Geometry of the Feigenbaum map.

by
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Abstract.

We show that the Feigenbaum-Cvitanović equation can be interpreted as a linearizing equation, and the domain of analyticity of the Feigenbaum fixed point of renormalization as a basin of attraction. There is a natural decomposition of this basin which enables to recover a result of local connectivity by Jiang and Hu for the Feigenbaum Julia set.

Keywords. Dynamics, puzzle, renormalization, Feigenbaum, local connectivity.

We have included in this article several computer drawn pictures. We would like to thank especially Henri Epstein who gave us the picture of the domain of analyticity of the Feigenbaum map, Louis Granboulan who helped us drawing some of the pictures using the program Pari, and Dan Sørensen who designed a wonderful program to draw Mandelbrot and Julia sets.

I am grateful to Adrien Douady for having taught me so much about dynamical systems and renormalization. Most of the tools to deal with the Feigenbaum equation were explained to me by Henri Epstein, and most of my ideas came during discussions with him in IHES, with Marguerite Flexor in Orsay and with John H. Hubbard in Cornell. I am greatly indebted to those four people.

Introduction.

The notion of renormalization for dynamical systems was introduced by Feigenbaum and Cvitanović. Landford and Sullivan have then proved the existence of fixed points of renormalization. Those fixed points satisfy a functional equation which has been studied by Eckmann, Epstein, Wittwer and others. We will make use of this functional equation, known as the Feigenbaum-Cvitanović equation, to give a new approach to a result by Jiang and Hu regarding to the local connectivity of the Julia set of the Feigenbaum polynomial.

We assume that the reader is familiar with the notion of renormalization and the notion of polynomial-like mappings. One can for example read [DH], [McM] or [S]. The central object in our study is the Feigenbaum polynomial. It is the most famous example of polynomial

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which is infinitely renormalizable (i.e. k-renormalizable for infinitely many k). It is the unique real quadratic polynomial $z^2 + c_{Feig}$ which is 2^k-renormalizable for all $k \geq 1$.

One can define the Feigenbaum polynomial as the unique real polynomial which is a fixed point of tuning by -1. Tuning is the inverse of renormalization. Given a parameter $c \in M$, such that 0 is a periodic point of period p, Douady and Hubbard [DH] have constructed a tuning map, $x \to c * x$, which is a homeomorphism of M into itself, sending 0 to c, and such that if $x \neq 1/4$, then f_{c*x} is p-renormalizable, and the corresponding renormalization is in the same inner class as f_x (where f_x is the complex polynomial $f_x(z) = z^2 + x$). This is the way they show there are small copies of the Mandelbrot set inside itself (cf figure 1).

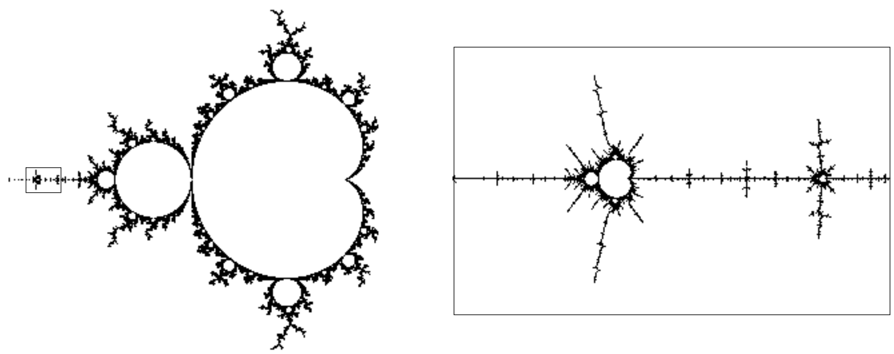

Figure 1: The Mandelbrot set and a small copy of it centered at a periodic point of period 3.

The Feigenbaum value, $c_{Feig} = -1.401155...$, is in the intersection of all the copies of M obtained by tuning by -1. This intersection is not known to consist of a single point, but its intersection with the real axis is exactly the point c_{Feig}.

By construction, the Feigenbaum polynomial, P_{Feig}, is 2-renormalizable. There are several mappings $g : U \to V$ such that $g = f^2|U$ is a polynomial-like map with connected Julia set. All those maps have the same Julia set, and are equal on this Julia set. We will say they define the same germ $[g]$ of polynomial-like map (cf [McM]).

We can define a renormalization operator $\mathcal{R}_2$ in the following way.

Definition 1 *Assume $[f]$ is a germ of a 2-renormalizable polynomial-like map. There exist open sets U and V such that the map $g : U \to V$ defined by $g = f^2|U$ is a polynomial-like map with connected Julia set. The renormalization operator $\mathcal{R}_2$ is defined by*

$$\mathcal{R}_2([f]) = [\alpha^{-1} \circ g \circ \alpha],$$

with $\alpha = g(0) = f^2(0)$, and $\alpha(z) = \alpha z$.

We have normalized the germ so that the critical value is 1.

Definition 2 *Two polynomial-like maps f and g representing germs are said to be hybrid equivalent if there is a quasi-conformal conjugacy between them with $\overline{\partial} h = 0$ almost everywhere on the filled Julia set $K(f)$.*

The equivalence classes are called inner classes. One of the main theorems in [DH] states that in the inner class any quadratic-like map with connected Julia set, there is a unique polynomial $P(z) = z^2 + c$, $c \in \mathbb{R}$.

Hence, if $[f]$ is a germ of Feigenbaum polynomial-like map, then $\mathcal{R}_2([f])$ is hybrid equivalent to a unique polynomial. This polynomial is 2^k-renormalizable for all $k \geq 1$. So it is the Feigenbaum polynomial. Hence $\mathcal{R}_2([f])$ is still a germ of Feigenbaum map, and we can iterate this process, defining in such a way a sequence of germs: $\mathcal{R}_2^n([P_{Feig}])$, $n \in \mathbb{N}$. The following result has been proved and can be found in [S] or [McM].

Theorem 1 *The sequence of germs $\mathcal{R}_2^n([P_{Feig}])$, $n \in \mathbb{N}$, converges to a point $[\phi]$. This point is a fixed point of renormalization:*

$$\mathcal{R}_2([\phi]) = [\phi],$$

and is in the inner class of the Feigenbaum polynomial. It is the unique fixed point of $\mathcal{R}_2$.

The properties we have mentioned still hold if the degree of the critical point is any even integer. In the following ℓ is an even integer and f is an analytic function which coincides in a neighborhood of 0 with the real fixed point of $\mathcal{R}_2$ with critical point of degree ℓ.

In section 1, we state some results about this fixed point of renormalization. It is just a germ of polynomial-like map. However, we will show that there exists a natural representative of this germ. This construction has already been made by Epstein in [E], but we think it is worth making again here, to help the reader get accustomed with the tools.

In section 2, we give a description of the domain of analyticity of the fixed point of renormalization. McMullen [McM] proved that it has a maximal analytic extension $\hat{f}$ to a dense, simply connected open set $\widehat{W} \subset \mathbb{C}$. In lemmas 1 and 2, we prove that it is the basin of attraction of the map $\hat{f}(\lambda z)$, where $\lambda = -f^2(0)$. In particular, this enables us to prove that $\widehat{W} \subset \widehat{W}/\lambda$ (theorem 2), and to give a dynamical interpretation of the intersection $\bigcap_{n\in\mathbb{N}} \lambda^n \widehat{W}$ (theorem 3).

The boundary of $\widehat{W}$ is a closed set with empty interior. We would like to try and give a description of it. We prove that the boundary of $\widehat{W}$ contains accessible points with at least two accesses (proposition 4). This proves, in particular, that this boundary does not have the structure of a Cantor Bouquet. Moreover, we think that both its Hausdorff dimension and its Lebesgue measure ought to be studied. The result of proposition 5 is a step in this direction.

In section 3, we show that the domain $\widehat{W}$ is naturally paved by puzzles that are homothetic to each other and cut the Julia set $K(f)$ in a connected way. This enables us to give a new proof of the local connectivity of the Julia set $K(f)$ at the critical point.

1 Cvitanović-Feigenbaum equation.

First of all, we would like to recall that f is a solution of the following system of equations (cf [B2] or [E]).

Definition 3 *The Cvitanović-Feigenbaum equation:*

$$\begin{cases} f(z) &= -1/\lambda f\circ f(\lambda z),\\ f(0) &= 1,\\ f(z) &= F(z^\ell),\ \textit{with } F^{-1} \textit{ univalent in } \mathbb{C}\setminus(]-\infty,-1/\lambda]\cup[1/\lambda^2,+\infty[).\end{cases}$$

We will first state some results by Henri Epstein in the following two propositions. Figure 2 illustrates proposition 1.

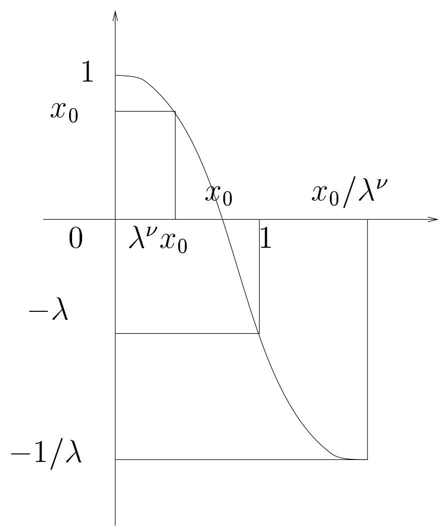

Figure 2: The graph of f on $\mathbb{R}^+$.

Proposition 1 *(cf [E]) Let f be a solution of the equation, and let x_0 be the first positive preimage of 0 by f. Then,*

- $f(1) = -\lambda$,
- $f(\lambda x_0) = x_0$, *and*
- *the first critical point in $\mathbb{R}^+$ is x_0/λ, with $f(x_0/\lambda) = -1/\lambda$.*

This graph enables us to deduce the relative positions of some points on the real axis.

Proposition 2 *(cf [E]) Univalent extension of F:*

- *it is possible to extend F^{-1} continuously to the boundary $\mathbb{R}$ of $\mathbb{H}_+$, and even analytically except at points $(-1/\lambda)^n$, $n\geq 1$, which are branching points of type $z^{1/\ell}$,*
- *the values of F^{-1} are never real except in $[-1/\lambda, 1/\lambda^2]$,*
- *the extension of F^{-1} to the closure of $\mathbb{H}_+$ is injective, and*

- *when z tends to infinity in $\mathbb{H}_+$, $F^{-1}(z)$ tends to a point in $\mathbb{H}_-$ which will be denoted by $F^{-1}(i\infty)$.*

By symmetry, similar statements hold in $\mathbb{H}_-$. Hence, $\mathcal{W}$ is a bounded domain of $\mathbb{C}$. Those results are summarized in figure 3.

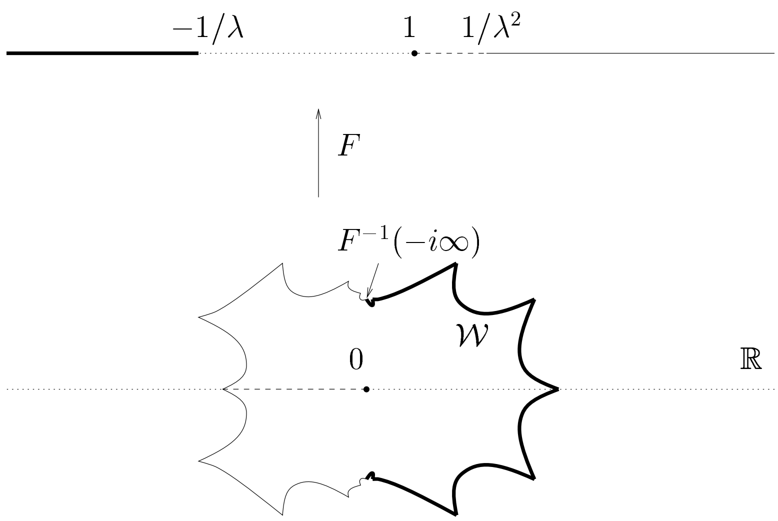

Figure 3: Maximal univalent extension of F.

In the following, we will use the notations:

$$\begin{aligned}
\mathbb{C}_\lambda &= \mathbb{C} \setminus (]-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, +\infty[), \\
\mathcal{W} &= F^{-1}(\mathbb{C}_\lambda), \\
W &= \{z \in \mathbb{C} \mid z^\ell \in \mathcal{W}\}, \\
W_+ &= W \cap \{z \in \mathbb{C} \mid 0 < \mathrm{Arg}(z) < \pi/\ell\}, \text{ and} \\
W_- &= W \cap \{z \in \mathbb{C} \mid -\pi/\ell < \mathrm{Arg}(z) < 0\}.
\end{aligned}$$

As $W_+ \cap \mathbb{R} = [0, x_0/\lambda[$, we can deduce that $W \cap \mathbb{R} =]-x_0/\lambda, x_0/\lambda[$ (here we use the fact that ℓ is an even degree).

Corollary 1 *(cf figure 4) The map $f : W \to \mathbb{C}_\lambda$ is a polynomial-like map of degree ℓ representing the renormalization fixed point.*

Proof. The graph of f (cf figure 2) enables us to conclude that $\overline{W} \subset \mathbb{C}_\lambda$, because $x_0 < 1$. Hence $f : W \to \mathbb{C}_\lambda$ is a polynomial-like map. Besides, f has a unique critical point in 0 of degree ℓ. Then by uniqueness of the fixed point, the corollary follows. ∎

Hence, this polynomial-like map is quasi-conformally conjugate to the Feigenbaum polynomial. Thus, to know the geometry of the Julia set of the Feigenbaum polynomial, it is enough to know the geometry of the Julia set $K(f)$ of this polynomial-like map (cf figure 4).

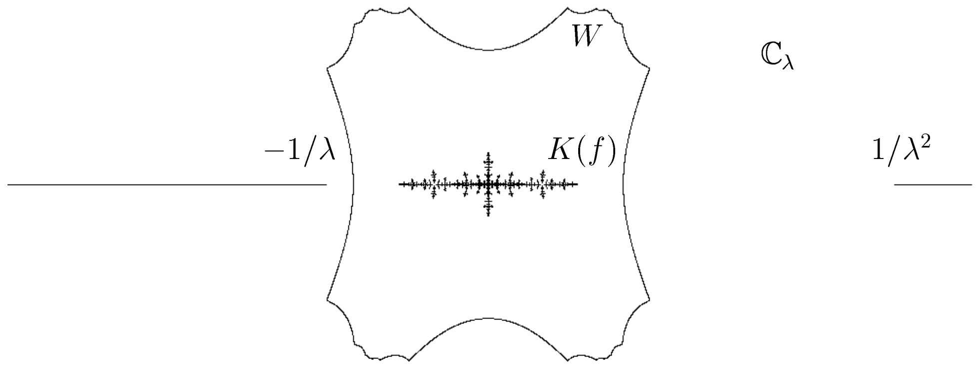

Figure 4: A representative of the Feigenbaum fixed point of renormalization, and the Julia set $K(f)$.

2 Geometry of the domain of analyticity of f.

Our first goal is to describe the geometry of the domain of analyticity of f. In [McM], McMullen proved that this domain exists, and is a dense open subset of $\mathbb{C}$. We will show that it is contained in one of its homothetics, and that it can be seen as the basin of attraction of a map related to f. This will enable us to introduce a chess board and some puzzles which we would like to use to prove the local connectivity of the Julia set $K(f)$.

2.1 The domain of analyticity.

Definition 4 *Let f and g be two holomorphic functions defined on open connected domains of $\mathbb{C}$: U_f and U_g. We say g is an analytic extension of f if $f = g$ on some nonempty open set. Moreover, if all such analytic extensions are restrictions of a single map*

$$\hat{f}:\ \widehat{W} \to \mathbb{C},$$

we will say that $\hat{f}$ is the unique maximal analytic extension of f

We will use a result by McMullen concerning the domain of analyticity of the Feigenbaum map f.

Proposition 3 *(cf [McM]). Let f be a solution of the Cvitanović-Fibonacci equation. There exists a unique maximal extension of f,*

$$\hat{f}:\ \widehat{W} \to \mathbb{C}.$$

where $\widehat{W}$ is a simply connected open set in $\mathbb{C}$.

Proof. First of all, recall that $f:\ W \to \mathbb{C}_\lambda$ is a polynomial-like map with non-escaping critical point. Hence, the set $W_n = f^{-2^n}(\mathbb{C}_\lambda)$ is a simply connected open set and the map $f^{2^n}:\ W_n \to \mathbb{C}_\lambda$ is a proper map. But the Cvitanović-Feigenbaum equation enables us to say that

$$1/\lambda^n f^{2^n}(\lambda^n z):\ W_n/\lambda^n \to \mathbb{C}_\lambda/\lambda^n$$

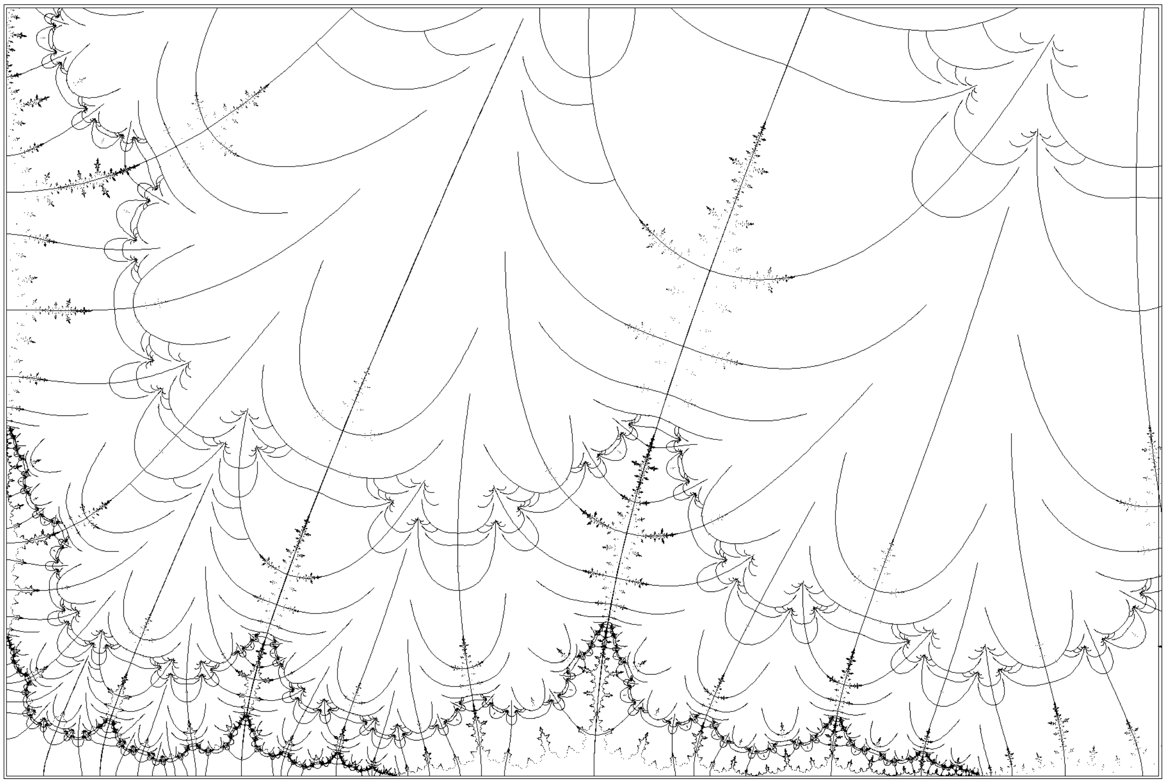

Figure 5: The skeleton of the domain $\widehat{W}$, and some points in its boundary. The reader should imagine the domain by putting some flesh on the skeleton.

is an extension of f. Moreover, as $\mathbb{C}_\lambda/\lambda^n \subset \mathbb{C}_\lambda/\lambda^{n+1}$,

$$W_n/\lambda^n \subset W_{n+1}/\lambda^{n+1}.$$

We can now define

$$\widehat{W} = \bigcup_{n\in\mathbb{N}} W_n/\lambda^n.$$

The map f has a unique maximal analytic extension on $\widehat{W}$ which coincides with $1/\lambda^n f^{2^n}(\lambda^n z)$, if $\lambda^n z \in W_n$. ■

Figure 5 was given to us by Henri Epstein. This picture should enable the reader to imagine the intersection of the domain $\widehat{W}$ with $\{z = x + iy \in \mathbb{C} \mid x \geq 0 \text{ and } y \geq 0\}$. The lines are preimages of the real axis under $\hat{f}$ and form the skeleton of $\widehat{W}$. The dots are points in the boundary of $\widehat{W}$. The reader should imagine the domain $\widehat{W}$ by putting some flesh between the skeleton and the dots. But one should remember that the domain is simply connected, and dense in the plane. Hence, each point in the boundary of $\widehat{W}$ is connected to infinity.

2.2 A linearizing equation.

Now, we will show that the set $\widehat{W}$ can be seen as a basin of attraction.

Lemma 1 *The map $f_\lambda(z) = f(\lambda z)$ has an attracting fixed point x_0 of multiplier $-\lambda$. Moreover f_λ exchanges the two sets W_+ and W_-.*

PROOF. The trick in the proof is that the Cvitanović-Feigenbaum equation can be translated on the following commutative diagram:

$$\begin{array}{ccc} (\widehat{W}, x_0) & \xrightarrow{f(\lambda z)} & (\widehat{W}, x_0) \\ f \downarrow & & \downarrow f \\ (\mathbb{C}, 0) & \xrightarrow[-\lambda z]{} & (\mathbb{C}, 0). \end{array}$$

It is a linearizing equation which proves the first part of the proposition. On the other hand,

$$\mathbb{H}_- = f(W_+) = -1/\lambda f(f(\lambda W_+)).$$

Hence $f(f_\lambda(W_+)) = \mathbb{H}_+$. The set $f_\lambda(W_+)$ is a preimage of $\mathbb{H}_+$ under f. As x_0 is in the closure of W_+ and $f_\lambda(x_0) = x_0 \in \overline{W_-}$, we can immediately deduce that $f_\lambda(W_+) = W_-$. We can use the same arguments to show that $f_\lambda(W_-) = W_+$.

∎

It is now possible to show the following result.

Lemma 2 *The set $\widehat{W}$ is the basin of attraction of the fixed point x_0 of $\hat{f}_\lambda = \hat{f}(\lambda z)$.*

PROOF. The commutative diagram we have written tells us that the linearizer of the map f_λ is f. But we know the domain of analyticity of the linearizer is the immediate basin of the attracting fixed point x_0. □

We will prove that $\widehat{W}$ is dense in $\mathbb{C}$. Hence, there cannot be any other component in the basin of $\hat{f}_\lambda$.

2.3 An inclusion of sets with their homothetics.

To study the geometry of $\widehat{W}$, McMullen proved the following result.

Lemma 3 *The sets $K(f)/\lambda^n$, $(n \in \mathbb{N})$, are all contained in $\widehat{W}$.*

PROOF. The domain $\widehat{W}$ is the union of the sets W_n/λ^n. But $K(f) \subset W_n = f^{-2^n}(\mathbb{C}_\lambda)$. Hence, for all $n \in \mathbb{N}$,

$$K(f)/\lambda^n \subset W_n/\lambda^n \subset \widehat{W}.$$

□

Theorem 2 *We have the following inclusion of sets.*

$$\lambda K(f) \subset K(f) \subset \bigcup_{n\in\mathbb{N}} \frac{K(f)}{\lambda^n} \subset \bigcap_{n\in\mathbb{N}} \lambda^n \widehat{W} \subset \widehat{W} \subset \frac{\widehat{W}}{\lambda}.$$

Proof. The set $\lambda K(f)$ is the Julia set $K(f^2)$ of the renormalization

$$f^2 \ : \ \lambda W \to \lambda \mathbb{C}_\lambda.$$

This Julia set is contained in $K(f)$. The inclusion

$$K(f) \subset \bigcup_{n\in\mathbb{N}} K(f)/\lambda^n$$

follows immediately. The inclusion $\widehat{W} \subset \widehat{W}/\lambda$ comes from lemma 2. The domain $\widehat{W}$ is the basin of attraction of $\hat{f}_\lambda$. Hence, it is contained in the domain of analyticity of $\hat{f}_\lambda$ which is $\widehat{W}/\lambda$. The inclusion

$$\bigcap_{n\in\mathbb{N}} \lambda^n \widehat{W} \subset \widehat{W}$$

follows immediately from this one. The remaining inclusion

$$\bigcup_{n\in\mathbb{N}} \frac{K(f)}{\lambda^n} \subset \bigcap_{n\in\mathbb{N}} \lambda^n \widehat{W}$$

comes from lemma 3. ■

McMullen proved that the union $\bigcup_{n\in\mathbb{N}} K(f)/\lambda^n$ is dense in $\mathbb{C}$. Hence $\widehat{W}$ is a dense open subset of $\mathbb{C}$. As $K(f)$ is a compact set with empty interior, the inclusions are strict because of Baire Theorem: a countable union of closed sets with empty interior cannot be equal to a countable intersection of dense open sets.

Finally, we would like to mention that the intersection of the homothetics $\lambda^n\widehat{W}$ can be defined dynamically.

Theorem 3 *The intersection of all the sets $\lambda^n\widehat{W}$ is the set of points the orbit of which under $\hat{f}$ stays in $\widehat{W}$:*

$$\bigcap_{n\in\mathbb{N}} \lambda^n \widehat{W} = \widehat{K} = \{z \in \widehat{W} \mid (\forall n \in \mathbb{N}) \ \hat{f}^n(z) \in \widehat{W}\}.$$

Proof. To show this equality, note that

$$\widehat{K} = \bigcap_{n\in\mathbb{N}} \hat{f}^{-n}(\widehat{W}).$$

We have noticed that $\hat{f}(\lambda\widehat{W}) = \hat{f}_\lambda(\widehat{W}) = \widehat{W}$. Moreover, in a neighborhood of the origin,

$$\hat{f}(z) = -\frac{1}{\lambda}\hat{f} \circ \hat{f}(\lambda z).$$

This equality has to be true whenever both sides are simultaneously defined. The left side is defined on $\widehat{W}$. When $z \in \widehat{W}$, $\lambda z \in \lambda\widehat{W}$. Hence, $f(\lambda z) \in \widehat{W}$, and the right side of the equality is defined. The Cvitanović-Feigenbaum equation holds on the whole domain $\widehat{W}$. As $\hat{f}^{-1}(\widehat{W}) = \lambda\widehat{W}$, it follows that $\hat{f}^{-2^n}(\widehat{W}) = \lambda^n\widehat{W}$, and the theorem holds. ■

2.4 Two remarks on the boundary of $\widehat{W}$.

The domain $\widehat{W}$ is dense in $\mathbb{C}$, hence its boundary has empty interior. However, it seems to have a very complex structure. Studying its topology, its Hausdorff dimension, or its Lebesgue measure could be interesting. But we do not have many tools at the moment. However, we will show that this boundary does not have the structure of a Cantor Bouquet, by proving that some points have at least two accesses in $\widehat{W}$. Moreover, we will prove that some points in the boundary of $\widehat{W}$ cannot be deep points of $\partial\widehat{W}$ in the sense of McMullen (cf [McM]).

First of all, recall that Proposition 2 tells that the following decomposition of the boundary of ∂W_+ is well defined.

Definition 5 *Let us denote by ∂W_+ the boundary of W_+. This boundary is the union of*

- *an arc γ_1 which is mapped by $\hat{f}$ to $\mathbb{R}_+$,*
- *an arc γ_2 which is mapped by $\hat{f}$ to $\mathbb{R}-$, and*
- *a point x_1 such that $(x_1)^\ell = F^{-1}(-i\infty)$.*

Proposition 4 *The point x_1/λ belongs to the boundary of $\widehat{W}$, and the arcs γ_1/λ and γ_2/λ which both land at x_1/λ belong to the domain of analyticity $\widehat{W}$. Moreover, they do not belong to the same access to x_1/λ in $\widehat{W}$.*

Proof. First of all, note that a slight improvement of Lemma 1 enables us to show that $(x_1, \overline{x}_1)$ is a repelling cycle of period 2 for the map f_λ. In fact, this is the way Epstein proves the last statement of Proposition 2 (cf [E]).

Hence, x_1 does not belong to the basin $\widehat{W}$. Thus, it is in the boundary $\partial\widehat{W}$. Moreover, according to theorem 2, $\lambda\widehat{W} \subset \widehat{W}$, thus x_1 belongs to the boundary of $\lambda\widehat{W}$, and the statement concerning x_1/λ is proved.

The Feigenbaum map f maps γ_1 and γ_2 to intervals of $\mathbb{R}$. Hence, $\hat{f}_\lambda$ maps γ_1/λ and γ_2/λ inside $\widehat{W}$. Now, recall that $\widehat{W}$ is the basin of attraction of $\hat{f}_\lambda$. Hence, γ_1/λ and γ_2/λ are in the basin of attraction of $\hat{f}_\lambda$, which concludes the first statement relative to γ_1 and γ_2.

To conclude the proof, note that the union $\gamma_1/\lambda \cup \gamma_2/\lambda \cup \{x_1/\lambda\}$ bounds the set W_+/λ which contains the point $x_1 \in \mathbb{C} \setminus \widehat{W}$ in its interior. ■

We will finally prove the following proposition.

Proposition 5 *The point $x_1 \in \mathbb{C} \setminus \widehat{W}$ cannot be a deep point of $\mathbb{C} \setminus \widehat{W}$.*

This means that there exists a constant $K < 1$ such that for all radii $r \in \mathbb{R}_+$, there exists a ball of radius Kr inside the intersection $\widehat{W} \cap B(x_1, r)$.
Proof. Recall that all the critical values of f_λ are on the real axis. Hence one can choose a branch of f_λ^{-2} which maps $\mathbb{H}_+$ into itself, having x_1 as attracting fixed point.

Finally, Lemma 2 proves that this inverse branch maps the boundary of $\widehat{W}$ into itself. As its interior is empty, a classical result enables us to conclude that x_1 cannot be a deep point of $\mathbb{C} \setminus \widehat{W}$. ■

The same result holds for all the points which eventually land under iteration of $\hat{f}_\lambda$ on x_1. Obviously, those points are in the boundary of the basin $\widehat{W}$ and they cannot be deep points of $\partial\widehat{W}$.

3 Local connectivity of $K(f)$.

We will now give a new proof of the result by Jiang and Hu [JH] relative to the local connectivity of the Julia set of the Feigenbaum polynomial, by using the fixed point of renormalization.

We have proved that $\widehat{W}$ is the basin of attraction of $\hat{f}_\lambda$. Moreover $\hat{f}_\lambda$ exchanges the two sets W_+ and W_-. Hence it is possible to define a chessboard for $\hat{f}_\lambda$.

Definition 6 *The squares of the chessboard are defined in the following way:*

- *The first two chess squares are the sets W_+ and W_-.*
- *The other chess squares are the preimages of those two by the map $\hat{f}_\lambda$.*

Some of those chess squares appear at the bottom left of figure 5.

It is now natural to take into account the following puzzles.

Definition 7 *We can define:*

- *the puzzle pieces of the puzzle $\mathcal{P}_0$ of depth 0 are the chess squares P, and*
- *the puzzle pieces of the puzzle $\mathcal{P}_n$ of depth n are the pieces $\lambda^n P$.*

On figure 6, we have drawn some pieces of the puzzles of depth 1, 2, and 3. A lot of pieces are missing (but drawing them with a computer is a far too difficult and long work), that is why the puzzles seem to be disconnected, which is not the case: remember that the chess squares cover a dense open subset $\widehat{W} \subset \mathbb{C}$ which is connected and simply connected.

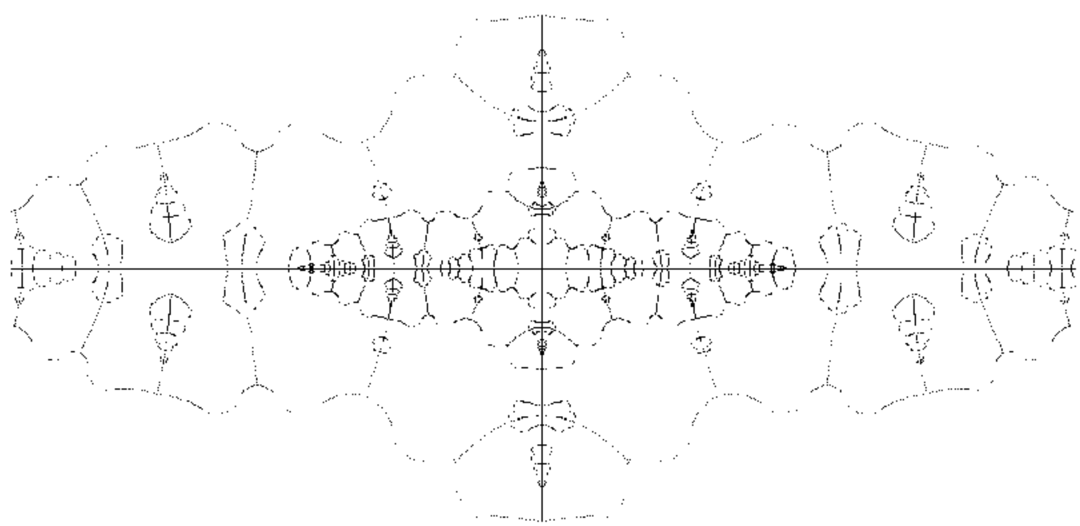

Figure 6: Some pieces of the puzzles $\mathcal{P}_i$, $i = 1, 2, 3$.

Lemma 3 enables us to conclude that for every $n \geq 0$, the puzzle $\mathcal{P}_n$ covers the Julia set $K(f)$. On the other hand, it is possible to obtain the following result which is crucial to show local connectivity.

Lemma 4 *If P is a chess square, then for all $n \geq 0$, the intersection $\lambda^n \overline{P} \cap K(f)$ is connected.*

Proof. Given a chess square P, there exists an integer $k \geq 0$ such that $f_\lambda^k(P) = W_\pm$. We will show lemma 4 by induction on the age k of the chess square P.

For $k = 0$, $\lambda^n \overline{W}_\pm \cap K(f)$ is connected, because $f^{2^n}(\lambda^n \overline{W}_\pm)$ is a half plane, which cuts $K(f)$ in two parts along the real axis, and because $K(f)$ is totally invariant by the polynomial-like map $f : W \to \mathbb{C}_\lambda$.

If P is a chess square of age $k+1 \geq 1$, we first notice that $K(f) \subset W$. Hence,

- either P is a copy of $W_\pm$ by rotation of angle $2\pi/\ell$, and we can easily conclude,
- either $P \cap K(f) = \emptyset$.

On the other hand, $f_\lambda(P) = Q$ is a chess square of age k. Hence, every $\lambda^n \overline{Q} \cap K(f)$, $n \in \mathbb{N}$, is connected. But,

$$f(\lambda P) = \left(-\frac{1}{\lambda}\right)^{n-1} f^{2^{n-1}}(\lambda^{n-1}\lambda P).$$

Hence,

$$f^{2^{n-1}}(\lambda^n P) = (-\lambda)^{n-1} f_\lambda(P) = (-\lambda)^{n-1} Q.$$

We can conclude that $\lambda^n \overline{P} \cap K(f)$ is connected. □

Lemma 5 *Let P be a chess square. Then $f(P) = \mathbb{H}_\pm$ and $f : P \to \mathbb{H}_\pm$ is an isomorphism.*

Proof. The commutative diagram shows that f is the linearizer of f_λ. Hence we can write:

$$f(z) = \left(-\frac{1}{\lambda}\right)^n f \circ f_\lambda^n(z).$$

But, if P is a chess square, there exists an integer n such that $f_\lambda^n : P \to W_\pm$ is an isomorphism. Hence,

$$f(P) = \left(-\frac{1}{\lambda}\right)^n f(W_\pm) = \mathbb{H}_\mp.$$

Finally, as $f : W_\pm \to \mathbb{H}_\mp$ is an isomorphism, we can conclude. □

We can now show that the puzzle $\mathcal{P}_n$ contains the puzzle $\mathcal{P}_{n+1}$. In other words, we will show the following lemma.

Lemma 6 *Every puzzle piece $\lambda P \in \mathcal{P}_1$ is contained in a chess square $Q \in \mathcal{P}_0$*

This is obvious on figure 6. Proof. Let z be a point in λP and let Q be the chess square which contains z. Using lemma 5, we can say that $f(Q) = \mathbb{H}_\pm$. As $f(\lambda P) = f_\lambda(P)$ is a chess square, we know that

$$f(\lambda P) \subset f(Q),$$

and as $f : Q \to \mathbb{H}_\pm$ is an isomorphism, we can deduce that

$$\lambda P \subset Q.$$

□

We would like to reprove the following theorem by Jiang and Hu.

Theorem 4 *The Julia set $K(f)$ is locally connected.*

Up to now, we can only prove local connectivity around the critical point 0 (cf figure 7):

- $0 \in \lambda^n W \subset \lambda^{n-1} W$, and
- $\lambda^n W \cap K(f)$ is connected.

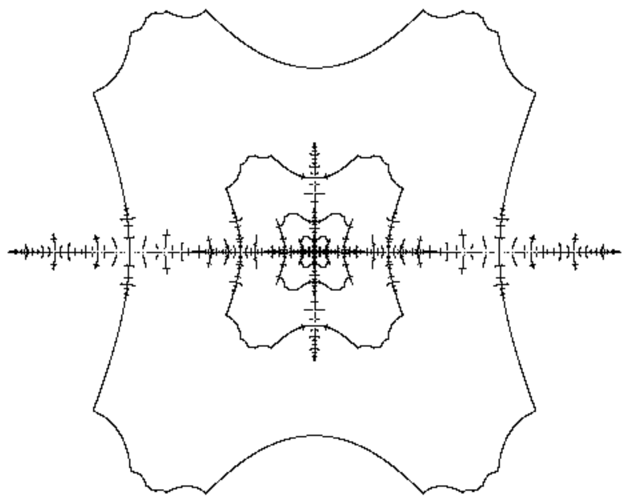

Figure 7: Local connectivity of the Feigenbaum Julia set around the critical point.

This local connectivity spreads around using classical arguments of recurrence, but those arguments do not exhibit a nice basis of connected neighborhoods. We would like to prove that if we consider the nested sequence of puzzle pieces $P_n(z) \in \mathcal{P}_n$ which contain a point $z \in K(f)$, then the diameter of the piece $P_n(z)$ decreases geometrically with n. We just need to show the following conjecture.

Conjecture 1 *There exists a constant $\varepsilon > 0$ such that if $P \in \mathcal{P}_0$ and $Q \in \mathcal{P}_0$ are chess squares and $\lambda P \subset Q$, then*

$$\frac{\text{diam}(Q)}{\text{diam}(\lambda P)} \geq 1 + \varepsilon.$$

We think it is possible to show this conjecture using some classical arguments of moduli of annuli. Moreover,we think that there is a stronger result, but we have no idea how to prove it.

Conjecture 2 *There exists a constant $K \in \mathbb{R}$ such that the diameter of every chess square $P \in \mathcal{P}_0$ is bounded by K:*

$$diam(P) \leq K.$$

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