Dynamical Cocycles with Values in the Artin Braid Group

J.-M. Gambaudo and E. E. Pécou

Abstract: By considering the way an n-tuple of points in the 2-disk are linked together under iteration of an orientation preserving diffeomorphism, we construct a dynamical cocycle with values in the Artin braid group. We study the asymptotic properties of this cocycle and derive a series of topological invariants for the diffeomorphism which enjoy rich properties.

1 Introduction

When one is concerned with the study of a group of automorphisms on a probability space $(X, \mathcal{B}, \mu)$, the knowledge of some cocycle associated to this group is useful. Indeed, cocycles carry into a simple, well understood, target group G the main dynamical features. By using subadditive functions on G, for instance left-invariant metrics, one is able to define asymptotic invariants with relevant dynamical properties. As an illustration, see the Oseledec theory of the Lyapounov exponents ([7]).

In this paper, we start by recalling some basic definitions related to cocycles in general and we state some nice asymptotic properties when a subadditive function on the target group is given. Namely, we show the existence of asymptotic invariants and give conditions for their topological invariance (section 2).
Section 3 deals with the study of orientation preserving C^1-diffeomorphisms of the 2-disk which preserve a non atomic measure. Given a n-tuple of distinct points in the disk, we construct on the group of diffeomorphisms we are considering a cocycle with values in the Artin braid group B_n. Indeed, we show that given a diffeomorphism there is a well defined way to associate a braid to a n-tuple of orbits. This construction generalizes a well-known construction for periodic orbits.
Despite the relative complexity of the Artin braid group, there are naturally defined subadditive functions on B_n. Therefore, we can consider the asymptotic invariants associated to these cocycles. These invariants turn to be topological invariants that we can relate to other well known quantities such as the Calabi invariant and the topological entropy (section 4). In this last case, we generalize a result of minoration of the entropy by Bowen (see [3]).

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2 Cocycles

2.1 Asymptotic behaviour

Let $(X, \mathcal{B}, \mu)$ be a probability space, $Aut(X, \mu)$, the group of all its automorphisms, i.e. invertible measure preserving transformations and G a topological group (with identity element e). For any subgroup Γ of $Aut(X, \mu)$, we say that a measurable map $\alpha : X \times \Gamma \to G$ is a *cocycle* of the dynamical system $(X, \mathcal{B}, \mu, \Gamma)$ with values in G if for all γ_1 and γ_2 in Γ and for μ-a.e. x in X:

$$\alpha(x, \gamma_1\gamma_2) \;=\; \alpha(x, \gamma_1)\,\alpha(\gamma_1 x, \gamma_2).$$

A continuous map $\theta : G \to \mathbb{R}^+$, which satisfies, for all g_1 and g_2 in G:

$$\theta(g_1 g_2) \;\leq\; \theta(g_1) \,+\, \theta(g_2)$$

is called a *subadditive function* on G.
For example, if the group G is equipped with a right (or left) invariant metric d, then it is clear that the map:

$$\begin{array}{ccc} G & \to & \mathbb{R}^+ \\ x & \mapsto & d(g, e) \end{array}$$

is a subadditive function.

The following Theorem is a straightforward application of the subadditive ergodic theorem (see for instance [8]).

Theorem 1 *Let α be a cocycle of the dynamical system $(X, \mathcal{B}, \mu, \Gamma)$ with values in a group G and θ a subadditive function on G. Assume that there exists an automorphism γ in Γ such that the map:*

$$\begin{array}{ccc} X & \to & \mathbb{R}^+ \\ x & \mapsto & \theta(\alpha(x, \gamma)) \end{array}$$

is integrable. Then:

1. *For μ-a.e. x in X, the sequence $\frac{1}{n}\theta(\alpha(x, \gamma^n))$ converges, when n goes to $+\infty$ to a limit denoted by $\Theta(x, \gamma, \alpha)$.*

2. *The map:*

$$\begin{array}{ccc} X & \to & \mathbb{R}^+ \\ x & \mapsto & \Theta(x, \gamma, \alpha) \end{array}$$

is integrable and we have :

$$\lim_{n \to +\infty} \frac{1}{n} \int \theta(\alpha(x, \gamma^n)) d\mu \;=\; \inf_{n \to +\infty} \frac{1}{n} \int \theta(\alpha(x, \gamma^n)) d\mu \;=\; \int \Theta(x, \gamma, \alpha) d\mu.$$

We denote by $\Theta_\mu(\gamma, \alpha)$ the quantity $\int \Theta(x, \gamma, \alpha) d\mu$.

2.2 Invariance property

Let α and β be two cocycles of the dynamical system $(X, \mathcal{B}, \mu, \Gamma)$ with value in a group G. We say that α and β are *cohomologous* if there exists a measurable function $\phi : X \to G$ such that for all γ in Γ and for μ-a.e. x in X:

$$\beta(x, \gamma) \;=\; \phi(x)\, \alpha(x, \gamma)\, (\phi(\gamma\, x))^{-1}.$$

Consider two probability spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ and an isomorphism $h : X_1 \to X_2$, (i.e. an invertible transformation which carries the measure μ_1 onto the measure μ_2). Let Γ_1 be a subgroup of $Aut(X_1, \mu_1)$ and $\Gamma_2 \;=\; h\Gamma_1 h^{-1}$. Given a cocycle α_1 on $(X_1, \mathcal{B}_1, \mu_1, \Gamma_1)$, we denote by $h \circ \alpha_1$ the cocycle defined on $(X_2, \mathcal{B}_2, \mu_2, \Gamma_2)$ by:

$$h \circ \alpha_1(x_2, \gamma_2) \;=\; \alpha_1(h^{-1}(x_2), h^{-1}\gamma_2 h),$$

for all $\gamma_2 \in \Gamma_2$ and for all $x_2 \in X_2$.

Given two dynamical systems $(X_1, \mathcal{B}_1, \mu_1, \Gamma_1)$ and $(X_2, \mathcal{B}_2, \mu_2, \Gamma_2)$ and given two cocycles α_1 and α_2 defined respectively on the first and the second system and with values in the same group G, we say that the two cocycles α_1 and α_2 are *weakly equivalent* if there exists an isomorphism $h : X_1 \to X_2$ which satisfies :

1. $\Gamma_2 \;=\; h\Gamma_1 h^{-1}$,
2. α_2 and $h \circ \alpha_1$ are cohomologous.

Theorem 2 *Let α_1 and α_2 be two cocycles defined on the dynamical systems $(X_1, \mathcal{B}_1, \mu_1, \Gamma_1)$ and $(X_2, \mathcal{B}_2, \mu_2, \Gamma_2)$ respectively and with values in a group equipped with a subadditive function θ . Assume that α_1 and α_2 are weakly equivalent through an isomorphism h.*
If $\gamma_1 \in \Gamma_1$ and $\gamma_2 = h\gamma_1 h^{-1}$ satisfy that both maps:

$$x_1 \mapsto \theta\,(\alpha_1(x_1, \gamma_1)) \qquad \text{and} \qquad x_2 \mapsto \theta\,(\alpha_2(x_2, \gamma_2))$$

are integrable respectively on X_1 and X_2, then:

$$\Theta_{\mu_1}(\gamma_1, \alpha_1) \;=\; \Theta_{\mu_2}(\gamma_2, \alpha_2).$$

Before proceeding to the proof of the theorem, let us recall a basic lemma in ergodic theory:

Lemma 1 *Let $(X, \mathcal{B}, \mu)$ be a probability space, γ an automorphism on X, G a group, θ a subadditive function on G and $\phi : X \to G$ a measurable map. Then, for μ-a.e. x in X:*

$$\liminf_{n \to +\infty} \theta(\phi(\gamma^n x)) \;<\; +\infty.$$

Proof: Let N be a positive integer and

$$\mathcal{E}_N \;=\; \{x \in X \mid \theta(\phi(x)) \leq N\}.$$

Since the map ϕ is measurable the set $\mathcal{E}_N$ is measurable and has a positive measure for N big enough. From the Poincaré recurrence theorem (see for instance [8]) we know that, for μ-a.e x in $\mathcal{E}_N$, the orbit $\gamma^n(x)$ visits $\mathcal{E}_N$ infinitely often and thus:

$$\liminf_{n\to+\infty} \theta(\phi(\gamma^n x)) \;\leq\; N.$$

Since the union $\cup_{N\geq 0}\mathcal{E}_N$ is a set of μ-measure 1 in X, the lemma is proved. □

Proof of Theorem 2: The weak equivalence of the cocycles α_2 and α_1 leads to the existence of some measurable function $\phi : X_2 \to G$ such that, for all $n \geq 0$, for all γ_2 in Γ_2 and for μ_2-a.e. x_2 in X_2:

$$\alpha_2(x_2, \gamma_2^n) \;=\; \phi(x_2)\, h \circ \alpha_1(x_2, \gamma_2^n)\, (\phi(\gamma_2^n x_2))^{-1}.$$

The subadditivity property of the map $\theta : G \to \mathbb{R}^+$ gives us:

$$|\theta(\alpha_2(x_2, \gamma_2^n)) \;-\; \theta(h \circ \alpha_1(x_2, \gamma_2^n))| \;\leq\; \theta(\phi(x_2)) \;+\; \theta(\phi(\gamma_2^n x_2)^{-1}).$$

From Lemma 1, it follows that for all γ_2 in Γ_2, and for μ_2 a.e. x_2 in X_2:

$$\liminf_{n\to+\infty} |\theta(\alpha_2(x_2, \gamma_2^n)) \;-\; \theta(h \circ \alpha_1(x_2, \gamma_2^n))| \;<\; +\infty.$$

Thus:

$$\liminf_{n\to+\infty} |\frac{1}{n}\theta(\alpha_2(x_2, \gamma_2^n)) \;-\; \frac{1}{n}\theta(h \circ \alpha_1(x_2, \gamma_2^n))| \;=\; 0.$$

Since the map $\theta\,(\alpha_i(\,.\,, \gamma_i))$ is integrable for $i = 1$ and $i = 2$, it results from Theorem 1 that for μ_i-a.e. x in X_i:

$$\lim_{n\to+\infty} \frac{1}{n}\theta\,(\alpha_i(x, \gamma_i^n)) = \Theta(x, \gamma_i, \alpha_i)$$

Recalling that the definition of $h \circ \alpha_1(x_2, \gamma_2^n)$ is $\alpha_1(h^{-1}(x_2), h^{-1}\gamma_2^n h)$, we get that, for μ_2-a.e. x_2 in X_2:

$$\Theta(h^{-1}(x_2), \gamma_1, \alpha_1) \;=\; \Theta(x_2, \gamma_2, \alpha_2).$$

By integrating this equality with respect to μ_2, we get:

$$\Theta_{\mu_1}(\gamma_1, \alpha_1) \;=\; \Theta_{\mu_2}(\gamma_2, \alpha_2).$$

3 Braids and Dynamics

3.1 The Artin braid group

For a given integer $n > 0$, the Artin braid group B_n is the group defined by the generators $\sigma_1, \ldots, \sigma_{n-1}$ and the relations:

$$\begin{array}{rcll} \sigma_i\sigma_j &=& \sigma_j\sigma_i, & |i-j| \geq 2, \quad 1 \leq i, j \leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i &=& \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n-2. \end{array}$$

An element in this group is called a *braid*.

A geometrical representation of the Artin braid group is given by the following construction. Let $\overline{\mathbb{D}}^2$ denote the closed unit disk in $\mathbb{R}^2$ centered at the origin and $\mathbb{D}^2$ its interior. Let $Q_n = \{q_1, \ldots, q_n\}$ be a set of n distinct points in $\mathbb{D}^2$ equidistributed on a diameter of $\mathbb{D}^2$. We denote by $\mathbb{D}_n$ the n-punctured disk $\overline{\mathbb{D}}^2 \setminus Q_n$.
We define a collection of n arcs in the cylinder $\mathbb{D}^2 \times [0,1]$:

$$\Gamma_i = \{(\gamma_i(t), t)\,,\, t \in\, [0,1]\}, \quad 1 \leq i \leq n$$

joining the points in $Q_n \times \{0\}$ to the points in $Q_n \times \{1\}$ and such that $\gamma_i(t) \neq \gamma_j(t)$ for $i \neq j$. We call $\Gamma = \cup_{i=0,\ldots,n} \Gamma_i$ a *geometrical braid* and say that two geometrical braids are equivalent if there exists a continuous deformation from one to the other through geometrical braids. The set of equivalence classes is the Artin braid group. The composition law is given by concatenation as shown in Figure 1 and a generator σ_i corresponds to the geometrical braid shown in Figure 2.

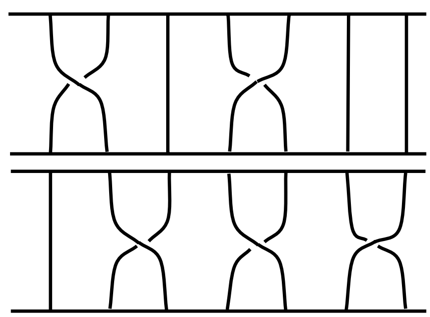

Figure 1: The concatenation of two geometrical braids

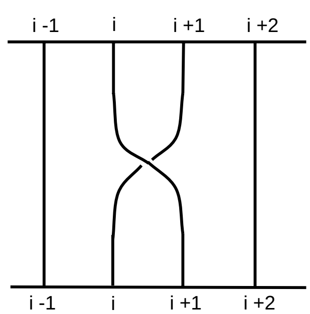

Figure 2: Representation of the i-th generator σ_i of the Artin braid group

Let $\text{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ be the subset of homeomorphisms from $\overline{\mathbb{D}}^2$ onto itself which are the iudentity map in a neighborhood on the boundary of the disk. The subgroup of elements which let the set Q_n globally invariant is denoted by $\text{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, Q_n)$. The following Theorem, due to J. Birman, shows the relation between braids and dynamics.

Theorem 3 ([2]) *The Artin braid group B_n is isomorphic to the group of automorphisms of $\pi_1(\mathbb{D}_n)$ which are induced by elements of* $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, Q_n)$*, that is to say the group of isotopy classes of* $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, Q_n)$.

This isomorphism $\mathcal{I}$ can be described as follows. Let $\{x_1, \ldots, x_n\}$ be a basis for the free group $\pi_1(\mathbb{D}_n)$, where x_i, for $1 \leq i \leq n$ is represented by a simple loop which encloses the boundary point q_i (see Figure 3). A generator σ_i of B_n induces on $\pi_1(\mathbb{D}_n)$ the following automorphism $\mathcal{I}(\sigma_i)$:

$$\mathcal{I}(\sigma_i) \begin{cases} x_i & \mapsto \; x_i \, x_{i+1} \, {x_i}^{-1} \\ x_{i+1} & \mapsto \; x_i \\ x_j & \mapsto \; x_j \quad \text{if} \;\; j \neq i \, , \, i+1. \end{cases}$$

The action of the Artin braid group on $\pi_1(\mathbb{D}_n)$ is a right action. We denote by wb the image of $w \in \pi_1(\mathbb{D}_n)$ under the automorphism induced by the braid b.

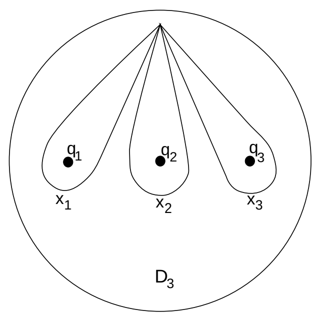

Figure 3: The generators of $\pi_1(\mathbf{D}_n)$

We now introduce two standard subadditive functions on the Artin braid group. Given a group presented by a finite number of generators and relations and g an element of the group, we denote by $L(g)$ the minimal length of the word g written with the generators and their inverses. We can define two subadditive functions on B_n by measuring lengths of words either in the Artin braid group or in the fundamental group of the punctured disk $\mathbb{D}_n$. More precisely, given an element b in B_n, the first subadditive function θ_1 is defined by:

$$\theta_1(b) \;=\; L(b).$$

Notice that by setting $d(b,c) \;=\; L(bc^{-1})$ for all b and c in B_n, we define a right invariant metric d on B_n.
The second subadditive function θ_2 is defined, for all b in B_n, by:

$$\theta_2(b) \;=\; \sup_{i=1,\ldots,n} \log L(x_i b),$$

where the x_i's are the generators of the fundamental group of $\mathbb{D}_n$ defined as above.

It is plain that the two subadditive functions are related as follows:

$$\theta_2(b) \ \leq \ (\log 3)\theta_1(b),$$

for all b in B_n.

In the sequel, we shall focus on a particular subgroup of the Artin braid group, that we now define. Let b be an element in B_n represented by a geometrical braid:

$$\Gamma = \cup_{i=0,\dots,n}\{(\gamma_i(t),t) \subset \mathbb{D}^2 \times [0,1], \ \ 1 \leq i \leq n\}.$$

We say that b is a *pure braid* if for all $i = 1,\dots,n$, $\gamma_i(0) = \gamma_i(1)$.
We denote by $\mathcal{F}_n$ the set of all pure braids in B_n (see for instance [10]) .

Remark: The pure braid group can be interpreted as follows. Consider a braid b in $\mathcal{F}_n$, and $\Gamma = \cup_{i=0,\dots,n}\{(\gamma_i(t),t), t \in [0,1]\}$ a geometrical braid whose equivalence class is b. We can associate to Γ the loop $(\gamma_1(t),\dots,\gamma_n(t))_{t\in[0,1]}$ in the space $\mathbf{D^2} \times \dots \times \mathbf{D^2} \setminus \Delta$, where Δ is the *generalized diagonal* :

$$\Delta \ = \ \{(p_1,\dots,p_n) \in \mathbf{D^2} \times \dots \times \mathbf{D^2}, \mid \ \exists\, i \neq j \ \text{ and } \ p_i = p_j\}.$$

In other words, there exist an isomorphism $\mathcal{J}$ between the pure braid group $\mathcal{F}_n$ and the fundamental group $\pi_1(\mathbf{D^2} \times \dots \times \mathbf{D^2} \setminus \Delta)$.

3.2 A cocycle with values in the Artin braid group

Let ϕ be an element in Homeo$(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, and let $P_n = (p_1,\dots,p_n)$ be a n-tuple of pairwise disjoint points in $\mathbb{D}^2$. The following procedure describes a natural way to associate a pure braid in $\mathcal{F}_n$ to the datas ϕ and P_n (see figure 4).

1. For $1 \leq i \leq n$, we join the point $(q_i,0)$ to the point $(p_i,\frac{1}{3})$ in the cylinder $\mathbb{D}^2 \times [0,\frac{1}{3}]$ with a segment.

2. For $1 \leq i \leq n$, we join the point $(p_i,\frac{1}{3})$ to the point $(\phi(p_i),\frac{2}{3})$ in the cylinder $\mathbb{D}^2 \times [\frac{1}{3},\frac{2}{3}]$ with the arc $(\phi_{(3t-1)}(p_i),t)_{t\in[\frac{1}{3},\frac{2}{3}]}$, where $(\phi_\tau)_{\tau\in[0,1]}$ is an isotopy from the identity to ϕ in Homeo$(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$.

3. For $1 \leq i \leq n$, we join the point $(\phi(p_i),\frac{2}{3})$ to the point $(q_i,1)$ in the cylinder $\mathbb{D}^2 \times [\frac{2}{3},1]$ with a segment.

The concatenation of this sequence of arcs provides a geometrical braid. The equivalence class of this geometrical braid is a braid in $\mathcal{F}_n$ that we denote by $\beta(P_n;\phi)$.

It is clear that the above procedure is well defined if and only if for all $1 \leq i < j \leq n$:

(i) the segment joining the points $(q_i,0)$ and $(p_i,\frac{1}{3})$ does not intersect the segment joining the points $(q_j,0)$ and $(p_j,\frac{1}{3})$, and

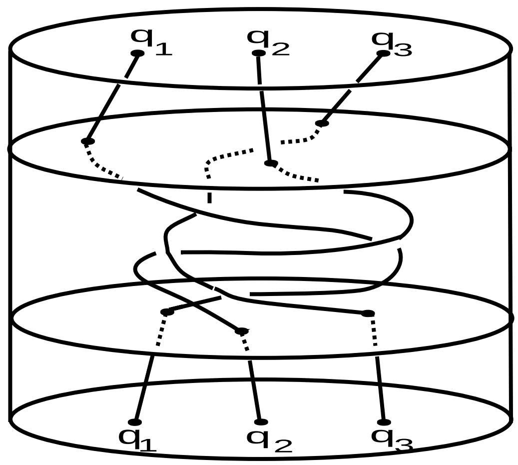

Figure 4: Construction of $\beta(P_n;\phi)$

(ii) the segment joining the points $(\phi(p_i), \frac{2}{3})$ and $(q_i, 1)$ does not intersect the segment joining the points $(\phi(p_j), \frac{2}{3})$ and $(q_j, 1)$.

For $1 \le i \ne j \le n$, consider in $\mathbb{R}^2 \times \ldots \times \mathbb{R}^2$ the codimension 2 plane $P_{i,j}$ of points $(z_1, \ldots, z_n)$ such that $z_i = z_j$. Let $H_{i,j}$ be the hyperplane which contains $P_{i,j}$ and the point $(q_1, \ldots, q_n)$. The plane $P_{i,j}$ splits the hyperplane $H_{i,j}$ in 2 components; we denote by $H^0_{i,j}$ the closure of the component which does not contain the point $(q_1, \ldots, q_n)$. Let Ω^{2n} be the open dense subset of $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$ defined by:

$$\Omega^{2n} \;=\; \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 \setminus \bigcup_{1\le i,j\le n} H^0_{i,j} \cap \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2.$$

We can reformulate conditions (i) and (ii) as follows : the braid $\beta(P_n;\phi)$ is defined if and only if both P_n and $(\phi, \ldots, \phi)(P_n)$ belong to Ω^{2n}.

Remark 1: This procedure is locally constant where it is defined. More precisely if $\beta(P_n;\phi)$ is defined and if P'_n is close enough to P_n in $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 \setminus \Delta$, then $\beta(P'_n;\phi)$ is also defined and $\beta(P'_n;\phi)$ is equal to $\beta(P_n;\phi)$.

Remark 2: Since the set $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ is contractible, this procedure does not depend on the isotopy ϕ_t.

In order to construct a cocycle with values in the Artin braid group, we also need to define the invariant measures we are going to consider. We say that a n-tuple of probability measures $\lambda_1, \ldots, \lambda_n$ on $\overline{\mathbb{D}}^2$ is *coherent* if the subset Ω^{2n} has measure 1 with respect to the product measure $\lambda_1 \times \ldots \times \lambda_n$. Notice that this is the case when the measures λ_i are non atomic.

Lemma 2 *Let ϕ be a map in* $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ *which preserves the coherent probability measures $\lambda_1, \ldots, \lambda_n$. Then, the map :*

$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 & \longrightarrow & \mathcal{F}_n \\ P_n & \longmapsto & \beta(P_n; \phi) \end{array}$$

is measurable with respect to the product measure $\lambda_1 \times \ldots \times \lambda_n$.

Proof: The map is continuous on Ω^{2n} which has measure 1 with respect to the product measure $\lambda_1 \times \ldots \times \lambda_n$. $\square$

There is a natural injection j from the set of maps from $\overline{\mathbb{D}}^2$ into itself into the set of maps from $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$ into itself. Namely:

$$j(\phi) = (\phi, \ldots, \phi).$$

Let $\lambda_1, \ldots, \lambda_n$ be coherent probability measures and $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \lambda_1, \ldots, \lambda_n)$ the set of maps in $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ which preserve the measures $\lambda_1, \ldots, \lambda_n$.
We call Γ the image of $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \lambda_1, \ldots, \lambda_n)$ by j in $Aut(\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2, \lambda_1 \times \ldots \times \lambda_n)$. The following proposition is straightforward:

Proposition 1 *The map:*

$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 \times \Gamma & \longrightarrow & \mathcal{F}_n \\ (P_n, j(\phi)) & \longmapsto & \beta(P_n; \phi) \end{array}$$

is a cocycle of the dynamical system $(\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2, \mathcal{B}(\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2), \lambda_1 \times \ldots \times \lambda_n, \Gamma)$ *with values in the group* $\mathcal{F}_n$ *(here* $\mathcal{B}(\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2)$ *stands for the Borel* σ*-algebra of* $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$*).*

3.3 Asymptotic limits and invariance

Let $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ denote the set of C^1-diffeomorphisms from $\overline{\mathbb{D}}^2$ into itself which are the identity in a neighborhood of the boundary. The following result is fundamental for our purpose.

Proposition 2 *If the map* ϕ *is in* $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ *then there exists a positive number* $K(\phi, n)$ *such that for all* P_n *where* $\beta(P_n; \phi)$ *is defined :*

$$\theta_1(\beta(P_n; \phi)) \leq K(\phi, n).$$

Proof: From the isomorphism $\mathcal{J}$ between $\mathcal{F}_n$ and $\pi_1(\mathbf{D^2} \times \ldots \times \mathbf{D^2} \setminus \Delta)$ (see section 3.1), we know that when the braid $\beta(P_n; \phi)$ is defined, it can be seen as a homotopy class of a loop in $\mathbf{D^2} \times \ldots \times \mathbf{D^2} \setminus \Delta$. The fundamental group $\pi_1(\mathbf{D^2} \times \ldots \times \mathbf{D^2} \setminus \Delta)$ possesses a finite set of generators (e_k). These generators can be expressed with the generators σ_i of the braid group B_n through the isomorphism $\mathcal{J}$. It follows that the proposition will be proved once we prove that the length of the word $\beta(P_n; \phi)$ (seen as a homotopy class written with the

system of generators (e_k)) is uniformely bounded when both P_n and $(\phi,\ldots,\phi)(P_n)$ are in Ω^{2n}.
Consider the blowing-up set $\mathcal{K}$ of the generalized diagonal Δ in $\overline{\mathbb{D}}^2\times\ldots\times\overline{\mathbb{D}}^2$. More precisely, $\mathcal{K}$ is the compact set of points:

$$(p_1,\ldots,p_i,\ldots,p_n;p'_1,\ldots,p'_j,\ldots p'_n;\Delta_{1,1},\ldots,\Delta_{i,j},\ldots,\Delta_{n,n})$$

where, for all $1\leq i,j\leq n$, $\Delta_{i,j}$ is a line containing p_i and p'_j. Obviously, if $p_i\neq p'_j$, then the line $\Delta_{i,j}$ is uniquely defined and thus $\overline{\mathbb{D}}^2\times\ldots\times\overline{\mathbb{D}}^2\setminus\Delta$ is naturally embedded in $\mathcal{K}$ as an open dense set.
A map ϕ in Homeo$(\overline{\mathbb{D}}^2,\partial\overline{\mathbb{D}}^2)$ yields a map $j(\phi)=(\phi,\ldots,\phi)$ defined and continuous on $\overline{\mathbb{D}}^2\times\ldots\times\overline{\mathbb{D}}^2$. Its restriction to $\mathbf{D^2}\times\ldots\times\mathbf{D^2}$ let Δ globally invariant. We claim that if the map ϕ is a C^1-diffeomorphism, then it can be extended to a continuous map on $\mathcal{K}$. This is done as follows: whenever for some $1\leq i,j\leq n$, we have $p_i=p'_j$, then a line $\Delta_{i,j}$ containing $p_i=p'_j$ is mapped to the line $d\phi(p_i)(\Delta_{i,j})$.
Furthermore, if the map ϕ is in Diff$^1(\overline{\mathbb{D}}^2,\partial\overline{\mathbb{D}}^2)$ we can choose an isotopy ϕ_t from the identity to ϕ in Diff$^1(\overline{\mathbb{D}}^2,\partial\overline{\mathbb{D}}^2)$. Thus, the map Ψ:

$$\begin{array}{rrcl}\Psi: & [0,1]\times\mathcal{K} & \longrightarrow & \mathcal{K}\\ & (t,p_1,\ldots,p_n) & \longmapsto & (\phi_t(p_1),\ldots,\phi_t(p_n))\end{array}$$

is continuous.
Let $\mathcal{K}^0$ be the universal cover of $\mathcal{K}$, $\pi:\mathcal{K}^0\to\mathcal{K}$ the standard projection, and $\Psi^0:[0,1]\times\mathcal{K}\to\mathcal{K}^0$ a lift of the map Ψ ($\Psi=\pi\circ\Psi^0$). The system of generators (e_k) can be chosen so that the projection π restricted to a fundamental domain of $\mathcal{K}^0$ is a homeomorphism onto Ω^{2n}. Since $\mathcal{K}$ is compact $\Psi^0([0,1]\times\mathcal{K})$ is also compact and consequently covers a bounded number $k(\phi,n)$ of fundamental domains of $\mathcal{K}^0$. If both $(p_1,\ldots,p_n)$ and $(\phi_1(p_1),\ldots,\phi(p_n))$ are in Ω^{2n}, the choice of the system of generators implies that the length of the word $\beta((p_1,\ldots,p_n);\phi)$ written with the system of generators (e_k) is also uniformly bounded by $k(\phi,n)$. This achieves the proof of the proposition. □

Lemma 3 *Let ϕ be a map in* Diff$^1(\overline{\mathbb{D}}^2,\partial\overline{\mathbb{D}}^2)$ *which preserves the coherent probability measures $\lambda_1,\ldots,\lambda_n$. Then, for $i=1$ and $i=2$ the map:*

$$\begin{array}{rcl}\overline{\mathbb{D}}^2\times\ldots\times\overline{\mathbb{D}}^2 & \longrightarrow & \mathbb{R}^+\\ P_n & \longmapsto & \theta_i(\beta(P_n;\phi))\end{array}$$

is integrable with respect to $\lambda_1\times\ldots\times\lambda_n$.

Proof: The integrability is a direct consequence of Proposition 2 and of the fact that $\theta_2(b)\leq(\log 3)\theta_1(b)$ for all b in B_n. □

By applying Theorem 1 to the cocycle β and using lemma 3, we immediately get:

Corollary 1 *Let ϕ be an element in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ which preserves the coherent probability measures $\lambda_1, \ldots, \lambda_n$. Then for $i = 1$ and $i = 2$:*

1. *For $\lambda_1 \times \ldots \times \lambda_n$ a.e. P_n in $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$ the quantity $\frac{1}{N}\theta_i(\beta(P_n;\phi^N))$ converges when N goes to $+\infty$ to a limit $\Theta^{(i)}(P_n;\phi)$.*

2. *The map*
$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 & \longrightarrow & \mathbb{R}^+ \\ P_n & \longmapsto & \Theta^{(i)}(P_n;\phi) \end{array}$$
is integrable on $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$ with respect to $\lambda_1 \times \ldots \times \lambda_n$.

We denote by $\Theta^{(i)}_{\lambda_1,\ldots,\lambda_n}(\phi)$ the integral of this function.

Corollary 2 *Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be two sets of coherent probability measures on $\mathbb{D}^2$, and let ϕ_1 and ϕ_2 be two elements in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ which preserve the probability measures $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ respectively.*
Assume there exists a map h in $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ such that :

1. *$h \circ \phi_1 \;=\; \phi_2 \circ h$,*

2. *$h * \lambda_j \;=\; \mu_j$, , for $j = 1, \ldots, n$.*

Then for $i = 1$ and $i = 2$:
$$\Theta^{(i)}_{\lambda_1,\ldots,\lambda_n}(\phi_1) \;=\; \Theta^{(i)}_{\mu_1,\ldots,\mu_n}(\phi_2).$$

Proof: Consider the cocycle α_1:
$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 \times j(\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \lambda_1, \ldots, \lambda_n)) & \longrightarrow & \mathcal{F}_n \\ (P_n, j(\phi_1)) & \longmapsto & \beta(P_n;\phi_1) \end{array}$$

and the cocycle α_2:
$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 \times j(\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \mu_1, \ldots, \mu_n)) & \longrightarrow & \mathcal{F}_n \\ (P_n, j(\phi_2)) & \longmapsto & \beta(P_n;\phi_2) \end{array}$$

Thanks to Theorem 2, the corollary will be proved once we show that α_1 and α_2 are weakly equivalent. Consider the homeomorphism $j(h)$ of $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$ into itself. It is clear that $j(h) * \lambda_1 \times \ldots \times \lambda_n = \mu_1 \times \ldots \times \mu_n$ and that:
$$j(\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \lambda_1, \ldots, \lambda_n)) = j(h) j(\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2, \mu_1, \ldots, \mu_n))(j(h))^{-1}.$$

Thus it remains to prove that α_2 and $j(h) \circ \alpha_1$ are cohomologous.
For $\lambda_1 \times \ldots \times \lambda_n$ a.e. P_n in $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$, we have:
$$\beta(P_n;\phi_2) \;=\; \beta(P_n;h^{-1})\; \beta(j(h^{-1})(P_n);\phi_1)\; \beta(j(\phi_1 \circ h^{-1})(P_n);h).$$

This reads :

$$\alpha_2(P_n, j(\phi_2)) \;=\; \beta(P_n; h^{-1})\; h \circ \alpha_1(P_n, j(\phi_2))\; \beta(j(\phi_1 \circ h^{-1})(P_n)); h).$$

Since the map h^{-1} is in Homeo$(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ we know from Lemma 2 that the map:

$$\begin{array}{rcl} \overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2 & \longrightarrow & \mathcal{F}_n \\ P_n & \longmapsto & \beta(P_n; h^{-1}) \end{array}$$

is measurable. This shows that α_2 and $j(h) \circ \alpha_1$ are cohomologous. □

4 A discussion about these invariants

4.1 The fixed points case

In the particular case when we consider a set of fixed points $P_n^0 = (p_1^0, \ldots, p_n^0)$ of $\phi \in$ Diff$^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, we get, for all $N \geq 0$:

$$\beta(P_n^0; \phi^N) \;=\; \beta(P_n^0; \phi)^N,$$

and consequently, for $i = 1$ and $i = 2$, we have:

$$\Theta^{(i)}(P_n^0; \phi) \;=\; \theta_i(\beta(P_n^0; \phi)).$$

For the set of Dirac measures $\delta_{p_1^0}, \ldots, \delta_{p_n^0}$, which are coherent, the invariant numbers $\Theta^{(i)}_{\delta_{p_1^0}, \ldots, \delta_{p_n^0}}$ have a clear meaning:

- The invariant number $\Theta^{(1)}_{\delta_{p_1^0}, \ldots, \delta_{p_n^0}}$ is the minimum number of generators σ_i which are necessary to write the word $\beta(P_n^0; \phi)$. In the particular case of a pair of fixed points (p_1^0, p_2^0), it is the absolute value of the linking number of these two fixed points.

- The map ϕ induces a map $\phi_\star$ on the first homotopy group of the punctered disk $\overline{\mathbb{D}}^2 \setminus P_n^0$. The invariant $\Theta^{(2)}_{\delta_{p_1^0}, \ldots, \delta_{p_n^0}}$ is the growth rate of the map $\phi_\star$. It has been shown by R. Bowen [3] that this growth rate is a lower bound of the topological entropy $h(\phi)$ of the map ϕ [1]. The assumption on the map ϕ to be a diffeomorphism is required in order to get a continuous map acting on the compact surface $\hat{\mathbb{D}}_n$ obtained from $\mathbb{D}_n = \overline{\mathbb{D}}^2 \setminus \{q_1, \ldots, q_n\}$ after blowing up the points $q_1, \ldots, q_n$ with the circles of directions.

4.2 The general case

Let ϕ be a map in Diff$^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$. From Proposition 2, there exists a positive number $K(\phi, n)$ such that for all P_n where $\beta(P_n; \phi)$ is defined:

$$\frac{1}{\log 3}\theta^{(2)}(\beta(P_n; \phi)) \leq \theta^{(1)}(\beta(P_n; \phi)) \leq K(\phi, n).$$

Let $\mathcal{M}_n(\phi)$ be the set of n-tuples of ϕ-invariant, coherent probability measures, and let $(\lambda_1, \dots, \lambda_n)$ be in $\mathcal{M}_n(\phi)$. By integrating with respect to $\lambda_1 \times \dots \times \lambda_n$, we get:

$$\frac{1}{\log 3}\Theta^{(2)}_{\lambda_1,\dots,\lambda_n}(\phi) \leq \Theta^{(1)}_{\lambda_1,\dots,\lambda_n}(\phi) \leq K(\phi, n).$$

It follows that the quantities:

$$\Theta^{(i)}_n(\phi) \;=\; \sup_{(\lambda_1,\dots,\lambda_n)\in\mathcal{M}_n(\phi)} \Theta^{(i)}_{\lambda_1,\dots,\lambda_n}(\phi),$$

are positive real numbers which are, by construction, topological invariants. More precisely, for any pair of maps ϕ_1 and ϕ_2 in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, which are conjugated by a map in $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, we have, for $i = 1$, $i = 2$ and for all $n \geq 2$:

$$\Theta^{(i)}_n(\phi_1) \;=\; \Theta^{(i)}_n(\phi_2).$$

It also results from the definitions that, for $i = 1$ and $i = 2$ the sequences $n \mapsto \Theta^{(i)}_n(\phi)$ are non decreasing sequences.
In the next two paragraphs, we give estimates of these invariants that generalize the fixed points case.

4.2.1 The Calabi invariant as a minoration of the sequence $(\Theta^{(1)}_n(\phi))_n$

Let ϕ be a map in $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ and consider an isotopy ϕ_t from the identity to ϕ in $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$. To a pair of distinct points p_1 and p_2 in $\mathbb{D}^2$, we can associate a real number $Ang_\phi(p_1, p_2)$ which is the angular variation of the vector $\overrightarrow{\phi_t(p_1)\phi_t(p_2)}$ when t goes from 0 to 1 (when normalizing to 1 the angular variation of a vector making one loop on the unit circle in the direct direction). Since the set $\mathrm{Homeo}(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ is contractible, it is clear that the map $(p_1, p_2) \mapsto Ang_\phi(p_1, p_2)$ does not depend on the choice of the isotopy. If ϕ is in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, using arguments similar to the ones we used in the proof of Proposition 2, it is easy to check (see [5]) that the function Ang_ϕ is bounded where it is defined, that is to say on $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \Delta$. Let (λ_1, λ_2) be a pair of ϕ-invariant, coherent probability measures. The function Ang_ϕ is integrable on $\overline{\mathbb{D}}^2 \times \overline{\mathbb{D}}^2$ with respect to $\lambda_1 \times \lambda_2$ and the *Calabi invariant* of the map ϕ with respect to the pair (λ_1, λ_2) is defined by the following integral:

$$\mathcal{C}_{\lambda_1,\lambda_2}(\phi) \;=\; \int\!\!\int_{\overline{\mathbb{D}}^2\times\overline{\mathbb{D}}^2} Ang_\phi(p_1, p_2)\, d\lambda_1(p_1)\, d\lambda_2(p_2)$$

Remark: In [4], E. Calabi defines a series of invariant numbers associated to symplectic diffeomorphisms on symplectic manifolds. In the particular case of area preserving maps of the 2-disk, only one of these invariant quantities is not trivial. In [5], it is shown that this invariant fits with our definition in the particular case when the invariant measures λ_i are all equal to the area.

Proposition 3 *Let ϕ be a map in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ which preserves a pair of coherent probability measures (λ_1, λ_2). Then:*

$$|\mathcal{C}_{\lambda_1,\lambda_2}(\phi)| \leq \Theta^{(1)}_{\lambda_1,\lambda_2}(\phi).$$

Consequently, for any map ϕ in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, and for all $n \geq 2$:

$$\sup_{(\lambda_1,\lambda_2)\in\mathcal{M}_2(\phi)} |\mathcal{C}_{\lambda_1,\lambda_2}(\phi)| \leq \Theta^{(1)}_2(\phi) \leq \Theta^{(1)}_n(\phi).$$

Proof: In order to compute the Calabi invariant of ϕ with respect to (λ_1, λ_2), we can use the Birkhoff ergodic theorem. The corresponding Birkhoff sums:

$$Ang_\phi(p_1,p_2) + Ang_\phi(\phi(p_1),\phi(p_2)) + \ldots + Ang_\phi(\phi^{N-1}(p_1),\phi^{N-1}(p_2)),$$

are equal to:

$$Ang_{\phi^N}(p_1,p_2).$$

It follows that for $\lambda_1 \times \lambda_2$ almost every points (p_1, p_2) the following limit:

$$\tilde{Ang}_\phi(p_1,p_2) = \lim_{N\to+\infty} \frac{1}{N} Ang_{\phi^N}(p_1,p_2),$$

exists and is integrable. Furthermore:

$$\int\int_{\overline{\mathbb{D}}^2\times\overline{\mathbb{D}}^2} \tilde{Ang}_\phi(p_1,p_2)d\lambda(p_1)d\lambda(p_2) = \int\int_{\overline{\mathbb{D}}^2\times\overline{\mathbb{D}}^2} Ang_\phi(p_1,p_2)d\lambda(p_1)d\lambda(p_2).$$

It is clear that for all $N \geq 0$ and for all pair of distinct points p and q in D^2, we have the following estimate:

$$|L(\beta(p,q;\phi^N)) - |Ang_{\phi^N}(p,q)|| \leq 3.$$

Dividing by N and making N going to ∞, we get for $\lambda_1 \times \lambda_2$ a.e (p_1, p_2):

$$|\tilde{Ang}_\phi(p_1,p_2)| = \Theta^{(1)}(p_1,p_2;\phi).$$

The integration gives:

$$|\mathcal{C}_{\lambda_1,\lambda_2}(\phi)| \leq \int\int_{\overline{\mathbb{D}}^2\times\overline{\mathbb{D}}^2} |\tilde{Ang}_\phi(p_1,p_2)|d\lambda_1(p_1)d\lambda_2(p_2) \leq \Theta^{(1)}_{\lambda_1,\lambda_2}(\phi).$$

□

4.2.2 The topological entropy as a majoration of the sequence $(\Theta_{2,n}(\phi))_n$

Let $\mathrm{Diff}^\infty(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ denote the subset of elements in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ which are C^∞-diffeomorphisms. The following theorem can be seen as an extension of the Bowen result ([3]) when we relax the hypothesis of invariance of finite sets.

Theorem 4 *Let ϕ be an element in $\mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2)$ with entropy $h(\phi)$ and $\lambda_1, \ldots, \lambda_n$ a set of coherent, ϕ-invariant probability measures. Then, for $\lambda_1 \times \ldots \times \lambda_n$-a.e. P_n in $\overline{\mathbb{D}}^2 \times \ldots \times \overline{\mathbb{D}}^2$, we have:*

$$\Theta^{(2)}(P_n; \phi) \; \leq \; h(\phi),$$

and consequently, for all $n \geq 2$:

$$\Theta^{(2)}_{\lambda_1 \times \ldots \times \lambda_n}(\phi) \; \leq \; h(\phi).$$

Proof: Let $P_n = (p_1, \ldots, p_n)$ be a point in Ω^{2n} and let h_{P_n} be an element in $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ with the following properties:

1. For each $1 \leq i \leq n$, h_{P_n} maps q_i on p_i.
2. There exists an isotopy $(h_{P_n;t})_{t \in [0,1]}$ from the identity to h_{P_n} such that, for each $1 \leq i \leq n$, the arc $\{(h_{P_n;t}, t) \;\; t \in [0,1]\}$ coincides with the segment joining $(q_i, 0)$ to $(p_i, 1)$ in $D^2 \times [0,1]$.
3. The map
$$\begin{array}{rcl} \Omega^{2n} & \longrightarrow & \mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2) \\ P_n & \longmapsto & h_{P_n} \end{array}$$
is continuous when $\mathrm{Diff}^1(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ is endowed with the C^1 topology.

Let ϕ be in $\mathrm{Diff}^\infty(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$ and assume that for $N \geq 0$, P_n and $j(\phi^N)(P_n)$ are in Ω^{2n}. From the above construction , it follows that the map:

$$\Psi^{(N)}_{P_n} \; = \; h^{-1}_{j(\phi^N)(P_n)} \circ \phi^N \circ h_{P_n}$$

let the points $q_1, \ldots, q_n$ invariant and that its restriction to $\mathbb{D}_n$ induces, through the Birman isomorphism (Theorem 3), the braid $\beta(P_n; \phi^N)$.
Let P_n be such that the limit $\Theta^{(2)}(P_n; \phi)$ exists (which is true for a $\lambda_1 \times \ldots \times \lambda_n$ full measure set of points). It follows that there exists $\epsilon > 0$ such that, for N big enough, we have:

$$\Theta^{(2)}(P_n; \phi) \, - \epsilon \leq \frac{1}{N} \theta^{(2)}(\beta(P_n; \phi^N))$$

which reads:

$$(*) \;\; e^{(\Theta^{(2)}(P_n;\phi) \, - \epsilon)N} \leq \sup_{i=1,\ldots,n} \; L(x_i \beta(P_n; \phi^N))$$

It is standard that there exists a constant $c(n) > 0$ such that for any differentiable loop τ in $\hat{\mathbb{D}}_n$:

$$L([\tau]) \; \leq \; c(n) l(\tau)$$

where l stands for the euclidian length and $[-]$ is the homotopy class in $\hat{\mathbb{D}}_n$. Combined with (*), this argument gives:

$$\frac{1}{c(n)} e^{(\Theta^{(2)}(P_n;\phi) \, - \epsilon)N} \leq \sup_{i=1,\ldots,n} \; l(\Psi^{(N)}_{P_n}(x_i)).$$

That is to say:

$$\frac{1}{c(n)} e^{(\Theta^{(2)}(P_n;\phi)-\epsilon)N} \leq \sup_{i=1,\dots,n} l(h^{-1}_{j(\phi^N)(P_n)} \circ \phi^N \circ h_{P_n}(x_i))$$

and consequently:

$$(**) \quad \frac{1}{c(n)} e^{(\Theta^{(2)}(P_n;\phi)-\epsilon)N} \leq \|h^{-1}_{j(\phi^N)(P_n)}\|_1 \sup_{i=1,\dots,n} l(\phi^N(h_{P_n}(x_i)),$$

where $\| - \|_1$ stands for the C^1 norm.
Assume from now on that the point P_n is a recurrent point of the map $j(\phi)$ (which is true for a $\lambda_1 \times \ldots \times \lambda_n$ full measure set of points) and let $\nu(N)$ be a sequence of integers so that:

$$\lim_{N\to\infty} j(\phi^{\nu(N)})(P_n) \;=\; P_n.$$

From (**) we deduce that there exists a subsequence $\hat{\nu}(N)$ of $\nu(N)$ such that for N big enough, and for some $1 \leq i_0 \leq n$, we have:

$$\frac{1}{c(n)K} e^{(\Theta^{(2)}(P_n;\phi)-\epsilon)N} \;\leq\; l(\phi^N(h_{P_n}(x_{i_0})),$$

where K is an upper bound of $\|h^{-1}_{P_n}\|_1$.
In conclusion, the euclidian length of the loop $h_{P_n}(x_{i_0})$ increases exponentially under iteration of the map ϕ with a growth rate which is at least $\Theta^{(2)}(P_n;\phi) - \epsilon$. Using a result by Y. Yomdim (see [9] and [6]), in the case of C^∞ maps, we know that this provides a lower estimate of the topological entropy. Namely, if ϕ is in $\mathrm{Diff}^\infty(\overline{\mathbb{D}}^2, \partial\overline{\mathbb{D}}^2)$, we get:

$$\Theta^{(2)}(P_n;\phi) - \epsilon \;\leq\; h(\phi).$$

Since, this is true for any $\epsilon > 0$, this yields:

$$\Theta^{(2)}(P_n;\phi) \;\leq\; h(\phi),$$

and by integrating:

$$\Theta^{(2)}_{\lambda_1\times\ldots\times\lambda_n}(\phi) \;\leq\; h(\phi).$$

□

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J.-M. Gambaudo - INSTITUT NON LINÉAIRE DE NICE, U.M.R. du C.N.R.S 129, 1361 route des Lucioles, 06560 Valbonne, France.

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E. E. Pécou - INSTITUTE FOR MATHEMATICAL SCIENCES, SUNY at Stony Brook, Stony Brook, NY 11794, U.S.A.

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