Rational Parameter Rays of the Mandelbrot Set

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Contents

Abstract

We give a new proof that all external rays of the Mandelbrot set at rational angles land, and of the relation between the external angle of such a ray and the dynamics at the landing point. Our proof is different from the original one, given by Douady and Hubbard and refined by Lavaurs, in several ways: it replaces analytic arguments by combinatorial ones; it does not use complex analytic dependence of the polynomials with respect to parameters and can thus be made to apply for non-complex analytic parameter spaces; this proof is also technically simpler. Finally, we derive several corollaries about hyperbolic components of the Mandelbrot set.

Along the way, we introduce partitions of dynamical and parameter planes which are of independent interest, and we interpret the Mandelbrot set as a symbolic parameter space of kneading sequences and internal addresses.

Nous donnons une nouvelle démonstration que tous les rayons externes à arguments rationels de l'ensemble Mandelbrot aboutissent, et nous montrons la relation entre l'argument externe d'un tel rayon et la dynamique au paramètre où le rayon aboutit. Notre démonstration est différente de l'originale, donnée par Douady et Hubbard et elaborée par Lavaurs, à plusieurs égards: elle remplace des arguments analytiques par des arguments combinatoires; elle n'utilise pas la dépendence analytique des polynômes par rapport au paramètre et peut donc être appliquée aux espaces de paramètres qui ne sont pas analytiques complexes; la démonstration est aussi techniquement plus facile. Finalement, nous démontrons quelques corollaires sur les composantes hyperboliques de l'ensemble Mandelbrot.

En route, nous introduisons des partitions du plan dynamique et de l'espace des paramètres qui sont intéressantes en elles-mêmes, et nous interprétons l'ensemble Mandelbrot comme un espace de paramètres symboliques contenant des *kneading sequences* et des adresses internes.

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1 Introduction

Quadratic polynomials, when iterated, exhibit amazingly rich dynamics. Up to affine conjugation, these polynomials can be parametrized uniquely by a single complex variable. The Mandelbrot set serves to organize the space of (conjugacy classes of) quadratic polynomials. It can be understood as a "table of contents" to the dynamical possibilities and has a most beautiful structure. Much of this structure has been discovered and explained in the groundbreaking work of Douady and Hubbard [DH1], and a deeper understanding of the fine structure of the Mandelbrot set is a very active area of research. The importance of the Mandelbrot set is due to the fact that it is the simplest non-trivial parameter space of analytic families of iterated holomorphic maps, and because of its universality as explained by Douady and Hubbard [DH2]: the typical local configuration in one-dimensional complex parameter spaces is the Mandelbrot set (see also [McM]).

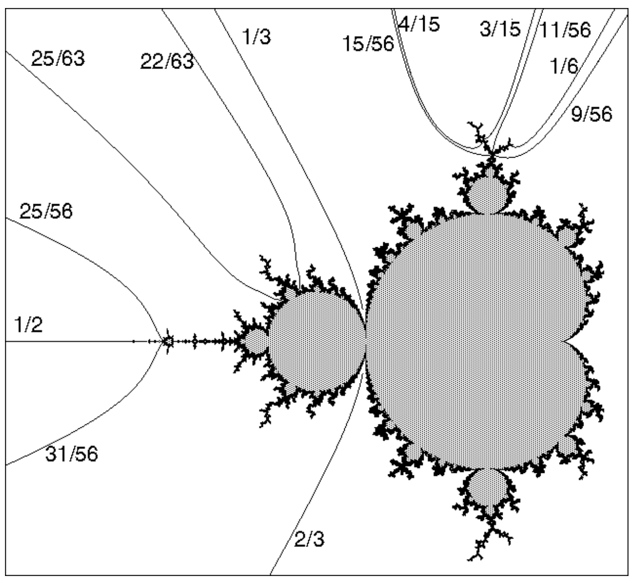

Figure 1: The Mandelbrot set and several of its parameter rays which are mentioned in the text. (Picture courtesy of Jack Milnor)

Unfortunately, most of the beautiful results of Douady and Hubbard on the structure of the Mandelbrot set are written only in preliminary form in the preprints [DH1]. The purpose of this article is to provide concise proofs of several of their theorems. Our proofs are quite different from the original ones in several respects: while Douady and Hubbard used elaborate perturbation arguments for many basic results, we introduce partitions of dynamical and parameter planes, describe them by symbolic dynamics, and reduce many of the questions to a combinatorial level. We feel that our proofs are technically significantly simpler than those of Douady and Hubbard. An important difference for certain applications is that our proof does not use complex analytic dependence of the maps with respect to the parameter and is therefore applicable in certain wider circumstances: the initial motivation for this research was a project with Nakane (see [NS] and the references therein) to understand the parameter space of antiholomorphic quadratic polynomials, which depends only real-analytically on the parameter. Of course, the "standard proof" using Fatou coordinates and

Ecalle cylinders, as developed by Douady and Hubbard and elaborated by Lavaurs [La2], is a most powerful tool giving interesting insights; it has had many important applications. Our goal is to present an alternative approach in order to enlarge the toolbox for applications in different situations.

The fundamental result we want to describe in this article is the following theorem about landing properties of external rays of the Mandelbrot set, a theorem due to Douady and Hubbard; for background and terminology, see the next section.

Theorem 1.1 (The Structure Theorem of the Mandelbrot Set)

1. *Every parameter ray at a periodic angle ϑ lands at a parabolic parameter such that, in its dynamic plane, the dynamic ray at angle ϑ lands at the parabolic orbit and is one of its two characteristic rays.*

2. *Every parabolic parameter c is the landing point of exactly two parameter rays at periodic angles. These angles are the characteristic angles of the parabolic orbit in the dynamic plane of c.*

3. *Every parameter ray at a preperiodic angle ϑ lands at a Misiurewicz point such that, in its dynamic plane, the dynamic ray at angle ϑ lands at the critical value.*

4. *Every Misiurewicz point c is the landing point of a finite non-zero number of parameter rays at preperiodic angles. These angles are exactly the external angles of the dynamic rays which land at the critical value in the dynamic plane of c.*

(The parameter $c = 1/4$ is the landing point of a single parameter ray, but this ray corresponds to external angles 0 and 1; we count this ray twice in order to avoid having to state exceptions.)

The organization of this article is as follows: in Section 2, we describe necessary terminology from complex dynamics and give a few fundamental lemmas. Section 3 contains a proof of the periodic part of the theorem, and along the way it shows how to interpret the Mandelbrot set as a parameter space of kneading sequences. The preperiodic part of the theorem is then proved in Section 4, using properties of kneading sequences. In the final Section 5, we show that the methods of Section 3 can be used to prove an Orbit Separation Lemma, which has interesting consequences about hyperbolic components of the Mandelbrot set. Most of the results and proofs in this paper work also for "Multibrot sets": these are the connectedness loci of the polynomials $z^d + c$ for $d \geq 2$.

This article is an elaborated version of Chapter 2 of my Ph.D. thesis [S1] at Cornell University, written under the supervision of John Hubbard and submitted in the summer of 1994. It is part of a mathematical ping-pong with John Milnor: it builds at important places on the paper [GM]; recently Milnor has written a most beautiful new paper [M2] investigating external rays of the Mandelbrot set from the point of view of "orbit portraits", i.e., landing patterns of periodic dynamic rays. I have not tried to hide how much both I and this paper have profited from many discussions with him, as will become apparent at many places. This paper, as well as Milnor's new one, uses certain global counting arguments to provide estimates, but in different directions. It is a current project to combine both approaches to provide a new, more conceptual proof without global counting. Proofs in a similar spirit of further fundamental properties of the Mandelbrot set can be found in [S2].

Acknowledgements. It is a pleasure to thank many people who have contributed to this paper in a variety of ways. I am most grateful to John Hubbard for so much help and friendship over the years. To John Milnor, I am deeply indebted for the ping-pong mentioned above, as well as for many discussions and a lot of encouragement along the way. Many more people have shared their ideas and understanding with me and will recognize their contributions; they include Bodil Branner, Adrien Douady, Karsten Keller, Eike Lau, Jiaqi Luo, Misha Lyubich, Shizuo Nakane, Chris Penrose, Carsten Petersen, Johannes Riedl, Mitsu Shishikura and others. I am also grateful to the Institute for Mathematical Sciences in Stony Brook for its inspiring environment and support. Finally, special thanks go to Katrin Wehrheim for a most helpful suggestion.

2 Complex Dynamics

In this section, we briefly recall some results and notation from complex dynamics which will be needed in the sequel. For details, the notes [M1] by Milnor are recommended and, of course, the work [DH1] by Douady and Hubbard which is the source of most of the results mentioned below.

By affine conjugation, quadratic polynomials can be written uniquely in the normal form $p_c : z \mapsto z^2 + c$ for some complex parameter c. For any such polynomial, the filled-in Julia set is defined as the set of points z with bounded orbits under iteration. The Julia set is the boundary of the filled-in Julia set. It is also the set of points which do not have a neighborhood in which the sequence of iterates is normal (in the sense of Montel). Julia set and filled-in Julia set are connected if and only if the only critical point 0 has bounded orbit; otherwise, these sets coincide and are a Cantor set. The Mandelbrot set $\mathbf{M}$ is the *quadratic connected locus:* the set of parameters c for which the Julia set is connected. Julia sets and filled-in Julia sets, as well as the Mandelbrot set, are compact subsets of the complex plane. The Mandelbrot set is known to be connected and full (i.e. its complement is connected).

Douady and Hubbard have shown that Julia sets and the Mandelbrot set can profitably be studied using external rays: for a compact connected and full set $K \subset \mathbb{C}$, the Riemann mapping theorem supplies a unique conformal isomorphism Φ_K from the exterior of K to the exterior of a unique disk $\overline{D}_R = \{z \in \mathbb{C} : |z| \leq R\}$ subject to the normalization condition $\lim_{z\to\infty} \Phi(z)/z = 1$. The inverse of the Riemann map allows to transport polar coordinates to the exterior of K; images of radial lines and centered circles are called *external rays* and *equipotentials*, respectively. For a point $z \in \mathbb{C} - K$ with $\Phi(z) = re^{2\pi i\vartheta}$, the number ϑ is called the *external angle* and $\log r$ is called the *potential* of z. External angles live in $\mathbb{S}^1$; we will always measure them in full turns, i.e., interpreting $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Sometimes, it will be convenient to count the two angles 0 and 1 differently and have external angles live in $[0, 1]$. Potentials are parametrized by the open interval $(\log R, \infty)$. An external ray at angle ϑ is said to *land* at a point z if $\lim_{r\searrow\log R} \Phi_K^{-1}(re^{2\pi i\vartheta})$ exists and equals z. For general compact connected full sets K, not all external rays need to land. By Carathéodory's theorem, local connectivity of K is equivalent to landing of all the rays, with the landing points depending continuously on the external angle.

For all the sets we consider here, it turns out that the conformal radius R is necessarily equal to 1. In order to avoid confusion, we will replace the term "external ray" by *dynamic ray* or *parameter ray* according to whether it is an external ray of a filled-in Julia set or of the Mandelbrot set.

For $c \in \mathbf{M}$, the filled-in Julia set K_c is connected. For brevity, we will denote the preferred Riemann map by φ_c, rather than Φ_{K_c}. A classical theorem of Böttcher asserts that this map conjugates the dynamics outside of K_c to the squaring map outside the closed unit disk: $\varphi_c \circ p_c = (\varphi_c)^2$. A dynamic ray is periodic or preperiodic whenever its external angle is periodic or preperiodic under the doubling map on $\mathbb{S}^1$. The periodic and preperiodic angles are exactly the rational numbers. More precisely, a rational angle is periodic iff, when written in lowest terms, the denominator is odd; if the denominator is even, then the angle is preperiodic. It is well known [M1, Section 18] that dynamic rays of connected filled-in Julia sets always land whenever their external angles are rational. The landing points of periodic (resp. preperiodic) rays are periodic (resp. preperiodic) points on repelling or parabolic orbits. Conversely, every repelling or parabolic periodic or preperiodic point of a connected Julia set is the landing point of one or more rational dynamic rays; preperiods and periods of all the rays landing at the same point are equal.

If a quadratic Julia set is a Cantor set, then there still is a Böttcher map φ_c near infinity conjugating the dynamics to the squaring map. One can try to extend the domain of definition of the Böttcher map by pulling it back using the conjugation relation. However, there are problems about choosing the right branch of a square root needed in the conjugation relation. The absolute value of the Böttcher map is independent of the choices and allows to define potentials outside of the filled-in Julia set. The set of points at potentials exceeding the potential of the critical point is simply connected and the map φ_c can be defined there uniquely. This domain includes the critical value. In particular, the external angle of the critical value is defined uniquely. Douady and Hubbard have shown that the preferred Riemann map $\Phi_{\mathbf{M}}$ of the exterior of the Mandelbrot set is given by $\Phi_{\mathbf{M}}(c) = \varphi_c(c)$.

For disconnected Julia sets, the map φ_c defines dynamic rays at sufficiently large potentials. If a dynamic ray at angle ϑ is defined for potentials greater than $t > 0$, then one can pull back by the dynamics and obtain the dynamic rays at angles $\vartheta/2$ and $(\vartheta+1)/2$ down to potential $t/2$, except if the ray at angle ϑ contains the critical value. In the latter case, the two pull-back rays will bounce into the critical point and the pull-back is no longer possible uniquely. This phenomenon has been studied by Goldberg and Milnor in the appendix of [GM]. Conversely, a dynamic ray at angle ϑ can be extended down to the potential $t > 0$ provided its image ray at angle 2ϑ can be extended down to the potential $2t$ and does not contain the critical value, or if the ray at angle 4ϑ can be extended down to the potential $4t$ without containing the critical value or its image, etc.. The ray can be defined for all potentials in $(0, \infty)$ if the external angle of the critical value is different from $2^k\vartheta$ for all $k = 1, 2, 3, \ldots$. This is the general situation, and in this case, the dynamic ray is known to land at a unique point of the Julia set, whether or not the angle ϑ is rational.

We rephrase these facts in a form which we will have many opportunities to use: if a parameter $c \notin \mathbf{M}$ has external angle ϑ, then the dynamic ray at angle ϑ for the parameter c will contain the critical value. If the angle ϑ is periodic, then this ray cannot possibly land: the ray must bounce into an inverse image of the critical point at a finite positive potential. The main focus of Sections 3 and 4 will be to transfer the landing properties of dynamic rays at rational angles into landing properties of parameter rays at rational angles: as so often in complex dynamics, the general strategy is "to plow in the dynamical plane and then to harvest in parameter space", as Douady phrased it.

When a periodic ray lands at a periodic point, the periods need not be equal: it is possible that the period of the ray is a proper multiple of the period of the point it is landing at. We will therefore distinguish *ray periods* and *orbit periods*. If only one ray lands at every periodic point on the orbit, then both periods are equal; in general, there is a relation between these periods and the number of rays landing at each point on the orbit; see Lemma 2.4. For our purposes, periodic orbits will be most interesting if at least two rays land at each of its points. Such periodic orbits have a distinguished point and two distinguished dynamic rays landing at this point; these play a prominent role in all the symbolic descriptions of the Mandelbrot set. Following the terminology of Milnor [M2], we will call the distinguished point and rays the *characteristic periodic point of the orbit* and the *characteristic rays* (see below), and the corresponding external angles will be the two *characteristic angles* of the orbit. In Thurston's fundamental preprint [T], the two characteristic rays and their common landing point are the "minor leaf" of a "lamination". We will not use or describe his notation here, but we note that it is very close in spirit to this article.

For our purposes, it will be sufficient to define characteristic points and rays only for parabolic periodic orbits.

Definition 2.1 (Characteristic Components, Points and Rays)
For a quadratic polynomial with a parabolic orbit, the unique Fatou component containing the critical value will be called the characteristic Fatou component*; the only parabolic periodic point on its boundary will be the* characteristic periodic point *of the parabolic orbit. It is the landing point of at least two dynamic rays, and the two of them closest to the critical value on either side will be the* characteristic rays.

The fact that every parabolic periodic point is the landing point of at least two dynamic rays will be shown after Lemma 3.6. Lemma 2.4 will describe the characteristic rays dynamically.

With hesitation, we use the term "Misiurewicz point" for a parameter c for which the critical point or, equivalently, the critical value, is (strictly) preperiodic. This terminology has been introduced long ago, but it is only a very special case of what Misiurewicz was investigating. In real dynamics, the term is used in a wider meaning. We have not been successful in finding an adequate substitution term and invite the reader for suggestions.

In this section, we provide two lemmas which are the engine of our proof: the first one is of analytical nature; it is a slight generalization of Lemma B.1 in Goldberg and Milnor [GM], guaranteeing stability in the Julia set at repelling (pre)periodic points. The second lemma will make counting possible by estimating the number of parabolic parameters with given ray periods.

Lemma 2.2 (Stability of Repelling Orbits)
Suppose that, for some parameter $c_0 \in \mathbb{C}$ (not necessarily in the Mandelbrot set), there is a repelling periodic point z_0 at which some periodic dynamic ray at angle ϑ lands. Then, for parameters c sufficiently close to c_0, the periodic point z_0 can be continued analytically as a function $z(c)$ and the dynamic ray at angle ϑ in the dynamic plane of c lands at $z(c)$.

Moreover, the dynamic ray and its landing point form a closed set which is canonically homeomorphic to $[0,\infty]$ *via potentials, and this parametrized ray depends continuously on the parameter.*

If z_0 *is repelling and preperiodic, the analogous statement holds provided that neither the point* z_0 *nor any point on its forward orbit is the critical point.*

PROOF. We first assume z_0 to be a periodic point. By the implicit function theorem, z_0 can be continued analytically as a function $z(c)$ in a neighborhood of c_0; the multiplier $\lambda(c)$ will also depend analytically on c so that the cycle is repelling sufficiently close to c_0. Let V be such a neighborhood of c_0 and denote the period of z_0 by n. Then for every $c \in V$ there exists a local branch g_c of the inverse map of $p_c^{\circ n}$ fixing $z(c)$. There is a neighborhood U of z_0 such that g_{c_0} maps $\overline{U}$ into U, and possibly by shrinking V, we may assume that all g_c have the same property for $c \in V$. Under iteration of g_c, any point in U then converges to $z(c)$. Let $t > 0$ be a potential such that, for the parameter c_0, the set U contains all the points of the dynamic ϑ-ray at potentials t and below, including the landing point.

Now we distinguish two cases, according to whether or not $c_0 \in \mathbf{M}$. If $c_0 \notin \mathbf{M}$, then the external angle of the parameter c_0 is well-defined and different from the finitely many angles $2^k\vartheta$ for $k = 1, 2, 3, \ldots$ because the dynamic ray at angle ϑ lands. If V is small enough so that all points in V are outside $\mathbf{M}$ and have their external angles different from all the $2^k\vartheta$, then for every $c \in V$, the dynamic ray at angle ϑ lands, and the point at potential t depends analytically on the parameter. It will therefore be contained in U for sufficiently small perturbations and thus converge to $z(c)$ under iteration of g_c, so the landing point of the ray is $z(c)$.

However, if $c_0 \in \mathbf{M}$, then we may assume V small enough so that all its points have potentials less than $t/2$ (with respect to the potential function of the Mandelbrot set). In the corresponding dynamic planes, the critical values then have potentials less than $t/2$, so every dynamic ray exists and depends analytically on the parameter for potentials greater than $t/2$. By shrinking V, we may then assume that for all $c \in V$, the segment between potentials $t/2$ and t in the dynamic ray at angle ϑ is contained in U. Iterating the map g_c, it follows that the dynamic ray at angle ϑ lands at $z(c)$. In both cases, rays and landing points depend continuously on the parameter, including the parametrization by potentials.

The statement about preperiodic points follows by taking inverse images and is straightforward, except if z_0 or any point on its forward orbit are the critical point. However, if some preperiodic dynamic ray lands at the critical value, then a small perturbation may bring the critical value onto this dynamic ray, and the inverse rays will bounce into the critical point (after that, both branches will land, and the landing points are two branches of an analytic function). $\square$

Lemma 2.3 (Counting Parabolic Orbits)
For every positive integer n*, the number of parabolic parameters in* $\mathbb{C}$ *having a parabolic orbit of exact ray period* n *is at most half the number of periodic angles in* $[0,1]$ *having exact period* n *under doubling modulo* 1.

PROOF. We can calculate the exact number of periodic angles. If an angle $\vartheta \in [0,1]$ satisfies $2^n\vartheta \equiv \vartheta$ modulo 1, then we can write $\vartheta = a/(2^n - 1)$ for some integer a, and there are 2^n

such angles in $[0,1]$. Only a subset of these angles has exact period n: denoting the number of such angles by s'_n, we have $\sum_{k|n} s'_k = 2^n$, which allows to determine the s'_n recursively or via the Möbius inversion formula. We have $s'_1 = 2$, and all the s'_n are easily seen to be even. In the sequel, we will work with the integers $s_n := s'_n/2$. The first few terms of the sequence (s_n), starting with s_1, are $1, 1, 3, 6, 15, 27, 63, \ldots$. The specified number of periodic angles in $[0,1]$ is then exactly $2s_n$.

We consider the curve $\{(c,z) \in \mathbb{C}^2 : p_c^{\circ n}(z) = z\}$ consisting of points z which are periodic under p_c with period dividing n. It factors as a product $\prod_{k|n} Q_k(c,z)$ according to exact periods. (The curves Q_k have been shown to be irreducible by Bousch [Bo] and by Lau and Schleicher [LS], a fact we will not use.) For $|c| > 2$, the filled-in Julia set of p_c is a Cantor set containing all the periodic points. For $|c| > 4$, it is easy to verify that points z with $|z| > |c|^{1/2} + 1$ escape to ∞, and so do points with $|z| < |c|^{1/2} - 1$. Periodic points therefore satisfy $|z| = |c|^{1/2}(1 + o(1))$ as $c \to \infty$. The multiplier of a periodic orbit of exact period n is the product of the periodic points on the orbit multiplied by 2^n, so it grows like $|4c|^{n/2}(1 + o(1))$.

For any parameter c, the number of points which are fixed under the n-th iterate is obviously equal to 2^n, counting multiplicities. These points have exact periods dividing n, so the number of periodic points of exact period n equals $2s_n$ by the same recursion formula as above. These periodic points are grouped in orbits, so the number of orbits is $2s_n/n$ (which implies that $2s_n$ is divisible by n). For bounded parameters c, the periodic points and thus the multipliers are bounded; since there are $2s_n/n$ orbits, the multipliers of which are analytic and behave like $|c|^{n/2}$ near infinity, and since every orbit contains n points, it follows that sufficiently large multipliers are assumed exactly $(2s_n/n)(n/2)n = ns_n$ times on Q_n (we do not have to count multiplicities here because multiple orbits always have multiplier $+1$). Consider the multiplier map on Q_n which assigns to every point (c,z) the multiplier $(\partial/\partial z)p_c^{\circ n}(z)$. It is a proper map and thus has a mapping degree, so (counting multiplicities) every multiplier in $\mathbb{C}$ is assumed equally often, including the value $+1$. The number of points (c,z) having multiplier $+1$ therefore equals ns_n, counting multiplicities. Projecting onto the c-coordinate and ignoring multiplicities, a factor n is lost because points on the same orbit project onto the same parameter, and we obtain an upper bound of s_n for the number of parameters. (In fact, it is not too hard to show at this point that s_n provides an exact count [M2]. We will show this in Corollary 3.4 by a global counting argument.)

Consider a parabolic orbit of exact period k and multiplier $\mu = e^{2\pi i p/q}$ with $(p,q) = 1$. Then the exact ray period is $qk =: n$, and qk is also the smallest period such that, when interpreting the orbit as an orbit of this period, the multiplier becomes $+1$. Therefore, the periodic points on this orbit are on Q_n, and the number of parabolic parameters having exact ray period n therefore is at most s_n. $\square$

A more detailed account of such counting arguments can be found in Section 5 of Milnor [M2].

The following standard lemma is folklore and at the base of every description of quadratic iteration theory. Our proof follows Milnor [M2]; compare also Thurston [T, Theorem II.5.3 case b) i) a)]. We do not assume the Julia set to have any particular property; it need not even be connected.

Lemma 2.4 (Permutation of Rays)
If more than two periodic rays land at a periodic point, or if the orbit period is different from the ray period, then the first return map of the point permutes the rays transitively.

PROOF. Denote the orbit period by k and the ray period by n. Since a periodic orbit has periodic rays landing only if the orbit is repelling or parabolic, the first return map of any of its periodic points is a local homeomorphism and permutes the rays landing there in such a way that their periods are all equal, and the number of rays landing at each point of the orbit is a constant s, say. If $s = 1$, then orbit period and ray period are equal. If $s = 2$, then either ray and orbit periods are equal, or the first return map of any point has no choice but to transitively permute the two rays landing at this point. We may hence assume $s \geq 3$. Then the s rays landing at any one of these periodic points separate the dynamic plane into s sectors. Every sector is bounded by two dynamic rays, so it has associated a width: the external angles of the two rays cut $\mathbb{S}^1$ into two open intervals, exactly one of which does not contain external angles of rays landing at the same point. The *width* of the sector will be the length of this interval (normalized so that the total length of $\mathbb{S}^1$ is 1).

Since the dynamics of the first return map is a local homeomorphism near the periodic point, every sector is periodic, and so is the sequence of the corresponding widths. More precisely, we will justify the following observations below: if a sector does not contain the critical point, then it maps homeomorphically onto its image sector (based at the image of its landing point), and the width of the sector doubles. However, if the sector does contain the critical point, then the sector maps in a two-to-one fashion onto the image sector, and it covers the remaining dynamic plane once. In this case, the width of the sector will decrease under this mapping, and the image sector contains the critical value. To justify these statements, first note that the rays bounding any sector are mapped to the rays bounding the image sector. Looking at external angles within the sector, it follows that either the sector maps forward homeomorphically, or it covers the entire complex plane once and the image sector twice. The latter must happen for the sector containing the critical point. Since all the sectors at any periodic point combined exactly cover the complex plane twice when mapped forward, all the other sectors must map homeomorphically onto the image sectors. We also see that among all the sectors based at any point, the sector containing the critical point must have width greater than $1/2$, and all the other sectors then have widths less than $1/2$ (the critical point cannot be on a sector boundary: if it is on a periodic dynamic ray, then this ray cannot land, and if it is on a periodic point, then this point is superattracting). The width of any sector doubles under the map if it does not contain the critical point; since the sum of the widths of all the sectors based at any point is 1, the width of the critical sector must decrease.

For each orbit of sectors, there must be at least one sector with minimal width. It must contain the critical value (or it would be the image of a sector with half the width), and it cannot contain the critical point (or its image sector would have smaller width). Therefore, all the shortest sectors of the various cycles of sectors must be bounded by pairs of rays separating the critical point from the critical value, and these sectors are all nested. Among them, there is one innermost sector S_1 based at some point z_1 of the periodic orbit. This sector S_1 cannot contain another point from the orbit of z_1: if there was such a point z', there would have to be a sector based at z' which was shorter than all the shortest sectors at points on the orbit of z_1, and this is obviously absurd.

If there is an orbit of sectors not involving S_1, then any shortest sector on this orbit must contain the critical value and thus S_1, but it cannot contain the critical point. This sector must then contain all sectors at z_1 except the one containing the critical point. The representative of this orbit of sectors at z_1 must then be the unique sector containing the critical point. Any cycle of sectors has then only two choices for its representative at z_1: the sector containing the critical point or the critical value. If there is more than one cycle, then it follows that there are just two cycles, and each of them has exactly one representative at each point of the periodic orbit, so $s = 2$ in contradiction to our assumption. □

Remark. This lemma is at the heart of the general definition of characteristic rays: the main part of the proof works when at least two rays land at each of the periodic points, and it shows that there is a unique sector of minimal width containing the critical value. The rays bounding this sector are called the characteristic rays. For the special case of a parabolic orbit, this definition agrees with the one we have given above.

3 Periodic Rays

In this section, we will be concerned with parameter rays at periodic angles. The proof of the following weak form of the theorem is due to Goldberg, Milnor, Douady, and Hubbard; see [GM, Theorem C.7].

Proposition 3.1 (Periodic Parameter Rays Land)
Every parameter ray at a periodic angle ϑ lands at a parabolic parameter c_0. In the dynamic plane of c_0, the dynamic ray at angle ϑ lands at the parabolic orbit.

Proof. Let c_0 be a parameter in the limit set of the parameter ray at angle ϑ and let n be the exact period of ϑ. In the dynamic plane of c_0, the dynamic ray at angle ϑ must land at a repelling or parabolic periodic point z of ray period n; see [M1, Theorem 18.1]. If z was repelling, Lemma 2.2 would imply that for parameters c sufficiently close to c_0, the dynamic ray at angle ϑ in the dynamic plane of c would land at a repelling periodic point $z(c)$, so it could not bounce off any precritical point. However, when c is on the parameter ray at angle ϑ, then the dynamic ray at angle ϑ must bounce off some precritical point, even infinitely often.

Therefore, c_0 is parabolic, and within its dynamics, the dynamic ray at angle ϑ lands at the parabolic orbit. Since limit sets are connected but parabolic parameters of given ray period form a finite set by Lemma 2.3, the parameter ray at angle ϑ lands and the statements follow. □

This proves half of the first assertion in Theorem 1.1. The remainder of the first and the second assertion will be shown in several steps. We want to show that at a parabolic parameter c_0, those two parameter rays land which have the same external angles as the two characteristic rays of the critical value Fatou component, and no other rational ray lands there. The first statement is usually shown using Ecalle cylinders. It turns out that it is much easier to show that some ray does *not* land at a given point, rather than to show where it does land. The idea in this paper will be to exclude all the wrong rays from landing at given parabolic parameters, using partitions in the dynamic and parameter planes. Using that the rays must land somewhere, a global counting argument will then prove the theorem.

Let c be a parabolic parameter and let Θ_c be the set of periodic angles ϑ such that the parameter ray at angle ϑ lands at c. A priori, it might be empty. We will prove the following two results later in this section.

Proposition 3.2 (Necessary Condition)
If an angle ϑ is in Θ_c, then the dynamic ray at angle ϑ lands at the characteristic point of the parabolic orbit in the dynamic plane of c.

Proposition 3.3 (At Most Two Rays)
If Θ_c contains more than one angle, then it consists of exactly those two angles which are the characteristic angles of the parabolic orbit in the dynamic plane of c.

These two propositions allow to prove the half of the theorem dealing with periodic rays; we will deal with the preperiodic half in the next section.

Proof of Theorem 1.1 (periodic case). By Lemma 2.3, the number of parabolic parameters of any given ray period is at most half the number of parameter rays at periodic angles of the same period. Since every ray lands at such a parabolic parameter by Proposition 3.1, and at most two rays may land at any such point by Proposition 3.3, it follows that exactly two rays land at every parabolic point, and Proposition 3.3 says which ones these are. It also follows that the number of parabolic parameters of any given period is largest possible as allowed by Lemma 2.3. □

Remark. Since we will complete the proof of Proposition 3.3 by induction on the period, using Theorem 1.1 for lower periods, it is important to note that in order to prove the Theorem for any period, it suffices to know Proposition 3.3 for the same period.

Corollary 3.4 (Counting Parabolic Orbits Exactly)
Let s_k be the number of parameters having a parabolic orbit of exact ray period k. These numbers satisfy the recursive relation $\sum_{k|n} s_k = 2^{n-1}$, which determines them uniquely. □

It remains to prove the two propositions. In both of them, we have to exclude that certain rays land at given parabolic parameters. We do that using appropriate partitions: first in the dynamic plane, then in parameter space. We start by discussing the topology of parabolic quadratic Julia sets and define a variant of the Hubbard tree on them. Hubbard trees have been introduced by Douady and Hubbard in [DH1] for postcritically finite polynomials. We will be interested in combinatorial statements about combinatorially described Julia sets, so these results could be derived in purely combinatorial terms. However, it will be more convenient to use topological properties of the Julia sets in the parabolic case, in particular that they are pathwise connected (which follows from local connectivity). This was originally proved by Douady and Hubbard [DH1]; proofs can also be found in Carleson and Gamelin [CG] and in Tan and Yin [TY].

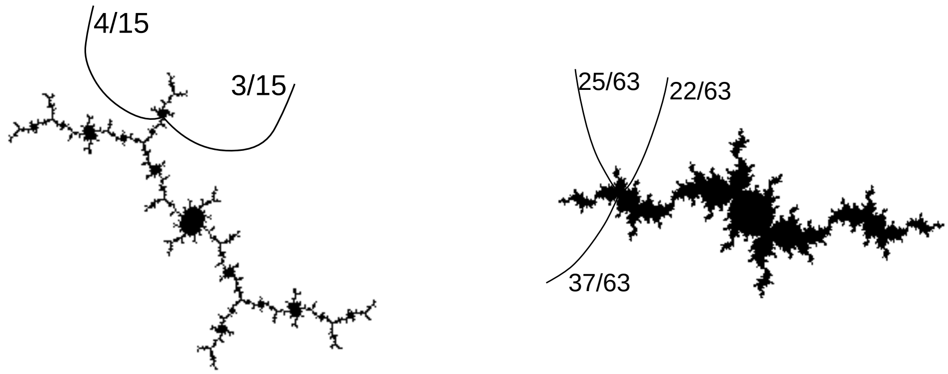

Figure 2: *Illustration of the theorem in the periodic case. The polynomials at the landing points of the parameter rays at angles* $3/15$ *and* $4/15$ *(left) and at angles* $22/63$ *and* $25/63$ *(right) are shown. In both pictures, the rays landing at the characteristic points are drawn. For the corresponding parameter rays, see Figure 1.*

In a quadratic polynomial with a parabolic orbit, let z be any point within the filled-in Julia set and let U be a bounded Fatou component. We then define a *projection* of z onto $\overline{U}$ as follows: If $z \in \overline{U}$, then the projection of z onto $\overline{U}$ is z itself. Otherwise, consider any path within the filled-in Julia set connecting z to an interior point of U (such a path exists because the filled-in Julia set is pathwise connected); then the projection of z onto $\overline{U}$ is the first point where this path intersects ∂U. There may be many such paths, but the projection is still well-defined: take any two paths from z to the interior of U and connect their endpoints within U. If the paths are different, they will bound some subset of $\mathbb{C}$, which must be in K because K is full. If the paths reach ∂U in different points, then these paths enclose part of the boundary of U, but the boundary of any Fatou component is always in the boundary of the filled-in Julia set. This contradiction shows that the projection is well-defined. Every parabolic periodic point is on the boundary of at least one periodic Fatou component, so the projection in this case is just the identity.

Lemma 3.5 (Projection Onto Periodic Fatou Components)
In a quadratic polynomial with a parabolic orbit, the projections of all the parabolic periodic points onto the Fatou component containing the critical value take images in the same point, which is the characteristic point of the parabolic periodic orbit. Projections of the parabolic periodic points onto any other bounded Fatou component take images in at most two boundary points, which are periodic or preperiodic points on the parabolic orbit.

Proof. Let n be the period of the periodic Fatou components and number them $U_0, U_1, \ldots, U_{n-1}, U_n = U_0$ in the order of the dynamics, so that $U_0 = U_n$ contains the critical point. Let a_k be the number of different images that the projections of all the parabolic periodic points onto $\overline{U}_k$ have, for $k = 0, 1, \ldots, n$ (with $a_0 = a_n$). We first show that $a_{k+1} \geq a_k$ for $k = 1, 2, \ldots, n-1$.

Let z be a parabolic periodic point and let $\pi(z)$ be its projection onto $\overline{U}_k$. We claim that $p(\pi(z))$ is the projection onto $\overline{U}_{k+1}$ of either $p(z)$ or the parabolic point on the boundary of the Fatou component containing the critical value (i.e. the characteristic point on the parabolic orbit). Indeed, if the path between z and $\pi(z)$ maps forward homeomorphically under p, then $\pi(p(z)) = p(\pi(z))$. If it does not, then the path must intersect the component containing the critical point, and $\pi(z) = \pi(0)$. But then $p(\pi(z))$ is the projection of the characteristic point on the parabolic orbit. Therefore, for $k \in \{0, 1, 2, \ldots, n-1\}$, all the a_k image points of the projections of parabolic periodic points onto $\overline{U}_k$ will be mapped under p to image points of the projection onto $\overline{U}_{k+1}$. Since for $k \neq 0$, the polynomial p maps $\overline{U}_k$ homeomorphically onto $\overline{U}_{k+1}$, we get $a_n \geq a_{n-1} \geq \ldots a_2 \geq a_1$. Similarly, since p maps $\overline{U}_0$ in a two-to-one fashion onto $\overline{U}_1$, we have $a_1 \geq a_0/2$.

Now we connect the parabolic periodic points by a tree: first, there is a path between the critical point and the critical value, and all the other parabolic periodic points which are not on this path can be connected, one by one, to the subtree which has been constructed thus far. We can require that every path which we are adding intersects the boundary of any bounded Fatou component in the least number of points (at most two). After finitely many steps, all the parabolic periodic points are connected by a finite tree, and all the endpoints of this tree are parabolic periodic points. It is not hard to check that this tree intersects the boundary of any bounded Fatou component exactly in the image points of the projections of the parabolic periodic points.

We now claim that there is a periodic Fatou component whose closure does not disconnect the tree. Indeed, any component which does disconnect the tree has at least one parabolic periodic point and thus at least one periodic Fatou component in each connected component of the complement. Pick one connected component, and within it pick a periodic Fatou component that is "closest" to the removed one (in the sense that the path between these two components does not contain further periodic Fatou components). Remove this component and continue; this process can be continued until we arrive at a component which does not disconnect the tree. Let $U_{\tilde{k}}$ be such a component.

It follows that $a_{\tilde{k}} \leq 2$: all the parabolic periodic points which are not on the boundary of $U_{\tilde{k}}$ must project to the same boundary point, say b, which may or may not be the parabolic periodic point on the boundary of $U_{\tilde{k}}$. We will now show that it will be, so $a_{\tilde{k}} = 1$.

Since b is the image of a projection onto $\overline{U}_{\tilde{k}}$ of some parabolic periodic point which is not in $\overline{U}_{\tilde{k}}$, it follows from the argument above that $p(b)$ is the image of a projection onto $\overline{U}_{\tilde{k}+1}$ of some parabolic periodic point which is not in $\overline{U}_{\tilde{k}+1}$. But since b is the only boundary point of $\overline{U}_{\tilde{k}}$ with this property, it follows that b is fixed under the first return map of this Fatou component. The point b must then be the unique parabolic periodic point on the boundary of $\overline{U}_{\tilde{k}}$, and we have $a_{\tilde{k}} = 1$.

Since $a_1 \leq a_2 \leq a_n \leq 2a_1$, it follows in particular that $a_1 = 1$ and all $a_k \leq 2$, and all projections onto the Fatou component containing the critical value take values in the same point, which is the characteristic point of the parabolic orbit. The remaining claims follow. □

Remark. The tree just constructed is similar to the *Hubbard tree* introduced in [DH1] for postcritically finite polynomials. An important difference is that our tree does not connect the critical orbit. Moreover, Hubbard trees in [DH1] are specified uniquely, while our trees still involve the choice of how to traverse bounded Fatou components. We will suggest a preferred tree below.

However, some properties are independent of the choice of the tree. Assume that two simple curves γ_1 and γ_2 within the filled-in Julia set connect the same two points z_1 and z_2, such that a point w is on one of the curves but not on the other. Then w is on the closure of a bounded Fatou component because the region which is enclosed by the two curves must be in the filled-in Julia set. The tree intersects the boundary of any bounded Fatou component in at most two points which are projection images and thus well-defined. Therefore, the choice for the curves and thus for the tree is only in the interior of bounded Fatou components.

For any point w in the Julia set (not in a bounded Fatou component), it follows that the number of branches of the tree (i.e. the number of components the point disconnects the tree into) is independent of the choice of the tree. Similarly, the number of branches is "almost" non-decreasing under the dynamics, so that $p(w)$ has at least as many branches as w: all the different branches at w will yield different branches at $p(w)$, except if w is on the boundary of the Fatou component containing the critical point. At such boundary points, only the branch leading into the critical Fatou component can get lost. (However, it does happen that $p(w)$ has extra branches in the tree.) It follows that the characteristic point on the parabolic orbit has at most one branch on any tree, and all the other parabolic periodic points can have up to two branches.

A branch point of a tree is a point w which disconnects the tree into at least three complementary components.

Lemma 3.6 (Branch Points of Tree)
Branch points of the tree between parabolic periodic points are periodic or preperiodic points on repelling orbits.

PROOF. Branch points are never on the parabolic orbit, as we have just seen. Therefore, the image of a branch point is always a branch point with at least as many branches. Since there are only finitely many branch points, every branch point is periodic or preperiodic and hence on a repelling orbit. □

We can now proceed to select a preferred tree, which we will call a *(parabolic) Hubbard tree*. We only have to specify how it will traverse bounded Fatou components. In fact, since every bounded Fatou component will eventually map homeomorphically onto the critical Fatou component, we only have to specify how the tree has to traverse this component; for the remaining components, we can pull back.

Let U be the critical Fatou component and let w be the parabolic periodic point on its boundary. First we want to connect the critical point in U to w by a simple curve which is forward invariant under the dynamics. We will use *Fatou coordinates* for the attracting petal of the dynamics [M1, Section 7]. In these coordinates, the dynamics is simply addition of $+1$, and our curve will just be a horizontal straight line connecting the critical orbit. This curve can be extended up to the critical point. The other point on the boundary of the critical Fatou component which we have to connect is $-w$, and we use the symmetric curve. With this choice, we have specified a preferred tree which is invariant under the dynamics, except that the image of the tree connects the characteristic periodic point on the parabolic orbit to the critical value. Removing this curve segment from the image tree, we obtain the same tree as before.

It is well known that, if a repelling or parabolic periodic point disconnects the Julia set into several parts, then this point is the landing point of as many dynamic rays as it disconnects the Julia set into. This is not hard to see once it is known that at least one ray lands. It follows that any branch point of the Hubbard tree has dynamic rays landing between any two branches; any periodic point on the interior of the tree is the landing point of at least two dynamic rays separating the tree. It now follows that the characteristic point on the parabolic orbit, and thus every parabolic periodic point, is the landing point of at least two dynamic rays. The two characteristic rays of the parabolic orbit are the two rays landing at the characteristic point of the orbit and closest possible to the critical value on either side. A different description of the characteristic rays has been given in Lemma 2.4.

Lemma 3.7 (Orbit Separation Lemma)
Any two different parabolic periodic points of a quadratic polynomial can be separated by two (pre)periodic dynamic rays landing at a common repelling (pre)periodic point.

PROOF. It suffices to prove the lemma when one of the two parabolic periodic points is the characteristic point of the orbit; this is also the only case we will need in this section. Let z be this characteristic point and let z' be a different parabolic periodic point. Consider the tree of the polynomial as constructed above. It contains a unique path connecting z and z'. We may assume that this path does not traverse a periodic Fatou component except at its ends; if it does, we replace z' by the parabolic periodic point on that Fatou component. Similarly, we may assume that the path does not traverse another parabolic periodic point. If the path from z to z' contains a branch point of the tree, then by Lemma 3.6, this branch point is periodic or preperiodic and repelling, and it is therefore the landing point of rational dynamic rays separating the parabolic orbit as claimed.

If the Hubbard tree does not have a branch point between z and z', then it takes a finite number k of iterations to map z' for the first time onto z. Denoting the path from z to z' by γ, then the k-th iterate of γ must traverse itself and possibly more in an orientation reversing way: denoting the k-th image of z by z'', then the image curve connects z and z''; it must start along the end of γ because z is an endpoint of the Hubbard tree, and it cannot branch off because we had assumed no branch point of the tree to be on γ. There must be a unique point z_f in the interior of γ which is fixed under the k-th image of γ. This point is a repelling periodic point, and it is the landing point of two dynamic rays with the desired separation properties. □

Now we can prove Proposition 3.2.

PROOF OF PROPOSITION 3.2. In Proposition 3.1, we have shown that Θ_c can contain only angles ϑ of dynamic rays landing at the parabolic cycle in the dynamic plane of c. By the Orbit Separation Lemma 3.7, all the rays not landing at the characteristic point of the parabolic orbit are separated from the critical value by a partition formed by two dynamic rays landing at a common repelling (pre)periodic point. This partition is stable in a neighborhood in parameter space by Lemma 2.2. But the parameter c being a limit point of the parameter ray at angle ϑ means that, for parameters arbitrarily close to c, the critical value is on the dynamic ray at angle ϑ. □

The set Θ_c of external angles of the parabolic parameter c can thus contain only such periodic angles which are external angles of the characteristic periodic point of the parabolic orbit in the dynamical plane of c. If there are more than two such angles, we want to exclude all those which are not characteristic. This is evidently impossible by a partition argument in the dynamic plane. In order to prove Proposition 3.3, we will use a partition of parameter space; for that, we have to look more closely at parameter space and incorporate some symbolic dynamics using kneading sequences. The partition of parameter space according to kneading sequences and, more geometrically, into internal addresses is of interest in its own right (see below) and has been investigated by Lau and Schleicher [LS]; related ideas can be found in Thurston [T], Penrose [Pr1] and [Pr2] and in a series of papers by Bandt and Keller (see [Ke1], [Ke2] and the references therein). This partition will also be helpful in the next section, establishing landing properties of preperiodic parameter rays.

Definition 3.8 (Kneading Sequence)
To an angle $\vartheta \in \mathbb{S}^1$, we associate its kneading sequence *as follows: divide $\mathbb{S}^1$ into two parts at $\vartheta/2$ and $(\vartheta+1)/2$ (the two inverse images of ϑ under angle doubling); the open part containing the angle 0 is labeled $\mathtt{0}$, the other open part is labeled $\mathtt{1}$ and the boundary gets the label $\star$. The kneading sequence of the angle ϑ is the sequence of labels corresponding to the angles ϑ, 2ϑ, 4ϑ, 8ϑ, ...*

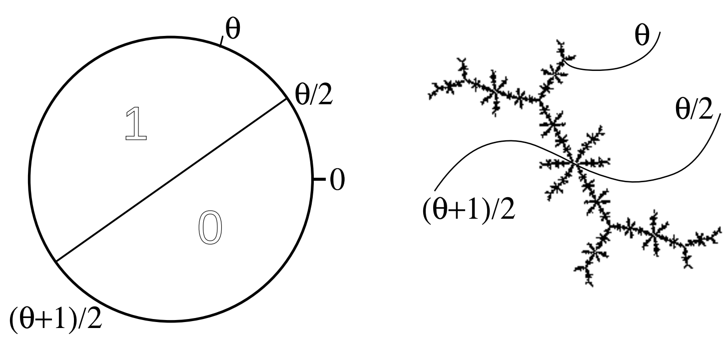

Figure 3: Left: the partition used in the definition of the kneading sequence. Right: a corresponding partition of the dynamic plane by dynamic rays, shown here for the example of a Misiurewicz polynomial.

It is easy to check that, for $\vartheta \neq 0$, the first position always equals $\mathtt{1}$. If ϑ is periodic of period n, then its kneading sequence obviously has the same property and the symbol $\star$ appears exactly once within this period (at the last position). The symbol $\star$ occurs only for periodic angles. However, it may happen that an irrational angle has a periodic kneading sequence (see e.g. [LS]). As the angle ϑ varies, the entry of the kneading sequence at any position n changes exactly at those values of ϑ for which $2^{n-1}\vartheta$ is on the boundary of the partition, i.e. where the kneading sequence has the entry $\star$. This happens if and only if the angle ϑ is periodic, and its exact period is n or divides n.

Another useful property which will be needed in Section 4 is that the pointwise limits $\mathtt{K}_-(\vartheta) := \lim_{\vartheta' \nearrow \vartheta} \mathtt{K}(\vartheta')$ and $\mathtt{K}_+(\vartheta) := \lim_{\vartheta' \searrow \vartheta} \mathtt{K}(\vartheta')$ exist for every ϑ. If ϑ is periodic, then $\mathtt{K}_\pm(\vartheta)$ is also periodic with the same period (but its exact period may be smaller). Both limiting kneading sequences coincide with $\mathtt{K}(\vartheta)$ everywhere, except that all the $\star$-symbols are replaced by 0 in one of the two sequences and by 1 in the other. The reason is simple: if ϑ' is very close to ϑ, then the orbits under doubling, as well as the partitions in the kneading sequences are close to each other, and any symbol 0 or 1 at any finite position will be unchanged provided ϑ' is close enough to ϑ. However, if the period of ϑ is n so that $2^{n-1}\vartheta$ is on the boundary of the partition in the kneading sequence, then $2^{n-1}\vartheta'$ will barely miss the boundary in its own partition, and the $\star$ will turn into a 0 or 1. As long as the orbit of ϑ' is close to the orbit of ϑ, all the symbols $\star$ will be replaced by the same symbol.

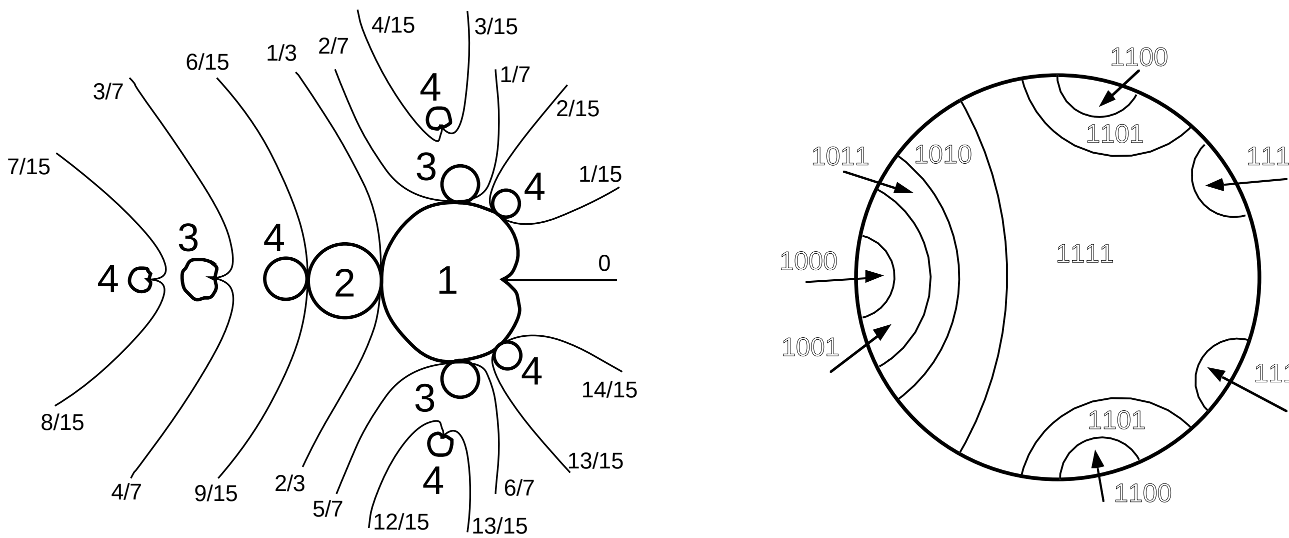

Figure 4: Left: the partition $\mathcal{P}_n$ used in the proof of Proposition 3.3, for $n = 4$. The corresponding hyperbolic components are drawn in for clarity and do not form part of the partition. Right: a corresponding symbolic picture, showing how the partition yields a parameter space of initial segments of kneading sequences. The same pairs of rays are drawn in as on the left hand side, but the angles are unlabeled for lack of space.

Proposition 3.1 asserts in particular that all the periodic parameter rays landing at the same parameter have equal period. All the rays of period at most $n-1$ divide the plane into finitely many pieces. We denote this partition by $\mathcal{P}_{n-1}$; it is illustrated in Figure 4. Parabolic parameters of ray period n and parameter rays of period n have no point in common with the boundary of this partition.

Lemma 3.9 (Kneading Sequences in the Partition)
Fix any period $n \geq 1$ and suppose that all the parameter rays of periods at most $n-1$ land in pairs. Then all parameter rays in any connected component of $\mathcal{P}_{n-1}$ have the property that the first $n-1$ entries in their kneading sequences coincide and do not contain the symbol $\star$. In particular, rays of period n with different kneading sequences do not land at the same parameter.

Proof. The first statement is trivial for two periodic rays which do not have rays of lower periods between them, i.e., for rays from the same “access to infinity” of the connected component in $\mathcal{P}_{n-1}$: the first $n-1$ entries in the kneading sequences are stable for angles within every such access. The claim is interesting only for a connected component with several "accesses to infinity".

The hypothesis of the theorem asserts that parameter rays of periods up to $n-1$ land in pairs. Therefore, whenever two rays at angles ϑ_1, ϑ_2 are in the same connected component of $\mathcal{P}_{n-1}$, the parameter rays of any period $k \le n-1$ on either side (in $\mathbb{S}^1$) between these two angles must land in pairs. The number of such rays is thus even, and the k-th entry in the kneading sequence changes an even number of times between `0` and `1`. $\square$

Remark. This lemma allows to interpret $\mathcal{P}_n$ as a parameter space of initial segments of kneading sequences. In Figure 4, the partition is indicated for $n = 4$, together with the initial four symbols of the kneading sequence. The entire parameter space may thus be described by kneading sequences, as noted above. To any parameter $c \in \mathbb{C}$, we may associate a kneading sequence as follows: it is a one-sided infinite sequence of symbols, and the k-th entry is `1` if and only if the parameter is separated from the origin by an even number of parameter ray pairs of periods k or dividing k; if the number of such ray pairs is odd, then the entry is `0`, and if the parameter is exactly on such a ray pair, then the entry is $\star$. Calculating the kneading sequence of any point is substantially simplified by the observation that, in order to know the entire kneading sequence at a parameter ray pair of some period n, it suffices to know the first $n-1$ entries in the kneading sequence, so we only have to look at ray pairs of periods up to $n-1$. This leads to the following algorithm: for any point $c \in \mathbb{C}$, find consecutively the parameter ray pairs of lowest periods between the previously used ray pair and the point c. The periods of these ray pairs will form a strictly increasing sequence of integers and allow to reconstruct the kneading sequence, encoding it very efficiently. If we extend this sequence by a single entry 1 in the beginning, we obtain the *internal address* of c. For details, see [LS]. In the context of real quadratic polynomials, this internal address is known as the sequence of *cutting times* in the Hofbauer tower.

The figure shows that certain initial segments of kneading sequences appear several times. This can be described and explained precisely and gives rise to certain symmetries of the Mandelbrot set; see [LS].

Lemma 3.10 (Different Kneading Sequences)
Let c be a parabolic parameter and let z_1 be the characteristic periodic point on the parabolic orbit. Among the dynamic rays landing at z_1, only the two characteristic rays can have angles with identical kneading sequences.

Proof. Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_s$ be the angles of the dynamic rays landing at z_1. If their number s is 2, then both angles are characteristic, and there is nothing to show. We may hence assume $s \ge 3$. All the rays ϑ_i are periodic of period n, say. By Lemma 2.4, the orbit period of the parabolic orbit is exactly $n/s =: k$. Let z_0 and z_0' be the two (different) immediate inverse images of z_1 such that z_0 is periodic. If any one of the rays $R(\vartheta_i)$ is chosen, its two inverse images, together with any simple path in the critical Fatou component connecting z_0 and z_0', form a partition of the complex plane into two parts. We label these parts again by `0` and `1` so that the dynamic ray at angle 0 is in part `0`, and we label the boundary by $\star$.

Now the labels of the parts containing the rays $R(\vartheta_i)$, $R(2\vartheta_i)$, $R(4\vartheta_i)$, ... again reflect the kneading sequence of ϑ_i because the partitions are bounded at the same angles.

Above, we have constructed a tree connecting the parabolic orbit. By Lemma 3.6, branch points of this tree are on repelling orbits, so z_0 and z'_0 have at most two branches of the tree. One branch always goes into the critical Fatou component to the critical point. For z_0 or z'_0, the other branch goes to the critical value which is always in the region labeled `1`. The second branch at the other point (z'_0 or z_0) must leave in the symmetric direction, so it will always lead into the region labeled `0` (if the second branch at z'_0 were to lead into a direction other than the symmetric one, then the common image point z_1 of z_0 and z'_0 could not be an endpoint of the tree, which it is by Lemma 3.6). It follows that the entire tree, except the part between z_0 and z'_0, is in a subset of the filled-in Julia set whose label does not depend on which of the angles ϑ_i have been used to define the partition and the kneading sequence. For positive integers l, let z_l be the l-th forward image of z_0. If l is not divisible by k, it follows that the label of z_l is independent of ϑ_i; since z_l is the landing point of all the dynamic rays at angles $2^{l-1}\vartheta_i$, it follows that the $l-1$-st entries of the kneading sequences of all the ϑ_i are the same. Therefore, we can restrict our attention to the rays at angles $\vartheta'_1, \vartheta'_2, \ldots, \vartheta'_s$ landing at z_0, where ϑ'_i is an immediate inverse image of ϑ_i. The first return dynamics among the angles is multiplication by 2^k; for the rays, this must be a cyclic permutation with combinatorial rotation number r/s for some integer r (Lemma 2.4); compare Figure 5.

Depending on which of the rays ϑ_i is used for the kneading sequence, i.e. which of the rays ϑ'_i defines the partition, a given ray ϑ'_j may have label `0` or label `1`. In particular, the total number among these rays which are in region `0` may be different. But then the number of symbols `0` within any period of the kneading sequence will be different and the kneading sequences cannot coincide. Two angles among the ϑ_i can thus have the same kneading sequence only if the corresponding partition has equally many rays in the region labeled `0`. This leaves only various pairs of angles at symmetric positions around the critical value as candidates to have identical kneading sequences. But it is not hard to verify, looking at the cyclic permutation of the rays ϑ'_i, that if two such angles define a partition in which at least one of the ϑ'_i is in region `0`, then the two corresponding kneading sequences are different at some position which is a multiple of k. The only two angles with identical kneading sequences are therefore those for which all the ϑ'_i are in region `1` (or on its boundary), so the partition boundary is adjacent to the Fatou component containing the critical point. The angles ϑ_i are hence those for which the dynamic rays land at z_1 adjacent to the critical value, so the corresponding rays are the characteristic rays of the parabolic orbit. They do in fact have identical kneading sequences. □

PROOF OF PROPOSITION 3.3. We will do the proof by induction on the period n. For $n = 1$, there are only two angles 0 and 1 which both describe the same parameter ray, and this ray lands at $c = 1/4$.

To show the statement of the proposition for period n, we may suppose that all parameter rays of periods up to $n-1$ land in pairs. We have a parabolic parameter c of ray period n and the set Θ_c contains the angles of parameter rays landing at c. All these angles have the same period n and, by Lemma 3.9, identical kneading sequences. Since the corresponding dynamic rays all land at the characteristic point of the parabolic periodic orbit by Proposi-

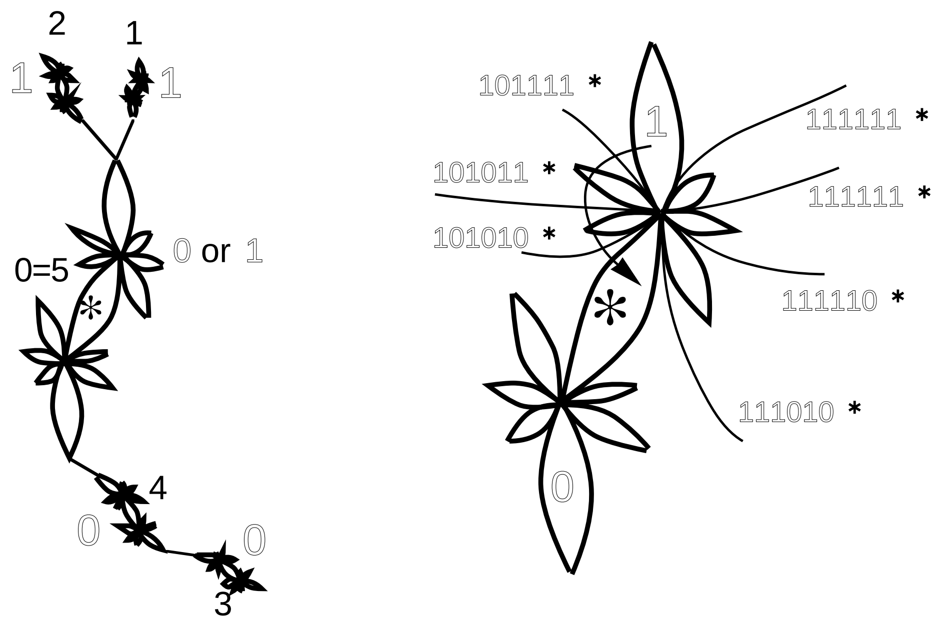

Figure 5: Illustration of the proof of Lemma 3.10. Left: coarse sketch of the entire Julia set; solid numbers describe the parabolic orbit (in this case, of period 5), and outlined numbers specify the corresponding entries in the kneading sequences of external angles of rays landing at the characteristic periodic point. Right: blow-up near the critical point (center, marked by $*$) with the periodic point z_0, considered as a fixed point of the first return map. In this case, seven rays land at z_0 with combinatorial rotation number 3/7. The rays are labeled by the corresponding kneading sequences. The symbols `0` *and* `1` *indicate regions of the Julia set which are always on the same side of the partition, independently of which ray is chosen.*

tion 3.2, Lemma 3.10 says that if Θ_c contains more than a single element, it contains the two characteristic angles. This proves the proposition for period n, which implies the Structure Theorem for period n, and the inductive step is completed. $\square$

We have now finished the proof of the periodic part of the theorem, describing which periodic parameter rays land at common points. This prepares the ground for combinatorial descriptions such as Lavaurs' algorithm [La1] or internal addresses.

4 Preperiodic Rays

In this section, we will turn to parameter rays at preperiodic angles and show at which Misiurewicz points they land. We will use again kneading sequences. Recall that if the angle ϑ is periodic of period n, then its kneading sequence $\mathtt{K}(\vartheta)$ will be periodic of the same period; it will have the symbol $\star$ exactly at positions $n, 2n, 3n, \ldots$. Moreover, the pointwise limits $\mathtt{K}_-(\vartheta) := \lim_{\vartheta' \nearrow \vartheta} \mathtt{K}(\vartheta')$ and $\mathtt{K}_+(\vartheta) := \lim_{\vartheta' \searrow \vartheta} \mathtt{K}(\vartheta')$ both exist; in one of them, all the symbols $\star$ are replaced by `1` and in the other by `0` throughout. Both are still periodic; in fact, their period is n (or a divisor thereof). More precisely, if the parameter rays at periodic angles ϑ_1 and ϑ_2 both land at the same parameter value, then $\mathtt{K}_\pm(\vartheta_1) = \mathtt{K}_\mp(\vartheta_2)$: it suffices to verify this statement for a single period within the kneading sequences, and this follows from Lemma 3.9. We can thus imagine every pair of periodic parameter rays being replaced by two pairs, infinitesimally close on either side to the given pair and having periodic kneading sequences without the symbol $\star$.

Kneading sequences of preperiodic rays are themselves preperiodic; the lengths of the preperiods of external angle and kneading sequence are equal (this is easy to verify; or see the proof of Lemma 4.1). However, the lengths of the periods do not have to be equal: the ray $9/56 = 0.001\,\overline{010}$ has kneading sequence $\texttt{110}\overline{\texttt{111}} = \texttt{110}\overline{\texttt{1}}$. This fact is directly related to the number of parameter rays landing at the same Misiurewicz point; see below.

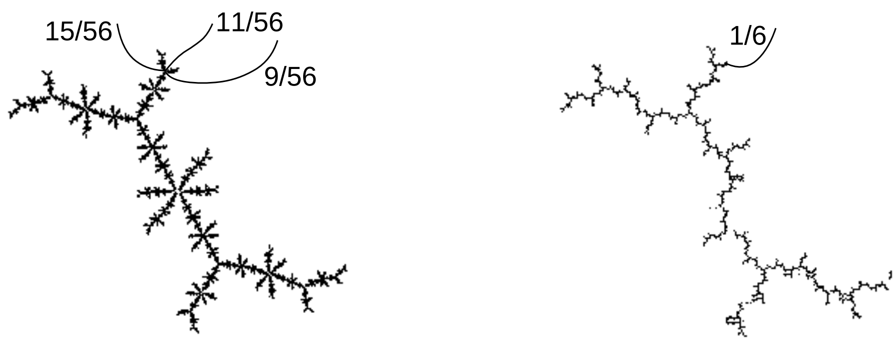

Figure 6: Illustration of the theorem in the preperiodic case. Shown are the Julia sets of the polynomials at the landing points of the parameter rays at angles $9/56$, $11/56$ *and* $15/56$ *(left) and at angle* $1/6$ *(right). In both pictures, the dynamic rays landing at the critical values are drawn.*

Proof of Theorem 1.1 (preperiodic case). Consider any preperiodic parameter ray at angle ϑ and let c be one of its limit points. First suppose that c is a parabolic parameter. We know that there are two parameter rays at periodic angles ϑ_1, ϑ_2 which land at c. We can imagine two parameter ray pairs infinitesimally close to the ray pair $(\vartheta_1, \vartheta_2)$ on both sides, and these two parameter ray pairs have periodic kneading sequences without symbols $\star$. Each of these two periodic kneading sequences must differ at some finite position from the preperiodic kneading sequence of ϑ. But we had seen in the previous section that the regions of constant initial segments of kneading sequences are bounded by pairs of parameter rays at periodic angles (see Lemma 3.9), so there is a pair of periodic parameter rays landing at the same point separating the parameter ray at angle ϑ from the two rays at angles $\vartheta_{1,2}$ and from the parabolic point c. Therefore, c cannot be a limit point of the parameter ray at angle ϑ. This contradiction shows that no limit point of a preperiodic parameter ray is parabolic.

Now we argue similarly as in the proof of Proposition 3.1. For the parameter c, there is no parabolic orbit, so the dynamic ray at angle ϑ lands at a repelling preperiodic point. We want to show that the landing point is the critical value. Since c is a limit point of the parameter ray at angle ϑ, there are arbitrarily close parameters for which the critical value is on the dynamic ray at angle ϑ.

If, for the parameter c, the dynamic ray at angle ϑ does not land at the critical point or at a point on the backwards orbit of the critical point, then the dynamic ray at angle ϑ and its landing point depend continuously on the parameter by Lemma 2.2, so the critical value must be the landing point of the dynamic ray at angle ϑ for the parameter c. If, however, the landing point of the dynamic ϑ-ray is on the backwards orbit of the critical value, then some finite forward image of this ray will depend continuously on the parameter, and pulling back may yield a dynamic ray bouncing once into the critical value or a point on its backward orbit, but after that the two continuations will land at well-defined points. The ray with both continuations and both landing points will still depend continuously on the parameter, so again the dynamic ϑ-ray must land at the critical value for the parameter c. (However, this contradicts the assumption that the landing point is on the backwards orbit of the critical value because that would force the critical value to be periodic.)

We see that, for any limit point c of the parameter ray at angle ϑ, the number c is preperiodic under $z \mapsto z^2 + c$ with fixed period and preperiod, and c is a Misiurewicz point. Since any such point c satisfies a certain polynomial equation, there are only finitely many such points. The limit set of any ray is connected, so the parameter ray at angle ϑ lands, and the landing point is a Misiurewicz point with the required properties. This shows the third part of Theorem 1.1.

For the last part, we have already shown that a Misiurewicz point cannot be the landing point of a periodic parameter ray, or of a preperiodic ray with external angle different from the angles of the dynamic rays landing at the critical value. It remains to show that, given a Misiurewicz point c_0 such that the critical value is the landing point of the dynamic ray at angle ϑ, then the parameter ray at angle ϑ lands at c_0. We will use ideas from Douady and Hubbard [DH1]. By Lemma 2.2, there is a simply connected neighborhood V of c_0 in parameter space such that c_0 can be continued analytically as a repelling preperiodic point, yielding an analytic function $z(c)$ with $z(c_0) = c_0$ such that the dynamic ray at angle ϑ for the parameter $c \in V$ lands at $z(c)$. The relation $z(c) = c$ is certainly not satisfied identically on all of V, so the solutions are discrete and we may assume that c_0 is the only one within V.

Now we consider the winding number of the dynamic ray at angle ϑ around the critical value, which is defined as follows: denoting the point on the dynamic ϑ-ray at potential $t \geq 0$ by z_t and decreasing t from $+\infty$ to 0, the winding number is the total change of $\arg(z_t - c)$ (divided by 2π so as to count in full turns). Provided that the critical value is not on the dynamic ray or at its landing point, the winding number is well-defined and finite and depends continuously on the parameter. When the parameter c moves in a small circle around c_0 and if the winding number is defined all the time, then it must change by an integer corresponding to the multiplicity of c as a root of $z(c) - c$. However, when the parameter returns back to where it started, the winding number must be restored to what it was before. This requires a discontinuity of the winding number, so there are parameters arbitrarily close to c_0 for which the critical value is on the dynamic ray at angle ϑ, and c_0 is a limit point of the parameter ray at angle ϑ. Since this parameter ray lands, it lands at c_0. This finishes the proof of Theorem 1.1. $\square$

Remark. There is no partition in the dynamic plane showing that preperiodic parameter rays can not land at parabolic parameters: there are countably many preperiodic dynamic rays landing at the boundary of the characteristic Fatou component, for example at preperiodic points on the parabolic orbit, and they cannot be separated by a stable partition.

For the final part of the theorem, we used that a repelling preperiodic point $z(c)$ depends analytically on the parameter. As mentioned before, this proof started with a need to describe parameter spaces of antiholomorphic polynomials like the Tricorn and Multicorns, and there we do not have analytic dependence on parameters. Here is another way to prove that every Misiurewicz point is the landing point of all the parameter rays whose angles are the external angles of the critical value in the dynamic plane. We start with any Misiurewicz point c_0 and external angle ϑ of its critical value. Let c_1 be the landing point of the parameter ray at angle ϑ. Then both parameters c_0 and c_1 have the property that in the dynamic plane, the ray at angle ϑ lands at the critical value. It suffices to prove that this property determines the parameter uniquely. This is exactly the content of the Spider Theorem, which is an iterative procedure to find postcritically finite polynomials with assigned external angles of the critical value. In Hubbard and Schleicher [HS], there is an easy proof for polynomials with a single critical point. While the existence part of that proof works only if the critical point is periodic, all we need here is the uniqueness part, and that works in the preperiodic case just as well, both for holomorphic and for antiholomorphic polynomials.

The last part could probably also be done in a more combinatorial but rather tedious way, using counting arguments like in the periodic case. This would, however, be quite delicate, as the number of Misiurewicz points and the number of parameter rays landing at them require more bookkeeping: the number of parameter rays landing at Misiurewicz points varies and can be any positive integer. The following lemma makes this more precise.

Lemma 4.1 (Number of Rays at Misiurewicz Points)
Suppose that a preperiodic angle ϑ has preperiod l and period n. Then the kneading sequence $\mathtt{K}(\vartheta)$ has the same preperiod l, and its period k divides n. If $n/k > 1$, then the total number of parameter rays at preperiodic angles landing at the same point as the ray at angle ϑ is n/k; if $n/k = 1$, then the number of parameter rays is 1 or 2.

In the example above, we had seen that the angle $9/56$ has period 3, while its kneading sequence has period 1. Therefore, the total number of rays landing at the corresponding Misiurewicz point is three: their external angles are $9/56, 11/56$ and $15/56$. If more than one ray lands at a given Misiurewicz point, it is not hard to determine all the angles knowing one of them, using ideas from the proof below.

Proof. In the dynamic plane of ϑ, the dynamic ray at angle ϑ lands at the critical value, so the two inverse image rays at angles $\vartheta/2$ and $(\vartheta+1)/2$ land at the critical point and separate the dynamic plane into two parts; this partition cuts the external angles of dynamic rays in the same way as in the partition defining the kneading sequence, see Definition 3.8 and Figure 3. We label the two parts by $\mathtt{0}$ and $\mathtt{1}$ in the analogous way, assigning the symbol $\star$ to the boundary. The partition boundary intersects the Julia set only at the critical point.

The critical value jumps after exactly l steps onto a periodic orbit of ray period n. Denote the critical orbit by $c_0, c_1, c_2, \dots$ with $c_0 = 0$ and $c_1 = c$, so that $c_{l+1} = c_{l+n+1}$, while $c_l = -c_{l+n}$. The points c_l and c_{l+n} are on different sides of the partition. The periodic part of the kneading sequence starts exactly where the periodic part of the external angles start, so the preperiods are equal.

We know that n is the ray period of the orbit the critical value falls onto. The orbit period is exactly k: periodic rays which have their entire forward orbits on equal sides of the partition land at the same point, for the following reason: we can connect the landing points of two such rays by a curve which avoids all the preperiodic rays landing at the critical point, all the finitely many rays on their forward orbits, and their landing points (if we have to cross some of these ray pairs, they must also visit the same sides of the partition, and we can reduce the problem). Now inverse images of the rays are connected by inverse images of the curve, which avoids the same rays. Continuing to take inverse images in this way, the periodic landing points must converge to each other, so they cannot be different.

The number of dynamic rays on the orbit of ϑ landing at every point of the periodic orbit is therefore n/k, and the critical value jumps onto this orbit as a local homeomorphism, so it is the landing point of equally many preperiodic rays. But these rays reappear in parameter space as the rays landing at the Misiurewicz point.

It remains to show that there are no extra rays at the periodic orbit. By Lemma 2.4, more than two dynamic rays can land at the same periodic point only if these rays are on the same orbit, i.e., the dynamics permutes the rays transitively. The number of dynamic rays can therefore be greater than n/k only if $n/k = 1$, and in that case, there can be at most two rays. $\square$

REMARK. It does indeed happen that $n/k = 1$ while the number of rays is two. An example is given by the two parameter rays at angles $25/56 = 0/011\,\overline{100}$ and $31/56 = 0.100\,\overline{011}$; their common kneading sequence is $\mathtt{100}\,\overline{\mathtt{101}}$, so $n = k = 3$, but these two rays land together at a point on the real axis. On the other hand, for the angle $1/2 = 0.0\overline{1} = 0.1\overline{0}$, the kneading sequence is $\mathtt{1}\overline{\mathtt{0}}$, so $n = k = 1$; the parameter ray at angle $1/2$ is the only ray landing at the leftmost antenna tip $c = -2$ of the Mandelbrot set. These rays are indicated in Figure 1.

For a related discussion of rays landing at common points, from the point of view of "Thurston obstructions", see [HS].

5 Hyperbolic Components

The Orbit Separation Lemma 3.7 has an important consequence: it helps to control the dynamics when a parabolic Julia set is perturbed. Perturbations of parabolics are a subtle issue because both the Julia set and the filled-in Julia set behave drastically discontinuously. We show that nonetheless the landing points of all the dynamic rays at rational angles behave continuously wherever the rays land. In a way, the following proposition is the parabolic analogue to Lemma 2.2, which dealt with repelling periodic points. However, we will explain below that the rays themselves do not depend continuously on the parameter.

Proposition 5.1 (Continuous Dependence of Landing Points)
For any rational angle ϑ, the landing point of the dynamic ray at angle ϑ depends continuously on the parameter on the entire subset of parameter space for which the ray lands.

The dynamic ray at angle ϑ for the polynomial p_c fails to land if and only if it bounces into the critical point or into a point on the inverse orbit of the critical point, which happens if and only if the parameter c is outside the Mandelbrot set on a parameter ray at one of the finitely many angles $\{2\vartheta, 4\vartheta, 8\vartheta, \ldots\}$. All these parameter rays land, and at the landing parameters, the dynamic ray at angle ϑ lands as well. These landing points are the interesting cases of the proposition.

PROOF. First we discuss the case of a periodic angle ϑ. If the landing point is repelling, then the proposition reduces to Lemma 2.2. We may thus assume the landing point to be parabolic. Under perturbation, any parabolic periodic point splits up into several periodic points which may be attracting, repelling, or indifferent, and these periodic points depend continuously on the parameter. We need to show that the landing point of the ray after perturbation is one of the continuations of the parabolic periodic point it was landing at.

Denote the parabolic parameter before perturbation by c_0, let n be its ray period and let $V \subset \mathbb{C}$ be a simply connected open neighborhood which does not contain further parabolics of equal or lower ray periods. Then analytic continuation of periodic points of ray periods up to n is possible in $V - \{c_0\}$. Let z be a repelling periodic point for the parameter c_0; it can then be continued analytically as a function $z(c)$ in a neighborhood of c_0 so that its orbit remains repelling. We may assume this to be the case in all of V, possibly by shrinking V. Then by Lemma 2.2, $z(c)$ will keep all the rays landing at it throughout all of V; since this point cannot lose rays in V, it cannot gain rays, either.

Since for the parameter c_0, the dynamic ray at angle ϑ lands at the parabolic orbit, the landing point of this ray after perturbation will be on a periodic orbit coming out of the parabolic orbit, and it remains to show that the landing point does not jump between continuations of different periodic points of the parabolic orbit. This is where the Orbit Separation Lemma 3.7 comes in: the parabolic periodic points are separated by pairs of rays landing at repelling periodic or preperiodic points, and this separation is stable under perturbations. The dynamic rays cannot cross this partition, so their landing points depend continuously on the angle, provided the rays land at all.

For preperiodic rays, the statement follows by taking inverse images because the pull-back is continuous. If the orbit visits the critical point along its preperiodic orbit, which happens at Misiurewicz points, then several preperiodic points may merge and split up with different rays, but this happens in a continuous way. □

REMARK. Unlike their landing points, the dynamic rays themselves may depend discontinuously on the perturbation. The simplest possible example occurs near the parabolic parameter $c_0 = 1/4$: for this parameter, the dynamic ray at angle $0 = 1$ lands at the parabolic fixed point $z = 1/2$, and the ray is the real line to the right of $1/2$. The critical point 0 is in the interior of the filled-in Julia set. Perturbing the parameter to the right on the real axis, i.e., on the parameter ray at angle $0 = 1$, the dynamic ray will bounce into the critical point and thus fail to land. But for arbitrarily small perturbations near this parameter ray, the dynamic ray at angle $0 = 1$ will get very close to the critical point before it turns back and lands near $1/2$. The closer the parameter is to $c_0 = 1/4$, the lower will the potential of the critical point be, and while the dynamic ray keeps reaching out near the critical point, it does so at lower and lower potentials, and in the limit the part of the ray at real parts less than $1/2$ will be squeezed off. Points at any potential $t > 0$ will depend continuously on the parameter, and so does the landing point at potential $t = 0$; however, this continuity is not uniform in t, and the dynamic ray as a whole can and does change discontinuously with respect to the Hausdorff metric.

Continuous dependence of landing points of rays requires a single critical point (of possibly higher multiplicity). It is false already for cubic polynomials; for an example, see the appendix in Goldberg and Milnor [GM].

Most, if not all, of the interior of the Mandelbrot set consists of what is known as hyperbolic components. Proposition 5.1 is one possible key to understanding many of their properties. First we discuss some necessary background.

A hyperbolic rational map is one where all the critical points are attracted by attracting or superattracting periodic orbits. The dynamical significance is that this is equivalent to the existence of an expanding metric in a neighborhood of the Julia set, which has many important consequences such as local connectivity of the Julia set (see Milnor [M1]). For a polynomial, the critical point at ∞ is always superattracting, and in the quadratic case, the polynomial is hyperbolic if the unique finite critical point either converges to ∞ or to a finite (super)attracting orbit. Hyperbolicity is obviously an open condition. A *hyperbolic component of the Mandelbrot set* is a connected component of the hyperbolic interior. The period of the attracting orbit is constant throughout the component and defines the *period of the hyperbolic component*. We will see below that every boundary point of a hyperbolic component is a boundary point of the Mandelbrot set, so a hyperbolic component is also a connected component of the interior of $\mathbf{M}$. There is no example known of a non-hyperbolic component; it is conjectured that there are none. A *center* of a hyperbolic component is a polynomial for which there is a superattracting orbit; a *root* of such a component of period n is a parabolic boundary point where the parabolic orbit has ray period n. We will show below that every hyperbolic component has a unique center and a unique root. It is easy to verify that the multiplier of the attracting orbit on a hyperbolic component is a proper map from the component to the open unit disk, so it has a finite mapping degree; we will see that this map is in fact a conformal isomorphism. The relation between centers and roots of hyperbolic components is important; the difficulty in establishing it lies in the discontinuity of Julia sets at parabolic parameters. Proposition 5.1 helps to overcome this difficulty.

Lemma 5.2 (Roots of Hyperbolic Components)
Every parabolic parameter with ray period n is a root of at least one hyperbolic component of period n. If the orbit period k is smaller than n, then this parameter is also on the boundary of a hyperbolic component of period k. In no case is such a parabolic parameter on the boundary of a hyperbolic component of different period.

PROOF. First suppose that orbit period and ray period are equal. Then the first return map of any parabolic periodic point z leaves all the dynamic rays landing at z fixed, so its multiplier is $+1$. In local coordinates, the map has the form $\zeta \mapsto \zeta+\zeta^{q+1}+\ldots$ for some integer $q \geq 1$. The point z then has q attracting and repelling petals each, and every attracting petal must absorb a critical orbit. Since there is a unique critical point, we have $q = 1$. Under perturbation, the parabolic orbit then breaks up into exactly two orbits of exact period n, and no further orbit is involved. Denote the parabolic parameter by c_0 and let V be a simply connected neighborhood of c_0 not containing further parabolics of equal ray period. In $V - \{c_0\}$, all periodic points of exact period n can be continued analytically because their multipliers are different from $+1$. Among these periodic points, those which are repelling at c_0 can be continued analytically throughout all of V, while the two colliding orbits might be interchanged by a simple loop in $V - \{c_0\}$ (in fact, they will be: see Corollary 5.7). Their multipliers are therefore defined on a two-sheeted covering of $V - \{c_0\}$ and are analytic, even when the point c_0 is put back in. By the open mapping principle, the parameter c_0 is on the boundary of at least one hyperbolic component of period n. Since, for the parameter c_0, all the orbits of periods not divisible by n are repelling, the parameter can be only on the boundary of hyperbolic components with periods divisible by n. If it was on the boundary of a hyperbolic component with period rn for some integer $r > 1$, then the rn-periodic orbit would have to be indifferent at c_0; since there can be only one indifferent orbit, it would have to merge with the indifferent orbit of period n, and this orbit would get higher multiplicity than 2, a contradiction. This contradiction shows that c_0 is not on the boundary of any hyperbolic component of period other than n.

If the orbit period strictly divides the ray period, so that $s := n/k \geq 2$, then the first return map of the orbit must permute the rays transitively by Lemma 2.4. The least iterate which fixes the rays must also be the least iterate for which the multiplier is $+1$: the landing point of a periodic ray is either repelling or has multiplier $+1$ (this is the Snail Lemma, see [M1]); conversely, whenever the multiplier is $+1$, then all the finitely many rays must be fixed. It follows that the multiplier of the first return map of any of the parabolic periodic points is an exact s-th root of unity. The periodic orbit can then be continued analytically in a neighborhood of the root. Since the multiplier map is analytic, the parabolic parameter is on the boundary of a hyperbolic component of period k. The s-th iterate of the first return map has multiplier $+1$ and hence again the form $\zeta \mapsto \zeta + \zeta^{q+1} + \ldots$ in local coordinates, for an integer $q \geq 1$. The number of coalescing fixed points of this iterate is then exactly $q + 1$.

Since there is only one critical orbit, the first return map of the parabolic orbit must permute the q attracting petals transitively and we have $q = s$. For the first, second, ..., $s-1$-st iterate of the first return map, the multiplier is different from $+1$, so the respective iterate has a single fixed point. The s-th iterate, however, corresponding to the $sk = n$-th iterate of the original polynomial, has a fixed point of multiplicity $q + 1 = s + 1$: exactly one of these points has exact period k; all the other points can have no lower periods than n, so they are on a single orbit of period n of which s points each are coalesced. There is no further orbit involved (or some iterate would have to have a parabolic fixed point of higher multiplicity with more attracting petals attached, as above). Since there is a single indifferent orbit of period n, its multiplier is well defined and analytic in a neighborhood of the parabolic parameter, which is hence on the boundary of a hyperbolic component of period n as well. □

A root of a hyperbolic component is called *primitive* if its parabolic orbit has equal orbit and ray periods, so it is the merger of two orbits of equal period. If orbit and ray periods are different, then the root is called *non-primitive* or a *bifurcation point*: at this parameter, an attracting orbit bifurcates into another attracting orbit of higher period (the terminology *bi*-furcation comes from the dynamics on the real line, where the ratio of the periods is always two). We will see below that every hyperbolic component has a unique root. It therefore makes sense to call a hyperbolic component primitive or non-primitive according to whether or not its root is primitive.

We can now draw a couple of useful conclusions.

Corollary 5.3 (Stability at Roots of Hyperbolic Components)
For any hyperbolic component, the landing pattern of periodic and preperiodic dynamic rays is the same for all polynomials from the component and at any of its roots.

PROOF. Again, it suffices to discuss periodic rays; the statement about preperiodic rays follows simply by taking inverse images because for the considered parameters, all the preperiodic dynamic rays land, and they never land at the critical value. Throughout the component, all the periodic rays land at repelling periodic points, so no orbit can lose a ray under perturbations, and consequently no orbit can gain a ray, either. The same rays at rational angles land at common points throughout the component.

When the dynamics at a primitive root is perturbed into the component, then the parabolic orbit breaks up into one attracting and one repelling orbit of equal period, and by continuous dependence of the landing points (Proposition 5.1), the repelling orbit must inherit all the dynamic rays. It cannot get any further rays because they have been attached to repelling orbits.

For a non-primitive root, denote orbit and ray periods by k and n, respectively. Perturbing the root into the component of period n, the orbit of period n becomes attracting, and the k-periodic orbit must take all the dynamic rays from the parabolic orbit without getting any further rays. □

REMARK. Perturbing a parabolic orbit with orbit period k and ray period $n > k$ into the component of period k changes the landing pattern of rational rays: the parabolic orbit creates an attracting orbit of period k, so a repelling orbit of period n remains, and all the dynamic rays from the parabolic orbit land at different points. Phrased differently, when moving from the component of period k into the one of period n, then n/k periodic rays each start landing at common periodic points; of course, this forces the obvious relations for the preperiodic rays. The landing pattern of all other periodic rays remains stable.

This discussion not only describes the landing patterns of periodic rays within hyperbolic components, but on the entire parameter space except at parameter rays at periodic angles: since the landing points depend continuously on the parameter and periodic orbits are simple except at parabolics, the pattern can change only at parabolic parameters or at parameter rays where the dynamic rays fail to land. The relation between landing patterns of periodic rays and the structure of parameter space has been investigated and described by Milnor in [M2]. The landing pattern of preperiodic rays also changes at Misiurewicz points and at preperiodic parameter rays.

Corollary 5.4 (The Multiplier Map)
The multiplier map on any hyperbolic component is a conformal isomorphism onto the open unit disk, and it extends as a homeomorphism to the closures. In particular, every hyperbolic component has a unique root and a unique center. The boundary of a hyperbolic component is contained in the boundary of the Mandelbrot set.

PROOF. The multiplier map is a proper analytic map from the hyperbolic component to the open unit disk and extends continuously to the boundary. Any point on the boundary has a unique indifferent orbit. If the multiplier at such a boundary point is different from

$+1$, then orbit and multiplier extend analytically in a neighborhood of the boundary point. The multiplier is obviously not constant. This shows in particular that parabolic parameters are dense on the boundary of the component; since parabolics are landing points of parameter rays and thus in the boundary of $\mathbf{M}$, the boundary of every hyperbolic component is contained in the boundary of the Mandelbrot set.

The number of parabolic parameters with fixed orbit period and multiplier $+1$ is finite, so the boundary of any hyperbolic component consists of a finite number of analytic arcs (which might contain critical points) limiting on finitely many parabolic parameters with multipliers $+1$. Since the multiplier map is proper onto $\mathbb{D}$, the component has at least one root.

By Corollary 5.3, the landing pattern of periodic dynamic rays has to be the same at all the roots of a given hyperbolic component. It follows that at all the roots, the angles of the characteristic rays of the parabolic orbits have to coincide: in every case, the characteristic ray pair lands at the Fatou component containing the critical value and separates the critical value from the rest of the parabolic orbit. Among all ray pairs separating the critical point from the critical value, the characteristic ray pair must be the one closest to the critical value. Hence the landing pattern of periodic dynamic rays determines the characteristic angles. Since any root of a hyperbolic component must be the landing point of the parameter rays at the characteristic angles of the parabolic orbit, every hyperbolic component has a unique root.

We can now determine the mapping degree d, say, of the multiplier map. The hyperbolic component is simply connected because the Mandelbrot set is full. Then the multiplier map μ has exactly $d-1$ critical points, counting multiplicities. If $d > 1$, let $v \in \mathbb{D}$ be a critical value of μ and connect v and $+1$ by a simple smooth curve $\gamma \subset \mathbb{D}$ avoiding further critical values. Then $\mu^{-1}(\gamma)$, together with the unique root of the component, contains a simple closed curve Γ enclosing an open subset of the hyperbolic component. This subset must map at least onto all of $\mathbb{D} - \gamma$, so Γ surrounds boundary points of the component and thus of $\mathbf{M}$. But this contradicts the fact that the Mandelbrot set is full, so the multiplier map is a conformal isomorphism onto $\mathbb{D}$ and the component has a unique center. It extends continuously to the closure and is surjective onto $\partial\mathbb{D}$ because it is surjective on the component. Near every non-root of the component, the boundary of the component is an analytic arc (possibly with critical points) and the multiplier is not locally constant on the arc, so it is locally injective. Global injectivity now follows from injectivity on the component. The multiplier map is thus invertible, and continuity of the inverse is a generality. □

Proposition 5.5 (No Shared Roots)
Every parabolic parameter is the root of a single hyperbolic component.

PROOF. Since the period of a component equals the ray period of its root, we can restrict attention to any fixed period n. We have well defined maps from centers to hyperbolic components (which we have just seen is a bijection) and from hyperbolic components to their roots. This gives a surjective map from centers of period n to parabolic parameters of ray period n. Denote the number of centers of period n by s_n.

A center of a hyperbolic component of period n is a point c such that 0 is periodic of exact period n under $z \mapsto z^2 + c$; therefore, c must satisfy a polynomial equation $(\ldots((c^2 + c)^2 +$

$c)\ldots)^2+c=0$ of degree 2^{n-1}. Since this polynomial is also solved by centers of components of periods k dividing n, we get the recursive relation $\sum_{k|n} s_k = 2^{n-1}$. By Lemma 2.3, this is exactly the number of parabolic parameters of ray period k. Since a surjective map between finite sets of equal cardinality is a bijection, every parabolic parameter is the root of a single hyperbolic component. □

Remark. This proposition shows even without resorting to Corollary 5.4 that every hyperbolic component has a unique center, so that the only critical point of the multiplier map (if it had mapping degree greater than one) could be the center. This is indeed what happens for the "Multibrot sets": the connectedness loci for the maps $z \mapsto z^d + c$ with $d \geq 2$.

Before continuing the study of hyperbolic components, we note an algebraic observation following from the proof we have just given.

Corollary 5.6 (Centers of Components as Algebraic Numbers)
Every center of a hyperbolic component of degree n is an algebraic integer of degree at most s_n. It is a simple root of its minimal polynomial. □

A neat algebraic proof for this fact has been given by Gleason; see [DH1]. As far as I know, the algebraic structure of the minimal polynomials of the centers of hyperbolic components is not known: when factored according to exact periods, are they irreducible? What are their Galois groups? Manning (unpublished) has verified irreducibility for $n \leq 10$, and he has determined that the Galois groups for the first few periods are the full symmetric groups. Giarrusso (unpublished) has observed that this induces a Galois action between the Riemann maps of hyperbolic components of equal periods, provided that their centers are algebraically conjugate.

Now we can describe the boundary hyperbolic components much more completely.

Corollary 5.7 (Boundary of Hyperbolic Components)
No non-parabolic parameter can be on the boundary of more than one hyperbolic component. Every parabolic parameter is either a primitive root of a hyperbolic component and on the boundary of no further component, or it is a "point of bifurcation": a non-primitive root of a hyperbolic component and on the boundary of a unique further hyperbolic component. In particular, if two hyperbolic components have a boundary point in common, then this point is the root of exactly one of them. The boundary of a hyperbolic component is a smooth analytic curve, except at the root of a primitive component. At a primitive root, the component has a cusp, and analytic continuation of periodic points along a small loop around this cusp interchanges the two orbits which merge at this cusp.

Proof. If two hyperbolic components have a non-parabolic parameter in their common boundary, then the landing patterns of periodic rays must be the same within both components. This must then also be true at their respective roots, which yields a contradiction: on the one hand, the roots must be different by Proposition 5.5; on the other hand, the parabolic orbits at the roots must have the same characteristic angles (compare the proof of Corollary 5.4), so they must be the landing points of the same parameter rays. It follows that the multiplier map of the indifferent orbit cannot have a critical point at c_0: if it had a critical point there, then c_0 would connect locally two regions of hyperbolic parameters which cannot belong to different hyperbolic components; however, if they belonged to the same component, then the closure of the component would separate part of its boundary from the exterior of the Mandelbrot set, a contradiction. Therefore, the boundary of every hyperbolic component is a smooth analytic curve near every non-parabolic boundary point.

Now let c_0 be a parabolic parameter of ray period n and orbit period k. We know that it is the root of a unique hyperbolic component of period n.

In the non-primitive case (when k strictly divides n), the point c_0 cannot be on the boundary of a hyperbolic component of period different from n and k by Lemma 5.2. It is on the boundary of a single hyperbolic component of period n (Proposition 5.5). In a small punctured neighborhood of c_0 avoiding further parabolics of ray period n, the multiplier map of the n-periodic orbit is analytic and cannot have a critical point at c_0, for the same reason as above, so the component of period n occupies asymptotically (on small scales) a half plane near c_0. The parameter c_0 is also on the boundary of a component of period k the multiplier of which is analytic near c_0. Since hyperbolic components cannot overlap, this component must asymptotically be contained in a half plane, so the multiplier map cannot have a critical point and must then be locally injective near c_0. The boundaries of both components must then be smooth analytic curves near c_0.

In the primitive case $k = n$, the parameter c_0 cannot be on the boundary of a hyperbolic component of period different from n by Lemma 5.2, and it cannot be on the boundary of two hyperbolic components of period n because otherwise it would have to be their simultaneous root, contradicting Proposition 5.5. In a small simply connected neighborhood V of c_0, analytic continuation of the two orbits colliding at c_0 is possible in $V - \{c_0\}$ (compare the proof of Lemma 5.2), so their multipliers can be defined on a two-sheeted cover of V ramified at c_0. If analytic continuation of these two orbits along a simple loop in V around c_0 did not interchange the two orbits, then both multipliers could be defined in V, and both would define different hyperbolic components intersecting V in disjoint regions, yielding the same contradiction as in the non-primitive case above. Therefore, small simple loops around c_0 do interchange the two orbits. In order to avoid the same contradiction again, the multiplier must be locally injective on the two-sheeted covering on V. Projecting down onto V, the component must asymptotically occupy a full set of directions, so the component has a cusp.
□

Remark. The basic motor for many of these proofs about hyperbolic components was uniqueness of parabolic parameters with given combinatorics, via landing properties of parameter rays at periodic angles. A consequence was uniqueness of centers of hyperbolic components with given combinatorics. One can also turn this discussion around and start with centers of hyperbolic components: the fact that hyperbolic components must have different combinatorics, and that they have unique centers, is a consequence of Thurston's topological characterization of rational maps, in this case most easily used in the form of the Spider Theorem [HS].

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