

Correspondence and Translation Principles for the Mandelbrot set

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Abstract

New insights into the combinatorial structure of the the Mandelbrot set are given by ‘Correspondence’ and ‘Translation’ Principles both conjectured and partially proved by E. Lau and D. Schleicher. We provide complete proofs of these principles and discuss results related to them.

Note: The Translation and Correspondence Principles stated in this note do not hold in the general case. Please see the errata at the end for details about which parts of the two statements are incorrect and which parts remain true.

1. The conjectures by Lau and Schleicher

Introduction. The detailed structure of the Mandelbrot set M is extremely complicated. However, much of the structure can be described by different kinds of symmetry and self-similarity. (For a listing of various symmetries in M , see [34].)

For example, each neighborhood of a boundary point of M contains infinitely many topological copies of M itself. This is a consequence of the (unpublished) tuning results by Douady and Hubbard (compare [25]).

Whereas symmetry in the dynamic plane can mostly be explained by the action of the quadratic map, the situation is more complicated in the parameter space. Often there is a correspondence between local structure in the dynamic plane and in the parameter space which helps to understand a special symmetry in the parameter space.

Typical examples are the local similarities about Misiurewicz points found by Tan Lei (see [36]): The neighborhoods of a Misiurewicz point c in the Mandelbrot set and in the corresponding Julia set are asymptotically similar in the Hausdorff metric.

Roughly speaking, this provides infinitely many points in M with a ‘local rotation symmetry’.

The present paper deals with symmetries whose nature is a combinatorial one. In particular, it proves two statements conjectured by Lau and Schleicher (see [23, 32]). The first of the conjectures, the *Correspondence Principle*, relates combinatorial structure in the dynamic plane and the parameter space and, as in the above example, it forces a symmetry in the parameter space. The latter symmetry has an exact description by the second conjecture of Lau and Schleicher, the *Translation Principle*.

To put us in a position to formulate the two principles, it is necessary to recall some quadratic iteration theory. Beside the standard paper [13] by Douady and Hubbard, the references are [5, 6, 9, 24, 26, 35, 28, 33].

Formulation of the Translation Principle. For a given complex parameter c , let p_c denote the quadratic map acting on the complex plane by $p_c(z) = z^2 + c$. The *filled-in Julia set* K_c of p_c is defined to consist of all points with bounded orbits, and its boundary $J_c = \partial K_c$ in the plane is said to be the *Julia set* of p_c .

The *Mandelbrot set* M is the set of all complex c with connected Julia set. It contains the set H of all c for which p_c possesses an *attractive* periodic orbit of some period m : The *multiplier* of the orbit, i.e. the derivative of the first return map p_c^m at any point of the orbit, has absolute value less than 1. If such an orbit exists for p_c , it is unique.

A connectedness component W of H is called *hyperbolic component* of the Mandelbrot set. Its period $\text{PER}(W)$ is defined to be the period of the unique attractive periodic orbit for $p_c; c \in W$, which indeed does not depend on $c \in W$. The only hyperbolic component of period 1 is called the *main hyperbolic component*.

The *multiplier map* which assigns to each point c of a given hyperbolic component W the multiplier of the attractive orbit for p_c forms a conformal isomorphism from W onto the unit disk and extends continuously to a homeomorphism from the closure of W onto the closed unit disk. The point in ∂W which is mapped to $e^{2\pi\nu i}$ is said to have *internal angle* $\nu \in [0, 1[$. Some special points in the closure of a hyperbolic component W of period m play a crucial role in understanding the combinatorial structure of M :

- P1) The *center* c_W of W is the unique point mapped to 0 by the multiplier map.
- P2) The *root* r_W of W is the point in ∂W with internal angle 0. If $m > 1$, at r_W the Mandelbrot set splits: $M \setminus \{r_W\}$ consists of two connectedness components.
- P3) A *bifurcation point* of W is a point in ∂W with internal angle $\frac{p}{q}$ (p and q are required to be relatively prime). Such a point and only such one is the root r_U of a hyperbolic component U bifurcating from W : $\text{PER}(U) = qm$, and the component of $M \setminus \{r_U\}$ containing U is called the $\frac{p}{q}$ -*sublimb* of W .

Definition 1 (Order and Visibility in parameter space)

Let W, U be hyperbolic components. Then $W \prec U$ if there exists a simple curve intersecting M in exactly one point and dividing the complex plane into two open parts, one containing U and the other one W and the main component.

U is said to be visible from W if $W \prec U$ and there is no hyperbolic component V of period less than the period of U with $W \prec V \prec U$.

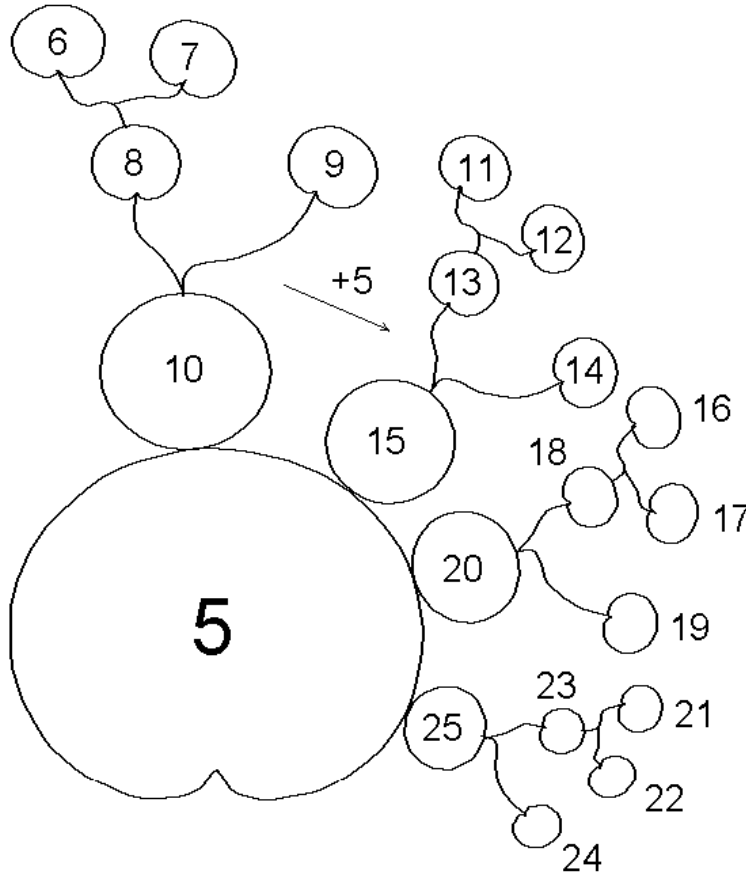


Figure 1: The Translation Principle

We want to formulate the Translation Principle now: In the trees we consider, the ‘nodes’ are hyperbolic components labeled by their periods and the ‘edges’ are defined by the successor relation induced by \prec . Equivalence of such trees includes the labeling.

Translation Principle: *Let W be a hyperbolic component of period m . Then the trees of hyperbolic components visible from W in any two sublimbs of denominators q_1 and q_2 coincide, including the embedding into the plane, when all periods of visible components in the first sublimb are increased by $(q_2 - q_1)m$.*

Figure 1 demonstrates the Translation Principle. A hyperbolic component of period 5 together with the trees of the visible hyperbolic components in sublimbs of denominators $q = 2, 3, 4, 5$ are shown.

The Correspondence Principle - a bridge between dynamic plane and parameter space.

The Translation Principle is forced by a similar statement in the dynamic plane which is caused by the action of a quadratic map, and Lau and Schleicher’s Correspondence Principle translates between dynamic objects and objects in the parameter space.

Fix a hyperbolic component W of period m and its center $c = c_W$. The quadratic map linking dynamic structure and structure in the parameter space is p_c , for which the *critical point* 0 and the *critical value* c lie on a common periodic orbit of period m . (We have the *superattractive case*, i.e. the multiplier of this orbit is 0.)

The interior of K_c decomposes into countably many connectedness components, the *bounded Fatou components*. Subsequently, ‘Fatou components’ should always read ‘bounded Fatou components’. There are the *critical (Fatou) component* and the *critical value (Fatou) component* A containing the critical point and the critical value, respectively. A is the dynamic object assigned to W .

More general, in the dynamic plane the Fatou components take the role of the hyperbolic components. The set of them is invariant under the action of p_c . Moreover, A has period m , and for each Fatou component F there exists an n with $p_c^n(F) = A$. The minimal n with this property is called the *STEP* of F . (Lau and Schleicher [23] defined the *STEP* of a precritical point instead of a Fatou component in an analogous way. This applies to more general cases, but in the context of our paper there is no substantial difference because each Fatou component contains exactly one backward iterate of c .)

We want to complete the analogy between W and A . There is a unique homeomorphism from the closure of A onto the closed unit disk which is conformal on A and conjugates p_c^m to the usual quadratic map p_0 . The unique point mapped to $e^{2\pi\eta i}$ is said to have the *internal angle* $\eta \in [0, 1[$.

D1) c can be thought as the *center* of A .

D2) The *dynamic root* r_A of A is the point in ∂A with internal angle 0. At r_A the filled-in Julia set splits: $K_c \setminus \{r_A\}$ consists of finitely many components.

D3) A *bifurcation point* x of A is a point in ∂A with internal angle $\frac{a}{2^q-1}$ for $q \geq 2$ and odd a : The point x is (first) mapped to r_A after $q-1$ iterates of p_c^m , the set $K_c \setminus \{x\}$ consists of finitely many components, and the union of those which do not contain A is called a $\frac{a}{2^q-1}$ -*sublimb* of A (compare [23, 32]).

Definition 2 (Order and Visibility in dynamic plane)

Let Q, R be Fatou components. Then $Q \prec R$ if there exists a simple curve intersecting K_c in exactly one point and dividing the complex plane into two open parts, one containing R and the other one Q and the critical component.

R is said to be *visible from* W if $Q \prec R$ and there is no Fatou component S of *STEP* less than the *STEP* of R with $Q \prec S \prec R$.

Let us formulate the Correspondence Principle. The trees considered for the dynamic plane are analogous to those for the parameter space, and they are labeled by the *STEP*. However, now we have trees in a generalized sense. They need not have a root.

Correspondence Principle: Let W be a hyperbolic component of period m , let c be its center and A be the critical value component of p_c . Then, for a given sublimb Subl_W of W and of denominator q and a given sublimb Subl_A of A and of denominator 2^{q-1} the following trees coincide, including the labeling and the embedding in the plane:

1. The tree of all hyperbolic components in Subl_W visible from W and different from the hyperbolic component bifurcating directly from W ,
2. the tree of all Fatou components visible from A in Subl_A of *STEP* less than mq .

2. Thinking in laminations

The nature of the Translation and Correspondence Principles is a combinatorial one, and there are different ways to extract the combinatorial part from quadratic dynamics. We want to use Thurston's concept of a lamination (see [37]).

By T we denote the unit circle, which we identify with the interval $[0, 1[$ ('=' R/Z) via $\beta \longleftrightarrow e^{2\pi\beta i}$. Further, h denotes the angle-doubling map $\beta \in T \longrightarrow 2\beta \bmod 1$ and $\beta_1\beta_2$ the chord with given ends β_1, β_2 in the unit circle T . By a chord we understand a straight line contained in the closed unit disk with ends in the unit circle, but without change of any statement below, a chord can alternately be considered as the Euclidean closure of a hyperbolic geodesic in the open unit disk. (For our illustrations we use 'hyperbolic' chords.)

By the *length* of a chord $B = \beta_1\beta_2$ and the *distance* of β_1 and β_2 we understand the minimum length of the two arcs in T connecting β_1 and β_2 , where the whole circle has length 1. The action of h is extended to chords $B = \beta_1\beta_2$ by $h(B) = h(\beta_1)h(\beta_2)$.

A chord $B = \beta_1\beta_2$ shorter than $\frac{1}{2}$ divides the unit disc into a smaller and a bigger open part. $\beta \in T$ is said to be *between* β_1 and β_2 if it lies in the smaller part, and a subset of the unit disc is said to be *behind* B if at least one of its points lies in the smaller, but no one in the bigger part of the disk. Similarly, a set is defined to be *between* two disjoint chords B_1, B_2 if it lies in the closed part of the disk bounded by B_1 and B_2 but at least one of its points does not belong to B_1 or to B_2 .

Finally, we shall use the following diction: Two chords S_1, S_2 *cross* each other if they are different, but have a common interior point. They are *separated* by a third chord S_3 if S_3 is between S_1 and S_2 but S_2 not between S_1 and S_3 .

complex plane	lamination model
hyperbolic component W of period $m > 1$	leaf S of period m in \mathcal{B}_*
Fatou component of STEP n	dynamic pair of STEP n
sublimb of denominator q in parameter space, therein: main antenna tip of the tuned Mandelbrot set corresponding to W	region behind immediately visible $B \in \mathcal{B}_*$ of period qm , preperiodic leaf R_B
sublimb of denominator 2^{q-1} in dynamic plane	region behind $L_{\mathbf{v}s_1\mathbf{v}s_2\dots\mathbf{v}s_{q-2}\mathbf{v}e}(S)$ for 0-1-word $s_1s_2\dots s_{q-2}$

Dictionary between complex plane and laminations

In the following, a *lamination* is a set of non-degenerate chords (i.e. chords with different ends) which do not cross each other. (Here the concept of a lamination is used in a more general way than Thurston's original one.) The elements of a lamination are called its *leaves*, and in the concrete situation their ends correspond to neighboring external rays landing at the same point of the Mandelbrot set or Julia set. By Lindelöf's theorem the curves in the Definitions 1 and 2 can be considered as the union of two external rays and their common landing point (but the concept of an external ray was not necessary to formulate

the principles.)

The above dictionary translating objects in the complex plane into the corresponding objects in the lamination setting should be an orientation for the reader.

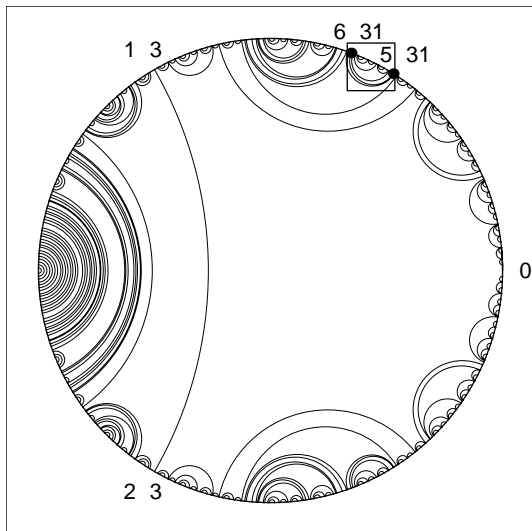


Figure 2: The periodic parameter lamination \mathcal{B}_*

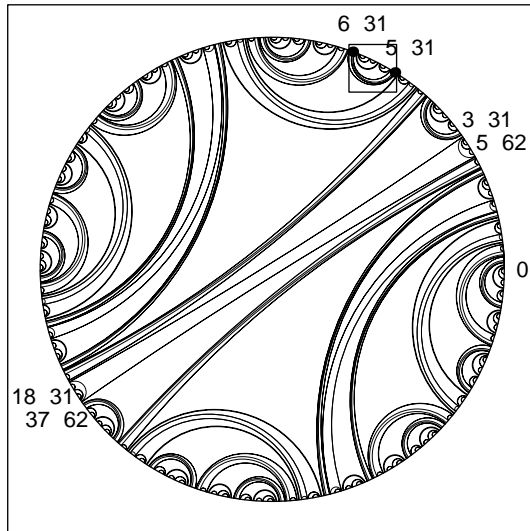


Figure 3: The generating dynamic lamination $\mathcal{B}_*(S)$ for $S = \frac{5}{31}, \frac{6}{31}$

a) The **periodic parameter lamination** \mathcal{B}_* (see Figure 2) consists of all *periodic parameter leaves* of some period $m > 1$: Their two ends are periodic of period m (with respect to h) and form the external angles of the root of a hyperbolic component having period m . (By Douady and Hubbard [13], such roots have exactly two periodic external angles and each periodic angle is an external angle of such a root. For this also compare [28, 33].)

So each leaf in \mathcal{B}_* of some period describes a non-main hyperbolic component of the same period, and vice versa. Let us express visibility of hyperbolic components within the lamination concept. ('immediately visible' indicates the direct bifurcation of a hyperbolic component.)

Definition 3 (Parameter visibility)

Let $S \in \mathcal{B}_*$. Then

- (i) $B \in \mathcal{B}_*$ is said to be *immediately visible* (from S) if B is behind S , and S, B are not separated by a leaf of \mathcal{B}_* .
- (ii) $B \in \mathcal{B}_*$ is said to be *visible* (from S) if B is behind S , and S, B are not separated by a leaf of \mathcal{B}_* whose period is less than the period of B .

The meaning of 'visible' is not changed when one substitutes the word 'less' by 'less or equal' in (ii). This follows from a well known result of Lavaurs [24] (compare also Section 5 in [4]):

Lavaurs' Lemma *Two leaves in \mathcal{B}_* of the same period are separated by a leaf of lower period.*

The **entire parameter lamination** \mathcal{B} consists of all *parameter leaves* defined to be an element or a (non-degenerate) accumulation chord of \mathcal{B}_* . By an accumulation chord of some set of chords \mathcal{A} we understand the limit of a converging sequence of mutually different elements taken from \mathcal{A} . Note that \mathcal{B} together with further one-point leaves is just Thurston's 'quadratic minor lamination' (see [37]). By (9) in [4], we have the simple geometric characterization of the elements of \mathcal{B} following now. (What in the present paper is denoted \mathcal{B}_* respectively \mathcal{B} , is \mathcal{S}_* respectively \mathcal{S} in [4].)

$$S = \alpha\gamma \in \mathcal{B} \iff \begin{array}{l} \text{The iterates of } S \text{ do not cross each other, and they} \\ \text{do not cross the diameters } \frac{\alpha\alpha+1}{2} \text{ and } \frac{\gamma\gamma+1}{2}. \end{array} \quad (1)$$

b) Fix a hyperbolic component W of period $m > 1$ with center c . It is well known that the periodic parameter leaf $S = \alpha\gamma \in \mathcal{B}_*$ corresponding to the root of W generates the lamination belonging to J_c :

S has a unique preimage with periodic ends denoted by \dot{S} and being equal to $h^{m-1}(S)$ and one with preperiodic ends denoted by \ddot{S} . The leaves \dot{S}, \ddot{S} are the longest in the lamination, and it holds $\{\dot{S}, \ddot{S}\} = \{\frac{\alpha\gamma+1}{2}, \frac{\alpha+1}{2}\frac{\gamma}{2}\}$. The **generating dynamic lamination** $\mathcal{B}_*(S)$ is defined to be the set of all chords R such that some iterate of R is equal to S , no iterate of R is longer than the leaves \dot{S} and \ddot{S} or crosses one of them, and if an iterate of R lies between the leaves \dot{S} and \ddot{S} , it has no end in common with them.

To J_c there corresponds the **entire dynamic lamination** $\mathcal{B}(S)$ defined to be the set $\mathcal{B}_*(S)$ together with its non-degenerate accumulation chords. It is important to note that no chord in $\mathcal{B}(S)$ between \dot{S} and \ddot{S} has a point in common with \dot{S} and \ddot{S} and that no iterate of S lies between \dot{S} and \ddot{S} .

Figure 3 shows the generating dynamic lamination $\mathcal{B}_*(S)$ for the leaf $S = \frac{5}{31}\frac{6}{31}$, which has period 5. To S there corresponds the hyperbolic component of period 5 sketched in Figure 1, and it holds $\dot{S} = \frac{3}{31}\frac{18}{31}$ and $\ddot{S} = \frac{5}{62}\frac{37}{62}$.

Some statements on the structure of $\mathcal{B}(S)$. We want to encode the leaves in the lamination $\mathcal{B}_*(S)$ belonging to J_c . For this we use *0-1-words*, i.e. words of symbols taken from the alphabet $\{0, 1\}$, including the empty word. As usual, and for any alphabet, the notations are as follows: $\mathbf{w}_1\mathbf{w}_2$ for the concatenation of two words $\mathbf{w}_1, \mathbf{w}_2$, further, \mathbf{w}^n for the n -fold concatenation of a word \mathbf{w} , and $\overline{\mathbf{w}}$ instead of $\mathbf{w}\mathbf{w}\mathbf{w}\dots$. Further, let α be the minimum of the ends of S (as elements of $[0, 1]$).

For a given 0-1-word $\mathbf{w} = w_1w_2\dots w_k$ define a map $L_{\mathbf{w}}$ on $\mathcal{B}(S)$ as follows: $L_0(R)$ and $L_1(R)$ denote the unique preimages of $R \in \mathcal{B}(S)$ in $\mathcal{B}(S)$ with at least one end in the open interval $]\frac{\alpha}{2}, \frac{\alpha+1}{2}[$ and at least one end in the open interval $]\frac{\alpha+1}{2}, \frac{\alpha}{2}[$, respectively, and $L_{\mathbf{w}}(R) := L_{w_1}(L_{w_2}(\dots(L_{w_k}(R)))$. (Intervals are taken in counter-clockwise direction.)

Further, let e^S be the unique symbol $e \in \{0, 1\}$ with $\dot{S} = L_e(S)$, and let \mathbf{v}^S be the unique 0-1-word \mathbf{v} of length $m-1$ with $S = L_{\mathbf{v}}(\dot{S})$. Then $\dot{S} = L_{1-e^S}(S)$ and $S = L_{\mathbf{v}^S e^S}(S)$.

There is a one-to-one correspondence between the Fatou components of K_c and the infinite gaps of $\mathcal{B}(S)$: The complement of $\bigcup \mathcal{B}(S)$ splits into connectedness components. The closure of such a component is said to be an *infinite gap* if it is bounded by infinitely many leaves

of $\mathcal{B}(S)$. In particular, there is the *critical value gap* bounded by S and lying behind S , and corresponding to the critical value component.

Now we provide two propositions which are basic for the following. The first one presents a well known description of bifurcation from the critical value component, but in the language of laminations (for a lamination proof see [22]).

Proposition 1 (Preperiodic boundary leaves of the critical value gap)

Let $S \in \mathcal{B}_*$ be of period m and let $\mathbf{v} = \mathbf{v}^S, e = e^S$. Then all boundary leaves of the critical value gap of $\mathcal{B}(S)$ different from S are preperiodic. They coincide with $L_{\mathbf{v}s_1\mathbf{v}s_2\dots\mathbf{v}s_l\mathbf{v}e}(S)$ for the 0-1-words $s_1s_2\dots s_l$ and are first mapped to S after $(l+1)m$ iterates. If $l \geq 1$, then the first $m-1$ -th iterates of the whole closed interval behind $L_{\mathbf{v}s_1\mathbf{v}s_2\dots\mathbf{v}s_l\mathbf{v}e}(S)$ are disjoint to the whole open interval behind S , and h^m maps the whole closed interval behind $L_{\mathbf{v}s_1\mathbf{v}s_2\dots\mathbf{v}s_l\mathbf{v}e}(S)$ to that behind $L_{\mathbf{v}s_2\mathbf{v}s_3\dots\mathbf{v}s_l\mathbf{v}e}(S)$ homeomorphically. \square

The left picture in Figure 4 illustrates $\mathcal{B}_*(S)$ for $S = \frac{5}{31}\frac{6}{31}$ in a neighborhood of the critical value gap (see window in Figure 3). Boundary leaves of that gap are drawn as thick arcs. Note that $\mathbf{v}^S = 0010$ and $e^S = 1$.

We come to the second proposition. Its first part gives a well known statement encoding all Fatou components, in the language of laminations (again see [22]), and the second part is not hard to show by using induction and the following simple statement for $S \in \mathcal{B}_*$: The leaves \dot{S} and \ddot{S} have length $a \geq \frac{1}{3}$ and span a ‘rectangle’ of ‘side’-lengths a and $\frac{1}{2} - a$. If some chord Q has length $a' \leq a$, then the length of $h(Q)$ is either $2a'$ or $1 - 2a' > \frac{1}{2} - a$.

Proposition 2 (Infinite gaps in $\mathcal{B}(S)$)

If $S \in \mathcal{B}_*$, then to each 0-1-word of length n not ending with $\mathbf{v} = \mathbf{v}^S$ there exists an infinite gap in $\mathcal{B}(S)$ different from the critical value one and vice versa.

The leaves $L_{\mathbf{w}0}(S)$ and $L_{\mathbf{w}1}(S)$ form the longest ‘sides’ of this gap: If S has length d , then $L_{\mathbf{w}0}(S)$ and $L_{\mathbf{w}1}(S)$ are not shorter than $\frac{d}{2^n-1}$ and the two shortest ‘sides’ of the ‘rectangle’ spanned by the ends of $L_{\mathbf{w}0}(S)$ and $L_{\mathbf{w}1}(S)$ have length $\frac{d}{2^n}$. \square

Proposition 2 shows that the Fatou components (different from the critical value component) can be described by two leaves of the corresponding lamination, and this justifies the following definition:

Definition 4 (Dynamic pairs and dynamic visibility)

Let $S \in \mathcal{B}_*$ and $S_1, S_2 \in \mathcal{B}_*(S)$. Then (S_1, S_2) is called a *dynamic pair* if S_1 is not shorter than S_2 and there exists a 0-1-word \mathbf{w} which does not end with \mathbf{v}^S , such that $\{S_1, S_2\} = \{L_{\mathbf{w}0}(S), L_{\mathbf{w}1}(S)\}$.

By the *STEP* of a dynamic pair (S_1, S_2) we understand the minimal number n with $h^n(S_1) = h^n(S_2) = S$. (It exceeds the length of the corresponding word \mathbf{w} by one.)

A dynamic pair (S_1, S_2) is said to be *visible* from a preperiodic boundary leaf R of the critical value gap if S_2 lies behind R and there exists no dynamic pair (Q_1, Q_2) whose Step is less than the Step of (S_1, S_2) such that $Q_1 = R$ or Q_1 separates R and S_1 .

Remark: The statements in Proposition 2 concerning lengths of chords show that no dynamic pair visible from a preperiodic boundary leaf lies between the members of a dynamic pair.

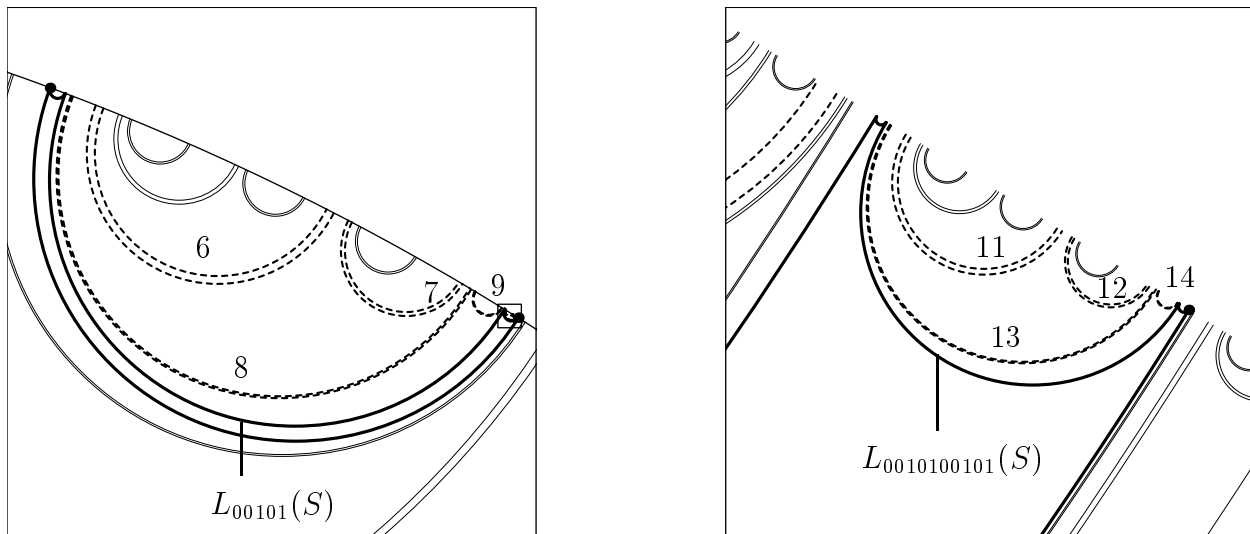


Figure 4: The Abstract Dynamical Translation Principle

What shall we show substantially? Showing the Translation and Correspondence Principles, we can assume that the hyperbolic component W is different from the main component. Namely, for the latter the statements are obvious: The filled-in Julia set for its center is the unit disk, and it is easy to see that the only hyperbolic components visible from the main component, bifurcate directly from it (see also [23]).

We shall give our proofs within the lamination models. (The necessary translations are given in the table above.) As mentioned at the beginning, the Translation Principle is forced by a similar statement in the dynamic plane. In the lamination setting, the latter is given by the following Theorem 1 being an immediate consequence of Proposition 1.

Theorem 1 (Abstract Dynamical Translation Principle)

Let $S \in \mathcal{B}_*$, let m be the period of S and $\mathbf{v} = \mathbf{v}^S, e = e^S$. Further, let $s_1 s_2 \dots s_l$ be a non-empty 0-1-word.

Then h^{lm} maps the whole closed interval behind $L_{\mathbf{v}s_1 \mathbf{v}s_2 \dots \mathbf{v}s_l \mathbf{v}e}(S)$ onto the whole closed interval behind $L_{\mathbf{v}e}(S)$ homeomorphically. In particular, h^{lm} transforms the set of all dynamic pairs behind $L_{\mathbf{v}s_1 \mathbf{v}s_2 \dots \mathbf{v}s_l \mathbf{v}e}(S)$ into the set of all those behind $L_{\mathbf{v}e}(S)$, decreasing the STEP by lm , and a dynamic pair (S_1, S_2) of STEP n is visible from $L_{\mathbf{v}s_1 \mathbf{v}s_2 \dots \mathbf{v}s_l \mathbf{v}e}(S)$ iff $(h^{lm}(S_1), h^{lm}(S_2))$ is visible from $L_{\mathbf{v}e}(S)$ and is of STEP $n - lm$. \square

Figure 4 illustrates Theorem 1. The magnification of the section in the window on the left side yields the picture on the right side, and behind $L_{0010100101}(S)$ one sees the same as behind $L_{00101}(S)$. Namely, h^5 enlarges all behind the first leaf by the factor 2^5 , in particular, the visible dynamic pairs, indicated by thick dashed double-arcs and labeled by their STEP's.

According to the following proposition to each parameter leaf $B \in \mathcal{B}_*$ visible from some $S \in \mathcal{B}_*$ there is assigned a special (dynamic) boundary leaf R_B of the critical value gap of $\mathcal{B}(S)$. We will prove that proposition in Section 3. For the reader familiar with the tuning construction by Douady and Hubbard [14] we refer to the characterization of R_B in the table above.

Proposition 3 (Correspondence between immediately visible parameter leaves and special boundary leaves of the critical value gap)

Let $S \in \mathcal{B}_*$ be of period m and let $q \geq 2$. Then for each $B \in \mathcal{B}_*$ immediately visible from S and of period qm there exists a unique boundary leaf R_B of the critical value gap satisfying the following properties: R_B is first mapped to S after $(q-1)m$ iterates, R_B lies behind B , and each leaf in \mathcal{B}_* behind B and visible from S lies behind R_B .

Theorem 1 shows that the trees of Fatou components in two sublimbs of the same denominator coincide, including the labeling and the embedding in the plane, and Proposition 3 establishes a relation between the sublimbs in the parameter plane and special sublimbs in the dynamic plane. So the proof of the Correspondence Principle can be reduced to showing the following

Theorem 2 (Main Result) (Abstract Correspondence Principle)

Let $S \in \mathcal{B}_*$ be of period m and let $B \in \mathcal{B}_*$ be immediately visible from S and of period qm for $q \geq 2$. Then for each dynamic pair (S_1, S_2) visible from R_B of STEP less than qm there is a unique leaf $Q(S_1, S_2) \in \mathcal{B}_*$ visible from S and separating S_1 and S_2 . Its period is equal to the STEP of (S_1, S_2) .

Moreover, by $(S_1, S_2) \mapsto Q(S_1, S_2)$ the set of all those dynamic pairs is mapped bijectively onto the set of all leaves of \mathcal{B}_* behind B which are visible but not immediately visible from S .

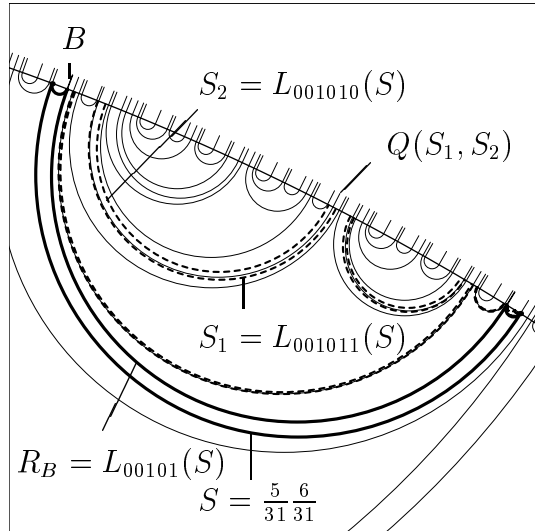


Figure 5: The Abstract Correspondence Principle

Figure 5 illustrates Proposition 3 and Theorem 2 for $S = \frac{5}{31} \frac{6}{31}$ and $q = 2$. Visible dynamic pairs are drawn in as in Figure 4. Beside them one sees some elements of \mathcal{B}_* . To distinguish the latter from the dynamic leaves, they are represented by curves whose end points lie a little outside the unit disk. B has period 10, and R_B lies behind B but is very near to B .

Obviously, the Translation Principle is an immediate consequence of the Theorems 1 and 2. So the substantial part of the paper is devoted to the proof of Theorem 2 (see Section 3). We shall refer to it as the main result of the paper.

3. Proofs

Kneading sequences. For a given point $\alpha \in T$, we call the sequence

$$s_1 s_2 s_3 \dots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{for } h^{i-1}(\alpha) \in]\frac{\alpha}{2}, \frac{\alpha+1}{2}[\\ 1 & \text{for } h^{i-1}(\alpha) \in]\frac{\alpha+1}{2}, \frac{\alpha}{2}[\\ * & \text{for } h^{i-1}(\alpha) \in \{\frac{\alpha}{2}, \frac{\alpha+1}{2}\} \end{cases}$$

the *kneading sequence* of α . The concept of a kneading sequence has been introduced by Milnor and Thurston for the investigation of real quadratic maps (see [29]), and its use in the context of complex quadratic maps (in more or less explicit form) is due to different authors (compare [1, 3, 8, 16], and note that the use of symbols in the kneading sequence is not uniform in literature).

The n -th symbol of the kneading sequence of $\alpha \in T$ is equal to $*$ iff α is periodic and the period of α divides n . An easy computation shows that this is satisfied iff α has the form $\frac{a}{2^n - 1}$ with $a \in \{0, 1, \dots, 2^n - 2\}$, and exactly in the points having this form with $a \neq 0$, the n -th symbol of the kneading sequence changes from 0 to 1. (In particular, the period of α is the minimal n for which α has a representation of the form $\frac{a}{2^n - 1}$.) So the following is obvious (compare [19, 23]):

Lemma 1 *Let β_1, β_2 be non-periodic points in T with $\beta_1 < \beta_2$. Then the n -th symbols of their kneading sequences coincide iff the number of points $\frac{a}{2^n - 1}$ with $a = 1, 2, \dots, 2^n - 2$ in $]\beta_1, \beta_2[$ is even. \square*

Fix some leaf $B = \alpha_1 \alpha_2 \in \mathcal{B}$. The number of points $\frac{a}{2^j - 1}$ between α_1 and α_2 is even for each j . This follows since the leaves in \mathcal{B}_* are mutually disjoint and since each $B \in \mathcal{B}$ with one periodic end belongs to \mathcal{B}_* (e.g., see [4], Lemma 3 and Theorem 2(a), also [13, 28, 33]).

Thus, if a sequence $(\alpha_i)_{i=1}^\infty$ of points behind B converges to an end of B , then also the corresponding kneading sequences converge, and the limit does not depend on $(\alpha_i)_{i=1}^\infty$ and the end of B . In particular, if $B \in \mathcal{B}_*$ has period m , then the limit is $\mathbf{v}^B e^B$, which can easily be deduced since the $m - 1$ -th iterate of α_i lies between $\overset{\cdot}{B}$ and $\overset{\cdot}{B}$ for sufficiently large i . This justifies the following definition (compare [23], Definition 5.3):

Definition 5 *The kneading sequence of a leaf $B \in \mathcal{B}$ is defined to be that of its ends. Moreover, if $B \in \mathcal{B}_*$ then $\mathbf{v}^B(1 - e^B)$ is called the kneading sequence just before B and $\mathbf{v}^B e^B$ the kneading sequence just behind B . If $B \in \mathcal{B} \setminus \mathcal{B}_*$, then the kneading sequences just before and just behind B are the kneading sequence of B .*

If $B \in \mathcal{B}$ is separated from some $Q \in \mathcal{B}$ behind B by a leaf of period m , and if m is assumed to be minimal with this condition, then the corresponding leaf is unique by Lavaurs' Lemma. This shows that the number of periodic points of period m in each of the two intervals between B and Q having one end in common with both B and Q is odd. So by Lemma 1 one easily obtains the following statement (compare [23], Proposition 5.4):

Lemma 2 (Symbolic description of parameter visibility)

Let $Q \in \mathcal{B}$ be a leaf behind some $B \in \mathcal{B}$. Then $B, Q \in \mathcal{B}$ are not separated by a leaf in \mathcal{B}_ of period less than or equal to n iff the initial subwords of length n of the kneading sequences just*

behind B and just before Q coincide. In particular, a leaf $Q \in \mathcal{B}_*$ of period m is visible from $B \in \mathcal{B}_*$ iff \mathbf{v}^B is equal to the initial subword of length $m - 1$ just behind Q . It is immediately visible if furthermore the period of B divides m . \square

If $S \in \mathcal{B}_*$, then the elements of the forward orbit of a given leaf $R = \alpha_1\alpha_2 \in \mathcal{B}(S)$ behind S do not cross \dot{S} and \ddot{S} and are not longer than \dot{S}, \ddot{S} . Thus $h^i(R)$ for $i = 0, 1, 2, \dots$ does not cross $\frac{\alpha_1 \alpha_1 + 1}{2}$ and $\frac{\alpha_2 \alpha_2 + 1}{2}$ iff $h^{i+1}(R)$ does not separate S and R . With (1) and statement (b) in Theorem 2 of [4] (compare also [28] and [22]) we have the following

Proposition 4 (Important dynamic chords lie in \mathcal{B}_*) *For $S \in \mathcal{B}_*$ the following statements are valid:*

(i) *Each boundary leaf of the critical value gap of $\mathcal{B}(S)$ and each leaf belonging to a dynamic pair visible from a boundary leaf of the critical value gap lies in \mathcal{B} .*

(ii) *If $B \in \mathcal{B}_*$ separates 0 and S , then B, \dot{B} and \ddot{B} are contained in $\mathcal{B}(S)$.* \square

Proof of Proposition 3. Let $S \in \mathcal{B}_*$ of period m and of length d with minimum end α be given. The boundary of the critical value gap of $\mathcal{B}(S)$ consists of the boundary leaves and further elements of T . Contracting each boundary leaf to a point, one gets a space homeomorphic to T .

There is a special map from T onto that space, which preserves orientation and conjugates h and h^m . That map, which is denoted by f^α here and was described in [4], is due to (the combinatorial description of tuning by) Douady (see [11]).

f^α is defined as follows (for equivalence to Douady's definition, see Proposition 2 in [20]): For $\beta \in T$ with binary expansion $.b_1b_2b_3\dots$, i.e. with $\beta = \sum_{i=1}^{\infty} b_i 2^{-i}$, let

$$f^\alpha(\beta) = \alpha + (2^m - 1) d \sum_{i=1}^{\infty} b_i 2^{-im}. \quad (2)$$

f^α assigns to each β with unique binary expansion a point different from all contracted boundary leaves. If $\beta \in T$ is first mapped to 0 after l iterates, then the two binary expansions $.b_1b_2\dots b_{l-1}1\bar{0}$ and $.b_1b_2\dots b_{l-1}0\bar{1}$ lead to the two endpoints of a boundary leaf $f^\alpha(\beta)$, which according to (2) has length

$$(2^m - 1) d \left(2^{-lm} - \sum_{i=2}^{\infty} 2^{-(l+i)m} \right) = (2^m - 1) d \left(\frac{1}{2^{lm}} - \frac{1}{2^{lm}(2^m - 1)} \right) = \frac{2^m - 2}{2^{lm}} d$$

in the case $l > 0$.

Now fix some $B \in \mathcal{B}_*$ of period qm being immediately visible from S . As well known, the ends of such B lie on a common orbit. Further, let $\gamma, \delta \in T$ be the unique points which are mapped by f^α to the ends of B . They must be periodic of period q and must lie on a common orbit. Moreover, the smaller closed interval I with ends γ and δ is not shorter than $\frac{1}{2^q - 1}$.

From this one easily deduces $T = \bigcup_{i=0}^{q-1} h^i(I)$, which implies the existence of some $\beta \in T$ first mapped to 0 after l iterates, where $0 < l \leq q - 1$. Let $R_B := f^\alpha(\beta)$.

Clearly, R_B lies behind the leaf B , which has length $\frac{(2^m-1)^2}{2^{qm}-1}d$ by a result of Lau and Schleicher (see Proposition 2.7 in [23]). This and the formula above show that $l = q - 1$. Since $h^{i-1}(R_B)$ lies between \dot{S} and \ddot{S} iff m divides i , one obtains that R_B is first mapped to S after $(q-1)m$ iterates and that the kneading sequence of R_B starts with $\mathbf{v}s_1\mathbf{v}s_2\dots s_{q-1}\mathbf{v}$, where $\mathbf{v} = \mathbf{v}^S$ and $s_1, s_2, \dots, s_{q-1} \in \{0, 1\}$.

Now let a leaf $Q \in \mathcal{B}_*$ behind B be visible from S . Then its period is less than qm . By Lemma 2 it cannot separate B and R_B , and by Proposition 4(i) it does not cross R_B . The difference between the lengths of B and R_B is

$$\left(\frac{(2^m - 1)^2}{2^{qm} - 1} - \frac{2^m - 2}{2^{(q-1)m}} \right) d = \frac{2^{qm} + 2^{2m} - 2^{m+1}}{2^{qm}(2^{qm} - 1)} d < \frac{2d}{2^{qm} - 1} < \frac{1}{2^{qm} - 1}.$$

Therefore Q must lie behind R_B .

Proof of the main result. We are going to prove Theorem 2 now. For the whole proof fix a leaf $S \in \mathcal{B}_*$ of a given period m , and let $\mathbf{v} = \mathbf{v}^S = v_1v_2\dots v_{m-1}$ and $e = e^S$. (For some possibly helpful illustrations, we refer to Figures 3, 4 and 5, which show the case $S = \frac{5}{31}\frac{6}{31}$ with period 5 and $\mathbf{v}^S = 0010, e^S = 1$.) At this place also recall the well known fact that the kneading sequence just behind S has period m (e.g., see [23]).

First of all, let us verify the following simple statement:

Lemma 3 *If for a given i the subword of $\overline{\mathbf{v}e}$ starting with the i -th symbol and having length $m - 1$ is equal to \mathbf{v} , then $i = jm + 1$ for some nonnegative integer j .*

Proof: Under the assumptions of the Lemma, the $i + m - 1$ -th symbol must be equal to e . This can be shown by counting the symbols in a sufficiently long initial subword of the periodic sequence $\overline{\mathbf{v}e}$. So, if i were different from all $jm + 1$ for $j = 0, 1, 2, \dots$, then the period of $\overline{\mathbf{v}e}$ would be less than m , which is false. \square

We shall use the following simple statement, which implies that for given 0-1-words \mathbf{u}, \mathbf{v} and \mathbf{w} and $s \in \{0, 1\}$, the leaf $L_{s\mathbf{u}}(S)$ separates $L_{s\mathbf{v}}(S)$ from $L_{s\mathbf{w}}(S)$ iff $L_{\mathbf{u}}(S)$ separates $L_{\mathbf{v}}(S)$ from $L_{\mathbf{w}}(S)$. (One implication is an immediate consequence of the following statement, and the other one can be shown indirectly, also applying that statement.)

Order-invariance Principle: *If three (or more) points are contained in an open semi-circle, then the action of h does not change their circular order.* \square

Before starting the proof of Theorem 2, let us make an agreement: Leaves S_1, S_2, \dots, S_k are said to be *in the arrangement* S_1, S_2, \dots, S_k if, for each $i = 3, 4, \dots, k$, the leaf S_{i-1} separates S_{i-2} and S_i , or $S_{i-2} = S_{i-1}$ or $S_{i-1} = S_i$.

I. Visible dynamic pairs determine (not immediately) visible periodic parameter leaves:

We start looking at the dynamic pairs visible from the longest preperiodic boundary leaf of the critical value gap of $\mathcal{B}(S)$.

Lemma 4 *Let (S_1^0, S_2^0) be a dynamic pair visible from $L_{\mathbf{v}e}(S)$ and having STEP $jm + k \geq 2$ with $j \in \{0, 1\}$ and $0 \leq k < m$.*

Then $k \neq 0$, the kneading sequence of S_1^0 is equal to $(\mathbf{ve})^j v_1 v_2 \dots v_k \overline{\mathbf{v}(1-e)}$, and the kneading sequence of S_2^0 is equal to $(\mathbf{ve})^j v_1 v_2 \dots v_{k-1} (1-v_k) \overline{\mathbf{v}(1-e)}$. (The kneading sequences of the leaves S_1^0 and S_2^0 are well-defined since by Proposition 4(i) they belong to \mathcal{B} .)

Proof: The arguments of the proof will be based on studying the mutual positions of the iterates of the leaves $S = L_{\mathbf{v}(1-e)}(S), L_{\mathbf{ve}}(S), S_1^0, S_2^0$, which are in the arrangement $S, L_{\mathbf{ve}}(S), S_1^0, S_2^0$. Further, we use the statement that for α behind S the i -th symbol of the kneading sequence is e if $h^{i-1}(\alpha)$ lies behind \check{S} , and $1-e$ if $h^{i-1}(\alpha)$ lies behind \dot{S} . This follows from $\check{S} = L_e(S), \dot{S} = L_{1-e}(S)$ and the fact that $\frac{\beta}{2} \frac{\beta+1}{2}$ separates \check{S} and \dot{S} .

We start with the case that $j = 0$. If then $0 \leq i < k-1$, the leaves $h^i(S_1^0), h^i(S_2^0)$ cannot be separated from $h^i(S)$ by $L_e(S) = \check{S}$ or $L_{1-e}(S) = \dot{S}$. Otherwise, by the Order-invariance Principle, S_1^0 or S_2^0 would be separated from S by $L_{v_1 v_2 \dots v_i e}(S)$ or $L_{v_1 v_2 \dots v_i (1-e)}(S)$, which is a contradiction to visibility of (S_1^0, S_2^0) . In particular, by the Order-invariance Principle, the $k-1$ -th iterates of S, S_1^0, S_2^0 are in the arrangement $h^{k-1}(S), \overline{h^{k-1}(S_1^0)}, h^{k-1}(S_2^0)$.

To see that S_1^0 and S_2^0 have kneading sequences $v_1 v_2 \dots v_k \overline{\mathbf{v}(1-e)}$ and $v_1 v_2 \dots v_{k-1} (1-v_k) \overline{\mathbf{v}(1-e)}$, one only needs that $\{h^{k-1}(S_1^0), h^{k-1}(S_2^0)\} = \{\dot{S}, \check{S}\}$. (Recall that the iterates of S do not lie between \dot{S} and \check{S} .)

$j = 1$ and $k = 0$ is impossible. Indeed, otherwise by use of the same arguments as in the case $j = 0$ one would obtain $\{S_1^0, S_2^0\} = \{S, L_{\mathbf{ve}}(S)\}$ from $\{h^{m-1}(S_1^0), h^{m-1}(S_2^0)\} = \{\dot{S}, \check{S}\}$, which contradicts the assumption that (S_1^0, S_2^0) is a (visible) dynamic pair.

Finally, let us come to the case $j = 1$ and $k > 0$. Again from the arguments in the case $j = 0$, it follows that the leaves S, S_1^0 and S_2^0 have kneading sequences with common initial subwords of length $m-1$.

To show that the m -th symbols of all kneading sequences considered are equal to e , also take into consideration the leaf $L_{\mathbf{ve}}(S)$. Using the Order-invariance Principle, one easily sees that the leaves $\dot{S} = h^{m-1}(S), \check{S} = h^{m-1}(L_{\mathbf{ve}}(S)), h^{m-1}(S_1^0), h^{m-1}(S_2^0)$ are in the arrangement $\dot{S}, \check{S}, h^{m-1}(S_1^0), h^{m-1}(S_2^0)$, which shows that both S_1^0 and S_2^0 have kneading sequences whose m -th symbols are equal to e .

For the rest, one argues as at the beginning of the proof: If $0 \leq i < k-1$, then the leaves $h^{m+i}(S_1^0), h^{m+i}(S_2^0)$ cannot be separated from $h^{m+i}(S) = h^i(S)$ by $L_e(S) = \check{S}$ or $L_{1-e}(S) = \dot{S}$. Otherwise, S_1^0 or S_2^0 would be separated from S by $L_{\mathbf{ve}v_1 v_2 \dots v_i e}(S)$ or $L_{\mathbf{ve}v_1 v_2 \dots v_i (1-e)}(S)$, but $L_{\mathbf{ve}v_1 v_2 \dots v_i e}(S)$ and by Lemma 3 $L_{\mathbf{ve}v_1 v_2 \dots v_i (1-e)}(S)$ belong to a dynamic pair having STEP less than $m+k$.

Moreover, $\{h^{m+k-1}(S_1^0), h^{m+k-1}(S_2^0)\} = \{\dot{S}, \check{S}\}$, and the $m+k-1$ -th iterates of S, S_1^0, S_2^0 are in the arrangement $h^{m+k-1}(S), h^{m+k-1}(S_1^0), h^{m+k-1}(S_2^0)$. \square

Now we are able to prove the first part of Theorem 2, which is covered by the following

Corollary 1 *Let $B \in \mathcal{B}_*$ be immediately visible from S and of period qm for some $q \geq 2$. Further, let (S_1, S_2) be a dynamic pair in $\mathcal{B}_*(S)$ visible from R_B and having STEP $jm+k \geq 2$ with $j < q$ and $0 \leq k < m$. Then the following statements are valid:*

(i) $j = q-2$ or $j = q-1$, and $k \neq 0$.

(ii) *The kneading sequence of S_1 is equal to $(\mathbf{ve})^j v_1 v_2 \dots v_k \overline{\mathbf{v}(1-e)}$, and the kneading sequence of S_2 is equal to $(\mathbf{ve})^j v_1 v_2 \dots v_{k-1} (1-v_k) \overline{\mathbf{v}(1-e)}$.*

(iii) *There exists a unique leaf $Q(S_1, S_2) \in \mathcal{B}_*$ of period $jm + k$ between S_1 and S_2 . It has kneading sequence $(\overline{\mathbf{ve}})^j v_1 v_2 \dots v_{k-1} *$ and it is the only visible leaf in \mathcal{B}_* separating S_1 and S_2 .*

Proof: $h^{(q-2)m}$ maps the leaf R_B onto $L_{\mathbf{ve}}(S)$. If (S_1, S_2) is a dynamic pair in $\mathcal{B}_*(S)$ visible from R_B and having STEP $n = jm + k$, then by Theorem 1 the dynamic pair (S_1^0, S_2^0) with $S_1^0 = h^{(q-2)m}(S_1)$ and $S_2^0 = h^{(q-2)m}(S_2)$ is visible from $L_{\mathbf{ve}}(S)$. So one easily obtains (i) and (ii) from Proposition 1 and Lemma 4.

(iii): Denote the two open intervals between S_1 and S_2 each having one end together with S_1 and one together with S_2 by I_1 and I_2 . By (ii) and Proposition 2, these intervals are shorter than $\frac{1}{2^n}$. Thus, for each $l \leq n$, both I_1 and I_2 contain at most one periodic point of period l (see above Lemma 1). By Lemma 1, there is one in I_1 and one in I_2 for $l = n$, since in this case the kneading sequences of S_1 and S_2 have different n -th symbols, but no one for $l < n$ since the l -th symbols coincide now.

Let $Q = Q(S_1, S_2)$ be the leaf connecting the two periodic points of period n we obtained. $S_1, S_2 \in \mathcal{B}$ forces that $Q \in \mathcal{B}_*$, and obviously the kneading sequence of Q is equal to $(\overline{\mathbf{ve}})^j v_1 v_2 \dots v_{k-1} *$. Moreover, by Lemma 2 it is visible and, since $0 < k < m$, not immediately visible from S .

What remains is to verify that there is no other leaf in \mathcal{B}_* visible from S and separating S_1 and S_2 . Assuming the opposite, the period of such a leaf must be greater than n and so by visibility it must separate S_1 and Q .

Let r be the minimum of periods of all leaves in \mathcal{B}_* separating S_1 and Q . Then by Lavaurs' Lemma there is a unique one of period r , and $r > n$. Let \mathbf{s} and \mathbf{q} be the kneading sequence of S_1 and the kneading sequence just before Q , respectively. Then by Lemma 2, the initial subwords of length $r - 1$ of \mathbf{s} , \mathbf{q} and $\overline{\mathbf{ve}}$ coincide, but the r -th symbols of \mathbf{s} and \mathbf{q} are different.

Thus from $\mathbf{s} = (\overline{\mathbf{ve}})^j v_1 v_2 \dots v_k \overline{\mathbf{v}(1-e)}$ and $\mathbf{q} = (\overline{\mathbf{ve}})^j v_1 v_2 \dots v_{k-1} v_k$ one easily obtains $r \geq (j + 1)m + k$, implying $(\overline{\mathbf{ve}})^j v_1 v_2 \dots v_k \mathbf{v} = (\overline{\mathbf{ve}})^{j+1} v_1 v_2 \dots v_k$. Therefore, we have $\mathbf{v} = v_{k+2} \dots v_{m-1} e v_1 \dots v_k$, which contradicts Lemma 3. \square

II. All visible but not immediately visible periodic parameter leaves are covered:

According to the remark below Definition 4 the map $Q(\cdot)$ is injective. Now we show its surjectivity. The first step is given by

Corollary 2 *Let $B \in \mathcal{B}_*$ be immediately visible from S . Further, let $Q \in \mathcal{B}_*$ of period n be visible from S and behind B , and let (S_1, S_2) be a dynamic pair of STEP $k \leq n$ in $\mathcal{B}_*(S)$. If S_1 separates S and Q , then $k = n$ and (S_1, S_2) is visible from R_B .*

Proof: Q lies behind R_B , and under the assumption that S_1 separates S and Q , the leaf S_2 must be behind R_B . If (S_1, S_2) were not visible from R_B , then one could find a dynamic pair (S_3, S_4) visible from R_B such that S_1 lies behind S_3 , and by the remark below Definition 4 also behind S_4 . So $Q(S_3, S_4)$ would be a visible leaf in \mathcal{B}_* of period less than n , which contradicts visibility of Q . \square

Obviously, the proof of Theorem 2 becomes complete by showing the following

Lemma 5 *Let $B \in \mathcal{B}_*$ be immediately visible from S . Further, let $Q \in \mathcal{B}_*$ be visible from S and behind B , and let $\mathbf{w} = \mathbf{v}^Q$.*

Then $L_{\mathbf{w}0}(S)$ and $L_{\mathbf{w}1}(S)$ are separated by Q and form a dynamic pair in $\mathcal{B}_(S)$.*

Proof: Let n be the period of Q and let $\mathbf{w} = w_1 w_2 \dots w_{n-1}$. Then by Lemma 2 and Lemma 3 the word \mathbf{w} does not end with \mathbf{v} , thus the two leaves $L_{w_k w_{k+1} \dots w_{n-1}}(\dot{S}), L_{w_k w_{k+1} \dots w_{n-1}}(\ddot{S})$ form a dynamic pair for all $k = 1, 2, \dots, n-1$. (The maps $L_{\mathbf{u}}$ for 0-1-words \mathbf{u} are taken with respect to S , i.e. all leaves considered belong to $\mathcal{B}_*(S)$.)

According to Proposition 4(ii), one has $S, \dot{S}, \ddot{S} \in \mathcal{B}(Q)$, and by induction one shows that for all $l = 1, 2, \dots, n-1$ both $L_{w_l w_{l+1} \dots w_{n-1}}(\dot{S})$ and $L_{w_l w_{l+1} \dots w_{n-1}}(\ddot{S})$ belong to $\mathcal{B}(Q)$.

The induction step is as follows: Assume that $L_{w_{k+1} w_{k+2} \dots w_{n-1}}(\dot{S}), L_{w_{k+1} w_{k+2} \dots w_{n-1}}(\ddot{S})$ are elements of $\mathcal{B}(Q)$ but at least one of the leaves $L_{w_k w_{k+1} \dots w_{n-1}}(\dot{S})$ and $L_{w_k w_{k+1} \dots w_{n-1}}(\ddot{S})$ does not belong to $\mathcal{B}(Q)$. Then the infinite gap in $\mathcal{B}(S)$ defined by the latter leaves lies between \dot{S} and \ddot{S} , but none of its boundary leaves separates \dot{S} and \ddot{S} .

Denote the longer of the leaves $L_{w_k w_{k+1} \dots w_{n-1}}(\dot{S})$ and $L_{w_k w_{k+1} \dots w_{n-1}}(\ddot{S})$ by R . Obviously, R crosses \dot{Q} or \ddot{Q} . If R crossed both \dot{Q} and \ddot{Q} , then $h(R) \in \{L_{w_{k+1} w_{k+2} \dots w_{n-1}}(\dot{S}), L_{w_{k+1} w_{k+2} \dots w_{n-1}}(\ddot{S})\}$ would separate S and Q , which is impossible by Corollary 2. Otherwise, by the Order-invariance Principle, $h(R)$ would cross Q , in contradiction to $h(R) \in \mathcal{B}(Q)$. This finishes the induction.

$\dot{Q} = h^{n-1}(Q)$ separates \dot{S} and \ddot{S} . So the Order-invariance Principle shows that $h^{n-2}(Q)$ separates $L_{w_{n-1}}(\dot{S})$ and $L_{w_{n-1}}(\ddot{S})$, then that $h^{n-3}(Q)$ separates the leaves $L_{w_{n-2} w_{n-1}}(\dot{S})$ and $L_{w_{n-2} w_{n-1}}(\ddot{S})$ etc. Finally, one obtains that the two leaves $L_{\mathbf{w}}(\dot{S})$ and $L_{\mathbf{w}}(\ddot{S})$ are separated by Q . \square

4. Some further remarks on the Translation Principle

To explain where the interest for the Translation Principle comes from, let us mention a problem, which originates in the combinatorial description of the Mandelbrot set. The central concept in the approach by Lau and Schleicher is the *internal address* of a hyperbolic component (see [23, 32]), which can be given recursively:

1. The main component has internal address 1.
2. If W is a hyperbolic component with internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m$, then a hyperbolic component visible from W and of period $n > m$ has internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = n_{k+1} = n$.

One easily sees that, by this definition, to each hyperbolic component there is assigned an internal address, but Lau and Schleicher have shown more: By the procedure defining the internal address, in each step one fixes a hyperbolic component. If one further fixes the internal angle $\frac{p}{q}$, for which the succeeding hyperbolic component is contained in the $\frac{p}{q}$ -sublimb of the given one, then one obtains the *angled internal address* (compare [23], Definition 6.1). Angled internal addresses are complete, i.e. different hyperbolic components have different angled internal addresses (see [23], Theorem 9.2).

However, not each ‘abstract internal address’ $1 \rightarrow n_2 \rightarrow \dots \rightarrow n_k$ occurs as the internal address of a hyperbolic component (compare [23, 32]), and so the following problem remains:

Problem (geometric version) *Which (angled) internal addresses are admissible by a hyperbolic component of the Mandelbrot set?*

In [4], Bandt and the author have given a description of the (abstract) Mandelbrot set by use of kneading sequences. This way of description, based on Thurston’s ideas in [37], allows a good insight into the relation between dynamical properties of Julia sets and properties of the Mandelbrot set (compare [20, 21], especially [22]).

Internal addresses can be turned into kneading sequences and vice versa (see [23]): Let $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m$ be the internal address of a hyperbolic component of period equal to m and let $\alpha \in T$ be given such that the corresponding external angle lands at its root. Then the kneading sequence of α starts with 0, its m -th symbol is *, and if its initial subword of length n_i is \mathbf{w} , then the initial subword of the length n_{i+1} taken from the kneading sequence and $\overline{\mathbf{w}}$ coincide with exception of exactly the last symbol. (This procedure becomes also clear from Lemma 2, and to find the inverse procedure from the kneading sequence to the internal address is easy.)

In so far, internal addresses are a tool equivalent to kneading sequences, but more compact and containing the geometric information in a direct form, and it is not hard to show that the following problem can be reduced to find a description of all admissible internal addresses (compare [22]).

Problem (symbolic version) *Which sequences are admissible by the kneading sequence of a point $\alpha \in T$?*

The formulation of this problem, which was noted by C. Penrose (see [30]) and Bandt and the author (see [3]), is on a rather elementary level, and at a first view the relation to complex quadratic iteration is surprising. But, perhaps, a ‘naive’ research of the structure of kneading sequences would lead to a structure like \mathcal{B}_* directly.

The idea behind internal addresses is not new. In one-dimensional real dynamics it appeared as the concept of *cutting times*. Different characterizations of admissible kneading sequences (or of the admissible internal addresses) for unimodal maps, and so for real quadratic maps, have been given, for example by Collet and Eckmann [10], Milnor and Thurston [29], Hofbauer and G. Keller [17, 18], and Bruin [7].

C. Penrose has considered an object more general than the abstract Mandelbrot set and consisting of 0-1-sequences (see [30, 31]). To investigate bifurcations there, and also to describe Julia sets by symbol sequences, he used the concept of a ‘principal nonperiodicity function’, a generalized form of an internal address. (We refer to his viewpoint concerning ‘non-admissible’ kneading sequences and to his rather deep Theorem 4.2, although the latter is only loosely related to the present subject.) Also the ‘combinatorics of initial subwords’ in [3] touches the idea of an internal address (compare [22]).

One can try to find all admissible internal addresses step by step, as the above definition suggests. However, in each step one has to know which number can be appended to a given admissible internal address to get a new one. In fact, this amounts to finding all hyperbolic

components visible from a given one, but of a greater period.

In this, one special case is simpler than the general one: in [23], Lau and Schleicher have called a hyperbolic component of period m *narrow* if there exists no hyperbolic component visible from the given one of period less than m . Obviously, a hyperbolic component W of that type is characterized by the fact that the leaf in \mathcal{B}_* corresponding to its root has length $\frac{1}{2^m-1}$.

By use of this, Lau and Schleicher have shown that in the narrow case there exist hyperbolic components visible from W for all $n > m$ (see Theorem 10.2 in [23]), and also for narrow hyperbolic components, they have proved the Translation Principle. (For partial proofs of the Translation and Correspondence Principle in the non-narrow case, compare Proposition 8.4 and Corollary 8.5 in [23].)

If W of period m fails to be narrow, the situation becomes more complicated, but the general validity of the Translation Principle shows that the existence of a visible hyperbolic component of a given period greater than m can be decided by looking at the $\frac{1}{2}$ -sublimb of W :

If this sublimb contains a visible component of period r or $m+r$ with $r = 1, 2, \dots, m-1$, then there exist visible components of period $jm+r$ for all $j = 1, 2, 3, \dots$. In the other direction, if there exists a visible component of period $jm+r$, then let j_0 be the least j with this property. The corresponding visible component lies in the $\frac{1}{2}$ -sublimb, and j_0 must be equal to 0 or to 1. So we have the following

Corollary 3 (Translation Principle for non-angled Internal Addresses)

If the internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m$ is admissible, then for each $j = 2, 3, 4, \dots$ the internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m \rightarrow jm$ is admissible. Moreover, for each $r = 1, 2, \dots, m-1$, the following holds: If one of the internal addresses $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m \rightarrow jm+r$ with $j = 1, 2, 3, \dots$ is admissible, then all these addresses are admissible.

□

Clearly, the first part of the theorem is trivial, and the theorem does not touch the embedding in the plane. However, it is no problem to say more about this embedding by use of angled internal addresses. This is left to the reader.

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Errata

The ‘Translation’ and ‘Correspondence’ Principles we gave earlier turned out to be false in the general case. The aim of this errata is to discuss which parts of the two statements are incorrect and which parts remain true.

A. Weaker statements

Some notations. We make use of **notations**, **statements** and **references** as given in the *Stony Brook IMS Preprint 1997/14*. Beyond this some further notations will be convenient.

By a *visibility tree* of a given hyperbolic component W we understand the tree of hyperbolic components visible from W in a sublimb of W . The visibility tree contained in a $\frac{p}{q}$ -sublimb of W is denoted by $\mathfrak{Vis}_{\frac{p}{q}}(W)$. Two visibility trees $\mathfrak{Vis}_{\frac{p_1}{q_1}}(W)$ and $\mathfrak{Vis}_{\frac{p_2}{q_2}}(W)$ are called *equivalent* if they coincide, including the embedding into the plane, when all periods in $\mathfrak{Vis}_{\frac{p_1}{q_1}}(W)$ are increased by $(q_2 - q_1)m$.

Further, we denote the hyperbolic component associated with a leaf $B \in \mathcal{B}_*$ by W_B . (Recall that the two end points of B are the unique external angles of the root of that hyperbolic component.)

The false statements and their modifications. The Translation Principle stated at page 3 does not hold in general. A counter-example is given by the hyperbolic component $W = W_{\frac{13}{31} \frac{18}{31}}$ of period $m = 5$ (see figure): $\mathfrak{Vis}_{\frac{1}{2}}(W)$ does not contain a hyperbolic component of period 6 (= 11 - 5), and beyond this the embedding of the hyperbolic components of periods 8 and 11 is different in $\mathfrak{Vis}_{\frac{1}{3}}(W)$ and $\mathfrak{Vis}_{\frac{2}{3}}(W)$.

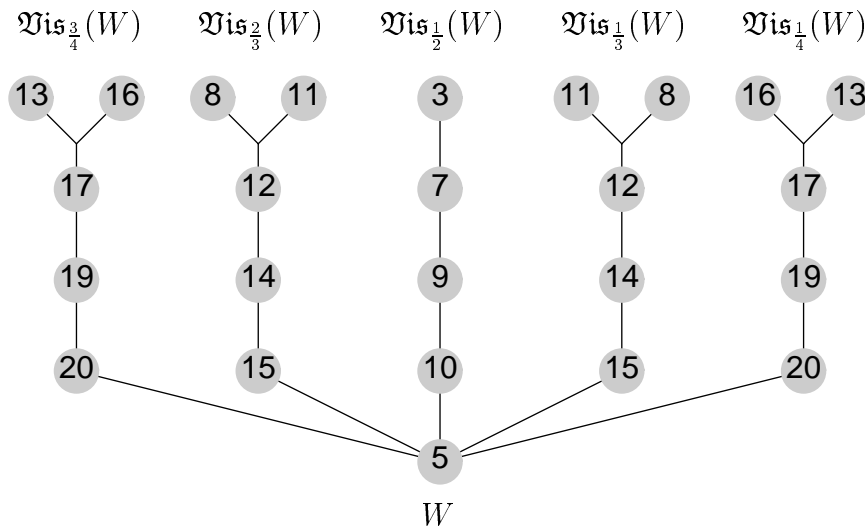


Figure 6: Counter example

The following theorem answers the question of which part of the Translation Principle can be saved.

Theorem I. (Partial Translation Principle)

Each visibility tree other than $\mathfrak{Vis}_{\frac{1}{2}}(W)$ for a hyperbolic component W is equivalent to $\mathfrak{Vis}_{\frac{1}{3}}(W)$ or to $\mathfrak{Vis}_{\frac{2}{3}}(W)$.

This errata is mainly devoted to the proof of Theorem I. Since the Translation Principle is not valid in general, Corollary 3 must be changed in the following way:

Corollary I. (Partial Translation Principle for Internal Addresses)

If the internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m$ is admissible, then for each $j = 2, 3, 4, \dots$ the internal address $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m \rightarrow jm$ is admissible. Moreover, for each $r = 1, 2, \dots, m-1$, the following holds: If one of the internal addresses $1 \rightarrow n_2 \rightarrow n_3 \dots \rightarrow n_k = m \rightarrow jm + r$ with $\boxed{j = 2, 3, 4, \dots}$ is admissible, then all these addresses are admissible.

In order to see that Corollary 3 is false, look at the above counter-example. The kneading sequence of $S = \frac{13}{31} \frac{18}{31}$ is $\overline{0100*}$ and so $W = W_S$ has internal address $1 \rightarrow 2 \rightarrow 4 \rightarrow 5$. Therefore, the hyperbolic components of period 11 visible from W have internal address $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 11$.

On the other hand, a hyperbolic component of internal address $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$ does not exist, providing a counter-example to the statement of Corollary 3. The failure of that internal address can easily be obtained by checking the kneading sequences of all points in (T, h) of period 6.

The source of all errors in the paper is that Corollary 2 is false. However, the statements in ‘3. Proofs’ before Corollary 2. guarantee the following: To visible Fatou components in the dynamic plane there correspond visible hyperbolic components in the parameter space, which do not bifurcate directly from W .

The problem, hidden by the false Corollary 2, is the existence of visible hyperbolic components which do not bifurcate directly from W and are not covered by visible Fatou components. Therefore, in the general case only the following can be shown:

Proposition I. (Partial Correspondence Principle)

Let W be a hyperbolic component of period m , let c be its center and A be the critical value component of p_c . Then, for relatively prime r, q and a given sublimb of A of denominator 2^{q-1} the following holds:

There exists an injective map from the tree of all Fatou components of STEP less than qm visible from A and contained in that sublimb, into $\mathfrak{Vis}_{\frac{r}{q}}(W)$, which preserves the embedding into the plane and which turns STEP’s into periods.

Roughly speaking, the Correspondence Principle transfers ‘dynamic translation’ to the parameter space; if this transfer is not complete, the Translation Principle can be incomplete.

The proof of Theorem I will base on a modification of the Correspondence Principle transferring more ‘dynamic translation’ to the parameter space than Proposition I (see Proposition II).

B. Proof of the Partial Translation Principle.

Let $S \in \mathcal{B}_*$ of period m be fixed, and let $\mathbf{v} = \mathbf{v}^S = v_1 v_2 \dots v_{m-1}$ and $e = e^S$. Among the boundary leaves of the critical value gap of $\mathcal{B}(S)$ there are the special leaves R_B for visible $B \in \mathcal{B}_*$. We denote such leaves R_B also by $R_{\frac{p}{q}}$ if the root of W_B has internal angle $\frac{p}{q}$ with respect to W_S . (W_B bifurcates directly from W and determines $\mathfrak{Vis}_{\frac{p}{q}}(W)$.) Note that $R_{\frac{1}{2}} = L_{\mathbf{ve}}(S)$ and $\{h^{m-1}(R_{\frac{1}{3}}), h^{m-1}(R_{\frac{2}{3}})\} = \{L_{e\mathbf{ve}}(S), L_{(1-e)\mathbf{ve}}(S)\}$.

By Propositions 1 and 2 and by Lemma 4 one easily gets the following

Lemma I. *For p, q relatively prime, the kneading sequence of $R_{\frac{p}{q}}$ is equal to $(\mathbf{ve})^{q-1} \overline{\mathbf{v}(1-e)}$. Moreover, the kneading sequence of each point α behind $R_{\frac{p}{q}}$ begins with $(\mathbf{ve})^{q-2}$.*

If $q \geq 3$, then it holds $h^{(q-2)m-1}(R_{\frac{p}{q}}) \in \{L_{e\mathbf{ve}}(S), L_{(1-e)\mathbf{ve}}(S)\}$, and $h^{(q-2)m-1}(\alpha)$ lies behind $L_{e\mathbf{ve}}(S)$ or $L_{(1-e)\mathbf{ve}}(S)$. ■

As already mentioned above there exist visible hyperbolic components which do not correspond to visible Fatou components in the dynamic plane. In order to get a better correspondence let us introduce the concept of semi-visibility. We do this in the lamination framework.

Definition I. (semi-visible dynamic pair)

Let $B \in \mathcal{B}_$ be immediately visible from S . Then a dynamic pair (S_1, S_2) of STEP n behind R_B is said to be semi-visible from R_B if $S_1, S_2 \in \mathcal{B}$, and if the sequence $\overline{\mathbf{ve}}$ and the kneading sequence of S_1 have the same initial subwords of length n .*

Remark: If B has period qm ; $q \geq 2$ and $n < qm$, then visibility of the dynamic pair (S_1, S_2) implies its semi-visibility (see Corollary 1(ii)). Moreover, semi-visibility of (S_1, S_2) provides $n < qm$. This follows from Lemma 2 and the fact that S_1 and S are separated by R_B .

The statements of Proposition 2 concerning lengths of leaves show that no iterate of a dynamic pair (S_1, S_2) of some STEP n can lie between its members. In particular, for $S_1 = \beta_1 \beta_2, S_2 = \beta_3 \beta_4$ with $\beta_1 < \beta_3 < \beta_2 < \beta_4$ no iterate of the points $\beta_1, \beta_2, \beta_3, \beta_4$ lies between $\frac{\beta_1}{2}$ and $\frac{\beta_2}{2}$ or between $\frac{\beta_3}{2}$ and $\frac{\beta_4}{2}$. This and $\{h^{n-1}(S_1), h^{n-1}(S_2)\} = \{\dot{S}, \dot{S}\}$ imply the following statement:

Lemma II. *Let $B \in \mathcal{B}_*$ of period qm be immediately visible from S and let (S_1, S_2) be a semi-visible dynamic pair of STEP n behind R_B . Then $n < qm$. Moreover, $\overline{\mathbf{ve}}$ and the kneading sequence of S_2 have the same initial subwords of length n with the exception of their n -th symbols. ■*

Let us now relate semi-visibility in the dynamic plane to visibility in the parameter space.

Proposition II. ('dynamic' semi-visibility and 'parameter' visibility)

Let $B \in \mathcal{B}_$ be immediately visible and of period qm ; $q \geq 3$. Then there exists a map $Q : (S_1, S_2) \mapsto Q(S_1, S_2)$ from the set of all dynamic pairs semi-visible from R_B onto the set of all visible leaves in \mathcal{B}_* behind B satisfying the following properties:*

- (i) *The STEP of (S_1, S_2) and the period of $Q(S_1, S_2)$ coincide.*
- (ii) *$Q(S_1, S_2)$ is the only visible leaf in \mathcal{B}_* separating S_1 and S_2 .*

Moreover, if (S_1, S_2) is a dynamic pair semi-visible from R_B , then the $(q-3)m$ -th iterate of (S_1, S_2) lies behind $R_{\frac{1}{3}}$ or $R_{\frac{2}{3}}$, and each higher iterate lying behind S is behind $R_{\frac{1}{2}}$.

Proof: Let (S_1, S_2) be semi-visible from R_B of STEP $n = jm + k$ with $j < q$ and $0 \leq k < m$. Denote the two open intervals between S_1 and S_2 each having one end together with S_1 and one together with S_2 by I_1 and I_2 .

By Proposition 2, these intervals are shorter than $\frac{1}{2^n}$. Thus, for each $l \leq n$, both I_1 and I_2 contain at most one periodic point of period l (see above Lemma 1). By Lemma 1, there is one in I_1 and one in I_2 for $l = n$ since in this case the kneading sequences of S_1 and S_2 have different n -th symbols, but no one for $l < n$ since the l -th symbols coincide now.

Let $Q = Q(S_1, S_2)$ be the leaf connecting the two periodic points of period n we have obtained. $S_1, S_2 \in \mathcal{B}$ forces that $Q \in \mathcal{B}_*$, and obviously the kneading sequence of Q is equal to $(\overline{ve})^j v_1 v_2 \dots v_{k-1} *$. Moreover, Q is visible by Lemma 2.

We now verify that there is no other leaf in \mathcal{B}_* visible from S and separating S_1 and S_2 . Assuming the opposite, the period of such a leaf must be greater than n and so by visibility it must separate S_1 and Q .

Let r be the minimum of periods of all leaves in \mathcal{B}_* separating S_1 and Q . Then by Lavaurs' Lemma there is a unique one of period r , and $r > n$. Let \mathbf{s} and \mathbf{q} be the kneading sequence of S_1 and the kneading sequence just before Q , respectively. Then by Lemma 2, the initial subwords of length $r-1$ of \mathbf{s}, \mathbf{q} and \overline{ve} coincide, but the r -th symbols of \mathbf{s} and \mathbf{q} are different.

Thus from $\mathbf{s} = (\overline{ve})^j v_1 v_2 \dots v_k \overline{\mathbf{v}}(1-e)$ and $\mathbf{q} = (\overline{ve})^j v_1 v_2 \dots v_{k-1} v_k$ one gets $r \geq (j+1)m + k$, implying $(\overline{ve})^j v_1 v_2 \dots v_k \overline{\mathbf{v}} = (\overline{ve})^{j+1} v_1 v_2 \dots v_k$. So, we have $\overline{\mathbf{v}} = v_{k+2} \dots v_{m-1} e v_1 \dots v_k$, which contradicts Lemma 3.

It remains to show the last statement of the proposition and that Q is surjective. For this, let $Q \in \mathcal{B}_*$ behind B be visible. Further, let n be the period of Q and let l be minimal with the property that $h^l(Q)$ separates \dot{S} and \ddot{S} . Clearly, $l+1 \leq n < qm$, and according to Proposition 4(ii), one has $S, \dot{S}, \ddot{S} \in \mathcal{B}(Q)$. Let $R = h^l(Q)$.

Denote the i -th symbol of \overline{ve} by w_i and fix symbols $s_1, s_2, \dots, s_l \in \{0, 1\}$ such that $Q = L_{s_1 s_2 \dots s_l}(R)$. By induction on k one shows that for all $k = 1, 2, \dots, l$ the leaves $L_{s_{l-k+1} s_{l-k+2} \dots s_l}(\dot{S})$ and $L_{s_{l-k+1} s_{l-k+2} \dots s_l}(\ddot{S})$ are separated by $L_{s_{l-k+1} s_{l-k+2} \dots s_l}(R) = h^{l-k}(Q)$. (One uses that $L_{s_{l-k+1} s_{l-k+2} \dots s_l}(\dot{S}), L_{s_{l-k+1} s_{l-k+2} \dots s_l}(\ddot{S})$ and $h^{l-k}(Q)$ do not separate \dot{S} and \ddot{S} .)

Let S_1 be the longer of the leaves $L_{s_1 s_2 \dots s_l}(\dot{S})$ and $L_{s_1 s_2 \dots s_l}(\ddot{S})$, and let S_2 be the other one. Clearly, by Propositions 1 and 2 the $(q-3)m$ -th iterates of S_1, S_2, Q lie behind $R_{\frac{1}{3}}$ or $R_{\frac{2}{3}}$ and their $(q-2)m$ -th iterates behind $R_{\frac{1}{2}}$.

If for some $j > 0$ the leaves $h^{(q-2)m+j}(S_1), h^{(q-2)m+j}(S_2)$ were behind $R_{\frac{1}{3}}$ or $R_{\frac{2}{3}}$, then also the leaf $h^{(q-2)m+j}(Q)$. So by Proposition 1 the word $w_{(q-2)m+j+1} w_{(q-2)m+j+2} \dots w_{(q-1)m+j-1}$ would coincide with $\overline{\mathbf{v}}$ and Lemma 3 would imply $j = m$. So $(h^{qm}(S_1), h^{qm}(S_2))$ would be behind $R_{\frac{1}{2}}$, what is obviously false.

Therefore, $h^{(q-2)m+j}(S_1), h^{(q-2)m+j}(S_2)$ cannot be behind $R_{\frac{1}{3}}$ or $R_{\frac{2}{3}}$. In particular, we get that no iterate of S_1, S_2 separates S_1 or S_2 from S . According to the considerations above Proposition 4, this implies $S_1, S_2 \in \mathcal{B}$.

To show that (S_1, S_2) forms a dynamic pair, assume the contrary. Then $s_{l-m+2} s_{l-m+3} \dots s_l =$

\mathbf{v} and $\{h^{l-m+1}(S_1), h^{l-m+1}(S_2)\} = \{S, R_{\frac{1}{2}}\}$. Therefore, $h^{l-m+1}(Q)$ separates the boundary chords S and $R_{\frac{1}{2}}$ of the critical value gap, implying $w_{l-m+2}w_{l-m+3}\dots w_l = \mathbf{v}$. Clearly, if $j \leq (q-2)m$, then $h^j(Q)$ is shorter than $R_{\frac{1}{2}}$. So we have $l-m+1 > (q-2)m$. Hence Lemma 3 provides $l-m+2 = (q-1)m+1$, leading to the contradiction $l+1 = qm$.

Since Q is visible, S and S_1 cannot be separated by a leaf in \mathcal{B}_* of period less than or equal to n . Thus by Lemma 2 (S_1, S_2) is semi-visible, and the first part of the proof shows $n = l+1$ and $Q = Q(S_1, S_2)$.

So we have surjectivity of Q and therefore the above also provides the last statement of the proposition. ■

With Propositions 1 and II and Lemma III following below, the proof of Theorem I is complete. Namely, Lemma III relates semi-visibility from different leaves R_B , and Theorem I can be specified as follows:

Theorem II. *Let W be a hyperbolic component, and let p_1, q_1 and p_2, q_2 be relatively prime with $q_1, q_2 \geq 3$. If $h^{(q_1-3)m}(R_{\frac{p_1}{q_1}}) = h^{(q_2-3)m}(R_{\frac{p_2}{q_2}})$, then the visibility trees $\mathfrak{Vis}_{\frac{p_1}{q_1}}(W)$ and $\mathfrak{Vis}_{\frac{p_2}{q_2}}(W)$ are equivalent.* ■

Lemma III. *Let p_1, q_1 and p_2, q_2 be relatively prime with $q_1, q_2 \geq 3$ and with $h^{(q_1-3)m}(R_{\frac{p_1}{q_1}}) = h^{(q_2-3)m}(R_{\frac{p_2}{q_2}})$. Further, let (S_1, S_2) and (S_3, S_4) be dynamic pairs of STEP's $k < q_1m$ and $l < q_2m$ behind $R_{\frac{p_1}{q_1}}$ and $R_{\frac{p_2}{q_2}}$, respectively.*

If $h^{(q_1-3)m}(S_1) = h^{(q_2-3)m}(S_3)$ and $h^{(q_1-3)m}(S_2) = h^{(q_2-3)m}(S_4)$, then it holds $l - k = (q_2 - q_1)m$, and (S_1, S_2) is semi-visible iff (S_3, S_4) is.

Proof: $l - k = (q_2 - q_1)m$ is obvious. So let us assume that (S_1, S_2) is semi-visible. Then by the last statement of Proposition II only the $(q_1 - 3)m$ -th iterate of (S_1, S_2) lies behind $R_{\frac{1}{3}}$ or $R_{\frac{2}{3}}$ and so only the $(q_2 - 3)m$ -th one of (S_3, S_4) (see also Proposition 1).

The first consequence of this fact is that $S_3, S_4 \in \mathcal{B}_*$. Namely, no iterate of S_3, S_4 separates S from S_3 or S_4 , and one can argue as in the second part of the proof of Proposition II. Moreover, together with Proposition 1 one obtains the following consequence: The leaf $h^{(q_1-2)m-1}(S_1) = h^{(q_2-2)m-1}(S_3)$ lies behind $L_{e\mathbf{v}e}(S)$ or $L_{(1-e)\mathbf{v}e}(S)$, and if some of its iterates is between \dot{S} and \ddot{S} , then again behind one of the leaves $L_{e\mathbf{v}e}(S)$ or $L_{(1-e)\mathbf{v}e}(S)$.

Since by Lemma I the kneading sequences of S_3 begins with $(\mathbf{v}e)^{q_2-2}$, the initial subwords of length l of $\overline{\mathbf{v}e}$ and the kneading sequence of S_3 coincide, and we are done. ■

Let us finish by noting the problem in the proof of the false Corollary 2: The dynamic pair (S_1, S_2) need not be visible because it can lie between S_3 and S_4 for some visible dynamic pair (S_3, S_4) , although S_3 and S_4 are not separated by S_1, S_2 (and Q).

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