which do not admit such a family: Pilgrim cites the example $g(z) = \frac{i\sqrt{3}}{2}(z + \frac{1}{z})$ and describes its Julia set as an *almost-Sierpinski carpet.* Such maps presumably lie in bounded hyperbolic components.

A good deal of what is known about hyperbolic quadratic rational maps - that Fatou components are usually Jordan domains [16], that polynomials can be mated if and only if they do not lie in conjugate limbs of the Mandelbrot set [21], that mating is discontinuous due to the existence of type D hyperbolic components whose closures are not homeomorphic to $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$ [2], that moduli space is isomorphic to $\mathbb{C}^2$ - is valid with minor changes for higher degree *bicritical maps* possessing two maximally degenerate critical points. Much of the discussion here extends similarly, and Milnor has recently generalized Lemma 3 to this larger setting: if $[g_k]$ is unbounded then the eigenvalues of all but at most two fixed points tend to infinity [13]. However, it is not immediately apparent how best to adapt the brute-force calculations of Section 4, or better yet, how to replace them with a more conceptual approach applicable to other degenerating families.

6 Intersection Theory

The results above yield preliminary information about the intersection theory at infinity of dynamically defined curves in moduli space. Milnor's isomorphism $j : \mathbf{Rat}_2 \to \mathbb{C}^2$ induces a natural compactification $\widehat{\mathbf{Rat}}_2 \cong \mathbb{P}^2$. Following the discussion in [12] we identify the line at infinity $\mathcal{L}$ with the set of unordered triples $[\alpha, \alpha^{-1}, \infty]$ where $\alpha \in \widehat{\mathbb{C}}$, so that $\alpha + \alpha^{-1}$ is the limiting ratio of $\frac{Y}{X-2}$ in the coordinates of Section 3; see [19] for a treatment in language of geometric invariant theory. With this convention, an unbounded sequence $[g_k] \in \mathbf{Rat}_2$ converges to the ideal point $[\alpha, \alpha^{-1}, \infty]$ if and only if $[\alpha_k, \beta_k, \gamma_k] \to [\alpha, \alpha^{-1}, \infty]$, where $\alpha_k, \beta_k, \gamma_k$ are the eigenvalues of the fixed points of g_k. Note that the degeneration described in Proposition 2 takes place in a parameter space where

$$\infty_{p/q} = [e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty] = \infty_{(q-p)/q}$$

has been blown up and replaced by a 2-fold branched cover of the line $\mathrm{Per}_1(1)$.

Recall that a *curve* C in $\mathbb{P}^2$ may be defined as an equivalence class of homogeneous polynomials in $\mathbb{C}[W, X, Y]$, where $H \sim \tilde{H}$ when $H = \lambda\tilde{H}$ for some $\lambda \in \mathbb{C}^*$. We write $C = \underline{H}$ and deg $C =$ deg H; a point $P \in \mathbb{P}^2$ with homogeneous coordinates $[w : x : y]$ *belongs to* $C = \underline{H}$ if and only if $H(w, x, y) = 0$, and we write $P \in C$. An *algebraic family* of degree d curves parametrized by a variety Λ is a regular map $\Lambda \to \mathcal{C}_d$, where the set $\mathcal{C}_d$ of all degree d curves is naturally regarded as the projective space $\mathbb{P}^{\frac{d(d+3)}{2}}$.

If H has no nontrivial factors then $C = \underline{H}$ is said to be *irreducible*. An irreducible curve $\hat{C} = \underline{\hat{H}}$ with $\hat{H}|H$ is a *component* of C, and curves C_1 and C_2 with no common component are said to *intersect properly*. Notice that $C = \underline{H}$ intersects $\mathcal{L}$ properly if and only if deg $H(1, X, Y) =$ deg H. Curves C_1, C_2 which intersect properly have finitely many points in common, and each such point can be assigned an appropriate *intersection multiplicity* $\mathcal{I}_{C_1,C_2}(P) > 0$; the *intersection cycle* is the formal sum

$$C_1 \bullet C_2 = \sum_{P \in C_1 \cap C_2} \mathcal{I}_{C_1,C_2}(P) \cdot P\ .$$

Bezout's Theorem asserts that the total intersection multiplicity is the product of the degrees

$d_i = \deg\ C_i$, whence $C_1 \bullet C_2$ may be regarded as an element of the symmetric product

$$\mathcal{S}_{d_1 d_2} \;=\; \mathrm{Sym}^{d_1 d_2}(\mathbb{P}^2).$$

Moreover, $(C_1, C_2) \rightsquigarrow C_1 \bullet C_2$ yields a regular map

$$\mathcal{C}_{d_1} \times \mathcal{C}_{d_2} \,-\, \mathcal{E}_{d_1,d_2} \to \mathcal{S}_{d_1,d_2}$$

where $\mathcal{E}_{d_1,d_2}$ is the set of pairs of curves with a common component; see [3] for further details. The *intersection cycle at infinity* is

$$C_1 \bullet_\infty C_2 \;=\; \sum_{P \in C_1 \cap \mathcal{L} \cap C_2} \mathcal{I}_{C_1,C_2}(P) \cdot P\ .$$

Consider the function $n \rightsquigarrow d(n)$ defined inductively by the relation

$$\sum_{m|n} d(m) = 2^{n-1};$$

equivalently, $d(n)$ is the number of period n hyperbolic components of the Mandelbrot set M. Milnor has shown the following [12]:

Lemma 9 *For each $n \geq 1$ there is a algebraic family of curves*

$$\mathbb{C} \ni \rho \rightsquigarrow \mathrm{Per}_n(\rho) \in \mathcal{C}_{d(n)}$$

uniquely determined by the condition that $[g] \in \mathrm{Per}_n(\rho)$ for $\rho \neq 1$ if and only if g has an n-cycle with eigenvalue ρ. The curves $\mathrm{Per}_n(1)$ are reducible for $n > 1$, indeed

$$\mathrm{Per}_n(1) = \mathrm{Per}_n^{\#}(1) \,\cup \bigcup_{1<q|n,\,(p,q)=1} \mathrm{Per}_{\frac{n}{q}}(e^{2\pi i p/q})$$

where the generic $[g] \in \mathrm{Per}_n^{\#}(1)$ has an n-cycle of eigenvalue 1.

Here are the defining polynomials for $n = 1, 2, 3$:

$$\begin{array}{ll}
\mathrm{Per}_1(\rho): & \rho^3 W - \rho^2 X + \rho Y - X + 2W \\
\mathrm{Per}_2(\rho): & \rho W - 2X - Y \\
\mathrm{Per}_3(\rho): & \rho^2 W^3 - \rho[WX(2X+Y) + 3W^2X + 2W^3] + (X+Y)^2(2X+Y) \\
 & -\,WX(X+2Y) + 12W^2X + 28W^3
\end{array}$$

Notice that

$$\mathrm{Per}_1(\rho) \bullet \mathcal{L} = [\rho, \rho^{-1}, \infty].$$

For $n > 1$, it follows from Proposition 1 that $\mathrm{Per}_n(\rho) \bullet \mathcal{L}$ consists of points of the form $\infty_{p/q}$ where $1 \leq p < q \leq n$; it is easily verified that

$$\mathrm{Per}_2(\rho) \bullet \mathcal{L} = \infty_{1/2} \tag{20}$$

and

$$\mathrm{Per}_3(\rho) \bullet \mathcal{L} = \infty_{1/2} + 2 \cdot \infty_{1/3} \tag{21}$$

for every $\rho \in \mathbb{C}$. The degeneration described in Proposition 2 takes place in a parameter space where $\infty_{p/q}$ has been blown up and replaced by a 2-fold branched cover of the line $\mathrm{Per}_1(1)$.

Recall that the p/q*−limb* of M is the set $L_{p/q}$ of all parameter values for which $P_c(z) = z^2 + c$ has a fixed point of combinatorial rotation number p/q; see for example [4]. Given $\alpha \in \mathbb{D}$, it follows from standard deformation considerations [8] that there is a unique class $[P_{c,(\alpha)}] \in \mathrm{Per}_1(\alpha)$ consisting of maps which are quasiconformally conjugate to P_c on a neighborhood of the filled-in Julia set $K(P_c)$ through conjugacies with vanishing dilatation on $K(P_c)$. Petersen [12, 14] showed the following by a modulus estimate similar to that in the proof of Yoccoz Inequality:

Lemma 10 *Let $P_c(z) = z^2 + c$ where $c \in M$. If $\alpha_k \in \mathbb{D}$ converges nontangentially to $e^{-2\pi i p/q} \neq 1$ then $[P_{c_k,(\alpha_k)}] \to \infty_{p/q} \in \mathcal{L}$ uniformly for $c_k \in L_{p/q}$.*

Each $c \in L_{p/q}$ determines an arc $\left\{[P_{c,(\alpha)}] : \; e^{2\pi i p/q}\alpha \in [0,1)\right\}$ with an endpoint at $[P_c]$. In view of Lemma 10 the other endpoint is $\infty_{p/q}$; in particular, each of the period n components in $L_{p/q}$ yields a branch of $\mathrm{Per}_n(\rho)$ at $\infty_{p/q}$, at least for $\rho \in \overline{\mathbb{D}}$. As distinct P_c lie on disjoint arcs, it follows from Lemma 9 and Bezout's Theorem that there are $d(n)$ such branches in total. Thus,

$$\mathrm{Per}_n(\rho) \bullet \mathcal{L} \;=\; \sum_{1<q|n,\,(p,q)=1} d_{p/q}(n) \cdot \infty_{p/q} \tag{22}$$

where $d_{p/q}(n)$ is the cardinality of

$$L_{p/q}(n) = \{W : \; W \subset L_{p/q} \text{ is a period } n \text{ component of } M\};$$

as $\rho \rightsquigarrow \mathrm{Per}_n(\rho)$ is continuous, (22) holds for every $\rho \in \mathbb{C}$, in accordance with Proposition 1. Moreover, it follows conversely that $[P_{c_k,(\alpha_k)}]$ is bounded in $\mathbf{Rat}_2$ if $c_k \in W \subset L_{p/q}$ and $\alpha_k \in \mathbb{D}$ is bounded away from $e^{-2\pi i p/q}$. Indeed, if $[P_{c_k,(\alpha_k)}] \in \mathrm{Per}_n(\rho_k)$ is unbounded then necessarily $[P_{c_k,(\alpha_k)}] \to \infty_{p/q}$, hence $\alpha_k \to e^{\pm 2\pi i p/q}$ after passage to a subsequence; the sign is determined by the fact the $d_{p/q}(n)$ nearby points of $\mathrm{Per}_1(\alpha_k)\bullet$ $\mathrm{Per}_n(\rho_k)$ are all deformations of polynomials P_c with $c \in L_{p/q}$ rather than $L_{-p/q}$.

As shown in [2], the intersection of $\mathrm{Per}_{n^+}(\rho^+)$ and $\mathrm{Per}_{n^-}(\rho^-)$ is generically proper. It is somewhat surprising that there are nontrivial exceptions: as observed in [12], it follows from (3) that

$$\mathrm{Per}_2(-3) = \mathrm{Per}_3^{\#}(1). \tag{23}$$

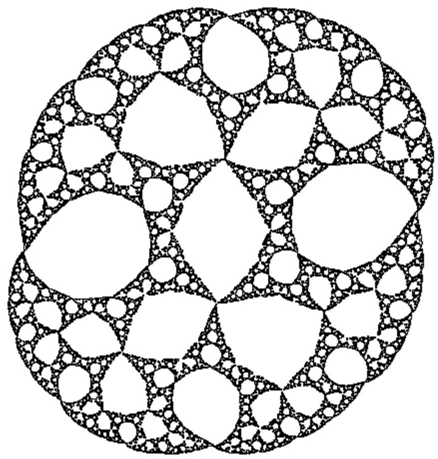

Figure 3: $J(f)$ for $f \in \mathcal{RAT}_2$ with critical points of periods 2 and 3.

In conjunction with the explicit expressions following Lemma 9, this coincidence yields a short independent proof of Theorem 1 in the special case $(n^+, n^-) = (2, 3)$. There are exactly two such hyperbolic components, a complex-conjugate pair obtained by mating the unique period 2 component in M with the period 3 components in $L_{1/3}$ and $L_{2/3}$; see Figure 3.

Recall that a quadratic rational map has precisely two 3-cycles counting multiplicity, whence

$$\mathrm{Per}_2(-3) \bullet \mathrm{Per}_3(\rho^-) = \mathrm{Per}_3^{\#}(1) \bullet \mathrm{Per}_3(\rho^-) = 3 \cdot \infty_{1/2}$$

for $\rho^- \neq 1$ by (20), (21), (23) and Bezout's Theorem; thus, $\mathrm{Per}_2(-3)$ and $\mathrm{Per}_3(\rho^-)$ are tangent at $\infty_{1/2}$. In view of the transversality of distinct lines $\mathrm{Per}_2(\rho^+)$, the curves $\mathrm{Per}_2(\rho^+)$ and $\mathrm{Per}_3(\rho^-)$ are transverse at $\infty_{1/2}$ provided that they intersect properly. Consequently,

$$\mathrm{Per}_2(\rho^+) \bullet_\infty \mathrm{Per}_3(\rho^-) = \infty_{1/2}$$

for $(\rho^+, \rho^-) \neq (-3, 1)$; in particular, for $(\rho^+, \rho^-) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ the points in

$$\mathrm{Per}_2(\rho^+) \bullet \mathrm{Per}_3(\rho^-) \;-\; \mathrm{Per}_2(\rho^+) \bullet_\infty \mathrm{Per}_3(\rho^-)$$

are uniformly bounded away from $\mathcal{L}$.

Conversely, it follows from Theorem 1 and the remarks after (22) that

$$\mathrm{Per}_{n^+}(\rho^+) \bullet_\infty \mathrm{Per}_{n^-}(\rho^-) \;=\; \sum_{\substack{1 \le p < q \le \min(n^+, n^-) \\ (p,q)=1}} \mathcal{I}_{p/q} \cdot \infty_{p/q}$$

independent of $\rho^\pm$, for $n^\pm > 1$ and generic $(\rho^+, \rho^-) \in \mathbb{C}^2$. Heuristic considerations supported by calculations in [20] suggest that

$$\mathcal{I}_{p/q} \;=\; \sum_{(W^+, W^-) \in L_{p/q}(n^+) \times L_{p/q}(n^-)} \iota(W^+, W^-)$$

where $\iota(W^+, W^-)$ measures the mutual combinatorial depth of $W^\pm$ in M. The language of *internal addresses* [5] may be useful in the formulation and proof of this assertion.

References

[1] A. Epstein, *Algebraic dynamics I: contraction and finiteness principles*, manuscript in preparation.

[2] A. Epstein, *Counterexamples to the quadratic mating conjecture I*, manuscript in preparation.

[3] W. Fulton, *Algebraic curves*, Benjamin/Cummings, Reading, Massachusetts, 1974.

[4] J. Hubbard, *Three theorems of J.-C. Yoccoz*, pp. 467-511 in *Topological Methods in Modern Mathematics*, Publish or Perish, Houston, 1992.

[5] E. Lau & D. Schleicher, *Internal addresses in the Mandelbrot set, and irreducibility of polynomials*, Stony Brook IMS Preprint 1994/19.

[6] P. Makienko, *Unbounded components in parameter space of rational maps*, ICTP Preprint IC/93/84, Miramar-Trieste, 1993.

[7] P. Makienko, *Pinching deformations of rational maps*, pp. 174-7 in *Proceedings of the International Conference on Dynamical Systems and Chaos*, World Scientific, Singapore, 1995.

[8] R. Mañe, P. Sad & D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Ec. Norm. Sup. **16** (1983), 193-217.

[9] C. McMullen, *The classification of conformal dynamical systems*, pp. 323-360 in *Current Developments in Mathematics, 1995*, International Press, 1995.

[10] J. Milnor, *Dynamics in one complex variable: Introductory lectures*, Stony Brook IMS Preprint 1990/5.

[11] J. Milnor, *Hyperbolic components in spaces of polynomial maps*, Stony Brook IMS Preprint 1992/3.

[12] J. Milnor, *Geometry and dynamics of quadratic rational maps*, Exp. Math. **2** (1993), 37-83.

[13] J. Milnor, *Geometry of bicritical rational maps*, manuscript in preparation.

[14] C. Petersen, *No elliptic limbs for quadratic rational maps*, Mat-Report #1994-45, Mathematical Institute, Technical Institute of Denmark, 1994.

[15] K. Pilgrim, *Cylinders for iterated rational maps*, Thesis, University of California at Berkeley, 1994.

[16] K. Pilgrim, *Rational maps whose Fatou components are Jordan domains*, Erg. Thy. & Dyn. Sys. **16** (1996), 1323-1343.

[17] M. Rees, *Components of degree two hyperbolic maps*, Inv. Math. **100** (1990), 357-382.

[18] M. Shishikura, *On the quasi-conformal surgery of rational functions*, Ann. Sci. Ec. Norm. Sup. 4^e Ser. **20** (1987), 1-29.

[19] J. Silverman, *The space of rational maps on* $\mathbb{P}^1$, preprint, 1996.

[20] J. Stimson, *Degree two rational maps with a periodic critical point*, Thesis, University of Liverpool, 1993.

[21] Tan Lei, *Matings of quadratic polynomials*, Erg. Thy. & Dyn. Sys. **12** (1992), 589-620.

Address:
Institute for Mathematical Sciences
Department of Mathematics
State University of New York at Stony Brook
Stony Brook, NY 11794-3651

E-mail: adame@math.sunysb.edu